# A Characterization of Reversible Markov Processes with Applications to Shared-Resource Environments

**Vassilios Koukoulidis**

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# ABSTRACT

A Characterization of Reversible Markov Processes with Applications to Shared-Resource Environments

Vassilios Koukoulidis, Ph.D.
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The connection between reversible stochastic processes and product-form queueing networks is examined from a graph-theoretic perspective. The result is a characterization of reversibility and a methodological tool called the *state multiplier*. The use of state multipliers, in modeling the state dependencies of arrival and departure rates, is demonstrated both theoretically and by example. Next, multi-server queues with multiple classes of customers and general service requirements are considered. Using the state multipliers and assuming work conservation, a non-egalitarian processor-sharing discipline is analyzed. This discipline is called the *extended shared-resource (ESR) model* and has a product-form solution under finite or infinite queue sizes. Applying a complete analysis, a computationally efficient algorithm for the normalization constant, the moments of the population and the blocking probabilities is derived. Finally, state multipliers and work conservation are used in the analysis of circuit-switched networks and further generalizations are suggested.

*TO GINA*

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# Contents

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# List of Symbols

Following is a list of the most important symbols and the page numbers of their definition or first reference.

$a(s)$, allocation distribution, 60

$\boldsymbol{b} \stackrel{\text{def}}{=} (b_1, b_2, \ldots, b_R)$, 54

$\chi(\cdot)$, characteristic function of a queueing discipline, 18

$D_r$, delay for type $r$ customers, 75

$\varepsilon(\boldsymbol{k}, j) \stackrel{\text{def}}{=} E\{n_1^{k_1} n_2^{k_2} \cdots n_R^{k_R} | j\} q(j)$, 63

$\mathcal{G} = (\Omega, E)$, graph with set of vertices $\Omega$ and set of edges $E$, 7

$\gamma(i|\boldsymbol{n}, r)$, probability that a calss $r$ customer enters station $(i, r)$ given that the population prior to the arrival was $\boldsymbol{n}$, 17

$G(\Omega)$, normalization constant over state-space $\Omega$, also defined as $G(\Omega)^{-1} = \pi(\mathbf{0})$, 19

$\mathcal{I}$, set of boundary states above which processor sharing begins, 59

$\mathbf{I}$, $|\Omega| \times |\Omega|$ identity matrix, 11

$\lambda_r$, arrival rate of class $r$ customers, 18

$(l)_r$, vector with $l$ in the $r$th position and 0 elsewhere, 64

$\mu(i, r)^{-1}$, mean service-time of customer at station $(i, k)$ rate of class $r$ customers, 18

$\mathcal{N} = (U, A)$, network with set of nodes $U$ and set of arcs $A$, 15

$\boldsymbol{n}$, population vector $(n_1, n_2, \ldots, n_R)$, 16

$\boldsymbol{n}_r^+ \stackrel{\text{def}}{=} (n_1, n_2, \ldots, n_{r-1}, n_r + 1, n_{r+1}, \ldots, n_R)$, 17

$\boldsymbol{n}_r^- \stackrel{\text{def}}{=} (n_1, n_2, \ldots, n_{r-1}, n_r - 1, n_{r+1}, \ldots, n_R)$, 17

$\Omega$, state-space of a process or queue, 9

$\mathbf{P}$, stochastic transition probability matrix $[p(i, j)]_{|\Omega| \times |\Omega|}$, 11

$P_{b_r}$, blocking probability for type $r$ customers, 54

$\phi_r(\boldsymbol{n})$, state-dependent factor of bandwidth reduction or service-rate deceleration, 56

$\boldsymbol{\pi}$, row vector $(\pi(i))_{i \in \Omega}$, 11

$\{\pi(i),\ i \in \Omega\}$, equilibrium probability distribution of a process or chain with state-space $\Omega$

$p(i, j)$, transition probability from state $i$ to state $j$, 9

$\Pi(\boldsymbol{z})$, probability generating function for $\pi(\boldsymbol{n})$, 62

$\mathbf{Q}$, stochastic transition rate matrix $[q(i, j)]_{|\Omega| \times |\Omega|}$, 12

$q(i, j)$, probability transition rate from state $i$ to state $j$, 12

$q(j)$, occupancy distribution, 59

$Q(z)$, probability generating function for $q(j)$, 63

$R$, number of customer classes, 16

$\mathbf{R}$, set of real numbers, 12

$\rho(i, j)$, weight of directed edge $(i, j)$, 33

$\mathcal{T}$, spanning tree, 8

$x(i)$, state multiplier for state $i$, 36

$x(\Psi)$, set multiplier associated with set $\Psi$, 37

$\xi(i,r|\boldsymbol{n})$, service rate for the customer at station $(i,r)$ when the population is $\boldsymbol{n}$, 17

$\Xi(r|\boldsymbol{n})$, total service rate for class $r$ customers, 17

$X(k),\ k \in bfZ$, discrete-time Markov chain, 9

$X(t),\ k \in bfR$, continuous-time Markov process, 12

**Z**, set of integers, 9

# Chapter 1

# Introduction

Efficient and effective use of a system's resource is generally accomplished through some means of sharing the available resources among the users of the system. Resource sharing has been a central issue in the design of computer and communications systems since their inception. It is a truism to say that the notion of sharing is generally directed at the most expensive resources and hence, the bottlenecks limiting a system's performance tend to focus around these critical resources. As technological innovation alleviates one problem, making it possible to engineer new systems with higher performance, the sharing of the previously crucial resource may be rendered unnecessary. However, the focus of the bottleneck typically shifts to some other system resources which are now considered crucial for sharing in order to achieve the higher performance objectives. This is clearly exemplified by the computation-communication bottlenecks encountered in computer networks. If communication is slow relative to the speed of computation then distributed processing is *communication bound.* On the other hand, the advent of fast communication can shift the focus of the bottleneck and distributed processing becomes *compute bound.* In both of these cases, sharing of the critical communicating or processing bandwidth is usually the preferred solution and it is, in fact, necessary in order to tune the performance of the system to balance any disparity between the communication and computation speeds inherent in the system. It is evident that efficient and effective sharing of resources is of considerable concern to computer communication network planners and operators.

From a dynamical point-of-view, the *optimal* or, at least, *intelligent* allocation of resources to users and/or processes is a complex problem, requiring sophisticated mathematical models of the system and of the demand for its resources, in order to quantify useful performance measures. Any such model can, of course, only approximate a system's behavior. For a system's engineer, it is sufficient and highly desirable to have a model which is capable of describing the relevant performance measures accurately enough for design tolerances, yet which is relatively easy from a computational perspective. Therein, lies a tradeoff between accuracy and computational simplicity.

The advent of large expensive mainframe computing facilities motivated a great deal of research in the area of computer resource sharing during the 1960s and 70s. Consequently, a rich body of results now exists for the analysis and control of systems' resources. Some of the deepest work in this area concerns the sharing of a single high-performance processor among many users of a mainframe computer. The analysis of such single-resource-sharing systems is well understood with characterizations of their achievable performance which are sufficient to determine optimal control strategies, many of which are now commonly implemented in existing computer operating systems. However, multiple-resource sharing systems are less understood and much more complex than single-resource-sharing systems. This is evident from the lack of a solid theoretical foundation for multiserver queueing systems. In general, queueing systems and networks are so complex that analytical results exist only for very restricted systems and only under certain, rather unrealistic, assumptions. As of yet, no solid characterization of achievable performance exists which would allow systems engineers to devise optimal control strategies for such systems.

The models applied to the analysis and design of resource-sharing systems range from deterministic to stochastic in nature and a wealth of knowledge on the topic can be found in the literature. The fundamental difficulty encountered in studying multiple resource systems is one of complexity. The complexity is introduced due to the dimansionality of the associated models. General queueing network models can typically be solved for their performance behaviors only approximately or, perhaps exactly, but only numerically and for systems of limited scale. It is virtually intractable to solve such models exactly and in a form suitable for further investigation. There is, however, a large class of Markovian queueing network models whose equilibrium state distribution factors into a product-form solution. These network models can be associated with efficient algorithms for the computation of performance measures of interest to the analyst.

The current state-of-the-art in product-form queueing network models of multiple resource sharing systems has culminated in the works of Gordon and Newell [10], Jackson [13, 14], Baskett, Chandy, Muntz and Palacios [2], Kelly [19] and others. These models are appealing from the two perspectives of a desirable model as cited above. Firstly, they represent queueing networks from an initially Markovian framework and they are, therefore, conceptually extensible to a large class of processes. Secondly, if they posses a factoring into a product-form solution for the state distribution in equilibrium, then they are at least *potentially tractable* from a computational point-of view. Emphasis should be placed here on the phrase "potentially tractable", as even with a product-form network, the computation of performance measures may still be extremely difficult, requiring state-space enumeration. However, these networks may posses efficient recursive algorithms for their solution and they are usually amenable to asymptotic analysis for further simplification.

The objective of this thesis is to explore the possibility of developing models of resource-sharing systems which are computationally efficient and yet sufficiently sophisticated to capture the relevant behaviors. The basic underlying approach is that of Markovian queueing theory and the focus is on product-form models.

A fundamental property that results in product-form queueing disciplines is the $M \Rightarrow M$ or quasi-reversibility property [19, 39]. The most important implications of quasi-reversibility are:

- Arrivals and departures from a quasi-reversible queue are Poisson-distributed.
- The equilibrium distribution of the occupancy for certain quasi-reversible queues is insensitive to the distribution of the service requirement.

These properties have lead researchers to study whether it is possible to parameterize the transition rates of a quasi-reversible queue. Several authors have studied the conditions under which quasi-reversibility is satisfied and discovered simple criteria for testing this fundamental property. Furthermore, they have shown that a queueing network of quasi-reversible queues can be described by a reversible process. In this thesis, I present the results of their work and produce a characterization of reversibility by viewing a Markov process from the perspective of graph theory. Then, I apply this characterization on a work-conserving system and produce a non-egalitarian processor-sharing discipline that admits product-form solution. This discipline is useful in the modeling and performance analysis of network problems such as resource allocation and multiplexing.

## 1.1 Overview

First, I examine the topology of the state-transition diagram of a reversible process from a graph-theoretic point of view. An immediate consequence of this approach is that reversibility can be verified by checking whether a well-known criterion, namely Kolmogorov's criterion, is satisfied around the fundamental circuits of the state-transition diagram. Even though this result seems trivial, it has an important consequence: It explains why it is possible to use state multipliers or weights for the modeling of state dependencies of the transition rates. State multipliers or weights have been widely used to model state-dependent arrival and service rates [2, 20, 28] and recently have been shown to be useful in controlling the arrival rate of different types of traffic in ATM networks [34]. I extend their range of applications by using them as a tool for parameterizing arrival and/or departure rates in a manner similar to raising the node potential in an electrical network. I extensively discuss the theoretical aspects of this technique and explain it with examples. Finally, I summarize the technique in a simple theorem called the Characterization Theorem.

Next, I consider the notion of work conservation for multi-server queues with multiple classes of customers. The setting is as follows. An arriving customer may request more than one server. Customer requests are granted for as long as there servers available or the sharing policy permits it. Otherwise, customers may have to share the servers. In this context, work conservation means that no server is idle when the total number of requests exceeds the number of servers and no customers depart before their service is completed. Using the Characterization Theorem and work conservation, I show how a non-egalitarian processor-sharing discipline can be derived. This discipline is called the *extended shared-resource (ESR) model* and has a product-form solution under finite or infinite queue sizes. A similar model has been proposed by Kaufman [17]. In Kaufman's model processor sharing is not allowed. If a customer's request cannot be met, then this customer is blocked and lost. The ESR model generalizes Kaufman's results in a natural and intuitive way. My analysis results in an efficient computational algorithm for the normalization constant, the moments of the population distribution and the blocking probabilities. I demonstrate the behavior of the ESR model with a simple asymptotic analysis and several numerical examples.

Finally, I apply once more the Characterization Theorem and work conservation, in order to expand the circuit-switched network model of Dziong and Roberts [8]. Here, when the offered load exceeds the capacity of the network, the bandwidth requirement of each traffic type is gradually reduced until all traffic is accommodated.

The analysis of the ESR model has some important implications:

- The infinite-server and processor-sharing disciplines, with multiple customer classes, are unified in a single model.

- It defines a non-egalitarian processor sharing discipline admitting product-form solution.

- As a node in a queueing network, the ESR model expands the class of queueing networks with product-form solution.

- It defines a practical method for approximating non-product-form disciplines or creating new reversible disciplines from known ones.

## 1.2 Plan of the Thesis

In Chapter 2, I present the basic concepts of graph theory, reversibility and queueing networks used in the thesis and give some motivating observations.

In Chapter 3, I show the thinking that lead to the development of the state multipliers, discuss their modeling capabilities and demonstrate their use with examples. Appendix A complements Chapter 3 and presents an interesting analogy between Kirchhoff's voltage law and Kolmogorov's criteria for reversibility.

Chapter 4 contains the major contributions of the thesis. Here, I analyze the extended shared-resource model and produce the algorithms needed to obtain performance measures. The behavior of the ESR model is also examined in Chapter 4. The proofs and algebraic manipulations that are not essential to the development of the model are given in Appendices B and C.

In Chapter 5, I present a further generalization of the ESR model and apply it to the analysis of circuit-switched networks. Closing this chapter, I show some problems raised by the generalization and requiring more research.

Finally, in the Epilogue, I discuss how the ESR model could prove useful in the synthesis of communications networks. This discussion defines a range of potential applications of the ESR model with theoretical and practical value.

# Chapter 2

# Background Material and Previous Work

The sources of theoretical results in performance analysis and modeling are Hayes [11] and Kleinrock [21, 22]. A more descriptive text is Bertsekas and Gallager [3]. Kelly [19] gives a rigorous treatment of reversibility.

The graph-theoretic notation and results used throughout the thesis follow Swamy and Thulasiraman [43].

A detailed presentation of queueing networks and their stochastic processes, accompanied with a rich bibliography, is given by the survey of Disney and König [7].

Some of the theoretical results appearing herein have been published or submitted for publication [4, 24, 25].

## 2.1 Graph Theory

A basic graph-theoretic notation and some elementary results are given next. More advanced material is needed for the proofs in Appendix A but is not necessary for understanding this thesis.

A *graph* $\mathcal{G} = (\Omega, E)$ is a set $\Omega$ of *vertices* and a set $E$ of *edges*, each edge connecting two vertices from $\Omega$. A *walk* $i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_k$ in a graph $\mathcal{G}$ is a finite sequence of vertices $i_1, i_2, \cdots, i_k$, such that $i_l$ and $i_{l+1}$, $1 \leq l \leq k-1$ are connected with an edge. The vertices $i_1$ and $i_k$ are called the *end vertices* of the walk. A walk is *closed* if its end vertices are identical; otherwise it is *open*. A walk is a *trail* if all its edges are distinct. An open trail is a *path* if all its vertices are distinct. A closed trail is a *circuit* if all its vertices except the end vertices are distinct. An *Euler trail* in a graph $\mathcal{G}$ is a closed trail containing all the edges of $\mathcal{G}$. A graph having an Euler trail is called *Eulerian*. A *spanning tree* $\mathcal{T}$ of a graph $\mathcal{G}$ is an acyclic subgraph of $\mathcal{G}$ having all the vertices of $\mathcal{G}$. The edges of $\mathcal{T}$ are called *branches* and the edges of $\mathcal{G} - \mathcal{T}$ are called *chords*. The circuit created by adding a chord $e$ to $\mathcal{T}$ is called the *fundamental circuit* of $\mathcal{G}$ with respect to the chord $e$ of the spanning tree $\mathcal{T}$.

A directed graph has a direction associated with each edge and therefore, each edge of a directed graph corresponds to an ordered pair of vertices. A *directed walk*, *directed trail*, *directed path*, or *directed circuit* is defined as a walk, trail, circuit, and path where the direction of traversal agrees with the direction of the edges. A vertex $v$ in a directed graph $\mathcal{G}$ is a *root* of $\mathcal{G}$ if there are directed paths from $v$ to all the remaining vertices of $\mathcal{G}$. A *directed spanning tree* of a directed graph $\mathcal{G}$ is a spanning tree of $\mathcal{G}$ having a root. The orientation of a fundamental circuit of $\mathcal{G}$ with respect to a chord $e$ of a spanning tree $\mathcal{T}$ agrees with the orientation of $e$.

A directed circuit $C$ of $\mathcal{G} = (\Omega, E)$ can be represented by a vector $[c_i]_{|E|}$, each element $c_i$, $i = 1, \ldots, |E|$, corresponding to an edge of $\mathcal{G}$ as follows. Assume that $C$ has an orientation assigned to it. Then,

$$c_i = \begin{cases} 0, & \text{if the } i\text{th edge is not in } C, \\ 1, & \text{if the orientations of the } i\text{th edge and } C \text{ agree,} \\ -1, & \text{if the orientations of the } i\text{th edge and } C \text{ disagree.} \end{cases} \tag{2.1}$$

The circuits of a graph form a space with rank $|E| - |\Omega| + 1$. The fundamental circuit vectors of a graph $\mathcal{G}$ with respect to the chords of a spanning tree of $\mathcal{G}$ form a basis of the circuit space.

## 2.2 Reversibility

In this section, I introduce and discuss the concept of reversibility. Theorems are given without proofs. However, I present an interpretation influenced by Bertsekas and Gallager [3]. The proofs can be found in Kelly [19].

Let $X(k)$, $k \in \mathbf{Z}$ be a time-homogeneous, stationary, irreducible, aperiodic and discrete-time Markov process with a countable state-space $\Omega$ and state-transition probabilities $p(i,j) = P(X(k+1) = j | X(k) = i)$, $i,j \in \Omega$, $k \in \mathbf{Z}$. In the rest of the thesis, the term Markov chain will be used for discrete-time Markov processes. The word process will be reserved for the continuous time Markov processes. Furthermore, I shall consider only time-homogeneous, irreducible and aperiodic processes, unless it is otherwise specified.

Process $X(k)$ has an equilibrium probability distribution if and only if there exists a set $\{\pi(i),\, i \in \Omega\}$ summing to unity and satisfying the *global balance* equations

$$\pi(i) = \sum_{j \in \Omega} \pi(j) p(j,i), \qquad i \in \Omega. \tag{2.2}$$

Then, the equilibrium probability distribution is $\{\pi(i),\, i \in \Omega\}$.

Assume that $X(k)$ is in equilibrium. Imagine that we trace the sequence of states of $X(k)$ in reverse time. Let $\{\pi_r(i),\, i \in \Omega\}$ and $p_r(i,j)$, $i,j \in \Omega$ be the equilibrium distribution and the transition probabilities of the reversed chain, respectively. Then

$$\begin{aligned} P(X(k) = j, X(k-1) = i) &= P(X(k-1) = i | X(k) = j) P(X(k) = j) \\ &= \pi_r(j) p_r(j,i). \end{aligned} \tag{2.3}$$

We also have

$$\begin{aligned} P(X(k) = j, X(k-1) = i) &= P(X(k) = j | X(k-1) = i) P(X(k) = i) \\ &= \pi(i) p(i,j). \end{aligned} \tag{2.4}$$

Since the chain is in equilibrium ($t \to \infty$), equilibrium probabilities equal time averages or, equivalently, $X(k)$ is *ergodic*. The concept of ergodicity is often used to provide intuitive interpretations of theoretical results. Then, $\pi(j)$ is the proportion of time the chain spends in state $j$. Hence,

$$\pi_r(j) = \pi(j). \tag{2.5}$$

Then, from equations (2.3), (2.4) and (2.5) we get

$$\pi(j)p_r(j,i) = \pi(i)p(i,j), \qquad i,j \in \Omega, \tag{2.6}$$

and the reverse transition probabilities are given by

$$p_r(j,i) = \frac{\pi(i)p(i,j)}{\pi(j)}, \qquad i,j \in \Omega. \tag{2.7}$$

If $p_r(j,i) = p(j,i), \forall i,j \in \Omega$, the chain is called *time-reversible*. So, for a reversible chain, equation (2.6) becomes

$$\pi(j)p(j,i) = \pi(i)p(i,j), \qquad i,j \in \Omega, \tag{2.8}$$

and we can say that the chain satisfies *detailed balance*.

The ergodicity of equilibrium distribution can help us derive an intuitive interpretation of detailed balance and reversibility. Since $\pi(i)$ is the fraction of time spent by $X(k)$ in state $i$ and $p(i,j)$ is the fraction of transitions from state $i$ to state $j$ given that $X(k)$ is in state $i$, the quantity $\pi(i)p(i,j)$ is the fraction of transitions from state $i$ to state $j$. Similarly, $\pi(j)p(j,i)$ is the fraction of transitions from state $j$ to state $i$. Hence, if detailed balanced is satisfied for every pair of adjacent states, the forward and reverse chains are indistinguishable. The following definition and Theorem 2.1 formally state the connection between reversibility and detailed balance.

**Definition 2.1 (Reversible Chain)** *A stationary chain $X(k)$, $k \in \mathbf{Z}$, is reversible if $(X(k_1), X(k_2), \ldots, X(k_n))$ has the same equilibrium distribution as $(X(k - k_1), X(k-k_2), \ldots, X(k-k_n))$, $\forall k_1, k_2, \ldots, k_n, k \in \mathbf{Z}$.*

**Theorem 2.1** *A stationary Markov chain is reversible if and only if there exists a set of positive numbers $\{\pi(i), i \in \Omega\}$, summing up to unity, that satisfy the detailed balance equations*

$$\pi(j)p(j,i) = \pi(i)p(i,j), \qquad i,j \in \Omega.$$

*When the set $\{\pi(i), i \in \Omega\}$ exists, it is the equilibrium distribution of the chain.*

The equilibrium distribution of a chain can be derived from the global balance equations and is the solution of the equation

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}, \tag{2.9}$$

where $\boldsymbol{\pi}$ is the row vector $(\pi(i))_{i\in\Omega}$, $\mathbf{I}$ is the $|\Omega| \times |\Omega|$ identity matrix, and $\mathbf{P}$ is the transition probability matrix $[p(i,j)]_{|\Omega|\times|\Omega|}$ with $p(i,i) = 1 - \sum_{j\in\Omega-\{i\}} p(i,j)$, $i \in \Omega$, and $p(i,j)$ being the transition probabilities. Consequently, we should be able to determine whether a chain is reversible from the transition probabilities alone. Kolmogorov's criteria, which are presented in the sequel, let us to do exactly that.

Let $C = (i_1, i_2, \ldots, i_k, i_1)$, be a finite sequence of adjacent states in $\Omega$. Without loss of generality, it is assumed that the length of $C$ is $k > 2$. If the chain is in state $i_1$, it will traverse $C$ with probability

$$p(i_1,i_2)p(i_2,i_3)\cdots p(i_{k-1},i_k)p(i_k,i_1). \tag{2.10}$$

Suppose that, while $X(k)$ is in equilibrium, we trace the sequence of states backwards in time. If the reversed chain is in state $i_1$, it will traverse $C$ backwards with probability

$$p_r(i_1,i_k)p_r(i_k,i_{k-1})\cdots p_r(i_3,i_2)p_r(i_2,i_1). \tag{2.11}$$

If $X(k)$ is reversible, expressions (2.10) and (2.11) are equal. Conversely, assume that

$$p(i_1,i_2)p(i_2,i_3)\cdots p(i_{k-1},i_k)p(i_k,i_1) = p(i_1,i_k)p(i_k,i_{k-1})\cdots p(i_3,i_2)p(i_2,i_1),$$

for any finite sequence $C = (i_1, i_2, \ldots, i_k, i_1)$ of adjacent states in $\Omega$. Then, the probability that $X(k)$ traverses circuit $C$ starting from state $i_1$ is independent of the direction of traversal. Hence, there is no net circulation which would make the forward chain distinguishable from the reversed chain. The relationship between reversibility and the transition probabilities around any closed finite sequence of states is formally established by the following theorem.

**Theorem 2.2 (Kolmogorov's criterion)** *A stationary Markov chain is reversible if and only if its transition probabilities satisfy*

$$p(i_1,i_2)p(i_2,i_3)\cdots p(i_{k-1},i_k)p(i_k,i_1) = p(i_1,i_k)p(i_k,i_{k-1})\cdots p(i_3,i_2)p(i_2,i_1), \quad (2.12)$$

*for any finite sequence of states* $i_1, i_2, \ldots, i_k \in \Omega$.

The previous presentation of Markov chains and reversibility can be extended to Markov processes. Let $X(t)$, $t \geq 0$, be a time-homogeneous, stationary, irreducible and aperiodic Markov process with a countable state-space $\Omega$. The necessary and sufficient condition that $X(t)$ has an equilibrium probability distribution $\{\pi(i), i \in \Omega\}$ is that the *global balance* equations

$$\pi(i)\sum_{j\in\Omega} q(i,j) = \sum_{j\in\Omega} \pi(j)q(j,i), \qquad i \in \Omega, \quad (2.13)$$

are satisfied and $\sum_{i\in\Omega}\pi(i) = 1$. Now, the transition probabilities are replaced by the transition rates $q(i,j)$, $i,j \in \Omega$, and equation (2.9) becomes

$$\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}, \quad (2.14)$$

where $\boldsymbol{\pi}$ is the row vector $(\pi(i))_{i\in\Omega}$ and $\mathbf{Q}$ is the transition rate matrix $[q(i,j)]_{|\Omega|\times|\Omega|}$ with $q(i,i) = -\sum_{j\in\Omega-\{i\}} q(i,j)$, $i \in \Omega$, and $q(i,j)$ being the transition rates. The rest of this work deals only with continuous time Markov processes. Definition 2.1 and Theorems 2.1 and 2.2 have analogues for continuous time Markov processes.

**Definition 2.2 (Reversible Process)** *A stationary process* $X(t)$, $t \in \mathbf{R}$, *is reversible if* $(X(t_1), X(t_2), \ldots, X(t_n))$ *has the same equilibrium distribution as* $(X(\tau -$

$t_1), X(\tau - t_2), \ldots, X(\tau - t_n)), \forall t_1, t_2, \ldots, t_n, \tau \geq 0.$

**Theorem 2.3** *A stationary Markov process is reversible if and only if there exists a set of positive numbers $\{\pi(i),\ i \in \Omega\}$, summing up to unity, that satisfy the detailed balance equations*

$$\pi(j)q(j,i) = \pi(i)q(i,j), \qquad i,j \in \Omega. \tag{2.15}$$

*When the set $\{\pi(i),\ i \in \Omega\}$ exists, it is the equilibrium distribution of the process.*

The quantity $\pi(i)q(i,j)$ is called the *probability flux* from state $i$ to state $j$ since it represents the fraction of transitions from state $i$ to state $j$ per time unit.

**Theorem 2.4 (Kolmogorov's criterion)** *A stationary Markov process is reversible if and only if its transition rates satisfy*

$$q(i_1,i_2)q(i_2,i_3)\cdots q(i_{k-1},i_k)q(i_k,i_1) = q(i_1,i_k)q(i_k,i_{k-1})\cdots q(i_3,i_2)q(i_2,i_1). \tag{2.16}$$

*for any finite sequence of states $i_1, i_2, \ldots, i_k \in \Omega$.*

A Markov process can be associated with an undirected graph $\mathcal{G} = (\Omega, E)$ whose set of vertices is $\Omega$, the state-space, and $E$ is a set of edges, each edge joining two vertices, say $i$ and $j$, if there is a positive transition probability from $i$ to $j$ or from $j$ to $i$. Irreducibility guarantees that the graph is connected. Then, the process can be viewed as a random walk on $\mathcal{G}$.

Consider a cut $\langle \Psi, \Omega - \Psi \rangle$ of $\mathcal{G}$, where $\Psi \subseteq \Omega$. During any time interval $(t, t+\tau]$, the number of transitions from $\Psi$ to $\Omega - \Psi$ differs from the number of transitions from $\Omega - \Psi$ to $\Psi$ by at most 1. So, in equilibrium, the fraction of transitions from $\Psi$ to $\Omega - \Psi$ is equal to the fraction of transitions from $\Omega - \Psi$ to $\Psi$. Hence, the following is proved.

**Lemma 2.5** *For a stationary Markov process, the total probability flux in one direction across a cut equals the total probability flux in the opposite direction. That is, for any $\Psi \subset \Omega$*

$$\sum_{i\in\Psi} \sum_{j\in\Omega-\Psi} \pi(i)q(i,j) = \sum_{i\in\Psi} \sum_{j\in\Omega-\Psi} \pi(j)q(j,i). \tag{2.17}$$

Lemma 2.5 implies that if the graph associated with a process is a tree, then balance of probability flux across a cut is equivalent to detailed balance. This proves lemma 2.6 and provides a sufficient condition for reversibility.

**Lemma 2.6** *If the graph associated with a Markov process is a tree, then the process is reversible.*

## 2.3 Queueing Networks

Next, I present the concept of queueing networks within the framework defined by the BCMP theorem and its characterizations and extensions. For traditional purposes, I shall call this framework *the BCMP framework*. In the original work of Baskett et al. [2], which extended the models of Jackson [13, 14] and Gordon and Newell [10], the BCMP framework included networks with four types of nodes. Each type is defined by the service discipline of the service center associated with the node:

1. First-Come-First-Served (FCFS)

2. Processor Sharing (PS)

3. Infinite Server (IS)

4. Last-Come-First-Served Preemptive-Resume (LCFS-PR)

The advantage of these service disciplines is that the equilibrium state probabilities of the network could be expressed as the product of the equilibrium state probabilities of the nodes multiplied by a normalization constant. The class of service disciplines yielding product-form probabilities was characterized by Kelly [19, 20] and Chandy et al. [37, 38] and further expanded by Noetzel [40] and Le Boudec [32, 33].

In Chapter 4, I shall analyze a queueing discipline that can be included in the BCMP framework. The rest of this section presents the formalism needed to prove that such inclusion satisfies the criteria for product-form equilibrium state probabilities.

### 2.3.1 Preliminaries

A *network*, $\mathcal{N} = (U, A)$, is a directed graph with a finite nonempty set $U$ of vertices, called *nodes*, and a set $A$ of edges, called *arcs*. The number of nodes is $|U|$ and nodes are numbered so that $U = \{1, 2, \ldots, |U|\}$. A *queueing network* is a network with a service center associated with each node, where customers arrive, receive service and flow over the arcs without delay. Customers can be of $K < \infty$ different *types*. A queueing network is said to be *open* if all the customers arriving at a node follow a directed walk of finite length through the network and then depart. A queueing network is said to be *closed* if there are no arrivals and no departures but only a fixed number of customers circulating in the network. If a network is open for some types of customers but closed for others, is called *mixed*. In order to unify open, closed and mixed queueing networks define a special node $\Delta$ is defined. All arrivals originate from $\Delta$, which is then called *source*, and all departures end to $\Delta$, then called *sink*.

Now, I introduce the notation needed to describe the circulation of customers in the network. A customer may change its type when it moves from one node to the next. Let $B = \{1, 2, \ldots, K\}$ be the set of all types and $B_k$ be the set of types that a customer of type $k$ may assume while moving through the network. Clearly, $B_1 \cup B_2 \cup \cdots \cup B_K = B$ but the sets $B_k$, $k = 1, 2, \ldots, K$, do not necessarily form a partition of $B$. A customer of type $k$ who is visiting node $i$ as type $r$ is said to be *in class* $r$. The probability that a type $k$ customer in class $r$ at node $i$ visits, upon completion of its service, node $j$ in class $q$, is $p_k(i, r; j, q)$. The transition probabilities $p_k(i, r; j, q)$ are called *routing probabilities* and form a stochastic matrix

$$\mathbf{P}_r = [p_k(i, r; j, q)], \qquad 1 \leq i, j \leq K, r \in B_{ik}, q \in B_{jk},$$

where $B_{ik}$ is the set of classes that a type $k$ customer may belong to while in node $i$. Matrix $\mathbf{P}_r$ is known as the *routing chain* for type $k$ customers.

In the sequel, I deal with a particular class of queueing networks called *product-form queueing networks*. In order to analyze such networks, it suffices to study the queueing behavior at each node separately. The presentation of concepts follows

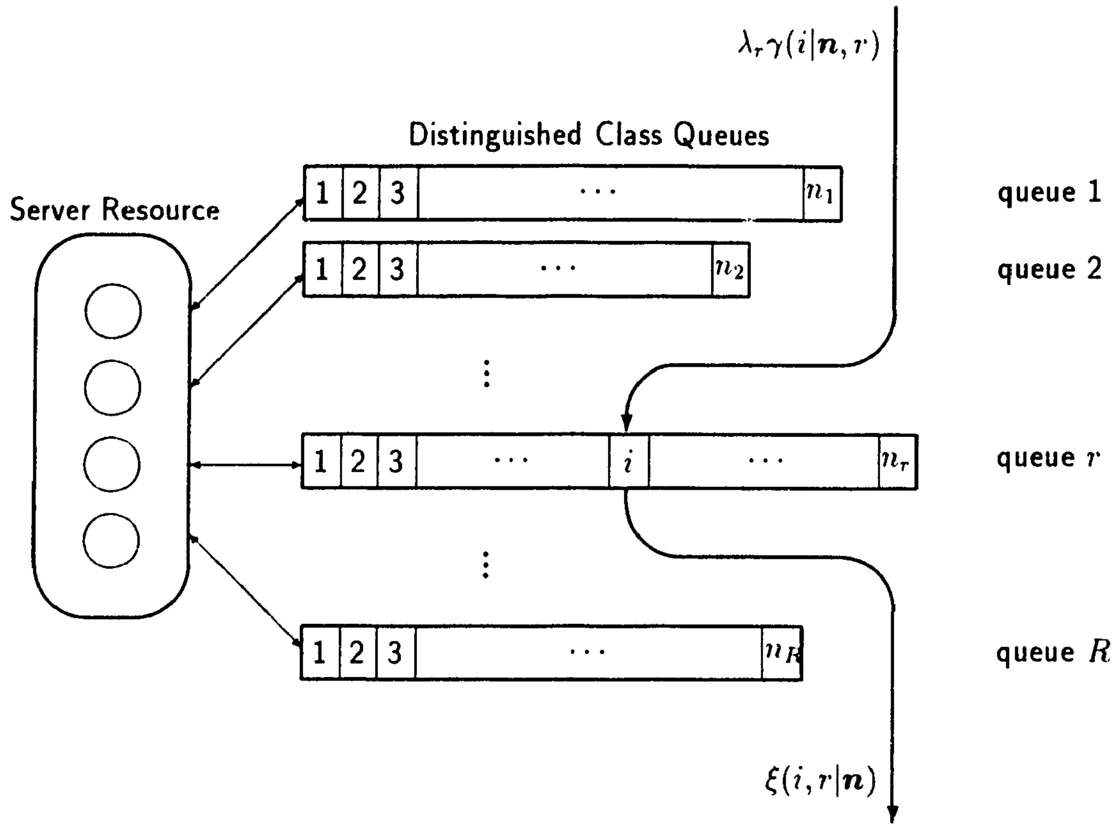



Figure 2.1: A service center as a set of distinguished class queues.

closely Chandy and Martin [38]. Only the notation has been changed in order to maintain consistency with the rest of this thesis.

### 2.3.2 Queueing at a Node

Assume that customers who visit node $i$ can be in $R$ distinguished classes. Without loss of generality, assume that there is a queue for each distinguished class. Each queue consists of stations and each station is occupied by at most one customer (Figure 2.1). The $i$th station of the $r$th queue is referred to as $(i, r)$. If there are $n_r$ customers in queue $r$, they occupy stations $(1, r), (2, r), \ldots, (n_r, r)$. The population of the service center is defined as

$$\boldsymbol{n} \stackrel{\text{def}}{=} (n_1, n_2, \ldots, n_R).$$

Let $\gamma(i|\boldsymbol{n},r)$ be the probability that an arriving customer enters station $(i,r)$ given that the population prior to the arrival was $\boldsymbol{n}$ and the customer is in class $r$. Furthermore, let $\xi(i,r|\boldsymbol{n})$ be the service rate for the customer at station $(i,r)$ when the population is $\boldsymbol{n}$. Then, given that the population is $\boldsymbol{n}$ the total service rate for customers in distinguished class $r$ is

$$\Xi(r|\boldsymbol{n}) = \sum_{i=1}^{n_r} \xi(i,r|\boldsymbol{n}).$$

The following notation is needed.

$$\begin{aligned} \boldsymbol{n}_r^- &\stackrel{\text{def}}{=} (n_1, n_2, \ldots, n_{r-1}, n_r - 1, n_{r+1}, \ldots, n_R), \\ \boldsymbol{n}_r^+ &\stackrel{\text{def}}{=} (n_1, n_2, \ldots, n_{r-1}, n_r + 1, n_{r+1}, \ldots, n_R). \end{aligned}$$

The fundamental property that results in BCMP-type queueing disciplines is the $M \Rightarrow M$ property. It was first identified by Muntz [39] and termed *quasi-reversibility* by Kelly [19]. The implications of quasi-reversibility are summarized as follows [20].

1. The state of a quasi-reversible queue at any time $t$, is independent of the future arrival times and past departure times.

2. The arrival times of class $r$ customers form independent Poisson processes.

3. The departure times of class $r$ customers form independent Poisson processes.

4. *Insensitivity property.* Assume that the queue is balancing or symmetric in the sense that, the relative frequency at which a class $r$ customer visits station $(i,r)$ is equal to the proportion of service allocated to this station. Then, the equilibrium distribution of the population depends on the the service requirement of a class only through its mean.

The following definitions and theorems are based on quasi-reversibility and define formally the BCMP framework.

**Definition 2.3 (Balanced)** *A queueing discipline is called* balanced *if there exists a positive function* $\chi(\boldsymbol{n})$, *defined over all feasible vectors* $\boldsymbol{n}$, *such that*

$$\Xi(r|\boldsymbol{n}) = \frac{\chi(\boldsymbol{n}_r^-)}{\chi(\boldsymbol{n})}, \qquad \forall \boldsymbol{n}, r : n_r > 0.$$

*Function* $\chi(\boldsymbol{n})$ *is called the* characteristic function *of the discipline and* $\chi(\mathbf{0}) = 1$.

**Definition 2.4 (Station Balancing)** *A queueing discipline is called* station balancing *for distinguished class* $r$ *if it is balanced and*

$$\Xi(r|\boldsymbol{n})\gamma(i|\boldsymbol{n}, r) = \xi(i, r|\boldsymbol{n}), \qquad \forall \boldsymbol{n}, i : n_r > 0 \wedge i = 1, 2, \ldots, n_r.$$

*A station balancing discipline is also called* symmetric *[19].*

Lemma 2.7 gives a simple criterion for the balanced property.

**Lemma 2.7** *A queueing discipline is balanced if and only if*

$$\Xi(r|\boldsymbol{n})\Xi(q|\boldsymbol{n}_r^-) = \Xi(q|\boldsymbol{n})\Xi(r|\boldsymbol{n}_q^-),$$

*for all feasible* $\boldsymbol{n}$, $\boldsymbol{n}_r^-$, $\boldsymbol{n}_q^-$.

**Corollary 2.8** *The characteristic function of a balanced discipline is unique.*

Next, I give a version of the product-form theorem for the equilibrium probabilities of the population process. The product-form theorem has a more general form and holds for systems whose state descriptor includes the remaining service time of each customer. However, in this thesis, as in most practical applications, it suffices to know the equilibrium distribution of the population only.

