A Network Approach to Layered Anisotropic Waveguides

Christos Baltassis

A Thesis

in

The Department

of

Electrical Engineering

Presented in Partial Fulfillment of the Requirements
for the degree of Master of Engineering at
Concordia University
Montréal, Québec, Canada

April 1984

ABSTRACT

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The transmission characteristics of layered anisotropic waveguides are investigated by means of matrix analysis and an equivalent network model. The anisotropy considered may be either inherent crystalline anisotropy, as in the case of uniaxial, biaxial and optically active crystals, or anisotropy induced by external fields, as in the case of the electro-optic effect and Faraday rotation. TE-TM mode coupling resulting from off-diagonal terms in the system matrix indicates that a four-port network is required to represent a single mode in the anisotropic waveguide. Criteria for losslessness, reciprocity, antireciprocity, transversal/bilateral symmetry and the newly developed condition of semireciprocity are discussed. Setting out with the normalized wave equation, expressions for the electromagnetic field distribution are obtained and the concept of the field and wave transfer matrix of a layer is developed. Reflection and transmission at an interface separating dissimilar media is also examined. By way of examples, three layered symmetric waveguides supporting symmetric and antisymmetric field distributions are examined; their characteristic equation, cutoff conditions, power flow and equivalent network is examined. Numerical examples are considered using the FORTRAN V program LAYER, which solves the dispersion equation for various values of the normalized frequency.

This program also computes the dispersion characteristic of the waveguide. Coupling between modes, that does not occur in three layer isotropic waveguides, is observed , pointing toward new applications and indicating regions free from such couplings.

ACKNOWLEDGEMENTS

The author wishes to express his deep gratitude to Dr. Otto Schwelb for his assistance and guidance during the entire preparation of this thesis. The support of National Sciences and Engineering Research Council of Canada is gratefully acknowledged. The author also wishes to express his thankfulness to Miss H. Monkiewicz for typing this thesis and Mr. M. Sifakis for drafting the figures.

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LIST OF IMPORTANT ABBREVIATIONS AND SYMBOLS

Greek

α_i	Transverse effective attenuation index (along x)
β	Longitudinal effective guide index
γ	Transverse effective guide index (along y)
Γ	Electric field reflection coefficient matrix (of $\bar{g}$)
Δ_{ij}	ij-th cofactor of the permittivity tensor
$\bar{\bar{\varepsilon}}(\bar{\bar{\varepsilon}}_p)$	Permittivity tensor in the device's (principal) system
η_o	Free space wave impedance
Θ	Electric field transmission coefficient matrix (of $\bar{g}$)
κ_i	Transverse effective guide index (along x)
κ_i'	Transverse film's effective guide index of a layered waveguide
Λ_g	Propagation factor matrix
Λ_R	Eigenvalue matrix
$\Lambda_f(\Lambda_b)$	Forward (backward) 2×2 partitions of Λ_g
μ_o	Free space permeability constant
Ξ_i	Dirac matrices
ρ	Wave-reflection matrix (of $\bar{a}$)
σ_i	Pauli matrices
Σ_i	Dirac matrices
τ	Wave-transmission matrix (of $\bar{a}$)
ω	Angular frequency
Ω	Impedance transformation matrix

<u>Latin</u>

$\overline{a}$	Wave vector
A	Rotation matrix
B^{-1}	Inverse of matrix B
B^T	Transpose of B
B^+	Hermitean of B
$\tilde{B}$	Tilde form of B
$\overline{c}$	Electromagnetic field amplitude-vector
C	Transfer matrix of $\overline{c}$
$\overline{D}$	Electric displacement vector
D_i	Denominators of the eigenvector forms
Det B	Determinant of B
$\overline{E}$	Electric field
$F_f(F_b)$	Forward (backward) matrix arising from partition of $\tilde{G}$
$\overline{g}$	Electromagnetic field vector
$g_i(\hat{g}_i)$	Longitudinal parameter of the <u>film</u> (cladding)
G	Impedance transfer matrix
$h_i(\hat{h}_i)$	Polar parameter of the film (cladding)
$\overline{H}$	Magnetic field
H_i	Projector matrices of $F_{f,b}$
$J_f(J_b)$	Forward (backward) matrix arising from partition of $\tilde{G}$
k_s	Propagation constant along the s direction
K_i	Projector matrices of $J_{f,b}$
ℓ	Normalized half-width of the film of a layered waveguide

M	Impedance transfer matrix of $\overline{a}$
n	Effective index of refraction
$\overline{p}$	Electric polarization vector
$P_f(P_b)$	Forward (backward) upper partitions of $\tilde{U}$
$\overline{q}$	Magetic polarization vector
$Q_f(Q_b)$	Forward (backward) lower partitions of $\tilde{U}$
r	Reflection coefficient matrix (of $\overline{c}$)
$R_a(R_g)$	Coupling matrix of $\overline{a}(\overline{g})$
R_{ij}	ij-th entry of R_g
S_x	Power flow density in a waveguide along x
t	Transmission coefficient matrix (of $\overline{c}$)
TrB	Trace of matrix B
U	Modal matrix of $\overline{g}$
$y_f(y_b)$	Forward (backward) admittance matrix
$z_f(z_b)$	Forward (backward) impedance matrix
Z_i	Normalized line impedances

CHAPTER 1

INTRODUCTION

Layered anisotropic waveguides are often employed in the design of microwave, millimeter wave and optoelectronic devices. The range of applications anticipated by anisotropic waveguides such as mode conversion, polarization control, nonlinear signal processing, laser beam modulation and optical filtering to name only a few, has given a great impulse to their development in integrated optics and related fields.

Wave propagation in anisotropic structures can be analyzed using either the ray-optics approach or the wave-optics approach. An investigation of plane wave propagation in a birefringent layered structure using both approaches was carried out by Wang and coworkers [30], [31]. They showed that the propagation is characterized by substantial TE-TM mode conversion and suggested the polarization control, frequency modulation of distributed feedback lasers and the design of waveguide modulators. At the same time, Schesser and Eichmann [32] examined propagation of plane waves in a three-layer structure consisting of biaxial crystals. Assuming oblique incidence of arbitrarily polarized waves, they evaluated the polarization vectors of the partial electric and magnetic fields using Snell's law and Fresnel's equation. Their results however, were not in form adaptable to treat propagation in multilayered media. Considering the off-diagonal entries of the permittivity tensor as a perturbation of the principal system, Yamamoto, et al.[6] used the Rayleigh-Ritz variational method to determine the normal modes in an anisotropic medium. Their method however yielded approximate expressions for the normal (propagating) modes. Practical considerations lead to the analysis of three simple uniaxial orientations, namely, the polar, the longitudinal and the equatorial. In the polar configuration the optic axis lies in the plane of the interface whereas in the longitudinal, in the plane normal to the axial direction of propagation. In the third configuration, the equatorial, the optic axis lies in the sagittal plane of the crystal [25], [26]. However, the results of the field analysis derived even for these simple configurations were in a form too cumbersome to be used for wave propagation in multilayered birefringent media. It was Berreman in 1972 who used a matrix method in a modal formulation [33]. Berreman investigated wave propagation in cholesteric liquid crystals, by reformulating Maxwell's equations into a set of coupled differential equations involving only those field components that lie in the plane of the interface. These field components must be continuous at a dielectric interface and therefore the method becomes eminently suited to multilayer analysis using modal and impedance transfer matrices. Later on, in 1978, Kishioka and Rokushima [34] presented a unified treatment of the radiation and guided modes of an anisotropic slab waveguide with arbitrary permittivity and permeability tensors. A simple application, namely the expression of the characteristic equations in the longitudinal, polar and equatorial configurations verified the results of many previous workers. At the same time, Yeh [35] generalized the wave approach and expressed the electromagnetic field distribution in each layer in a closed form. Although this method is computationally advantageous it does not proceed beyond the introduction of the propagation factor and field transfer matrices.

This thesis develops a field analysis in a lossless birefringent medium characterized by an arbitrary (complex) permittivity tensor. It uses 4x4 matrix formalism, which lends itself to the study of wave propagation in stratified media. The concept of matrix wave-impedance will be introduced and used to express the impedance transfer and the reflection/transmission coefficient matrices. The generalization of the reflection coefficient matrix which quantifies the reflection from a stack of layers permits formulation of the transverse resonance condition (TRC) which, in turn, provides a means to obtain the characteristic equation. Propagation characteristics, lateral confinement and power flow in three-layer symmetric waveguides in the polar and longitudinal configurations, will also be analyzed.

In the second Chapter, the classical method of treating wave propagation in an anisotropic medium [3] is treated. It is found that the propagating eigenmodes are two linearly polarized eigenmodes, whose effective indices of refraction are the principal solutions of the quartic Fresnel's equation. The index ellipsoid is employed to relate our arbitrary direction of propagation with the corresponding refractive indices. The linear electro-optic effect, optical activity and Faraday rotation phenomena are outlined and are shown to be represented by off-diagonal terms in the principal permittivity tensor. Finally, the TE-TM mode coupling is discussed, arising from off-diagonal dielectric tensor components and the concept of coupling/modal matrix, normalized wave equation and polarization vector is introduced.

The third Chapter begins with a review of the propagation characteristics of a symmetric isotropic slab. The results are compared with those of the corresponding anisotropic slab waveguide and the similarities are noted. The wave impedance concept is introduced in birefringent media; the impedance and admittance matrices are expressed in terms of the entries of the modal matrix. The impedance transfer matrix, defined as the transfer matrix of the field vector from one interface to the next, is expressed in terms of the partitioned form of the modal matrix and the impedance/admittance matrices. Wave vectors are defined as linear combinations of the field components and a four-port network is employed to represent an anisotropic layer in a composite waveguide structure. Criteria for losslessness, bilateral/transversal symmetry, reciprocity, antireciprocity and the newly developed concept of semi-reciprocity are derived in terms of the coupling and impedance transfer matrices. The Chapter concludes with a treatment of the reflection and transmission characteristics of an interface between two dissimilar anisotropic layers. Three different vector representations are used to formulate the transmission and reflection matrix. The TRC generalized for any number of layers is also derived in terms of these three vectors.

In the fourth Chapter, the previous results are applied to analyze media in polar, longitudinal or equatorial configuration. The field distribution, the characteristic equation and other pertinent data is presented for three-layer symmetric waveguides fabricated with dielectrics in the polar and/or longitudinal configuration. Bounds for the effective transverse guide index are determined. These characterize the mode cutoff frequency and provide the asymptotic phase and group velocity in the waveguide at the very high frequencies. Power flow expressions are found and are compared with the corresponding ones of the isotropic case.

Finally in the Appendices, the power expressions of all the guiding structures examined are listed along with two computer programs. The first program deals with propagation characteristics of a bulk birefringent medium, whereas the second with a symmetric three-layer waveguide. The latter program solves numerically the characteristic equation and displays the result both in an appropriate ω-β diagram and in a transverse vs. axial wavenumber plot.

CHAPTER 2

ELECTROMAGNETIC WAVE PROPAGATION IN ANISOTPOPIC MEDIA

2.1 Fundamental Concepts

The macroscopic response of a crystalline dieletric to an external electromagnetic wave propagating through it, is described by its permittivity, which relates the displacement $\overline{D}$ to the applied electric field $\overline{E}$. Anisotropic dielectrics are those characterized by tensor permittivity and scalar permeability. The constitutive relations are

$$\overline{D} = \varepsilon_o \overline{\overline{\varepsilon}}\, \overline{E} \qquad (2.1.1a)$$

$$\overline{B} = \mu_o \overline{H} \qquad (2.1.1b)$$

where $\overline{\overline{\varepsilon}}$ is the relative permittivity tensor. Due to the anisotropy expressed in (2.1.1a), the electric field $\overline{E}$ is not parallel to the displacement $\overline{D}$.

The condition of losslessness requires that $\overline{\overline{\varepsilon}}$ must be a Hermitean tensor [1]

$$\overline{\overline{\varepsilon}} = \overline{\overline{\varepsilon}}^{\dagger} \qquad (2.1.2)$$

that is, the diagonal entries must be real and the off-diagonal ones appearing in symmetric positions must be complex conjugate pairs. There always exists a coordinate transformation converting $\overline{\overline{\varepsilon}}$ to a diagonal $\overline{\overline{\varepsilon}}_p$. In this new coordinate system, called the principal system, $\overline{\overline{\varepsilon}}_p$ is of the form

$$\bar{\bar{\varepsilon}}_p = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \qquad (2.1.3)$$

The axes of this system, are the principal dielectric axes of the crystal. The crystal is isotropic if $\varepsilon_x = \varepsilon_y = \varepsilon_z$, uniaxial if two of the three parameters are equal, and biaxial if all three are different.

To find the field configuration for propagation along an arbitrary direction, one has to write Maxwell's equations

$$\nabla\times\bar{E} = -\mu_o\frac{\partial\bar{H}}{\partial t} \quad ; \quad \nabla\times\bar{H} = \varepsilon_o\bar{\bar{\varepsilon}}\frac{\partial\bar{E}}{\partial t} \qquad (2.1.4)$$

and assume monochromatic plane wave solutions of radian frequency ω, propagating in the crystal with phase factor

$$\exp[j\omega(t-\varepsilon_o\eta_o n\bar{s}\cdot\bar{r})] = \exp[j\omega(t - \frac{\mu_o}{\eta_o}n\bar{s}\cdot\bar{r})] \qquad (2.1.5)$$

where n is the effective index of refraction for an electromagnetic wave propagating in the crystal in the $\bar{s}$ direction, i.e., $\bar{V}_p = \frac{c}{n}\bar{s}$ is the phase velocity of the monochromatic wave in the $\bar{s}$ direction, η_o is the free space wave impedance and $\bar{s}$ the unit vector normal to the wavefront. It is then found,

$$\bar{D} = \varepsilon_o n^2[\bar{E} - \bar{s}(\bar{s}\cdot\bar{E})] \qquad (2.1.6a)$$

$$\bar{H} = \frac{1}{\eta_o} n\bar{s}\times\bar{E} \qquad (2.1.6b)$$

and

$$\bar{s}\times\bar{D} = \varepsilon_o\eta_o n\bar{H} \qquad (2.1.7)$$

The above equations show that $\bar{s}$, $\bar{D}$ and $\bar{H}$ ($\bar{B}$) are mutually orthogonal vectors and $\bar{D}$ is not parallel to $\bar{E}$. These results are shown in Fig. 2.1.

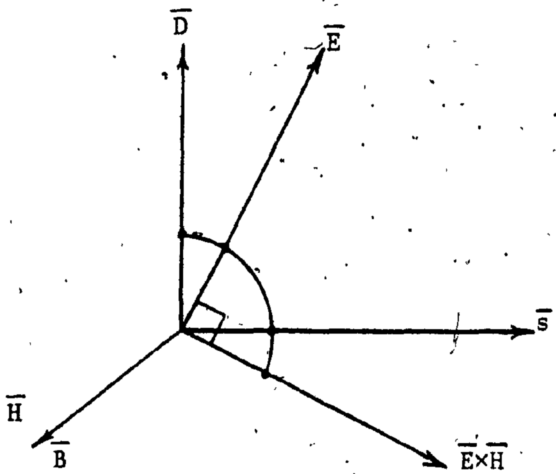

Fig. 2.1 The relative orientation of $\overline{E}$, $\overline{D}$, $\overline{H}$, $\overline{B}$, $\overline{E}\times\overline{H}$ and $\hat{s}$ in an anisotropic medium. Vectors $\overline{D}$, $\overline{E}$, $\overline{s}$ and $\overline{E}\times\overline{H}$ lie in the same plane.

To take into account the anisotropy of the crystal in the solution of Maxwell's equations, one has to substitute (2.1.1a) into (2.1.6). The resulting equation expressed in the principal system, can then be recast in the form [2]

$$\frac{s_x^2}{n^2 - \varepsilon_x} + \frac{s_y^2}{n^2 - \varepsilon_y} + \frac{s_z^2}{n^2 - \varepsilon_z} = \frac{1}{n^2} \qquad (2.1.8)$$

This is the Fesnel's equation of wave normals. It is a quadratic equation in n^2 and therefore the four solutions are $\pm n_1$ and $\pm n_2$, where the $\pm$ signs correspond to the signs of phase velocity (bidirectional wave propagation). Substitution of these values in the ratio of the electric field components, arising from the substitution of (2.1.1a) into (2.1.6a), yields

$$\frac{E_{i,m}}{E_{j,m}} = \frac{s_i[(n_m^2 - \varepsilon_j]}{s_j[(n_m^2 - \varepsilon_i]} \quad ; \; i,j = x,y,z \qquad (2.1.9)$$

where m corresponds to the independent solution ($m = 1, 2$). For real n, that is, for nonabsorbing, nonamplifying (nonactive) media, the above ratio is real and hence the two independent monochromatic plane waves that can propagate in an anisotropic crystal are linearly polarized. For propagation along z, for example, the two modes have their polarizations along x and y, where x,y and z are the principal axes.

Propagation along an arbitrary direction in the crystal, is a more complicated problem. In this case, the determination of the two effective refractive indices $n_{1,2}$ and the corresponding directions of the $\overline{D}$ field polarizations, is accomplished using the index ellipsoid (or optical indicatrix) [3]. This is an ellipsoid described by the equation

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1 \qquad (2.1.10)$$

where $n_\ell^2 = \varepsilon_\ell$ (ℓ = x,y,z) are the principal indices of refraction. The axes of the ellipsoid are parallel to the principal axes x,y,z and their respective lengths are $2\sqrt{\varepsilon_x}$, $2\sqrt{\varepsilon_y}$ and $2\sqrt{\varepsilon_z}$. For an arbitrary direction of propagation $\overline{s}$, the intersection between the index ellipsoid and a plane through the origin that is normal to $\overline{s}$, is generally an ellipse whose major and minor axes give the directions of polarization of $\overline{D}$ and their respective length is $2n_1$ and $2n_2$, where n_1 and n_2 are the two allowed solutions of (2.1.8). Since the axes of an ellipse are perpendicular to each other, the directions of polarization of the two vectors $\overline{D}$ corresponding to an arbitrary direction of propagation $\overline{s}$, are also perpendicular to each other.

In a uniaxial crystal, the index ellipsoid is cylindrically symmetric (spheroid). If z is the axis of symmetry, equ. (2.1.10) simplifies to

$$\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1 \qquad (2.1.11)$$

For $n_e < n_o$ the crystal is called negative; for $n_e > n_o$, positive. Since in a uniaxial crystal, the intersection ellipse normal to the direction of propagation $\bar{s}$ remains the same under rotation of $\bar{s}$ around the z axis, one can choose without loss of generality, the $\bar{s}$ vector to coincide with one of the principal planes, e.g., y-z plane. In this case, the major axis of the intersection ellipse lies in the principal y-z plane, normal to $\bar{s}$ and the minor axis is parallel to the x axis (see Fig. 2.2). Upon a change in the angle θ between the z axis and the direction of propagation $\bar{s}$, one of the directions of polarization of $\bar{D}$, namely the one parallel to the x axis, remains fixed and its refractive index is equal to the radius of the equatorial plane of the

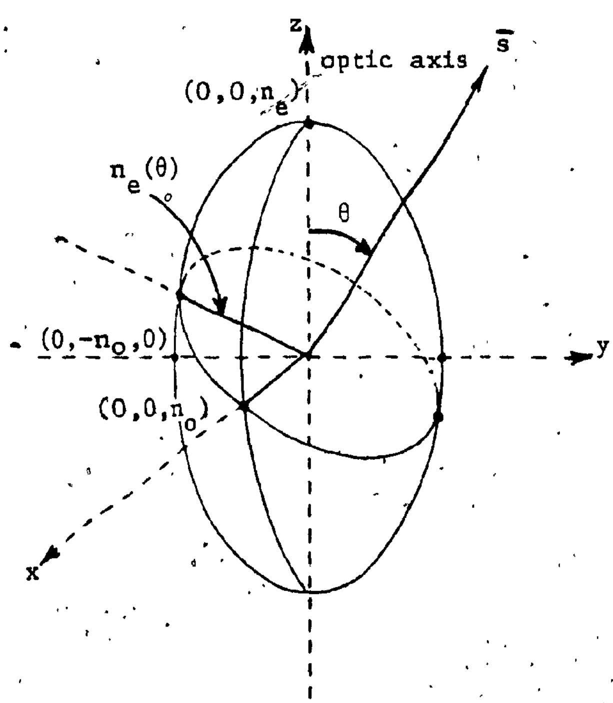

Fig. 2.2 Index ellipsoid of a positive uniaxial crystal. The optic axis is the z axis and the wave normal unit vector $\bar{s}$ lies on the y-z plane.

spheroid (ordinary wave). However, the other direction of polarization of $\overline{D}$, namely the one in the s-z plane, depends on θ (extraordinary wave).

The direction of propagation for which the extraordinary refractive index $n_e(\theta)$ becomes equal to the ordinary one, is the optic axis of the crystal. A biaxial crystal has two optic axes, because there are two circular sections passing through the center of an ellipsoid, whose normals $\overline{s}_1$ and $\overline{s}_2$ are coplanar with its largest and shortest principal axes. Wave propagation along the optic axis of an anisotropic crystal is characterized by the same equations as those that characterize propagation in an isotropic medium (spherical index ellipsoid). Propagation along any other direction than the optic axis results in birefringence, that is, the separation of an electromagnetic field distribution into two: an ordinary and an extraordinary. Ordinary modes derive their name from the property that the effective refractive index n_o associated with them is independent of the angle θ. In biaxial no ordinary-extraordinary distinction can be made.

Wave propagation in stratified anisotropic waveguides, as in isotropic ones, can be conveniently analyzed using either the geometrical techniques of the zig-zag ray model [4,5] or the method of normal modes [6]. In the zig-zag ray model, modal properties of waveguides are analyzed by defining the wave normal as the direction normal to the equiphase surface of the wave and the ray as the direction of power flow. It is interesting to note that the ray description of the modes allows only discrete angles of incidence of the wave normal and each angle represents a specific waveguide mode [7]. In the normal mode method, the effective refractive indices and field configurations are the eigenvalues and eigenvectors respectively, of the matrix eigenvalue problem derived from Maxwell's equations, as will be seen in a following section.

