A Family of Real Cubic Fields

Fares Fares

A Thesis

in

The Department

of

Mathematics

Presented in Partial Fulfillment of the Requirements
for the Degree of Master of Science at
Concordia University
Montréal, Québec, Canada

August 1992

ISBN 0-315-90830-0

ABSTRACT

A Family of Real Cubic Fields

Fares Fares

This thesis deals with a family of real cubic extensions K_t/Q where t is a rational integer. This family is paramatrized by the cubic polynomials

$$f_t = x^3 - 3(t^2 + t + 1)x - (t^2 + t + 1)(2t + 1).$$

The ring of integers O_{K_t} of K_t is computed for all t and the unit group U_t is obtained under certain conditions (when $O_{K_t} = \mathbb{Z}[\alpha_t]$). In the general case, a bound on the index of a subgroup of U_t is given. Also we investigate the arithmetic invariants of the family K_t, and get bounds for the regulator and the class number.

ACKNOWLEDGEMENT

I thank Dr. Kisilevsky for accepting to be my supervisor and for all his help to finish this thesis (everything, the contents, the style ...). I thank also Dr. Ford for introducing me to ALGEB and VAX2, Dr. Cummins for many helpful conversations and Prof. Cohen, who was the program director for most of the duration of my M.Sc., for all his advice.

TABLE OF CONTENTS

1. Introduction 1

2. Ring of Integers of K_t 3

3. Group of Units of K_t 10

4. Class Number of K_t 17

5. References 23

1. Introduction

Consider the polynomial

$$f = x^3 - 3\phi x - \phi\phi' \in \mathbb{Z}[\tau][x]$$

where $\phi = \tau^2 + \tau + 1$, $\phi' = 2\tau + 1 \in \mathbb{Z}[\tau]$ and τ, x are indeterminates. Denote by α one root of f in $\overline{\mathbb{Q}(\tau)}$ the algebraic closure of $\mathbb{Q}(\tau)$ and let $K = \mathbb{Q}(\tau)(\alpha)$. For $t \in \mathbb{Z}$, ϕ_t, ϕ'_t, f_t and α_t will refer to specializations of ϕ, ϕ', f and α at $\tau = t$ (hence $f_t(\alpha_t) = 0$) and let $K_t = \mathbb{Q}(\alpha_t)$.

We list certain properties of f:

Proposition 1.1. *f is irreducible over $\mathbb{Q}(\tau)$, and $[K : \mathbb{Q}(\tau)] = 3$.*

Proof: For all rational integers t, $\phi_t \equiv 1 \pmod 2$, and $\phi'_t \equiv 1 \pmod 2$, therefore $f_t \equiv x^3 - x - 1 \pmod 2$. The latter polynomial has no linear factors in $\mathbb{Z}/2\mathbb{Z}$, hence f_t is irreducible over $\mathbb{Z}$ and hence also over $\mathbb{Q}$ for all t. ∎

Proposition 1.2. *$K_t/\mathbb{Q}$ is an abelian (cyclic) extension for all t. In particular, K_t is a real field.*

Proof: The discriminant of f_t is $(9\phi_t)^2$, which is a perfect square. It follows that the cubic extension $(K_t/\mathbb{Q})$ is a Galois extension for all rational integers t. It is also abelian since its Galois group has order 3. ∎

Proposition 1.3. **(i)** *ϕ_t is not divisible by 9 for all $t \in \mathbb{Z}$, and ϕ_t is divisible by 3 if and only if $t \equiv 1 \pmod 3$.*

(ii) *ϕ'_t is divisible by 9 if and only if $t \equiv 4 \pmod 9$*

(iii) *$\gcd(\phi_t, \phi'_t)$ equals 3 if $t \equiv 1 \pmod 3$, and 1 otherwise.*

Proof: The congruence $\phi_t \equiv 0 \pmod 3$ is satisfied iff $t \equiv 1 \pmod 3$, but $\phi_t \not\equiv 0 \pmod 9$ for $t \equiv 1, 4, 7 \pmod 9$. Thus 9 does not divide ϕ_t for all t and (i) follows. To prove (ii) consider the reduction of ϕ'_t modulo 9. Finally, (iii) follows from (i) and the observation that $4\phi_t - \phi_t'^2 = 3$. ∎

2. Ring of integers of K_t

Denote by O_{K_t} the ring of integers of the number field $K_t = \mathbb{Q}(\alpha_t)$.

Proposition 2.1. *If $\phi_t = 3^{n_0} p_1{}^{n_1} \ldots p_s{}^{n_s}$, where $3, p_1, \ldots, p_s$ are distinct primes, $n_0, n_1, \ldots, n_s \in \mathbb{Z}$, and δ_t is disc (K_t), the discriminant of K_t, then for $1 \leq i \leq s$*

$$n_i \not\equiv 0 \pmod 3 \text{ implies that } p_i | \delta_t$$

and

$$t \equiv 1 \pmod 3 \;\text{ and } t \not\equiv 4 \pmod 9 \text{ imply that } 3|\delta_t.$$

Proof: Since $f_t(\alpha_t) = 0$ we see that $\alpha_t \in O_{K_t}$ and that $\alpha_t^3 = 3\phi_t\alpha_t + \phi_t\phi_t'$. Taking ideals in O_{K_t},

$$(\alpha_t)^3 = (\phi_t)(3\alpha_t + \phi_t'). \tag{1}$$

Let $n_i \not\equiv 0 \pmod 3$. Take $q = p_i$, where $1 \leq i \leq s$, and let Q be a prime ideal in O_{K_t} above q ($\mathrm{Q} \cap \mathbb{Z} = q\mathbb{Z}$), with ramification index e. Q divides (ϕ_t) implies that Q divides (α_t). But q does not divide ϕ_t' (Proposition 1.3), thus Q does not divide (ϕ_t') (otherwise, $\mathrm{Q} \cap \mathbb{Z} = q\mathbb{Z}$ would include $(\phi_t') \cap \mathbb{Z} = \phi_t'\mathbb{Z}$) and hence Q does not divide $(3\alpha_t + \phi_t')$ ($3\alpha_t \in \mathrm{Q}$ and $\phi_t' \notin \mathrm{Q}$). In terms of Q-valuation,

$$3\nu_{\mathrm{Q}}(\alpha_t) = \nu_{\mathrm{Q}}(\phi_t) \tag{2}$$

where $\nu_{\mathrm{Q}}(\alpha_t)$, $\nu_{\mathrm{Q}}(\phi_t)$ are the exponents of Q in the decomposition of (α_t) and (ϕ_t) into primes in O_{K_t}. But $\nu_{\mathrm{Q}}(\phi_t) = en_i$. Then by equation (2), 3 does not divide n_i implies that 3 divides e, and therefore that q ramifies in O_{K_t}. This proves that in this case, q divides δ_t the discriminant of K_t.