**Theorem 2.9 (Product-Form Theorem)** *Let the following assumptions be true for a queueing network node.*

1. *The service discipline depends only on the population vector* $\boldsymbol{n}$.
2. *The process of arrivals of class* $r$ *customers is Poisson with mean* $\lambda_r$.

3. *The routing probabilities in the queueing network are constant and independent of system population.*

4. *The service-time distribution of the customer at station $(i,k)$ is differentiable and its mean is $\mu(i,r)^{-1}$.*

*Then, the equilibrium probabilities $\pi(\boldsymbol{n})$ of the population process satisfy the product-form equation*

$$\pi(\boldsymbol{n}) = G(\Omega)^{-1}\chi(\boldsymbol{n})\prod_{r=1}^{R}\prod_{i=1}^{n_r}\frac{\lambda_r\gamma(i|\boldsymbol{n},r)}{\mu(i,r)}, \tag{2.18}$$

*where $G(\Omega)$ is the normalization constant, if and only if the queueing discipline is balanced and for a distinguished class $r$*

(i) *the service times of all customers of this distinguished class have the same exponential density, or*

(ii) *the queueing discipline is station balancing for distinguished class $r$.*

**Theorem 2.10 (Preservation Theorem)** *Let $\mathcal{D}_j$, $j = 1,2$, be a queueing discipline with distinguished classes $1,2,\ldots,R_j$, population $\boldsymbol{n}_j$ and parameters $\gamma(i|\boldsymbol{n}_j,r)$ and $\xi(i,r|\boldsymbol{n}_j)$. Let $\mathcal{D}$ be a queueing discipline obtained by combining $\mathcal{D}_1$ and $\mathcal{D}_2$ so that the population of $\mathcal{D}$ is $\boldsymbol{n} = (\boldsymbol{n}_1,\boldsymbol{n}_2)$,*

$$\gamma(i|\boldsymbol{n},r) \;=\; \begin{cases} \gamma(i|\boldsymbol{n}_1,r), & \text{for } r = 1,2,\ldots,R_1, \\ \gamma(i|\boldsymbol{n}_2,r-R_1), & \text{for } r = R_1+1,\ldots,R_1+R_2, \end{cases}$$

*and*

$$\xi(i,r|\boldsymbol{n}) \;=\; \begin{cases} \xi(i,r|\boldsymbol{n}_1), & \text{for } r = 1,2,\ldots,R_1, \\ \xi(i,r-R_1|\boldsymbol{n}_2), & \text{for } r = R_1+1,\ldots,R_1+R_2. \end{cases}$$

*Then $\mathcal{D}$ is balanced if and only if $\mathcal{D}_1$ and $\mathcal{D}_2$ are both balanced. The characteristic function of $\mathcal{D}$ is the product of the characteristic functions of $\mathcal{D}_1$ and $\mathcal{D}_2$. Furthermore, if $\mathcal{D}$ is balanced and $\mathcal{D}_1$ or $\mathcal{D}_2$ is station balancing for a distinguished class, then $\mathcal{D}$ is station balancing for the corresponding distinguished class.*

The equilibrium state probabilities at each node yield the equilibrium state probabilities of the queueing network in the way shown by the next theorem. The queueing-network state considered is the population vector

$$\mathbf{n} \stackrel{\mathrm{def}}{=} (\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots, \boldsymbol{n}_{|U|}),$$

where $\boldsymbol{n}_i$, is the population vector of node $i$.

**Theorem 2.11** *Assume that the queueing discipline at each node satisfies the assumptions of Theorem 2.9 or can be presented as limits of a sequence of queueing disciplines satisfying these assumptions. Then the equilibrium state probabilities* $\pi(\mathbf{n})$ *of the network have the product form*

$$\pi(\mathbf{n}) = G(\Omega)^{-1} \prod_{i \in U'} \pi_i(\boldsymbol{n}_i), \tag{2.19}$$

*where* $U' = U$ *for closed networks,* $U' = U \cup \{\Delta\}$ *for open and mixed networks,* $\pi_i(\boldsymbol{n}_i)$ *are the equilibrium state probabilities at node* $i$ *assuming Poisson input and* $G(\Omega)$ *is the normalization constant.*

**Comment.** It is obvious that the population distribution can be easily obtained if the characteristic function can be determined. However, this is not trivial. The main difficulty is the size of the state space associated with the population vector. The next chapter shows a technique that helps to determine the characteristic function of a queueing discipline. This technique is based on the fact that the population process of a balanced discipline is reversible. Additionally, reversible processes can be combined in order to yield a reversible process whose equilibrium distribution can be easily derived from the component processes. This is analogous to the preservation theorem. Our result is more general because it allows us to modulate the arrival process while maintaining the balanced property. ■

## 2.4 Motivation

The goal of this section is to present informally the observations that lead this work to the study of reversibility.

First, the graph-theoretic framework for treating the state transition diagram of a queueing network is presented. This framework has been sketched by previous authors and is outlined next. Lazar [30, 31], Robertazzi [41], and Wang [44] have discovered that the state transition diagram of a product-form queueing network can be decomposed into elementary subgraphs. Conversely, they also showed that local balance in a queueing network leads to graphs whose geometric replication forms the state transition diagram. These subgraphs can be further simplified thus producing a building block or cell of the state transition diagram. I explain the concept of building blocks by means of an example and give the basic theorems that characterize this concept. Next, I show that the probability flux around an elementary subgraph can be expressed by a flow graph and provide an algorithm that solves the local balance equations corresponding to such a subgraph. The result is a building block whose geometric replication derives a reversible process. So far, there is no graph-theoretic method for identifying cyclic flows in a queueing network and their corresponding balance equations.

The relationship between reversibility and product-form solution is well-known in the queueing network literature [38, 5, 7, 19, 37]. Conway and Georganas [5], for example, show how a reversible process, equivalent to a closed BCMP network, can be constructed. The approach presented in this section is not concerned with the type of queueing disciplines comprising the network, but rather with the relationship between probability flows and product-form solution.

In the context of our research, the decomposition of the state transition diagram, resulting from the independent cyclic flows, has two advantages:

1. It facilitates the analytic solution of a product form network.

2. It results in a reversible process whose solution is the marginal equilibrium distribution of the queueing network.

The reason for seeking an equivalent reversible process is justified in Section 3.1. There, a practical tool, the state multiplier, is developed and used for transforming a reversible process. Using the state multipliers, we can create new reversible systems from known ones or approximate non-product-form networks with reversible processes. The advantage of using state multipliers is that they allow a gradual modification of the initial solution until the desired level of approximation has been reached.

### 2.4.1 The Geometry of the Transition Diagram of Product-Form Markov Processes

Consider the closed queueing network in Figure 2.2. The state of the network is described by $(i, j, k)$ where $i$, $j$ and $k$ denote the numbers of customers in queues 1, 2 and 3, respectively. There is only one class of customers. The service time of a customer is exponentially distributed and each server has a rate $\mu_i$, $i = 1, 2, 3$. After departing from server $i$, a customer is routed to server $j$ with probability $p_{ij}$. Generally, if local balance is satisfied, the probability flux into a state due to the arrival of a customer equals the probability flux out of this state due to a departure of the same type of customer. Let us examine a customer arriving at queue 3 and the corresponding probability flux into state $(i, j, k)$ of the state transition diagram (Figure 2.3). An arrival to queue 3 could come from queue 1 or queue 2, two events corresponding to transitions

$$(i+1, j, k-1) \rightarrow (i, j, k) \text{ and } (i, j+1, k-1) \rightarrow (i, j, k)$$

and bringing into state $(i, j, k)$ an amount of flux equal to

$$\pi(i+1, j, k-1)\mu_1 p_{13} + \pi(i, j+1, k-1)\mu_2 p_{23}.$$

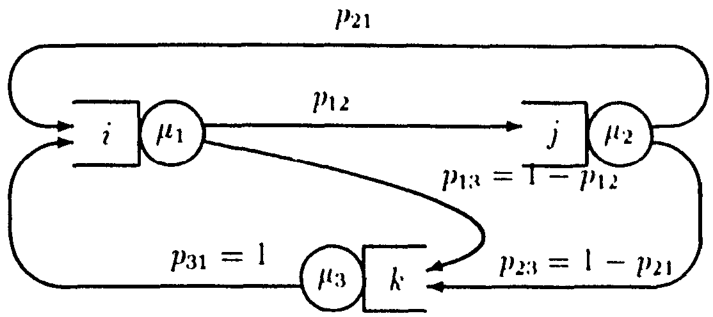



Figure 2.2: A closed queueing network.

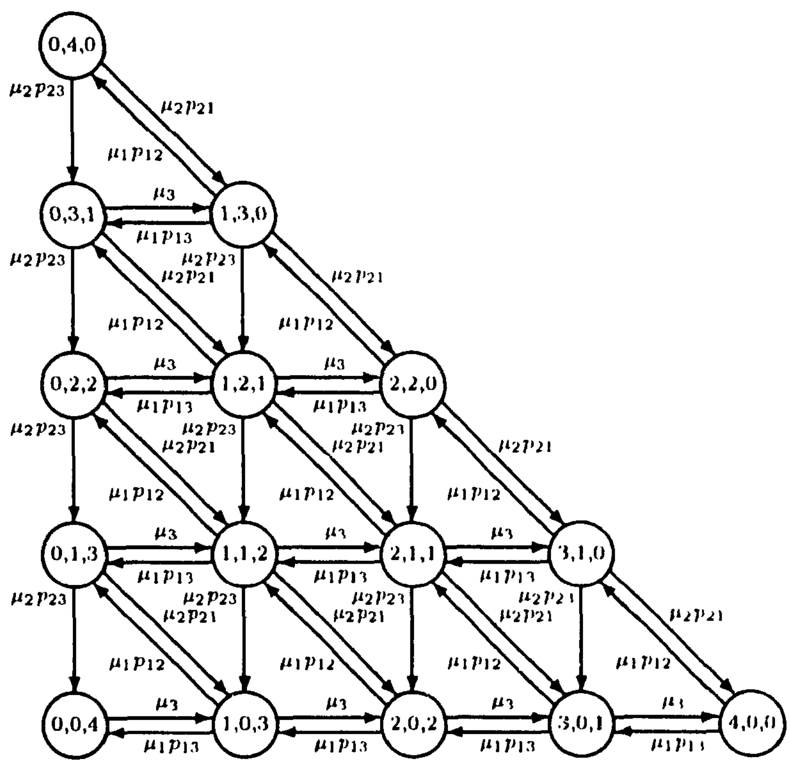



Figure 2.3: The state transition diagram for the network in Figure 2.2 with $N = 4$ customers.

When a customer departs from queue 3 the flux of state $(i, j, k)$ is reduced by the same amount. Then,

$$\pi(i+1, j, k-1)\mu_1 p_{13} + \pi(i, j+1, k-1)\mu_2 p_{23} = \pi(i, j, k)\mu_3. \qquad (2.20)$$

Now this flux is directed towards state $(i+1, j, k-1)$ and corresponds to a customer leaving queue 3 and going to queue 1. Local balance at queue 1 means

$$\pi(i, j+1, k-1)\mu_2 p_{21} + \pi(i, j, k)\mu_3 = \pi(i+1, j, k-1)\mu_1. \qquad (2.21)$$

A departure from queue 1 can either lead to queue 3, where local balance is described by equation (2.20), or to queue 2, where local balance implies that

$$\pi(i+1, j, k-1)\mu_1 p_{12} = \pi(i, j+1, k-1)\mu_2. \qquad (2.22)$$

Following a departure from queue 2 for any number of transitions, would result in one of the equations (2.20)-(2.22). These three equations define an elementary subgraph whose geometric replication reproduces the state transition diagram. Since equations (2.20)-(2.22) are not linearly independent, we can only solve them to derive the probability of a state with respect to a neighbor state. Such a solution is given in Figure 2.4. The graph associated with this solution is called a *building block* or a *cell*. The graph derived from the geometric replication of a building block is called a *consistency graph* (Figure 2.5). If the following theorem holds, the consistency graph can be used to derive that equilibrium distribution [31].

**Theorem 2.12 (The consistency condition)** *A system of local balance equations is consistent if and only if the product of any circuit of the consistency graph is equal to one.*

Not every circuit has to be checked, as the following theorem shows [31].

**Theorem 2.13** *The minimum number of independent circuits needed to verify the consistency condition is given by Betti's number.*

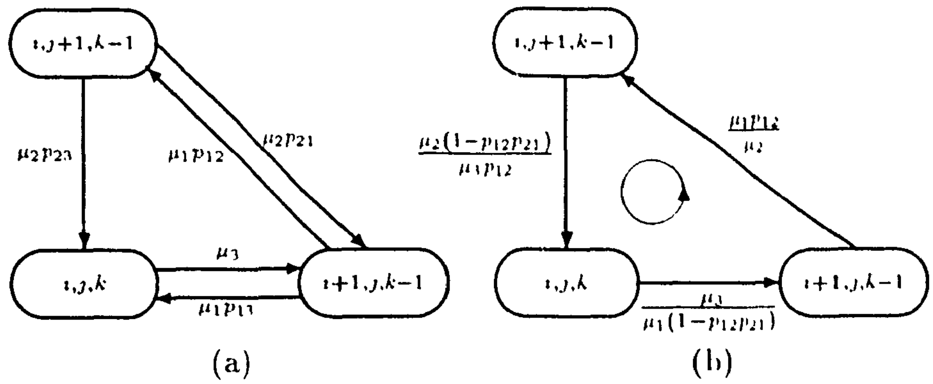



Figure 2.4: (a) A building block of the state transition diagram in Figure 2.3 and (b) the subgraph associated with its solution.

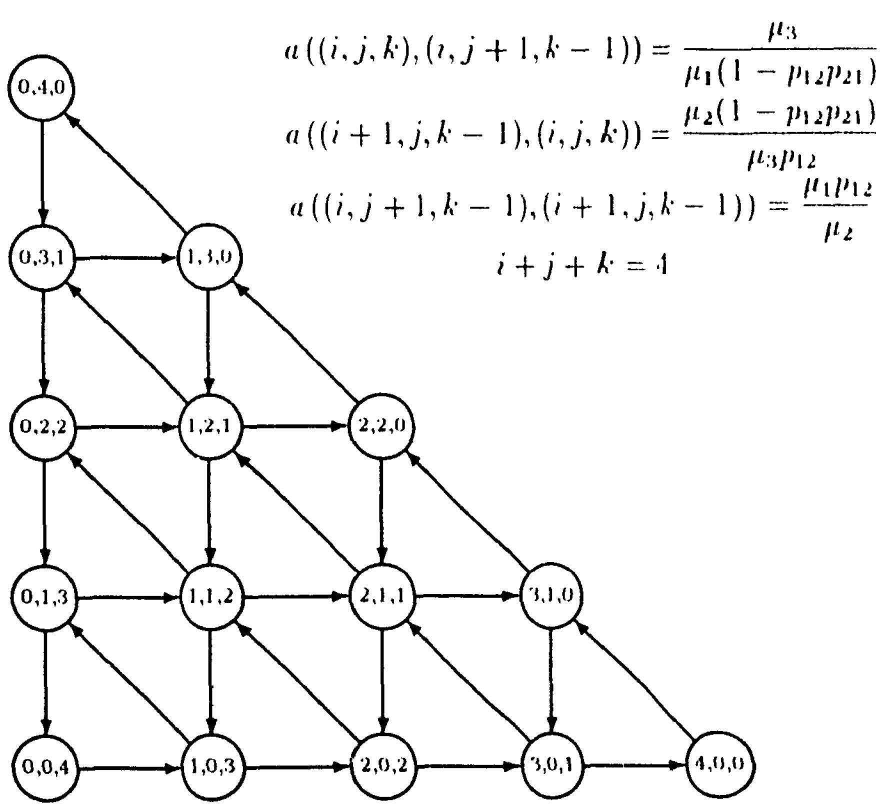



Figure 2.5: The consistency graph of the state transition diagram in Figure 2.3.

Note that Betti's number of a connected graph is $|E| - |\Omega| + 1$ (the nullity of the graph) The relationship between cyclic flows in the state transition diagram and circuits in the queueing network is formally stated in the next theorem [44].

**Theorem 2.14** *Consider a Markovian queueing network with its state described by the total number of customers in each queue and having a consistent set of local balance equations. Then, a circuit of $N$ queues corresponds to a cyclic flow of length $N$ and a path of $N$ queues corresponds to a cyclic flow of length $N+1$.*

Wang and Robertazzi conclude that there is an one to one correspondence between the existence of local balance in the queueing network and the presence of isolated circulations in the state transition diagram [44].

### 2.4.2 Derivation of Building Blocks

The general form of the local balance equations associated with an elementary subgraph is

$$\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}, \tag{2.23}$$

where, $\mathbf{Q}$ is the transition rate matrix of the graph representing the subgraph and $\boldsymbol{\pi}$ is the un-normalized probability distribution of the subgraph vertices with respect to a reference vertex, say $r$. Equation (2.23) is equivalent to

$$\begin{aligned}
& \sum_{i\in\Omega} \pi(i)q(i,j) = 0, \qquad\qquad j \in \Omega, \\
\Leftrightarrow\ & \pi(j)[-q(j,j)] = \sum_{i\in\Omega-\{j\}} \pi(i)q(i,j), \quad j \in \Omega, \\
\Leftrightarrow\ & \left\{ \begin{array}{l} \pi(j) = \displaystyle\sum_{i\in\Omega-\{j\}} \pi(i)\frac{q(i,j)}{-q(j,j)}, \quad j \in \Omega - \{r\}, \\ \pi(r) = \pi(r) \end{array} \right\}.
\end{aligned} \tag{2.24}$$

Consider the state transition graph $\mathcal{G} = (\Omega, E)$ defined by (2.24). Each vertex $i \in \Omega$ has a probability $\pi(i)$. The set $E$ is a set of directed edges $(i,j)$ with weights $a(i,j) = q(i,j)/[-q(j,j)]$. Then, the probability $\pi(i)$ of vertex $i$ makes a contribution of $\pi(i)a(i,j)$ to the probability $\pi(j)$ of vertex $j$. Thus, the graph associated with

(2.24) is a Mason flow graph [43]. Therefore, the solution of (2.24) can be obtained by the following method [43].

*Let $\mathcal{G}$ be the Mason graph associated with a system of linear equations. Then,*

*(a) we can remove the self-loop of weight $a(k,k) \neq 1$ at vertex $k$ simply by multiplying the weight of every edge incident into $k$ by the factor $1/[1 - a(k,k)]$, and*

*(b) we can remove a vertex $p$ with no self-loop by doing the following: for all $i \neq p$ and $k \neq p$, add $a(k,p)a(p,i)$ to the weight of the edge $(k,i)$.*

A question that naturally arises is whether a self-loop removal is always possible. Initially, our Mason graph does not contain any self-loops. After the removal of a vertex $j \in \Omega - \{r\}$, such that $a(j,i)a(i,j) \neq 0$ for an $i \in \Omega - \{r\}$, the resulting graph will contain a self-loop of weight $a(k,i)a(i,j)$ on vertex $i$. Note that, the matrix $\mathbf{A} = [a(i,j)]$, $i,j \in \Omega$ is a transition rate matrix and remains a transition rate matrix after a vertex removal. Additionally, $a(i,i) = 1$, $\forall i \in \Omega$. Then,

$$a(j,i)a(i,j) = p(j,i)p(i,j)$$

where, $p(i,j)$, $i,j \in \Omega$ are the jump chain probabilities. The Markovian property implies that the probability of the process moving to state $i$ from state $j$ and then back to state $i$ is $p(j,i)p(i,j)$. Then, $0 < p(j,i)p(i,j) < 1$, because of the irreducibility assumption, and the self-loop removal is possible.

So, it is proved that a unique solution to (2.23) can be obtained by the algorithm in Figure 2.6. This algorithm is not claimed to be optimal. The optimization or replacement with a more efficient algorithm is presently under study.

The consistency graph of the state transition diagram is produced by the geometric replication of the building blocks resulting from the above algorithm. To prove that, simply note that the probability $\pi(i)$ at any vertex $i$ is always $\pi(i) = \pi(j)a(j,i)$, where $j$ is a vertex adjacent to $i$. Lazar's consistency condition [30] is equivalent to Kolmogorov's criterion for the consistency graph. Thus the original process, which in

**Definitions**

Let $\mathcal{G} = (\Omega, E)$ be a weighted directed graph with edge weights $a(i,j)$.
Define $N^+(v) = \{u | (v,u) \in E \wedge a(v,u) \neq 0\}$ and
$N^-(v) = \{u | (u,v) \in E \wedge a(u,v) \neq 0\}$.

**Input**

A Mason flow graph $\mathcal{G} = (\Omega, E)$ with the following properties:

1. It is quasi-strongly connected and the reference vertex is a root.
2. $a(i,i) \neq 1,\ \forall i \in \Omega - \{r\}$.

**Output**

A row vector $(a(r,i))_{i \in \Omega - \{r\}}$ such that $\pi(i) = \pi(r)a(r,i)$.

**The Algorithm**

*Initialization*

```
V ← N+(r);
Ω ← Ω − V;
do {  remove(V, W);
      Ω ← Ω − W;
      V ← W;
} while (Ω ≠ ∅);
end;
```

*Main routine*

```
remove(V, W)
W ← ∅;
for each i ∈ V do
      for each (i, j) ∈ E : a(i, j) ≠ 0 do {
            if (j ∉ V) W ← W ∪ {j};
            for each k ∈ N−(i) do {
                  a(k, j) ← a(k, j)/[1 − a(j, j)];
                  a(j, j) ← 0;
                  a(k, j) ← a(k, j) + a(k, i)a(i, j);
            }
            a(i, j) ← 0;
      }
return;
```

Figure 2.6: An algorithm for computing the weights of a building block.

general is not reversible, is equivalent to a reversible one. Therefore, the consistency graph can be further reduced to a tree which spans the state space and has the equilibrium distribution of the queueing network.

**Example 2.1 (A Closed Queueing Model of a Multiprocessor Architecture)** Consider the simplified queueing network model of Figure 2.7 (more detailed models of parallel and multiprocessor architectures are given by Hwang and Briggs [12] and Ajmone Marsan, Balbo, and Conte [1]). Two processors service requests at rates $\mu_1$ and $\mu_2$. Local memory requests have probability $\alpha$. Global memory requests are allocated to the appropriate memory by a memory allocator with service rate $\lambda_1$. A global request is directed to the first or second memory with probability $\beta$ or $1-\beta$, respectively. The service rate of each memory is $\nu_i$, $i = 1,2$. Serviced requests can returned to the local processor with probability $\gamma$, or directed to a server with rate $\lambda_2$ (representing the users) with probability $1-\gamma$. Users request processors 1 and 2 with probabilities $\delta$ and $1-\delta$, respectively. The general system state is $(n_1, n_2, n_3, n_4, n_5, n_6)$, where $n_i$ is the number in queue $i$, $i = 1, \ldots, 6$. An elementary subgraph of the state transition diagram and the corresponding building block are given in Figures 2.8 and 2.9, respectively. ■

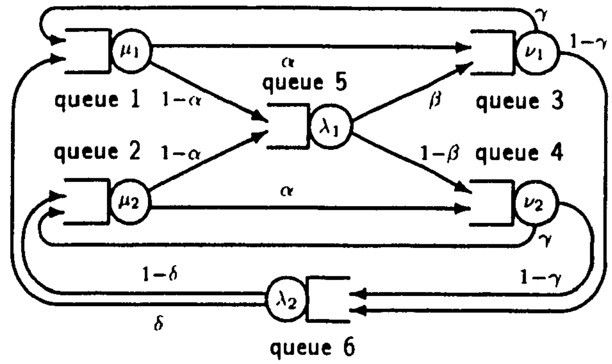



Figure 2.7: A multiprocessor queueing model.

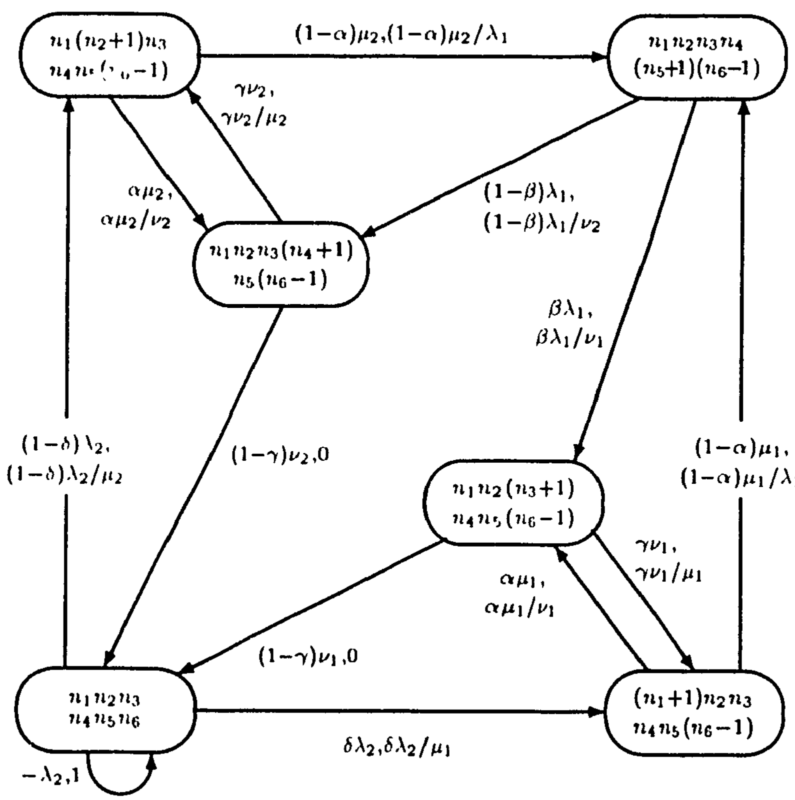


Figure 2.8: A building block of the multiprocessor model in Figure 2.7. The first edge weight is the transition rate and the second one is the weight defined by equations (2.24).

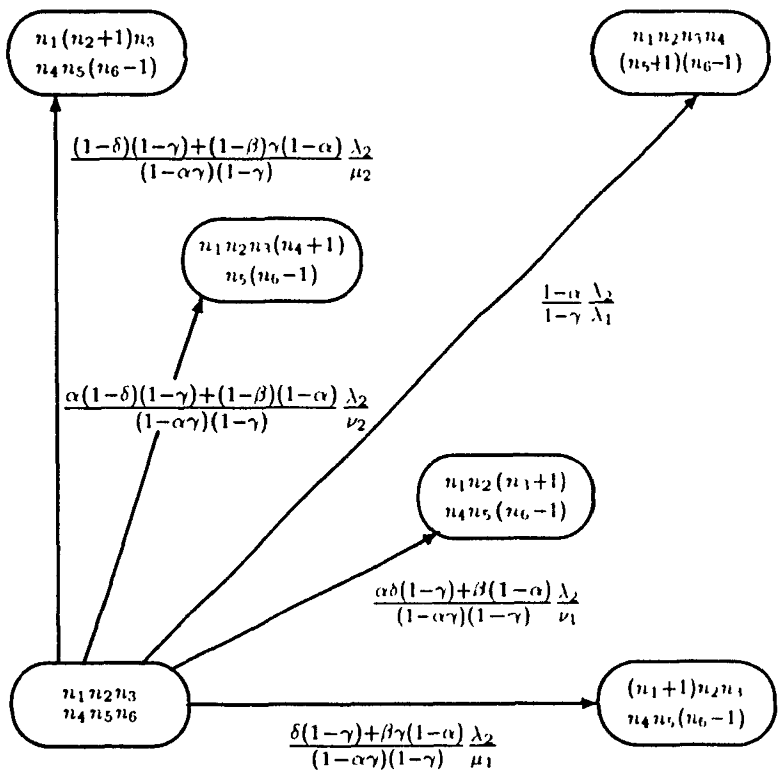



Figure 2.9: The Mason method solution of the building block in Figure 2.8.

# Chapter 3

# Characterization of Reversibility

## 3.1 The Class of Reversible Processes with a Given State Graph

Let $X(t), t \geq 0$, be a time-homogeneous, stationary, irreducible and aperiodic Markov process with a countable state-space $\Omega$ and transition rates $q(i,j)$, $i,j \in \Omega$. Assume that $X(t)$ has an equilibrium probability distribution $\{\pi(i),\ i \in \Omega\}$. Irreducibility implies that $\pi(i) > 0$, $\forall i \in \Omega$. Then, if the process is reversible, from the detailed balance equations we have

$$q(i,j) > 0 \ \Leftrightarrow\ q(j,i) > 0, \qquad i,j \in \Omega. \tag{3.1}$$

Equation (3.1) is a necessary (but not sufficient) condition for a process to be reversible. Hence, in order to establish a characterization of reversibility, we need to consider only the class of processes that satisfy (3.1). Note that, if process $X(t)$ describes an ergodic subchain of a BCMP queueing network, (3.1) is equivalent to Lam's sufficient condition for product-form solution [28].

Let $\mathcal{G}_u = (\Omega, E_u)$ be the undirected graph associated with the process. Then, the process can be fully described by a weighted directed graph, the *state transition graph*, $\mathcal{G} = (\Omega, E)$ defined as follows. Derive $E$ from $E_u$ by assigning an arbitrary orientation to each edge in $E_u$. A directed edge *incident out of* vertex $i$ and *incident* *into* vertex $j$ is denoted by the ordered pair $(i,j)$. For each edge in $E$, define a weight function $\rho(i,j)$ such that

$$\rho(i,j) = \begin{cases} q(i,j)/q(j,i), & \text{if } (i,j) \in E, \\ q(j,i)/q(i,j), & \text{if } (j,i) \in E, \\ 1, & \text{if } i = j. \end{cases} \tag{3.2}$$

Notice that $\mathcal{G}$ has neither parallel nor anti-parallel edges. Equation (3.1) guarantees that, for each pair of adjacent vertices $i$ and $j$, $\rho(i,j)$ and $\rho(j,i)$ are positive. The state transition graph should not be confused with the ***state transition diagram*** of a process. The state transition diagram is a weighted directed graph with adjacency matrix $\mathbf{P}$, if the process is discrete, or $\mathbf{Q}$, if the process is continuous.

The weight function $\rho(i,j)$ will help us understand the intuitive ideas that lead to Characterization Theorem 3.4. One such idea is that Kolmogorov's criterion is analogous to Kirchhoff's voltage law, as shown in Appendix A. The definition of edge weights allows us to provide an alternative statement of Kolmogorov's criterion.

**Theorem 3.1 (Kolmogorov's Criterion)** *A stationary Markov process, with a state transition graph $\mathcal{G}$, is reversible if and only if the product of weights around every closed walk $i_1 \to i_2 \to \cdots \to i_k \to i_1$ in $\mathcal{G}$ is one.*

**Corollary 3.2** *A stationary Markov process is reversible if and only if the equilibrium distribution satisfies the product-form equation*

$$\pi(i_k) = \pi(i_1) \prod_{j=1}^{k-1} \frac{q(i_j, i_{j+1})}{q(i_{j+1}, i_j)} = \pi(i_1) \prod_{j=1}^{k-1} \rho(i_j, i_{j+1}), \tag{3.3}$$

*for any walk $i_1 \to i_2 \to \cdots \to i_k$ in the state graph $\mathcal{G}$.*

**Proof**

*Necessity.* Repeated application of the detailed balance equations along the walk results in equation (3.3).

*Sufficiency.* From equation (3.3) follows that Kolmogorov's criterion is satisfied for any closed walk. **Q.E.D.**

**Theorem 3.3** *A stationary Markov process, with a state transition graph $\mathcal{G}$, is reversible if and only if the product of weights around every fundamental circuit of a spanning tree $T$ of $\mathcal{G}$ is one.*

**Proof**

*Necessity.* Necessity follows from Theorem 3.1.

*Sufficiency.* Let $w$ be a closed walk $i_1 \to i_2 \to \cdots \to i_k \to i_{k+1} \equiv i_1$ in $\mathcal{G}$. Walk $w$ defines a new graph $\mathcal{G}' = (V', E')$. Notice that $\mathcal{G}'$ is not necessarily a subgraph of $\mathcal{G} = (V, E)$, since $\mathcal{G}'$ may contain parallel edges. The direction of each edge in $\mathcal{G}'$ agrees with the direction of traversal of $w$. Then, the weight $\rho'(i,j)$ of a directed edge from vertex $i$ to vertex $j$, in $\mathcal{G}'$ is

$$\rho'(i,j) = \begin{cases} \rho(i,j), & \text{if } (i,j) \in E, \\ 1/\rho(i,j), & \text{otherwise.} \end{cases} \tag{3.4}$$

By construction, $\mathcal{G}'$ is directed Eulerian and therefore, it is the union of some edge-disjoint directed circuits $C_1, C_2, \ldots, C_l$. So, the product of weights around $w$ can be expressed as

$$\prod_{n=1}^{k} \rho(i_n, i_{n+1}) = \prod_{m=1}^{l} \left[ \prod_{(i,j) \in C_m} \rho'(i,j) \right]. \tag{3.5}$$

A directed circuit of $\mathcal{G}'$ can have length at least two. The product around a directed circuit of length two is one. Each directed circuit of length greater than two, corresponds to a circuit in $\mathcal{G}$ and has an orientation that agrees with the direction of the traversal of $w$. Every such circuit can be represented by a vector $C$ in the circuit subspace of $\mathcal{G}$.

A circuit vector $C$ can be expressed as a linear combination of the fundamental circuit vectors $\beta_i$, $i = 1, \ldots, |E| - |\Omega| + 1$, with respect to a spanning tree of $\mathcal{G}$. That is,

$$C = \sum_{i=1}^{|E|-|\Omega|+1} a_i \beta_i, \tag{3.6}$$

where,

$$a_i = \begin{cases} 0, & \text{if } \beta_i \cap C = \emptyset, \\ 1, & \text{if the orientations of } \beta_i \text{ and } C \text{ agree}, \\ -1, & \text{if the orientations of } \beta_i \text{ and } C \text{ disagree}. \end{cases} \tag{3.7}$$

Then,

$$\prod_{(i,j)\in C} \rho(i,j) = \prod_{i=1}^{|E|-|\Omega|+1} \left[ \prod_{(j,k)\in\beta_i} \rho(j,k) \right]^{a_i} = 1, \tag{3.8}$$

where, $(i,j) \in C$ or $(i,j) \in \beta_i$ means that circuit $C$ or $\beta_i$, respectively, has a non-zero entry for edge $(i,j)$.

Thus, the left-hand side of (3.5) is one. Sufficiency now follows from Theorem 3.1. **Q.E.D.**

Theorem 3.3 suggests that reversibility can be verified by checking whether Kolmogorov's criterion is satisfied for each fundamental circuit of a spanning tree $\mathcal{T}$ of $\mathcal{G}$. The number of equations needed is $|E| - |\Omega| + 1$, the nullity of $\mathcal{G}$.

Theorem 3.3 allows us to generate all the reversible processes associated with the same graph $\mathcal{G} = (\Omega, E)$. Let $\mathcal{T}$ be a spanning tree of $\mathcal{G}$. A reversible process can be derived by assigning $|\Omega| - 1$ arbitrary weights $\rho(i,j)$ to the branches $(i,j)$ of $\mathcal{T}$. The weights of the chords of $\mathcal{T}$ can be uniquely identified using Theorem 3.3. That is, if $C_{(i,j)}$ is the fundamental circuit with respect to chord $(i,j)$ and is arbitrarily oriented, the weight of chord $(i,j)$ is

$$\rho(i,j) = \left[ \prod_{(k,l)\in C_{(i,j)}-(i,j)} \rho(k,l) \right]^{-1}. \tag{3.9}$$

It is assumed that all the subscripts in the above product have the orientation of the circuit. Now, from Corollary 3.2, we can calculate the equilibrium probabilities. There is a unique path $P_{ri}$ in $\mathcal{T}$ from a reference vertex $r$ to every vertex $i$. Then,

$$\pi(i) = \pi(r) \prod_{(k,l)\in P_{ri}} \rho(k,l), \tag{3.10}$$

where the path is traversed towards $i$ and $\pi(r)$ is the normalizing constant.

So, we have derived a reversible Markov process with state transition graph $\mathcal{G}$ and calculated its equilibrium distribution. Now, we can move to a new process with the same state graph by assigning an arbitrary positive multiplier $x(i)$ to each vertex $i$. Let

$$\pi'(i) = G''(\Omega)^{-1}x(i)\pi(i), \qquad i \in \Omega, \tag{3.11}$$

where $G''(\Omega)$ is a normalization constant. Define new edge weights $\rho'(i,j)$, so that detailed balance is satisfied for each edge of of $\mathcal{G}$. Then,

$$\pi'(i)\rho'(i,j) = \pi'(j), \qquad (i,j) \in \mathcal{T}. \tag{3.12}$$

Since detailed balance is satisfied for the original process, we have

$$x(i)\rho'(i,j) = x(j)\rho(i,j), \qquad (i,j) \in \Omega. \tag{3.13}$$

The new chord weights can be chosen to satisfy Kolmogorov's criterion for each fundamental circuit of $\mathcal{T}$. The new equilibrium distribution is $\{\pi'(i), i \in \Omega\}$ with $G''(\Omega)^{-1}$ being the normalization constant.

The transformation of a process to a new one, with the same associated graph, can be done with more than one set of multipliers. Thus, different kinds of state dependencies of the transition rates can be modeled. For example, in a birth-death process, we can model state dependent birth rates with one set of multipliers $\{x(i), i \in \Omega\}$ and state dependent death rates with another set $\{y(i), i \in \Omega\}$. Then, equations (3.11) and (3.13) become

$$\pi'(i) = G''(\Omega)^{-1}x(i)y(i)\pi(i), \qquad i \in \Omega, \tag{3.14}$$

and

$$x(i)y(i)\rho'(i,j) = x(j)y(j)\rho(i,j), \qquad (i,j) \in \mathcal{T}. \tag{3.15}$$

Now, I extend the scope of a state multiplier to a set of states $\Psi$. Define the state multipliers as

$$x(\Psi) = x > 0, \text{ for every state in } \Psi, \tag{3.16}$$

where $x$ is the same for every state in $\Psi$. We call $x(\Psi)$, $\Psi \subseteq \Omega$, the *state multiplier associated with the set* $\Psi$, or simply the *set multiplier* $x(\Psi)$. The state multipliers $x(i)$ are related to the set multipliers through

$$x(i) = \prod_{\{\Psi|\Psi\subseteq\Omega\wedge i\in\Psi\}} x(\Psi). \tag{3.17}$$

If we substitute $x(i)$ according to (3.17), equations (3.11), (3.12), and (3.13) are still valid. Equations (3.11)-(3.15) are also valid when some of the state multipliers are zero. This corresponds to truncating the state space as shown later in this section. So, we can generalize the discussion that lead to equations (3.11) (3.15) with the following theorem. The proof is based on detailed balance, thus making the theorem self-contained.