2.2 Linear Birefringence, Linear Electro-optic effect, Optical (or Rotary) Activity and Faraday Rotation

The electro-optic effect and the phenomena of optical activity and Faraday rotation must be clearly distinguished from that of linear birefringence. They involve rotation of the plane of linearly polarized waves in some sense, rather than conversion to elliptical polarization [8].

(1) Linear Birefringence

As it was seen in the previous section, in anisotropic media Fresnel's equation (2.1.8) provides two values for the magnitude of the wave vector $\overline{k}$. This, together with (2.1.9) leads to the result that a monochromatic plane electromagnetic wave entering an anisotropic crystal is refracted into two partial waves linearly polarized, characterized by perpendicular planes of polarization. Mathematically speaking, these two wave solutions form a complete set and thus an arbitrary wave in the crystal can be considered as a superposition of them. Since the phases and amplitudes of these two waves, as seen from (2.1.5) and (2.1.9), are different in all directions but the optic axis, the resulting wave is an elliptically polarized wave. The conversion of a linearly polarized wave to an elliptically polarized one, is typical of linear birefringence (double refraction).

(2) Linear Electro-optic Effect

Application of a bias electric field in anisotropic media, results in field-atomic interactions which cause distortion of the index ellipsoid [9]. The two linearly polarized wave solutions become elliptically polarized, whose ellipses have the same shape but opposite senses of rotation.

The amount of distortion of the index ellipsoid varies with direction. Since the undistorted index ellipsoid is described by the equation (2.1.10), i.e.

$$\left\{\frac{1}{n_x^2}\right\} x^2 + \left\{\frac{1}{n_y^2}\right\} y^2 + \left\{\frac{1}{n_z^2}\right\} z^2 = 1 \qquad (2.2.1)$$

the perturbed index ellipsoid under an applied field, will be described by

$$\left\{\frac{1}{n^2}\right\}_1 x^2 + \left\{\frac{1}{n^2}\right\}_2 y^2 + \left\{\frac{1}{n^2}\right\}_3 z^2 + 2\left\{\frac{1}{n^2}\right\}_4 yz + 2\left\{\frac{1}{n^2}\right\}_5 xz + 2\left\{\frac{1}{n^2}\right\}_6 xy = 1 \qquad (2.2.2)$$

Using Voigt's convention: 1-x, 2-y, 3-z, 4-yz, 5-xz, 6-xy, agreement of (2.2.1) and (2.2.2) under zero applied field is expressed by

$$\left\{\frac{1}{n^2}\right\}_i \Bigg|_{E=0} = \frac{1}{n_i^2} \quad ; \quad i = x,y,z$$

$$\left\{\frac{1}{n^2}\right\}_4 \Bigg|_{E=0} = \left\{\frac{1}{n^2}\right\}_6 \Bigg|_{E=0} = \left\{\frac{1}{n^2}\right\}_6 \Bigg|_{E=0} = 0 \qquad (2.2.3)$$

The linear electro-optic effect (Pockel's) is associated with a linear change of the coefficients $\left\{\frac{1}{n^2}\right\}_i$; $i = 1$ to 6 with the applied field. Thus

$$\Delta\left\{\frac{1}{n^2}\right\}_i = \sum_{j=1}^{3} r_{ij} E_j \quad ; \quad \begin{matrix} i=1 \text{ to } 6 \\ 1=x,\ 2=y,\ 3=z \end{matrix} \tag{2.2.4}$$

where r_{ij} are the entries of a 6x3 matrix, called electro-optic tensor and E_j is the j^{th} component of the dc applied field. These r_{ij} coefficients give rise to off-diagonal terms in the permittivity tensor $\bar{\bar{\varepsilon}}_p$, which are proportional to the applied electric field. For most practical cases, these terms are all zero but one, namely, the one resulting from nonzero coefficient in a direction normal to the optic axis, provided the field is applied along the optic axis. Thus

$$\bar{\bar{\varepsilon}}_p = \begin{bmatrix} \varepsilon_x & r_{xy}E_o & 0 \\ r_{xy}E_o & \varepsilon_x & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \tag{2.2.5}$$

is the principal dielectric tensor of a uniaxial crystal whose optic axis is along z. To find the eigenmodes of such a crystal, one must first derive the characteristic equation and substitute its solutions, that is, the effective guide indices in the normalized wave equation. This will be done for a complex off-diagonal ε_{xy}, in order the formulae to be used in the cases to follow.

Consider a uniaxial crystal characterized by a dielectric tensor

$$\bar{\bar{\varepsilon}}_p = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & 0 \\ \varepsilon_{xy}^* & \varepsilon_x & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \tag{2.2.6}$$

and supporting wave propagation in the z direction. Assuming plane wave solutions, Maxwell's equations (2.1.4) yield: $\beta^2(\bar{a}_z \times \bar{E}) \times \bar{a}_z = \bar{\bar{\varepsilon}}\, E$ where β is the longitudinal effective guide index. Decomposition of this in the transverse field components, yields the normalized wave equations

$$(\beta^2 - \varepsilon_x)\, E_x - \varepsilon_{xy}\, E_y = 0 \tag{2.2.7a}$$

and

$$\varepsilon_{xy}^*\, E_x - (\beta^2 - \varepsilon_x)\, E_y = 0 \tag{2.2.7b}$$

The resulting characteristic equation is a quadratic in β^2 and its solutions are given by $\beta_1, \beta_2 = \beta_1$, β_3, and $\beta_4 = -\beta_3$ where

$$\beta_{1,3}^2 = \varepsilon_x \pm |\varepsilon_{xy}| \tag{2.2.8}$$

with subscripts 1(3) corresponding to plus (minus). The two normalized forward eigenmodes are found from either (2.2.7a) or (2.2.7b) to be:

$$\left.\frac{E_y}{E_x}\right|_{\beta_1} = \frac{|\varepsilon_{xy}|}{\varepsilon_{xy}} \quad \text{and} \quad \left.\frac{E_y}{E_x}\right|_{\beta_3} = -\frac{|\varepsilon_{xy}|}{\varepsilon_{xy}}. \tag{2.2.9}$$

In the case of the linear electro-optic effect, ε_{xy} is real. The guide indices $\beta_{1,3}$ are

$$\beta_{1,3}^2 = \varepsilon_x \pm \varepsilon_{xy} \tag{2.2.10}$$

and the corresponding eigenvectors satisfy

$$\frac{E_y}{E_x} = 1 \quad \text{and} \quad \frac{E_y}{E_x} = -1$$

This result can be written in the form

$$\overline{E}_o = E \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \overline{E}_e = E \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{2.2.11}$$

where $\overline{E}_o$ and $\overline{E}_e$ are the transverse field eigenmodes (see Fig. 2.3). It is seen that a monochromatic plane wave passing through the crystal

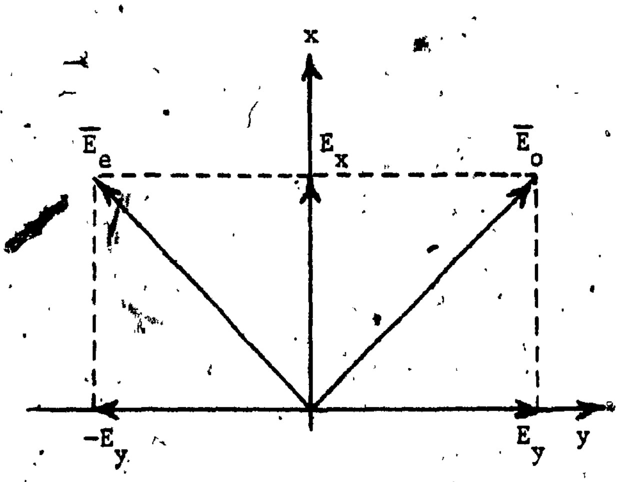

Fig. 2.3. Linear electro-optic effect in a uniaxial crystal. The bias electric field and the optic axis lie in the z-direction. $\overline{E}_o$ and $\overline{E}_e$ are the ordinary and extraordinary transverse eigenvectors.

is decomposed in two linearly polarized at right angles field configurations. One of the components of these in the principal dielectric axes remains fixed, whereas the other oscillates around the origin. The resulting field at a distance z is

$$\overline{E}(z) = \begin{bmatrix}1\\1\end{bmatrix} e^{-j\beta_1 z} + \begin{bmatrix}1\\-1\end{bmatrix} e^{-j\beta_3 z} = 2(\overline{a}_x \cos\frac{\Delta\beta}{2} z - j\overline{a}_y \sin\frac{\Delta\beta}{2} z)\exp(-j\beta_o z), \tag{2.2.12}$$

where $\Delta\beta$ and β_o are defined by:

$$\Delta\beta = \beta_1 - \beta_3 \quad \text{and} \quad \beta_o = (\beta_1 + \beta_3)/2 \tag{2.2.13}$$

For backward waves, where $z < 0$, the $\overline{E}$ field is found by changing z to $-z$, i.e.,

$$\overline{E}(z) = 2(\overline{a}_x \cos\frac{\Delta\beta}{2} z + j\overline{a}_y \sin\frac{\Delta\beta}{2} z)\exp(j\beta_o z). \tag{2.2.14}$$

As it can be seen from (2.2.12) and (2.2.14), the waves undergo the same sense of polarization rotation with respect to the their propagation direction. Thus the linear electro-optic effect is a reciprocal phenomenon.

(3) Optical (or Rotary) Activity

The rotation of the plane of polarization of a linearly polarized wave passing through an anisotropic crystal, with no electric or magnetic bias fields applied, is known to result from first-order spatial dispersion contributions to the dielectric tensor [10].

When the $\overline{D}$ vector at a given point in the crystal depends on $\overline{E}$ not only at that point, but at neighboring points as well, a dependence of the dielectric constant on the wave vector occurs. This dependence known as spatial dispersion, is expressed by writing the functional form of the dielectric constant $\varepsilon_{ij}(\omega,\overline{k})$. When the magnitude of the nonlocal dependence is small, $\varepsilon_{ij}(\omega,\overline{k})$ may be expanded in a power series in $\overline{k}$, namely

$$\varepsilon_{ij}(\omega,\overline{k}) = \varepsilon_{ij}^{\circ}(\omega) + jg_{ij\ell}(\omega)k_\ell + h_{ij\ell m}(\omega)k_\ell k_m + \ldots \tag{2.2.15}$$

Optical activity arises from the first order terms $jg_{ij\ell}(\omega)k_\ell$, where $g_{ij\ell}(\omega)$, or simply g, is the optical gyrotropic tensor. Note that in dispersive, lossless media [11]: $\varepsilon_{ij}^{\circ}(\omega,\overline{k}) = \varepsilon_{ji}^{*}(\omega,\overline{k})$ and the third rank coefficient g is real. Therefore, the dielectric tensor $\overline{\overline{\varepsilon}}_p$ of an optically active crystal supporting z-propagating waves with $k_z = k_o\beta$, is

$$\overline{\overline{\varepsilon}}_p = \begin{bmatrix} \varepsilon_x & jg\beta & 0 \\ -jg\beta & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \tag{2.2.16}$$

Physically, the imaginary factor j is due to the fact that $\overline{D}$, at any given point, depends also on the derivative of $\overline{E}$ at the direction of propagation [12]. Thus, optical activity is a nonlinear effect.

The axial guide indices β are found by rearranging and squaring (2.2.8), to be:

$$\beta^2_{1,3} = \varepsilon_x\{1+\delta [2\delta \pm \sqrt{1+\delta^2}]\} \tag{2.2.17}$$

where the parameter δ is defined by: $\delta = g/2\sqrt{\varepsilon_x}$.

Due to the β dependence of $\bar{\bar{\varepsilon}}_p$, that is, the complexity of (2.2.8) shown in (2.2.17), the eigenvectors are no longer given by (2.2.9), but (2.2.7a), i.e.,

$$\frac{E_y}{E_x} = \frac{\beta^2 - \varepsilon_x}{jg\beta}$$

Substituting β_1 and β_3, the two ratios become

$$\left.\frac{E_y}{E_x}\right|_{\beta_1} = -ja \quad \text{and} \quad \left.\frac{E_y}{E_x}\right|_{\beta_3} = jb ,$$

where a and b are defined by:

$$a = \frac{g}{4\delta\beta_1}(2\delta + \sqrt{1+\delta^2}) \quad \text{and} \quad b = \frac{g}{4\delta\beta_3}(\sqrt{1+\delta^2} - 2\delta) \tag{2.2.18}$$

Thus, the ordinary and extraordinary modes are:

$$\overline{E}_o = E\begin{bmatrix}1\\-ja\end{bmatrix} \quad \text{and} \quad \overline{E}_e = E\begin{bmatrix}1\\jb\end{bmatrix} \qquad (2.2.19)$$

The resulting electric field at a distance z is an elliptically polarized wave described by

$$\overline{E}^+(z) = E\begin{bmatrix}1\\-ja\end{bmatrix}e^{-j\beta_1 z} + E\begin{bmatrix}1\\jb\end{bmatrix}e^{-j\beta_3 z}$$

$$= E\left\{\overline{a}_x 2\cos\frac{\Delta\beta}{2}z - \overline{a}_y\left[(a+b)\sin\frac{\Delta\beta}{2}z + j(a-b)\cos\frac{\Delta\beta}{2}z\right]\right\}\exp(-j\beta_o z)$$

(2.2.20)

where $\Delta\beta$ and β_o were defined in (2.2.14). The electric field can now be decomposed into two partial field vectors, namely

$$\overline{E}_1^+ = E[\overline{a}_x\cos\frac{\Delta\beta}{2}z - \overline{a}_y(a+b)\sin\frac{\Delta\beta}{2}z]\exp(-j\beta_o z) \qquad (2.2.21a)$$

and

$$\overline{E}_2^+ = E\cos\frac{\Delta\beta}{2}z[\overline{a}_x - j\overline{a}_y(a-b)]\exp(-j\beta_o z) \qquad (2.2.21b)$$

Similarly, the field propagating in the -z direction can be decomposed into:

$$\overline{E}_1^-(z) = E[\overline{a}_x\cos\frac{\Delta\beta}{2}z + \overline{a}_y(a+b)\sin\frac{\Delta\beta}{2}z]\exp(j\beta_o z) \qquad (2.2.22a)$$

and

$$\overline{E}_2^-(z) = E\cos\frac{\Delta\beta}{2}z[\overline{a}_x - j\overline{a}_y(a-b)]\exp(j\beta_o z) \qquad (2.2.22b)$$

Note that, if the input field is x-polarized, then $\overline{E}_1^+$, $\overline{E}_2^+$ and

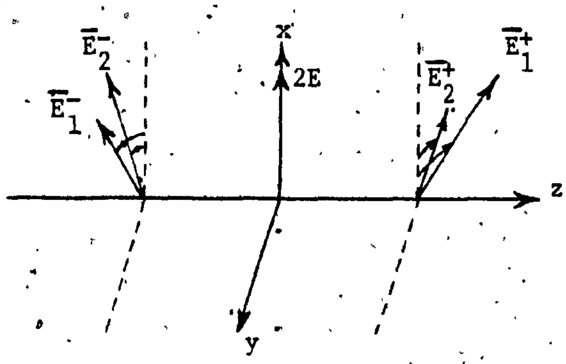

Fig. 2.4 Polarization rotation of the $\overline{E}_1^{\pm}$ and $\overline{E}_2^{\pm}$ components of the field vector propagating in the $\pm z$ directions.

$\overline{E}_1^-$, $\overline{E}_2^-$ are tilted in the same sense at $\pm z$ respectively, with respect to their propagation direction. Hence, optical activity is a reciprocal phenomenon.

Since the resulting field vectors $\overline{E}^{\pm}$ are tilted with respect the input field $\overline{E}(0)$ at a distance z, optical activity is circular birefringence. Physically, the decomposition in the two elliptically polarized waves along with the reciprocity, is due to the helical structure of the optically active substance [13]. It is worthnoting that, although the permittivity tensor $\overline{\overline{\varepsilon}}_p$ is nonsymmetric, Lorentz' reciprocity law still applies because the off-diagonal entries of $\overline{\overline{\varepsilon}}_p$ are not constant [24].

(4) Faraday Rotation

It has been found experimentally that the plane of polarization of a light wave passing through certain crystals in which a longitudinal bias magnetic field is applied, rotates by an angle Θ given by

$$\Theta(z) = VBz \qquad (2.2.23)$$

where z is the length of the crystal and V is the Verdet constant. Theoritically, this has been found to be the result of temporal (or frequency) dispersion {9}. Faraday rotation is a linear effect and can be considered as the magnetic equivalent of the linear electro-optic effect. The dielectric tensor $\bar{\bar{\varepsilon}}_p$ of a crystal supporting wave propagation along z with a bias magnetic field also along z, is of the form {14}

$$\varepsilon_p = \begin{bmatrix} \varepsilon_x & j\varepsilon & 0 \\ -j\varepsilon & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \qquad (2.2.24)$$

It is understood that if the bias field and the direction of wave propagation are along x or y, the off-diagonal terms appear in the yz or xz positions respectively.

Assuming $\varepsilon_x = \varepsilon_y$ and $\varepsilon_{xy} = j\varepsilon$, the longitudinal guide indices are

$$\beta_{1,3}^2 = \varepsilon_x \pm \varepsilon \qquad (2.2.25)$$

and from (2.2.9) the normalized eigenmodes satisfy

$$\left.\frac{E_y}{E_x}\right|_{\beta_1} = -j \quad \text{and} \quad \left.\frac{E_y}{E_x}\right|_{\beta_3} = j \qquad (2.2.26)$$

or in other words

$$\bar{E}_o = E\begin{bmatrix}1\\-j\end{bmatrix} \quad \text{and} \quad \bar{E}_e = E\begin{bmatrix}1\\j\end{bmatrix} \qquad (2.2.27)$$

where $\bar{E}_o$ and $\bar{E}_e$ are the ordinary and extraordinary transverse field eigenmodes. They are left- and right-circularly polarized waves,

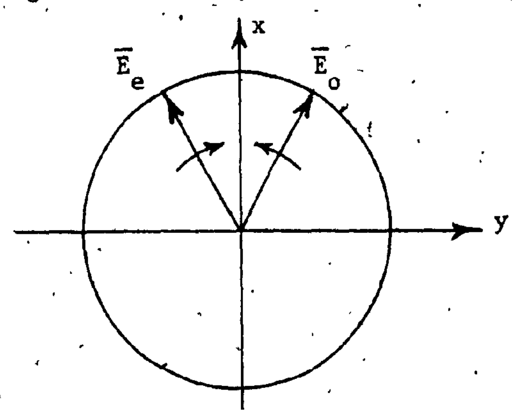

Fig. 2.5 Transverse field eigenmodes in a magneto-optic crystal. $\bar{E}_o$ and $\bar{E}_e$ are left- and right-circularly polarized waves, respectively.

respectively. For propagation in the ±z-directions, similar field expansion as before reveals that the rotation angle is in the same sense irrespective of the direction of propagation along z. Thus Faraday rotation is a nonreciprocal phenomenon. This is to be expected since the entries of the permittivity tensor are constant - independent of the effective guide indices.

In this section, the linear electro-optic effect, optical activity and Faraday rotation were discussed and distinguished from the phenomenon of linear birefringence. The linear electro-optic effect and the nonlinear optical activity were found to be reciprocal phenomena, whereas Faraday rotation nonreciprocal.

2.3 TE-TM Coupling, Coupling and Modal Matrices R_g, U and Polarization Vectors $\overline{p}$ and $\overline{q}$

The study of EM wave propagation in anisotropic media, as discussed in Section 2.1, led to the result that an arbitrary EM wave in such media is represented as an expansion of linearly (and orthogonally) polarized plane wave eigenmodes. This result can be used for the field expansion in the case of propagation in active media [15].

Consider a uniaxial crystal whose principal dielectric axes are x-y-z, with z the optic axis. The two $\overline{D}$ vectors are along the t (extraordinary) and u (ordinary) directions, shown in Fig. 2.6. Since the optic axis is along z, axis u is coplanar with x and y (see

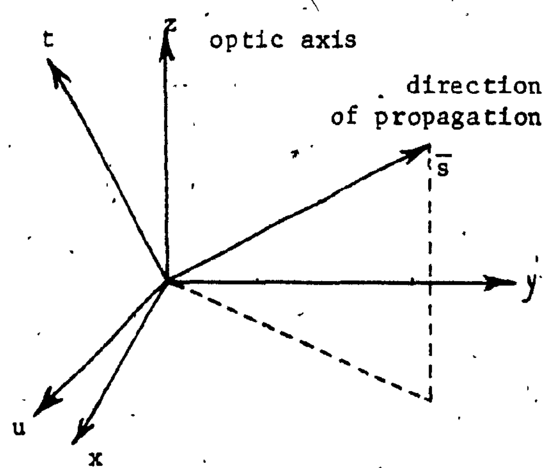

Fig. 2.6 Coordinate orientation of an optically active uniaxial crystal.

Fig. 2.2). From the geometry of Fig. 2.2, it is seen that the permittivity indices ε_t and ε_u along t and u respectively, are given by

$$\frac{1}{\varepsilon_t} = \frac{\sin^2\theta}{n_e^2} + \frac{\cos^2\theta}{n_o^2} \quad ; \quad \varepsilon_u = n_o^2 \tag{2.3.1}$$

The uniaxial crystal is assumed to be an unbounded medium and hence the propagation constants along directions t and u are

$$k_t = k_o\sqrt{\varepsilon_t} \quad , \quad k_u = k_o\sqrt{\varepsilon_u} \tag{2.3.2}$$

Allowing now the uniaxial medium to be an electro-optic crystal or a Faraday cell, the permittivity tensor is perturbed by off-diagonal terms and hence the constitutive equation of the transverse field components becomes

$$\begin{bmatrix} D_t \\ D_u \end{bmatrix} = \begin{bmatrix} \varepsilon_t & \varepsilon \\ \varepsilon^* & \varepsilon_u \end{bmatrix} \begin{bmatrix} E \\ E_u \end{bmatrix} \tag{2.3.3}$$

where real values of ε account for the electro-optic effect and imaginary for Faraday rotation.