Next, let $t \equiv 1 \pmod 3$, $t \not\equiv 4 \pmod 9$, then both ϕ_t and ϕ'_t are divisible by 3 but not by 9, and so $n_0 = 1$. Let U be a prime ideal in O_{K_t} above 3, with ramification index e_3. Equation (1) becomes

$$(\alpha_t)^3 = (\phi_t)(3)(\alpha_t + \frac{\phi'_t}{3}).$$

Since U divides α_t and does not divide $\frac{\phi'_t}{3}$, U does not divide $(\alpha_t + \frac{\phi'_t}{3})$. Hence $3\nu_{\mathrm{U}}(\alpha_t) = e_3 + e_3 = 2e_3$, thus 3 divides e_3 and therefore 3 ramifies in O_{K_t}. This proves that 3 divides δ_t. ∎

The next proposition produces a $\mathbb{Z}$-submodule of O_{K_t}, whose index is determined by Proposition 2.3. Let $\phi_t = 3^{n_0}{p_1}^{n_1} \ldots {p_s}^{n_s}$, where $3, p_1, \ldots, p_s$ are distinct primes, and $n_0, \ldots, n_s$ nonnegative integers. Write $n_i = 3k_i + r_i$, where r_i, k_i are natural numbers, $0 \leq r_i < 3$ and $1 \leq i \leq s$. Note that n_0 is either 1 (if $t \equiv 1 \pmod 3$) or 0 (otherwise). For $1 \leq i \leq s$, let

$$j_i = \begin{cases} 0 & \text{if } r_i = 0 \\ r_i - 1 & \text{otherwise} \end{cases} \qquad j_0 = \begin{cases} 1 & \text{if } t \equiv 4 \pmod 9 \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$j_i = \begin{cases} 0 & \text{if } n_i \equiv 0, 1 \pmod 3 \\ 1 & \text{if } n_i \equiv 2 \pmod 3. \end{cases}$$

Proposition 2.2. *(i)* $\xi_t = (\alpha_t / 3^{j_0} p_s^{k_s} \ldots p_n^{k_n})$ *and* $\xi'_t = (\alpha_t^2 / 3^{n_0 + j_0} p_1^{2k_1 + j_1} \ldots p_s^{2k_s + j_s})$ *are algebraic integers.*

(ii) $\frac{1}{n}\xi_t \in O_{K_t}$ *($n \in \mathbb{Z}$) implies that* $n = \pm 1$

$\frac{1}{n}\xi'_t \in O_{K_t}$ *($n \in \mathbb{Z}$) implies that* $n = \pm 1$

Proof: If $\omega \in O_{K_t}$ has a minimal polynomial $h(x) = x^3 + m_2 x^2 + m_1 x + m_0$, where m_0, m_1 and m_2 are rational integers, and if $a \in \mathbb{Z}$, then $(\omega / a) \in O_{K_t}$ if and only if a divides m_2, a^2 divides m_1 and a^3 divides m_0. Now since α_t has the minimal polynomial $x^3 - 3\phi_t x - \phi_t\phi'_t$, it follows that $\frac{\alpha_t}{3} \in O_{K_t}$ if and only if $27|\phi_t\phi'_t$ and $9|3\phi_t$. This is equivalent to $3|\phi_t$ and $9|\phi'_t$, which is equivalent to $t \equiv 4 \pmod 9$

Also, $\frac{\alpha_t}{9}$ does not belong to O_{K_t} for any t since 9 does not divide ϕ_t. Hence,

$$\begin{cases} \frac{\alpha_t}{3} \in O_{K_t} & \text{if and only if } t \equiv 4 \pmod 9 \\ \frac{\alpha_t}{9} \notin O_{K_t} & \text{for all } t \in \mathbb{Z}. \end{cases} \tag{3}$$

Taking q to be a prime different from 3, then for $m \in \mathbb{Z}$, $\frac{\alpha_t}{q^m} \in O_{K_t}$ if and only if $q^{3m}|\phi_t\phi'_t$ and $q^{2m}|3\phi_t$. Thus

$$\frac{\alpha_t}{q^m} \in O_{K_t} \text{ if and only if } q^{3m}|\phi_t. \tag{4}$$

Similarly, α_t^2 satisfies the minimal polynomial : $x^3 - 6\phi_t x^2 + 9\phi_t^2 x - {\phi'_t}^2\phi_t^2$. Consequently, $\frac{\alpha_t^2}{3} \in O_{K_t}$ if and only if $3|\phi_t$ (since $3|\phi_t$ if and only if $3|\phi'_t$). Also $\frac{\alpha_t^2}{3^m} \in O_{K_t}$, $m \in \mathbb{Z}$, $m > 1$ if and only if $m = 2$, $3|\phi_t$, $9|\phi'_t$. Hence

$$\begin{cases} \frac{\alpha_t^2}{3} \in O_{k_t} & \text{if and only if } t \equiv 1 \pmod 3 \\ \frac{\alpha_t^2}{3^m} \in O_{K_t}\ ,\ m \in \mathbb{Z}\ ,\ m > 1 & \text{if and only if } m = 2 \text{ and } t \equiv 4 \pmod 9. \end{cases} \tag{5}$$

For q a rational prime different from 3, $m \in \mathbb{Z}$

$$\frac{\alpha_t^2}{q^m} \in O_{K_t} \text{ if and only if } q^{3m}|\phi_t^2. \tag{6}$$

In the above computations, we've used repeatedly the fact that $\gcd(\phi_t, \phi'_t) = 1$ or 3.