**Theorem 3.4 (Characterization Theorem)** *Let* $\{\pi(i),\ i \in \Omega\}$ *be the equilibrium distribution of a reversible Markov process with transition rates* $q(i,j)$, $i,j \in \Omega$, $\Psi$ *be a subset of* $\Omega$ *and* $x(\Psi) \geq 0$ *the set multiplier associated with the set* $\Psi$. *If*

$$x(i) = \prod_{\{\Psi|\Psi\subseteq\Omega\wedge i\in\Psi\}} x(\Psi), \tag{3.18}$$

*the process with transitions rates* $q'(i,j)$, $i,j \in \Omega$, *satisfying*

$$x(i)\frac{q'(i,j)}{q'(j,i)} = x(j)\frac{q(i,j)}{q(j,i)} \tag{3.19}$$

*is also reversible and its equilibrium distribution is given by*

$$\pi'(i) = G''(\Omega)^{-1}x(i)\pi(i), \tag{3.20}$$

*where*

$$G''(\Omega) = \sum_{i\in\Omega} x(i)\pi(i) \tag{3.21}$$

*is the normalization constant.*

**Proof**

The numbers $\pi'(i)$, $i \in \Omega$, sum to unity. They also satisfy the detailed balance equations:

$$\begin{aligned} \pi(i)q(i,j) &= \pi(j)q(j,i) \\ \Leftrightarrow \quad \pi(i)x(i)\frac{q'(i,j)}{q'(j,i)} &= \pi(j)\frac{q(i,j)}{q(j,i)}x(i)\frac{q'(i,j)}{q'(j,i)} \\ &= \pi(j)x(j) \\ \Leftrightarrow \quad \pi'(i)q'(i,j) &= \pi'(j)q'(j,i) \end{aligned} \tag{3.22}$$

Hence, $\{\pi'(i),\ i \in \Omega\}$, is the equilibrium distribution of a reversible process. **Q.E.D.**

The next result is a direct consequence of the Characterization Theorem and gives the constructive means for determining which transition rates are affected when a set multiplier is applied.

**Corollary 3.5** *Let $\langle \Psi, \Omega - \Psi \rangle$ be a cut of the state transition graph $G = (\Omega, E)$ of a reversible process $X(t)$ and $x(\Psi)$ a set multiplier associated with $\Psi$. Then,*

$$\frac{q'(i,j)}{q'(j,i)} = \begin{cases} \dfrac{1}{x(\Psi)}\dfrac{q(i,j)}{q(j,i)}, & \text{if } i \in \Psi \wedge j \in \Omega - \Psi, \\ x(\Psi)\dfrac{q(i,j)}{q(j,i)}, & \text{if } j \in \Psi \wedge i \in \Omega - \Psi, \\ \dfrac{q(i,j)}{q(j,i)}, & \text{otherwise.} \end{cases} \tag{3.23}$$

### 3.1.1 Discussion

Given a set of multipliers $\{x(i) | i \in \Omega\}$, there are infinitely many choices of $q'(i,j)$s that satisfy equation (3.19). Recall that state multipliers are introduced in order to model state-dependent transition rates and they could be superimposed. In many cases of practical interest, it suffices to model one state dependency at a time, thus affecting transition rates in one direction only, and then to superimpose the resulting multipliers. In this context, it is reasonable to assume that if at least one of $x(i)$ and $x(j)$ in (3.19) is non-zero, the fraction $q'(i,j)/q'(j,i)$ does not have an indeterminate form $0/0$ or $\infty/\infty$. Nevertheless, such forms satisfy reversibility trivially.

What makes the state multipliers useful as an analytic tool is that they can be used to modify the state space of any reversible process without removing the reversibility property. For example, truncating of the state space $\Omega$ of a Markov process can be modeled as follows. Let $\Psi$ be the subset of $\Omega$ to be truncated. Define $x(\Psi) = 0$. Then, by Theorem 3.4, $\pi'(i) = 0$. Let's verify this result and see how the transition rates are affected from the introduction of the state multipliers. From equation (3.19) we have

$$x(i)\frac{q'(i,j)}{q'(j,i)} = x(j)\frac{q(i,j)}{q(j,i)}, \; i \in \Psi, \; j \in \Omega - \Psi, \tag{3.24}$$

with $x(j)$, $q(i,j)$ and $q(j,i)$ being positive and finite quantities. Since $x(i) = 0$, equation (3.24) can be satisfied only in two cases:

1. $0 < q'(i,j) < \infty$ and $q'(j,i) = 0$.
2. $q'(i,j) = \infty$ and $0 < q'(j,i) < \infty$.

From the global balance equations

$$\pi'(i) \sum_{k \in \Omega} q'(i,k) = \sum_{k \in \Omega} \pi'(k) q'(k,i), \qquad i \in \Psi, \tag{3.25}$$

we can verify that $\pi'(i)$ is zero in both cases. Furthermore, the jump chain probabilities $q'(j,i)/\sum_{k\in\Omega} q'(i,k)$ are zero in both cases.

In the first case the behavior of the process is clear.

Now consider the second case. Since the residency time in state $i$ is exponentially distributed with parameter $\sum_{k\in\Omega} q'(i,k)$, the mean residency time in any state $i \in \Psi$ is zero. Then, ergodicity implies that $\pi'(i) = 0$. It seems as if a transition to a non-permissible state is allowed as long as the residency time there is zero. However, this is not true. The transition rates of the process describing such a queueing model are generally proportional to the arrival and service rates. Furthermore, the basic assumption behind the definition of transition rates is that only one transition is possible during an infinitesimal time period. Then, if it is assumed that a finite capacity system is in a boundary state when a new arrival occurs, the probability that the system moves to a non-permissible state at a finite rate and returns to a permissible state infinitely fast is zero. In other words, during equilibrium the transition from a boundary state to a non-permissible state is blocked an infinite number of times. So, setting $q'(i,j) = \infty$ is equivalent to a customer arriving at the system and immediately departing with no service.

Additionally, the symmetry condition (3.1) is satisfied by the truncated process for any pair of permissible states.

## 3.2 Examples

**Example 3.1 (A Movable Boundary Model)** Consider the following variant of a well-known traffic access control strategy [26]. Two types of traffic arrive at a service facility at Poisson rates $\lambda$ and $\omega$. Their bandwidth requirements are $b_1 = 2$ and $b_2 = 1$ and their service times are exponentially distributed with means $1/\mu$ and $1/\nu$. The facility has one server dedicated to the narrow-band ($b_2 = 1$) customers and six servers for the wide-band customers. All the servers have the same speed. Queueing is allowed only for the narrow-band customers. Narrow-band traffic may spill into unoccupied wide-band servers but it could be preempted by wide-band traffic if necessary, Figure 3.1. Preempted narrow-band customers wait in queue until their service can resume. The Markov process for this system is not reversible, Figure 3.2. The reversibility property is destroyed by transitions causing preemption of the narrow-band traffic.

Since the arrival rate and service requirement of the wide-band traffic cannot be altered, I will try to develop an approximate reversible process by modulating the arrival rate of the narrow-band traffic. Let's start with a two dimensional birth-death process with birth rates $\lambda$ and $\omega$, death rates $\mu$ and $\nu$ and equilibrium distribution

$$\pi(n_1, n_2) = \pi(0,0) \left(\frac{\lambda}{\mu}\right)^{n_1} \left(\frac{\omega}{\nu}\right)^{n_2}. \tag{3.26}$$

Using the Characterization Theorem 3.4 and Corollary 3.5, the departure rates for

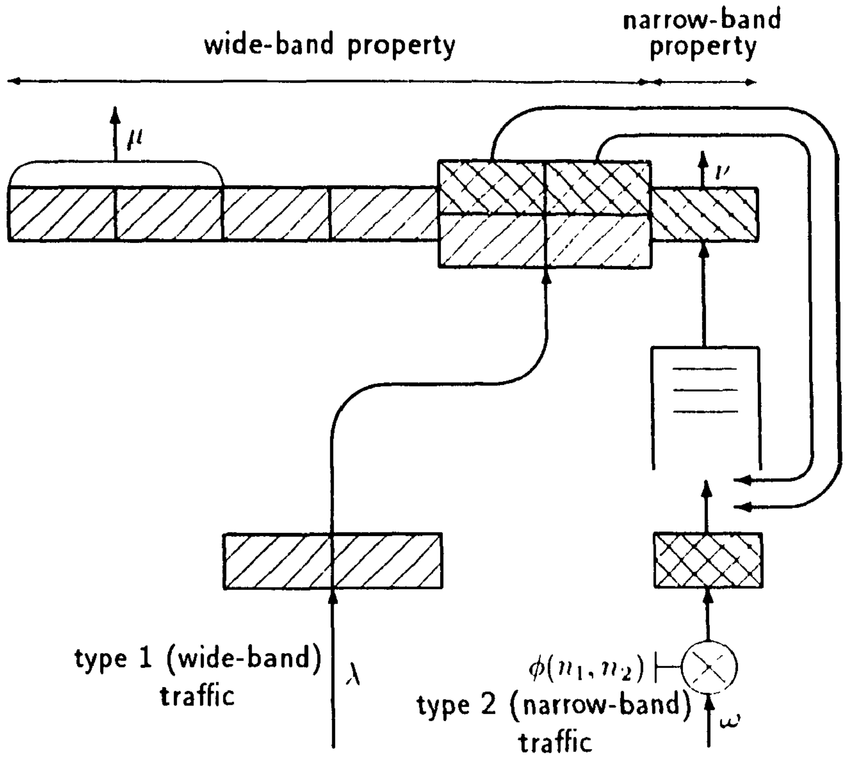



Figure 3.1: A movable boundary model. Wide-band traffic has preemptive priority over narrow-band traffic. The modulating function $\phi$ may be used to reduce the narrow-band arrival rate if preemption occurs. If $\phi(n_1, n_2) = 1, \forall n_1, n_2$, the model is not reversible.

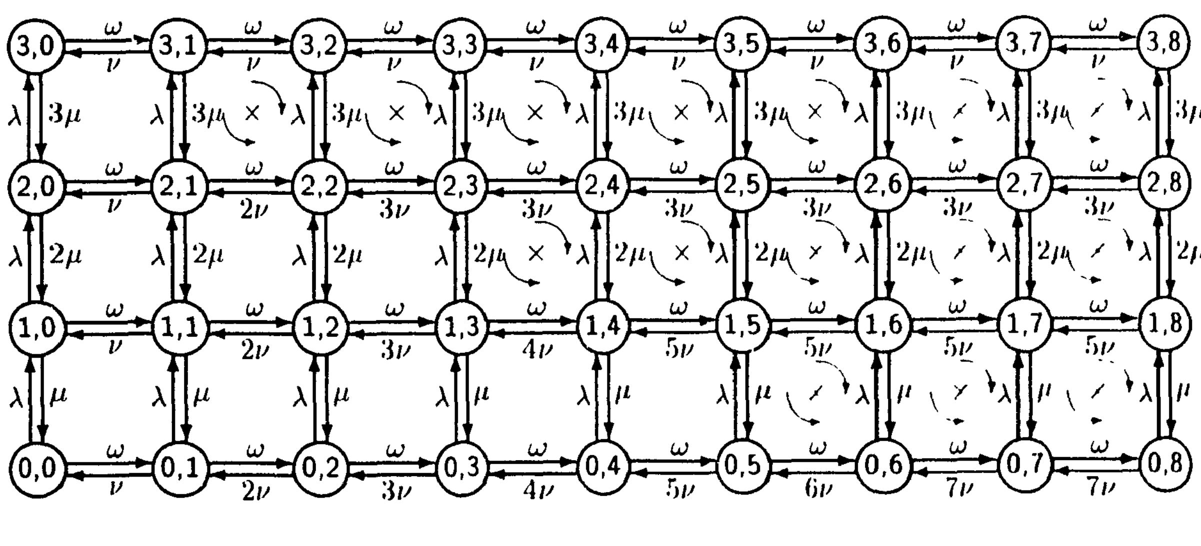


Figure 3.2: The state transition diagram of the movable boundary model. This process is not reversible.

the wide-band traffic can be adjusted to become

$$q'((n_1,n_2),(n_1-1,n_2)) = n_1\mu, \qquad 0 \le n_1 \le 3,$$

with the set of state multipliers

$$\left\{x(n_1,n_2)|x(n_1,n_2) = \frac{1}{n_1!}, 0 \le n_1 \le 3\right\}.$$

Then, the state space can be truncated by setting

$$y(n_1,n_2) = \begin{cases} 1, & \text{for } 0 \le n_1 \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

A first adjustment for the departure rates of the narrow-band traffic is

$$q'((n_1,n_2),(n_1,n_2-1)) = \begin{cases} n_2\nu, & \text{for } 0 \le n_2 \le 7, \\ 7\nu, & \text{for } n_2 > 7, \end{cases}$$

which yields the multipliers

$$z(n_1,n_2) = \begin{cases} \dfrac{1}{n_2!}, & \text{for } 0 \le n_2 \le 7, \\ \dfrac{1}{7!7^{n_2-7}}, & \text{for } n_2 > 7. \end{cases}$$

But for $2n_1 + n_2 > 7$ we can only have

$$q'((n_1,n_2),(n_1,n_2-1)) = (7-2n_1)\nu$$

and we cannot use the multipliers $z(n_1,n_2)$ in this region for adjusting the narrow-band departure rates. Applying the Characterization Theorem 3.4, for $2n_1 + n_2 > 7$,

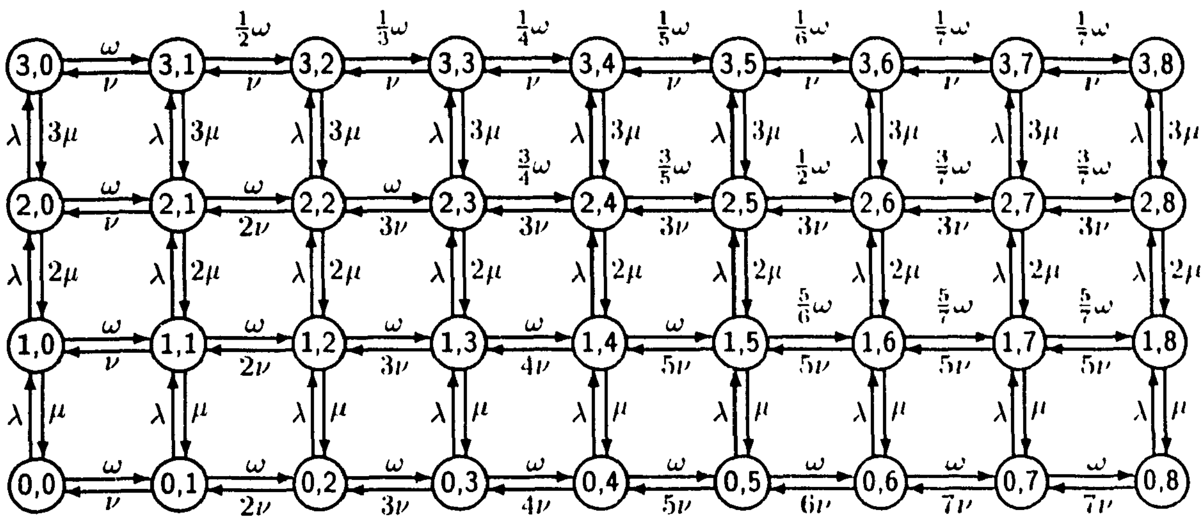



Figure 3.3: The state transition diagram of the modulated-arrival model. This process is reversible.

we get

$$
\begin{aligned}
\frac{q'((n_1,n_2-1),(n_1,n_2))}{(7-2n_1)\nu} &= \frac{z(n_1,n_2)}{z(n_1,n_2-1)}\frac{\omega}{\nu} \\
\Leftrightarrow \quad q'((n_1,n_2-1),(n_1,n_2)) &= \begin{cases} \dfrac{(7-2n_1)}{n_2}\omega, & \text{for } 2n_1+n_2>7 \wedge n_2 \le 7, \\ \dfrac{(7-2n_1)}{7}\omega, & \text{for } 2n_1+n_2>7 \wedge n_2 > 7. \end{cases}
\end{aligned}
$$

The resulting reversible process is shown in Figure 3.3. Its equilibrium distribution is

$$
\pi'(n_1,n_2) = \begin{cases} \pi'(0,0)\dfrac{1}{n_1!n_2!}\left(\dfrac{\lambda}{\mu}\right)^{n_1}\left(\dfrac{\omega}{\nu}\right)^{n_2}, & \text{for } 0 \le n_1 \le 3 \wedge 0 \le n_2 \le 7, \\ \pi'(0,0)\dfrac{1}{n_1!7!7^{n_2-7}}\left(\dfrac{\lambda}{\mu}\right)^{n_1}\left(\dfrac{\omega}{\nu}\right)^{n_2}, & \text{for } 0 \le n_1 \le 3 \wedge n_2 > 7, \\ 0, & \text{otherwise}, \end{cases}
$$

where $\pi'(0,0)$ is determined by the normalizing condition $\sum_{(n_1,n_2)} \pi'_i(n_1,n_2) = 1$. ■

**Example 3.2 (A Load Balancing Queueing Discipline)** In this example, I show how set multipliers can be used to approximate a non-reversible process with a reversible one.

Two types of traffic with Poisson rates $\lambda$ and $\omega$ and service requirements $\mu$ and $\nu$ arrive at two infinite length queues, each queue being dedicated to one type of

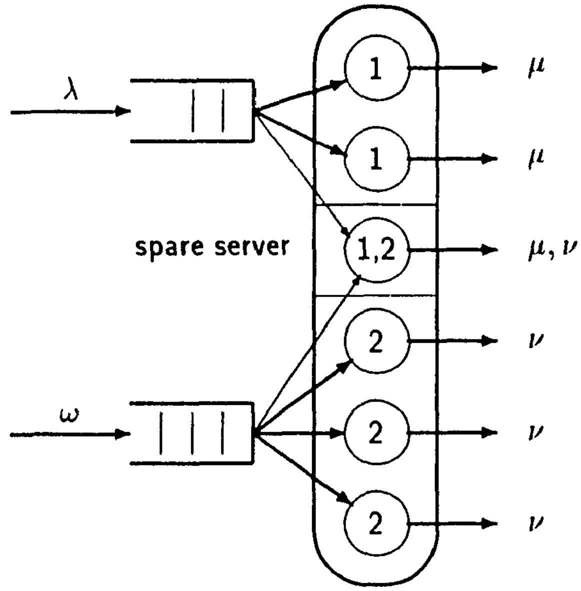



Figure 3.4: A load balancing multi-server facility. The number of servers corresponds to the desired load. In case of imbalance the spare server is activated.

traffic only. The queues are serviced by a multi-server facility, which allocates a fixed number of servers, $s_1$ and $s_2$, to each type. Let $n_1$ and $n_2$ be the number in the system for the two types. When queueing begins the facility tries to maintain the population ratio of the two types at $s_1/s_2$. For this purpose, the facility maintains a number of spare servers which are used to remedy any imbalance.

Figure 3.4 depicts such a system, where the desired population ratio is $2/3$ and there is one spare. When an imbalance occurs, the spare is allocated to the queue that causes this imbalance. As shown in Figure 3.5, the occupancy process of this system is not reversible. The meshes that do not satisfy Kolmogorov's criterion are the ones containing the almost load-balanced states These meshes lie on the line $3n_1 = 2n_2$, $n_1 \geq 2 \wedge n_2 \geq 3$, and their common characteristic is that they contain pairs of neighboring states with opposite imbalances. These states cause abrupt changes in the transition rates. For example, the service rates for type 2 traffic at states $(4,5)$ and $(3,5)$ are $3\nu$ and $4\nu$, respectively. Such changes increase the probability that the process circulates in one direction while do not affect the probability of circulation in the other direction.

Next, I develop an approximation which "smooths out" the abrupt changes in

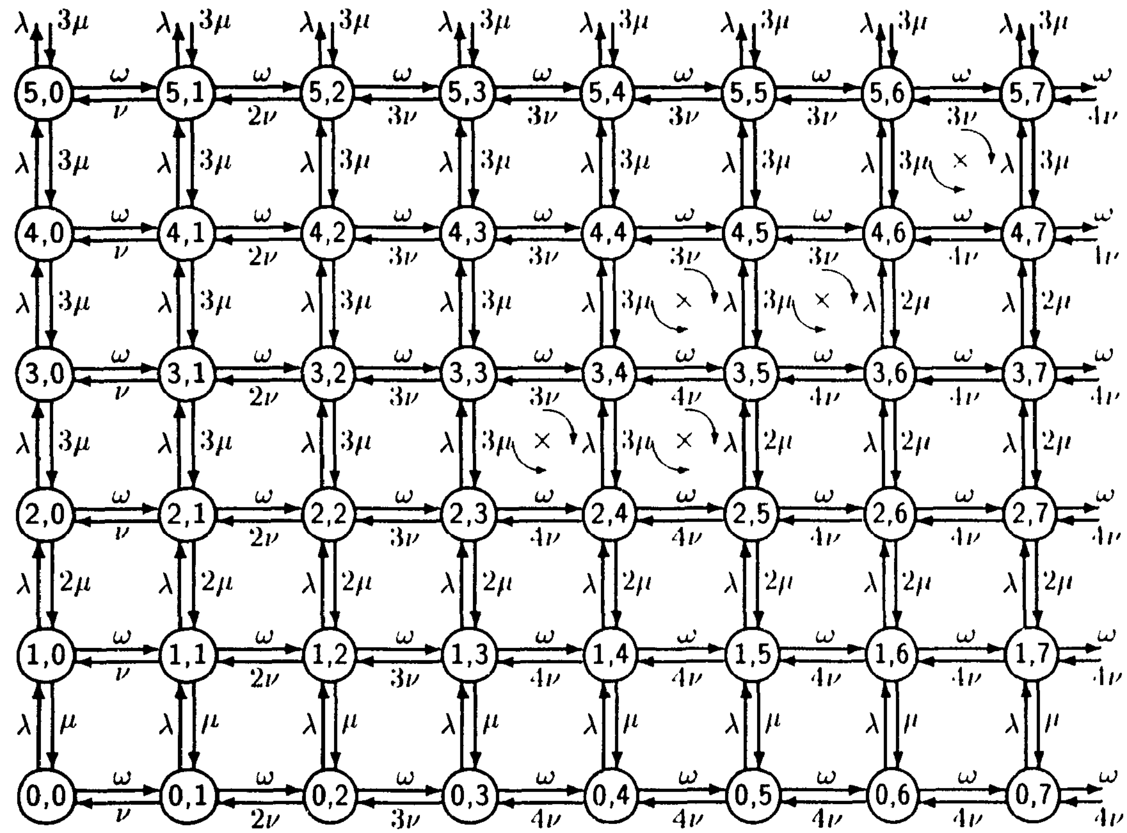



Figure 3.5: The state transition diagram of the queueing model in Figure 3.4. Kolmogorov's criterion is not satisfied only in the marked meshes, namely the meshes on the line $3n_1 = 2n_2$, $n_1 \geq 2 \wedge n_2 \geq 3$.

the transition rates by balancing the probability flux around each mesh. Consider a reversible system, e.g. a two-dimensional birth-death process with birth rates $\lambda$ and $\omega$ and death rates $\mu$ and $\nu$ and equilibrium distribution

$$\pi(n_1, n_2) = \pi(0,0)\left(\frac{\lambda}{\mu}\right)^{n_1}\left(\frac{\omega}{\nu}\right)^{n_2}. \tag{3.27}$$

I shall adjust the transition rates of this process, by overlaying sets of multipliers, so that the resulting reversible process approximates the one in Figure 3.5.

For $0 \leq n_1 \leq 2$ there is no queueing for type 1. Similarly, for $0 \leq n_2 \leq 3$ there is no queueing for type 2. So, the first obvious adjustment is to speed up the departure rates in these regions, that is

$$q'((n_1, n_2), (n_1 - 1, n_2)) = n_1\mu, \quad \text{for } 0 \leq n_1 \leq 2,$$
$$q'((n_1, n_2), (n_1, n_2 - 1)) = n_2\nu, \quad \text{for } 0 \leq n_2 \leq 3.$$

Then, the Characterization Theorem 3.4 and Corollary 3.5 yield the sets of multipliers

$$\left\{x(n_1, n_2) | x(n_1, n_2) = \frac{1}{n_1!},\ 0 \leq n_1 \leq 2\right\},\ \left\{y(n_1, n_2) | y(n_1, n_2) = \frac{1}{n_2!},\ 0 \leq n_2 \leq 3\right\}.$$

Now, I adjust the departure rates, in the regions of queueing for either types, to become

$$q'((n_1, n_2), (n_1 - 1, n_2)) = 3\mu, \quad \text{for } n_1 > 2,$$
$$q'((n_1, n_2), (n_1, n_2 - 1)) = 4\nu, \quad \text{for } n_2 > 3.$$

These rates are achieved by the sets of multipliers

$$\left\{u(n_1, n_2) | u(n_1, n_2) = \frac{1}{3!3^{n_1-3}},\ n_1 > 2\right\},\ \left\{v(n_1, n_2) | v(n_1, n_2) = \frac{1}{4!4^{n_2-4}},\ n_2 > 3\right\}.$$

The adjusted process is shown in Figure 3.6. Observe that for $n_1 > 2 \wedge n_2 > 3$, the transition rates of the process obtained so far, cannot be met by our load balancing system. The transition rates imply that our system has seven servers. A reduction

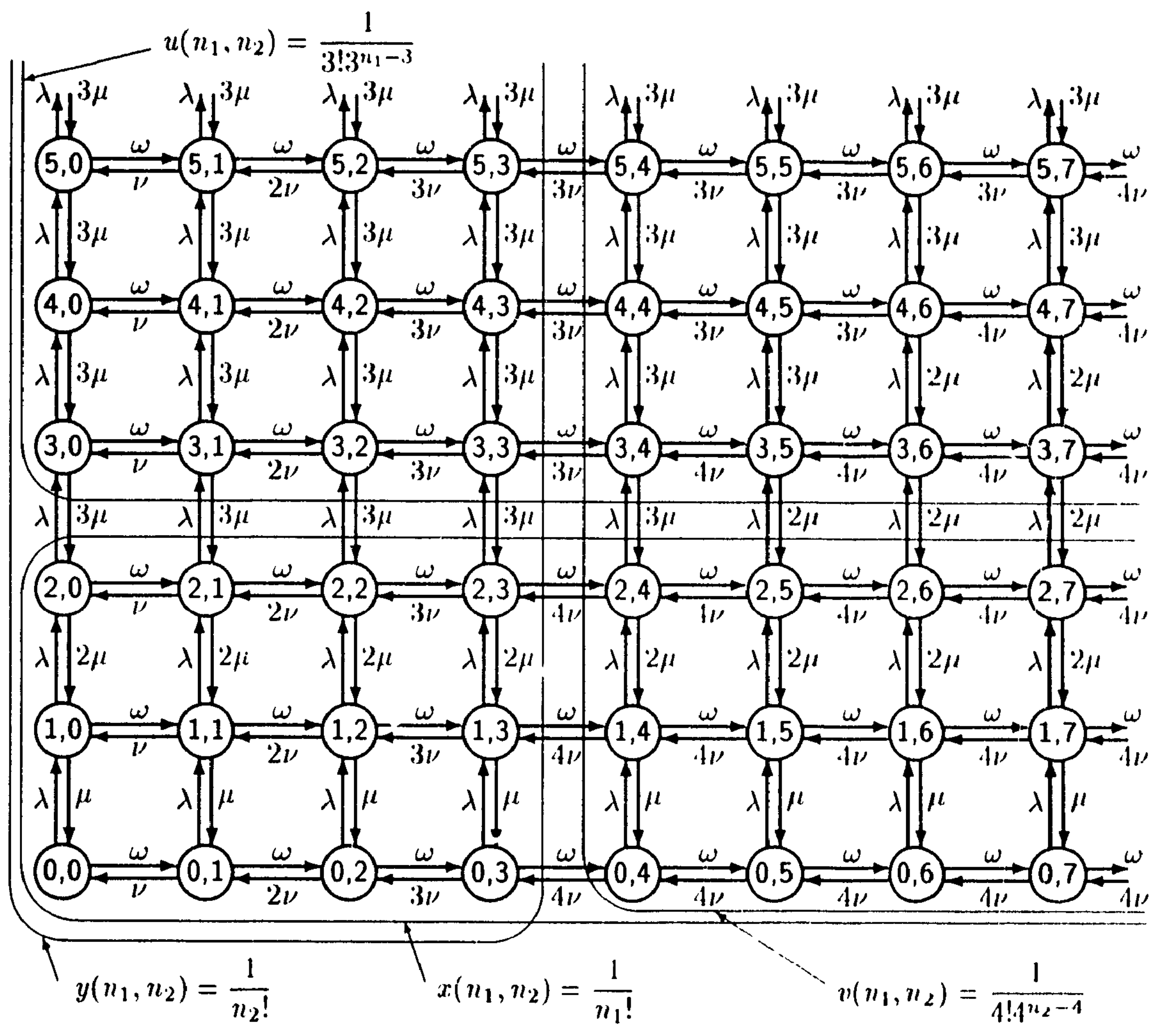



Figure 3.6: A first (reversible) approximation of the non reversible process in Figure 3.5.

in the service rates is necessary by a factor $\phi_i(n_1, n_2)$, $i = 1, 2$, so that

$$3\phi_1(n_1, n_2) + 4\phi_2(n_1, n_2) = 6, \qquad \forall n_1, n_2 : n_1 > 2 \wedge n_2 > 3.$$

By the Characterization Theorem 3.4, the reduction factors $\phi_i(n_1, n_2)$, $i = 1, 2$, can be expressed as

$$\phi_1(n_1, n_2) = \frac{z(n_1 - 1, n_2)}{z(n_1, n_2)}, \ \phi_2(n_1, n_2) = \frac{z(n_1, n_2 - 1)}{z(n_1, n_2)}.$$

Thus, the equations

$$z(n_1, n_2) = \begin{cases} \frac{1}{6}[3z(n_1 - 1, n_2) + 4z(n_1, n_2 - 1)], & \text{for } n_1 > 2 \wedge n_2 > 3, \\ 1, & \text{otherwise,} \end{cases} \tag{3.28}$$

define another set of multipliers. Equation (3.28) represents work conservation and is examined, in a more general form, in Chapter 4.

The process obtained so far corresponds to a system that allows the spare server to be shared when both types of traffic are queued and uses it as before in any other case. This system is reversible, Figure 3.7, and its equilibrium distribution is

$$\pi'(n_1, n_2) = \begin{cases} \pi'(0,0)\frac{1}{n_1! n_2!}\left(\frac{\lambda}{\mu}\right)^{n_1}\left(\frac{\omega}{\nu}\right)^{n_2}, & \text{for } 0 \le n_1 \le 2 \wedge 0 \le n_2 \le 3, \\ \pi'(0,0)\frac{z(n_1, n_2)}{3!3^{n_1-3}4!4^{n_2-4}}\left(\frac{\lambda}{\mu}\right)^{n_1}\left(\frac{\omega}{\nu}\right)^{n_2}, & \text{for } n_1 > 2 \wedge n_2 > 3, \\ 0, & \text{otherwise,} \end{cases}$$

where $\pi'(0,0)$ is obtained from the normalizing condition $\sum_{(n_1,n_2)} \pi_i'(n_1, n_2) = 1$.

**Comment.** Depending on which region we want our approximation to be better, we can introduce another set of multipliers to adjust the rates locally. If, for example, we are interested in the region near the line $3n_1 = 2n_2$ and it is imperative that type 2 gets only three servers for $3n_1 \ge 2n_2$ and type 1 gets only two servers for $3n_1 < 2n_2$,

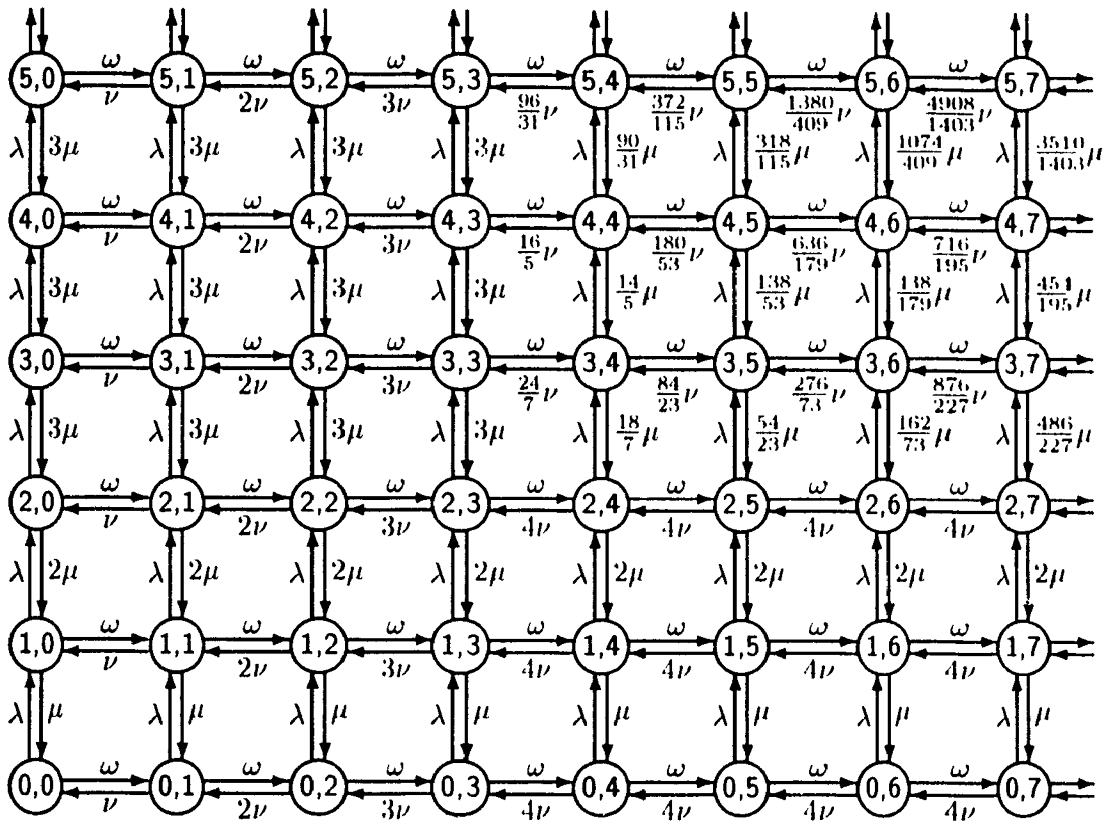


Figure 3.7: The state transition diagram of the approximation system. This process is reversible.

we may set

$$z'(4,6) = \frac{636}{3 \cdot 179},\ z'(5,6) = \frac{1380}{3 \cdot 409},\ z'(5,7) = \frac{1380}{3 \cdot 409} \cdot \frac{4908}{3 \cdot 1403}, \dots$$

and

$$z'(3,6) = \frac{162}{2 \cdot 73},\ z'(4,7) = \frac{454}{2 \cdot 195}, \dots.$$

In other words, we define another boundary affecting the behavior of the process along the line $3n_1 = 2n_2$. When such boundaries are defined, we may have to check whether the physical constraints of our system are satisfied. Formal methods for the development of approximations based on the physical constraints of a system, as well as analytical and software tools for their solution and evaluation of errors, constitute an interesting research topic. Unfortunately, they are beyond the scope of this thesis. ■■

# Chapter 4

# Extension of the Shared Resource Model

The shared-resource (SR) model presented by Kaufman [17] possesses a product-form solution for the equilibrium distribution of the population of the system. This product-form admits a simple one-dimensional recursion for the normalization constant and the blocking probabilities under the complete-sharing policy. It is this feature of the SR model which makes it particularly attractive when state-space explosion is an issue.

This feature is further exploited by extending the model through the use of state-dependent service rates to allow sharing of the resource between customers that would otherwise be blocked and customers already in service. The resulting model is called the *extended shared-resource (ESR)* model. The analysis of the ESR model results in the equilibrium distributions and the probability generating functions for the population and occupancy of the system. Even though these results are in closed from, it is the computation of the normalization constant that makes the problem intractable. This issue is resolved by a one-dimensional recursion for the occupancy distribution with the same computational requirements as Kaufman's recursion. Additionally, I present a recursive algorithm for the moments of the population and discuss its space requirements. Finally, I show the behavior of the ESR model by taking limiting cases for the population, studying several examples and comparing my results with those of similar existing models.

## 4.1 The Shared-Resource (SR) Model

Kaufman considered a shared resource of finite capacity $S$ (Figure 4.1), where the $S$ units of the resource are shared by customers with different spatial and/or temporal requirements according to a resource sharing policy. Customers are derived from

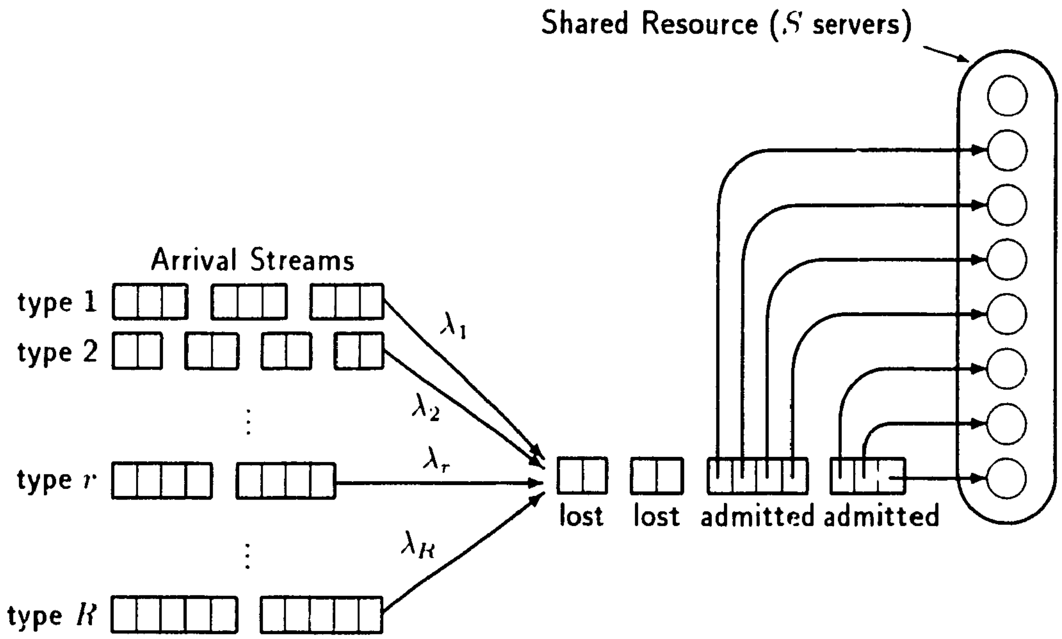



Figure 4.1: The shared-resource model. Type $r$ customers arrive at a rate $\lambda_r$, require $b_r$ units of the resource and have a random service requirement with mean $\mu_r^{-1}$, $r = 1, 2, \ldots, R$.

a Poisson process of stationary rate $\lambda$. There are $R$ customer types. An arriving customer is of type $r$ with probability $\lambda_r/\lambda$, $r = 1, 2, \ldots, R$. Type $r$ customers require $b_r$ units of the resource and have a random service requirement $\tau_r$ with mean $1/\mu_r$. The distribution of $\tau_r$ may have an arbitrary rational Laplace transform.

A customer's spatial requirement $b_r$ can be satisfied if and only if at least $b_r$ units of the resource are available upon arrival. An arriving customer whose spatial requirement can be satisfied is admitted into the system and allocated any $b_r$ units available. Otherwise, the customer is blocked and lost. Thus the model assumes, for instance, that a message can be distributed in non-contiguous storage units or that a reshuffling is possible and permissible without a time penalty. The performance measures of interest in this model are the blocking probabilities for customers with spatial requirements $b_r$, $r = 1, 2, \ldots, R$.