The transverse field distribution assumes the form

$$\overline{E}_T(t,u,s) = \overline{t}A_s(s)\exp[j(\omega t - k_t s)] + \overline{u}A_u(s)\exp[j(\omega t - k_u s)] \tag{2.3.4}$$

This must satisfy the wave equation

$$\nabla x \nabla x \cdot \overline{E} = k_o^2 \overline{\overline{\varepsilon}}\,\overline{E} \tag{2.3.5}$$

which, in the case being examined, takes the form

$$\frac{\partial^2 E_t}{\partial s^2} = k_t^2 E_t + k_o^2 \varepsilon E_u$$

$$\frac{\partial^2 E_u}{\partial s^2} = k_o^2 \varepsilon^* E_t + k_u^2 E_u \tag{2.3.6}$$

Substitution of the component E_t and E_u from (2.3.4) into (2.3.6), yields

$$\frac{dA_t}{ds} = \frac{k_o^2 \varepsilon}{2jk_t} A_u \exp[-j(k_u - k_t)s]$$

$$\frac{dA_u}{ds} = \frac{k_o^2 \varepsilon^*}{2jk_u} A_t \exp[j(k_u - k_t)s] \tag{2.3.7}$$

where a slow variation, i.e., $\left|\frac{d^2 A_{u,t}}{ds^2}\right| \ll \left|k^2_{t,u} A_{t,u}\right|$, was assumed. It is seen that in the absence of the electro-optic effect or Faraday rotation ($\varepsilon = 0$), the amplitudes A_t and A_u of the eigenmodes are independent of s. Furthermore, the coupled set of equations (2.3.7) indicate that, the presence of optical activity results in coupling between the TE(H_t, E_u, H_s) and TM(E_t, H_u, E_s) modes.

Summarizing, TE-TM coupling is always the result of electromagnetic wave propagation in media characterized by dielectric tensors with off diagonal terms and in anisotropic media when the coordinate axes do not coincide with the principal dielectric axes.

Generally, the problem of wave propagation in such media can be handled by one of the two equivalent methods:

(1) In the first method, solution of the coupled mode equations pertaining to the particular structure of the crystal (θ, ε) is obtained,

subject to the boundary conditions at $s = 0$. This method is a continuation of the method treated above.

(2) In the second method, which will be followed hereafter, Maxwell's equations are solved and the eigenmode components are found. The total field then resulting at any point, is the vector sum of the corresponding components.

Assume now an anisotropic layer characterized by a permittivity tensor whose off-diagonal elements are generally complex. The direction of wave propagation is chosen to be along the z-axis, whereas the direction normal to the interfaces along the x-axis.

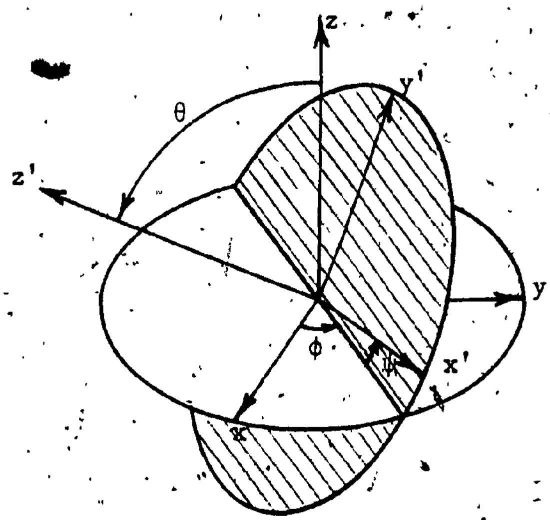

Fig. 2.7 The three Euler's angles ϕ, ψ and θ between the device's coordinate axes and the crystal's x'-y'-z'.

The relations between the principal axes of the crystal x'-y'-z' and the device's coordinate axes x-y-z, is described by Euler's angles ϕ,ψ and θ, shown in Fig. 2.7. The dielectric tensor in the x-y-z coordinate system is given by

$$\bar{\bar{\varepsilon}} = A\bar{\bar{\varepsilon}}_p A^T \tag{2.3.8}$$

where A is the orthogonal rotation matrix [16]:

$$A = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix} \quad (2.3.9)$$

Since A is orthogonal and $\bar{\bar{\varepsilon}}_p$ Hermitean, the dielectric tensor $\bar{\bar{\varepsilon}}$ is also Hermitean, i.e.,

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon^*_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon^*_{xz} & \varepsilon^*_{yz} & \varepsilon_{zz} \end{bmatrix} \quad (2.3.10)$$

where ε_{ii} $(i = x,y,z)$ is real.

The propagation vector $\bar{k}$ associated with propagating modes in the layer, is

$$\bar{k} = k_o(\bar{a}_x\kappa + \bar{a}_y\gamma + \bar{a}_z\beta) \quad (2.3.11)$$

where κ, γ and β are effective guide indices. Assuming the anisotropic layer to be homogeneous in the y-z plane, guide indices γ and β are the same for all layers.

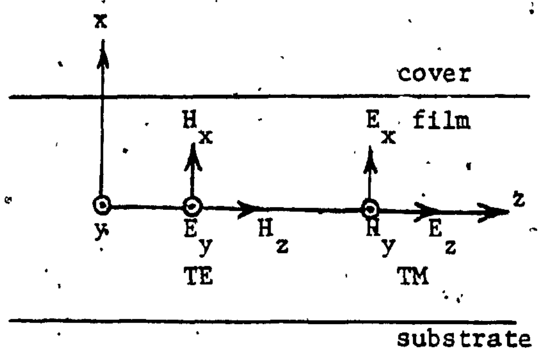

Fig. 2.8 Orientation of an anisotropic layer's coordinate system and TE, TM mode polarizations.

In a nonmagnetizable dielectric, Maxwell's equations assume the form:

$$\nabla\times(\eta_o^{-\frac{1}{2}}\,\overline{E}) = -jk_o(\eta_o^{\frac{1}{2}}\,\overline{H}) \quad , \quad \nabla\times(\eta_o^{\frac{1}{2}}\,\overline{H}) = jk_o\overline{\overline{\varepsilon}}(\eta_o^{-\frac{1}{2}}\,\overline{E}) \qquad (2.3.12)$$

Taking into account that $\frac{\partial}{\partial y} = -jk_o\gamma$ and $\frac{\partial}{\partial z} = -jk_o\beta$, eliminating the E_x and H_x components, and defining the normalized length x along the x-direction as $k_o x$, equations (2.3.12) become

$$\frac{d}{dx}\,\overline{g}(x) = -jR_g\overline{g}(x) \qquad (2.3.13)$$

where vector $\overline{g}(x)$ is given by

$$\overline{g}(x) = [\eta_o^{-\frac{1}{2}}E_y,\ \ \eta_o^{\frac{1}{2}}H_z,\ \ \eta_o^{-\frac{1}{2}}E_z,\ \ -\eta_o^{\frac{1}{2}}H_y]^T \qquad (2.3.14)$$

and the coupling matrix R_g by

$$R_g = \begin{bmatrix} -\gamma\varepsilon_{xy}/\varepsilon_{xx} & 1-\gamma^2/\varepsilon_{xx} & -\gamma\varepsilon_{xz}/\varepsilon_{xx} & -\beta\gamma/\varepsilon_{xx} \\ \Delta_{zz}/\varepsilon_{xx}-\beta^2 & -\gamma\varepsilon^*_{xy}/\varepsilon_{xx} & \beta\gamma-\Delta_{zy}/\varepsilon_{xx} & -\beta\varepsilon^*_{xy}/\varepsilon_{xx} \\ -\beta\varepsilon_{xy}/\varepsilon_{xx} & -\beta\gamma/\varepsilon_{xx} & -\beta\varepsilon_{xz}/\varepsilon_{xx} & 1-\beta^2/\varepsilon_{xx} \\ \beta\gamma-\Delta_{yz}/\varepsilon_{xx} & -\gamma\varepsilon^*_{xz}/\varepsilon_{xx} & \Delta_{yy}/\varepsilon_{xx}-\gamma^2 & -\beta\varepsilon^*_{xz}/\varepsilon_{xz} \end{bmatrix} \qquad (2.3.15)$$

with Δ_{ij} (i,j = x,y,z) the ij-th cofactor of the permittivity tensor (2.3.10). The first two components of $\overline{g}(x)$ correspond to TE, whereas the last two to TM polarization. Note that all entries of the 4x4 coupling matrix are dimensionless. Finally, the E_x and H_x components expressed with respect to $\overline{g}(x)$ take the form

$$\begin{bmatrix} \eta_o^{-\frac{1}{2}} E_x \\ \eta_o^{\frac{1}{2}} H_x \end{bmatrix} = -\begin{bmatrix} \varepsilon_{xy}/\varepsilon_{xx} & \gamma/\varepsilon_{xx} & \varepsilon_{xz}/\varepsilon_{xx} & \beta/\varepsilon_{xx} \\ \beta & 0 & -\gamma & 0 \end{bmatrix} \overline{g}(x). \qquad (2.3.16)$$

Assuming now that propagation takes place only along the z-direction, i.e., $\frac{\partial}{\partial y} = -jk_o\gamma = 0$, the coupling matrix reduces to

$$R_g = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \Delta_{zz}/\varepsilon_{xx}-\beta^2 & 0 & -\Delta_{zy}/\varepsilon_{xx} & -\beta\varepsilon^*_{xy}/\varepsilon_{xx} \\ -\beta\varepsilon_{xy}/\varepsilon_{xx} & 0 & -\beta\varepsilon_{xz}/\varepsilon_{xx} & 1-\beta^2/\varepsilon_{xx} \\ -\Delta_{yz}/\varepsilon_{xx} & 0 & \Delta_{yy}/\varepsilon_{xx} & -\beta\varepsilon^*_{xz}/\varepsilon_{xx} \end{bmatrix} \qquad (2.3.17)$$

from which it is seen that

$$R_{23} = R^*_{41} \quad , \quad R_{24} = R^*_{31} \quad , \quad R_{44} = R^*_{33} \, . \qquad (2.3.18)$$

The off-diagonal 2x2 blocks of R_g characterize TE-TM coupling, whereas the diagonal 2x2 blocks, pure mode propagation. This is summarized by writing

$$R_g = \begin{bmatrix} R_{TE} & R_{TE-TM} \\ R_{TM-TE} & R_{TM} \end{bmatrix} \tag{2.3.19}$$

When $R_{31} = R_{41} = 0$, R_g becomes block diagonal, indicating that the conditions of pure TE and TM modes is: $\varepsilon_{xy} = \Delta_{yz} = 0$.

The eigenvalue equation

$$(R_g - \kappa I)\,\bar{g}(x) = 0 \tag{2.3.20}$$

yields four transverse guide indices κ_i, $i = 1$ to 4. These are obtained from the characteristic equation

$$\kappa^4 + a\kappa^3 + b\kappa^2 + c\kappa + d = 0 \tag{2.3.21a}$$

where

$$a = -2\mathrm{Re}(R_{33})$$

$$b = |R_{33}|^2 - R_{34}R_{43} - R_{21}$$

$$c = 2[R_{21}\mathrm{Re}(R_{33}) - \mathrm{Re}(R_{31}R_{41}^*)]$$

$$d = R_{21}(R_{34}R_{43} - |R_{33}|^2) + 2\mathrm{Re}(R_{31}R_{41}^*R_{33}^*) - |R_{31}|^2R_{43} - |R_{41}|^2R_{34} \tag{2.3.21b}$$

All four coefficients are real and hence the four admissible solutions of κ are either real or complex conjugate pairs. The condition for bidirectionality, that is, that the κ_i's come in pairs of opposite signature, is $a = c = 0$, that is, $\mathrm{Re}(R_{33}) = 0$ and $R_{31} = 0$ or $R_{41} = 0$. Knowledge of the four κ's obtained in (2.3.21) enables one to solve the eigenvalue equation (2.3.20), which in normalized form is written

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ R_{21} & 0 & R^*_{41} & R^*_{31} & u_{2i} \\ R_{31} & 0 & R_{33} & R_{34} & u_{3i} \\ R_{41} & 0 & R_{43} & R_{33} & u_{4i} \end{bmatrix} = \kappa_i \begin{bmatrix} 1 \\ u_{2i} \\ u_{3i} \\ u_{4i} \end{bmatrix} \quad ; \ i=1 \text{ to } 4. \tag{2.3.22}$$

The (right) eigenvectors are found to have the form

$$u_{2i} = \kappa_i$$

$$u_{3i} = \frac{1}{D_i} [R_{31}(\kappa_i - R^*_{33}) + R_{34}R_{41}]$$

$$u_{4i} = \frac{1}{D_i} [R_{41}(\kappa_i - R_{33}) + R_{31}R_{43}] \tag{2.3.23}$$

where the denominator D_i is given by

$$D_i = (R_{33} - \kappa_i)(R^*_{33} - \kappa_i) - R_{34}R_{43} \tag{2.3.24}$$

The modal matrix, formed by the column form of the four eigenvectors of R_g, is given by

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \kappa_1 & \kappa_2 & \kappa_3 & \kappa_4 \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{bmatrix}. \tag{2.3.25}$$

The eigenvalue equation (2.3.20) can now be written in the matrix form

$$R_g U = \Lambda_R U \qquad (2.3.26)$$

where

$$\Lambda_R = \mathrm{Diag}[\kappa_1 \;\; \kappa_2 \;\; \kappa_3 \;\; \kappa_4]. \qquad (2.3.27)$$

Another way to obtain the components of the electromagnetic field, is from the wave equation, by use of the $\nabla\times = -j\bar{k}\times$ operator. Thus

$$\bar{k}\times(\bar{k}\times\bar{E}) + k_o^2\bar{\bar{\varepsilon}}\,\bar{E} = 0 \qquad (2.3.28a)$$

which, expanded in matrix form yields

$$\begin{bmatrix} \varepsilon_{xx}-\beta^2 & \varepsilon_{xy} & \varepsilon_{xz}+\beta\kappa \\ \varepsilon_{xy}^{*} & \varepsilon_{yy}-\beta^2-\kappa^2 & \varepsilon_{yz} \\ \varepsilon_{xz}^{*}+\beta\kappa & \varepsilon_{yz}^{*} & \varepsilon_{zz}-\kappa^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0. \qquad (2.3.28b)$$

The polarization vector $\bar{p}$ is an electric field vector (normalized in some sense or not) which satisfies(2.3.28b). Considering on the one hand that the determinant of the matrix in (2.3.28b) must vanish and on the other hand recalling that the sum of the products of the entries of one row with the corresponding cofactors of another row must be zero, the components of $\bar{p}$ are simply proportional to the cofactors of any row of the matrix in (2.3.28b). The corresponding polarization vectors $\bar{q}$ of the magnetic field, can be obtained from the reduced Maxwell equation

$$k_o\bar{q}_i = \bar{k}_i\times\bar{p}_i \qquad (2.3.29)$$

where $\bar{k}_i = k_o(\bar{a}_x\kappa_i + \bar{a}_z\beta)$. In this thesis the $\bar{p}$ vector will be normalized so that $p_{iy} = 0$.

With $\bar{p}_i$ and $\bar{q}_i$ known, the total electric and magnetic field at x, is expressed as the sum of all partial waves, viz.,

$$\eta_o^{-\frac{1}{2}} \bar{E}(x) = \sum_{i=1}^{4} c_i \bar{p}_i e^{-j\kappa_i x}$$

$$\eta_o^{\frac{1}{2}} \bar{H}(x) = \sum_{i=1}^{4} c_i \bar{q}_i e^{-j\kappa_i x} \tag{2.3.30}$$

where the amplitudes c_i (i = 1 to 4) are generally complex numbers. Retaining only the y and z components, equation (2.3.30) can be cast in the matrix form

$$\bar{g}(x) = U\Lambda_g(x)\bar{c} \tag{2.3.31}$$

where U is the modal matrix expressed in terms of the components of $\bar{p}$ and $\bar{q}$

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ q_{1z} & q_{2z} & q_{3z} & q_{4z} \\ p_{1z} & p_{2z} & p_{3z} & p_{4z} \\ -q_{1y} & -q_{2y} & -q_{3y} & -q_{4y} \end{bmatrix}, \tag{2.3.32}$$

$\Lambda_g(x)$ is the propagation factor matrix

$$\Lambda_g(x) = \mathrm{diag}[\exp(-j\kappa_1 x), \exp(-j\kappa_2 x), \exp(-j\kappa_3 x), \exp{-j\kappa_4 x})] \tag{2.3.33}$$

and $\bar{c}$ is the amplitude vector referred to x=0.

It will be found advantageous to use a permuted form of U and Λ_g. For this reason, the "tilde" transformation is introduced, defined by

$$\tilde{A} = P_{23} A P_{23} \tag{2.3.34}$$

where

$$P_{23} = P_{23}^{T} = P_{23}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.3.35}$$

The tilde-transform when applied to (2.3.31) results in

$$P_{23}\bar{g}(x) = \begin{bmatrix} \eta_o^{-\frac{1}{2}} \bar{E}_\tau \\ \sigma^T \eta_o^{-\frac{1}{2}} \bar{H}_\tau \end{bmatrix} = \tilde{U}\tilde{\Lambda}_g(x) P_{23}\bar{c}$$

$$= \begin{bmatrix} P_f & P_b \\ Q_f & Q_b \end{bmatrix} \begin{bmatrix} \Lambda_f(x) & 0 \\ 0 & \Lambda_b(x) \end{bmatrix} \begin{bmatrix} \bar{c}_f \\ \bar{c}_b \end{bmatrix} \tag{2.3.36}$$

where the definition and properties of σ matrix are

$$\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} , \quad \sigma^{-1} = \sigma^{T} , \quad \sigma^{2} = -I; \tag{2.3.37}$$

the P_f, P_b, Q_f and Q_b matrices are defined by

$$P_f = \begin{bmatrix} 1 & 1 \\ p_{1z} & p_{3z} \end{bmatrix} , P_b = \begin{bmatrix} 1 & 1 \\ p_{2z} & p_{4z} \end{bmatrix} , Q_f = \begin{bmatrix} q_{1z} & q_{3z} \\ -q_{1y} & -q_{3y} \end{bmatrix} , Q_b = \begin{bmatrix} q_{2z} & q_{4z} \\ -q_{2y} & -q_{4y} \end{bmatrix} , \tag{2.3.38}$$

the $\Lambda_f(x)$ and $\Lambda_b(x)$ by

$$\Lambda_f(x) = \text{Diag}(e^{-j\kappa_1 x}, e^{-j\kappa_3 x}), \; \Lambda_b(x) = \text{Diag}(e^{-j\kappa_2 x}, e^{-j\kappa_4 x}) \qquad (2.3.39)$$

and finally the $\overline{c}_f$ and $\overline{c}_b$ by

$$\overline{c}_f = \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} ; \; \overline{c}_b = \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} \qquad (2.3.40)$$

Note that when the quartic (2.3.31) permits bidirectional wave propagation i.e., when $a = c = 0$ $(\kappa_2 = -\kappa_1, \kappa_4 = -\kappa_3)$, the f and b subscripts in (2.3.36) refer to forward and backward submatrices associated uniquely with forward and backward propagating waves respectively.

In this chapter the concepts of inherent and induced anisotropy were introduced and wave propagation in anisotropic media was investigated. TE-TM mode coupling was shown to result from off-diagonal termsof the permittivity tensor. The Maxwell's equations have been reduced to a set of four linear coupled differential and two algebraic equations. Finally, the electric and magnetic polarization vectors were introduced and used to express the electromagnetic field distribution in a birefringent medium.

CHAPTER 3

NETWORK ANALYSIS OF ANISOTROPIC LAYERS

3.1 Dispersion Equation and Transverse Resonance for Isotropic Lossless Waveguides

In this section, properties of waves in a lossless symmetric isotropic slab waveguide will be reviewed, so as one will be able to assess the changes caused by anisotropy in the following sections.

A symmetric isotropic slab waveguide is illustrated in Fig. 3.1. Assume z to be the direction of propagation and $\hat{n} < n$ (careted quantities refer to the cladding regions). The propagation vector of regions I and II is

$$\overline{k} = \overline{a}_x k_x + \overline{a}_z k_z \; ; \; k_x = k_o \kappa, \; k_z = k_o \beta \qquad (3.1.1)$$

and

$$\hat{\overline{k}} = \overline{a}_x k_x + \overline{a}_z k_z \quad ; \quad k_x = -jk_o \alpha_x \qquad (3.1.2)$$

respectively, where k_z is common to core and cladding.

A waveguide is characterized by its dispersion equation, relating the effective guide index β to the frequency. This relationship is usually obtained by substituting the reduced wave equation into the so-called characteristic equation of the mode. In the isotropic case, the general form of the wave equation given in (2.3.28a), is

$$(\overline{k} \cdot \overline{k})\overline{E} - (\overline{k} \cdot \overline{E})\overline{k} = k_o^2 \varepsilon \overline{E}$$

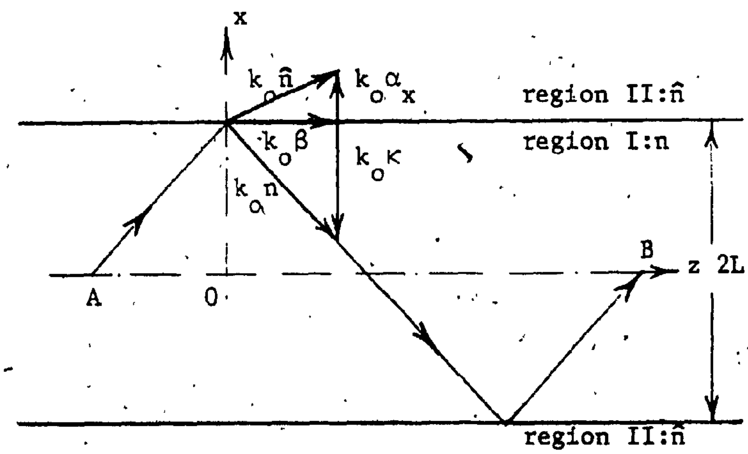

Fig. 3.1 Geometry of a symmetric isotropic slab waveguide. The internal region is called film, whereas the substrate and cover (region II), cladding.

where ε is the relative permittivity of the slab. Due to the orthogonality of $\bar{k}$ and $\bar{E}$ and the arbitrary nature of $\bar{E}$, this is further simplified to

$$\bar{k}\cdot\bar{k} = k_o^2\varepsilon. \tag{3.1.3}$$

Taking now into account (3.1.1) and (3.1.2) in their general form, the real part of (3.1.3) in region I and II becomes

$$\kappa = \sqrt{\varepsilon-\beta^2} \tag{3.1.4}$$

and

$$\alpha_x = \sqrt{\beta^2-\hat{\varepsilon}} \tag{3.1.5}$$

respectively.