By equation(3) $\alpha_t/3^{j_0} \in O_{K_t}$ and $\alpha_t/9 \notin O_{K_t}$ for any $t \in \mathbb{Z}$. Also equation (5) implies that $\frac{\alpha_t^2}{3^{n_0+j_0}}$ is an integer (since $t \equiv 1 \pmod 3$ if and only if $n_0 = 1$, and $t \equiv 4 \pmod 9$ is equivalent to $n_0 + j_0 = 2$). By equation(4) we have that for $m \in \mathbb{Z}$ and p_i one of the primes of ϕ_t with exponent n_i, $1 \leq i \leq s$ $\alpha_t/p_i^m \in O_{K_t}$ if

p_i^{3m} divides ϕ_t that is $3m \le n_i = 3k_i + r_i$ or $m \le k_i + \frac{r_i}{3}$. Therefore, $\alpha_t / p_i^{k_i}$ is an integer. Also by equation (6), we have that $\frac{\alpha_t^2}{p_i^m} \in O_{K_t}$ is equivalent to $3m \le 2n_i$. This is true if and only if $m \le 2k_i + \frac{2r_i}{3}$. Hence,

$$\begin{cases} \frac{\alpha_t^2}{p_i^{2k_i}} \in O_{K_t} & \text{if } r_i = 0, 1 \\ \frac{\alpha_t^2}{p_i^{2k_i+1}} \in O_{K_t} & \text{if } r_i = 2 \end{cases}.$$

Therefore,

$$\frac{\alpha_t^2}{p_i^{2k_i+j_i}} \in O_{K_t} \quad \text{where } j_i = \begin{cases} 0 & \text{if } n_i \equiv 0, 1 \pmod 3 \\ 1 & \text{if } n_i \equiv 2 \pmod 3 \end{cases}.$$

To complete the proof of *(i)*, let $\theta \in O_{K_t}$, c_1, c_2, relatively prime rational integers be such that $\frac{\theta}{c_1}$ and $\frac{\theta}{c_2}$ belong to O_{K_t}. Then there exist $l_1, l_2 \in \mathbb{Z}$ such that $l_1 c_1 + l_2 c_2 = 1$. Therefore $\frac{\theta}{c_1 c_2} = l_2 \frac{\theta}{c_1} + l_1 \frac{\theta}{c_2} \in O_{K_t}$.

To prove *(ii)* note that the conditions we obtained on the primes of the denominators of an integer of the form $\frac{\alpha_t}{n}$ or $\frac{\alpha_t^2}{n}$ (equations (3),(4), (5) and (6)) are necessary conditions. ∎

We describe the ring of integers of K_t.

Proposition 2.3. *With the same notation as in proposition 2.2, an integral basis for* O_{K_t} *is given by* $\left\{1, (\alpha_t/3^{j_0} p_1^{k_1} \dots p_s^{k_s}), (\alpha_t^2/3^{n_0+j_0} p_1^{2k_1+j_1} \dots p_s^{2k_s+j_s})\right\}$

Proof: Put $\xi_t = (\alpha_t/3^{j_0} p_1^{k_1} \dots p_s^{k_s})$, $\xi_t' = (\alpha_t^2/3^{n_0+j_0} p_1^{2k_1+j_1} \dots p_s^{2k_s+j_s})$, $A = \mathbb{Z}[\alpha_t]$ and $B = \mathbb{Z}[1, \xi_t, \xi_t']$. From Proposition 2.2*(i)*, it follows that the index of the $\mathbb{Z}$-module A in B is $[B : A] = 3^{n_0'} p_1^{n_1'} \dots p_s^{n_s'}$ where

$$n_0' = \begin{cases} 3 & \text{if } n_0 = 1 \text{ and } t \equiv 4 \pmod 9 \\ 1 & \text{if } n_0 = 1 \text{ and } t \not\equiv 4 \pmod 9 \\ 0 & \text{if } n_0 = 0. \end{cases}$$

and for $1 \leq i \leq s$,

$$\begin{cases} n'_i = n_i & \text{if } 3|n_i \\ n'_i = n_i - 1 & \text{otherwise.} \end{cases}$$

By proposition 2.1, the number $(3^{n''_0} p_1^{n''_1} \dots p_s^{n''_s})^2$ divides disc (K_t) (the power 2 appears because the discriminant of K_t is a perfect square) where

$$n''_0 = \begin{cases} 1 & \text{if } n_0 = 1 \text{ and } t \not\equiv 4 \pmod 9 \\ 0 & \text{otherwise} \end{cases}$$

and for $1 \leq i \leq s$

$$\begin{cases} n''_i = 0 & \text{if } 3|n_i \\ n''_i = 1 & \text{otherwise.} \end{cases}$$

Now disc $(f_t) = [O_{K_t} : B]^2 [B : A]^2 \text{disc}(K)$ where $\text{disc}(f_t) = (9\phi_t)^2$. Hence

$$[O_{K_t} : B] = \begin{cases} 1 & \text{if } t \equiv 4 \pmod 9 \\ 1, 3, \text{ or } 9 & \text{otherwise.} \end{cases}$$

This means that $O_{K_t} = B$ if $t \equiv 4 \pmod 9$. We prove that this is also true when $t \not\equiv 4 \pmod 9$.

We remark that $[O_{K_t} : B] = 3$ or 9 implies that there exists $\epsilon \in B$ such that $(\epsilon/3) \in O_{K_t} - B$. Let $\epsilon = \lambda_0 + \lambda_1 \xi_t + \lambda_2 \xi'_t$ where $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{Z}$ be such that $\epsilon/3 \in O_{K_t} - B$, suppose that $t \not\equiv 4 \pmod 9$. We will show that this leads to a contradiction, which in turn shows that the index of B in O_{K_t} cannot be 3 or 9. This shows that the index is 1 according to our proposition.

Observe that 3 cannot divide $\lambda_0, \lambda_1, \lambda_2$ simultaneously (since $\epsilon/3 \not\in B$) nor can it divide exactly 2 of those integers because $1/3$, $\xi_t/3$ and $\xi'_t/3$ are not integers (Proposition 2.2*(ii)*).