The resource may be shared by customers according to an arbitrary resource-sharing policy in the sense that the policy may give rise to an arbitrary connected set of allowable states. This follows from Lam's symmetry condition [28] on the loss and trigger functions for networks with population size constraints.

When all residency times are exponentially distributed, the SR model is a multi dimensional birth-death process whose state

$$\boldsymbol{n} \stackrel{\text{def}}{=} (n_1, n_2, \ldots, n_R) \tag{4.1}$$

is the population vector of the system; $n_r$ is the number of type $r$ customers in the system, $r = 1, 2, \ldots, R$.

Depending on the resource-sharing policy in effect, a customer may be blocked from departing when the customer's residency time expires. If this happens, then it is assumed that the customer commences another residency-time realization from the same exponential distribution or, equivalently, that the customer's departure at the end of his first residency-time realization triggers the immediate injection of another type $r$ customer into the system. This *triggered disposition* assumption allows us to view the SR model as a single-node queueing network with population-size constraints [28] and hence to assert that the model possesses the insensitivity property of being valid for all residency-time distributions with a rational Laplace transform.

The equilibrium distribution is

$$\pi(\boldsymbol{n}) = G(\Omega)^{-1} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!}, \tag{4.2}$$

where $\rho_r = \lambda_r / \mu_r$, $\Omega$ is the state space (defined by the resource sharing policy) and

$G(\Omega)$ is the normalization constant

$$G(\Omega) = \sum_{n\in\Omega} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!}. \tag{4.3}$$

In the case of complete sharing policy, the number of busy servers $s$ is equal to the occupancy of the system $j$, i.e.

$$s = j = \sum_{r=1}^{R} n_r b_r,$$

and its distribution $a(s)$ is given by the one-dimensional recursion

$$sa(s) = \sum_{r=1}^{R} b_r \rho_r a(s - b_r), \qquad j = 1, 2, \dots, S \wedge a(k) = 0, \forall k < 0. \tag{4.4}$$

This recursion can be easily computed and yields the normalization constant and blocking probabilities $P_{b_r}$ for type $r$ customers:

$$G(\Omega)^{-1} = a(0), \tag{4.5}$$

$$P_{b_r} = \sum_{i=0}^{b_r - 1} a(S - i). \tag{4.6}$$

## 4.2 The Extended Shared-Resource (ESR) Model

The following notation is needed.

$$\boldsymbol{b} \stackrel{\text{def}}{=} (b_1, b_2, \dots, b_R),$$

$$\boldsymbol{n} \cdot \boldsymbol{b} \stackrel{\text{def}}{=} \sum_{r=1}^{R} n_r b_r.$$

Now, I give a simple and intuitive extension of the SR model. The main idea is to allow processor-sharing for customers that would be blocked in Kaufman's model (Figure 4.2). Then, the set of permissible states is

$$\Omega = \{\boldsymbol{n} | \boldsymbol{n} \geq \boldsymbol{0},\ r = 1, \dots, R\}. \tag{4.7}$$

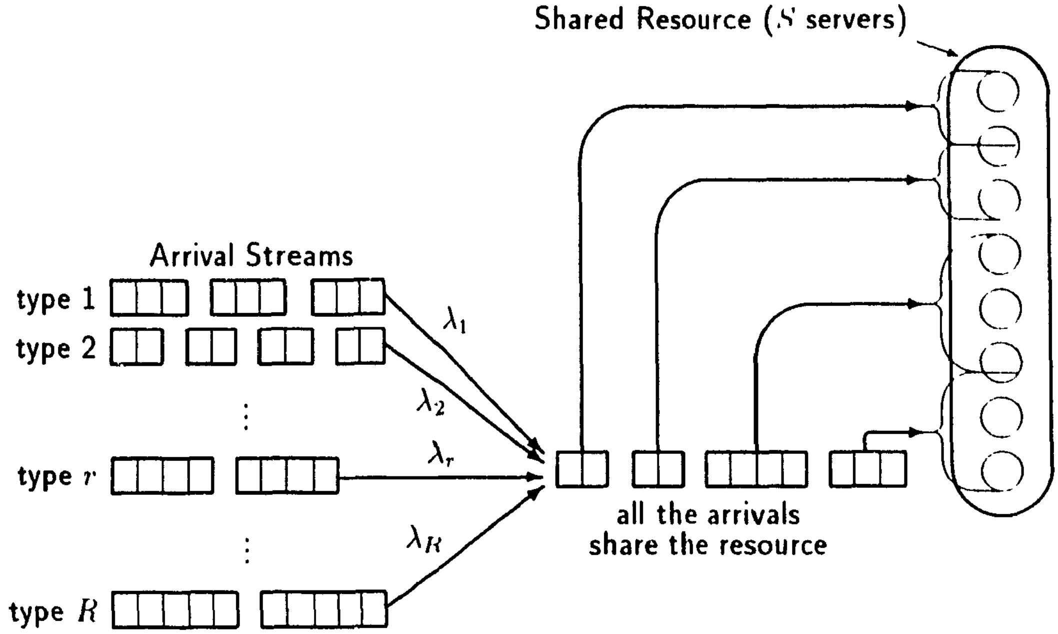



Figure 4.2: The extended shared-resource model.

Since processor-sharing is equivalent to time sharing when the quantum of time goes to zero, our extension is a reasonable one and it is based on the work conserving constraint introduced next. Our use of work conservation is first explained by means of two example models.

**Example 4.1** Consider a message transmission system with $S$ output channels. Messages of $R$ different types arrive according to a Poisson process with total rate $\lambda$. An arriving message is of type $r$, $r = 1, 2, \ldots, R$, with probability $q_r$. A type $r$ message occupies $b_r$ channels for a time period $\tau_r$ drawn from a probability distribution with rational Laplace transform and mean $\mu_r$. The maximum number of packets that can be transmitted at the same time is $S$. Assume that the system is in state $\boldsymbol{n}$ with $\boldsymbol{n} \cdot \boldsymbol{b} \leq S$ and a new message of type $k$ arrives. If $\boldsymbol{n}_k^+ \cdot \boldsymbol{b} > S$, the new message is not blocked but it shares the channels with the messages already in service. This can be done by reducing the number of channels allocated to a type $r$ message to $b_r \phi_r(\boldsymbol{n}) < b_r$, $r = 1, 2, \ldots, R$. The transmission rate for type $r$ messages slows down to $\mu_r \phi_r(\boldsymbol{n}) < \mu_r$. ■

**Example 4.2** A time-sharing transmission system exhibits similar model behavior. Messages of $R$ different types arrive at a Poisson rate $\lambda$. Messages of type $r$ occupy the transmitter for $b_r$ time units during each transmission cycle. The time $\tau_r$ required to transmit a message of type $r$ has a probability distribution with rational Laplace transform and mean $\mu_r$. Let $S$ be the duration of a transmission cycle. Under increased load or noise in the communication channel, the system can reach a state $\boldsymbol{n}$ with $\boldsymbol{n} \cdot \boldsymbol{b} > S$. The transmitter devotes a time $b_r\phi_r(\boldsymbol{n}) < b_r$ to each message of type $r$ during each cycle, until the system reaches a state $\boldsymbol{n}$ such that $\boldsymbol{n} \cdot \boldsymbol{b} \leq S$. The reduction of the time devoted to a type of messages can be implemented by reducing the size of the message packets dynamically (i.e. according to the load). Then, the average time required to transmit a message is increased to $[\mu_r\phi_r(\boldsymbol{n})]^{-1}$. ■

Whenever $\boldsymbol{n} \cdot \boldsymbol{b} > S$, the resource requirement of type $r$ customers is reduced by a positive factor $\phi_r(\boldsymbol{n})$ and their service rate is decelerated by the same factor $\phi_r(\boldsymbol{n})$ so that, $\sum_{r=1}^{R} n_r b_r \phi_r(\boldsymbol{n}) = S$ resource units are occupied. The reduction factor $\phi_r(\boldsymbol{n})$ can be modeled using a set $\{x(\boldsymbol{n})|\boldsymbol{n} \in \Omega\}$ of state multipliers. The above discussion implies that the state multipliers should be defined so that, the transition rates from state $\boldsymbol{n}_r^-$ to state $\boldsymbol{n}$ (arrival rates) remain unaffected and the transition rates from state $\boldsymbol{n}$ to state $\boldsymbol{n}_r^-$ (service rates) are reduced by a factor $\phi_r(\boldsymbol{n})$. According to theorem 3.4, we ' ave

$$\begin{aligned} \frac{q'(\boldsymbol{n},\boldsymbol{n}_r^-)}{q'(\boldsymbol{n}_r^-,\boldsymbol{n})} &= \frac{x(\boldsymbol{n}_r^-)}{x(\boldsymbol{n})}\frac{q(\boldsymbol{n},\boldsymbol{n}_r^-)}{q(\boldsymbol{n}_r^-,\boldsymbol{n})} \\ \Leftrightarrow \quad \frac{n_r\mu_r\phi_r(\boldsymbol{n})}{\lambda_r} &= \frac{x(\boldsymbol{n}_r^-)}{x(\boldsymbol{n})}\frac{n_r\mu_r}{\lambda_r} \\ \Leftrightarrow \quad \phi_r(\boldsymbol{n}) &= \frac{x(\boldsymbol{n}_r^-)}{x(\boldsymbol{n})}. \end{aligned} \tag{4.8}$$

Then, the set $\{x(\boldsymbol{n})|\boldsymbol{n} \in \Omega\}$ of state multipliers that satisfies the work conserving constraint $\sum_{r=1}^{R} n_r b_r \phi_r(\boldsymbol{n}) = S$, if $\boldsymbol{n} \cdot \boldsymbol{b} > S$, is unique and can be computed according to the multidimensional recursion

$$x(\boldsymbol{n}) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \boldsymbol{n} \cdot \boldsymbol{b} \leq S \wedge \boldsymbol{n} \geq \boldsymbol{0}, \\ \dfrac{1}{S}\displaystyle\sum_{r=1}^{R} n_r b_r x(\boldsymbol{n}_r^-), & \text{if } \boldsymbol{n} \cdot \boldsymbol{b} > S \wedge \boldsymbol{n} \geq \boldsymbol{0}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.9}$$

The function $x(\boldsymbol{n})$ acts as a service rate slowing function, attenuating the service rate of all customers with increasing occupancy. Using these rates, the global balance equations for the equilibrium distribution are

$$\pi(\boldsymbol{n})\sum_{r=1}^{R}\left[\lambda_r + n_r\mu_r\frac{x(\boldsymbol{n}_r^-)}{x(\boldsymbol{n})}\right] = \sum_{r=1}^{R}\lambda_r\pi(\boldsymbol{n}_r^-) + \sum_{r=1}^{R}(n_r+1)\mu_r\frac{x(\boldsymbol{n})}{x(\boldsymbol{n}_r^+)}\pi(\boldsymbol{n}_r^+). \quad (4.10)$$

### 4.2.1 The Distribution of the Population

Since the process is reversible, the detailed balance equations

$$q(\boldsymbol{n},\boldsymbol{n}_r^-)\pi(\boldsymbol{n}) = q(\boldsymbol{n}_r^-,\boldsymbol{n})\pi(\boldsymbol{n}_r^-) \quad (4.11)$$

are satisfied. Thus,

$$\begin{aligned} n_r\mu_r\frac{x(\boldsymbol{n}_r^-)}{x(\boldsymbol{n})}\pi(\boldsymbol{n}) &= \lambda_r\pi(\boldsymbol{n}_r^-) \\ \Leftrightarrow\quad n_r x(\boldsymbol{n}_r^-)\pi(\boldsymbol{n}) &= \rho_r x(\boldsymbol{n})\pi(\boldsymbol{n}_r^-), \end{aligned} \quad (4.12)$$

where $\rho_r = \lambda_r/\mu_r$. The solution is

$$\pi(\boldsymbol{n}) = G(\Omega)^{-1}x(\boldsymbol{n})\prod_{r=1}^{R}\frac{\rho_r^{n_r}}{n_r!}, \quad (4.13)$$

where the normalization constant is given by

$$G(\Omega) = \sum_{\boldsymbol{n}\in\Omega}x(\boldsymbol{n})\prod_{r=1}^{R}\frac{\rho_r^{n_r}}{n_r!}. \quad (4.14)$$

Notice that the equilibrium distribution can be directly derived from the equilibrium distribution of an infinite server (type 3 BCMP node [2]) with the use of theorem 3.4.

The product-form equation (4.13), although appealing, suffers from the fact that $x(\boldsymbol{n})$ is not in product-form. Thus, at the outset, it looks as if the entire state space must be enumerated in order to compute the normalization constant $G(\Omega)$. However, this is not the case, as it is demonstrated next.

Multiplying the detailed balance equations (4.12) by $b_r$ and summing over $r$, we obtain

$$\sum_{r=1}^{R} b_r n_r x(\boldsymbol{n}_r^-)\pi(\boldsymbol{n}) = \sum_{r=1}^{R} b_r \rho_r x(\boldsymbol{n})\pi(\boldsymbol{n}_r^-). \tag{4.15}$$

When $\boldsymbol{n}$ satisfies $\boldsymbol{n}\cdot\boldsymbol{b} \le S$ (i.e. $x(\boldsymbol{n}_r^-) = x(\boldsymbol{n}) = 1$), equation (4.15) reduces to

$$\begin{aligned} \sum_{r=1}^{R} b_r n_r \pi(\boldsymbol{n}) &= \sum_{r=1}^{R} b_r \rho_r \pi(\boldsymbol{n}_r^-), \\ \Leftrightarrow \quad (\boldsymbol{n}\cdot\boldsymbol{b})\pi(\boldsymbol{n}) &= \sum_{r=1}^{R} b_r \rho_r \pi(\boldsymbol{n}_r^-), \quad \boldsymbol{n}\cdot\boldsymbol{b} \le S. \end{aligned} \tag{4.16}$$

When $\boldsymbol{n}$ satisfies $\boldsymbol{n}\cdot\boldsymbol{b} > S$ (i.e. $x(\boldsymbol{n}) = \frac{1}{S}\sum_{r=1}^{R} b_r n_r x(\boldsymbol{n}_r^-)$), equation (4.15) becomes

$$\begin{aligned} S x(\boldsymbol{n})\pi(\boldsymbol{n}) &= x(\boldsymbol{n})\sum_{r=1}^{R} b_r \rho_r \pi(\boldsymbol{n}_r^-) \\ \Leftrightarrow \quad S\pi(\boldsymbol{n}) &= \sum_{r=1}^{R} b_r \rho_r \pi(\boldsymbol{n}_r^-). \end{aligned} \tag{4.17}$$

Therefore, the following theorem is proved.

**Theorem 4.1** *The equilibrium distribution $\pi(\boldsymbol{n})$ of the population in the ESR model satisfies the multidimensional recursion*

$$\pi(\boldsymbol{n}) = \frac{1}{\min\{\boldsymbol{n}\cdot\boldsymbol{b}, S\}} \sum_{r=1}^{R} b_r \rho_r \pi(\boldsymbol{n}_r^-). \tag{4.18}$$

In Appendix B a solution for $x(\boldsymbol{n})$ is given based on the boundary defined by $S$. This solution yields the equilibrium distribution

$$\pi(\boldsymbol{n}) = \begin{cases} G(\Omega)^{-1} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!}, & \text{if } \boldsymbol{n}\cdot\boldsymbol{b} \le S, \\ G(\Omega)^{-1} \sum_{\boldsymbol{m}\in\mathcal{I}^+} \left[\sum_{r=1}^{R}(n_r - m_r)\right]! \prod_{r=1}^{R} \left(\frac{b_r}{S}\right)^{n_r - m_r} \frac{\rho_r^{n_r}}{m_r!(n_r - m_r)!}, & \text{if } \boldsymbol{n}\cdot\boldsymbol{b} > S, \end{cases} \tag{4.19}$$

where the normalization constant is

$$G(\Omega) = \sum_{\boldsymbol{n}\in\Omega} \sum_{\boldsymbol{m}\in\mathcal{I}^+} \left[\sum_{r=1}^{R}(n_r - m_r)\right]! \prod_{r=1}^{R} \left(\frac{b_r}{S}\right)^{n_r - m_r} \frac{\rho_r^{n_r}}{m_r!(n_r - m_r)!}. \tag{4.20}$$

and the boundary is

$$\mathcal{I} = \bigcup_{i=0}^{b_{\max}-1} \{\boldsymbol{m} | \boldsymbol{m} \cdot \boldsymbol{b} = S - i\} \tag{4.21}$$

The set

$$\mathcal{I}^+ = \bigcup_{i=1}^{b_{\max}} \{\boldsymbol{m} | \boldsymbol{m} \cdot \boldsymbol{b} = S + i\} \tag{4.22}$$

contains the immediate neighbors of $\mathcal{I}$ in the direction of increasing population. It is only used for technical purposes, as explained in Appendix B.

Equations (4.18), (4.19) and (4.20) are of limited practical value since their use for the computation of the normalization constant would have significant time and space requirements. As Kaufman [17] shows, the Buzen-type recursion

$$\begin{aligned} G(j,r) &= \sum_{l=0}^{\lceil j/b_r \rceil} \frac{\rho_r^l}{l!} G(j - lb_r, r-1), \quad r = 1,2,\ldots,R, j = 0,1,\ldots,S, \\ G(j,1) &= \sum_{l=0}^{\lceil j/b_1 \rceil} \frac{\rho_1^l}{l!}, \qquad\qquad\qquad j = 0,1,\ldots,S, \end{aligned}$$

which applies to his model, requires that the elements of a $S \times R$ matrix are computed recursively column by column. In the extended model, the problem can only worsen. However, as we will see later, the ESR model cannot only be truncated, but also the normalization constant and the blocking probabilities can be computed by a simple one-dimensional recursion.

### 4.2.2 The Occupancy Distribution

The occupancy distribution for the ESR model is defined as

$$\begin{aligned} q(j) &\stackrel{\text{def}}{=} P\{n_1 b_1 + n_2 b_2 + \cdots + n_R b_R = j\} \\ &= \sum_{\{\boldsymbol{n} | \boldsymbol{n} \cdot \boldsymbol{b} = j\}} \pi(\boldsymbol{n}). \end{aligned} \tag{4.23}$$

The terms $n_r b_r$, $r = 1,2,\ldots,R$, are random variables representing the number of resource units occupied by customers of type $r$. Then, the number of servers allocated to customers follows the distribution

$$a(s) \stackrel{\text{def}}{=} \begin{cases} P\{\boldsymbol{n} \cdot \boldsymbol{b} = s\} & \text{if } s < S, \\ \sum_{\{j|j \geq S\}} P\{\boldsymbol{n} \cdot \boldsymbol{b} = j\}, & \text{if } s = S, \end{cases} \tag{4.24}$$

$$= \begin{cases} q(s) & \text{if } s < S, \\ \sum_{\{j|j \geq S\}} q(j), & \text{if } s = S. \end{cases} \tag{4.25}$$

Note that Kaufman's definition of occupancy is different from the one given here and corresponds to the allocation distribution $a(s)$. Our definition of occupancy is consistent with the BCMP framework (section 2.3) and represents the space, in bandwidth units, occupied by the customers in the system.

Let's sum the multidimensional recursion (4.18) for $\pi(\boldsymbol{n})$ over the set of states $\{\boldsymbol{n}|\boldsymbol{n} \cdot \boldsymbol{b} = j\}$. Then,

$$\begin{aligned} \sum_{\{\boldsymbol{n}|\boldsymbol{n} \cdot \boldsymbol{b}=j\}} \min\{\boldsymbol{n} \cdot \boldsymbol{b}, S\}\pi(\boldsymbol{n}) &= \sum_{\{\boldsymbol{n}|\boldsymbol{n} \cdot \boldsymbol{b}=j\}} \sum_{r=1}^{R} b_r \rho_r \pi(\boldsymbol{n}_r^-) \\ \Leftrightarrow \quad \min\{j, S\} \sum_{\{\boldsymbol{n}|\boldsymbol{n} \cdot \boldsymbol{b}=j\}} \pi(\boldsymbol{n}) &= \sum_{r=1}^{R} b_r \rho_r \sum_{\{\boldsymbol{n}|\boldsymbol{n} \cdot \boldsymbol{b}=j\}} \pi(\boldsymbol{n}_r^-). \end{aligned} \tag{4.26}$$

The quantity $\sum_{\{\boldsymbol{n}|\boldsymbol{n} \cdot \boldsymbol{b}=j\}} \pi(\boldsymbol{n})$ is, by definition, the probability $q(j)$. A little manipulation shows

$$\sum_{\{\boldsymbol{n}|\boldsymbol{n} \cdot \boldsymbol{b}=j\}} \pi(\boldsymbol{n}_r^-) = \sum_{\{\boldsymbol{n}|\boldsymbol{n} \cdot \boldsymbol{b}=j-b_r\}} \pi(\boldsymbol{n}) = q(j - b_r). \tag{4.27}$$

Therefore, the multidimensional recursion (4.18) leads to a simple one-dimensional recursion for the occupancy distribution $q(j)$, as stated in the next theorem.

**Theorem 4.2** *The occupancy distribution of the ESR model satisfies the one-dimensional recursion*

$$\min\{j, S\}q(j) = \sum_{r=1}^{R} b_r \rho_r q(j - b_r), \tag{4.28}$$

*where* $q(0) = \pi(\mathbf{0})$.

Equation (4.28) is a natural extension of Kaufman's recursion.

## 4.3 Blocking in the ESR Model

Next, the state-space of the ESR model is truncated to permit processor-sharing up to a certain occupancy. Again, a complete sharing policy is assumed: a customer is blocked and lost if and only if its arrival increases the occupancy above a limit $T < \infty$. Then, the set of permissible states becomes

$$\Omega = \{\boldsymbol{n} | 0 \leq \boldsymbol{n} \cdot \boldsymbol{b} < T \wedge \boldsymbol{n} \geq \boldsymbol{0}\}. \tag{4.29}$$

The following results are valid for $T \geq S$.

The state space $\Omega$ can be truncated to $\Omega'$ by introducing the state multipliers

$$y(\boldsymbol{n}) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \boldsymbol{n} \cdot \boldsymbol{b} \leq T \wedge \boldsymbol{n} \geq \boldsymbol{0}, \\ 0, & \text{otherwise}. \end{cases} \tag{4.30}$$

Then, the equilibrium distribution of the blocking model is given by

$$\pi(\boldsymbol{n}) = \begin{cases} \dfrac{1}{\boldsymbol{n} \cdot \boldsymbol{b}} \displaystyle\sum_{r=1}^{R} b_r \rho_r \pi(\boldsymbol{n}_r^-), & \text{if } \boldsymbol{n} \cdot \boldsymbol{b} \leq S \wedge \boldsymbol{n} \geq \boldsymbol{0}, \\ \dfrac{1}{S} \displaystyle\sum_{r=1}^{R} b_r \rho_r \pi(\boldsymbol{n}_r^-), & \text{if } S < \boldsymbol{n} \cdot \boldsymbol{b} \leq T \wedge \boldsymbol{n} \geq \boldsymbol{0}, \\ 0, & \text{otherwise}. \end{cases} \tag{4.31}$$

The occupancy distribution can be derived as in the previous section. The result is

$$q(j) = \begin{cases} \dfrac{1}{j} \displaystyle\sum_{r=1}^{R} b_r \rho_r q(j - b_r), & \text{if } j \leq S, \\ \dfrac{1}{S} \displaystyle\sum_{r=1}^{R} b_r \rho_r q(j - b_r), & \text{if } S < j \leq T, \\ 0, & \text{otherwise}, \end{cases} \tag{4.32}$$

with $q(0) = \pi(\boldsymbol{0})$. The normalization constant can be easily derived from

$$G(\Omega) = q(0)^{-1} = \sum_{j=0}^{T} q'(j), \tag{4.33}$$

where

$$q'(j) = \begin{cases} \frac{1}{j}\sum_{r=1}^{R} b_r\rho_r q'(j-b_r), & \text{if } j \le S, \\ \frac{1}{S}\sum_{r=1}^{R} b_r\rho_r q'(j-b_r), & \text{if } S < j \le T, \\ 0, & \text{otherwise,} \end{cases} \tag{4.34}$$

and $q'(0) = 1$.

The blocking probability $P_{b_r}$ is the probability that a type $r$ arrival is blocked. Then,

$$P_{b_r} = \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}>T-b_r\}} \pi(\boldsymbol{n}). \tag{4.35}$$

Since $\pi(\boldsymbol{n}) = 0$ for $\boldsymbol{n}\cdot\boldsymbol{b} > T$, it suffices to sum the equilibrium probabilities over the set

$$\{\boldsymbol{n}|T - b_r < \boldsymbol{n}\cdot\boldsymbol{b} \le T\} \quad = \quad \bigcup_{i=0}^{b_r-1}\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b} = T-i\}.$$

Then, equation (4.35) becomes

$$P_{b_r} = \sum_{i=0}^{b_r-1}\sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=T-i\}} \pi(\boldsymbol{n}) = \sum_{i=0}^{b_r-1} q(T-i). \tag{4.36}$$

## 4.4 Performance Measures

Theorem 4.3 provides the probability generating function $\Pi(\boldsymbol{z})$ for $\pi(\boldsymbol{n})$, where

$$\Pi(\boldsymbol{z}) \stackrel{\text{def}}{=} \sum_{\boldsymbol{n}\in\Omega} z_1^{n_1} z_2^{n_2}\cdots z_R^{n_R}\pi(\boldsymbol{n}) \tag{4.37}$$

and

$$\boldsymbol{z} \stackrel{\text{def}}{=} (z_1, z_2, \ldots, z_R).$$

**Theorem 4.3** *The probability generating function of the equilibrium population distribution* $\{\pi(\boldsymbol{n}), \boldsymbol{n} \in \Omega\}$ *is*

$$\Pi(\boldsymbol{z}) = \frac{S\Pi_{0,S}(\boldsymbol{z}) - \sum_{r=1}^{R} z_r b_r \rho_r \left[\Pi_{0,S-b_r}(\boldsymbol{z}) + \Pi_{T-b_r+1,T}(\boldsymbol{z})\right]}{S - \sum_{r=1}^{R} z_r b_r \rho_r}, \tag{4.38}$$

*where*

$$\Pi_{u,v}(\boldsymbol{z}) \stackrel{\text{def}}{=} \sum_{\{\boldsymbol{n} \mid u \leq \boldsymbol{n} \cdot \boldsymbol{b} \leq v\}} z_1^{n_1} z_2^{n_2} \cdots z_R^{n_R} \pi(\boldsymbol{n}). \tag{4.39}$$

The proof is given in Appendix C.

The probability generating function $Q(z)$ of $q(j)$ has a similar form.

**Theorem 4.4** *The probability generating function of the equilibrium occupancy distribution* $\{q(j), j = 0, 1, 2, \ldots, T\}$ *is*

$$Q(z) = \frac{SQ_{0,S}(z) - \sum_{r=1}^{R} z^{b_r} b_r \rho_r \left[Q_{0,S-b_r}(z) + Q_{T-b_r+1,T}(z)\right]}{S - \sum_{r=1}^{R} z^{b_r} b_r \rho_r}, \tag{4.40}$$

*where*

$$Q_{u,v}(z) \stackrel{\text{def}}{=} \sum_{j=u}^{v} z^j q(j). \tag{4.41}$$

The proof is given in Appendix C.

The form of the probability generating functions is not convenient for computing the moments of the population or the occupancy. The reasons are explained in detail in Appendix C. Fortunately, the joint moments of the population can be efficiently computed from the recursion given next.

**Theorem 4.5** *For* $\boldsymbol{k} = (k_1, k_2 \ldots, k_R)$, *let*

$$\varepsilon(\boldsymbol{k}, j) \stackrel{\text{def}}{=} E\{n_1^{k_1} n_2^{k_2} \cdots n_R^{k_R} | j\} q(j), \qquad 0 \leq j \leq T,\ k_r = 0, 1, \ldots,\ r = 1, 2, \ldots, R, \tag{4.42}$$

*and*

$$\boldsymbol{k} \pm (l)_r \stackrel{\text{def}}{=} (k_1, \ldots, k_{r-1}, k_r \pm l, k_{r+1}, \ldots, k_R). \tag{4.43}$$

*Then,*

$$\varepsilon(\boldsymbol{k}, j) = \begin{cases} \frac{1}{j} \sum_{r=1}^{R} b_r \rho_r q(j - b_r), & \textit{if } 0 \leq j \leq S \wedge \boldsymbol{k} = \boldsymbol{0}, \\ \rho_r \sum_{l=0}^{k_r - 1} \binom{k_r - 1}{l} \varepsilon(\boldsymbol{k} - (k_r - l)_r, j - b_r), & \textit{if } 0 \leq j \leq S \wedge k_r \neq 0, \\ \frac{1}{S} \sum_{r=1}^{R} b_r \rho_r \left[ \varepsilon(\boldsymbol{k}, j - b_r) + \sum_{l=0}^{k_r - 1} \binom{k_r}{l} \varepsilon(\boldsymbol{k} - (k_r - l)_r, j - b_r) \right], & \\ & \textit{if } S < j \leq T, \\ 0, & \textit{if } j < 0 \textit{ or } j > T, \end{cases} \tag{4.44}$$

*where* $\varepsilon(\boldsymbol{0}, j) = qj$.

## Proof

For $k_r = 0$, for all $r = 1, 2, \ldots, R$, the result follows from (4.42) and (4.32).

Without loss of generality, let's assume that $k_r \neq 0$ for $r = 1, 2, \ldots, i$, $i \leq R$, and $k_r = 0$ for $r = i + 1, i + 2, \ldots, R$. I shall prove equation (4.44) in the dimensions of nonzero $k_r$.

First, notice that

$$\begin{aligned} E\{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r} | j\} q(j) &= \sum_{\{\boldsymbol{n} | \boldsymbol{n} \cdot \boldsymbol{b} = j\}} n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r} P\{\boldsymbol{n} | \boldsymbol{n} \cdot \boldsymbol{b} = j\} q(j) \\ &= \sum_{\{\boldsymbol{n} | \boldsymbol{n} \cdot \boldsymbol{b} = j\}} n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r} \pi(\boldsymbol{n}) \end{aligned} \tag{4.45}$$

and

$$n_r^k = (n_r - 1 + 1)^k = \sum_{l=0}^{k} \binom{k}{l} (n_r - 1)^l. \tag{4.46}$$

For $b_r \leq j \leq S$, let's multiply the detailed balance equations

$$n_r \pi(\boldsymbol{n}) = \rho_r \pi(\boldsymbol{n}_r^-) \tag{4.47}$$

by $n_1^{k_1}\cdots n_{r-1}^{k_{r-1}} n_r^{k_r-1} n_{r+1}^{k_{r+1}}\cdots n_i^{k_i}$ and sum over $\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}$. Then,

$$\sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_1^{k_1}\cdots n_i^{k_i}\pi(\boldsymbol{n}) = \rho_r \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_1^{k_1}\cdots n_{r-1}^{k_{r-1}} n_r^{k_r-1} n_{r+1}^{k_{r+1}}\cdots n_i^{k_i} n_r^{k_r-1}\pi(\boldsymbol{n}_r^-). \tag{4.48}$$

Substituting equations (4.45) and (4.46) into (4.48) gives

$$\begin{aligned}
\varepsilon(\boldsymbol{k},j) &= \rho_r \sum_{l=0}^{k_r-1}\binom{k_r-1}{l} \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}(n_r-1)^l n_{r+1}^{k_{r+1}}\cdots n_i^{k_i}\pi(\boldsymbol{n}_r^-)\\
&= \rho_r \sum_{l=0}^{k_r-1}\binom{k_r-1}{l} \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j-b_r\}} n_1^{k_1}\cdots n_{r-1}^{k_{r-1}} n_r^l n_{r+1}^{k_{r+1}}\cdots n_i^{k_i}\pi(\boldsymbol{n})\\
&= \rho_r \sum_{l=0}^{k_r-1}\binom{k_r-1}{l}\varepsilon(\boldsymbol{k}-(k_r-l)_r, j)
\end{aligned} \tag{4.49}$$

For $j<0$ or $j>T$, $q(j)=0$ and the result follows.

For $S<j\le T$, substituting the recursion $\pi(\boldsymbol{n}) = \frac{1}{S}\sum_{r=1}^{R} b_r\rho_r\pi(\boldsymbol{n}_r^-)$, into (4.45) gives

$$\begin{aligned}
\varepsilon(\boldsymbol{k},j) &= \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_1^{k_1}n_2^{k_2}\cdots n_i^{k_i}\frac{1}{S}\sum_{r=1}^{R} b_r\rho_r\pi(\boldsymbol{n}_r^-)\\
&= \frac{1}{S}\sum_{r=1}^{i} b_r\rho_r \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_1^{k_1}\cdots(n_r-1+1)^{k_r}\cdots n_i^{k_i}\pi(\boldsymbol{n}_r^-)\\
&\quad+\frac{1}{S}\sum_{r=i+1}^{R} b_r\rho_r \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_1^{k_1}\cdots n_i^{k_i}\pi(\boldsymbol{n}_r^-)\\
&\overset{(4.46)}{=} \frac{1}{S}\sum_{r=1}^{i} b_r\rho_r \sum_{l=0}^{k_r}\binom{k_r}{l} \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}(n_r-1)^l n_{r+1}^{k_{r+1}}\cdots n_i^{k_i}\pi(\boldsymbol{n}_r^-)\\
&\quad+\frac{1}{S}\sum_{r=i+1}^{R} b_r\rho_r \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_1^{k_1}\cdots n_i^{k_i}\pi(\boldsymbol{n}_r^-)\\
&= \frac{1}{S}\sum_{r=1}^{i} b_r\rho_r \sum_{l=0}^{k_r-1}\binom{k_r}{l} \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j-b_r\}} n_1^{k_1}\cdots n_{r-1}^{k_{r-1}} n_r^l n_{r+1}^{k_{r+1}}\cdots n_i^{k_i}\pi(\boldsymbol{n})\\
&\quad+\frac{1}{S}\sum_{r=1}^{R} b_r\rho_r \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j-b_r\}} n_1^{k_1}\cdots n_i^{k_i}\pi(\boldsymbol{n})\\
&= \frac{1}{S}\sum_{r=1}^{R} b_r\rho_r \sum_{l=0}^{k_r-1}\binom{k_r}{l}\varepsilon(\boldsymbol{k}-(k_r-l)_r, j-b_r)
\end{aligned}$$

$$+\frac{1}{S}\sum_{r=1}^{R} b_r\rho_r\varepsilon(\boldsymbol{k}, j-b_r) \tag{4.50}$$

The proof is now complete. **Q.E.D.**

The joint moments of $n_r$, $r = 1, 2, \ldots, R$, are

$$E\{n_1^{k_1}\cdots n_R^{k_R}\} = \sum_{j=0}^{T}\varepsilon(\boldsymbol{k}, j). \tag{4.51}$$

Let's derive some special cases. For $\boldsymbol{k} = (1)_r$, after some algebraic manipulation, equations (4.51) and (4.44) give the average number for type $r$:

$$\begin{aligned} E\{n_r\} &= \rho_r + \frac{\rho_r}{S - \sum_{i=1}^{R} b_i\rho_i}\left\{\sum_{j=S-b_r+1}^{S-1}(j-S)q(j) + b_r(P_{s_r} - P_{b_r})\right. \\ &\left. + \sum_{i=1}^{R} b_i\rho_i \sum_{j=T-b_i+1}^{T}\left[q(j) - \frac{1}{\rho_r}\varepsilon((1)_r, j)\right]\right\}, \end{aligned} \tag{4.52}$$

where

$$P_{s_r} = 1 - \sum_{j=0}^{S-b_r} q(j) \tag{4.53}$$

is the probability that a type $r$ arr val has to share the resource units (i.e. reduce its bandwidth). For $\boldsymbol{k} = (2)_r$, the result is the second moment of the population for type $r$:

$$\begin{aligned} E\{n_r^2\} &= \rho_r(1+\rho_r) + \frac{\rho_r}{S - \sum_{i=1}^{R} b_i\rho_i}\left\{\sum_{j=S-b_r+1}^{S-1}(j-S)q(j) + \sum_{j=S-2b_r+1}^{S-1}(j-S)q(j)\right. \\ &+ b_r(P_{s_r} - P_{b_r}) + 2b_r\left[E\{n_r\} - \rho_r\sum_{j=0}^{S-2b_r} q(j) - \sum_{j=T-b_i+1}^{T}\varepsilon((1)_r, j)\right] \\ &\left. + \sum_{i=1}^{R} b_i\rho_i \sum_{j=T-b_i+1}^{T}\left[(1+\rho_r)q(j) - \frac{1}{\rho_r}\varepsilon((2)_r, j)\right]\right\}. \end{aligned} \tag{4.54}$$

**Implementation Issues.** I close this section by presenting an algorithm and discussing the space requirements for the computation of the occupancy distribution.

the normalization constant, the moments and the blocking probabilities.

The algorithm is given in Figures 4.3 and 4.4. The values of $q(j)$ and $\varepsilon(\boldsymbol{k}, j)$ can be computed in the same iteration as the algorithm loops from $j = 0$ to $j = T$. For the computation of $q(j)$ the algorithm needs the $b_{\max}$ most recently computed values, where $b_{\max} = \max\{b_r, r = 1, 2, \ldots, R\}$. For the computation of $\varepsilon(\boldsymbol{k}, j)$:

- If $0 \leq j \leq S - b_{\max}$, the algorithm needs the $k_r$ values of the terms $\varepsilon(\boldsymbol{k}(k_r - l)_r, j - b_r)$, $l = 0, 1, \ldots, k_r$, for each $r = 1, 2, \ldots, R$. These values require $(b_{\max} + 1)k_r$ memory positions.
- If $S < j \leq T$, the algorithm needs the $b_{\max}$ most recently computed values of $\varepsilon(\boldsymbol{k}, j)$ and all the values $\varepsilon(\boldsymbol{k}', j - b_r)$, for each $r = 1, 2, \ldots, R$, such that $k'_r \leq k_r - 1$.

So starting from $\varepsilon(\boldsymbol{u}, j)$ at $\boldsymbol{u} = \boldsymbol{0}$ and incrementing $\boldsymbol{u}$ until $\boldsymbol{u} = \boldsymbol{k}$, the maximum space requirement is $(b_{\max} + 1)\prod_{r=1}^{R}(k_r + 1)$. Additionally they are needed:

- One variable to accumulate the sum $\sum_{j'=0}^{j} q'(j')$, for $j = 0, 1, \ldots, T$, (the normalization constant).
- One variable to accumulate the sum $\sum_{j'=0}^{j} \varepsilon(\boldsymbol{k}, j')$, for $j = 0, 1, \ldots, T$.
- $R$ variables to accumulate the sums $\sum_{j'=T-b_r+1}^{j} q(j')$, $j = T - b_r + 1, \ldots, T$, $r = 1, 2, \ldots, R$ (the blocking probabilities).

It should be noted that even though the moment generating recursion is multidimensional, its space requirement is constant for given bandwidth requirements. In most cases of practical interest it suffices to compute only the variance for the population of a single customer type or the covariance for populations of different types. In those cases the worst-case space-requirement is $\Theta(2^R b_{\max})$

The time complexity grows linearly with the occupancy. The issue of time optimization is not addressed in this thesis.