The characteristic equation is the link between κ and α_x needed in (3.1.4) and (3.1.5) to obtain the $\beta = \beta(\omega)$ relation. It is obtained either from the continuity of the field components at the boundary [17], or from the transerve resonsance condition (TRC) [18] derived from the zig-zag ray model. According to the latter, the isotropic layer can be considered as a resonant transmission line in the x-direction of length 2L, terminated at both ends by infinitely long, below cutoff transmission lines representing the semi-infinite external regions. The condition for ray propagation along z, is that the phase difference between points A and B (see Fig. 3.1) must be integral multiple of 2π[19]. This means that in the transverse direction, the wave profile in the core must be a standing wave. Since resonance in a transmission line of finite length occurs at discrete wavelengths [20], that is, discrete values of κ, the dielectric slab supports only a discrete set of propagating modes.

The condition for resonance requires that the wave must be restored after a complete round trip in the transverse direction, i.e.,

$$\Gamma^2 \exp(-j4\kappa\ell) = 1 \tag{3.1.6}$$

where $2\ell = 2k_o L$ is the normalized width of the layer and $\Gamma = \rho \exp(j\theta)$ is the complex reflection coefficient at the boundary. The argument of (3.1.6) provides the phase condition

$$2\kappa\ell - \theta = N\pi \tag{3.1.7}$$

where N is an integer. The expression for the reflection coefficient Γ, in terms of the wave impedance of the film Z_c and the load impedance $\hat{Z}_c$, is given by: $\Gamma = (\hat{Z}_c - Z_c)\ (\hat{Z}_c \pm Z_c)$. The wave impedance for TE and TM modes is

$$Z_c^{TE} \cdot j\eta_o\ (\alpha \pm j\beta)\ , \qquad Z_c^{TM}\ \eta_o(\alpha \pm j\beta)\ j\varepsilon$$

respectively. Substitution of the appropriate impedance expression into Γ results in

$$\Gamma^{TE} = \exp(j\theta^{TE})\ , \qquad \theta^{TE} = 2\tan^{-1}\left(\frac{\alpha_x}{\kappa}\right) \tag{3.1.8}$$

or

$$\Gamma^{TM} = \exp(j\theta^{TM}) \quad , \quad \theta^{TM} = -2\tan^{-1}\left(\frac{\hat{\varepsilon}\kappa}{\varepsilon\alpha_x}\right) \qquad (3.1.9)$$

as the case may be.

The characteristic equations of the TE and TM modes in the isotropic layer, are obtained by substituting θ of (3.1.8) and (3.1.9) into the phase condition (3.1.7)

$$\kappa\ell - \tan^{-1}\left(\frac{\alpha_x}{\kappa}\right) = \frac{N\pi}{2} \quad \text{(TE modes)} \qquad (3.1.10a)$$

$$\kappa\ell - \tan^{-1}\left(\frac{\hat{\varepsilon}\kappa}{\varepsilon\alpha_x}\right) = \frac{N\pi}{2} \quad \text{(TM modes)} \qquad (3.1.10b)$$

Finally, substitution of the reduced wave equations (3.1.4), (3.1.5) into the characteristic equations (3.1.10) yields the dispersion equations

$$\ell\sqrt{\varepsilon - \beta^2} - \tan^{-1}\sqrt{\frac{\beta^2 - \hat{\varepsilon}}{\varepsilon - \beta^2}} = \frac{N\pi}{2} \quad \text{(TE modes)} \qquad (3.1.11a)$$

$$\ell\sqrt{\varepsilon - \beta^2} - \tan^{-1}\left[\frac{\hat{\varepsilon}}{\varepsilon}\sqrt{\frac{\varepsilon - \beta^2}{\beta^2 - \hat{\varepsilon}}}\right] = \frac{N\pi}{2} \quad \text{(TM modes)} \qquad (3.1.11b)$$

These transcedental equations give the functional relation between the longitudinal effective guide index β of the N-th mode and the frequency ($\ell = k_o L$). Note that for propagating modes (real β),

$$\hat{\varepsilon} < \beta^2 < \varepsilon \qquad (3.1.12)$$

must be satisfied.

Considering symmetric and antisymmetric configurations for E_y (TE) and H_y (TM), the field components evaluated by solving Maxwell's equations are listed in Table 3.1.

The power flow density in the external region is

$$S_x = \tfrac{1}{2}\mathrm{Re}[(\eta_o^{-\frac{1}{2}} E_y)(\eta_o^{\frac{1}{2}} H_z)^* - (\eta_o^{-\frac{1}{2}} E_z)(\eta_o^{\frac{1}{2}} H_y)^*]. \qquad (3.1.13)$$

Substitution of the corresponding field components from Table 3.1 yields $S_x = 0$, that is, the eigensolutions whose propagation characteristics are given by the dispersion equations (3.1.11), correspond to propagating modes.

To find the z-directed power flow per unit width (in the y direction), the integral

$$P = \tfrac{1}{2} \int_{-\infty}^{\infty} [(\eta_o^{-\frac{1}{2}} E_x)(\eta_o^{\frac{1}{2}} H_y)^* - (\eta_o^{-\frac{1}{2}} E_y)(\eta_o^{\frac{1}{2}} H_x)^*]dx \qquad (3.1.14)$$

must be evaluated. For TE modes, the first term in the brackets vanishes, whereas for TM modes, the second. For these modes, after the substitution of $\eta_o^{\frac{1}{2}} H_x$ and $\eta_o^{-\frac{1}{2}} E_x$ from Table 3.1, P becomes

$$P^{TE} = \beta \int_0^{\infty} |\eta_o^{-\frac{1}{2}} E_y|^2 dx \qquad (3.1.15)$$

and

$$P^{TM} = \frac{\beta}{\varepsilon} \int_0^{\infty} |\eta_o^{\frac{1}{2}} H_y|^2 \, dx, \qquad (3.1.16)$$

respectively. Substitution now of the appropriate field components yields

$$k_o P^{TE}_{sym} = \tfrac{1}{2}\beta\ell |A_s|^2 \left\{1 + \frac{\sin 2\kappa\ell}{2\kappa\ell} + \frac{\cos^2\kappa\ell}{\alpha_x \ell}\right\} \qquad (3.1.17a)$$

$$k_o P^{TE}_{ant} = \tfrac{1}{2}\beta\ell |A_a|^2 \left\{1 - \frac{\sin 2\kappa\ell}{2\kappa\ell} + \frac{\sin^2\kappa\ell}{\alpha_x \ell}\right\} \qquad (3.1.17b)$$

$$k_o P^{TM}_{sym} = \frac{1}{2\varepsilon}\beta\ell |B_s|^2 \left\{1 + \frac{\sin 2\kappa\ell}{2\kappa\ell} + \frac{\cos^2\kappa\ell}{\alpha_x \ell}\right\} \qquad (3.1.17c)$$

$$k_o P^{TM}_{ant} = \frac{1}{2\varepsilon}\beta\ell |B_a|^2 \left\{1 - \frac{\sin 2\kappa\ell}{\kappa\ell} + \frac{\sin^2\kappa\ell}{\alpha_x \ell}\right\} \qquad (3.1.17d)$$

where the expressions $\hat{A}_s(\hat{B}_s) = A_s(B_s)\cos\kappa\ell$ and $\hat{A}_a(\hat{B}_a) = A_a(B_a)\sin\kappa\ell$ arising from the boundary conditions, were used.

In this section, the dispersion equation of the symmetric isotropic slab waveguide was derived using its network (transmission line) equivalent. The symmetric and antisymmetric field distributions of propagating modes were given for both the core and cladding. Finally, the expression of the z-directed power flow per unit width was found for each field distribution.

3.2 The Wave Impedance Matrix

The wave impedance, defined as the ratio of electric to magnetic field components pertaining to the same electromagnetic field and evaluated at a given location, is known to be a scalar in the isotropic

Symmetric TE mode	Antisymmetric TE mode	Symmetric TM mode	Antisymmetric TM mode
$\eta_o^{-\frac{1}{2}} E_y = A_s \cos\kappa x$	$\eta_o^{-\frac{1}{2}} E_y = A_a \sin\kappa x$	$\eta_o^{-\frac{1}{2}} E_x = \frac{\beta}{\varepsilon} B_s \cos\kappa x$	$\eta_o^{-\frac{1}{2}} E_x = \frac{\beta}{\varepsilon} B_a \sin\kappa x$
$\eta_o^{\frac{1}{2}} H_x = -\beta A_s \cos\kappa x$	$\eta_o^{\frac{1}{2}} H_x = -\beta A_a \sin\kappa x$	$\eta_o^{-\frac{1}{2}} E_z = j \frac{\kappa}{\varepsilon} B_s \sin\kappa x$	$\eta_o^{-\frac{1}{2}} E_z = -j \frac{\kappa}{\varepsilon} B_a \cos\kappa x$
$\eta_o^{\frac{1}{2}} H_z = -j\kappa A_s \sin\kappa x$	$\eta_o^{\frac{1}{2}} H_z = j\kappa A_a \cos\kappa x$	$\eta_o^{\frac{1}{2}} H_y = B_s \cos\kappa x$	$\eta_o^{\frac{1}{2}} H_y = B_a \sin\kappa x$
$\eta_o^{-\frac{1}{2}} E_y = \hat{A}_s \exp[-\alpha_x(\lvert x\rvert-\ell)]$	$\eta_o^{-\frac{1}{2}} E_y = \frac{x}{\lvert x\rvert} \hat{A}_a \exp[-\alpha_x(\lvert x\rvert-\ell)]$	$\eta_o^{-\frac{1}{2}} E_x = \frac{\beta}{\hat{\varepsilon}} \hat{B}_s \exp[-\alpha_x(\lvert x\rvert-\ell)]$	$\eta_o^{-\frac{1}{2}} E_x = \frac{x}{\lvert x\rvert} \frac{\beta}{\hat{\varepsilon}} \hat{B}_a \exp[-\alpha_x(\lvert x\rvert-\ell)]$
$\eta_o^{\frac{1}{2}} H_x = -\beta \hat{A}_s \exp[-\alpha_x(\lvert x\rvert-\ell)]$	$\eta_o^{\frac{1}{2}} H_x = -\beta \frac{x}{\lvert x\rvert} \hat{A}_a \exp[-\alpha_x(\lvert x\rvert-\ell)]$	$\eta_o^{-\frac{1}{2}} E_z = j \frac{x}{\lvert x\rvert} \frac{\alpha_x}{\hat{\varepsilon}} \hat{B}_s \exp[-\alpha_x(\lvert x\rvert-\ell)]$	$\eta_o^{-\frac{1}{2}} E_z = j \frac{x}{\lvert x\rvert} \frac{\alpha_x}{\hat{\varepsilon}} \hat{B}_a \exp[-\alpha_x(\lvert x\rvert-\ell)]$
$\eta_o^{\frac{1}{2}} H_z = -j \frac{x}{\lvert x\rvert} \alpha_x \hat{A}_s \exp[-\alpha_x(\lvert x\rvert-\ell)]$	$\eta_o^{\frac{1}{2}} H_z = -j\alpha_x \hat{A}_a \exp[-\alpha_x(\lvert x\rvert-\ell)]$	$\eta_o^{\frac{1}{2}} H_y = \hat{B}_s \exp[-\alpha_x(\lvert x\rvert-\ell)]$	$\eta_o^{\frac{1}{2}} H_y = \frac{x}{\lvert x\rvert} \hat{B}_a \exp[-\alpha_x(\lvert x\rvert-\ell)]$
$\hat{A}_s = A_s \cos\kappa\ell$	$\hat{A}_a = A_a \sin\kappa\ell$	$\hat{B}_s = B_s \cos\kappa\ell$	$\hat{B}_a = B_a \sin\kappa\ell$

Table 3.1 Field distribution of even and odd modes in the isotropic slab waveguide. The factor $\exp[j(\omega t - k_o \beta z)]$ is assumed in all configurations above and the lengths x and ℓ are normalized, i.e., $x = k_o x$ and $\ell = k_o L$.

case. However in anisotropic media, due to TE-TM coupling, the wave impedance is expected to have a matrix form [21].

The wave impedance concept provides a method for expressing the field components in a homogeneous anisotropic medium. In analogy to the isotropic case, the normalized wave-impedance is defined by

$$\eta_o^{-\frac{1}{2}}\bar{E}_\tau = z(\eta_o^{\frac{1}{2}}\bar{H}_\tau\times\bar{a}_x) \qquad (3.2.1)$$

where τ stands for the vector components perpendicular to the x. The σ matrix defined in (2.3.37) allows one to express the cross product with a unit vector in the matrix form:

$$\bar{a}_x\times\bar{A}_\tau = \sigma\bar{A}_\tau\ , \qquad \bar{A}_\tau\times\bar{a}_x = \sigma^T\bar{A}_\tau\ . \qquad (3.2.2)$$

With the aid of (3.2.2), (3.2.1) is written

$$\eta_o^{-\frac{1}{2}}\bar{E}_\tau = z\sigma^T\ \eta_o^{\frac{1}{2}}\ \bar{H}_\tau\ . \qquad (3.2.3)$$

Upon the definition now of the wave admittance y, as: $y = z^{-1}$, straightforward algebra shows that

$$\eta_o^{\frac{1}{2}}\bar{H}_\tau = \sigma y\sigma^T(\bar{a}_x\times\eta_o^{-\frac{1}{2}}\ \bar{E}_\tau) = \sigma y\eta_o^{-\frac{1}{2}}\ \bar{E}_\tau \qquad (3.2.4)$$

holds. Since U and z relate field components, there ought to be some relationship between them. To find this relationship assume a = c = 0 in (2.3.21), that is, two forward propagating waves characterized by transverse effective guide indices κ_1,κ_3 and two backward ones, characterized by $\kappa_2 = -\kappa_1$ and $\kappa_4 = -\kappa_3$. By use of (2.3.38) and (3.2.3), it then follows that

$$\begin{bmatrix} z \\ I \end{bmatrix} \sigma^{\tau} \eta_o^{\frac{1}{2}} \overline{H}_{\tau} = \begin{bmatrix} P_f & P_b \\ Q_f & Q_b \end{bmatrix} \begin{bmatrix} \Lambda_f & 0 \\ 0 & \Lambda_b \end{bmatrix} \begin{bmatrix} \overline{c}_f \\ \overline{c}_b \end{bmatrix}$$

which can be decomposed into

$$z\sigma^{\tau} \eta_o^{\frac{1}{2}} \overline{H}_{\tau} = [P_f \quad P_b] \begin{bmatrix} \Lambda_f & 0 \\ 0 & \Lambda_b \end{bmatrix} \begin{bmatrix} \overline{c}_f \\ \overline{c}_b \end{bmatrix} \qquad (3.2.5)$$

and

$$\sigma^{\tau} \eta_o^{\frac{1}{2}} \overline{H}_{\tau} = [Q_f \quad Q_b] \begin{bmatrix} \Lambda_f & 0 \\ 0 & \Lambda_b \end{bmatrix} \begin{bmatrix} \overline{c}_f \\ \overline{c}_b \end{bmatrix} \qquad (3.2.6)$$

Substitution of (3.2.6) into (3.2.5) yields

$$z[Q_f \quad Q_b] = [P_f \quad P_b]$$

from where, one obtains the definition of the forward and backword normalized wave impedance:

$$z_f = P_f\, Q_f^{-1} \quad \text{and} \quad z_b = P_b\, Q_b^{-1}, \qquad (3.2.7)$$

respectively. Taking the inverse of z_f and z_b, the wave admittance in terms of P and Q is given by

$$y_f = Q_f\, P_f^{-1} \quad \text{and} \quad y_b = Q_b\, P_b^{-1} . \qquad (3.2.8)$$

Finally, expansion of (3.2.7) and (3.2.8) using (2.3.38) gives the analytical expressions of z_f, z_b, y_f and y_b:

$$z_f = \frac{1}{\mathrm{Det}Q_f} (\overline{p}_{3\tau}\, \overline{q}_{1\tau}^T - \overline{p}_{1\tau}\, \overline{q}_{3\tau}^T) \qquad (3.2.9)$$

$$z_b = \frac{1}{DetQ_b} \left(\overline{p}_{4\tau}\,\overline{q}_{2\tau}^T - \overline{p}_{2\tau}\,\overline{q}_{4\tau}^T\right) \quad (3.2.10)$$

$$y_f = \frac{1}{DetP_f} \left(\sigma^T\,\overline{q}_{1\tau}\,\overline{p}_{3\tau}^T\sigma - \sigma^T\overline{q}_{3\tau}\,\overline{p}_{1\tau}^T\sigma\right) \quad (3.2.11)$$

$$y_b = \frac{1}{DetP_b} \left(\sigma^T\,\overline{q}_{2\tau}\,\overline{p}_{4\tau}^T\sigma - \sigma^T\overline{q}_{4\tau}\,\overline{p}_{2\tau}^T\sigma\right) \quad (3.2.12)$$

where $\overline{p}_{i\tau}$ and $\overline{q}_{i\tau}$ ($i = 1$ to 4) are defined by:

$$\overline{p}_i = \begin{bmatrix} 1 \\ p_{iz} \end{bmatrix} \quad , \quad \overline{q}_i = \begin{bmatrix} q_{iy} \\ q_{iz} \end{bmatrix} \quad ; i = 1 \text{ to } 4; \quad (3.2.13)$$

It is worthnoting that upon the definition of z_1, z_3, y_1 and y_3 matrices as:

$$z_1 = \frac{\overline{p}_{1\tau}\,\overline{q}_{3\tau}^T}{\overline{q}_{3\tau}^T\,\overline{p}_{1\tau}} \quad , \quad z_3 = \frac{\overline{p}_{3\tau}\,\overline{q}_{1\tau}^T}{\overline{q}_{1\tau}^T\,\overline{p}_{3\tau}} \quad (3.2.14)$$

$$y_1 = \frac{\sigma^T\,\overline{q}_{1\tau}\,\overline{p}_{3\tau}^T\sigma}{\overline{p}_{3\tau}^T\,\overline{q}_{1\tau}} \text{ and } y_3 = \frac{\sigma^T\,\overline{q}_{3\tau}\,\overline{p}_{1\tau}^T\sigma}{\overline{p}_{1\tau}^T\,\overline{q}_{3\tau}} \quad (3.2.15)$$

z_f and z_b' from (3.2.9) and (3.2.10) take the form:

$$z_f = \frac{\overline{q}_{1\tau}^T\,\overline{p}_{3\tau}}{DetQ_f}\,z_3 - \frac{\overline{q}_{3\tau}^T\,\overline{p}_{1\tau}}{DetQ_f}\,z_1 \quad (3.2.16)$$

and

$$z_b = \frac{\bar{p}_{3\tau}^T \bar{q}_{1\tau}}{\mathrm{Det}P_f} y_1 - \frac{\bar{p}_{1\tau}^T \bar{q}_{3\tau}}{\mathrm{Det}P_f} y_3 \qquad (3.2.17)$$

The properties of z_1, z_3, y_1 and y_3 are the following:

$$\mathrm{Det}z_1 = \mathrm{Det}z_3 = \mathrm{Det}y_1 = \mathrm{Det}y_3 = 0 \qquad (3.2.18)$$

$$\mathrm{Tr}z_1 = \mathrm{Tr}z_3 = \mathrm{Tr}y_1 = \mathrm{Tr}y_3 = 1 \qquad (3.2.19)$$

$$z_1 + y_1 = z_3 + y_3 = I \qquad (3.2.20)$$

$$z_i^2 = z_i \quad ; \quad i = 1, 3 \qquad (3.2.21a)$$

$$y_i^2 = y_i \quad ; \quad i = 1, 3 \qquad (3.2.21b)$$

where TrA is the trace of a matrix A.

The relations between z, y and P, Q can be used to express the partitioned form of the tilde transform of the modal matrix, i.e., $\tilde{U}$ and its inverse $\tilde{U}^{-1}$. Using (2.3.36), (3.2.7) and (3.2.8), one can show that

$$\tilde{U} = \begin{bmatrix} I & I \\ y_f & y_b \end{bmatrix} \begin{bmatrix} P_f & 0 \\ 0 & P_b \end{bmatrix} = \begin{bmatrix} z_f & z_b \\ I & I \end{bmatrix} \begin{bmatrix} Q_f & 0 \\ 0 & Q_b \end{bmatrix} \qquad (3.2.22)$$

Considering that

$$\begin{bmatrix} I & I \\ y_f & y_b \end{bmatrix}^{-1} = \begin{bmatrix} -\nu y_b & \nu \\ \nu y_f & -\nu \end{bmatrix} \text{ and } \begin{bmatrix} z_f & z_b \\ I & I \end{bmatrix}^{-1} = \begin{bmatrix} \zeta & -\zeta z_b \\ -\zeta & \zeta z_f \end{bmatrix} \qquad (3.2.23)$$

where

$$\nu = (y_f - y_b)^{-1} \quad \text{and} \quad \zeta = (z_f - z_b)^{-1} \qquad (3.2.24)$$

the inverse of $\tilde{U}$ takes the form

$$\tilde{U}^{-1} = \begin{bmatrix} P_f^{-1} & 0 \\ 0 & P_b^{-1} \end{bmatrix} \begin{bmatrix} -\nu \mathbf{y}_b & \nu \\ \nu \mathbf{y}_f & -\nu \end{bmatrix} = \begin{bmatrix} Q_f^{-1} & 0 \\ 0 & Q_b^{-1} \end{bmatrix} \begin{bmatrix} \zeta & -\zeta \mathbf{z}_b \\ -\zeta & \zeta \mathbf{z}_f \end{bmatrix} \quad (3.2.25)$$

Summarizing, in this section the wave impedance was defined in a similar manner as in the isotropic case and it was seen to have a matrix form, with off-diagonal terms accounting for the TE-TM mode coupling. In bidirectional wave propagation, the forward and backward wave impedances were expressed in terms of the components of the polarization vectors of the corresponding field distributions and were used to express the partitioned form of the tilde transform of the modal matrix and its inverse.