In the remaining cases 3 divides none of the integers λ_0, λ_1, λ_2, or only one of them. Write $\mathrm{D}_1 = p_1^{k_1} \dots p_s^{k_s}$ and $\mathrm{D}_2 = p_1^{2k_1+j_1} \dots p_s^{2k_s+j_s}$. Since $j_0 = 0$ ($t \not\equiv 4$

(mod 9)) it follows that $\xi_t = (\alpha_t/\mathrm{D}_1)$ and $\xi'_t = (\alpha^2/3^{n_0}\mathrm{D}_2)$. Note that D_1 divides D_2 and 3 does not divide D_2. Now

$$\frac{\epsilon}{3}\mathrm{D}_2 = \frac{\lambda_0 \mathrm{D}_2}{3} + \frac{\lambda_1 \mathrm{D}_2}{3\mathrm{D}_1}\alpha_t + \frac{\lambda_2}{3^{n_0+1}}\alpha_t^2. \tag{7}$$

First consider the case where $n_0 = 0$ ($t \not\equiv 1 \pmod 3$). Reading $\frac{\epsilon}{3}\mathrm{D}_2$ modulo $\mathbb{Z}[\alpha_t]$ we conclude that one of the following elements: $\pm\frac{1}{3} \pm \frac{\alpha_t}{3}$, $\pm\frac{1}{3} \pm \frac{\alpha_t^2}{3}$, $\pm\frac{\alpha_t}{3} \pm \frac{\alpha_t^2}{3}$, $\pm\frac{1}{3} \pm \frac{\alpha_t}{3} \pm \frac{\alpha_t^2}{3}$ belongs to O_{K_t}. This list of elements could be restricted further to $(\frac{1}{3} \pm \frac{\alpha_t}{3})$, $(\frac{1}{3} \pm \frac{\alpha_t^2}{3})$, $(\frac{\alpha_t}{3} \pm \frac{\alpha_t^2}{3})$, $(\frac{1}{3} \pm \frac{\alpha_t}{3} \pm \frac{\alpha_t^2}{3})$. We use the norms of these elements to prove that none of the above is an integer.

The following table was produced using Maple:

(integer × 3)	*norm*	*norm* ≡ *0* (mod *27*)
$(1+\alpha_t)$	$2t^3 - 1$	no solution
$(1-\alpha_t)$	$-2t^3 - 6t^2 - 6t - 3$	no solution
$(1-\alpha_t^2)$	$(1-2t^3)(2t^3+6t^2+6t+3)$	no solution
$(1+\alpha_t^2)$	$17+30t+48t^2+40t^3+30t^4+12t^5+4t^6$	no solution
$(\alpha_t+\alpha_t^2)$	$(2t^3+3t^2+3t+1)(2t^3-1)$	$t \equiv 4, 13, 22 \pmod{27}$
$(\alpha_t-\alpha_t^2)$	$-(2t^3+3t^3+3t+1)(2t^3+6t^2+6t+3)$	$t \equiv 4, 13, 22 \pmod{27}$
$(1+\alpha_t+\alpha_t^2)$	$15t^4+12t^3+18t^2+9t+9+4t^6+6t^5$	no solution
$(1+\alpha_t-\alpha_t^2)$	$-27t^4-20t^3-6t^2+3t+1-18t^5-4t^6$	no solution
$(1-\alpha_t-\alpha_t^2)$	$3t^4+12t^3+12t^2+3t-1-4t^6-6t^5$	no solution
$(1-\alpha_t+\alpha_t^2)$	$45t^4+68t^3+72t^2+45t+19+18t^5+4t^6$	no solution

The only possible solution namely $t \equiv 4, 13, 22 \pmod{27}$ leads to $t \equiv 4 \pmod 9$, a case that we excluded at the start. Note that the above proves that this list of numbers are not integers when $n_0 = 1$ also.

Next consider the case when $n_0 = 1$. If equation (7) is reduced modulo $\mathbb{Z}[\alpha_t]$ it follows that one of the elements: $\pm\frac{\alpha}{3} + k\frac{\alpha^2}{9}$, $\pm\frac{1}{3} + k\frac{\alpha^2}{9}$, $\pm\frac{1}{3} \pm \frac{\alpha}{3}$, $\pm\frac{1}{3} \pm \frac{\alpha}{3} + k\frac{\alpha^2}{9}$, where $1 \leq k \leq 8$, should be an integer. Since $\alpha_t^2/3$ is now an integer, k can be restricted to 1,2 (read the above list mod $\mathbb{Z}[\alpha_t, \alpha_t^2/3]$). Finally use the norms of these numbers to prove that they are not integers. ∎

3. Group Of Units Of O_{K_t}

We compute the roots of the polynomial f_t.

Proposition 3.1. *If α_t is one root of f_t and*

$$\begin{cases} \beta_t = -\alpha_t^2 + t\alpha_t + 2\phi_t \\ \gamma_t = \alpha_t^2 - \frac{(\phi_t'+1)}{2}\alpha_t - 2\phi_t \end{cases}$$

then β_t and γ_t are the other two roots of f_t.

Proof: From $\beta_t + \gamma_t = -\alpha_t$ and $\beta_t\gamma_t = (\phi_t\phi_t'/\alpha_t) = \alpha_t^2 - 3\phi_t$, we see that

$$(\beta_t - \gamma_t)^2 = -3\alpha_t^2 + 12\phi_t. \tag{1}$$

On the other hand, $\mathrm{disc}(f_t) = (9\phi_t)^2$, thus

$$(\alpha_t - \beta_t)(\alpha_t - \gamma_t)(\beta_t - \gamma_t) = \pm 9\phi_t. \tag{2}$$

But $(\alpha_t - \beta_t)(\alpha_t - \gamma_t) = \alpha_t^2 - (\beta_t + \gamma_t)\alpha_t + \gamma_t\beta_t$, and so

$$(\alpha_t - \beta_t)(\alpha_t - \gamma_t) = 3\alpha_t^2 - 3\phi_t. \tag{3}$$

Equations (1), (2) and (3) imply that $(3\alpha_t^2 - 3\phi_t)(-3\alpha_t^2 + 12\phi_t) = \pm 9\phi_t(\beta_t - \gamma_t)$. Using the fact that $\alpha_t^3 = 3\phi_t\alpha_t + \phi_t\phi_t'$, the last equation gives $9\phi_t(\beta_t - \gamma_t) = \pm(18\phi_t\alpha^2 - 9\phi_t'\phi_t\alpha - 36\phi_t^2)$. The following system of equations

$$\begin{cases} \beta_t - \gamma_t = \pm(2\alpha_t^2 - \phi_t'\alpha_t - 4\phi_t) \\ \beta_t + \gamma_t = -\alpha_t \end{cases}$$

results in the expressions for β_t and γ_t. (Choosing either the plus or minus sign on the right side of the first equation interchanges the solutions for β_t and γ_t.) In what follows α_t refers to the root of largest absolute value (no two roots have the same absolute value as K_t is real) and β_t and γ_t are fixed by the statement of the proposition. ∎