**Input**

The parameters of the extended shared resource model:

- The bandwidth requirement $b_r$ and load $\rho_r = \frac{\lambda_r}{\mu_r}$ for type $r$ customers, $r = 1, 2, \ldots, R$.
- A vector $\boldsymbol{k} = (k_1, k_2, \ldots, k_R)$ defining the joint moment $E\{n_1^{k_1} n_2^{k_2} \cdots n_R^{k_R}\}$ to be computed.

**Output**

- The normalization constant $G$.
- The joint moment $\mathcal{E}(\boldsymbol{k}) = E\{n_1^{k_1} n_2^{k_2} \cdots n_R^{k_R}\}$.
- The blocking probabilities $P_{b_r}$ for type $r$ customers, $r = 1, 2, \ldots, R$.

**The Algorithm**

$b \leftarrow \max\{b_1, b_2, \ldots, b_R\} + 1;$

$G \leftarrow 0;\ \mathcal{E}(\boldsymbol{k}) \leftarrow \boldsymbol{0};\ \varepsilon(\boldsymbol{0}, 0) = 1;$

**for** $j \leftarrow 0$ **to** $S$ **do** {

  $\boldsymbol{u} \leftarrow \boldsymbol{0};$

  $\varepsilon(\boldsymbol{u}, j \bmod b) \leftarrow \frac{1}{j} \sum_{r=1}^{R} b_r \rho_r \varepsilon(\boldsymbol{u}, (j - b_r) \bmod b);$

  $G \leftarrow G + \varepsilon(\boldsymbol{u}, j \bmod b);$

  **for** $r \leftarrow 1$ **to** $R$ **do** {

    $\boldsymbol{u} \leftarrow \boldsymbol{u}_r^+;$

    **while** (incr($\boldsymbol{u}, r, \boldsymbol{k}$)) **do** {

      $\varepsilon(\boldsymbol{u}, j \bmod b) \leftarrow \rho_r \sum_{l=1}^{u_r - 1} \binom{u_r - 1}{l} \varepsilon(\boldsymbol{u} - (u_r - l)_r, (j - b_r) \bmod b);$

    }

  }

  $\mathcal{E}(\boldsymbol{k}) \leftarrow \mathcal{E}(\boldsymbol{k}) + \varepsilon(\boldsymbol{k}, j \bmod b);$

}

**for** $j \leftarrow S + 1$ **to** $T$ **do** {

  $\boldsymbol{u} \leftarrow \boldsymbol{0};$

  $\varepsilon(\boldsymbol{u}, j \bmod b) \leftarrow \frac{1}{j} \sum_{r=1}^{R} b_r \rho_r \varepsilon(\boldsymbol{u}, (j - b_r) \bmod b);$

  $G \leftarrow G + \varepsilon(\boldsymbol{u}, j \bmod b);$

  **while** (incr($\boldsymbol{u}, 1, \boldsymbol{k}$)) **do** {

    $\varepsilon(\boldsymbol{u}, j \bmod b) \leftarrow \frac{1}{S} \sum_{r=1}^{R} b_r \rho_r [\varepsilon(\boldsymbol{u}, j \bmod b)$
      $+ \sum_{l=1}^{u_r - 1} \binom{u_r - 1}{l} \varepsilon(\boldsymbol{u} - (u_r - l)_r, (j - b_r) \bmod b)];$

  }

  $\mathcal{E}(\boldsymbol{k}) \leftarrow \mathcal{E}(\boldsymbol{k}) + \varepsilon(\boldsymbol{k}, j \bmod b);$

}

$\mathcal{E}(\boldsymbol{k}) \leftarrow G^{-1} \mathcal{E}(\boldsymbol{k});$

**for** $r \leftarrow 1$ **to** $R$ **do** {

  $P_{b_r} \leftarrow G^{-1} \sum_{i=1}^{b_r - 1} \varepsilon(\boldsymbol{0}, (T - i) \bmod b);$

}

Figure 4.3: An algorithm for the computation of performance measures for the extended shared resource model. The function *incr()* is defined in Figure 4.4.

**Input**

Two vectors $\boldsymbol{u} = (u_1, u_2, \ldots, u_R)$ and $\boldsymbol{k} = (k_1, k_2, \ldots, k_R)$ and a subscript $i$, $1 \leq i \leq R$.

**Effect**

If the occupancy $j$ is $j \leq S - b_{\max}$, it increments $u_i$ by 1; this causes a lexicographic increase in $\boldsymbol{u}$.
Otherwise, it increments $u_i$ in modulo $k_i$ arithmetic and passes the quotient to $k_{i+1}$.
It returns **true** if an increment is possible and **false** otherwise.

```
function incr(u, i, k):boolean
    Let r be the subscript of the first nonzero element of k;
    if (i < r or i > R) return(false);
    Increment u_i lexicographically,
    if (j ≤ S − b_max) {
        if (u_i = k_i) return(false);
        else u_i ← u_i + 1;
        return(true);
    }
    or, increment u_i in modulo k_i arithmetic,
    else if (u_i = k_i and incr(u, r + 1, k); {
        u_i ← (u_i + 1)mod(k_i + 1);
        return(true);
    }
    or, fail (u_i cannot be incremented).
    return(false);
return;
```

Figure 4.4: The function incrementing the vector associated with the joint moment to be computed.

## 4.5 The Behavior of the ESR Model

### 4.5.1 Special Cases

Next, I show how the ESR model can reduce to some well-known queueing disciplines. The results can be verified from Lavenberg [29].

#### $M/G/S$ with Processor-Sharing and Loss

For $b_r = 1$, $r = 1, 2, \ldots, R$, the boundary defined by the capacity of the resource becomes

$$\mathcal{I} = \{\boldsymbol{m} | m_1 + m_2 + \cdots + m_R = S\}.$$

Let $n = \sum_{r=1}^{R} n_r$. Then, for $n > S$,

$$\begin{aligned} x(\boldsymbol{n}) &= \frac{(n-S)!}{S^{n-S}} \sum_{\{\boldsymbol{m}|m_1+\cdots+m_R=S\}} \prod_{r=1}^{R} \binom{n_r}{m_r} \\ &= \frac{(n-S)!}{S^{n-S}} \binom{n}{S} \\ &= \frac{n!}{S!S^{n-S}}. \end{aligned} \tag{4.55}$$

Then, the population and occupancy distributions are

$$\pi(\boldsymbol{n}) = \begin{cases} G(\Omega)^{-1} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!} & \text{if } n \le S, \\ G(\Omega)^{-1} \frac{n!}{S!S^{n-S}} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!}, & \text{if } S < n \le T. \end{cases} \tag{4.56}$$

$$q(j) = \begin{cases} G(\Omega)^{-1} \frac{\rho^j}{j!}, & \text{if } j \le S, \\ G(\Omega)^{-1} \left(\frac{\rho}{S}\right)^j \frac{S^S}{S!}, & \text{if } S < j \le T, \end{cases} \tag{4.57}$$

where $\rho = \sum_{r=1}^{R} \rho_r$. The normalization constant is

$$G(\Omega) = \sum_{j=0}^{T} q(j)$$

$$= \sum_{j=0}^{S-1} \frac{\rho^j}{j!} + \frac{S^{S+1}}{S!(S-\rho)} \left[ \left(\frac{\rho}{S}\right)^S - \left(\frac{\rho}{S}\right)^{T+1} \right]. \tag{4.58}$$

The probability of blocking becomes

$$P_b = G(\Omega)^{-1} \frac{S^S}{S!} \left(\frac{\rho}{S}\right)^T. \tag{4.59}$$

Equation (4.44) reduces to

$$\varepsilon((1)_r, j) = \begin{cases} G(\Omega)^{-1} \rho_r \dfrac{\rho^{j-1}}{(j-1)!}, & \text{if } j \leq S, \\ G(\Omega)^{-1} \dfrac{\rho_r}{S!} \left(\dfrac{\rho}{S}\right)^{j-1} [1 + S^{S-1}(j-S)], & \text{if } S < j \leq T. \end{cases} \tag{4.60}$$

Then, the average number in the system, for class $r$, is

$$E\{n_r\} = \rho_r \sum_{j=0}^{S-1} \frac{\rho^j}{j!} + \rho_r \frac{\left(\frac{\rho}{S}\right)^S (S+1-\rho) - \left(\frac{\rho}{S}\right)^T (T+1-\frac{T}{S}\rho)}{S-\rho} \cdot \frac{S^S}{S!}. \tag{4.61}$$

### $M/G/S$ with Processor-Sharing (Infinite Population)

For $T \to \infty$, the previous results are further simplified to

$$\begin{aligned} \pi(\boldsymbol{n}) &= \begin{cases} G(\Omega)^{-1} \displaystyle\prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!} & \text{if } n \leq S, \\ G(\Omega)^{-1} \dfrac{n!}{S! S^{n-S}} \displaystyle\prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!}, & \text{if } n > S. \end{cases} \\ G(\Omega) &= \sum_{j=0}^{S-1} \frac{\rho^j}{j!} + \frac{S\rho^S}{S!(S-\rho)}, \\ E\{n_r\} &= \rho_r \left[ 1 + \frac{S\rho^S}{S!(S-\rho)^2 G(\Omega)} \right]. \end{aligned}$$

The probability of queueing is equal to the probability $P_s$ that new arrivals will initiate processor-sharing. Then, for $R = 1$,

$$P_s = \sum_{n=S}^{\infty} \pi(n) = G(\Omega)^{-1} \frac{(S\rho)^S}{S!(1-\rho)}, \qquad \text{(Erlang C Formula)}.$$

**$M/G/S$ with Loss (The Truncated Infinite Server)**

For $T = S$, the results for $M/G/S$ with processor-sharing and loss reduce to

$$\begin{aligned} \pi(\boldsymbol{n}) &= G(\Omega)^{-1} \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!} \\ G(\Omega) &= \sum_{j=0}^{S} \frac{\rho^j}{j!}, \\ E\{n_r\} &= \rho_r \left[ 1 - \frac{\rho^S}{S! G(\Omega)} \right], \\ P_b &= G(\Omega)^{-1} \frac{\rho^S}{S!}, \qquad \text{(Erlang B Formula)}. \end{aligned}$$

### 4.5.2 The Limiting Behavior as $T \to \infty$

The ESR model behaves as an infinite server for as long as the occupancy does not exceed its capacity $S$ and as non-egalitarian processor-sharing when the capacity is exceeded. Next, I show that, as the population increases, the ESR model behaves asymptotically as an egalitarian processor-sharing model.

The egalitarian processor-sharing model corresponds to the cases where $b_r = 1$, $r = 1, 2, \ldots, R$ or $\mathcal{I} = \{\mathbf{0}\}$, i.e. when the customers share all the available servers. The first case has already been examined. In the second case,

$$x(\boldsymbol{n}) = \left( \sum_{r=1}^{R} n_r \right)! \prod_{r=1}^{R} \left( \frac{b_r}{S} \right)^{n_r}.$$

Then, for either case, the fraction of the bandwidth requirement allocated to type $r$ customers is

$$\phi_r(\boldsymbol{n}) = \frac{x(\boldsymbol{n}_r^-)}{x(\boldsymbol{n})} = \frac{S}{\left( \sum_{q=1}^{R} n_q \right) b_r}.$$

Now consider the behavior of the ESR model far from the sharing bandwidth. For $n_r \gg m_r$, $r = 1, 2, \ldots, R$,

$$\begin{aligned} (n_r - m_r)! &\approx n_r! n_r^{-m_r} \\ \text{and} \quad \binom{n_r}{m_r} &\approx \frac{n_r^{-m_r}}{m_r!}. \end{aligned} \tag{4.62}$$

Then,

$$\begin{aligned} x(\boldsymbol{n}) &\approx \left(\sum_{q=1}^{R} n_q\right)! \prod_{q=1}^{R} \left(\frac{b_q}{S}\right)^{n_q} \vartheta(\boldsymbol{n}) \\ \text{and} \quad x(\boldsymbol{n}_r^-) &\approx \left(\sum_{q=1}^{R} n_q - 1\right)! \frac{S}{b_q} \prod_{q=1}^{R} \left(\frac{b_q}{S}\right)^{n_q} \vartheta(\boldsymbol{n}_r^-), \end{aligned} \tag{4.63}$$

where

$$\vartheta(\boldsymbol{n}) = \sum_{\boldsymbol{m} \in \mathcal{I}^+} \left(\sum_{q=1}^{R} n_q\right)^{-\sum_{q=1}^{R} m_q} \prod_{q=1}^{R} \frac{n_q^{m_q}}{m_q!} \left(\frac{b_q}{S}\right)^{-m_q}. \tag{4.64}$$

The reduction factor, for type $r$ customers, becomes

$$\phi_r(\boldsymbol{n}) = \frac{x(\boldsymbol{n}_r^-)}{x(\boldsymbol{n})} \approx \frac{S}{\left(\sum_{q=1}^{R} n_q\right) b_r} \cdot \frac{\vartheta(\boldsymbol{n}_r^-)}{\vartheta(\boldsymbol{n})}. \tag{4.65}$$

Since $n_r \gg m_r$, $r = 1, 2, \ldots, R$, $\vartheta(\boldsymbol{n}_r^-) \approx \vartheta(\boldsymbol{n})$. Thus, we finally get

$$\phi_r(\boldsymbol{n}) \approx \frac{S}{\left(\sum_{q=1}^{R} n_q\right) b_r}. \tag{4.66}$$

So, in the vicinity of the sharing boundary $\mathcal{I}$, the ESR model behaves according to its original specification, i.e. non-egalitarian service. This behavior is defined by the irregularities in the shape of the boundary. As we move far from $\mathcal{I}$, i.e. under heavy loading conditions, the effect of these irregularities diminishes and the system behaves as if $\mathcal{I}$ is a single point in the origin of $\mathbf{Z}^R$.

## 4.6 The ESR Model in the BCMP Framework

The next theorem shows that the ESR model can be included in the BCMP framework and satisfies the criteria for network product-form solution. The proof is based on the Product-Form Theorem (Section 2.3, Theorem 2.9).

**Theorem 4.6** *The queueing discipline of the ESR model is station balancing and its characteristic function is given by*

$$\chi(\boldsymbol{n}) = x(\boldsymbol{n}) \prod_{r=1}^{R} \frac{1}{n_r!}, \qquad \forall \boldsymbol{n} \in \Omega. \tag{4.67}$$

**Proof**

The parameters of the discipline are

$$\gamma(i|\boldsymbol{n},r) = \frac{1}{n_r},\ \xi(i,r|\boldsymbol{n}) = \phi_r(\boldsymbol{n}) \text{ and } \Xi(r|\boldsymbol{n}) = n_r\phi_r(\boldsymbol{n}),\ \forall \boldsymbol{n} \in \Omega \wedge i = 1,2,\ldots,n_r.$$

If the characteristic function exists it must satisfy the equation

$$\frac{\chi(\boldsymbol{n}_r^-)}{\chi(\boldsymbol{n})} = n_r \frac{x(\boldsymbol{n}_r^-)}{x(\boldsymbol{n})}$$

and, therefore, should have the form

$$\chi(\boldsymbol{n}) = f(\boldsymbol{n})x(\boldsymbol{n}),$$

where $f$ is a function of the occupancy $\boldsymbol{n}$. Then,

$$f(\boldsymbol{n}) = \frac{1}{n_r} f(\boldsymbol{n}_r^-) = \prod_{r=1}^{R} \frac{1}{n_r!}$$

and equation (4.67) follows. Since the discipline also satisfies the definition of station balancing, the proof is complete. **Q.E.D.**

The work conserving constraint introduced in Section 4.2 and expressed by equation (4.9) results in a unique solution for the state multipliers and guarantees the existence of the characteristic function.

## 4.7 Numerical Examples

In the following, I show the behavior of the ESR model by means of examples and compare it with similar models that exist in the literature. The measures of performance used are the throughput, average number in the system, delay and average number of busy servers. The throughput for each type is $\lambda_r(1 - P_{b_r})$, $r = 1,2$. Then, the average delay for type $r$ is derived from Little's formula as

$$D_r = \frac{E\{n_r\}}{\lambda_r(1 - P_{b_r})}, \qquad r = 1, 2, \ldots, R.$$

Further design issues are addressed by studying the effect of the occupancy and buffer size on the bandwidth and blocking probability of each customer type.

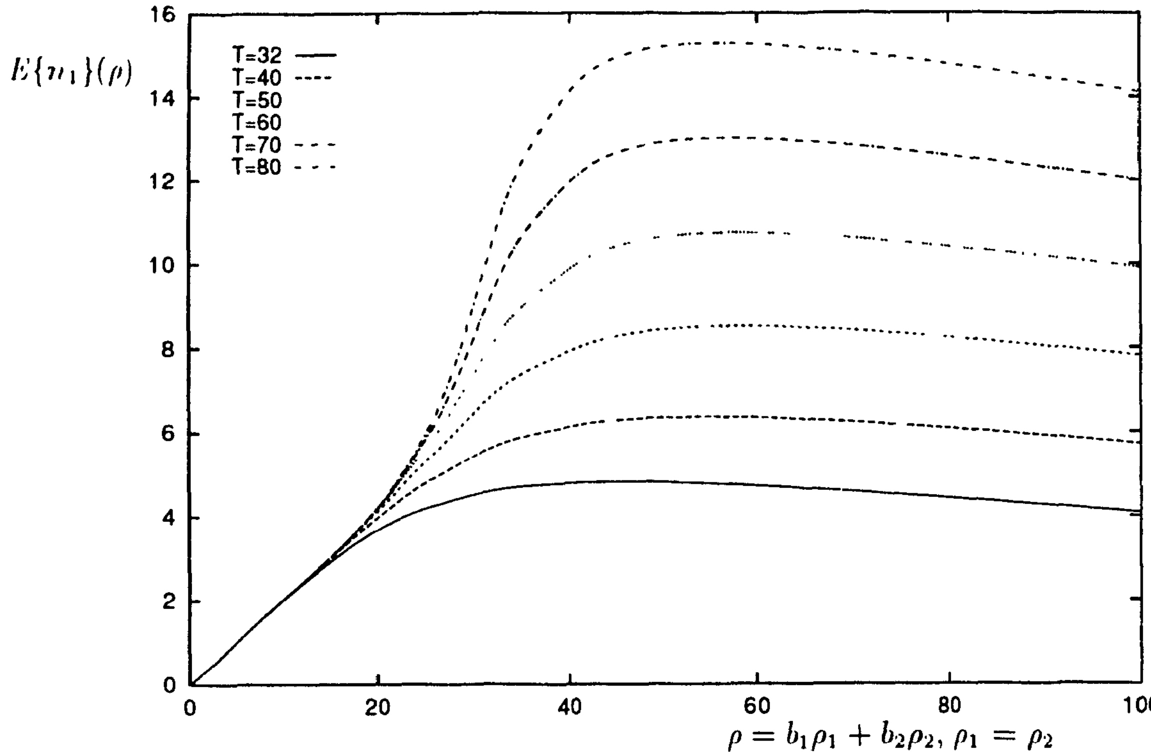
**Example 4.3** Consider a resource with $S = 32$ servers. Two types of customers arrive at rates $\lambda_1$ and $\lambda_2 = 2\lambda_1$. Their bandwidth requirements are $b_1 = 4$ and $b_2 = 1$ and their residency times are $\mu_1^{-1} = 2$ and $\mu_2^{-1} = 1$. The load on the system is $\rho = b_1\rho_1 + b_2\rho_2$, where $\rho_r = \lambda_r/\mu_r$, $r = 1, 2$.

The average number of each customer type versus the offered load, for different occupancy boundaries $T$, is plotted in Figure 4.5. The average occupancy and average number of busy servers, for various occupancy boundaries, are given in Figure 4.6. The blocking probability, as a function of the load, is shown in Figure 4.7. The delay-versus-load curves are drawn in Figure 4.8.

As expected, the average number of type 1 customers decreases as the load passes the capacity of the resource. The effect of heavy load on the average number in the system and the delay is shown in Figure 4.9. This behavior is due to the fact that customers of type 1 have larger bandwidth requirements and therefore experience higher blocking probability than the customers of type 2. Subsequently, the throughput for type 1 customers can only reach a maximum value and then start dropping. In Figure 4.10, I show the average delay as a function of the throughput and indicate the values of $\rho$ that maximize the throughput for type 1. The effect of different mixes of load on the delay and the blocking probabilities is shown in Figures 4.11, 4.12 and 4.13.

Finally, in Figures 4.14 and 4.15, I show the fraction of the bandwidth allocated to each customer class and the fraction of the bandwidth that would be allocated if the discipline were egalitarian processor-sharing. The effect of processor sharing on bandwidth reduction is felt more by the customers with large bandwidth requirements.

■



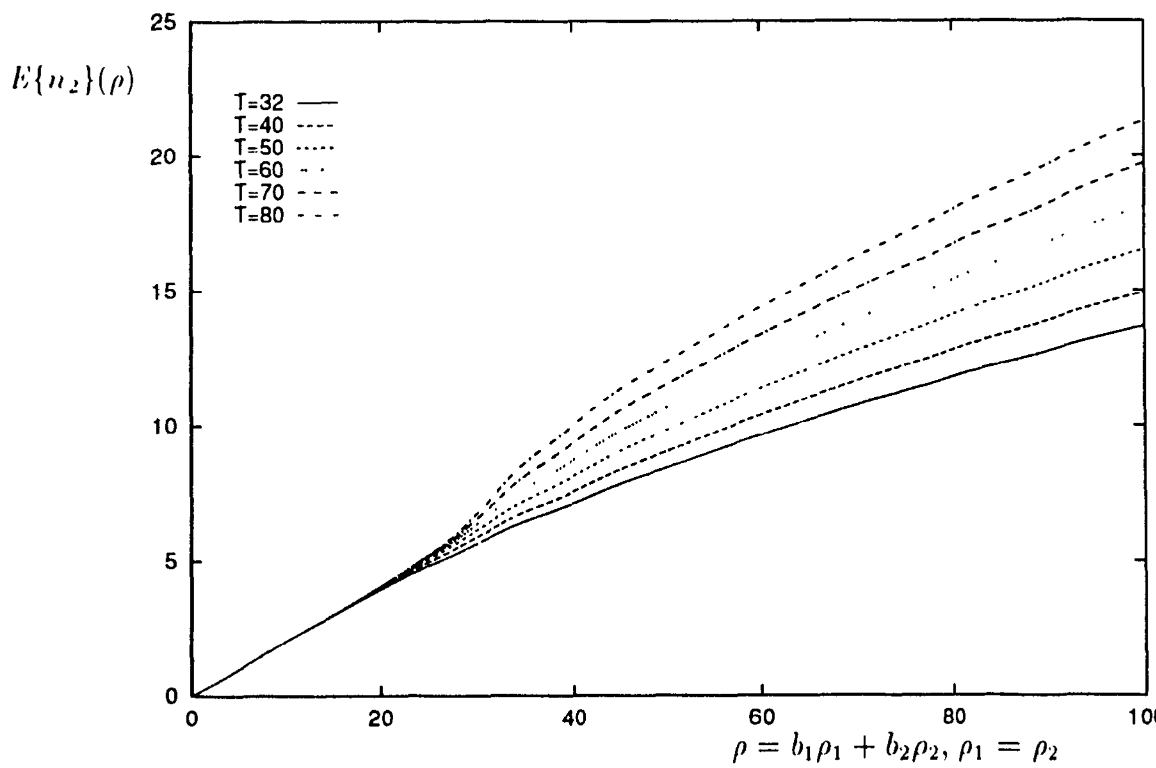




Figure 4.5: The effect of load on the average number in the system for different occupancy boundaries.

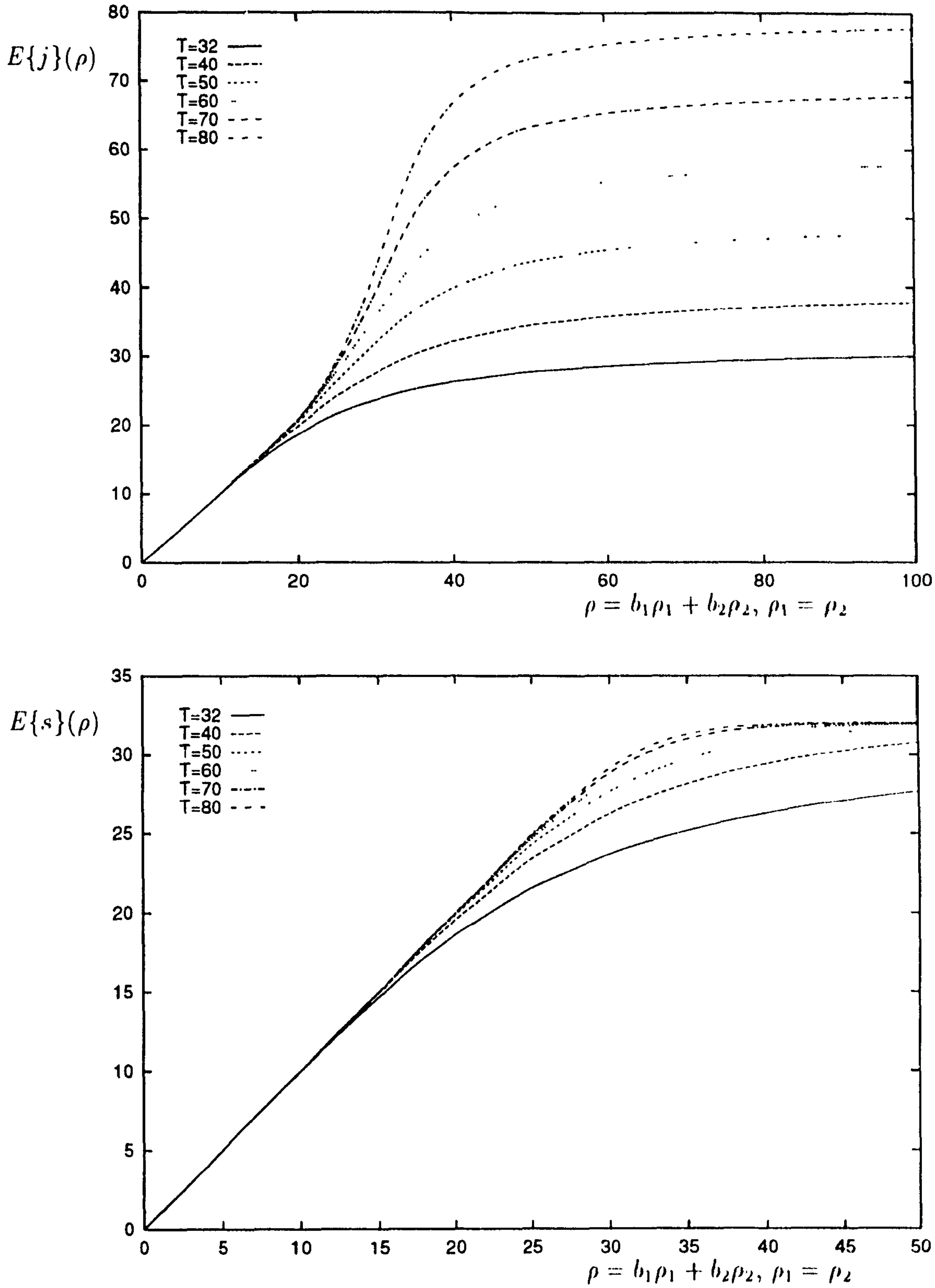
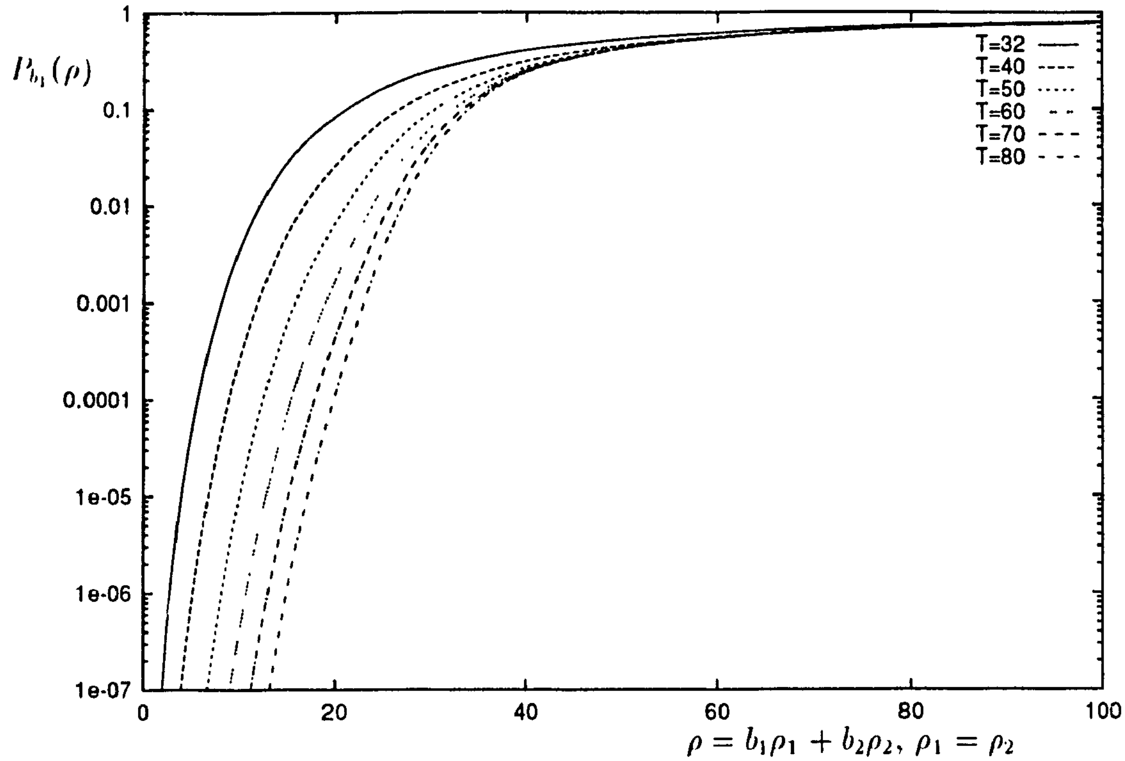



Figure 4.6: The effect of load on the average occupancy $E\{j\}$ and the average number of busy servers $E\{s\}$.



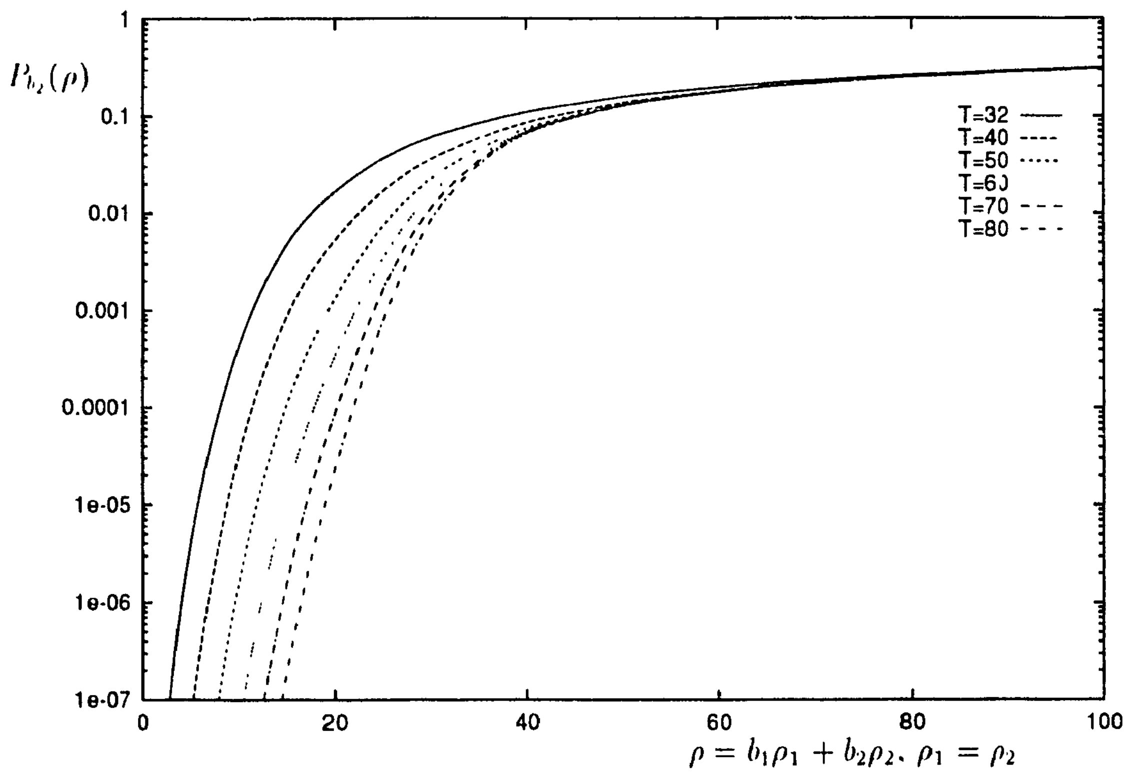
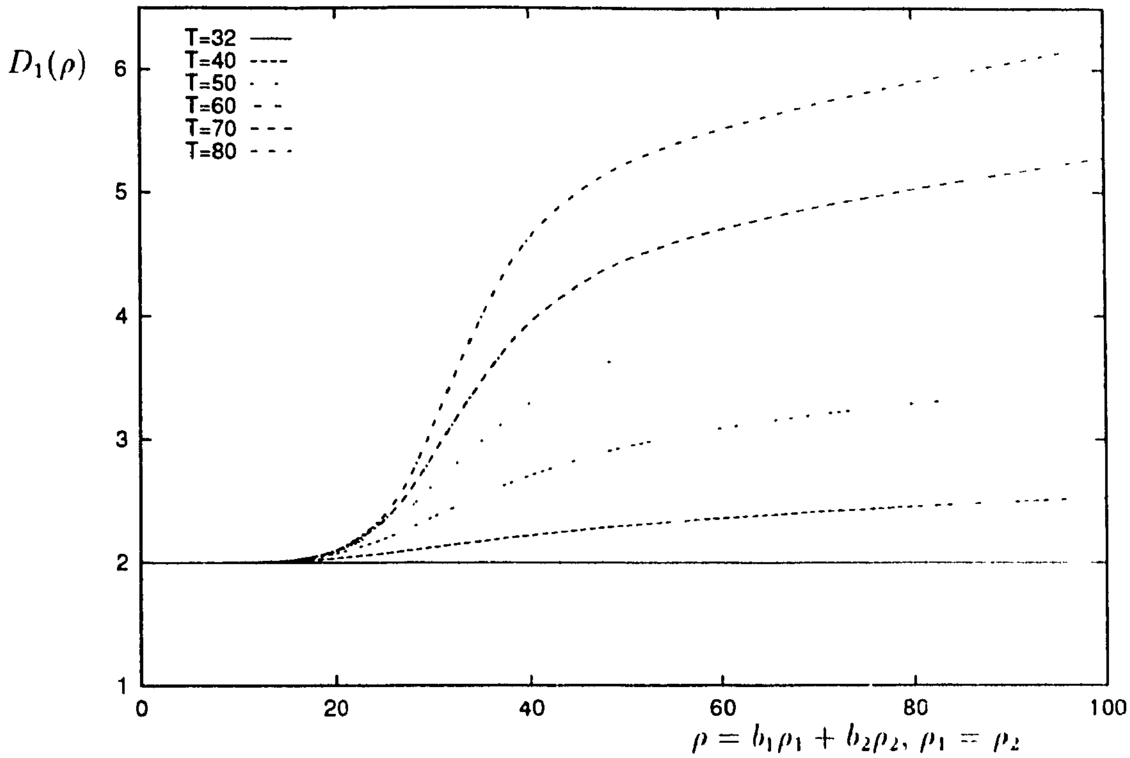




Figure 4.7: The effect of load on the blocking probabilities for different occupancy boundaries.



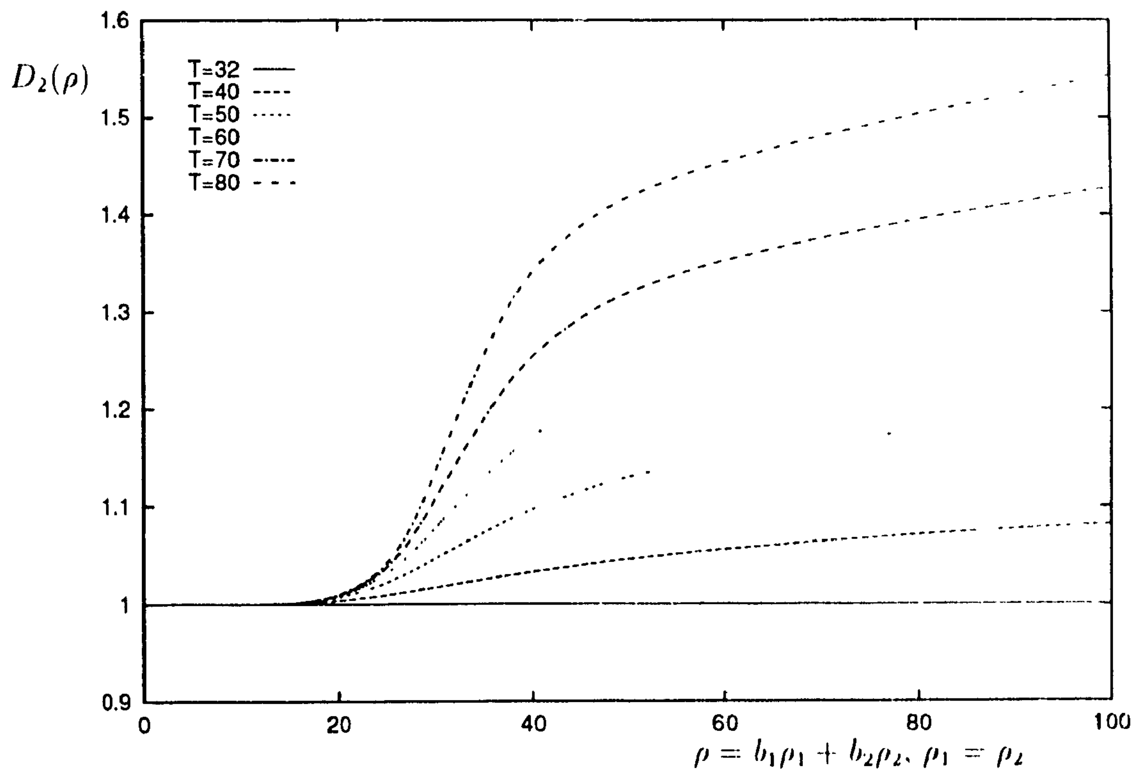




Figure 4.8: The effect of load on the average delay for different occupancy boundaries.