3.3 The Impedance Transfer Matrix

The transfer properties of $\overline{g}(x)$ along the x-direction of an anisotropic layer, are expressed by the impedance transfer matrix $G(x)$. As was seen in (2.3.31),

$$\overline{g}(x) = U \cdot \Lambda_g(x) \cdot \overline{c} \quad (3.3.1)$$

where $\overline{c}$ is the field amplitude vector at $x = 0$. At the origin,

$$\overline{g}(0) = U \cdot \overline{c} \quad (3.3.2)$$

and thus the expression between $\overline{g}(x)$ and $\overline{g}(0)$ takes the form

$$\overline{g}(x) = G(x) \cdot \overline{g}(0) \quad (3.3.3)$$

where

$$G(x) = U \cdot \Lambda_g(x) \cdot U^{-1} \quad (3.3.4)$$

is the transfer impedance matrix. Recalling from the definition of $\Lambda_g(x)$ that $\Lambda_g^{-1}(x) = \Lambda_g(-x)$,

$$G^{-1}(x) = G(-x). \tag{3.3.5}$$

Other forms of $G(x)$ can be found by taking the tilde transform of (3.3.4) and making use of equations (3.2.22) and (3.2.25) expressing $\tilde{U}$ and $\tilde{U}^{-1}$ in terms of $\mathbf{y}$, ν, $\mathbf{z}$ and ζ. The equivalent forms of $\tilde{G}(x)$ are

$$\tilde{G}(x) = \begin{bmatrix} J_f(x) & J_b(x) \\ \mathbf{y}_f J_f(x) & \mathbf{y}_b J_b(x) \end{bmatrix} \begin{bmatrix} -\nu \mathbf{y}_b & \nu \\ \nu \mathbf{y}_f & -\nu \end{bmatrix} \tag{3.3.6}$$

and

$$\tilde{G}(x) = \begin{bmatrix} \mathbf{z}_f F_f(x) & \mathbf{z}_b F_b(x) \\ F_f(x) & F_b(x) \end{bmatrix} \begin{bmatrix} \zeta & -\zeta \mathbf{z}_b \\ -\zeta & \zeta \mathbf{z}_f \end{bmatrix} \tag{3.3.7}$$

where the 2x2 matrices J and F are defined by:

$$J_f(x) = P_f \cdot \Lambda_f(x) \cdot P_f^{-1} \quad , \quad J_b(x) = P_b \cdot \Lambda_b(x) \cdot P_b^{-1} \tag{3.3.8}$$

$$F_f(x) = Q_f \cdot \Lambda_f(x) \cdot Q_f^{-1} \quad , \quad F_b(x) = Q_b \cdot \Lambda_b(x) \cdot Q_b^{-1} \tag{3.3.9}$$

Note that P_f is 2x2 modal matrix and $\Lambda_f(x)$ is the diagonalized form of $J_f(x)$. The inverse of J and F is

$$J_f^{-1}(x) = J_f(-x) \quad , \quad J_b^{-1}(x) = J_b(-x) \tag{3.3.10}$$

and

$$F_f^{-1}(x) = F_f(-x) \quad , \quad F_b^{-1}(x) = F_b(-x). \tag{3.3.11}$$

Furthermore, from (3.2.7) considering (3.3.8) and (3.3.9) the following symmetry relations are valid

$$J_f = z_f F_f y_f \qquad \text{and} \qquad J_b = z_b F_b y_b \tag{3.3.12}$$

Recall that the subscripts f and b pertain to forward and backward propagating modes, respectively, only when $\kappa_2 = -\kappa_1$ and $\kappa_4 = -\kappa_3$. The relation between the J,F matrices and the corresponding modes, is given by

$$J_f(x) = K_1 e^{-j\kappa_1 x} + K_3 e^{-j\kappa_3 x}$$

$$J_b(x) = K_2 e^{-j\kappa_2 x} + K_4 e^{-j\kappa_4 x} \tag{3.3.13}$$

and

$$F_f(x) = H_1 e^{-j\kappa_1 x} + H_3 e^{-j\kappa_3 x}$$

$$F_b(x) = H_2 e^{-j\kappa_2 x} + H_4 e^{-j\kappa_4 x} \tag{3.3.14}$$

where the analytical form of the 2x2 projectors K_i and H_i is:

$$K_1 = \frac{1}{\mathrm{Det}P_f} \bar{p}_{1\tau} \bar{p}_{3\tau}^T \sigma \quad , \quad K_2 = \frac{1}{\mathrm{Det}P_b} \bar{p}_{2\tau} \bar{p}_{4\tau}^T \sigma$$

$$K_3 = \frac{1}{\mathrm{Det}P_f} \bar{p}_{3\tau} \bar{p}_{1\tau}^T \sigma^T \quad , \quad K_4 = \frac{1}{\mathrm{Det}P_b} \bar{p}_{4\tau} \bar{p}_{2\tau}^T \sigma^T \tag{3.3.15}$$

and

$$H_1 = \frac{1}{\mathrm{Det}Q_f}\,\sigma\bar{q}_{1\tau}\,\bar{q}_{3\tau}^T \quad , \quad H_2 = \frac{1}{\mathrm{Det}Q_b}\,\sigma\,\bar{q}_{2\tau}\,\bar{q}_{4\tau}^T$$

$$H_3 = \frac{1}{\mathrm{Det}Q_f}\,\sigma^T\bar{q}_{3\tau}\,\bar{q}_{1\tau}^T \quad , \quad H_4 = \frac{1}{\mathrm{Det}Q_b}\,\sigma^T\bar{q}_{4\tau}\,\bar{q}_{2\tau}^T \qquad (3.3.16)$$

respectively. The projection matrices K_i and H_i ($i = 1$ to 4) satisfy the properties:

$$K_iK_j = H_iH_j = 0 \qquad (i \neq j) \qquad (3.3.17a)$$

$$K_i^2 = K_i \quad , \quad H_i^2 = H_i \qquad (3.3.17b)$$

$$\mathrm{Det}K_i = \mathrm{Det}H_i = 0 \qquad ;\ i = 1 \text{ to } 4 \qquad (3.3.17c)$$

$$K_1 + K_3 = K_2 + K_4 = H_1 + H_3 = H_2 + H_4 = I \qquad (3.3.17d)$$

The inter-relationship between J and F in (3.3.12) leads to a corresponding one between K and H, namely,

$$K_{1,3} = z_f H_{1,3} y_f \quad , \quad K_{2,4} = z_b H_{2,4} y_b \qquad (3.3.18)$$

Finally, one can relate the forms of K,H with respect to $\bar{p}_{i\tau}$, $\bar{q}_{i\tau}$ ($i = 1,3$), with the z_i, y_i ($i = 1,3$) arising from the impedance and admittance matrices. By use of (3.2.22), (3.2.23) and (3.3.15), (3.3.16) the following relations are valid

$$z_i K_i = K_i y_i = K_i \quad , \quad H_i z_i = y_i H_i = H_i \tag{3.3.19a}$$

$$z_i H_i = K_i z_i = z_i \quad , \quad y_i K_i = H_i y_i = y_i \tag{3.3.19b}$$

$$z_i H_j = H_i y_j = y_i K_j = K_i z_j = 0 \quad , \quad (i \neq j) \tag{3.3.19 c}$$

where $i,j = 1,3$. The rest of the products between z,y and K,H, i.e., $z_i K_j$, $K_i y_j$, $y_i H_j$ and $H_i z_j$ $(i \neq j)$ are nonzero. The symmetry of the relations (3.3.18) is demonstrated in tables 3.2 and 3.3.

3.4 Conservation Principles

This section is devoted to establish conditions for losslessness, reciprocity and symmetry pertaining to an anisotropic layer. These conditions will be expressed in terms of the coupling matrix R_g and the terminal parameter transfer matrix $G(x)$.

An anisotropic layer can be viewed as a four port, distributed in the x direction, as shown in Fig. 3.2. In this Fig. two sets of terminal parameters, namely, the by now well established $\overline{g}(x)$ basis and the new basis

$$\overline{a}(x) = [a_1^+(x) \;\; a_1^-(x) \;\; a_2^+(x) \;\; a_2^-(x)]^T . \tag{3.4.1a}$$

These are related via the linear transformation

$$\overline{g}(x) = \Omega \overline{a}(x) \tag{3.4.1b}$$

where Ω and its inverse is given by

	K_1	K_3	H_1	H_3
z_1	K_1	*	z_1	0
z_3	*	K_3	0	z_3
y_1	y_1	0	H_1	*
y_3	0	y_3	*	H_3

Table 3.2

	z_1	z_3	y_1	y_3
K_1	z_1	0	K_1	*
K_3	0	z_3	*	K_3
H_1	H_1	*	y_1	0
H_3	*	H_3	0	y_3

Table 3.3

The above Tables demonstrate the symmetry of equations (3.3.19). The quantities of the first column premultiply the corresponding ones of the first row and the product is shown in the corresponding column-row intersection. The asterisks have the meaning of nonzero product which cannot be expressed in a simple form. The diagonal entries are the same as the corresponding of the first row, whereas the non-zero, non-starred off-diagonal are the same as the corresponding in the first column. The asterisks of the one table are zeros of the other and vice versa.

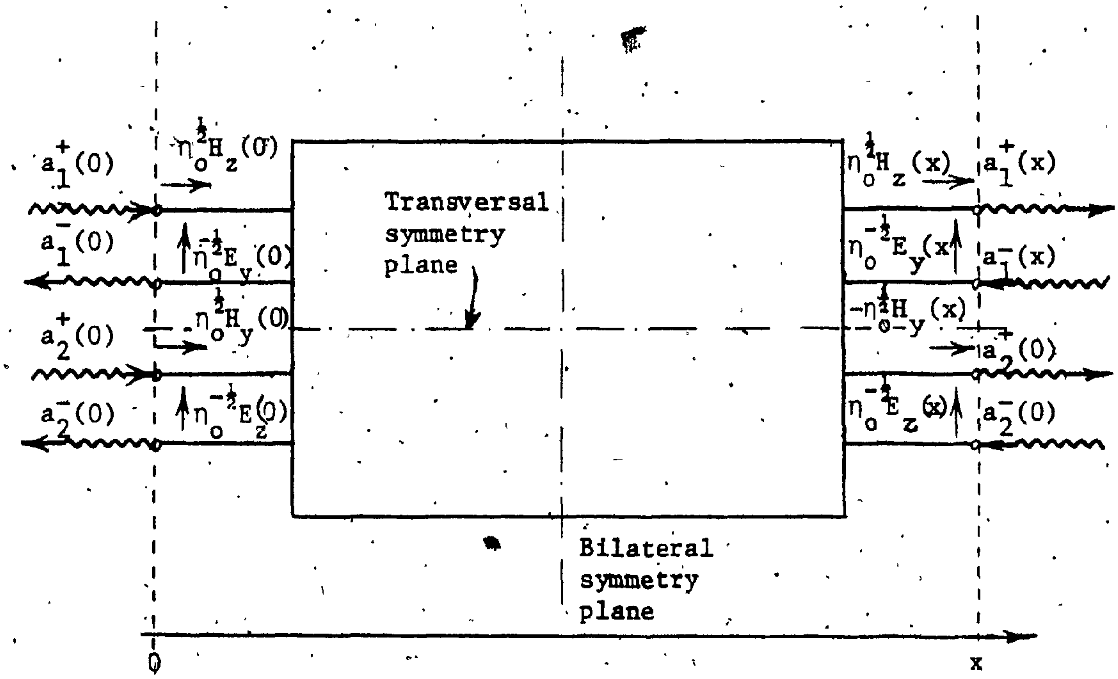

Fig. 3.2 Network representation of an anisotropic layer of (normalized) length x. Due to continuity of the field components at the boundary, the $\overline{g}(x)$ and $\overline{a}(x)$ vectors are the same at both sides of each interface and the network is cascadable.

$$\Omega = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{Z_1} & \sqrt{Z_1} & 0 & 0 \\ \frac{1}{\sqrt{Z_1}} & -\frac{1}{\sqrt{Z_1}} & 0 & 0 \\ 0 & 0 & \sqrt{Z_2} & \sqrt{Z_2} \\ 0 & 0 & \frac{1}{\sqrt{Z_2}} & -\frac{1}{\sqrt{Z_2}} \end{bmatrix}, \quad \Omega^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{Z_1}} & \sqrt{Z_1} & 0 & 0 \\ \frac{1}{\sqrt{Z_1}} & -\sqrt{Z_1} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{Z_2}} & \sqrt{Z_2} \\ 0 & 0 & \frac{1}{\sqrt{Z_2}} & -\sqrt{Z_2} \end{bmatrix} \qquad (3.4.2)$$

and Z_1 and Z_2 are suitable normalized line impedances. The reason of choosing the Ω in this form is to emulate the normalized ~~travelling~~ wave representation which establishes a relationship between incident and reflected waves on the one hand, with normalized voltage and current on the other.

The corresponding coupled mode equation of the state vector $\overline{a}(x)$, resulting from substitution of (3.4.3) into (3.4.1), has the form

$$\frac{d}{dx}\,\overline{a}(x) = -jR_a\overline{a}(x) \qquad (3.4.3)$$

with

$$R_a = \Omega^{-1}R_g\Omega \qquad (3.4.4)$$

The latter means that coupling matrices R_a and R_g are similar, that is, the eigenvalues of R_a are the same as those of R_g given by (2.3.20) while the modal matrix of R_a is $\Omega^{-1}U$ [22]. The analytical form of R_a is

$$R_a = \frac{1}{2}\begin{bmatrix} \frac{1}{Z_1} + Z_1R_{21} & -\frac{1}{Z_1} + Z_1R_{21} \\ \frac{1}{Z_1} - Z_1R_{21} & -\frac{1}{Z_1} - Z_1R_{21} \\ \sqrt{\frac{Z_1}{Z_2}}R_{31} + \sqrt{Z_1Z_2}\,R_{41} & \sqrt{\frac{Z_1}{Z_2}}R_{31} + \sqrt{Z_1Z_2}\,R_{41} \\ \sqrt{\frac{Z_1}{Z_2}}R_{31} - \sqrt{Z_1Z_2}\,R_{41} & \sqrt{\frac{Z_1}{Z_2}}R_{31} - \sqrt{Z_1Z_2}\,R_{41} \end{bmatrix}$$

$$\left.\begin{matrix} \sqrt{\frac{Z_1}{Z_2}}\ R_{31}^{*} + \sqrt{Z_1 Z_2}\ R_{41}^{*} & -\sqrt{\frac{Z_1}{Z_2}}\ R_{31}^{*} + \sqrt{Z_1 Z_2}\ R_{41}^{*} \\ -\sqrt{\frac{Z_1}{Z_2}}\ R_{31}^{*} - \sqrt{Z_1 Z_2}\ R_{41}^{*} & \sqrt{\frac{Z_1}{Z_2}}\ R_{31}^{*} - \sqrt{Z_1 Z_2}\ R_{41}^{*} \\ 2\mathrm{Re}(R_{33}) + \frac{R_{34}}{Z_2} + Z_2 R_{43} & 2j\mathrm{Im}(R_{33}) - \frac{R_{34}}{Z_2} + Z_2 R_{43} \\ 2j\mathrm{Im}(R_{33}) + \frac{R_{34}}{Z_2} - Z_2 R_{43} & 2\mathrm{Re}(R_{33}) - \frac{R_{34}}{Z_2} - Z_2 R_{43} \end{matrix}\right] \tag{3.4.5}$$

Note that the $(R_a)_{12}$, $(R_a)_{21}$, $(R_a)_{34}$ and $(R_a)_{43}$ entry refer to reflected waves in the homogeneous medium and must therefore vanish. This constraint is used to determine the arbitrary normalized wave impedances Z_1 and Z_2. Setting $(R_a)_{12}$ to zero yields: $Z_1 = \pm(R_{21})^{-\frac{1}{2}}$ while setting $(R_a)_{34}$ to zero yields: $Z_2 = \sqrt{R_{34}/R_{43}}$ provided that $\mathrm{Im}(R_{33}) = 0$. If $\mathrm{Im}(R_{33}) \neq 0$, such as in the case of Faraday rotation in a uniaxial dielectric in the equatorial configuration, one cannot obtain a real Z_2 indicating reactive TM mode power flow in the axial direction.

The corresponding impedance transfer matrix $M(x)$ of the state vector $\overline{a}(x)$ is found by substituting (3.4.3) into (3.3.3), i.e.,

$$\overline{a}(x) = M(x)\overline{a}(0). \tag{3.4.6}$$

$M(x)$ is found to be related to $G(x)$ via the similarity transformation

$$M(x) = \Omega^{-1} G(x)\Omega. \tag{3.4.7}$$

Having introduced the R_g, R_a, G and M matrices of the anisotropic layer, one can then examine the conditions for losslessness, reciprocity and symmetry.

(1) Losslessness

The condition of losslessness in terms of R_g, R_a and M is expressed by [23]

$$R_g = \Sigma_2 R_g^+ \Sigma_2 \quad (3.4.8a)$$

$$R_a = \Sigma_1 R_a^+ \Sigma_1 \quad (3.4.8b)$$

and

$$M^{-1} = \Sigma_1 M^+ \Sigma_1 \quad (3.4.8c)$$

where Σ_i are the Dirac matrices given by the Kronecker product

$$\Sigma_i = I_2 \times \sigma_i \quad ; \quad i = 0 \text{ to } 3, \quad (3.4.9)$$

and σ_i (i = 0 to 3) are 2x2 Pauli matrices defined by

$$\sigma_o = I_2 \quad , \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \sigma_3 = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} . \quad (3.4.10)$$

Recalling (3.4.7), the losslessness condition for the impedance transfer matrix G is

$$G^{-1} = \Sigma_2 G^+ \Sigma_2 \quad (3.4.11a)$$

or in expanded form

$$\begin{bmatrix} G_{11}(-x) & G_{13}(-x) & G_{12}(-x) & G_{14}(-x) \\ G_{31}(-x) & G_{33}(-x) & G_{32}(-x) & G_{34}(-x) \\ G_{21}(-x) & G_{23}(-x) & G_{22}(-x) & G_{24}(-x) \\ G_{41}(-x) & G_{43}(-x) & G_{42}(-x) & G_{44}(-x) \end{bmatrix} = \begin{bmatrix} G^*_{22}(x) & G^*_{42}(x) & G^*_{12}(x) & G^*_{32}(x) \\ G^*_{24}(x) & G^*_{44}(x) & G^*_{14}(x) & G^*_{34}(x) \\ G^*_{21}(x) & G^*_{41}(x) & G^*_{11}(x) & G^*_{31}(x) \\ G^*_{23}(x) & G^*_{43}(x) & G^*_{13}(x) & G^*_{33}(x) \end{bmatrix} \quad (3.4.11b)$$

where note has been taken of the fact that $\Omega\Sigma_1\Omega^+ = \Sigma_2$ and that $G^{-1}(x) = G(-x)$.

(2) <u>Reciprocity</u>

The condition of reciprocity originates from Loretz' reciprocity theorem, according to which the response of a system to a source is unchanged when source and measurer are interchanged. In the case of layered structures, the validity of this theorem depends on the optic axis orientation and the kind of intrinsic activity characterizing the crystal.

Mathematically, the reciprocity condition is expressed by [23]

$$R_g = - \Sigma_3 R_g^T \Sigma_3 \quad (3.4.12a)$$

$$R_a = - \Sigma_3 R_a^T \Sigma_3 \quad (3.4.12b)$$

and

$$M^{-1} = \Sigma_3 M^T \Sigma_3 \quad . \quad (3.4.12a)$$

Recalling (3.4.7), the reciprocity condition in terms of G is

$$G^{-1} = \Sigma_3 G^T \Sigma_3 \quad (3.4.13a)$$

or in expanded form

$$\begin{bmatrix} G_{11}(-x) & G_{13}(-x) & G_{12}(-x) & G_{14}(-x) \\ G_{31}(-x) & G_{33}(-x) & G_{32}(-x) & G_{34}(-x) \\ G_{21}(-x) & G_{23}(-x) & G_{22}(-x) & G_{24}(-x) \\ G_{41}(-x) & G_{43}(-x) & G_{42}(-x) & G_{44}(-x) \end{bmatrix} = \begin{bmatrix} G_{22}(x) & G_{42}(x) & -G_{12}(x) & -G_{32}(x) \\ G_{24}(x) & G_{44}(x) & -G_{14}(x) & -G_{34}(x) \\ -G_{21}(x) & -G_{41}(x) & G_{11}(x) & G_{31}(x) \\ -G_{23}(x) & -G_{43}(x) & G_{13}(x) & G_{33}(x) \end{bmatrix} \quad (3.4.13b)$$

where the identity $\Omega\Sigma_3\Omega^T = -\Sigma_3$ has been used.

(3) <u>Antireciprocity</u>

In the case a network represents an anisotropic medium whose dielectric tensor is non-symmetric, Lorentz theorem is no longer valid and its equivalent network is non-reciprocal [24]. Antireciprocal networks form a special category of non-reciprocal ones, which from the circuit point of view are composed by at least one gyrator, a device characterized by an antisymmetric impedance matrix.