Now we turn to the problem of finding the group $U_t = O^*_{K_t}$ of units. We know that K_t is a real field. Therefore all of its conjugates are real, and the group of roots of unity of K is $W(K) = \{-1, 1\}$. Hence the structure of U_t is as such:

$$U_t = \{-1, 1\} \times \mathbb{Z}^2.$$

Proposition 3.2. *$u_t = \alpha + t$ and $v_t = \alpha + t + 1$ are two independent units of K_t for all $t \in \mathbb{Z}$.*

Proof: Since $f_t(-t) = -1$, it follows that norm $(\alpha_t + t) = (\alpha_t + t)(\beta_t + t)(\gamma_t + t) = -(-t - \alpha_t)(-t - \beta_t)(-t - \gamma_t) = -f(-t) = 1$. This proves that $u_t = \alpha_t + t$ is a unit. Also, note that

$$v_t^{-1} = \alpha_t^2 - (1/2)(2t + 2)\alpha_t - 2t - t - 2 = \gamma_t + t = u_t^{(1)}$$

where $u_t^{(1)}$ is a conjugate of u_t. Hence, v_t is a unit.

It is sufficient to prove that u_t and $u_t^{(1)}$ are independent. Consider $V_t =< u_t, u_t^{(1)} >$ the subgroup of U_t generated by u_t and $u_t^{(1)}$. Let $\{1, \sigma, \sigma^2\}$ be the Galois group of $K_t/\mathbb{Q}$ where $\sigma(\alpha_t) = \gamma_t$. Let $\sigma(u_t) = u_t^{(1)}$ and $\sigma^2(u_t) = w_t$. From $w_t = 1/u_t u_t^{(1)}$, it follows that the Galois group acts on V_t. Suppose that u_t and $u_t^{(1)}$ are dependent. Then V_t has rank 1 (as a free abelian group). This makes it isomorphic to $\mathbb{Z}$, which admits only 2 isomorphisms (as an additive group), one is the identity and the other of order 2. But σ and σ^2 act nontrivially on u_t ($u_t \notin \mathbb{Q}$). This contradiction proves that u_t and v_t are independent. ∎

To explicitly determine the units of K_t we will consider the following quantity

$$S(\epsilon) = \frac{1}{2} \sum_{\substack{i,j=1\ldots 3 \\ i\neq j}} (\epsilon^{(i)} - \epsilon^{(j)})^2$$

where $\epsilon^{(i)}$, $i = 1 \ldots 3$, are the conjugates of ϵ, $\epsilon \in O_{K_t}$ ($\epsilon^{(1)} = \epsilon$, $\epsilon^{(2)} = \sigma(\epsilon)$ and $\epsilon^{(3)} = \sigma^2(\epsilon)$). If an integer θ is given by $\theta = m\alpha_t^2 + r\alpha_t + s$ (m,r,s being rational numbers) then one calculates using proposition 3.1 that

$$S(\theta) = 9\phi_t(r^2 + \phi_t' mr + \phi_t m^2).$$

(Compute, on Maple, recursively the powers of α_t using the equality $\alpha_t^3 = 3\phi\alpha_t + \phi_t\phi_t'$.)

Proposition 3.3. *Let t be such that ϕ_t is square free and not divisible by 3, and let $|t| > 4$ then*

$$U_t = < -1, u_t, v_t > .$$

Proof: In this case $O_{K_t} = \mathbb{Z}[\alpha_t]$. Hence $m, r, s \in \mathbb{Z}$ and so $m = 0$ and $r = 1$ would give the non-zero integers of lowest possible values for S (($S(\theta)/9\phi_t$) is here a positive rational integer since its discriminant (as a polynomial in r) is $-3m^2 \leq 0$). Explicitly

$$S(u_t = \alpha_t + t) = 9\phi_t \quad \text{and} \quad S(v_t = \alpha_t + t + 1) = 9\phi_t.$$

Also, v_t is not a power of u_t by the previous proposition.

[Godwin's theorem [1] is stated here for reference (with his own notations). Any integer λ of a (totally real) cubic field (K) is of the form $p + q\theta + rQ(\theta)$, where

θ is a defining integer (i.e. 1, θ, $Q(\theta)$ is an integral basis) $Q(\theta)$ is a quadratic in θ with rational coefficients and p, q, r are rational integers. $S(\lambda) = S(q\theta + rQ(\theta))$ is a positive definite form in q, r and we can find the values of q, r, which make $S(\lambda)$ least and then see if for any values of p, $p + q\theta + rQ(\theta)$ is a unit. If not, we repeat this with the next lowest possible value of $S(\lambda)$, and so on until a unit ϵ_1 is reached. We then continue until another unit ϵ_2, not a power of ϵ_1 is reached.

Theorem. *If every unit of the field has $S(\epsilon) > 34$ and if $S(\epsilon_2) \geq 122$ then either ϵ_1, ϵ_2 are a pair of fundamental units or there exists*

$$\textit{(i) a unit } \eta = \epsilon_1^{\frac{1}{2}}\epsilon_2^{\frac{1}{2}} \textit{ such that } S(\eta) < (81S(\epsilon_1)S(\epsilon_2)/2)^{\frac{1}{2}},$$

or

$$\textit{(ii) a unit } \eta = \epsilon_1^{\frac{2}{3}}\epsilon_2^{\frac{1}{3}} \textit{ such that } S(\eta) < (243S(\epsilon_1^2)S(\epsilon_2)/2)^{\frac{1}{3}}.$$

End of quote].

According to the above theorem, (and restricting $|t|$ to values greater than 4 so that $S(\theta) > 122$ for all units), the above information leads to one of the following possibilities

1. u_t and v_t form a fundamental system of units,
2. $\eta_1 = u_t^{\frac{1}{2}}v_t^{\frac{1}{2}}$ is a unit and $S(\eta_1) \leq (6)(9\phi_t)$,
3. $\eta_2 = u_t^{\frac{2}{3}}v_t^{\frac{1}{3}}$ is a unit such that $S(\eta_2) \leq (2)(9\phi_t)$.

The following table gives a list of all possible candidates for η_1 or η_2 (under the restriction that $S(\eta_1) \leq (6)(9\phi_t)$, and $S(\eta_2) \leq (2)(9\phi_t)$).