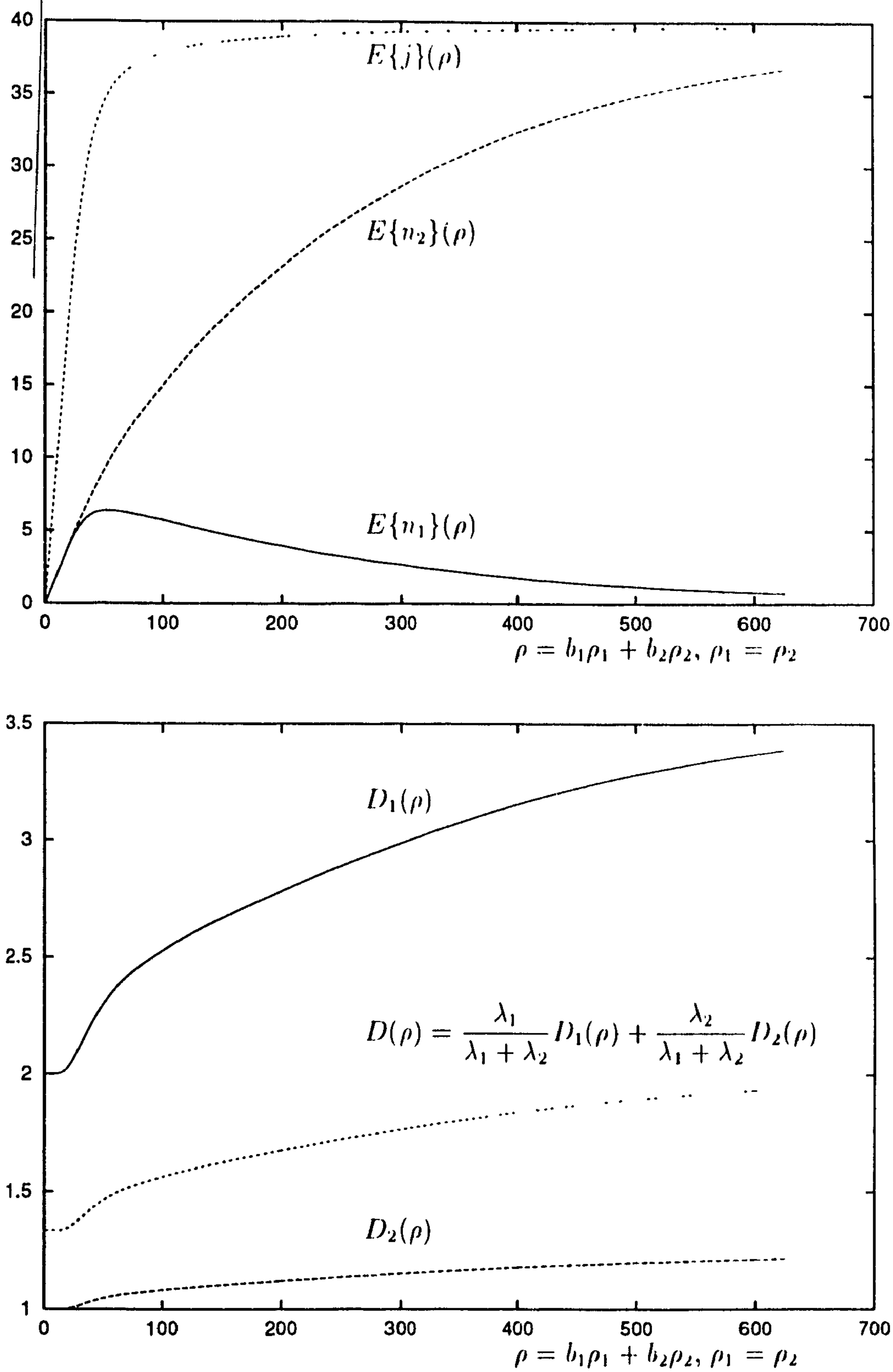



Figure 4.9: The effect of load on the average number in the system and the delay. $D$ is the delay that any customer may experience.

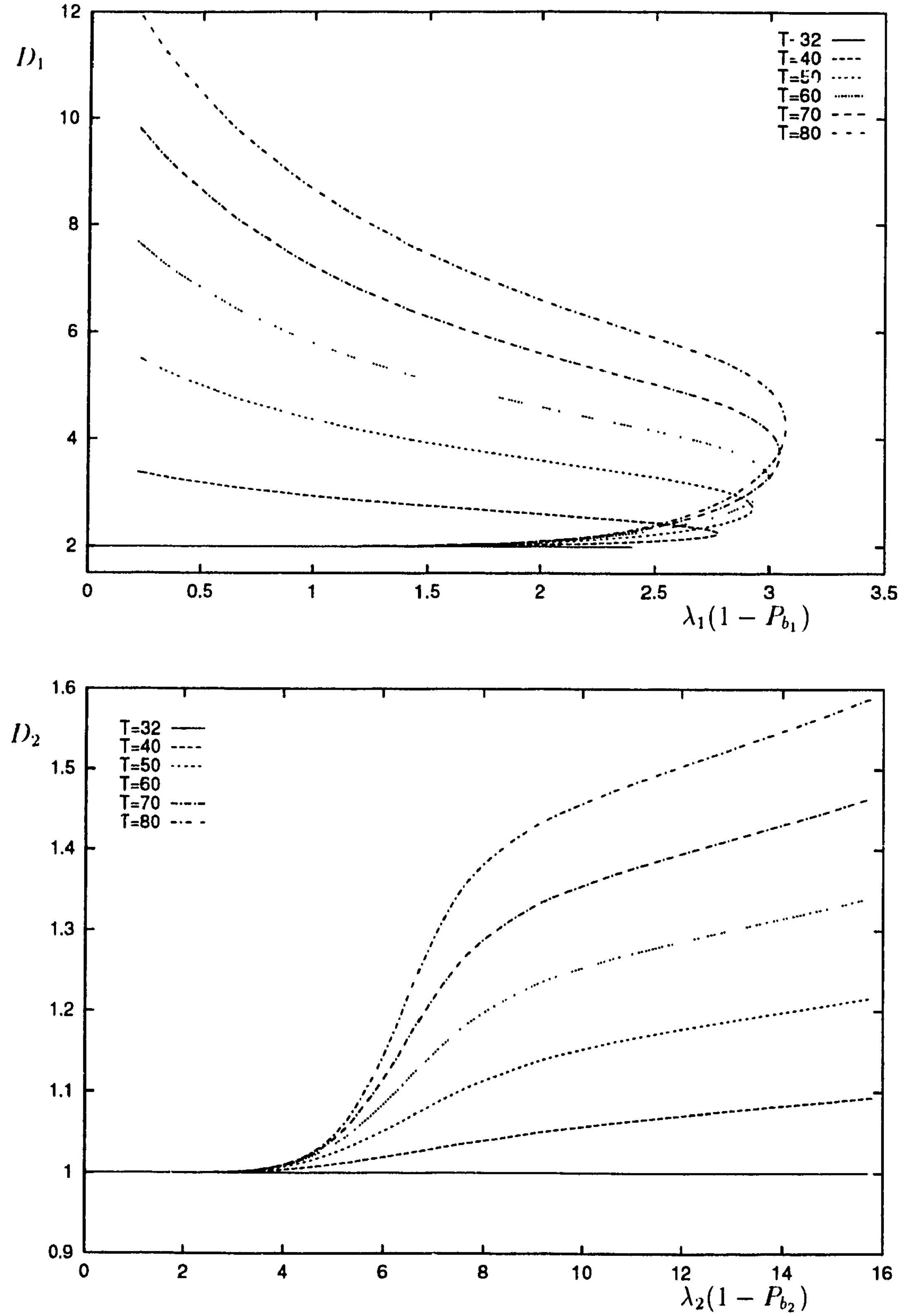



Figure 4.10: Delay versus throughput for different occupancy boundaries. It is assumed that $\rho_1 = \rho_2$. The curves were produced by increasing $\rho$.

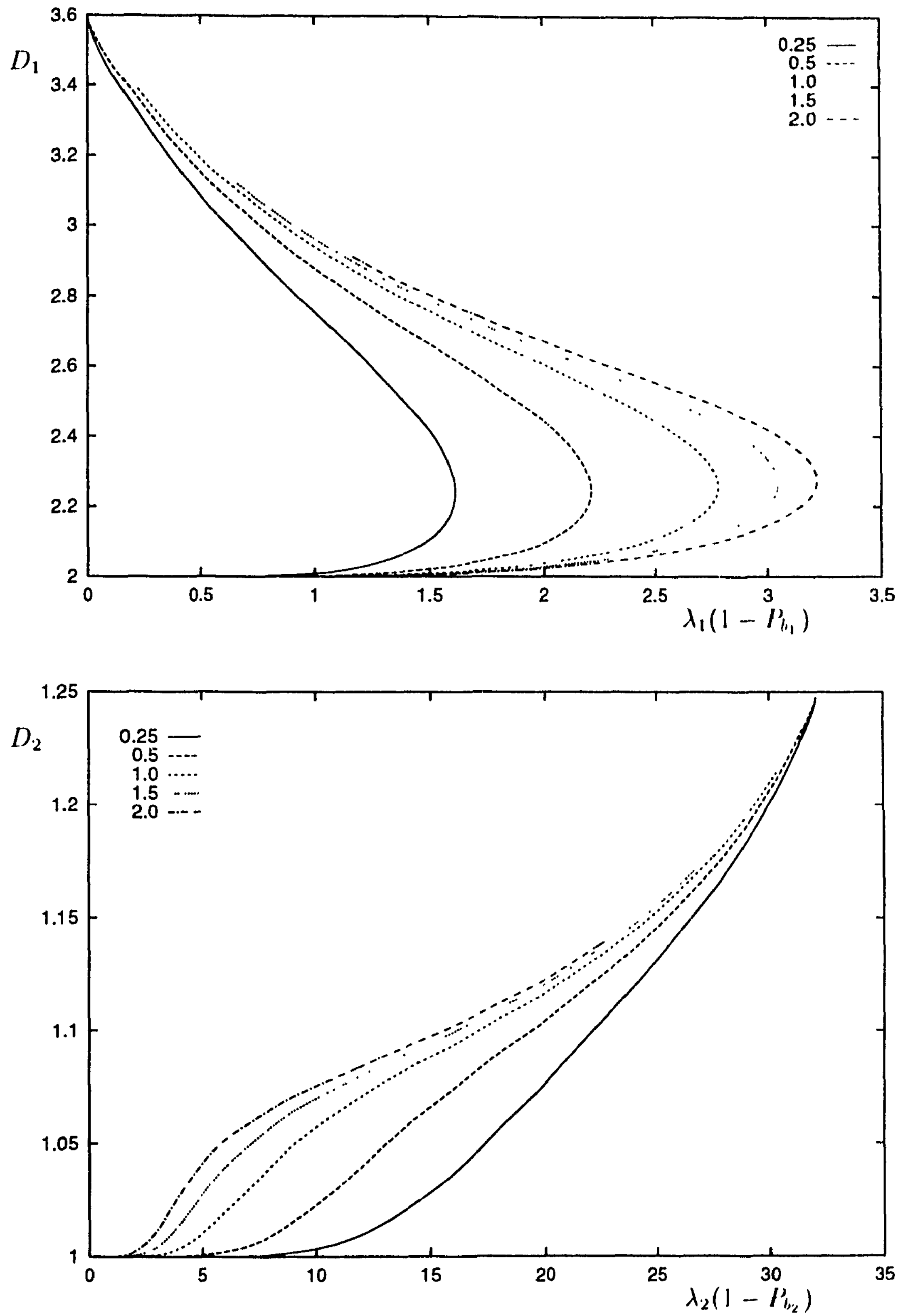
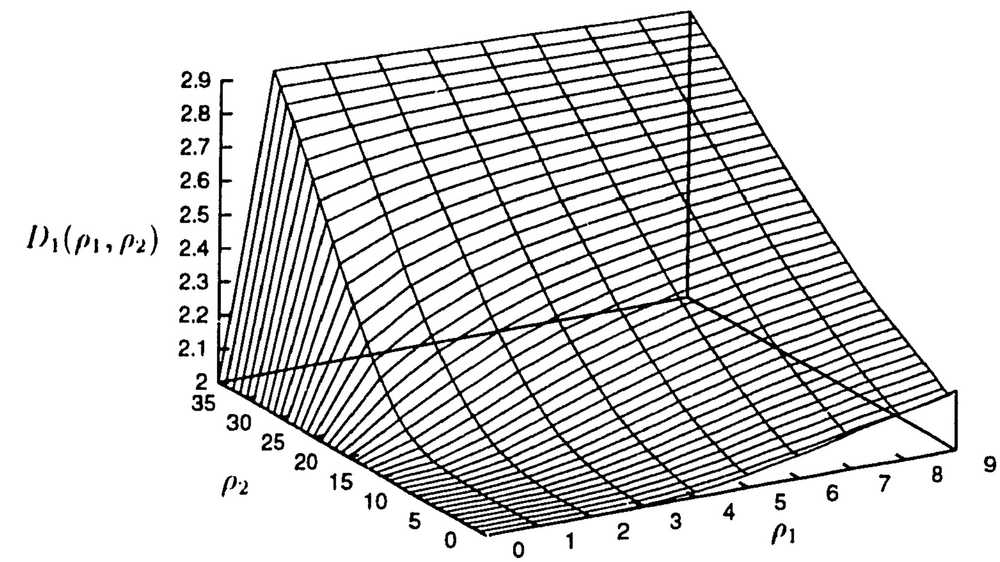



Figure 4.11: Delay versus throughput for different ratios $\rho_1/\rho_2$ of load. The occupancy limit is $T = 40$.



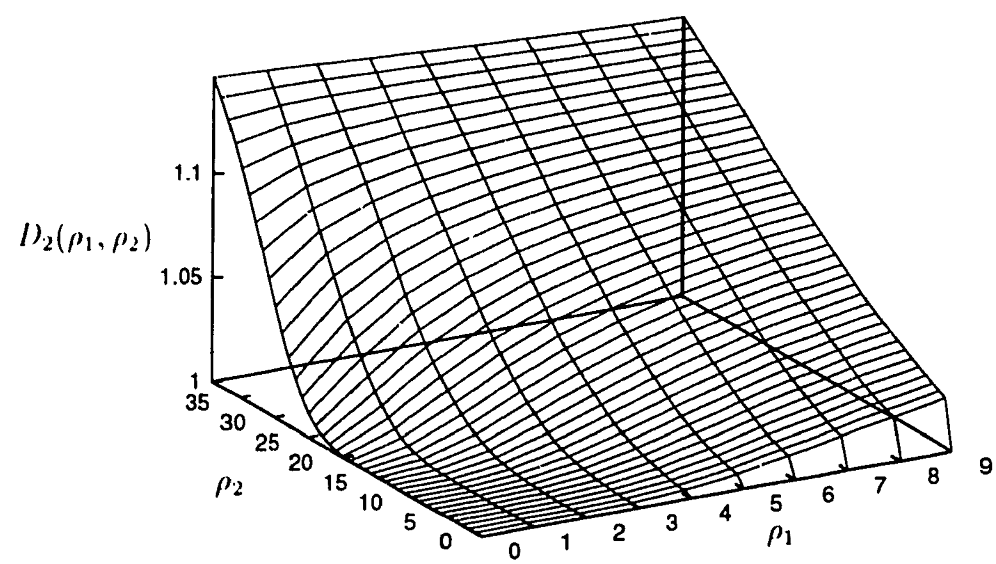
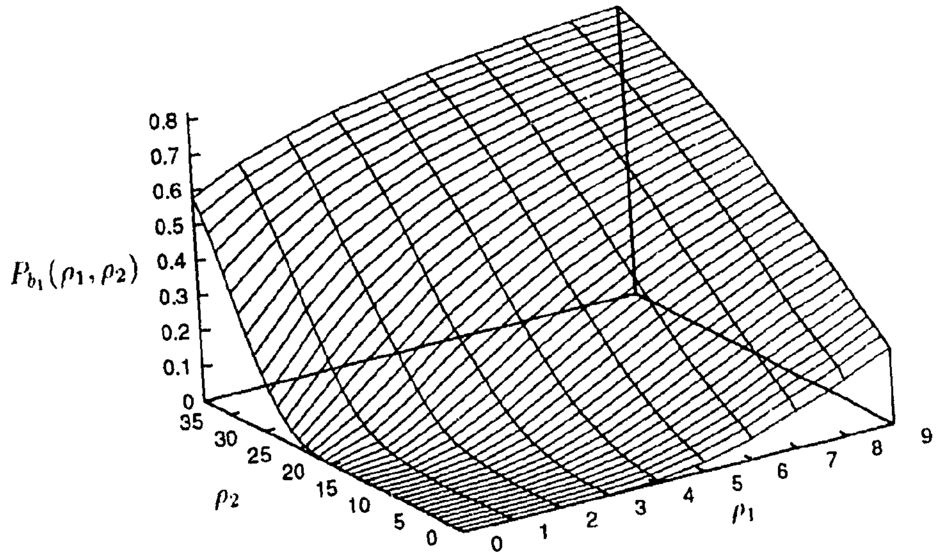




Figure 4.12: The average delays as functions of the load. The occupancy limit is $T = 40$.



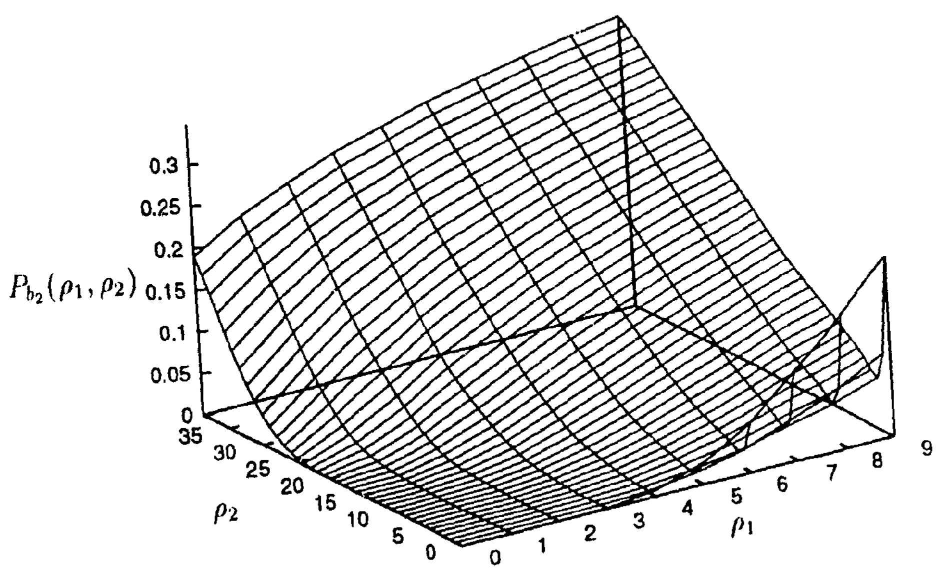




Figure 4.13: The blocking probabilities as functions of the load. The occupancy limit is $T = 40$.

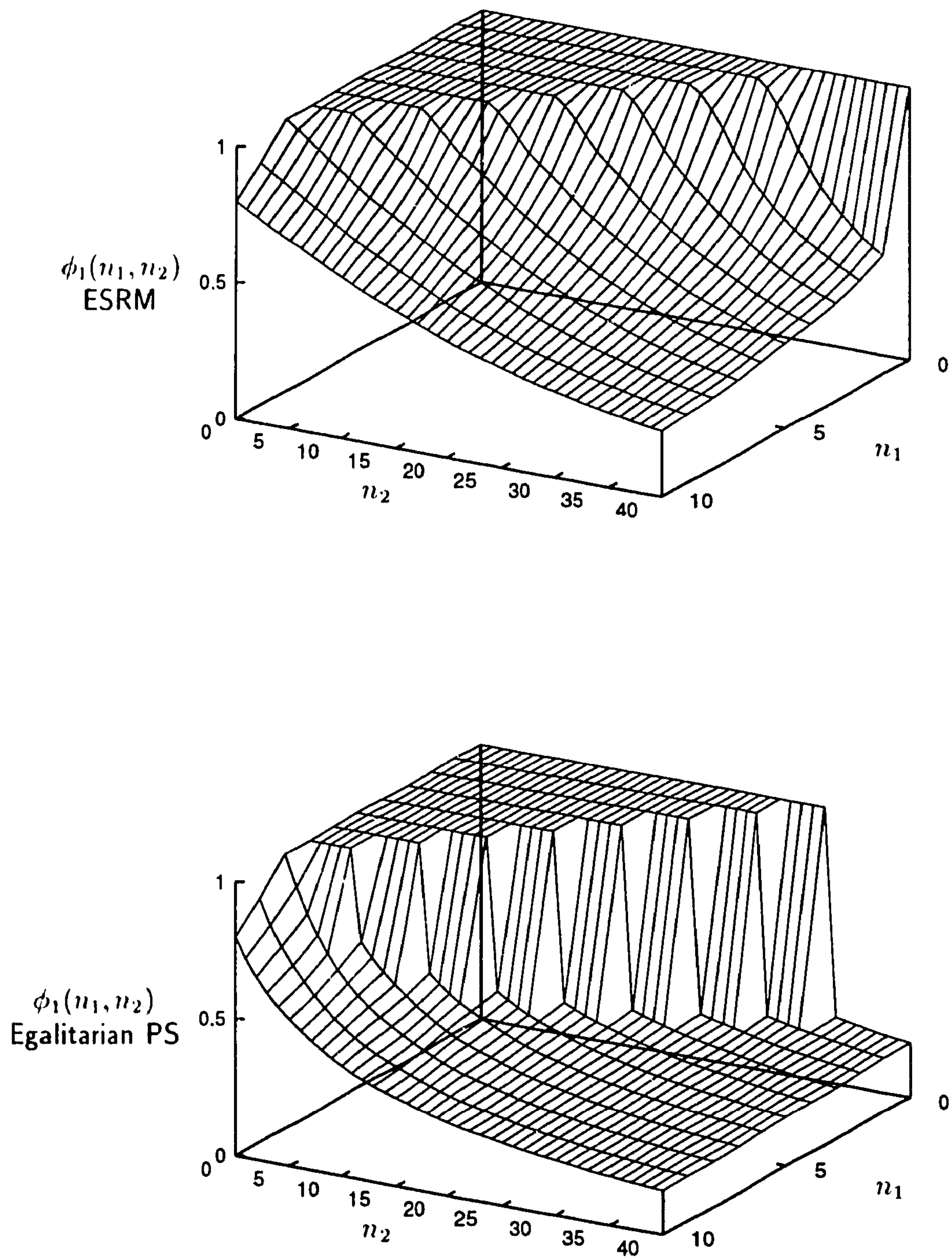
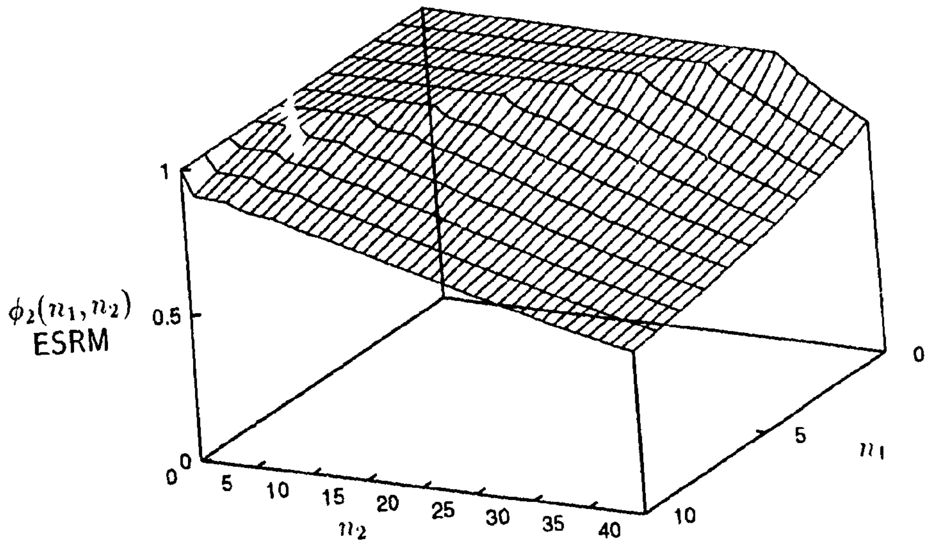



Figure 4.14: The fraction of bandwidth requirement allocated to type 1 customers for the extended shared resource (ESR) model and the egalitarian processor sharing (PS).



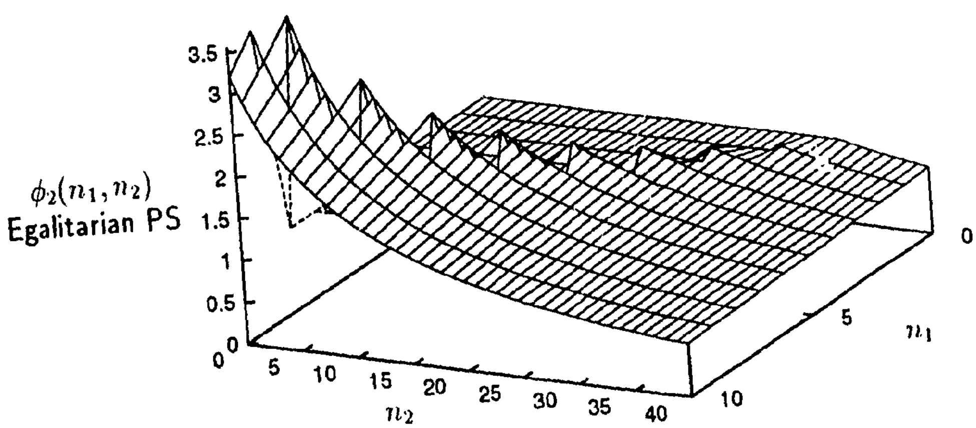




Figure 4.15: The fraction of bandwidth requirement allocated to type 2 customers for the extended shared resource (ESR) model and the egalitarian processor sharing (PS).

In the following examples, we compare the extended shared-resource model with the shared resource [17] and retry models [18]. The retry model is based on the idea that a blocked customer retries to enter service with reduced bandwidth requirements. A customer of type $r$, with bandwidth requirement $b_r$, may retry only a fixed number $N$ of times to enter service. Each time, its bandwidth requirement is reduced to a fixed amount $b_{r,l}$, $l = 1, 2, \ldots, N$. There is no time penalty associated with each retry. Generally, such a model does not accept a product-form solution. Nevertheless, Kaufman produces a product-form approximation assuming that the occupancy in the system due to retrials is negligible compared to the total occupancy. A variant of this model is to introduce a threshold in order to model state dependent bandwidth requirements. Then, a customer has a bandwidth requirement $b_r$ that is satisfied if $j \leq J_0$ and a contingency bandwidth requirement $b_{rc}$ to be used if $j > J_0$. Generally, multiple thresholds and contingency requirements may be applied.

The examples that follow consider the single-threshold case. The performance results for the retry model are the simulation results provided by Kaufman [18]. The limits $T$ for the occupancy have been chosen so that the different models result in approximately equal values for the average number of busy servers. In general, the extended model gives better utilization of the resource and lower blocking probabilities but reduces substantially the bandwidth of large customers.

Choosing the appropriate values for $S$ and $T$ is a design issue that depends on how well customers of different sizes are packed in the memory of the system. The effect of packing is associated with the periodic shape of the blocking probabilities and the average fraction of bandwidth when drawn as functions of the buffer size $T$. Recall that, the work conserving constraint requires that the product

$$\text{(bandwidth)} \times \text{(residency time)}$$

remains constant. Let $\phi_r$ be the average fraction of the original bandwidth requirement given to type $r$. Then, the average bandwidth allocated to type $r$ is $b_r\phi_r$ and

| System Parameters: $S = 32$, $\boldsymbol{b} = (1, 24)$, $\boldsymbol{\lambda} = (12, 0.3)$, $\boldsymbol{\mu}^{-1} = (1, 1)$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Model | $E\{s\}$ | $P_{b_1}$ | $P_{b_2}$ | $P_{b_{2,1}}$ | $P_{b_{2,2}}$ | $P_{b_{2,3}}$ | $P_{b_{2,4}}$ | $P_{b_{2,5}}$ |
| SR | 12.84 | 0.019 | 0.852 | | | | | |
| Basic Retry, $b_{2,1} = 4$ | 18.54 | 0.014 | 0.964 | 0.063 | | | | |
| Multiple Retry, $b_{2,l} = 4(6-l), l = 1, \ldots, 5$ | 16.61 | 0.067 | 0.883 | 0.642 | 0.367 | 0.291 | 0.282 | 0.279 |
| ESR, $T = 40$ | 16.72 | 0.018 | 0.315 | | | | | |
| ESR, $T = 68$ | 18.53 | 0.002 | 0.090 | | | | | |
| System Parameters: $S = 32$, $\boldsymbol{b} = (1, 24)$, $\boldsymbol{\lambda} = (6, 0.3)$, $\boldsymbol{\mu}^{-1} = (1, 1)$ | | | | | | | | |
| Model | $E\{s\}$ | $P_{b_1}$ | $P_{b_2}$ | $P_{b_{2,1}}$ | $P_{b_{2,2}}$ | $P_{b_{2,3}}$ | $P_{b_{2,4}}$ | $P_{b_{2,5}}$ |
| SR | 10.72 | 0.025 | 0.325 | | | | | |
| Basic Retry, $b_{2,1} = 4$ | 12.44 | 0.026 | 0.551 | 0.190 | | | | |
| ESR, $T = 40$ | 11.52 | $10^{-4}$ | 0.233 | | | | | |
| ESR, $T = 55$ | 12.41 | 0.009 | 0.102 | | | | | |

Table 4.1: Example 4.4.

must satisfy the equation

$$
\begin{aligned}
(b_r \phi_r) \cdot D_r &= \frac{b_r}{\mu_r} \\
\Rightarrow \qquad \phi_r &= \frac{\rho_r (1 - P_{b_r})}{E\{n_r\}}.
\end{aligned}
$$

**Example 4.4** The system parameters, the average number of busy servers and the blocking probabilities are presented in Table 4.1. $P_{b_{r,l}}$ refers to the probability of blocking for the $l$th retry of type $r$ customers. The system is examined for two different values for $\lambda_1$. The average delay, average fraction of bandwidth and blocking probabilities are shown in Figures 4.16 and 4.17, respectively. The plots explain the high variation in the blocking probability of type 1 customers and show the effect of buffer size on packing. ■

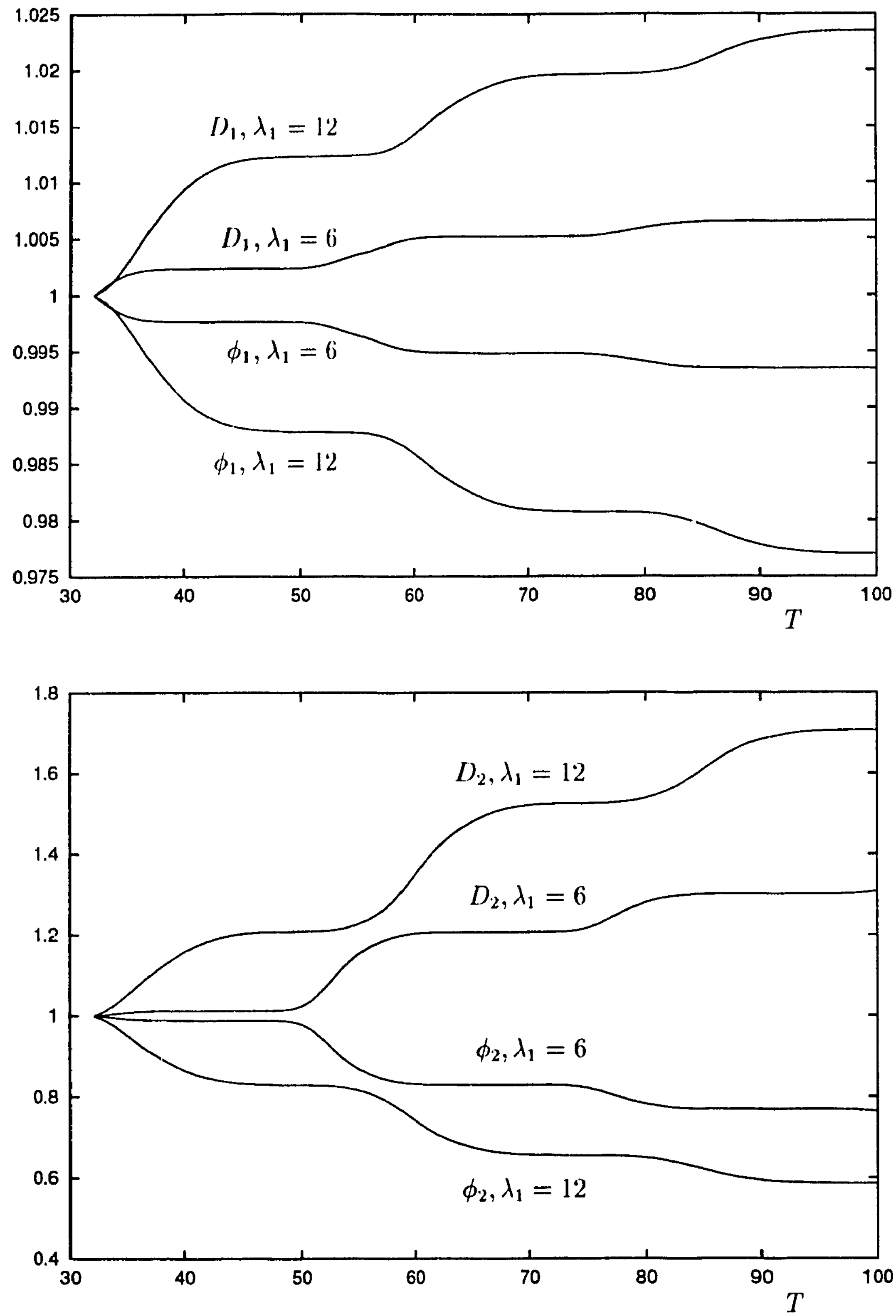



Figure 4.16: Delay and average bandwidth-fraction allocated versus occupancy limit for the system in Table 4.1.

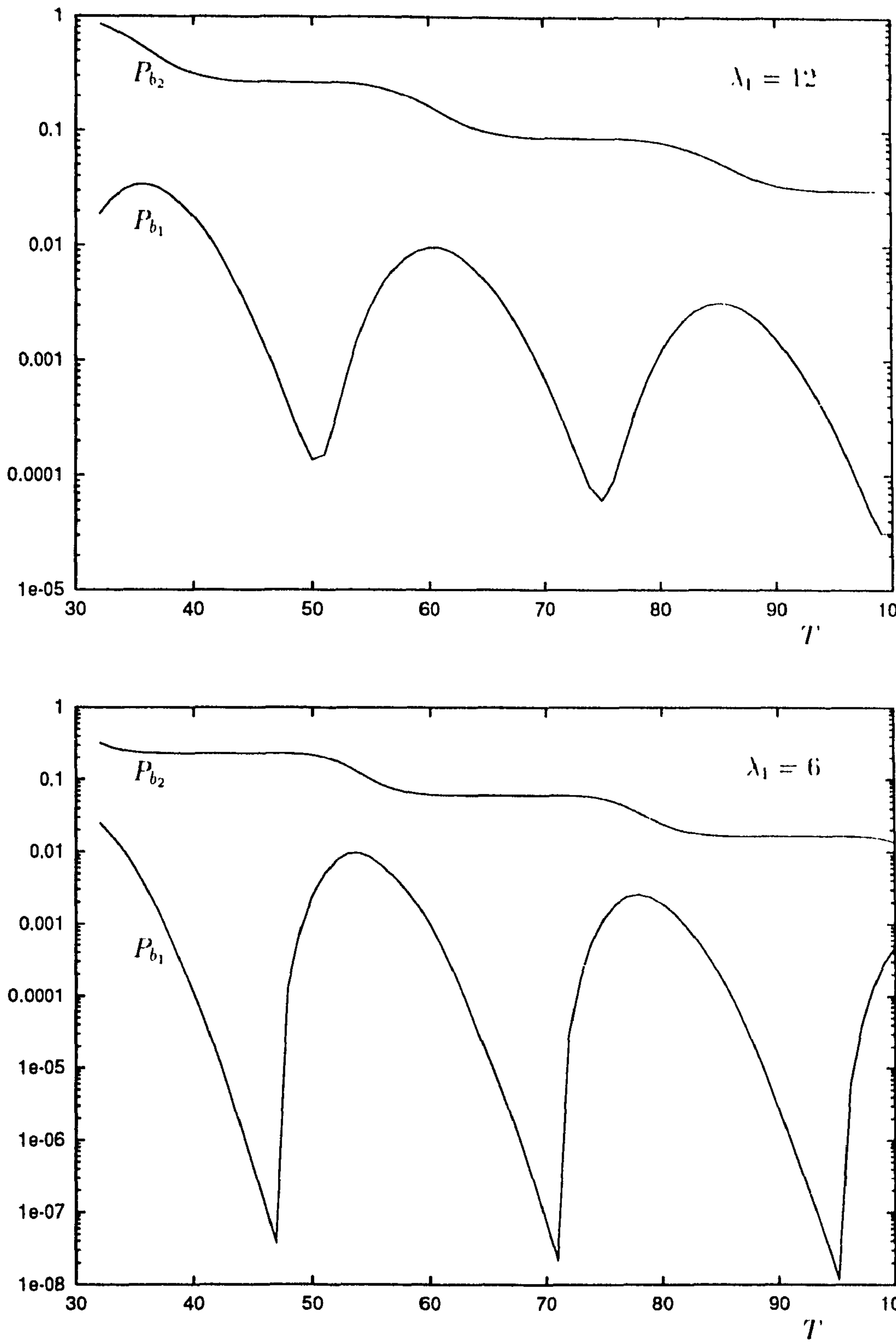



Figure 4.17: Blocking probability versus occupancy limit for the system in Table 4.1.

| System Parameters: $S = 30$, $\boldsymbol{b} = (3, 7)$, $\boldsymbol{\lambda} = (2, 2)$, $\boldsymbol{\mu}^{-1} = (2.5, 1)$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| Model | $E\{s\}$ | $P_{b_1}$ | $P_{b_2}$ | $P_{b_{2,1}}$ | $P_{b_{2,2}}$ | $P_{b_{2,3}}$ | $P_{b_{2,4}}$ |
| SR | 20.89 | 0.18 | 0.39 | | | | |
| Basic Retry, $b_{2,1} = 3$ | 21.74 | 0.22 | 0.44 | 0.83 | | | |
| Multiple Retry, $b_{2,l} = 7 - l, l = 1, \ldots, 4$ | 21.83 | 0.247 | 0.431 | 0.940 | 0.854 | 0.707 | 0.577 |
| State Dependent, $J_0 = 18$, $b_{2c} = 5$ | 24.09 | 0.296 | $P\{j > S - b_{2c}\} = 0.478$ | | | | |
| ESR, $T = 32$ | 21.79 | 0.141 | 0.364 | | | | |
| ESR, $T = 40$ | 27.07 | 0.106 | 0.238 | | | | |
| System Parameters: $S = 30$, $\boldsymbol{b} = (3, 7)$, $\boldsymbol{\lambda} = (10, 10)$, $\boldsymbol{\mu}^{-1} = (2.5, 1)$ | | | | | | | |
| Model | $E\{s\}$ | $P_{b_1}$ | $P_{b_2}$ | $P_{b_{2,1}}$ | $P_{b_{2,2}}$ | $P_{b_{2,3}}$ | $P_{b_{2,4}}$ |
| SR | 27.99 | 0.668 | 0.956 | | | | |
| Basic Retry, $b_{2,1} = 3$ | 28.30 | 0.738 | 0.950 | 0.962 | | | |
| ESR, $T = 32$ | 28.94 | 0.691 | 0.918 | | | | |
| ESR, $T = 40$ | 29.95 | 0.664 | 0.932 | | | | |

Table 4.2: Example 4.5.