The condition of antireciprocity in terms of R_g, R_a and M is expressed by [23]

$$R_g = -\Sigma_2 R_g^T \Sigma_2 \quad (3.4.13a)$$

$$R_a = -\Sigma_1 R_a^T \Sigma_1 \quad (3.4.13b)$$

$$M^{-1} = \Sigma_1 M^T \Sigma_1 \quad (3.4.13c)$$

and from (3.4.7) in terms of G by

$$G^{-1} = \Sigma_2 G^T \Sigma_2 \tag{3.4.15a}$$

or in expanded form

$$\begin{bmatrix} G_{11}(-x) & G_{12}(-x) & G_{13}(-x) & G_{14}(-x) \\ G_{21}(-x) & G_{22}(-x) & G_{23}(-x) & G_{24}(-x) \\ G_{31}(-x) & G_{32}(-x) & G_{33}(-x) & G_{34}(-x) \\ G_{41}(-x) & G_{42}(-x) & G_{43}(-x) & G_{44}(-x) \end{bmatrix} = \begin{bmatrix} G_{22}(x) & G_{12}(x) & G_{42}(x) & G_{32}(x) \\ G_{21}(x) & G_{11}(x) & G_{41}(x) & G_{31}(x) \\ G_{24}(x) & G_{14}(x) & G_{44}(x) & G_{44}(x) \\ G_{23}(x) & G_{13}(x) & G_{43}(x) & G_{43}(x) \end{bmatrix} \tag{3.4.15b}$$

where use has been made of the identity: $\Omega\Sigma_1\Omega^T = \Sigma_2$.

(4) Semireciprocity

Semireciprocity is defined for a 2n-port network through a Z matrix satisfying the relationship $Z_{ji} = (-1)^{i+j} Z_{ij}$. In the case of a four-port this reduces to : $Z = \Sigma_1 Z^T \Sigma_1$ [23]. The semireciprocity condition can be realized by a series connection of two four-ports, one representing a reciprocal and the other an antireciprocal medium.

Semireciprocity requires the following conditions to be satisfied

$$R_g = \Xi_1 \Sigma_3 R_g^T \Sigma_3 \Xi_1 \tag{3.4.16a}$$

$$R_a = -\Xi_1\Sigma_3 R_a^T \Sigma_3 \Xi_1 \qquad (3.4.16b)$$

$$M^{-1} = \Xi_1\Sigma_3 M^T \Sigma_3 \Xi_1 \qquad (3.4.16c)$$

and

$$G^{-1} = \Xi_1\Sigma_3 G^T \Sigma_3 \Xi_1 \qquad (3.4.17a)$$

or in expanded form

$$\begin{bmatrix} G_{11}(-x) & G_{13}(-x) & G_{12}(-x) & G_{14}(-x) \\ G_{31}(-x) & G_{33}(-x) & G_{32}(-x) & G_{34}(-x) \\ G_{21}(-x) & G_{23}(-x) & G_{22}(-x) & G_{24}(-x) \\ G_{41}(-x) & G_{43}(-x) & G_{42}(-x) & G_{44}(-x) \end{bmatrix} = \begin{bmatrix} G_{22}(x) & -G_{42}(x) & -G_{12}(x) & G_{32}(x) \\ -G_{24}(x) & G_{44}(x) & G_{14}(x) & -G_{34}(x) \\ -G_{21}(x) & G_{41}(x) & G_{11}(x) & -G_{31}(x) \\ G_{23}(x) & -G_{43}(x) & -G_{13}(x) & G_{33}(x) \end{bmatrix}$$

(3.4.17b)

where Ξ_1 is one of the Dirac matrices defined by the Kronecker product

$$\Xi_i = \sigma_i \times I_2 \qquad ; \qquad i = 0 \text{ to } 3.$$

Note that in deriving (3.4.17a) from (3.4.16) and (3.4.7), the identity $\Omega\,\Xi_1\Sigma_3\Omega^T = \Xi_1$ has been used.

(5) Bilateral Symmetry

A network is bilaterally symmetric when its transfer properties remain invariant under inversion with respect to the bilateral symmetry plane, that is, the plane parallel to the interfaces and equidistant from them, as shown in Fig. 3.2.

The condition of bilateral symmetry is expressed by [23]

$$R_a = -\Sigma_1 R_g \Sigma_1 \qquad (3.4.18a)$$

$$R_a = -\Sigma_2 R_a \Sigma_2 \qquad (3.4.18b)$$

$$M^{-1} = \Sigma_2 M \Sigma_2 \qquad (3.4.18c)$$

and by use of the identity: $\Omega\Sigma_2\Omega^{-1} = \Sigma_1$, in terms of G by

$$G^{-1} = \Sigma_1 G \Sigma_1 \qquad (3.4.19a)$$

or by the expanded version

$$\begin{bmatrix} G_{11}(-x) & G_{13}(-x) & G_{12}(-x) & G_{14}(-x) \\ G_{31}(-x) & G_{33}(-x) & G_{32}(-x) & G_{34}(-x) \\ G_{21}(-x) & G_{23}(-x) & G_{22}(-x) & G_{24}(-x) \\ G_{41}(-x) & G_{43}(-x) & G_{42}(-x) & G_{44}(-x) \end{bmatrix} = \begin{bmatrix} G_{11}(x) & G_{13}(x) & -G_{12}(x) & -G_{14}(x) \\ G_{31}(x) & G_{33}(x) & -G_{32}(x) & -G_{34}(x) \\ -G_{21}(x) & -G_{23}(x) & G_{22}(x) & G_{24}(x) \\ -G_{41}(x) & -G_{43}(x) & G_{42}(x) & G_{44}(x) \end{bmatrix}$$

(3.4.19b)

(6) Transversal Symmetry

A transversally symmetric network, in analogy to the bilaterally symmetric, is one whose transfer properties remain invariant under inversion with respect to a transverse symmetry plane, as shown in

Fig. 3.2. Note that an important requirement for transversal symmetry is: $Z_1 = Z_2$. Thus, if $\mathrm{Im}(R_{33}) = 0$, recalling (3.4.6)

$$R_{21} = R_{43}/R_{34} \tag{3.4.20}$$

has to be satisfied.

The condition of transversal symmetry is expressed by [23]

$$R_g = \Xi_2 R_g \Xi_2 \tag{3.4.21a}$$

$$R_a = \Xi_2 R_a \Xi_2 \tag{3.4.21b}$$

$$M = \Xi_2 M \Xi_2 \tag{3.4.21c}$$

and using (3.4.7) and the identity $\Omega\, \Xi_2 \Omega^{-1} = \Xi_2$, in terms of G by

$$G = \Xi_2 G \Xi_2 \tag{3.4.22a}$$

or

$$\begin{bmatrix} G_{11}(x) & G_{13}(x) & G_{12}(x) & G_{14}(x) \\ G_{31}(x) & G_{33}(x) & G_{32}(x) & G_{34}(x) \\ G_{21}(x) & G_{23}(x) & G_{22}(x) & G_{24}(x) \\ G_{41}(x) & G_{43}(x) & G_{42}(x) & G_{44}(x) \end{bmatrix} = \begin{bmatrix} G_{33}(x) & G_{31}(x) & G_{34}(x) & G_{32}(x) \\ G_{13}(x) & G_{11}(x) & G_{14}(x) & G_{12}(x) \\ G_{43}(x) & G_{41}(x) & G_{44}(x) & G_{42}(x) \\ G_{23}(x) & G_{21}(x) & G_{24}(x) & G_{22}(x) \end{bmatrix} \tag{3.4.22b}$$

In this section, conditions for losslessness, reciprocity and symmetry for a layer of anisotropic medium were given in terms of its terminal matrix and system coupling matrix representation.

3.5 Reflection, Transmission and the Input Impedance of a Homogeneous Layer

The concepts of transmission and reflection coefficient and of the input impedance are well known in microwave device analysis utilizing isotropic media. In this section these concepts will be generalized to encompass anisotropic dielectrics.

Consider an interface referenced to x, between media 1 and 2, as shown in Fig. 3.3. Due to the continuity of the transverse field components, $\bar{g}(x^-) = \bar{g}(x^+)$. If the field amplitude vector in medium 1: $\bar{c}(1)$ is referenced to x^- and that of medium 2 : $\bar{c}(2)$ is referenced to x^+, the above boundary condition leads to

$$\bar{c}(1) = U^{-1}(1)U(2)\bar{c}(2) \tag{3.5.1a}$$

or in other words

$$\begin{bmatrix} \bar{c}_f(1) \\ \bar{c}_b(1) \end{bmatrix} = \tilde{U}^{-1}(1)\tilde{U}(2) \begin{bmatrix} \bar{c}_f(2) \\ \bar{c}_b(2) \end{bmatrix} \tag{3.5.1b}$$

where

$$\tilde{U}^{-1}(1)\tilde{U}(2) = \begin{bmatrix} P_f^{-1}(1)\nu(1)[\mathbf{y}_f(2)-\mathbf{y}_b(1)]P_f(2) & P_f^{-1}(1)\nu(1)[\mathbf{y}_f(2)-\mathbf{y}_b(1)]P_f(2) \\ P_b^{-1}(1)\nu(1)[\mathbf{y}_f(1)-\mathbf{y}_f(2)]P_f(2) & P_b^{-1}(1)\nu(1)[\mathbf{y}_f(1)-\mathbf{y}_b(2)]P_b(2) \end{bmatrix}$$

(3.5.2)

Similarly to the isotropic case, the field-amplitude reflection and transmission coefficient matrix is defined by

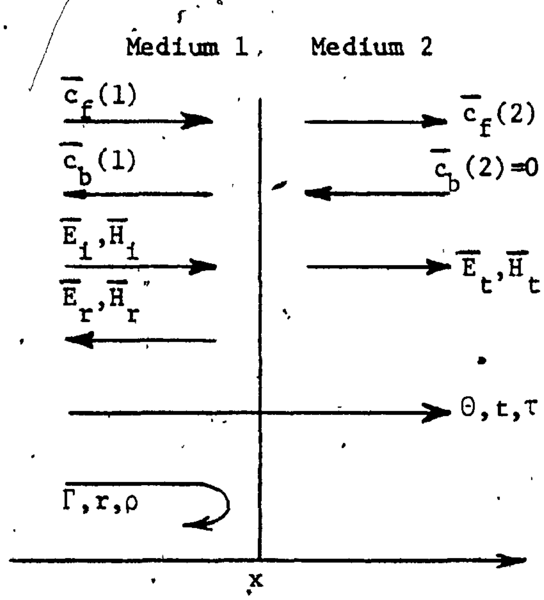

Fig. 3.3 An interface of two anisotropic layers. The arrows indicate the direction of propagation of the transverse field.

$$\overline{c}_b(1) = r\overline{c}_f(1) \tag{3.5.3}$$

and

$$\overline{c}_f(2) = t\overline{c}_f(1) \tag{3.5.4}$$

respectively. Using r and t in (3.5.1b), taking into account (3.5.2) and noting that $\overline{c}_b(2) = 0$, the following forms for r and t can be derived:

$$t = \{P_f^{-1}(1)\nu(1)[y_f(2)-y_b(1)]P_f(2)\}^{-1} \tag{3.5.5}$$

$$r = \{P_f^{-1}(1)\nu(1)[y_f(1)-y_f(2)]P_f(2)\}^{-1} \tag{3.5.6}$$

The (electric) field reflection and transmission coefficient matrix is similarly defined:

$$\eta_o^{-\frac{1}{2}} E_{\tau r} = \Gamma(\eta_o^{-\frac{1}{2}} \overline{E}_{\tau i}) \tag{3.5.7}$$

and

$$\eta_\Theta^{-\frac{1}{2}}\,\overline{E}_{\tau t} = \Theta(\eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau i}) \quad (3.5.8)$$

where the transcripts τ, i, r and t refer to transverse component and incident, reflected and transmitted field respectively. The boundary conditions require on the one hand that $\eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau i} + \eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau r} = \eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau t}$, that is,

$$I + \Gamma = \Theta \quad (3.5.9)$$

and on the other, that $\eta_o^{\frac{1}{2}}\,\overline{H}_{\tau i} + \eta_o^{\frac{1}{2}}\,\overline{H}_{\tau r} = \eta_o^{\frac{1}{2}}\,\overline{H}_{\tau t}$, where $\eta_o^{\frac{1}{2}}\,\overline{H}_{\tau i} = \sigma\mathbf{y}_f(1)\eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau i}$, $\eta_o^{\frac{1}{2}}\,\overline{H}_{\tau r} = \sigma\mathbf{y}_b(1)\eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau r}$ and $\eta_o^{\frac{1}{2}}\,\overline{H}_{\tau t} = \sigma\mathbf{y}_f(2)\eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau f}$. The latter yields

$$\mathbf{y}_f(1) + \mathbf{y}_b(1)\Gamma = \mathbf{y}_f(2)\Theta \quad (3.5.10)$$

The last two equations when solved with respect to Γ and θ result in expressions that are seen to be straightforward generalizations of their isotropic equivalent:

$$\Gamma = [\mathbf{y}_f(2) - \mathbf{y}_b(1)]^{-1}\,[\mathbf{y}_f(1) - \mathbf{y}_f(2)] \quad (3.5.11)$$

$$\Theta = [\mathbf{y}_f(2) - \mathbf{y}_b(1)]^{-1}\,[\mathbf{y}_f(1) - \mathbf{y}_b(1)]. \quad (3.5.12)$$

Finally, the inter-relationship between r, t and Γ, Θ resulting from the combination of (3.5.5), (3.5.6) and (3.5.11), (3.5.12) is given by

$$r = P_b^{-1}(1)\Gamma P_f(1) \quad , \quad \Gamma = P_b(1) r P_f^{-1}(1) \tag{3.5.13}$$

and

$$t = P_f^{-1}(2)\Theta P_f(1) \qquad \Theta = P_f(2) t P_f(1) \tag{3.5.14}$$

The reflection coefficient expression (3.5.11) is valid also when the medium to the right of the interface is a composite rather than a uniform semi-infinite region. In that case $y_f(2)$ must be replaced by $\vec{y}$, the surface admittance as seen from x^+ looking in the positive x direction. Thus

$$\vec{\Gamma} = [\vec{y} - y_b(1)]^{-1}[y_f(1) - \vec{y}] \tag{3.5.15}$$

In addition to $\vec{\Gamma}$ one also needs $\overleftarrow{\Gamma}$, defined in Fig. 3.4. To obtain $\overleftarrow{\Gamma}$ one simply replaces the arrows, the subscripts and the media designators as required:

$$\overleftarrow{\Gamma} = [\overleftarrow{y} - y_f(2)]^{-1}[y_b(2) - \overleftarrow{y}] \tag{3.5.16}$$

Solving (3.5.15) with respect to $\vec{y}$, one obtains:

$$\vec{y} = [y_f(1) + y_b(1)\vec{\Gamma}][I + \vec{\Gamma}]^{-1} \tag{3.5.17}$$

The corresponding expression for z is found using (3.5.11) and taking into account that $(y_f - y_b)^{-1} = z_f(z_b - z_f)^{-1}z_b$. Thus

$$\vec{z} = [I + \vec{\Gamma}]z_f(1)[z_b(1) + \vec{\Gamma}z_f(1)]^{-1}z_b(1). \tag{3.5.18}$$

The last two equations give the input admittance and impedance of a homogeneous medium, in a multilayered structure.

Finally, reflection and transmission can also be expressed in terms of the state vector $\overline{a}$, by defining

$$\overline{a}_b(1) = \rho\overline{a}_f(1) \tag{3.5.19}$$

and

$$\overline{a}_f(2) = \tau\overline{a}_f(1) \tag{3.5.20}$$

where ρ and τ are the wave-reflection and transmission matrices. The expression relating ρ, τ and Γ is found from (3.4.1b):

$$\tilde{\Omega} \begin{bmatrix} \overline{a}_f \\ \overline{a}_b \end{bmatrix} = \begin{bmatrix} \eta_o^{-\frac{1}{2}} \overline{E}_\tau \\ y\eta_o^{\frac{1}{2}} \overline{H}_\tau \end{bmatrix} \tag{3.5.21}$$

and (3.2.5):

$$\begin{bmatrix} \eta_o^{-\frac{1}{2}} \overline{E}_\tau \\ \sigma^T\eta_o^{\frac{1}{2}} \overline{H}_\tau \end{bmatrix} = \begin{bmatrix} I \\ y \end{bmatrix} \eta_o^{-\frac{1}{2}} \overline{E}_\tau \tag{3.5.22}$$

which result in

$$\tilde{\Omega} \begin{bmatrix} I \\ \vec{\rho} \end{bmatrix} \overline{a}_f(1) = \begin{bmatrix} I \\ \vec{y} \end{bmatrix} \eta_o^{-\frac{1}{2}} \overline{E}_\tau \tag{3.5.23}$$

After decomposition into 2x2 block form this equation reduces to

$$\tilde{\Omega}_{11} [I + \vec{\rho}]\overline{a}_f(1) = [I + \Gamma]\eta_o^{-\frac{1}{2}} \overline{E}_{\tau i} \tag{3.5.24}$$

and

$$\tilde{\Omega}_{21}[I - \vec{\rho}]\,\overline{a}_f(1) = \vec{y}[I + \Gamma]\eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau i} \qquad (3.5.25)$$

respectively, where $\tilde{\Omega}_{11}$ and $\tilde{\Omega}_{21}$ are defined from the partitioned $\tilde{\Omega}$:

$$\tilde{\Omega} = \begin{bmatrix} \tilde{\Omega}_{11} & \tilde{\Omega}_{11} \\ \tilde{\Omega}_{21} & -\tilde{\Omega}_{21} \end{bmatrix} \quad ; \quad \tilde{\Omega}^{-1} = \begin{bmatrix} \tilde{\Omega}_{21} & \tilde{\Omega}_{11} \\ \tilde{\Omega}_{21} & -\tilde{\Omega}_{11} \end{bmatrix} \qquad (3.5.26)$$

and are given by

$$\tilde{\Omega}_{11} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{Z_1} & 0 \\ 0 & \sqrt{Z_2} \end{bmatrix} \quad , \quad \tilde{\Omega}_{21} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{Z_1}} & 0 \\ 0 & \frac{1}{\sqrt{Z_2}} \end{bmatrix} \qquad (3.5.27)$$

The expression of $\vec{\rho}$ is found from (3.5,24) and (3.5.25) to be

$$\vec{\rho} = [\tilde{\Omega}_{21}(1) + \vec{y}\tilde{\Omega}_{11}(1)]^{-1}[\tilde{\Omega}_{21}(1) - \vec{y}\tilde{\Omega}_{11}(1)] \qquad (3.5.28)$$

and the corresponding expression of $\vec{y}$ with respect to $\vec{\rho}$

$$\vec{y} = \tilde{\Omega}_{21}(1)[I - \vec{\rho}]\,[I + \vec{\rho}]^{-1}\,\tilde{\Omega}_{11}^{-1}(1) \qquad (3.5.29)$$

Finally, to find the expression of $\vec{\tau}$, one has to rewrite (3.5.21) and (3.5.22) for region 2, i.e.

$$\tilde{\Omega}(2)\begin{bmatrix} \vec{\tau} \\ 0 \end{bmatrix}\overline{a}_f(1) = \begin{bmatrix} I \\ \vec{y} \end{bmatrix}\eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau t}$$

and solve for τ, to find

$$\vec{\tau} = \tilde{\Omega}_{11}^{-1}(2)\tilde{\Omega}_{11}(1)[I + \vec{\rho}] \tag{3.5.30}$$

Equations (3.5.28) and (3.5.30) express the wave reflection and transmission matrices $\vec{\rho}$ and $\vec{\tau}$ at an interface characterized by a right admittance $\vec{y}$.

3.6 Transverse Resonance

In the beginning of this chapter the transverse resonance condition was introduced and used to obtain the characteristic equation of isotropic layered structures. In this section the transverse resonance principle will be extended and applied to layered anisotropic media.

Consider a layered birefringent medium, as shown in Fig. 3.4, with interfaces at x = 0 and x = ℓ. Each one of the three regions in Fig. 3.4 may be a composite, or a single uniform anisotropic (or isotropic) layer. The state at x = 0 and ℓ can be described by the $\bar{c}$, the $\bar{g}$ or the $\bar{a}$ vector. Accordingly, the transverse resonance principle can be formulated in terms of three different transfer matrices.

As was shown before, the following transfer properties for a single layer are valid:

$$\bar{c}(\ell) = \Lambda_g(\ell)\bar{c}(0)$$

$$\bar{g}(\ell) = G(\ell)\bar{g}(0)$$

$$\bar{a}(\ell) = M(\ell)\bar{a}(0) \tag{3.6.1}$$

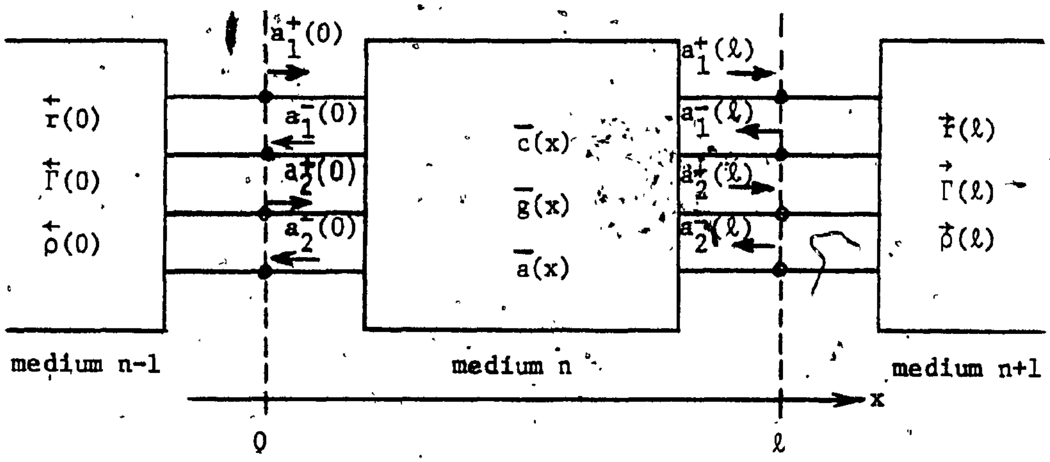

Fig. 3.4 Network representation of a medium, characterized by $\bar{c}(x)$, $\bar{g}(x)$, $\bar{a}(x)$. The left interface ($x = 0$) results in the introduction of $\overleftarrow{r}(0)$, $\overleftarrow{\Gamma}(0)$, $\overleftarrow{\rho}(0)$ and the right one ($x = \ell$) in $\vec{r}(\ell)$, $\vec{\Gamma}(\ell)$, $\vec{\rho}(\ell)$ respectively.