η_1		$S(\theta)/9\phi_t$
θ_1	$\pm[\alpha_t + c_1]$	1
θ_2	$\pm[\alpha_t^2 - t\alpha_t + c_2]$	1
θ_3	$\pm[\alpha_t^2 + (-1-t)\alpha_t + c_3]$	1
θ_4	$\pm[\alpha_t^2 + (-t-2)\alpha_t + c_4]$	3
θ_5	$\pm[\alpha_t^2 + (-t+1)\alpha_t + c_5]$	3
θ_6	$\pm[2\alpha_t^2 + (-2t-1)\alpha_t + c_6]$	3
θ_7	$\pm[2\alpha_t + c_7]$	4
θ_8	$\pm[2\alpha_t^2 - 2t\alpha_t + c_8]$	4
θ_9	$\pm[2\alpha_t^2 + (-2t-2)\alpha_t + c_9]$	4

where $c_1 \ldots c_9 \in \mathbb{Z}$

The squares of the above integers are

η_1	η_1^2
θ_1	$\alpha^2 + 2c_1\alpha + c_1^2$
θ_2	$(4t^2 + 2c_3 + 3t + 3)\alpha^2 - (2c_3 t + 3t + 4t^3 + 3t^2 - 1)\alpha + c_3^2 - 6t^2 - 2t - 4t^4 - 6t^3$
θ_3	$(4+5t+4t^2+2c_4)\alpha^2 - (5+4t^3+9t^2+9t+2c_4+2c_4t)\alpha + c_4^2 - 8t - 2 - 4t^4 - 10t^3 - 12t^2$
θ_4	$(2c_5 + 4t^2 + t + 4)\alpha^2 - (2c_5 t - 2c_5 + 4t^3 - 7 - 3t^2 - 3t)\alpha + c_5^2 - 4t^4 - 2t^3 + 4t + 2$
θ_5	$(4t^2+7t+7+2c_6)\alpha^2 - (11+4t^3+15t^2+15t+2c_6t+4c_6)\alpha + c_6^2 - 14t - 4 - 4t^4 - 14t^3 - 18t^2$
θ_6	$(16t^2+16t+13+4c_7)\alpha^2 - 2(2t+1)(4t^2+4t+4+c_7)\alpha + c_7^2 - 20t - 4 - 16t^4 - 32t^3 - 36t^2$
θ_7	$4\alpha^2 + 4c_8\alpha + c_8^2$
θ_8	$4(4t^2 + c_{10} + 3t + 3)\alpha^2 - 4(c_{10}t + 3t + 4t^3 + 3t^2 - 1)\alpha + c_{10}^2 - 24t^2 - 8t - 16t^4 - 24t^3$
θ_9	$4(4t^2+5t+4+c_{11})\alpha^2 - 4(5+4t^3+9t^2+9t+c_{11}t+c_{11})\alpha + c_{11}^2 - 32t - 8 - 16t^4 - 40t^3 - 48t^2$

The cubes of the candidates for the unit η_2 are

η_2	η_2^3
θ_1	$\pm[3c_1\alpha^2 + 3(c_1^2 + t^2 + t + 1)\alpha + c_1^3 + 1 + 2t^3 + 3t^2 + 3t]$
θ_2	$\pm[3(9t^2 + 5t + 6t^3 + 3c_3 + 3 + 3c_3t + 4c_3t^2 + c_3^2 + 4t^4)\alpha^2 - 3(6t^4 + 3t^2 + t - 2 + 4t^5 + 9t^3 + 3c_3t + c_3^2t + 4c_3t^3 - c_3 + 3c_3t^2)\alpha + 1 - 36t^5 - 6c_3t - 18c_3t^3 - 18c_3t^2 - 12c_3t^4 - 54t^4 - 42t^3 - 3t - 21t^2 - 16t^6 + c_3^3]$
θ_3	$\pm[3(5 + 4c_4t^2 + 10t^3 + 11t + c_4^2 + 4t^4 + 15t^2 + 4c_4 + 5c_4t)\alpha^2 - 3(4c_4t^3 + 25t^3 + 7 + 28t^2 + 9c_4t + c_4^2t + 4t^5 + c_4^2 + 5c_4 + 18t + 9c_4t^2 + 14t^4)\alpha - 9 - 30c_4t^3 - 36c_4t^2 - 60t^5 + c_4^3 - 16t^6 - 114t^4 - 99t^2 - 6c_4 - 134t^3 - 45t - 24c_4t - 12c_4t^4]$

Thus the candidates for η_1 are $\theta_1 \dots \theta_9$, and those for η_2 are $\theta_1 \dots \theta_3$. Using $\eta_1^2 = u_tv_t$ eliminates possibility 1. ($S(\theta_i^2)$ is quadratic in c_i for $1 \le i \le 9$. We put $S(\theta_i^2) = S(u_tv_t)$, and solve for c_i in terms of t. Then we check whether any of those 2 values for c_i results in $\theta_i^2 = u_tv_t$ for some t). Now use $\eta_2^3 = u_t^2v_t$ to eliminate possibility 2. (Reduce the terms not involving α_t on both sides of the equation modulo 3. The resulting equation has no solution for any t. As c_i is a rational integer for $1 \le i \le 3$, it follows that $\theta_i^3 \ne u_tv_t$ for $1 \le i \le 3$ and for all t).

Thus u_t and v_t form a system of fundamental units whenever $t \in \mathbb{Z}$ is such that $|t| > 4$ and $O_{K_t} = \mathbb{Z}[\alpha_t]$. ∎

The next proposition gives a bound on the index of $< u_t, v_t >$ in the group of units U_t of K_t. It is convenient to write $\phi_t = A_tB_tC_t$ where C_t is a perfect cube, B_t is a square and cube free and A_t square free. Also $\gcd(A, B) = 1$. Let μ_0 be 0 if $t \not\equiv 1 \pmod 3$, 2 if $t \equiv 4 \pmod 3$ and 1 otherwise.