**Example 4.5** In the next example, Table 4.2, the bandwidth requirements, of both customer types, are prime numbers and cannot be together in any factorization of $S = 30$. Notice that, in the diagram for the blocking probability, Figure 4.19, local minima for one type correspond to local maxima for the other. This effect subsides as $T$ increases and the model's behavior approaches the behavior of egalitarian processor-sharing. The delay and bandwidth fraction curves are drawn in Figure 4.18. ■

**Example 4.6** In the final example, Table 4.3, the values of the buffer size and the bandwidths as well as the light loading remedy possible packing problems. The Figures with the diagrams of interest are 4.20 and 4.21. ■

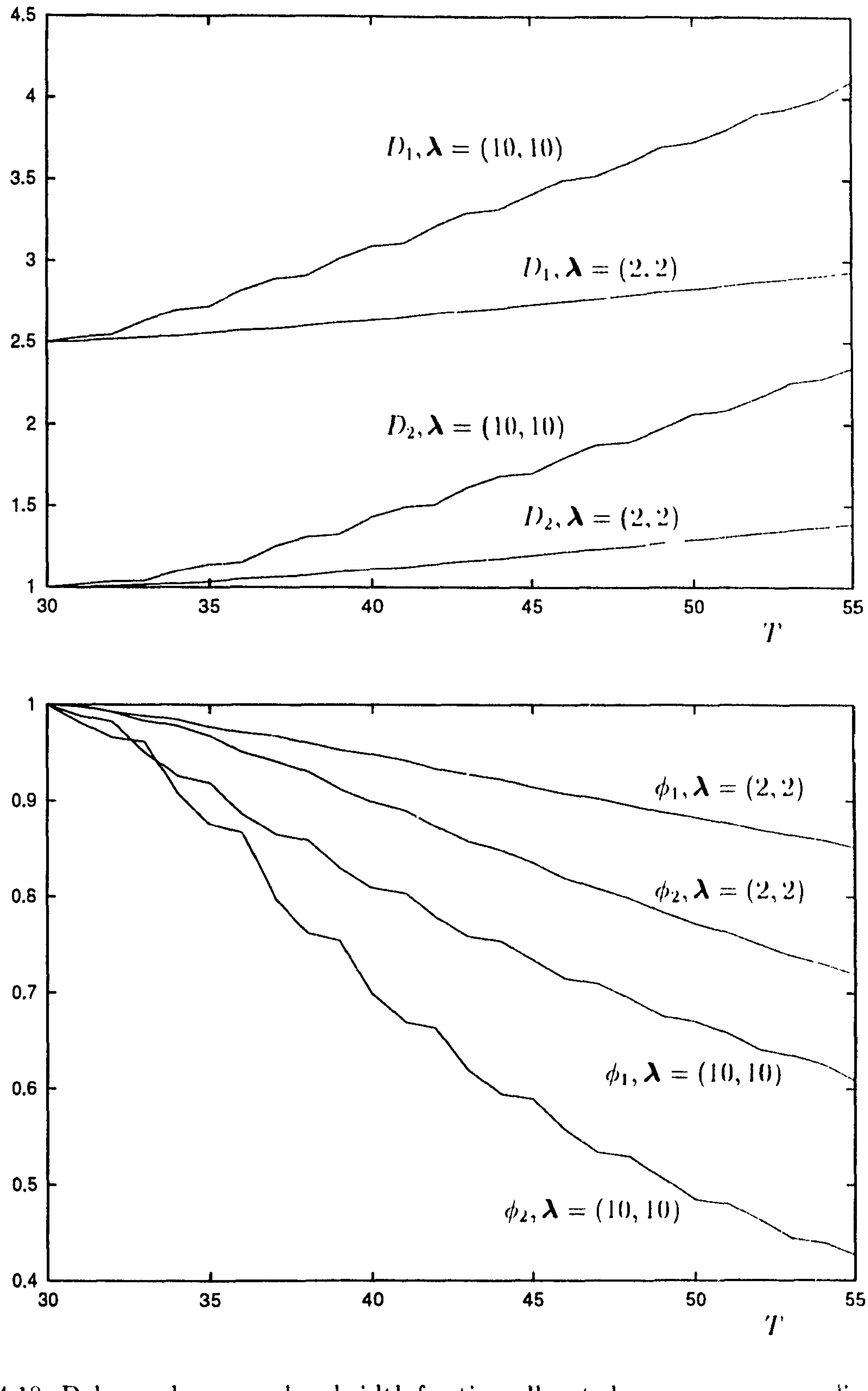



Figure 4.18: Delay and average bandwidth-fraction allocated versus occupancy limit for the system in Table 4.2.

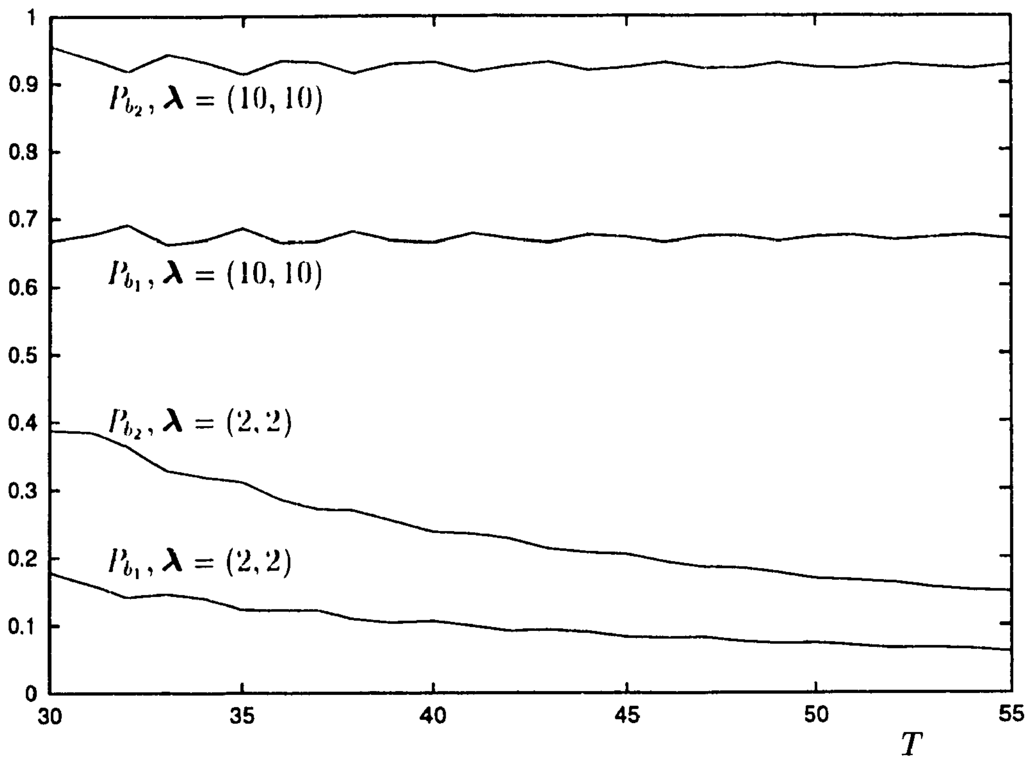



Figure 4.19: Blocking probability versus occupancy limit for the system in Table 4.2.

| System Parameters: $S = 128$, $\boldsymbol{b} = (1,4,8,16)$, $\boldsymbol{\lambda} = (5,2,1,1)$, $\boldsymbol{\mu}^{-1} = (6,4,3,1)$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| Model | $E\{s\}$ | $P_{b_1}$ | $P_{b_2}$ | $P_{b_3}$ | $P_{b_4}$ | $P_{b_{3,1}}$ | $P_{b_{4,1}}$ |
| SR | 95.10 | 0.0095 | 0.041 | 0.088 | 0.201 | | |
| Basic Retry, $b_{3,1} = 6$, $b_{4,1} = 2$ | 96.1 | 0.016 | 0.062 | 0.117 | 0.224 | 0.534 | 0.512 |
| State Dependent, $J_0 = 104$, $b_{3c} = 4$, $b_{4c} = 8$ | 97.24 | 0.0097 | 0.042 | $P\{j > S - b_{3c}\} = 0.041$<br>$P\{j > S - b_{4c}\} = 0.092$ | | | |
| State Dependent, $J_0 = 112$, $b_{3c} = 4$, $b_{4c} = 8$ | 96.67 | 0.013 | 0.055 | $P\{j > S - b_{3c}\} = 0.56$<br>$P\{j > S - b_{4c}\} = 0.112$ | | | |
| ESR, $T = 132$ | 96.21 | 0.0080 | 0.034 | 0.073 | 0.170 | | |
| ESR, $T = 137$ | 97.32 | 0.0065 | 0.027 | 0.059 | 0.138 | | |

Table 4.3: Example 4.6.

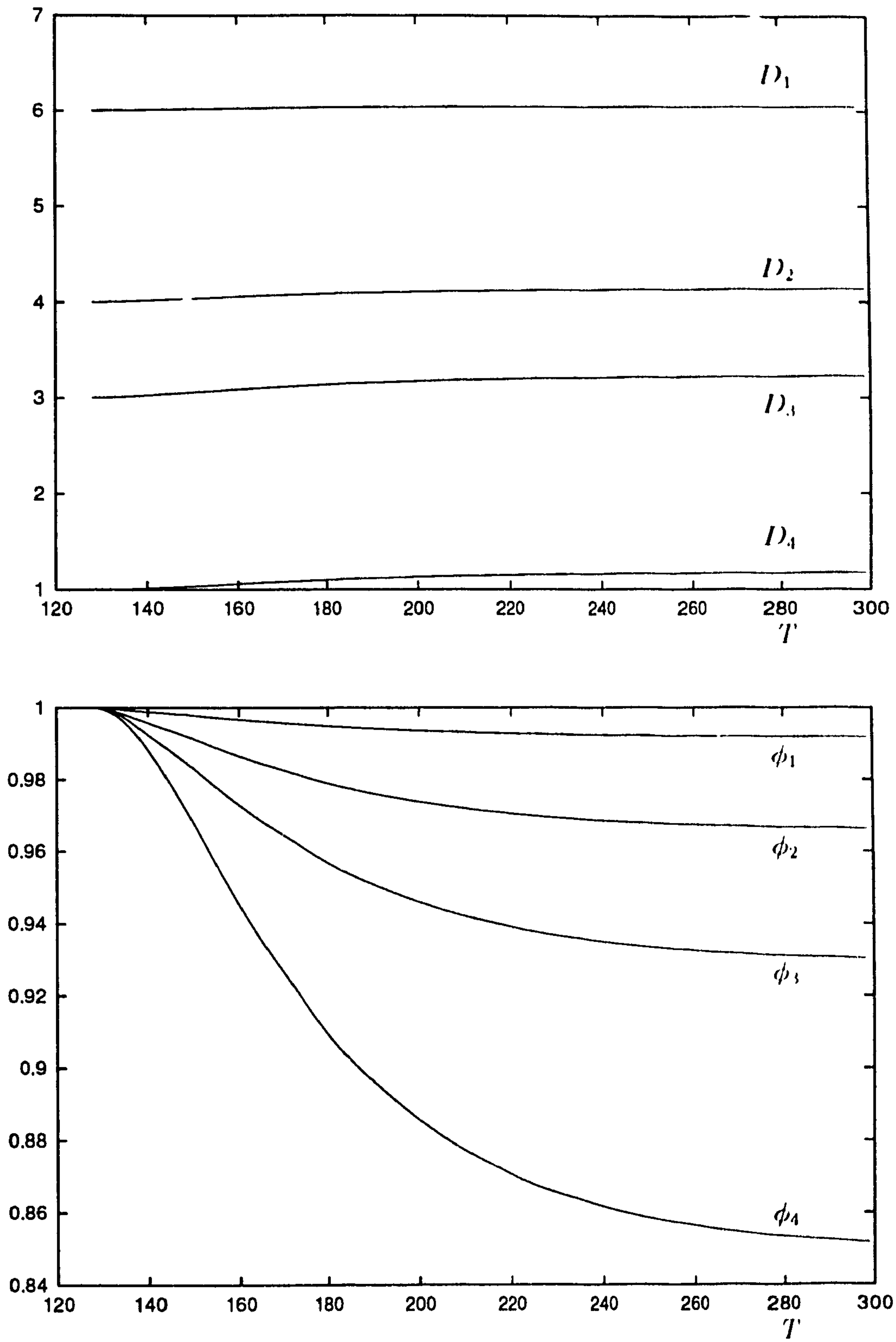



Figure 4.20: Delay and average bandwidth-fraction allocated versus occupancy limit for the system in Table 4.3.

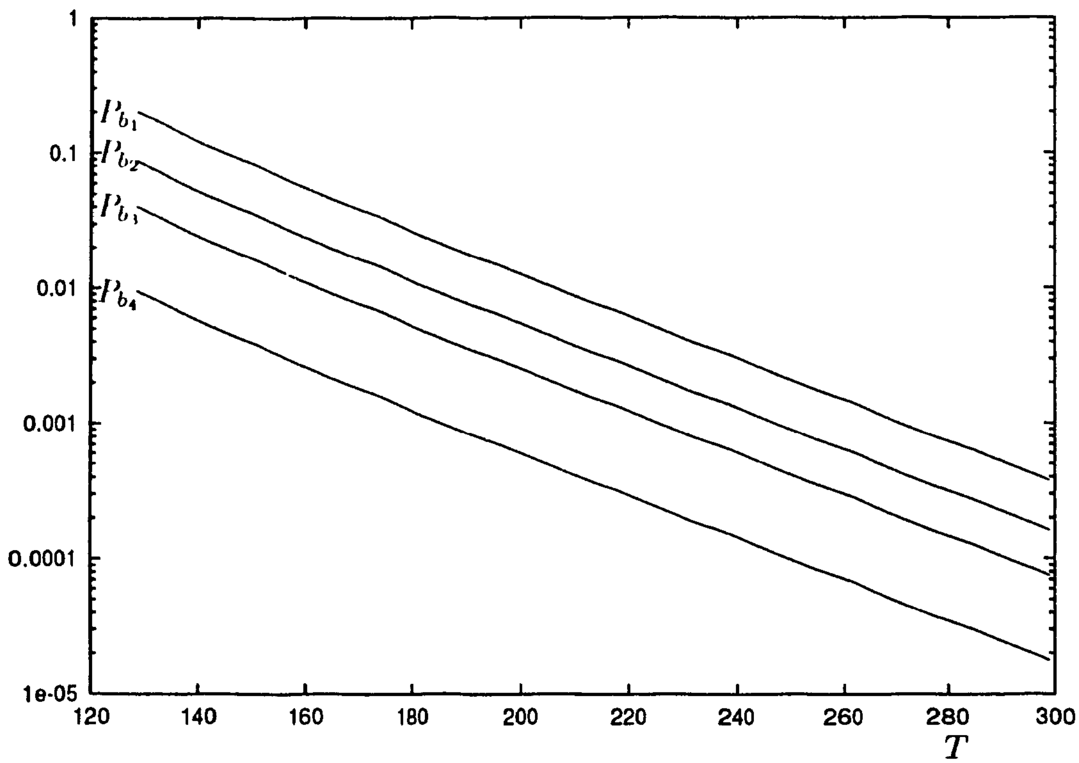



Figure 4.21: Blocking probability versus occupancy limit for the system in Table 4.3.

# Chapter 5

# Further Generalizations

## 5.1 Analysis of Circuit-Switched Networks

Kaufman's SR model introduced in Section 4.2 was extended by Dziong and Roberts [8] for circuit-switched networks with finite capacity links and, independently, by Stamatelos [42] for circuit-switching in broadband networks. In the sequel, I present a further generalization, where time-sharing replaces blocking when the capacity of a link is exceeded.

### 5.1.1 Definitions and Notation

Consider a network of $L$ links, each link having a capacity of $S_l$ channels, $l = 1, 2, \ldots, L$ (Figure 5.1). There are $R$ independent traffic streams of calls. Let $n_r$ be the number of type $r$ calls, $r = 1, 2, \ldots, R$. Call arrivals of type $r$ form a Poisson process with rate $\lambda_r$, $r = 1, 2, \ldots, R$. The residency times (call holding times) of type $r$ calls are independent identically distributed random variables with mean $1/\mu_r$. Note that the residency times are not necessarily exponentially distributed. Each call requires $b_{rl}$ channels on link $l$ through its holding time.[1] Define the $R \times L$ matrix

$$\mathbf{B} \stackrel{\text{def}}{=} [b_{rl}]_{R \times L}. \tag{5.1}$$

[1]In a circuit-switched network, each call requires the same number of channels in each link. However, this condition is not necessary for my analysis.

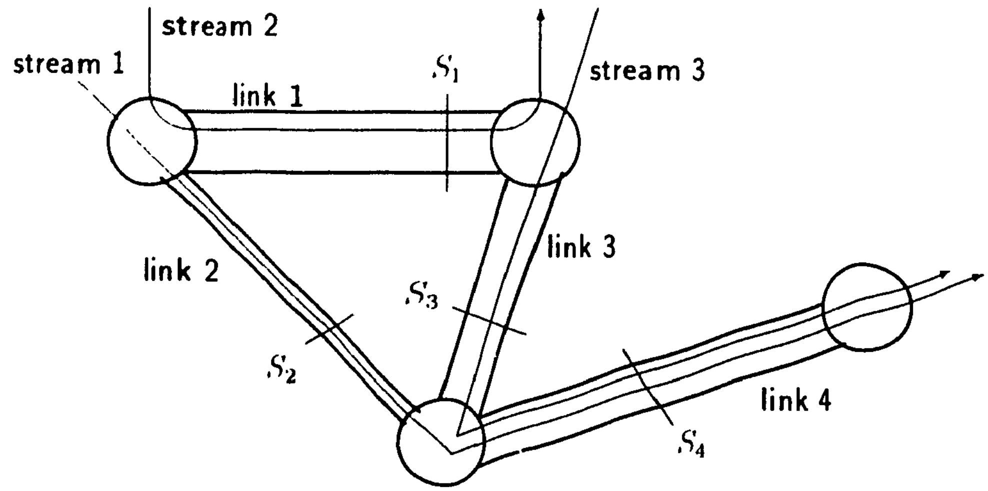



Figure 5.1: A circuit-switched network with different types of traffic streams

The row and column vectors of $\mathbf{B}$ will be denoted by

$$\boldsymbol{b}_{r\cdot} \stackrel{\text{def}}{=} (b_{r1}, b_{r2}, \ldots, b_{rL}), \qquad r = 1, 2, \ldots, R,$$
$$\text{and} \quad \boldsymbol{b}_{\cdot l} \stackrel{\text{def}}{=} (b_{1l}, b_{2l}, \ldots, b_{Rl}), \qquad l = 1, 2, \ldots, L.$$

The state of the system is defined by the vector

$$\boldsymbol{n} \stackrel{\text{def}}{=} (n_1, n_2, \ldots, n_R). \tag{5.2}$$

The state space is

$$\Omega = \{\boldsymbol{n} | \boldsymbol{n} \geq \mathbf{0} \wedge \boldsymbol{n}\mathbf{B} \leq \boldsymbol{S}\}, \tag{5.3}$$

where

$$\boldsymbol{S} \stackrel{\text{def}}{=} (S_1, S_2, \ldots, S_L). \tag{5.4}$$

The occupancy in link $l$ is defined as

$$j_l = \sum_{r=1}^{R} n_r b_{rl}, \qquad l = 1, 2, \ldots, L. \tag{5.5}$$

If

$$\boldsymbol{j} \stackrel{\text{def}}{=} (j_1, j_2, \ldots, j_L). \tag{5.6}$$

the occupancy distribution is

$$\begin{aligned} q(\boldsymbol{j}) &= P\{\boldsymbol{n} \cdot \boldsymbol{b}_{\cdot l} = j_l,\ l = 1, 2, \ldots, L\} \\ &= P\{\boldsymbol{n}\mathbf{B} = \boldsymbol{j}\} \\ &= \sum_{\{\boldsymbol{n} | \boldsymbol{n}\mathbf{B} = \boldsymbol{j}\}} \pi(\boldsymbol{n}), \end{aligned}$$

where $\{\pi(\boldsymbol{n}) | \boldsymbol{n} \in \Omega\}$ is the equilibrium distribution.

### 5.1.2 The Extended Model

Assume, for a moment, that each link has infinite capacity (type 3 BCMP node). The state space of such a system is

$$\Omega_I = \{\boldsymbol{n} | \boldsymbol{n} \geq \mathbf{0},\ r = 1, 2, \ldots, R\}. \tag{5.7}$$

Local balance states that, the probability flux due to an arrival of a call is equal to the probability flux due to a departure of a call of the same type (i.e. a call from the same traffic stream). Given our definition of the system state, local balance implies detailed balance. That is,

$$\pi_I(\boldsymbol{n}_r^-)\lambda_r = \pi_I(\boldsymbol{n}) n_r \mu_r, \tag{5.8}$$

where $\{\pi_I(\boldsymbol{n}) | \boldsymbol{n} \in \Omega_I\}$ is the equilibrium distribution of the infinite capacity model. Recall that

$$\pi_I(\boldsymbol{n}) = \pi_I(\mathbf{0}) \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!}. \tag{5.9}$$

Assume that the links are arbitrarily numbered $1, 2, \ldots, L$. Starting from link 1 and ending at link $L$, gradually reduce the bandwidth requirements of each traffic stream in order to satisfy the capacity constraints. At each link, the infinite process is modified by defining a set of state multipliers $\{x_l(\boldsymbol{n}) | \boldsymbol{n} \in \Omega_I\}$, $l = 0, 1, 2, \ldots, L$, and using it to model deceleration of the service rate or reduction of the bandwidth. The set $\{x_0(\boldsymbol{n}) | \boldsymbol{n} \in \Omega_I\}$ corresponds to the process of the initial infinite capacity system. Again, the principle behind the definition of state multipliers is work conservation.

The sequence $\{m_l, l = 0, 1, 2, \ldots, L\}$ contains the history of the modifications. Term $m_l$ is updated at the $l$th step and assigned the number of the link where congestion was most recently alleviated. This sequence is needed for the derivation of the recursion for the occupancy distribution.

Let's initialize:

$$x_0(\boldsymbol{n}) = 1, \ \forall \boldsymbol{n} \in \Omega_I \ \text{ and } \ m_0 = 0. \tag{5.10}$$

At link 1, reduce the bandwidth allocated to each type of calls by a positive factor $\phi_{r1}(\boldsymbol{n})$ such that

$$\begin{array}{llll} \text{if,} & \text{for some } q,\ b_{q1} = 0 & \text{then} & \phi_{q1}(\boldsymbol{n}) = 1, \\ \text{if} & \sum_{r=1}^{R} n_r b_{r1} \le S_1 & \text{then} & \phi_{r1}(\boldsymbol{n}) = 1, \\ \text{if} & \sum_{r=1}^{R} n_r b_{r1} > S_1 & \text{then} & \phi_{r1}(\boldsymbol{n}) \ \text{satisfies} \ \sum_{r=1}^{R} n_r b_{r1} \phi_{r1}(\boldsymbol{n}) = S_1. \end{array} \tag{5.11}$$

Reduction in bandwidth should be associated with an equal increase in residency time so that the product

$$\text{(bandwidth)} \times \text{(residency time)} \tag{5.12}$$

remains the same. Assuming that call set up is instantaneous, the total residency time equals the residency time in each link. Then the reduction factor for each class must be the same for all links. This condition is satisfied by (5.11).

Using the Characterization Theorem 3.4 and work conservation, define a set of state multipliers $\{x_1(\boldsymbol{n}) | \boldsymbol{n} \in \Omega_I\}$, such that

$$\phi_{r1}(\boldsymbol{n}) = \frac{x_1(\boldsymbol{n}_r^-)}{x_1(\boldsymbol{n})}. \tag{5.13}$$

Then,

$$x_1(\boldsymbol{n}) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \boldsymbol{n} \cdot \boldsymbol{b}_{\cdot 1} \le S_1 \wedge \boldsymbol{n} \ge \boldsymbol{0}, \\ \dfrac{1}{S_1} \sum_{r=1}^{R} n_r b_{r1} x_1(\boldsymbol{n}_r^-), & \text{if } \boldsymbol{n} \cdot \boldsymbol{b}_{\cdot 1} > S_1 \wedge \boldsymbol{n} \ge \boldsymbol{0}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.14}$$

Note that, for all the states $\boldsymbol{n}$ along the $q$th dimension such that $b_{q1} = 0$ recursion

(5.14) will produce the same value. This means that $\phi_{q1}(\boldsymbol{n}) = 1$, i.e. no slowing occurs for streams that do not use link 1. Also set

$$m_1 = \begin{cases} m_0, & \text{if } \boldsymbol{n} \cdot \boldsymbol{b}_{\cdot 1} \leq S_1 \wedge \boldsymbol{n} \geq \boldsymbol{0}, \\ 1, & \text{if } \boldsymbol{n} \cdot \boldsymbol{b}_{\cdot 1} > S_1 \wedge \boldsymbol{n} \geq \boldsymbol{0}. \end{cases} \tag{5.15}$$

In link 2 the demand $\sum_{r=1}^{R} n_r b_{r2} x_1(\boldsymbol{n}_r^-)/x_1(\boldsymbol{n})$ may still exceed the available capacity $S_2$. Let's remedy the situation with another set of multipliers $\{x_2(\boldsymbol{n}) | \boldsymbol{n} \in \Omega_l\}$, such that

$$\sum_{r=1}^{R} n_r b_{r2} \frac{x_1(\boldsymbol{n}_r^-)}{x_1(\boldsymbol{n})} \frac{x_2(\boldsymbol{n}_r^-)}{x_2(\boldsymbol{n})} = S_2 \tag{5.16}$$

if link 2 is congested. Using the same reasoning as for link 1, we get

$$x_2(\boldsymbol{n}) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \delta_2 \leq S_2 \wedge \boldsymbol{n} \geq \boldsymbol{0}, \\ \dfrac{1}{S_2 x_1(\boldsymbol{n})} \displaystyle\sum_{r=1}^{R} n_r b_{r2} x_1(\boldsymbol{n}) x_2(\boldsymbol{n}_r^-), & \text{if } \delta_2 > S_2 \wedge \boldsymbol{n} \geq \boldsymbol{0}, \\ 0, & \text{otherwise} \end{cases} \tag{5.17}$$

and

$$m_2 = \begin{cases} m_1, & \text{if } \delta_2 \leq S_2 \wedge \boldsymbol{n} \geq \boldsymbol{0}, \\ 2, & \text{if } \delta_2 > S_2 \wedge \boldsymbol{n} \geq \boldsymbol{0}, \end{cases} \tag{5.18}$$

where

$$\delta_l \stackrel{\text{def}}{=} \sum_{r=1}^{R} n_r b_{rl} \prod_{i=1}^{l-1} \frac{x_i(\boldsymbol{n}_r^-)}{x_i(\boldsymbol{n})} \quad l = 1, 2, \ldots, L. \tag{5.19}$$

Generalizing, the state multipliers obtained at the $l$th link are

$$x_l(\boldsymbol{n}) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \delta_l \leq S_l \wedge \boldsymbol{n} \geq \boldsymbol{0}, \\ \dfrac{1}{S_l \prod_{i=1}^{l-1} x_i(\boldsymbol{n})} \displaystyle\sum_{r=1}^{R} n_r b_{rl} \prod_{i=1}^{l-1} x_i(\boldsymbol{n}_r^-) & \text{if } \delta_l > S_l \wedge \boldsymbol{n} \geq \boldsymbol{0}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.20}$$

$$m_l = \begin{cases} m_{l-1}, & \text{if } \delta_l \leq S_l \wedge \boldsymbol{n} \geq \boldsymbol{0}, \\ l, & \text{if } \delta_l > S_l \wedge \boldsymbol{n} \geq \boldsymbol{0}. \end{cases} \tag{5.21}$$

The initial values are

$$x_0(\boldsymbol{n}) = 1, \ \forall \boldsymbol{n} \in \Omega \ \text{ and } m_0 = 0. \tag{5.22}$$

The state multipliers at each link can be analytically derived using the results of Appendix B.

Thus, a sequence of recursive terms has been defined and used to modify the original process without loosing the reversibility property. The detailed balance equations of the modified process are

$$\pi(\boldsymbol{n}_r^-)\lambda_r = \pi(\boldsymbol{n}) n_r \mu_r \prod_{l=1}^{m_L} \frac{x_l(\boldsymbol{n}_r^-)}{x_l(\boldsymbol{n})}, \qquad r = 1, 2, \ldots, R. \tag{5.23}$$

Observe that

$$S_{m_L} = \sum_{r=1}^{R} n_r b_{r m_L} \prod_{l=1}^{m_L} \frac{x_l(\boldsymbol{n}_r^-)}{x_l(\boldsymbol{n})}, \qquad r = 1, 2, \ldots, R.$$

Then, algebraic manipulation of the detailed balance equations (5.23) results in the equilibrium and occupancy distributions.

$$\pi(\boldsymbol{n}) = \begin{cases} \dfrac{1}{\boldsymbol{n} \cdot \boldsymbol{b}_{\cdot l}} \displaystyle\sum_{r=1}^{R} b_{rl} \rho_r \pi(\boldsymbol{n}_r^-), & \text{if } \boldsymbol{n}\mathbf{B} \leq \boldsymbol{S} \wedge \boldsymbol{n} \geq \mathbf{0}, \\ \dfrac{1}{S_{m_L}} \displaystyle\sum_{r=1}^{R} b_{r m_L} \rho_r \pi(\boldsymbol{n}_r^-), & \text{if } \boldsymbol{n}\mathbf{B} \not\leq \boldsymbol{S} \wedge \boldsymbol{n} \geq \mathbf{0}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.24}$$

$$q(\boldsymbol{j}) = \begin{cases} \dfrac{1}{j_l} \displaystyle\sum_{r=1}^{R} b_{rl} \rho_r q(\boldsymbol{j} - \boldsymbol{b}_{r\cdot}), & \text{if } \mathbf{0} \leq \boldsymbol{j} \leq \boldsymbol{S}, \\ \dfrac{1}{S_{m_L}} \displaystyle\sum_{r=1}^{R} b_{r m_L} \rho_r q(\boldsymbol{j} - \boldsymbol{b}_{r\cdot}), & \text{if } \boldsymbol{j} \not\leq \boldsymbol{S}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.25}$$

## 5.2 State-Dependent Arrival Rates

Let arrivals of type $r$ form a Poisson process with state-dependent rate

$$\lambda_r(n_r) = \alpha_r + \beta_r n_r, \qquad r = 1, 2, \ldots, R. \tag{5.26}$$

This type of process has been termed by Delbrouck [6] as Bernoulli-Poisson-Pascal (BPP). For $\beta_r < 0$ and $\alpha_r/\beta_r$ being a negative integer, the arrival process is Bernoulli. For $\beta_r = 0$, the process is Poisson and for $\beta_r > 0$ it is Pascal. Bernoulli arrivals have been shown to approximate smooth traffic. Such traffic is offered by a video source generating scenes without abrupt movement [35]. Pascal arrivals provide a good approximation of peaked traffic. The Poisson distribution has a limiting relationship with Bernoulli and Pascal distributions and is used for modeling the so-called regular traffic. The state multipliers that allow us to move from Poisson arrivals with rate $\lambda_r$ to BPP arrivals are

$$z(\boldsymbol{n}) = \frac{\alpha_r + \beta_r(n_r - 1)}{\lambda_r} = \frac{1}{\lambda_r^{n_r}} \prod_{i=0}^{n_r - 1} (\alpha_r + \beta_r i). \tag{5.27}$$

Equations (5.24) and (5.25) can be generalized for BPP arrivals when $\mathbf{0} \leq \boldsymbol{j} \leq \boldsymbol{S}$ [8]. In this case,

$$\pi(\boldsymbol{n}) = \frac{1}{\boldsymbol{n} \cdot \boldsymbol{b}_{\cdot l}} \sum_{r=1}^{R} \frac{\lambda_r(n_r - 1)}{\mu_r} b_{rl} \pi(\boldsymbol{n}_r^-) \tag{5.28}$$

and

$$q(\boldsymbol{j}) = \frac{1}{j_l} \sum_{r=1}^{R} b_{rl} \frac{\alpha_r}{\mu_r} \sum_{k} \left(\frac{\beta_r}{\mu_r}\right)^{k-1} q(\boldsymbol{j} - k\boldsymbol{b}_{r\cdot}). \tag{5.29}$$

where

$$k = 1, 2, \ldots, \max\{\lceil j_l / b_{rl} \rceil, l = 1, \ldots, L\}. \tag{5.30}$$

For $\boldsymbol{j} \not\leq \boldsymbol{S}$, the equilibrium distribution of the population becomes

$$\pi(\boldsymbol{n}) = \frac{1}{S_{m_L}} \sum_{r=1}^{R} \frac{\lambda_r(n_r - 1)}{\mu_r} b_{rl} \pi(\boldsymbol{n}_r^-). \tag{5.31}$$

**Comment.** Since, for $\boldsymbol{j} \not\leq \boldsymbol{S}$, the residency time is also state-dependent, the derivation of a simple recursion for the occupancy distribution is not easy. Such a recursion would expand the range of applications for our model and permit efficient computation of the normalization constant. Consider, for example, the extended shared-resource model. A buffer with size $aS$, $a > 0$, increases the state-space by $O((a^R - 1)S^R)$. Then, the space requirement for the computation of the normalization constant grows at least linearly with the size of the resource. ■

# Chapter 6

# Epilogue

## 6.1 Summary of Contributions

In this thesis, a structural characterization of reversibility is developed and used to synthesize a non-egalitarian processor-sharing queueing discipline that admits a product-form solution. Let's recall and summarize this procedure.

The first key result is Theorem 3.3. It states that a stationary Markov process, with a state transition diagram $\mathcal{G}$, is reversible if and only if the product of transition rates balances around every fundamental circuit of a spanning tree $\mathcal{T}$ of $\mathcal{G}$. This result defines a minimal set of equations needed to test reversibility and is derived from Kolmogorov's criterion. Its importance lies on the following three facts.

1. It bridges reversibility and graph theory.
2. It permits the study of a process only on a minimal set of edges.
3. It is essential to the understanding of the Characterization Theorem since it provides the intuition needed to explain why the state probabilities of a reversible process may be modified in a manner analogous to raising or dropping the node potential in an electrical network.

The natural consequences of this graph-theoretic point of view are the Characterization Theorem 3.4 and the concept of state or set multiplier. Their significance is due to the following.

1. The state or set multipliers, when used according to the Characterization Theorem, maintain reversibility.

2. Multipliers are used to parameterize the transition rates and the equilibrium distribution of all the reversible processes associated with the same transition diagram.

3. The state or set multipliers are very powerful in modeling blocking and state dependencies of the transition rates.

4. Superposition of multipliers allows the definition of different sets of multipliers for different kinds of dependencies.

5. Non-product-form queues can be approximated through successive overlays of multipliers.

6. When the Characterization Theorem is applied on a with work conserving queueing discipline, the existence of the characteristic function is guaranteed. The queueing discipline can be included in the BCMP framework.

An application of the Characterization Theorem is the analysis of shared-resource models. The result, the extended shared-resource model, is a generalization and extention of existing models. The use of state multipliers facilitates the analysis and provides the intuition needed for the development of the model. The normalization constant and the moments of the population distribution can be efficiently computed by a recursive algorithm which is also contributed by this work. The most important implications of the ESR model are:

1. It unifies the infinite server and processor sharing disciplines.

2. It defines a non-egalitarian processor sharing discipline admitting product-form solution.

3. As a node in a queueing network, the ESR model expands the class of queueing networks with product-form solution.

Given the applicability of the shared resource model in problems such as multiplexing, message storage and network problems, the ESR model can be used to analyze variants of practical interest.

## 6.2 Further Research

### 6.2.1 Extending the Analytical Results

A reading of this thesis may raise a number of questions:

- Can decomposition and aggregation techniques be associated with the topology of the state transition diagram and treated as graph-theoretic problems?
- Is it possible to define state multipliers for a non-reversible process?
- Besides work conservation, what other constraints preserve reversibility?
- Which resource-sharing policies can be modeled by state multipliers, have practical value and remain computationally tractable?
- How do performance measures for different scheduling strategies relate to each other?

The answers to these questions constitute intriguing research topics. Further, I describe an approach that might lead to a solution of these problems.

Given the performance requirements of a communications system the designer faces the following problems:

- Specification of the physical characteristics
- Traffic characteristics
- Scheduling strategy
- Resource allocation policy

Optimization of the scheduling strategy is a synthesis problem [22, 9] which is defined as follows.

> Assuming certain physical characteristics for a system with $R$ traffic types and the freedom to chose the scheduling strategy, what are the achievable delay vectors $\boldsymbol{D} = (D_1, D_2, \ldots, D_R)$?

A similar question concerning also the blocking probability and throughput can be posed for the resource-allocation policy.

Such issues have been sufficiently analyzed and resolved for single-resource systems. The basis of this analysis is the conservation law stated next [9].

**Theorem 6.1 (General Conservation Law)** *For a work conserving scheduling strategies $S$ applicable to a single-server queueing system, there exists a constant $V$ depending on the arrival and service-time processes, such that*

$$\sum_{r=1}^{R} V_S(r) = V, \tag{6.1}$$

*where $V_S(r)$ is the steady-state average unfinished work of type $r$ customers under scheduling strategy $S$.*

Assume that scheduling strategy depends on the residual service-times of type $r$ customers only through their means $\nu_r$. Then, Kleinrock [22] shows that for a multiclass, work-conserving and non-preemptive $G/G/1$ system the general conservation law becomes

$$\sum_{r=1}^{R} \rho_r D_r = V + \sum_{r=1}^{R} \rho_r \left( \frac{1}{\mu_r} - \nu_r \right). \tag{6.2}$$

Based on the information regarding the service-time process, a system designer may use the conservation laws to obtain an optimal scheduling strategy.

The conservation laws do not apply for multiple-resource systems. Such systems are difficult to analyze under general scheduling strategies. Consider for example a FCFS queue with customers of different bandwidth requirements and a multiple-server resource. This system is not work-conserving since servers may be idle in the presence of work in the queue.

It may be possible to develop a special conservation law for a class of multiple-resource systems, as I explain next. Consider the service center given in Figure 2.1 of the Introduction; each queue is served at a variable service rate depending on the occupancy and the scheduling strategy. The whole system is not necessarily work-conserving even if all the queues are. In the ESR model for example, some servers are idle when the occupancy is less than the system capacity. Therefore, it is possible that a conservation law could be established conditioned on the occupancy or the population. The necessary constants, such as the work load accumulated in a queue, can be determined from the ESR model.

### 6.2.2 Applications

State or set multipliers allow the development of reversible queues that approximate non-product-form queues. The method of successively applying layers of multipliers, until a certain degree of accuracy is reached, is appealing; the reason is that it reflects the physical differences between the exact and the approximate models. This is clearly demonstrated by Example 3.2. In this context, further study is required to formalize a method for developing such approximations and establish error margins.

Systems with large populations can be approximated by a continuous state-space. The development of continuous analogues for the state multipliers and equilibrium distribution and their them in asymptotic approximations are important research problems. Their solution has both theoretical and practical value since it can lead to efficient algorithms for performance metrics in large population systems.