In the case the four-port network represents two homogeneous regions $n-1$ and n of widths ℓ_1 and ℓ_2 ($\ell_1 + \ell_2 = \ell$) respectively, recalling that

$$\bar{g}_n(\ell) = U(n)\Lambda_{g,n}(\ell)\bar{c}(\ell_1) = U(n)\bar{c}(\ell)$$

$$\bar{g}_n(\ell_1) = \bar{g}_{n-1}(\ell_1) = U(n-1)\Lambda_{g,n-1}(\ell_1)\bar{c}(0) = U(n)\bar{c}(\ell_1)$$

$$\bar{g}_n(\ell) = G(n)\bar{g}_n(\ell_1) = G(n)G(n-1)\bar{g}_{n-1}(0)$$

$$\bar{a}_n(\ell_1) = \Omega^{-1}(n)\bar{g}_{n-1}(\ell_1) = \Omega^{-1}(n)G_n(n-1)\Omega(n-1)\bar{a}_{n-1}(0),$$

(3.6.2)

the corresponding transfer matrices take the form:

$$C(\ell) = \Lambda_g(n)U^{-1}(n)U(n-1)\Lambda_g(n-1)$$

$$G(\ell) = U(n)\Lambda_{g,n}(\ell_2)U(n-1)\Lambda_{g,n-1}(\ell_1)U^{-1}(n-1)$$

$$M(\ell) = \Omega^{-1}(n)G(n)G(n-1)\Omega(n-1), \qquad (3.6.3)$$

where $C(x)$ is the generalized propagation factor matrix $\Lambda_g(x)$.

(1) TRC in terms of the $\bar{c}$ vector

The tilde transform of $C(\ell)$ satisfies the following equation:

$$\begin{bmatrix} \bar{c}_f(\ell) \\ \bar{c}_b(\ell) \end{bmatrix} = \begin{bmatrix} \tilde{C}_{ff}(\ell) & \tilde{C}_{fb}(\ell) \\ \tilde{C}_{bf}(\ell) & \tilde{C}_{bb}(\ell) \end{bmatrix} \begin{bmatrix} \bar{c}_f(0) \\ \bar{c}_b(0) \end{bmatrix} \qquad (3.6.4)$$

Recalling (3.5.3) the $\bar{c}_b(\ell)$ and $\bar{c}_f(0)$ can be written as:

$$\bar{c}_b(\ell) = \vec{r}(\ell)\bar{c}_f(\ell) \qquad (3.6.5)$$

and

$$\bar{c}_f(0) = \overleftarrow{r}(0)\bar{c}_b(0) \qquad (3.6.6)$$

Substitution of $\bar{c}_b(\ell)$ and $\bar{c}_f(0)$ in (3.6.4), results in:

$$\vec{r}(\ell)[\tilde{C}_{ff}(\ell)\overleftarrow{r}(0) + \tilde{C}_{fb}(\ell)]\bar{c}_b(0) = [\tilde{C}_{bf}(\ell)\overleftarrow{r}(0) + \tilde{C}_{bb}(\ell)]\bar{c}_b(0) \qquad (3.6.7)$$

The condition for nontrivial solution, i.e.,

$$\mathrm{Det}[\vec{r}(\ell)\tilde{C}_{ff}(\ell)\overleftarrow{r}(0) + \vec{r}(\ell)\tilde{C}_{fb}(\ell) - \tilde{C}_{bf}(\ell)\overleftarrow{r}(0) - \tilde{C}_{bb}(\ell)] = 0 \qquad (3.6.8)$$

expresses the transverse resonance in terms of the $\bar{c}$ vector.

(2) TRC in terms of the g vector

To find the corresponding expression working with the field vector $\overline{g}(x)$, one has to recall from (3.5.7) that

$$\eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau f}(0^+) = \overleftarrow{\Gamma}(0)\cdot\eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau r}(0^+)$$

$$\eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau r}(\ell^-) = \overrightarrow{\Gamma}(\ell)\cdot\eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau f}(\ell^-),$$

where from (3.5.15) and (3.5.16)

$$\overleftarrow{\Gamma}(0) = [\overleftarrow{y}(0) - y_f(2)]^{-1}\,[y_b(2) - \overleftarrow{y}(0)] \qquad (3.6.9)$$

$$\overrightarrow{\Gamma}(\ell) = [\overrightarrow{y}(\ell) - y_b(2)]^{-1}\,[y_f(2) - \overrightarrow{y}(\ell)]. \qquad (3.6.10)$$

The tilde transform of (3.3.3), in view of (3.6.9) and (3.6.10) results in

$$\begin{bmatrix} I + \overrightarrow{\Gamma}(\ell) \\ \overrightarrow{y}(\ell)[I + \overrightarrow{\Gamma}(\ell)] \end{bmatrix} \eta_\rho^{-\frac{1}{2}}\,\overline{E}_{\tau f}(\ell) = \tilde{G}(\ell) \begin{bmatrix} I + \overleftarrow{\Gamma}(0) \\ \overleftarrow{y}(0)[I + \overleftarrow{\Gamma}(0)] \end{bmatrix} \eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau b}(0).$$

(3.6.11)

Matrix $\tilde{G}(\ell)$ can be decomposed in the 2x2 block form

$$\tilde{G}(\ell) = \begin{bmatrix} \tilde{G}_{11}(\ell) & \tilde{G}_{12}(\ell) \\ \tilde{G}_{21}(\ell) & \tilde{G}_{22}(\ell) \end{bmatrix}, \qquad (3.6.12)$$

substitution of which in (3.6.11), yields

$$\vec{\theta}(\ell)\cdot\eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau f}(\ell) = [\tilde{G}_{11}(\ell) + \tilde{G}_{12}(\ell)\overleftarrow{y}(0)]\overleftarrow{\theta}(0)\cdot\eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau b}(0)$$

and

$$\vec{y}(\ell)\vec{\theta}(\ell)\cdot\eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau f}(\ell) = [\tilde{G}_{21}(\ell) + \tilde{G}_{22}(\ell)\overleftarrow{y}(0)]\overleftarrow{\theta}(0)\cdot\eta_o^{-\frac{1}{2}}\,\overline{E}_{\tau b}(0).$$

Note that each one of these represents a single-trip between the interfaces. In order the field distribution reproduce itself after a complete round trip:

$$\vec{y}(\ell)[\tilde{G}_{11}(\ell) + \tilde{G}_{12}(\ell)\overleftarrow{y}(0)] = \tilde{G}_{21}(\ell) + \tilde{G}_{22}(\ell)\overleftarrow{y}(0) \qquad (3.6.13)$$

has to be satisfied. Thus the TRC takes the form

$$\mathrm{Det}\{\vec{y}(\ell)[\tilde{G}_{11}(\ell) + \tilde{G}_{12}(\ell)\overleftarrow{y}(0)] - [\tilde{G}_{21}(\ell) + \tilde{G}_{22}(\ell)\overleftarrow{y}(0)]\} = 0. \qquad (3.6.14)$$

(3) <u>TRC in terms of the $\overline{a}$ ve[illegible]</u>

The procedure is similar when waves instead of fields are used. Recall from (3.4.10) and (3.5.19) that

$$\begin{bmatrix} I \\ \vec{\rho}(\ell) \end{bmatrix} \overline{a}_f(\ell) = \begin{bmatrix} \tilde{M}_{ff}(\ell) & \tilde{M}_{fb}(\ell) \\ \tilde{M}_{bf}(\ell) & \tilde{M}_{bb}(\ell) \end{bmatrix} \begin{bmatrix} \overleftarrow{\rho}(0) \\ I \end{bmatrix} \overline{a}_b(0)$$

(3.6.15)

where $\tilde{M}(\ell)$ refers to medium 2, and from (3.5.28)

$$\overleftarrow{\rho}(0) = [\tilde{\Omega}_{21}(2) + \overleftarrow{y}(0)\tilde{\Omega}_{11}(2)]^{-1}\,[\tilde{\Omega}_{21}(2) - \overleftarrow{y}\,\tilde{\Omega}_{22}(2)] \qquad (3.6.16)$$

and

$$\vec{\rho}(\ell) = [\tilde{\Omega}_{21}(2) + \vec{y}(\ell)\tilde{\Omega}_{11}(2)]^{-1}\,[\tilde{\Omega}_{21}(2) - \vec{y}(\ell)\tilde{\Omega}_{22}(2)] \qquad (3.6.17)$$

Matrix equation (3.6.15) now is decomposed in the forms

$$\overline{a}_f(\ell) = [\tilde{M}_{ff}(\ell)\overleftarrow{\rho}(0) + \tilde{M}_{fb}(\ell)]\overline{a}_b(0)$$

and

$$\overrightarrow{\rho}(\ell)\overline{a}_f(\ell) = [\tilde{M}_{bf}(\ell)\overleftarrow{\rho}(0) + \tilde{M}_{bb}(\ell)]\overline{a}_b(0)$$

which, for round-trip field reproduction, result in

$$\overrightarrow{\rho}(\ell)[\tilde{M}_{ff}(\ell)\overleftarrow{\rho}(0) + \tilde{M}_{fb}(\ell)] = \tilde{M}_{bf}(\ell)\overleftarrow{\rho}(0) + \tilde{M}_{bb}(\ell) \quad (3.6.18)$$

Hence, the corresponding TRC is of the form

$$\mathrm{Det}\{\overrightarrow{\rho}(\ell)[\tilde{M}_{ff}(\ell)\overleftarrow{\rho}(0) + \tilde{M}_{fb}(\ell)] - [\tilde{M}_{bf}(\ell)\overleftarrow{\rho}(0) + \tilde{M}_{bb}(\ell)]\} = 0 \quad (3.6.19)$$

Equations (3.6.8), (3.6.14) and (3.6.19) mathematically state the same principle, expressed in terms of $\overline{c}$, $\overline{g}$ and $\overline{a}$. It should be noted that the dispersion equation obtained by use of either expression, is identical, that is, all three forms are equivalent.

In this chapter, wave propagation in a lossless symmetric isotropic slab waveguide was reviewed. The characteristic equation was obtained using the TRC in the transmission line equivalent of the isotropic film. The concepts of the wave impedance and field transfer matrices were introduced and expressions of the latter in terms of the forward/backward wave impedance/admittance under wave bidirectionality were investigated. Mode bidirectionality in birefringent media lead to a 4-port network representation of a layer. Conditions for losslessness, reciprocity and symmetry were formulated in terms of the coupling and transfer matrices. Finally, a generalized TRC for a multilayered anisotropic structure has been introduced and the characteristic equation of the waveguide has been derived.

CHAPTER 4

APPLICATIONS

In this chapter wave propagation in lossless uniaxial and quasi-uniaxial anisotropic media will be analyzed. Two further applications, namely liquid crystal twist cells and determination of the permittivity matrix of a crystal from the electric reflection coefficient matrix will also be outlined.

In uniaxial media the optic axis will be assumed to lie in one of the coordinate planes of the device, i.e., that the medium is either in the polar, the longitudinal or the equatorial configuration. The permittivity tensor of a biaxial medium rotated around one of its principal axes can also be brought into a uniaxial configuration. The treatment of this chapter includes this quasi-uniaxial case. Since the off-diagonal elements of the permittivity matrix are generally complex, Faraday rotation is one of the phenomena to which this analysis applies. Nonlinear effects, such as the electro-optic effect where elements of $\bar{\bar{\varepsilon}}$ depend explicitly on the electric field strength, or optical activity where the elements of $\bar{\bar{\varepsilon}}$ depend on the axial wavenumber β have not been explicitly investigated.

Finally, the results are applied to three layered symmetric waveguides fabricated using four different combinations of anisotropic dielectrics. The transmission characteristics of inhomogeneous uniaxial crystals are examined and an application on the determination of the permittivity tensor of biaxial crystals is discussed.

4.1 Polar Configuration

In polar configuration the optic axis lies in the y-z plane, as shown in Fig.4.1. The relative permittivity tensor is the same as in the case of a biaxial medium that suffered a rotation around its crystalline x-axis and is given by

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & \varepsilon_{yz} \\ 0 & \varepsilon^*_{yz} & \varepsilon_{zz} \end{bmatrix} \tag{4.1.1}$$

This can be seen from the matrix product

$$\bar{\bar{\varepsilon}} = A \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} A^T \tag{4.1.2}$$

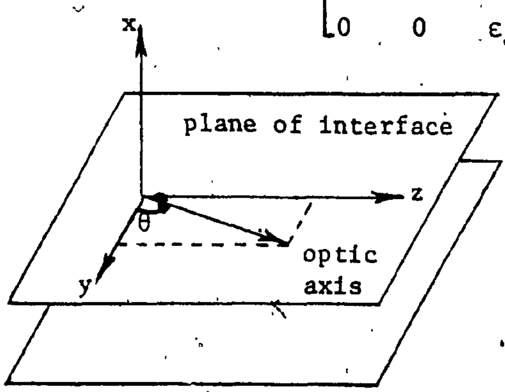

Fig. 4.1. Polar configuration. Optic axis lies on the plane of interface, forming an angle θ with the y axis.

where the rotation matrix A is evaluated from (2.3.9) by taking the optic axis to lie along the y' direction, that is, θ be the angle between z and z' and $\phi=\psi=0$ (see Fig.2.7). Thus

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{4.1.3}$$

By allowing $\bar{\bar{\varepsilon}}$ to be complex while restricting it to be Hermitean, one includes a larger class of anisotropy, such as that causing Faraday rotation and -potentially- the nonlinear effects:optical activity and electro-optic effect.

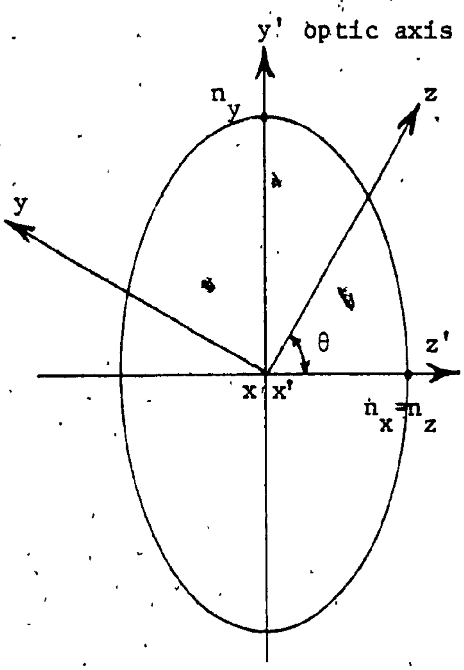

Fig. 4.2. Intersection of the index ellipsoid of a uniaxial crystal with the interface (plane y-z). The direction of propagation is along z and the optic axis the y' axis. Since the equatorial plane is circle: $\varepsilon_x = \varepsilon_z$.

The coupling matrix R_g given in (2.3.16) after the substitution of $\bar{\bar{\varepsilon}}$ takes the form

$$R_g = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \varepsilon_{yy}-\beta^2 & 0 & \varepsilon_{yz} & 0 \\ 0 & 0 & 0 & 1-\beta^2/\varepsilon_{xx} \\ \varepsilon_{yz}^* & 0 & \varepsilon_{zz} & 0 \end{bmatrix} \qquad (4.1.4)$$

where ε_{ij} (i,j=x,y,z) refer to the entries of $\bar{\bar{\varepsilon}}$. Since $R_{33}=R_{31}=0$, the fourth order characteristic equation (2.3.21a) reduces in a quadratic in κ^2, whose solution is given by

$$\kappa^2_{1,3}=\frac{1}{2}(R_{34}R_{43}+R_{21})\pm\frac{1}{2}\sqrt{(R_{34}R_{43}-R_{21})^2+4R_{34}|R_{41}|^2} \qquad (4.1.5)$$

where sub-index 1(3) refers to the upper(lower) sign. Note that $\kappa_2=-\kappa_1$ and $\kappa_4=-\kappa_3$, that is, the polar configuration assures wave bidirectionality. The modal matrix U, following (2.3.23) and (2.3.25), is written in the form

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \kappa_1 & -\kappa_1 & \kappa_3 & -\kappa_3 \\ h_1 & h_1 & h_3 & h_3 \\ h_1\kappa_1 a & -h_1\kappa_1 a & h_3\kappa_3 a & -h_3\kappa_3 a \end{bmatrix} \qquad (4.1.6)$$

where $a=1/R_{34}$ is a real number,

$$h_i=\frac{R_{41}}{aD_i} \qquad (4.1.7)$$

and D_i is the denominator of the eigenmode expressions in (2.3.32), given by

$$D_i=\kappa_i^2-R_{43}/a \qquad (4.1.8)$$

and satisfying

$$D_1D_3=-|R_{41}|^2/a \qquad (4.1.9)$$

Due to bidirectionality and to the form of D_i, the corresponding h's of the modes characterized by κ_2 and κ_4 are h_1 and h_3 respectively. Parameters h_1 and h_3 satisfy

$$h_1-h_3=\frac{1}{R_{41}^*}(\kappa_1^2-\kappa_3^2) \qquad (4.1.10)$$

and

$$h_1h_3=-\frac{R_{41}}{R_{41}^*}\frac{1}{a} \quad , \qquad h_1h_3^*=h_1^*h_3=-\frac{1}{a} \qquad (4.1.11)$$

i.e., $h_1h_3^*=h_1^*h_3$ is a real number. The inverse of the modal matrix U is

$$U^{-1}=\frac{1}{2(h_3-h_1)}\begin{bmatrix} h_3 & h_3/\kappa_1 & -1 & -1/\kappa_1 a \\ h_3 & -h_3/\kappa_1 & -1 & 1/\kappa_1 a \\ -h_1 & -h_1/\kappa_3 & 1 & 1/\kappa_3 a \\ -h_1 & h_1/\kappa_3 & 1 & -1/\kappa_3 a \end{bmatrix} \qquad (4.1.12)$$

and thus the impedance matrix G, given by $U\Lambda_g(x)U^{-1}$, takes the form

$$G(x) = \frac{1}{(h_3-h_1)} \begin{bmatrix} h_3\cos\kappa_1x-h_1\cos\kappa_3x & -j(\frac{h_3}{\kappa_1}\sin\kappa_1x-\frac{h_1}{\kappa_3}\sin\kappa_3x) & -(\cos\kappa_1x-\cos\kappa_3x) & \frac{j}{a}(\frac{1}{\kappa_1}\sin\kappa_1x-\frac{1}{\kappa_3}\sin\kappa_3x) \\ -j(h_3\kappa_1\sin\kappa_1x-h_1\kappa_3\sin\kappa_3x) & h_3\cos\kappa_1x-h_1\cos\kappa_3x & j(\kappa_1\sin\kappa_1x-\kappa_3\sin\kappa_3x) & -\frac{1}{a}(\cos\kappa_1x-\cos\kappa_3x) \\ h_1h_3(\cos\kappa_1x-\cos\kappa_3x) & -jh_1h_3(\frac{1}{\kappa_1}\sin\kappa_1x-\frac{1}{\kappa_3}\sin\kappa_3x) & -(h_1\cos\kappa_1x-h_3\cos\kappa_3x) & \frac{j}{a}(\frac{h_1}{\kappa_1}\sin\kappa_1x-\frac{h_3}{\kappa_3}\sin\kappa_3x) \\ -jh_1h_3a(\kappa_1\sin\kappa_1x-\kappa_3\sin\kappa_3x) & h_1h_3a(\cos\kappa_1x-\cos\kappa_3x) & ja(h_1\kappa_1\sin\kappa_1x-h_3\kappa_3\sin\kappa_3x) & -(h_1\cos\kappa_1x-h_3\cos\kappa_3x) \end{bmatrix} \tag{4.1.13}$$

Recalling that $\bar{g}(x)=G(x)\bar{g}(0)$, note from the form of $G(x)$ that while all parameters are normalized, κ_i is "similar" to an admittance whereas h_i and a are similar to a unitless parameter. The form of G satisfies (3.4.11b) provided κ_i is not complex (with nonzero real and imaginary parts), and (3.4.19b). Note that when ε_{yz} is purely real: $h_1h_3a = -1$ and G satisfies (3.4.13b), whereas when ε_{yz} is purely imaginary, $h_1h_3a = 1$ and G satisfies (3.4.17b). Hence an active anisotropic layer whose optic axis orientation corresponds to the polar configuration is lossless, bilaterally symmetric and in the case of real off-diagonal terms of $\bar{\bar{\varepsilon}}$: reciprocal, whereas in the case of imaginary off-diagonal terms: semireciprocal.

Taking now the tilde transform of the modal matrix U

$$\tilde{U} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ h_1 & h_3 & h_1 & h_3 \\ \kappa_1 & \kappa_3 & -\kappa_1 & -\kappa_3 \\ h_1\kappa_1 a & h_3\kappa_3 a & -h_1\kappa_1 a & -h_3\kappa_3 a \end{bmatrix} \qquad (4.1.14)$$

it is seen that $P_f=P_b$, $Q_f=-Q_b$ and hence from the definition of $z_{f,b}$, i.e., $z_{f,b}=P_{f,b}Q_{f,b}^{-1}$

$$z_b=-z_f \qquad (4.1.15)$$

where the wave impedance z_f is

$$z_f=\frac{1}{a\kappa_1\kappa_3(h_3-h_1)}\begin{bmatrix} a(h_3\kappa_3-h_1\kappa_1) & \kappa_1-\kappa_3 \\ ah_1h_3(\kappa_3-\kappa_1) & h_3\kappa_1-h_1\kappa_3 \end{bmatrix} \qquad (4.1.16)$$

In the case of real R_{41}, that is, real ε_{yz}, z_f assumes symmetric form. Finally, taking into account the form of $\bar{\bar{\varepsilon}}$ in the wave equation (2.3.28b) and expanding the cofactors of the second row, the polarization vector $\bar{p}$ assumes the form

$$\bar{p} = \begin{bmatrix} -\kappa\beta ha/\varepsilon_{xx} \\ 1 \\ h \end{bmatrix} \qquad (4.1.17)$$

The corresponding form of $\bar{q}$ obtains from the expression $\bar{q}=(\bar{k}\times\bar{q})/k_o$, where $\bar{k}$ is the propagation vector. It is found that

$$\bar{q} = \begin{bmatrix} -\beta \\ -\kappa\beta a \\ \kappa \end{bmatrix} \qquad (4.1.18)$$

With $\bar{p}$ and $\bar{q}$ known, the electromagnetic field distribution corresponding to the polar configuration can be found using (2.3.30). The field distribution is the result of distributed coupling between the two forward eigenmodes, as indicated by (4.1.5), with field amplitudes given by the components of the polarization vectors, and their corresponding backward ones.