Proposition 3.4.

$$[U_t :< u_t, v_t >] \le 3^{6\mu_0} C_t^4 B_t^3$$

Proof: Let $\rho_t = 3^{\nu_0} C_t^{2/3} B_t^{1/2}$. Then by proposition 2.2 $\rho_t O_{K_t} \subset \mathbb{Z}[\alpha_t]$. The canonical homomorphism from O_{K_t} to $O_{K_t}/\rho_t O_{K_t}$ sends U_t into the group of units of $O_{K_t}/\rho_t O_{K_t}$. Also, the ring $O_{K_t}/\rho_t O_{K_t}$ has ρ_t^3 elements. Hence for a unit $\epsilon \in O_{K_t}$ there exists a positive rational integer $n < \rho_t^3$ such that $\epsilon^n = 1 + \rho_t z$ where $z \in O_{K_t}$. Thus ϵ^n belongs to $\mathbb{Z}[\alpha]$. This proves that the index $[U_t :< u_t, v_t >] \leq \rho_t^6$. ∎

4. Class Number Of K_t

Let $\phi_t = p_0^{n_0} p_1^{n_1} \dots p_s^{n_s}$, where $n_0, \dots, n_s \in \mathbb{Z}$ and $s \geq 0$. We know from proposition 2.3 that the discriminant of K_t is $(3^\lambda q_1 \dots q_k)^2$ where $k \leq s$ is a non-negative integer, and q_i for $1 \leq i \leq k$ is such that $q_i = p_j$ for some j and n_j is not a multiple of 3. Also $\lambda = 0$ if $t \equiv 4 \pmod 9$ and $\lambda = 2$ otherwise. Now let $m_t = q_1 \dots q_k$ if $t \equiv 4 \pmod 9$ and $m_t = 9q_1 \dots q_k$ otherwise. Thus m_t is the conductor of K_t, and is equal to the square root of the discriminant of K_t. By the Kronecker-Weber Theorem, K_t is contained in the cyclotomic field $\mathbb{Q}[\zeta]$ where $\zeta = e^{\frac{2\pi i}{m_t}}$. Moreover the primes of $\mathbb{Z}$ which are ramified in K_t are exactly those which divide m_t.

By the class number formula we have the following (cf [3]) :

$$h_t = \frac{1}{4} \frac{\sqrt{\operatorname{disc}(K_t)}}{R_t} \prod_{p|m} (1 - \frac{1}{p})(1 - \frac{1}{p^{f_p}})^{-r_p} \prod_{\substack{\chi \in \widehat{G} \\ \chi \neq 1}} L(1, \chi).$$

where p is a rational prime, f_p the inertial degree of any prime ideal P of K_t over p, and r_p is the number of primes of K_t over p, χ is a character mod m , and $\widehat{G}$ is the group of characters of the galois group G of $K_t/\mathbb{Q}$, considered as a subgroup of $\widehat{\mathbb{Z}_m^*}$ the group of characters of $\mathbb{Z}_m^*$ (identified with the galois group of $\mathbb{Q}[\zeta]/\mathbb{Q}$).

We consider bounds on the class number h_t of K_t. We first obtain some bounds on the regulator R_t. To do this we need to find bounds on the roots of f_t.

Proposition 4.1. *(i)* $K_t = K_{-t-1}$ *for all* t.

(ii) If t is nonnegative then the roots of f_t are such that

$$-t-2 < \beta_t < -t-1 < \gamma_t < -t < 2t+1 < \alpha_t < 2t+2$$

and

$$|\beta_t + t + 1| < \frac{1}{10} \text{ for } t \geq 3$$

Proof: We first note that $f_{-t-1}(x) = x^3 - 3\phi x + \phi\phi'$. Hence $f_{-t-1}(-x) = f_t(x)$. It follows that the roots of the polynomial f_t are the opposite of the roots of f_{-t-1}, and that $K_t = K_{-t-1}$. We can therefore restrict t to positive integers.

Also,

$$f_t(-t-2) = -6t - 3 < 0$$

$$f_t(-t-1.1) = -.33t + .969 < 0 \text{ if } t \geq 3$$

$$f_t(-t-1) = 1 > 0$$

$$f_t(-t) = -1 < 0$$

$$f_t(2t+1) = -6t - 3 < 0$$

$$f_t(2t+2) = 9t^2 + 9t + 1 > 0$$

Finally, since α_t is the root of f_t of largest absolute value the inequalities $2t + 1 < \alpha_t < 2t + 2$ follow. It remains to show that $\gamma_t > \beta_t$. By Proposition 3.1

$$\gamma_t - \beta_t = 2\alpha_t^2 - \phi_t'\alpha_t - 4\phi_t.$$

Thus it is sufficient to prove that α_t is greater than the positive root of the second degree polynomial $2x^2 - \phi_t' x - 4\phi_t$, i.e. that $\alpha_t > z = \frac{\phi_t' + \sqrt{36t^2+36t+33}}{4}$. This is true because $z > \phi_t'$ and $f(z) = \frac{-3\phi_t' - 3\sqrt{36t^2+36t+33}}{16} < 0$. ∎

Proposition 4.2. *If ϕ_t is not divisible by 3 and is square free, and if $t > 3$ then*

$$\log(3t+3) < R_t < [\log(6t+6)]^2$$

Proof: In this case $\phi_t = p_1 \dots p_s$ and discriminant of the field is just $(9\phi_t)^2$. The regulator R_t of K_t is the determinant of

$$\begin{pmatrix} \log|\alpha_t + t| & \log|\beta_t + t| \\ \log|\alpha_t + t + 1| & \log|\beta_t + t + 1| \end{pmatrix}.$$

Note that with α_t, β_t and γ_t referring to the specific roots described above, we have

$$\log|\alpha_t + t + 1| > \log|\alpha_t + t|$$

$$\log|\beta_t + t| > \log|\beta_t + t + 1|$$

since $\beta_t + t$ and $\beta_t + t + 1$ are both negative.

Hence,

$$\begin{aligned}
R_t =& \big|\log|\alpha_t + t + 1|\log|\beta_t + t| - \log|\beta_t + t + 1|\log|\alpha_t + t|\big| \\
=&(\log|\alpha_t + t + 1|)(\log|\beta_t + t|) - (\log|\beta_t + t + 1|)(\log|\alpha_t + t|) \\
>&(\log|\alpha_t + t|)(\log|\beta_t + t|) - (\log|\alpha + t|)(\log|\beta_t + t + 1|) \\
=& \log|\alpha_t + t|(\log|\beta_t + t| - \log|\beta_t + t + 1|) \\
=& \log|\alpha_t + t|\log\frac{|\beta_t + t|}{|\beta_t + t + 1|} \\
=& \log|\alpha_t + t|\log\left|1 - \frac{1}{\beta_t + t + 1}\right| \\
=& \log|\alpha_t + t|\log(1 + \frac{1}{|\beta + t + 1|}) \\
>& \log|\alpha_t + t| \quad \text{if } t \geq 3 \text{ (since } |\beta_t + t + 1| < \tfrac{1}{10}) \\
>& \log(3t + 1) \\
>& \log t \quad (t \geq 3)
\end{aligned}$$