The ESR model and its variants are applicable in a wide range of problems in operating systems and communications networks. Even though the ESR model can be adapted to such environments by additional sets of multipliers, computational tractability remains an issue. This problem appears in the analysis of circuit-switched networks when the arrival rates are state-dependent (Section 5.2). Other state dependencies that need to be considered are class priorities and/or preemptions.

Finally, the analytic results of the ESR model can prove useful in the design of a shared-rescurce system, since they may be used in determining system parameters associated with delay, blocking probability and throughput. Such parameters are the total number of bandwidth units ($S$), the buffer size (occupancy limit) and maximum service degradation. Examples, requiring study, of such applications of the ESR model is the issue of providing quality of service guarantees in a high-speed network [27] or cell multiplexing in ATM networks. The latter example is presented in more detail since it is of practical importance and provides the testing ground for several of the research issues presented so far.

**Example 6.1 (Cell Multiplexing in ATM Networks)** Fiber-optic channels enable broadband point-to-point communication. Furthermore, since memory is inexpensive, *congestion* and *buffer overflow* are *not* interrelated issues in network flow control [3]. Then, a call acceptance is based primarily on the the delay constraints for this call. In this context, point-to-point single-access links and virtual paths offer better utilization of the available bandwidth than datagrams [15]. Asynchronous transfer mode (ATM) networks implement virtual paths with packet-switching. Cells that flow along a virtual path are stored at the switching nodes of the network and forwarded through the appropriate link if contention for the link permits. The method of buffering and multiplexing the incoming traffic at a switching node involves issues such as fairness, link utilization and delay constraints. Katevenis [15] shows that the allocation of the bandwidth achieved by a round-robin multiplexing method is fairer than FIFO multiplexing. In this method, each congested node buffers incoming calls allocating one buffer to each virtual-circuit passing through the node. The round-robin method distributes equally the available link bandwidth to the virtual paths that can use it. Also, if some virtual paths cannot use their share of bandwidth, they are allocated as much bandwidth as permitted and the rest is distributed to the other virtual paths. The round-robin method can be further extended to weighted round-robin that supports multiple types of traffic with different service requirements [16]. Since, the ESR model behaves as an egalitarian processor-sharing model under heavy load and large buffer size, it is expected to provide an approximation for the performance of the round-robin method. Further work is required to determine the load composition, buffer size and bandwidth requirements that minimize the approximation error. ■

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# Appendix A

# A Graph-Theoretic Interpretation of Reversibility

Let $X(t), t \geq 0$, be a time-homogeneous, stationary, irreducible and aperiodic Markov process with a countable state-space $\Omega$ and transition rates $q(i,j)$, $i,j \in \Omega$. Furthermore, let $G = (\Omega, E)$ be the state transition graph of $X(t)$, defined as in Section 3.1.

The weight function $\rho(i,j)$ can be used to show that Kolmogorov's criterion for reversibility is analogous to Kirchhoff's voltage law (KVL). This analogy is derived from the results of Chapters 6 and 11 of Swamy and Thulasiraman [43]. Electrical network theory has been previously used by Chandy, Herzog and Woo [36] in the hierarchical decomposition of queueing networks. Their work resulted in a method of obtaining performance metrics for a network of queues. In this Appendix, electrical network theory is used to explore the structural properties of a reversible process that explain why the Characterization Theorem 3.4 exists.

Define the *voltage* of directed edge $(i,j) \in E$ and the *relative potential* (or simply *potential*) of vertex $i \in \Omega$, respectively, as

$$v(i,j) \stackrel{\text{def}}{=} \ln \rho(i,j), \qquad (i,j) \in E, \tag{A.1}$$

$$\varphi(i) \stackrel{\text{def}}{=} \ln \frac{\pi(i)}{\pi(r)}, \qquad i \in \Omega. \tag{A.2}$$

where $r \in \Omega$ is a reference vertex and ln is the function of natural logarithms. Then, the detailed balance equations (2.15) are equivalent to the *voltage balance equations*

$$\varphi(i) + v(i,j) = \varphi(j), \qquad (i,j) \in E. \tag{A.3}$$

Hence, $v(i,j)$ represents a voltage drop from vertex $j$ to vertex $i$.

The definitions of edge voltage and vertex potential allow us to provide an alternative statement and proof of Kolmogorov's theorem.

**Theorem A.1 (Kolmogorov's Balance Law, KBL)** [1] *A stationary Markov process, with a state transition graph $G$, is reversible if and only if the algebraic sum of voltages around every circuit of $G$ is zero. That is,*

$$B_c V_e = 0, \tag{A.4}$$

*where $B_c$ is the circuit matrix of $G$ and $V_e$ is the column vector of edge voltages.*

**Proof**

*Necessity.* Let $C$ be an arbitrarily oriented circuit of $G$. The voltage balance equations around $C$ can be written

$$\begin{aligned} \varphi(i) + v(i,j) = \varphi(j), &\quad \text{if the orientation of } C \text{ agrees with } (i,j), \\ \varphi(i) - v(i,j) = \varphi(j), &\quad \text{if the orientation of } C \text{ disagrees with } (i,j). \end{aligned} \tag{A.5}$$

Summing up equations (A.5), for all $i,j \in C$, we see that the algebraic sum of voltages around $C$ is zero. Since $C$ is an arbitrarily chosen circuit, the result is true for every circuit of $G$.

*Sufficiency.* Since $B_c V_e = 0$, the cutset transformation theorem applies. So, there exists a column vector $\Phi = (\varphi(i))_{i \in \Omega - \{r\}}$, such that

$$V_e = A^{\mathsf{T}} \Phi,$$

[1] The term "Kolmogorov's Balance Law" is a successful suggestion by Marc Comeau of Concordia University.

where $A$ is the incidence matrix of $G$ with reference vertex $r$. Consider a row vector $[(i,j)]$ of $A^{\mathsf{T}}$ that corresponds to edge $(i,j) \in E$. Clearly, only the $i$th and $j$th entries of $[(i,j)]$ can be non-zero. If $(i,j)$ is not incident to $r$, then the $i$th and $j$th elements of the row are 1 and $-1$, respectively. Otherwise, only the entry corresponding to the vertex adjacent to $r$ is non-zero; its value, 1 or $-1$, depends on the direction of $(i,j)$. Thus, if we set $\varphi(r) = 0$, we have

$$\begin{aligned} v(i,j) &= \varphi(i) - \varphi(j), \\ \Leftrightarrow \quad e^{-\varphi(i)}\rho(i,j) &= e^{-\varphi(j)}, \\ \Leftrightarrow \quad \pi(i)q(i,j) &= \pi(j)q(j,i), \end{aligned}$$

where $(i,j) \in E$ and $\pi(i) = \pi(r)e^{-\varphi(i)}$, $i \in \Omega$, with $\pi(r)$ being a constant. Since the numbers $\pi(i)$ satisfy local balance, they also satisfy the equilibrium equations. Their sum is finite because the process is stationary. Therefore, we can choose $\pi(r)$ so that $\sum_{i\in\Omega} \pi(i) = 1$. Sufficiency now follows from theorem 2.3. **Q.E.D.**

The next result allows us to continue the analogy between Kolmogorov's criterion and KVL.

**Corollary A.2** *A stationary Markov process, with a state transition graph $G$, is reversible if and only if the algebraic sum of voltages around every fundamental circuit of a spanning tree $T$ of $G$ is zero. That is,*

$$B_f V_e = 0, \tag{A.6}$$

*where $B_f$ is the fundamental circuit matrix of $G$ with respect to $T$ and $V_e$ is the column vector of edge voltages.*

### Proof

*Necessity.* Since the process is reversible, equation (A.4) implies that, the inner product of each row of $B_c$ with $V_e$ is zero. Necessity now follows because $B_f$ consists of $|E| - |\Omega| + 1$ rows of $B_c$.

*Sufficiency.* Every circuit vector $\beta$ of $G$ can be expressed as a linear combination of the fundamental circuit vectors with respect to a spanning tree. Then, equation (A.6) implies that $\beta V_e = 0$, $\forall \beta \in G$. So, $B_c V_e = 0$ and the process is reversible. **Q.E.D.**

Kolmogorov's Balance Law allows us to generate all the reversible processes associated with the same graph $G = (\Omega, E)$. This can be done in a manner similar to raising or dropping the potential of a node or a set of nodes, as explained in Section 3.1.

# Appendix B

# Solution for the State Multipliers

A solution for the multidimensional recursion for the state multipliers is presented next. The analysis followed can be applied to any sharing policy and is based on the results of Section 6.11 of Swamy and Thulasiraman [43].

Consider the system of linear equations

$$x(\boldsymbol{k}) = \sum_{r=1}^{R} w(\boldsymbol{k}_r^-, \boldsymbol{k}) x(\boldsymbol{k}_r^-), \qquad \boldsymbol{m} \leq \boldsymbol{k}_r^- < \boldsymbol{n},\ r = 1, 2, \ldots, R, \tag{B.1}$$

where $w(\boldsymbol{k}_r^-, \boldsymbol{k})$ is the coefficient of the unknown $x(\boldsymbol{k}_r^-)$ and $\boldsymbol{k}$, $\boldsymbol{m}$, $\boldsymbol{n}$ are vectors in $\mathbf{Z}^R$. The relation $\boldsymbol{k} < \boldsymbol{n}$ is defined so that

$$\boldsymbol{k} < \boldsymbol{n} \Leftrightarrow \boldsymbol{k} \leq \boldsymbol{n} \wedge \exists k_r : k_r < n_r, \tag{B.2}$$

where $\leq$ is the usual partial order in $\mathbf{Z}^R$. I shall use $\boldsymbol{l}$ to denote a low bound of a finite subset of $\mathbf{Z}^R$. Assume that the sum $\sum_{r=1}^{R} w(\boldsymbol{k}_r^-, \boldsymbol{k}) x(\boldsymbol{k}_r^-)$ contains only those terms $w(\boldsymbol{k}_r^-, \boldsymbol{k}) x(\boldsymbol{k}_r^-)$ satisfying $\boldsymbol{m} \leq \boldsymbol{k}_r^- < \boldsymbol{n}$. Then, system (B.1) has $\prod_{r=1}^{R}(n_r - m_r) - 1$ equations and $\prod_{r=1}^{R}(n_r - m_r)$ unknowns.

Define a weighted directed graph $G_c = (V, E)$ with

$$\begin{aligned} V &= \{\boldsymbol{k} | \boldsymbol{m} \leq \boldsymbol{k} \leq \boldsymbol{n}\} \\ E &= \{(\boldsymbol{k}_r^-, \boldsymbol{k}) | \boldsymbol{m} \leq \boldsymbol{k}_r^- < \boldsymbol{n},\ r = 1, 2, \ldots, R\} \end{aligned}$$

$$w(e) \;=\; w(\boldsymbol{k}_r^-, \boldsymbol{k}), \text{ the weight of edge } e = (\boldsymbol{k}_r^-, \boldsymbol{k}) \in E.$$

Then, $G_c$ is the Coates flow graph associated with system (B.1).

**Lemma B.1** *A solution of the system of linear equations*

$$x(\boldsymbol{k}) = \sum_{r=1}^{R} w(\boldsymbol{k}_r^-, \boldsymbol{k}) x(\boldsymbol{k}_r^-), \qquad \boldsymbol{m} \le \boldsymbol{k}_r^- < \boldsymbol{n},\ r = 1, 2, \ldots, R, \tag{B.3}$$

*is given by*

$$x(\boldsymbol{k}) = x(\boldsymbol{m}) \sum_{p \in \mathcal{P}_{\boldsymbol{m},\boldsymbol{k}}} w(p), \qquad \boldsymbol{m} \le \boldsymbol{k} \le \boldsymbol{n},\ r = 1, 2, \ldots, R, \tag{B.4}$$

*where* $\mathcal{P}_{\boldsymbol{m},\boldsymbol{k}}$ *is the set of directed paths in* $G_c$ *from vertex* $\boldsymbol{m}$ *to vertex* $\boldsymbol{k}$, *and* $w(p)$ *is the weight-product of path* $p$.

**Proof**

Since for each edge $(\boldsymbol{i}, \boldsymbol{j}) \in E$, $\boldsymbol{i} < \boldsymbol{j}$, $G_c$ is directed acyclic. Then, the weight of a 1-factor of $G_c$ is one and the weight of a 1-factorial connection of $G_c$ from vertex $\boldsymbol{m}$ to vertex $\boldsymbol{k}$ is equal to the weight-product of the path from $\boldsymbol{m}$ to $\boldsymbol{k}$ contained in the 1-factorial connection. Note that, by definition, the weight of an empty graph is one, while the weight-product of an empty path is zero. Thus,

$$x(\boldsymbol{k}) = x(\boldsymbol{m}) \sum_{p \in \mathcal{P}_{\boldsymbol{m},\boldsymbol{k}}} w(p), \qquad \boldsymbol{m} \le \boldsymbol{k} \le \boldsymbol{n},\ r = 1, 2, \ldots, R. \tag{B.5}$$

**Q.E.D.**

**Theorem B.2** *Let* $G_c$ *be the Coates flow graph associated with the system of linear equations*

$$\left\{ \begin{array}{rcll} x(\boldsymbol{m}) & = & x(\boldsymbol{m}), & \boldsymbol{m} \in \mathcal{I}, \\ x(\boldsymbol{k}) & = & \sum_{r=1}^{R} w(\boldsymbol{k}_r^-, \boldsymbol{k}) x(\boldsymbol{k}_r^-), & \boldsymbol{k} \notin \mathcal{I},\ \boldsymbol{l} \le \boldsymbol{k}_r^- < \boldsymbol{n},\ r = 1, 2, \ldots, R. \end{array} \right\} \tag{B.6}$$

*Then,*

$$x(\boldsymbol{k}) = \sum_{\boldsymbol{m}\in\mathcal{I}} x(\boldsymbol{m}) \sum_{p\in\mathcal{P}_{\boldsymbol{m},\boldsymbol{k}}} w(p) \qquad \boldsymbol{v} \leq \boldsymbol{k} \leq \boldsymbol{n}. \tag{B.7}$$

*is a solution of the system.*

**Proof**

Add a vertex $\boldsymbol{v} < \boldsymbol{m}$, $\forall \boldsymbol{m} \in \mathcal{I}$, to $G_c$ and connect it with a directed edge $(\boldsymbol{v}, \boldsymbol{k})$ to every element $\boldsymbol{k}$ of $\mathcal{I}$. Furthermore, let $w(\boldsymbol{v}, \boldsymbol{m}) = x(\boldsymbol{m})/x(\boldsymbol{v})$ and $x(\boldsymbol{v}) \neq 0$. Then, $G_c + \boldsymbol{v}$ is the Coates flow graph associated with the system

$$\left\{\begin{array}{rcl} x(\boldsymbol{m}) & = & w(\boldsymbol{v},\boldsymbol{m})x(\boldsymbol{v}), \qquad \boldsymbol{m} \in \mathcal{I}, \\ x(\boldsymbol{v}) & = & x(\boldsymbol{v}), \\ x(\boldsymbol{k}) & = & \sum_{r=1}^{R} w(\boldsymbol{k}_r^-, \boldsymbol{k})x(\boldsymbol{k}_r^-), \quad \boldsymbol{k} \notin \mathcal{I}, \ \boldsymbol{l} \leq \boldsymbol{k}_r^- < \boldsymbol{n}, \ r = 1, 2, \ldots, R. \end{array}\right\} \tag{B.8}$$

From Lemma B.1,

$$x(\boldsymbol{k}) = x(\boldsymbol{v}) \sum_{q\in\mathcal{P}_{\boldsymbol{v},\boldsymbol{k}}} w(q), \qquad \boldsymbol{v} \leq \boldsymbol{k} \leq \boldsymbol{n}. \tag{B.9}$$

Since the sets $\{(\boldsymbol{v}, \boldsymbol{k}) \cup p | p \in \mathcal{P}_{\boldsymbol{m},\boldsymbol{k}}\}$, $\boldsymbol{m} \in \mathcal{I}$, form a partition of $\mathcal{P}_{\boldsymbol{v},\boldsymbol{k}}$,

$$\begin{aligned} x(\boldsymbol{k}) &= x(\boldsymbol{v}) \sum_{\boldsymbol{m}\in\mathcal{I}} \sum_{p\in\mathcal{P}_{\boldsymbol{m},\boldsymbol{k}}} w((\boldsymbol{v}, \boldsymbol{m}) \cup p) \\ &= x(\boldsymbol{v}) \sum_{\boldsymbol{m}\in\mathcal{I}} \sum_{p\in\mathcal{P}_{\boldsymbol{m},\boldsymbol{k}}} w(\boldsymbol{v}, \boldsymbol{m})w(p) \\ &= \sum_{\boldsymbol{m}\in\mathcal{I}} \sum_{p\in\mathcal{P}_{\boldsymbol{m},\boldsymbol{k}}} w(p)x(\boldsymbol{m}). \end{aligned}$$

**Q.E.D.**

Next, I define formally the set $\mathcal{I}$ of boundary states for the complete sharing policy.

**Lemma B.3** *A maximal set $\mathcal{I}$ of states $\boldsymbol{m}$ such that $\boldsymbol{m} \cdot \boldsymbol{b} \leq S$ and $\boldsymbol{n} \cdot \boldsymbol{b} > S$, for $\boldsymbol{n} > \boldsymbol{m}$, is defined by the customer class with the greatest bandwidth requirements.*

*That is,*

$$\mathcal{I} = \bigcup_{i=0}^{b_{\max}-1} \{\boldsymbol{m}|\boldsymbol{m}\cdot\boldsymbol{b} = S - i\}, \tag{B.10}$$

*where* $b_{\max} = \max\{b_r,\ r = 1,2,\ldots,R\}$.

**Proof**

A maximal set of states $\boldsymbol{m}$ such that $\boldsymbol{m}_r^+\cdot\boldsymbol{b} > S$ is

$$\begin{aligned}\mathcal{I}_r &= \{\boldsymbol{m}|S - b_r < \boldsymbol{m}\cdot\boldsymbol{b} \leq S\}\\ &= \{\boldsymbol{m}|\boldsymbol{m}\cdot\boldsymbol{b} = S - b_r + 1, S - b_r + 2,\ldots,S\}\\ &= \{\boldsymbol{m}|\boldsymbol{m}\cdot\boldsymbol{b} = S - i, i = 1,2,\ldots,b_r - 1\}\\ &= \bigcup_{i=0}^{b_r-1}\{\boldsymbol{m}|\boldsymbol{m}\cdot\boldsymbol{b} = S - i\},\end{aligned} \tag{B.11}$$

where the sets $\{\boldsymbol{m}|\boldsymbol{m}\cdot\boldsymbol{b} = S-i\}$, $i = 1,2,\ldots,b_r-1$, form a partition of $\{\boldsymbol{m}|S-b_r < \boldsymbol{m}\cdot\boldsymbol{b} \leq S\}$. For a customer class $q$ with $b_q < b_r$,

$$\begin{aligned}\mathcal{I}_r &= \mathcal{I}_q \cup \left(\bigcup_{i=b_r-b_q}^{b_r-1}\{\boldsymbol{m}|\boldsymbol{m}\cdot\boldsymbol{b} = S - i\}\right)\\ \Rightarrow \mathcal{I}_q &\subset \mathcal{I}_r.\end{aligned} \tag{B.12}$$

Then $\mathcal{I}_r \subseteq \mathcal{I}$, for all $r = 1,2,\ldots,R$. **Q.E.D.**

Let $\mathcal{I}^+$ be the set of the immediate neighbors of $\mathcal{I}$ in the direction of increasing occupancy; that is,

$$\begin{aligned}\mathcal{I}^+ &\stackrel{\text{def}}{=} \{\boldsymbol{m}|\boldsymbol{m}\cdot\boldsymbol{b} > S \wedge (\exists r = 1,2,\ldots,R : \boldsymbol{m}_r^- \in \mathcal{I})\}\\ &= \bigcup_{i=1}^{b_{\max}}\{\boldsymbol{m}|\boldsymbol{m}\cdot\boldsymbol{b} = S + i\}.\end{aligned} \tag{B.13}$$

The need for the definition of $\mathcal{I}^+$ will become apparent in the proof of Corollary B.4, where I use the combinatorial properties of $\mathbf{Z}^R$ to collectively treat the paths from $\boldsymbol{m}$ to $\boldsymbol{n}$. There, we need set $\mathcal{I}^+$ in order to avoid counting paths that begin from a point $\boldsymbol{m} \in \mathcal{I}$, touch the boundary and end to $\boldsymbol{n}$. It should be clear, that in order to calculate $x(\boldsymbol{n})$ at any point $\boldsymbol{n} : \boldsymbol{n}\cdot\boldsymbol{b} > S$, we only need to consider the values at the elements of $\mathcal{I}$ as it is defined by (B.10). Set $\mathcal{I}^+$ is introduced for purely technical reasons.

**Corollary B.4** *For the complete sharing policy, the solution of the recursion*

$$x(\boldsymbol{n}) = \begin{cases} 1, & \text{if } \boldsymbol{n} \in \mathcal{I} \text{ and } \boldsymbol{n} \geq \boldsymbol{0} \\ \frac{1}{S}\sum_{r=1}^{R} b_r n_r x(\boldsymbol{n}_r^-), & \text{if } \boldsymbol{n} \notin \mathcal{I} \text{ and } \boldsymbol{n} \geq \boldsymbol{0}, \\ 0, & \text{otherwise}, \end{cases} \quad \text{(B.11)}$$

*is*

$$\begin{aligned} x(\boldsymbol{n}) &= \sum_{\boldsymbol{m}\in\mathcal{I}^+} \binom{n_1 - m_1 + n_2 - m_2 + \cdots + n_R - m_R}{n_1 - m_1, n_2 - m_2, \ldots, n_R - m_R} \frac{\prod_{r=1}^{R} \frac{n_r!}{m_r!} b_r^{n_r - m_r}}{S^{\sum_{r=1}^{R}(n_r - m_r)}} x(\boldsymbol{m}) \\ &= \sum_{\boldsymbol{m}\in\mathcal{I}^+} \left[\sum_{r=1}^{R}(n_r - m_r)\right]! \prod_{r=1}^{R} \binom{n_r}{m_r} \left(\frac{b_r}{S}\right)^{n_r - m_r} x(\boldsymbol{m}), \end{aligned} \quad \text{(B.15)}$$

*where*

$$\binom{n_1 - m_1 + n_2 - m_2 + \cdots + n_R - m_R}{n_1 - m_1, n_2 - m_2, \ldots, n_R - m_R} = \frac{\left[\sum_{r=1}^{R}(n_r - m_r)\right]!}{\prod_{r=1}^{R}(n_r - m_r)!} \quad \text{(B.16)}$$

*is a multinomial coefficient.*

## Proof

The number of directed paths from $\boldsymbol{m}$ to $\boldsymbol{n}$ in $\mathbf{Z}^R$ is given by the multinomial coefficient [23]

$$\binom{n_1 - m_1 + n_2 - m_2 + \cdots + n_R - m_R}{n_1 - m_1, n_2 - m_2, \ldots, n_R - m_R} = \frac{\left[\sum_{r=1}^{R}(n_r - m_r)\right]!}{\prod_{r=1}^{R}(n_r - m_r)!}. \quad \text{(B.17)}$$

If $\boldsymbol{m} \in \mathcal{I}$, a path from $\boldsymbol{m}$ to $\boldsymbol{n}$ may contain vertices in $\mathcal{I}$, thus not contributing to $x(\boldsymbol{n})$. This cannot happen if $\mathcal{I}^+$ is used instead of $\mathcal{I}$ as a set of initial values.

The factor in the weight of a path from $\boldsymbol{m}$ to $\boldsymbol{n}$ due to type $r$ transitions is

$$\prod_{k_r=1}^{n_r-m_r} \frac{(m_r+k_r)b_r}{S} = \frac{n_r!}{m_r!}\left(\frac{b_r}{S}\right)^{n_r-m_r}. \tag{B.18}$$

So, each path from $\boldsymbol{m} \in \mathcal{I}^+$ to $\boldsymbol{n}$ has weight

$$\prod_{r=1}^{R} \frac{n_r!}{m_r!}\left(\frac{b_r}{S}\right)^{n_r-m_r} x(\boldsymbol{m}). \tag{B.19}$$

The result now follows from Theorem B.2. **Q.E.D.**

Then, for $\boldsymbol{n} \cdot \boldsymbol{b} > S$, the equilibrium distribution is

$$\begin{aligned}\pi(\boldsymbol{n}) &= G(\Omega)^{-1} x(\boldsymbol{n}) \prod_{r=1}^{R} \frac{\rho_r^{n_r}}{n_r!} \\ &= G(\Omega)^{-1} \sum_{\boldsymbol{m}\in\mathcal{I}^+} \left[\sum_{r=1}^{R}(n_r-m_r)\right]! \prod_{r=1}^{R}\left(\frac{b_r}{S}\right)^{n_r-m_r} \frac{\rho_r^{n_r}}{m_r!(n_r-m_r)!},\end{aligned} \tag{B.20}$$

where

$$G(\Omega) = \sum_{\boldsymbol{n}\in\Omega}\sum_{\boldsymbol{m}\in\mathcal{I}^+} \left[\sum_{r=1}^{R}(n_r-m_r)\right]! \prod_{r=1}^{R}\left(\frac{b_r}{S}\right)^{n_r-m_r} \frac{\rho_r^{n_r}}{m_r!(n_r-m_r)!}. \tag{B.21}$$

# Appendix C

# The Probability Generating Function of the ESR Model

**Theorem C.1** *The probability generating function of the equilibrium population distribution* $\{\pi(\boldsymbol{n}), \boldsymbol{n} \in \Omega\}$ *is*

$$\Pi(\boldsymbol{z}) = \frac{S\Pi_{0,S}(\boldsymbol{z}) - \sum_{r=1}^{R} z_r b_r \rho_r \left[\Pi_{0,S-b_r}(\boldsymbol{z}) + \Pi_{T-b_r+1,T}(\boldsymbol{z})\right]}{S - \sum_{r=1}^{R} z_r b_r \rho_r}, \tag{C.1}$$

*where*

$$\Pi_{u,v}(\boldsymbol{z}) \stackrel{\text{def}}{=} \sum_{\{\boldsymbol{n} | u \le \boldsymbol{n}\cdot\boldsymbol{b} \le v\}} z_1^{n_1} z_2^{n_2} \cdots z_R^{n_R} \pi(\boldsymbol{n}). \tag{C.2}$$

**Proof**

Multiplying both sides of equation (4.18) by $z_1^{n_1} z_2^{n_2} \cdots z_R^{n_R}$ and summing over

$\{\boldsymbol{n}|S < \boldsymbol{n}\cdot\boldsymbol{b} \leq T\}$, we get

$$\begin{aligned}
\sum_{\{\boldsymbol{n}|S<\boldsymbol{n}\cdot\boldsymbol{b}\leq T\}} z_1^{n_1} z_2^{n_2} \cdots z_R^{n_R} \pi(\boldsymbol{n}) &= \frac{1}{S} \sum_{\{\boldsymbol{n}|S<\boldsymbol{n}\cdot\boldsymbol{b}\leq T\}} z_1^{n_1} z_2^{n_2} \cdots z_R^{n_R} \sum_{r=1}^{R} b_r \rho_r \pi(\boldsymbol{n}_r^-) \\
\Leftrightarrow \quad \Pi(\boldsymbol{z}) - \Pi_{0,S}(\boldsymbol{z}) &= \frac{1}{S} \sum_{r=1}^{R} z_r b_r \rho_r \sum_{\{\boldsymbol{n}|S<\boldsymbol{n}\cdot\boldsymbol{b}\leq T\}} z_1^{n_1} z_2^{n_2} \cdots z_{r-1}^{n_{r-1}} z_r^{n_r - 1} z_{r+1}^{n_{r+1}} \cdots z_R^{n_R} \pi(\boldsymbol{n}_r^-) \\
&= \frac{1}{S} \sum_{r=1}^{R} z_r b_r \rho_r \sum_{\{\boldsymbol{n}|S-b_r<\boldsymbol{n}\cdot\boldsymbol{b}\leq T-b_r\}} z_1^{n_1} z_2^{n_2} \cdots z_R^{n_R} \pi(\boldsymbol{n}_r^-) \\
&= \frac{1}{S} \sum_{r=1}^{R} z_r b_r \rho_r \left[\Pi(\boldsymbol{z}) - \Pi_{0,S-b_r}(\boldsymbol{z}) - \Pi_{T-b_r+1,T}(\boldsymbol{z})\right]
\end{aligned}$$

and equation (C.1) follows. **Q.E.D.**

The probability generating function $Q(z)$ of $q(j)$ is derived in a similar manner.

**Theorem C.2** *The probability generating function of the equilibrium occupancy distribution* $\{q(j),\ j = 0, 1, 2, \ldots, T\}$ *is*

$$Q(z) = \frac{SQ_{0,S}(z) - \sum_{r=1}^{R} z^{b_r} b_r \rho_r \left[Q_{0,S-b_r}(z) + Q_{T-b_r+1,T}(z)\right]}{S - \sum_{r=1}^{R} z^{b_r} b_r \rho_r}, \qquad \text{(C.3)}$$

*where*

$$Q_{u,v}(z) \stackrel{\text{def}}{=} \sum_{j=u}^{v} z^j q(j). \qquad \text{(C.4)}$$

**Proof**

Multiplying both sides of equation (4.28) by $z^j$ and summing over $j = S+1, S+2, \ldots, T$, we get

$$\begin{aligned}
\sum_{j=S+1}^{T} z^j q(j) &= \frac{1}{S} \sum_{j=S+1}^{T} z^j \sum_{r=1}^{R} b_r \rho_r q(j - b_r) \\
\Leftrightarrow \quad Q(z) - Q_{0,S}(z) &= \frac{1}{S} \sum_{r=1}^{R} z^{b_r} b_r \rho_r \sum_{j>S} z^{j-b_r} q(j - b_r) \\
&= \frac{1}{S} \sum_{r=1}^{R} z^{b_r} b_r \rho_r \left[Q(z) - Q_{0,S-b_r}(z) - Q_{T-b_r+1,T}(z)\right]
\end{aligned}$$

and equation (C.3) follows. **Q.E.D.**

The first and second moments of $n_r$ are

$$E\{n_r\} = \frac{\partial^2}{\partial z_r^2}\Pi(\boldsymbol{z})\bigg|_{z_i=1}$$

and

$$E\{n_r^2\} = \frac{\partial^2}{\partial z_r^2}\Pi(\boldsymbol{z})\bigg|_{z_i=1} + E\{n_r\}.$$

Let's derive them. Equation (C.1) can be rewritten as

$$(S - \sum_{i=1}^{R} z_i b_i \rho_i)\Pi(\boldsymbol{z}) = S\Pi_{0,S}(\boldsymbol{z}) - \sum_{i=1}^{R} z_i b_i \rho_i \left[\Pi_{0,S-b_i}(\boldsymbol{z}) + \Pi_{T-b_i+1,T}(\boldsymbol{z})\right]. \qquad \text{(C.5)}$$

Differentiating both sides of (C.5), with respect to $z_r$, we get

$$\begin{aligned}(S - \sum_{i=1}^{R} z_i b_i \rho_i)\frac{\partial}{\partial z_r}\Pi(\boldsymbol{z}) \;=\;& S\frac{\partial}{\partial z_r}\Pi_{0,S}(\boldsymbol{z}) - \sum_{i=1}^{R} z_i b_i \rho_i \frac{\partial}{\partial z_r}\Pi_{0,S-b_i}(\boldsymbol{z}) \\ &+ b_r\rho_r\left[\Pi(\boldsymbol{z}) - \Pi_{0,S-b_r}(\boldsymbol{z}) - \Pi_{T-b_r+1,T}(\boldsymbol{z})\right] \\ &- \sum_{i=1}^{R} z_i b_i \rho_i \frac{\partial}{\partial z_r}\Pi_{T-b_i+1,T}(\boldsymbol{z}). \qquad \text{(C.6)}\end{aligned}$$

Differentiating once more, with respect to $z_r$, we have

$$\begin{aligned}(S - \sum_{i=1}^{R} z_i b_i \rho_i)\frac{\partial^2}{\partial z_r^2}\Pi(\boldsymbol{z}) \;=\;& S\frac{\partial^2}{\partial z_r^2}\Pi_{0,S}(\boldsymbol{z}) - \sum_{i=1}^{R} z_i b_i \rho_i \frac{\partial^2}{\partial z_r^2}\Pi_{0,S-b_i}(\boldsymbol{z}) \\ &+ 2b_r\rho_r\left[\frac{\partial}{\partial z_r}\Pi(\boldsymbol{z}) - \frac{\partial}{\partial z_r}\Pi_{0,S-b_r}(\boldsymbol{z}) - \frac{\partial}{\partial z_r}\Pi_{T-b_r+1,T}(\boldsymbol{z})\right] \\ &- \sum_{i=1}^{R} z_i b_i \rho_i \frac{\partial^2}{\partial z_r^2}\Pi_{T-b_i+1,T}(\boldsymbol{z}). \qquad \text{(C.7)}\end{aligned}$$

The quantities of interest are $\frac{\partial}{\partial z_r}\Pi_{u,v}(\boldsymbol{z})\big|_{z_r=1}$ and $\frac{\partial^2}{\partial z_r^2}\Pi_{u,v}(\boldsymbol{z})\big|_{z_i=1}$.

$$\begin{aligned}\frac{\partial}{\partial z_r}\Pi_{u,v}(\boldsymbol{z})\bigg|_{z_r=1} \;=\;& \frac{1}{z_r}\sum_{\{\boldsymbol{n}|u\le \boldsymbol{n}\cdot\boldsymbol{b}\le v\}} n_r z_1^{n_1} z_2^{n_2}\cdots z_R^{n_R}\pi(\boldsymbol{n})\bigg|_{z_i=1} \\ =\;& \sum_{j=u}^{v}\sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_r\pi(\boldsymbol{n}). \qquad \text{(C.8)}\end{aligned}$$

$$
\begin{aligned}
\left.\frac{\partial^2}{\partial z_r^2}\Pi_{u,v}(\boldsymbol{z})\right|_{z_r=1} &= \left.\frac{1}{z_r^2}\sum_{\{\boldsymbol{n}|u\le \boldsymbol{n}\cdot\boldsymbol{b}\le v\}} n_r^2 z_1^{n_1} z_2^{n_2}\cdots z_R^{n_R}\pi(\boldsymbol{n})\right|_{z_r=1} \\
&\quad - \left.\frac{1}{z_r^2}\sum_{\{\boldsymbol{n}|u\le \boldsymbol{n}\cdot\boldsymbol{b}\le v\}} n_r z_1^{n_1} z_2^{n_2}\cdots z_R^{n_R}\pi(\boldsymbol{n})\right|_{z_r=1} \\
&= \sum_{j=u}^{v}\sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_r^2\pi(\boldsymbol{n}) - \sum_{j=u}^{v}\sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_r\pi(\boldsymbol{n}). \qquad \text{(C.9)}
\end{aligned}
$$

For $j \le S$, using the detailed balance equations, we get

$$
\begin{aligned}
\sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_r\pi(\boldsymbol{n}) &= \rho_r \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} \pi(\boldsymbol{n}_r^-) \\
&= \rho_r q(j-b_r). \qquad \text{(C.10)} \\
\sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_r^2\pi(\boldsymbol{n}) &= \rho_r \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_r\pi(\boldsymbol{n}_r^-) \\
&= \rho_r \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} (n_r-1)\pi(\boldsymbol{n}_r^-) + \rho_r \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} \pi(\boldsymbol{n}_r^-) \\
&= \rho_r q(j-2b_r) + \rho_r q(j-b_r). \qquad \text{(C.11)}
\end{aligned}
$$

For $S < j \le T$, using the recursion $\pi(\boldsymbol{n}) = \frac{1}{S}\sum_{i=1}^{R} b_i\rho_i\pi(\boldsymbol{n}_i^-)$, we get

$$
\begin{aligned}
\sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_r\pi(\boldsymbol{n}) &= \frac{1}{S}\sum_{i=1}^{R} b_i\rho_i \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_r\pi(\boldsymbol{n}_i^-) \\
&= \frac{1}{S}\sum_{\substack{i=1\\ i\ne r}}^{R} b_i\rho_i \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j-b_i\}} n_r\pi(\boldsymbol{n}) + \frac{b_r\rho_r}{S}\sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_r\pi(\boldsymbol{n}_r^-) \\
&= \frac{1}{S}\sum_{i=1}^{R} b_i\rho_i \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j-b_i\}} n_r\pi(\boldsymbol{n}) + \frac{b_r\rho_r}{S} q(j-b_r). \qquad \text{(C.12)} \\
\sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_r^2\pi(\boldsymbol{n}) &= \frac{1}{S}\sum_{i=1}^{R} b_i\rho_i \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} n_r^2\pi(\boldsymbol{n}_i^-) \\
&= \frac{1}{S}\sum_{i=1}^{R} b_i\rho_i \Bigg[\sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} (n_r-1)^2\pi(\boldsymbol{n}_i^-) \\
&\quad + \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} 2(n_r-1)\pi(\boldsymbol{n}_i^-) + \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j\}} \pi(\boldsymbol{n}_r^-)\Bigg] \\
&= \frac{1}{S}\sum_{i=1}^{R} b_i\rho_i \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j-b_i\}} n_r^2\pi(\boldsymbol{n}) + \frac{2}{S} b_r\rho_r \sum_{\{\boldsymbol{n}|\boldsymbol{n}\cdot\boldsymbol{b}=j-b_r\}} n_r\pi(\boldsymbol{n})
\end{aligned}
$$

$$+\frac{1}{S}b_r\rho_r q(j-b_r). \tag{C.13}$$

As equations (C.12) and (C.13) indicate, for general bandwidth requirements, we can compute $\sum_{\{n|n\cdot b=j\}} n_r\pi(\boldsymbol{n})$ and $\sum_{\{n|n\cdot b=j\}} n_r^2\pi(\boldsymbol{n})$ recursively only. Since, for $j > S$, these sums need to be evaluated in the interval $T - b_r + 1 \leq j \leq T$, the recursions would have to scan the entire state space. Then, the use of the probability generating function is unnecessary:

$$E\{n_r^k\} = \sum_{j=0}^{T} \sum_{\{n|n\cdot b=j\}} n_r^k\pi(\boldsymbol{n}), \qquad k = 0,1,2,\ldots.$$

Thus, for finite population systems, the use of $\Pi(\boldsymbol{z})$ to generate the moments of $n_r$, $r = 1,2,\ldots,R$ is of limited value. A more efficient approach is the recursive computation of the sum $\sum_{\{n|n\cdot b=j\}} n_r^k\pi(\boldsymbol{n})$. As a matter of fact, we can exploit the structure of our model even further and produce a recursive formula that is used to derive the joint moments $E\{n_1^{k_1}n_2^{k_2}\cdots n_r^{k_r}\}$, $r \leq R$, of the number in the system for different customer types. The recursion is given by Theorem 4.5 in Section 4.4, where its computational requirements are also discussed.

**Comment.** For the extended shared resource problem, the probability generating function cannot generate the equilibrium distribution efficiently. Furthermore, it assumes knowledge of the normalization constant. However, studying $\Pi(\boldsymbol{z})$ and its drawbacks guided us to develop a space-efficient computational recursion for the moments of the population. As a result, we can get the derivatives of $\Pi(\boldsymbol{z})$ at $\boldsymbol{z} = \boldsymbol{1}$. Given the generality of applications for the probability generating function, this result may prove useful. ■