Also,

$$R_t = \big|(\log|\alpha_t + t + 1|)(\log|\beta_t + t|) - (\log|\beta_t + t + 1|)(\log|\alpha_t + t|)\big|$$

$$\leq \log(\alpha_t + t + 1)\left(\log|\beta_t + t| + \big|\log|\beta_t + t + 1|\big|\right) \tag{4.1}$$

$$\leq \log(3t + 3)\left[\log 2 + \big|\log|\beta_t + t + 1|\big|\right.$$

But $\beta_t + t + 1$ is a unit, so

$$
\begin{aligned}
1 > |\beta_t + t + 1| &= \frac{1}{|\alpha_t + t + 1||\gamma_t + t + 1|} \\
&\geq \frac{1}{(3t+3)} \quad (|\gamma_t + t + 1| < 1)
\end{aligned}
\tag{4.2}
$$

Equations (4.1) and (4.2) give that $R_t < \log(3t+3)\log(6t+6)$ Therefore

$$\log(3t+3) < R_t < [\log(6t+6)]^2$$

Now we obtain some bounds on the class number of K_t. We know that ([4])

$$|L(1,\chi)| \leq 3\log m_t.$$

Also, [2]

$$\prod_{\substack{\chi \in \widehat{G} \\ \chi \neq 1}} L(1,\chi) \geq \frac{E}{\log(\text{ disc } (K_t))}$$

where E is a positive constant independent of t (K_t has no quadratic subfield and hence κ_t the residue of its ζ-function $\zeta_{K_t}(s)$ at $s = 1$ is such that $\kappa^{-1} = O(3! \log(\text{ disc } (K_t)))$.

Proposition 4.3. *If $t > 3$, and ϕ_t is square free and not divisible by 3 then*

$$\frac{9\phi_t}{4} \frac{E}{\log(9\phi_t)^2} \frac{1}{[\log(6t+6)]^2} < h_t < \frac{81\phi_t}{4} \frac{(\log 9\phi_t)^2}{\log(3t+3)}$$

In particular under those circumstances, h_t tends to infinity when t gets larger.

Proof: In this case, the discriminant of K_t is $(9\phi_t)^2$. Looking at the class-number formula, we note that for all $p|m$, we have $f_p = 1$ and $r_p = 1$ since p ramifies in the extension K_t/Q. Also, G the Galois group of K_t/Q admits two non-trivial characters χ and $\overline{\chi}$, those characters being complex conjugates of each other. It follows that the corresponding L functions are complex conjugates, and that

$$L(1,\chi)L(1,\overline{\chi}) = |L(1,\chi)|^2$$

where χ is any nontrivial character of G.

Hence

$$h_t = \frac{9\phi_t}{4}\frac{|L(1,\chi)|^2}{R_t}$$

and

$$\frac{9\phi_t}{4}\frac{E}{\log(\text{ disc }(K_t))}\frac{1}{[\log(6t+6)]^2} < h_t < \frac{81\phi_t}{4}\frac{(\log m_t)^2}{\log(3t+3)}$$

Putting $m_t = 9\phi_t$ gives the desired result. ∎

Using Propositions 3.4 and 4.2 one gets the following Proposition.

Proposition 4.4. *Let $t > 3$. Then*

$$\frac{3^{\mu_1}A_tB_t^{\frac{1}{3}}}{4}\frac{E}{\log(3^{\mu_1}A_tB_t^{\frac{1}{3}})^2}\frac{1}{[\log(6t+6)]^2} < h_t < 3^{6\mu_0}C_t^4B_t^3\frac{3^{\mu_1}A_tB_t^{\frac{1}{3}}}{4}\frac{(\log 3^{\mu_1}A_tB_t^{\frac{1}{3}})^2}{\log(3t+3)}$$

where $\mu_1 = 0$ if 3 does not divide A_t and 1 otherwise.

Proof: The regulator $m_t = 3^{\mu_1}A_tB_t^{\frac{1}{3}} = \sqrt{\text{disc }(K_t)}$. Also if R_t' is the regulator computed from u_t and v_t then $R_t = (R_t'/[U_t :< u_t, v_t >])$ and so by Propositions 4.2 and 3.4 the inequalities

$$\frac{\log(3t+3)}{3^{6\mu_0}C_t^4B_t^3} < R_t < [log(6t+6)]^2$$

follow. The class-number formula completes the proof. ∎

If $t \equiv 4 \pmod 9$ and $\phi_t/3$ is a prime then the conductor of K_t is that prime, and the computation of the L function is easy (the princinpal character is primitive and the cubes modulo m_t are exactly its kernel). Also using proposition 3.4 and 4.2 we could get 'close' bounds on R_t. In fact in this case, $[U_t :< u_t, v_t >] \leq 27^2$. Call R'_t the regulator computed from u_t and v_t. Then

$$\frac{R_t}{27^2} \leq R'_t \leq R_t.$$

We compute using maple the following special cases. The class number h_t is such that $h'_t \leq h_t \leq 27^2 h'_t$.

t	ϕ_t	h'_t
4	(3) (7)	(.0769230788)
13	(3) (61)	1 (1.000000036)
31	(3) (331)	1 (.9999999541)
40	(3) (547)	4 (4.000000137)
76	(3) (1951)	4 (4.000000259)
85	(3) (2437)	7 (6.999999461)
103	(3) (3571)	7 (6.999985858)
112	(3) (4219)	28 (27.99999542)

References

[1] *H.J. Godwin,* The determination of units in totally real cubic fields, Proc. Cambridge Philos. Soc **56** (1960), 318-21.

[2] *H. Heilbronn,* On real zeros of Dedekind ζ-functions, Can. J. Math. **25** (1973), 870-873.

[3] *D.A. Marcus,* Number Fields. Springer-Verlag, 1977.

[4] *W. Narkiewicz,* Elementary and Analytic Theory of Algebraic Numbers. Springer-Verlag, 1990.

[5] *D. Shanks,* The simplest cubic fields, Math comp. **128** (1974), 1137-1152.

[6] *G.W. Smith,* Generic cyclic polynomials and some applications. Ph.D. thesis. U.C. Berkeley

[7] *L. Washington,* Class numbers of the simplest cubic fields, Math. Comp. **177** (1987), 371-384.