Coupling of the waves propagating in a polar crystal is described by the coupling matrix R_a, whose form was given in (3.4.5). Taking into account that $R_{31}=R_{33}=0$, R_a now assumes the form

$$R_a=\begin{bmatrix} \sqrt{R_{21}} & 0 & \frac{1}{2}\sqrt{Z_1Z_2}R_{41}^* & \frac{1}{2}\sqrt{Z_1Z_2}R_{41}^* \\ 0 & -\sqrt{R_{21}} & -\frac{1}{2}\sqrt{Z_1Z_2}R_{41}^* & -\frac{1}{2}\sqrt{Z_1Z_2}R_{41}^* \\ \frac{1}{2}\sqrt{Z_1Z_2}R_{41} & \frac{1}{2}\sqrt{Z_1Z_2}R_{41} & \sqrt{R_{34}R_{43}} & 0 \\ -\frac{1}{2}\sqrt{Z_1Z_2}R_{41} & -\frac{1}{2}\sqrt{Z_1Z_2}R_{41} & 0 & -\sqrt{R_{34}R_{43}} \end{bmatrix} \tag{4.1.19}$$

where the normalized line impedances Z_1 and Z_2 are

$$Z_1=(\varepsilon_{yy}-\beta^2)^{-\frac{1}{2}} \quad \text{and} \quad Z_2=\left[(1-\frac{\beta^2}{\varepsilon_{xx}})/\varepsilon_{zz}\right]^{\frac{1}{2}} \tag{4.1.20}$$

The diagonal entries of R_a are the axial guide indices of the uncoupled transmission lines {36}, whereas the off-diagonal entries are the forward and backward coupling coefficients, given by

$$\kappa_f=\kappa_b=\frac{1}{2}R_{41}^*(Z_1Z_2)^{\frac{1}{2}} \tag{4.1.21}$$

From their wavenumber and characteristic impedance an equivalent circuit

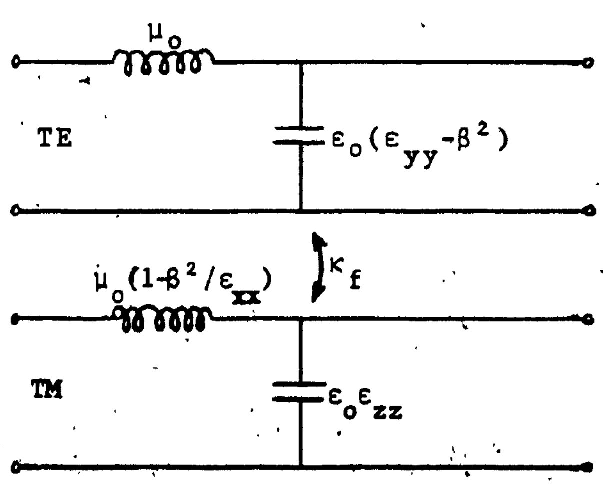

Fig. 4.3. Equivalent 4-port network of an anisotropic crystal in the polar configuration.

can be established for each transmission line. The result is shown in Fig. 4.3 indicating the per unit length inductance and capacitance of the dimensionless coupling coefficient.

4.2 Longitudinal Configuration

One speakes of a longitudinal configuration when the optic axis of a uniaxial crystal lies in the plane normal to the direction of propagation (x-y plane), or when a biaxial medium has been rotated about its crystalline z axis.

The permittivity tensor $\bar{\bar{\varepsilon}}$ results from the product

$$\bar{\bar{\varepsilon}} = A \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} A^T \qquad (4.2.1)$$

where the rotation matrix A is found from (2.3.9) by taking the optic axis to be along x', with $\psi=\theta=0$. Thus

$$A = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (4.2.2)$$

where the angle ϕ is measured from the x axis. A ccordingly, the permittivity tensor $\bar{\bar{\varepsilon}}$ will assume the general form:

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon^*_{xy} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \qquad (4.2.3)$$

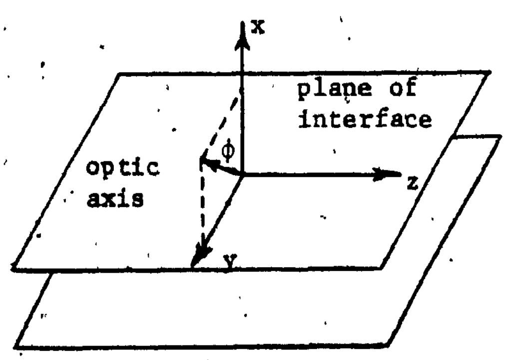

Fig. 4.4 Longitudinal configuration. Optic axis lies on x-y plane, forming an angle θ with the x axis.

where the off-diagonal entries account for Faraday rotation, optical activity and the electro-optic effect.

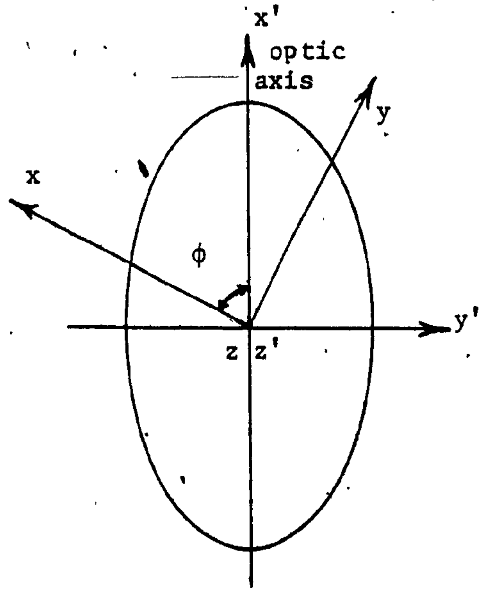

Fig. 4.5 Intersection of the index ellipsoid of a uniaxial crystal with the x-y plane of the device's coordinate system. The direction of propagation is along z and the optic axis along x'. Since the equatorial plane is circle : $\varepsilon_y=\varepsilon_z$.

The coupling matrix R_g is found from substitution of $\bar{\bar{\varepsilon}}$ into (2.3.16) to be :

$$R_g = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \Delta_{zz}/\varepsilon_{xx}-\beta^2 & 0 & 0 & -\beta\varepsilon^*_{xy}/\varepsilon_{xx} \\ -\beta\varepsilon_{xy}/\varepsilon_{xx} & 0 & 0 & 1-\beta^2/\varepsilon_{xx} \\ 0 & 0 & \varepsilon_{zz} & 0 \end{bmatrix} \qquad (4.2.4)$$

Due to $R_{33}=R_{41}=0$, the characteristic equation (2.3.21a) is again a quadratic in κ^2 equation. In other words, the eigenmodes in a crystal of longitudinal configuration are two forward, characterized by κ_1 and κ_3, and two backward, characterized by $\kappa_2=-\kappa_2$ and $\kappa_4=-\kappa_3$. The transverse effective guide indices κ_1 and κ_3 are given by

$$\kappa^2_{1,3}=\frac{1}{2}(R_{34}R_{43}+R_{21})\pm\frac{1}{2}\sqrt{(R_{34}R_{43}-R_{21})^2+4R_{43}|R_{31}|^2} \qquad (4.2.5)$$

where the plus (minus) sign refers to 1 (3). With R_g and κ's known, the modal matrix is found to be :

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \kappa_1 & -\kappa_1 & \kappa_3 & -\kappa_3 \\ g_1\kappa_1 b & -g_1\kappa_1 b & g_3\kappa_3 b & -g_3\kappa_3 b \\ g_1 & g_1 & g_3 & g_3 \end{bmatrix} \tag{4.2.6}$$

where b is a real number, defined as $b=1/R_{43}$,

$$g_i = \frac{R_{31}}{bD_i} \quad ; \; i=1,3 \tag{4.2.8}$$

and D_i, the denominator of the eigenmode solutions (2.3.23), is given by

$$D_i = \kappa_i^2 - R_{34}/b \tag{4.2.8}$$

Note that D_i and g_i satisfy the following properties

$$D_1 D_3 = -|R_{31}|^2/b \tag{4.2.9}$$

$$g_1 - g_3 = (\kappa_1^2 - \kappa_3^2)/R_{31}^* \tag{4.2.10}$$

and

$$g_1 g_3 = -\frac{R_{31}}{bR_{31}^*}, \qquad g_1 g_3^* = g_1^* g_3 = -1/b \tag{4.2.11}$$

To find the impedance transfer matrix G, one has first to evaluate U^{-1}.

This, is found to be

$$U^{-1} = \frac{\quad}{2(g_1-g_1)} \begin{bmatrix} g_3 & g_3/\kappa_1 & -1/\kappa_1 b & -1 \\ g_3 & -g_3/\kappa_1 & 1/\kappa_1 b & -1 \\ -g_1 & -g_1/\kappa_3 & 1/\kappa_3 b & 1 \\ -g_1 & g_1/\kappa_3 & -1/\kappa_3 b & 1 \end{bmatrix} \tag{4.2.12}$$

and hence

$$G(x) = \frac{1}{g_3 - g_1} \begin{bmatrix} g_3\cos\kappa_1 x - g_1\cos\kappa_3 x & -j\left(\frac{g_3}{\kappa_1}\sin\kappa_1 x - \frac{g_1}{\kappa_3}\sin\kappa_3 x\right) \\ -j(g_3\kappa_1\sin\kappa_1 x - g_1\kappa_3\sin\kappa_3 x & g_3\cos\kappa_1 x - g_1\cos\kappa_3 x \\ -jg_1g_3 b(\kappa_1\sin\kappa_1 b - \kappa_3\sin\kappa_3 x) & g_1g_3 b(\cos\kappa_1 x - \cos\kappa_3 x) \\ g_1g_3(\cos\kappa_1 x - \cos\kappa_3 x) & -jg_1g_3\left(\frac{\sin\kappa_1 x}{\kappa_1} - \frac{\sin\kappa_3 x}{\kappa_3}\right) \end{bmatrix}$$

$$\begin{bmatrix} \frac{j}{b}\left(\frac{1}{\kappa_1}\sin\kappa_1 x - \frac{1}{\kappa_3}\sin\kappa_3 x\right) & -\cos\kappa_1 x+\cos\kappa_3 x \\ \frac{-1}{b}(\cos\kappa_1 x - \cos\kappa_3 x) & j(\kappa_1\sin\kappa_1 x-\kappa_3\sin\kappa_3 x) \\ -(g_1\cos\kappa_1 x - g_3\cos\kappa_3 x) & jb(g_1\kappa_1\sin\kappa_1 x-g_3\kappa_3\sin\kappa_3 x) \\ \frac{j}{b}\left(\frac{g_1}{\kappa_1}\sin\kappa_1 x - \frac{g_3}{\kappa_3}\sin\kappa_3 x\right) & -(g_1\cos\kappa_1 x-g_3\cos\kappa_3 x) \end{bmatrix} \quad (4.2.13)$$

Comparing this form with the corresponding ones of Section 3.4, it is found that a crystal in longitudinal configuration is lossless, not bilaterally symmetric and semireciprocal (reciprocal) when ε_{xy} **is real (imaginary).**

The expressions of the forward and backward wave impendances occur from $\tilde{U}$, where

$$\tilde{U} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ g_1\kappa_1 b & g_3\kappa_3 b & -g_1\kappa_1 b & -g_3\kappa_3 b \\ \kappa_1 & \kappa_3 & -\kappa_1 & -\kappa_3 \\ g_1 & g_3 & g_1 & g_3 \end{bmatrix} \quad (4.2.14)$$

i.e., $P_b=\sigma_1 P_f$ and $Q_b=-\sigma_1 Q_f$ where σ_1 is a Pauli matrix defined in (3.4.21). Hence

$$z_b=-\sigma_1 z_f \sigma_1 \quad (4.2.15)$$

where the forward wave impedance is given by

$$z_f = \frac{1}{\kappa_1 g_3-\kappa_3 g_1}\begin{bmatrix} g_3-g_1 & \kappa_1 - \kappa_3 \\ \frac{R_{31}}{R_{31}^*}(\kappa_3-\kappa_1) & \kappa_1\kappa_3 b(g_3-g_1) \end{bmatrix} \quad (4.2.16)$$

Finally, the polarization vector $\overline{p}$, expanding the cofactors of the second row of the normalized wave equation (2.3.30), is

$$\overline{p} = \begin{bmatrix} g(b\kappa^2-1)/\beta \\ 1 \\ gb\kappa \end{bmatrix} \tag{4.2.17}$$

and $\overline{q}$, from expression (2.3.31), is

$$\overline{q} = \begin{bmatrix} -\beta \\ -g \\ \kappa \end{bmatrix} \tag{4.2.18}$$

Accordingly, the electromagnetic field distribution in the longitudinal configuration can be found using (2.3.30) by properly choosing the coefficients c_i.

The coupling matrix R_a of an anisotropic layer in longitudinal orientation is

$$R_a = \begin{bmatrix} \sqrt{R_{21}} & 0 & \frac{1}{2}\sqrt{\frac{Z_1}{Z_2}}R_{31}^* & -\frac{1}{2}\sqrt{\frac{Z_1}{Z_2}}R_{31}^* \\ 0 & -\sqrt{R_{21}} & -\frac{1}{2}\sqrt{\frac{Z_1}{Z_2}}R_{31}^* & \frac{1}{2}\sqrt{\frac{Z_1}{Z_2}}R_{31}^* \\ \frac{1}{2}\sqrt{\frac{Z_1}{Z_2}}R_{31} & \frac{1}{2}\sqrt{\frac{Z_1}{Z_2}}R_{31} & \sqrt{R_{34}R_{43}} & 0 \\ \frac{1}{2}\sqrt{\frac{Z_1}{Z_2}}R_{31} & \frac{1}{2}\sqrt{\frac{Z_1}{Z_2}}R_{31} & 0 & -\sqrt{R_{34}R_{43}} \end{bmatrix} \tag{4.2.19}$$

where the normalized line impedances are given by

$$Z_1 = \left(\frac{\Delta_{zz}}{\varepsilon_{xx}} - \beta^2\right)^{-\frac{1}{2}} \quad \text{and} \quad Z_2 = \left[\left(1 - \frac{\beta^2}{\varepsilon_{xx}}\right)/\varepsilon_{zz}\right]^{\frac{1}{2}} \tag{4.2.20}$$

The forward and backward coupling coefficients are equal in magnitude and are given by

$$\kappa_f = -\kappa_b^* = \frac{1}{2}R_{31}^*(Z_1 Z_2)^{\frac{1}{2}} \tag{4.2.21}$$

From considerations similar to those applied to the polar case an equiva-

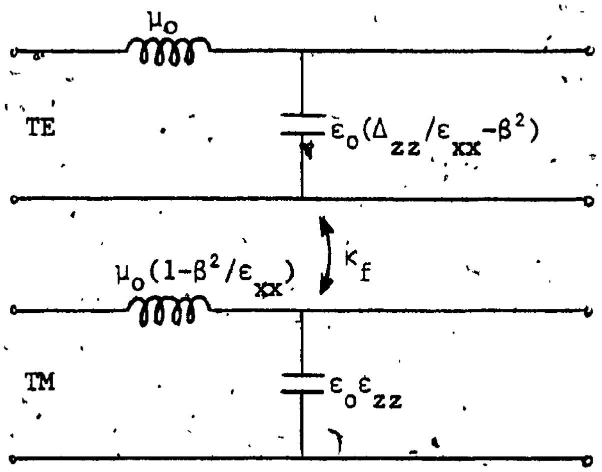

Fig. 4.6. Equivalent 4-port network of an anisotropic crystal in the longitudinal configuration.

lent circuit is obtained. The result is depicted in Fig. 4.6.

4.3 Equatorial Configuration

In what is called the equatorial configuration, the optic axis of the uniaxial crystal is in the sagittal (x-z) plane and for this reason this configuration permits pure TE and TM mode solutions. Similar considerations apply to a biaxial medium rotated around its crystalline y axis.

With reference to Fig.2.6, optic axis is the z' axis lying between x and z, and y, y' are coincident, that is, $\phi=\frac{\pi}{2}$ and $\psi=\frac{\pi}{2}$. The corresponding matrix (2.3.9) is

$$A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{4.3.1}$$

where θ is measured from the z axis (see Fig. 4.7). The permittivity tensor $\bar{\bar{\varepsilon}}$ is given by the product

$$\bar{\bar{\varepsilon}} = A \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} A^T \tag{4.3.2}$$

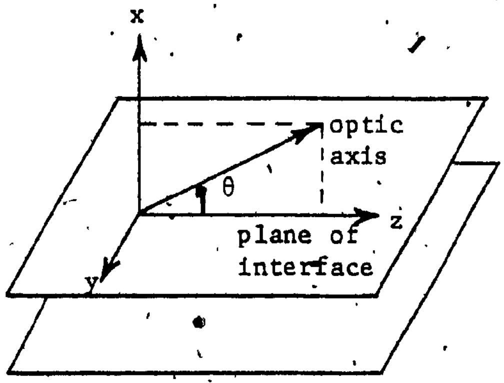

Fig. 4.7 Equatorial configuration. Optic axis lies in the x-z plane, forming an angle θ with the z axis.

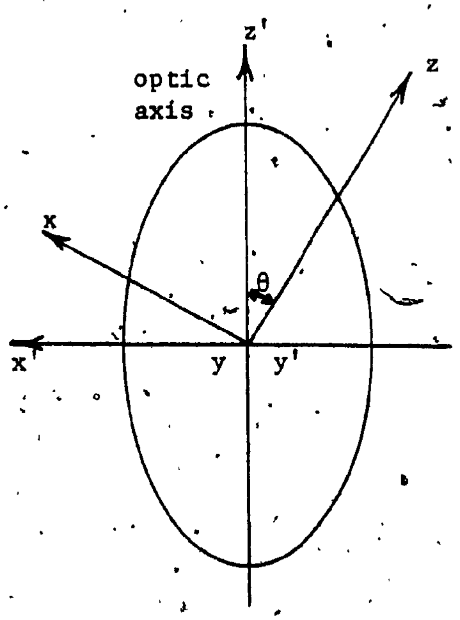

Fig. 4.8 Intersection of the index ellipsoid of a uniaxial crystal with the x-z plane of the device's coordinate system. The direction of propagation is along z and the optic axis along z'. Since the equatorial plane is circle : $\varepsilon_x = \varepsilon_z$

and thus assumes the general form :

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & 0 & \varepsilon_{xz} \\ 0 & \varepsilon_{yy} & 0 \\ \varepsilon_{xz}^* & 0 & \varepsilon_{zz} \end{bmatrix} \qquad (4.3.3)$$

The coupling matrix is

$$R_g = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \varepsilon_{xx}-\beta^2 & 0 & 0 & 0 \\ 0 & 0 & -\beta\varepsilon_{xz}/\varepsilon_{xx} & 1-\beta^2/\varepsilon_{xx} \\ 0 & 0 & \Delta_{yy}/\Delta_{xx} & -\beta\varepsilon_{xz}^*/\varepsilon_{xx} \end{bmatrix} \qquad (4.3.4)$$

The form of R_g is block diagonal, permitting the existence of pure TE and TM modes. The upper-left block pertains to TE and the lower-right one to TM modes. Whereas in the polar and longitudinal cases the trace of R_g is zero, in the equatorial case : $TrR_g=-2\beta Re(\varepsilon_{xz})/\varepsilon_{xx}$. This nonzero trace of R_g results in tilded phase fronts of the modes with respect to the axis z {25}. Note also that due to nonzero R_{33}, the characteristic equation (2.3.21) is a fourth order one with all four coefficients nonzero. In other words, mode bidirectionality ceases to exist in the equatorial configuration. The transverse effective guide indices κ^{TE} and κ^{TM} are obtained by eliminating E_x, E_z and E_y, respectively, in the wave equation (2.3.28b). It is of particular interest to note that $\kappa_2^{TE}=-\kappa_1^{TE}$, whereas $\kappa_4^{TM}=-\kappa_3^{TM}$ only when $Re(\varepsilon_{xz})=0$, that is, bidirectionality of the TM wave requires an imaginary ε_{xz} {26}.

To find the polarization vectors one has to expand the cofactors of the first and third rows of the coefficient matrix in the wave equation (2.3.28b). These are found to have the form

$$\bar{p}^{TE}_{1,2}=\begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \bar{p}^{TM}_{3,4}=\begin{bmatrix}-(\varepsilon_{xz}+\beta\kappa_{3,4})\\0\\\varepsilon_{xx}-\beta^2\end{bmatrix} \tag{4.3.5}$$

and

$$\bar{q}^{TE}_{1,2}=\begin{bmatrix}-\beta\\0\\\kappa_{1,2}\end{bmatrix} \qquad \bar{q}^{TM}_{3,4}=\begin{bmatrix}0\\-(\varepsilon_{xx}\kappa_{3,4}+\beta\varepsilon_{xz})\\0\end{bmatrix} \tag{4.3.6}$$

It is worthnoting that the TE modes, characterized by (4.3.5a) and (4.3.6a) are identical to those of the isotropic case,whereas the TM modes differ due to the presence of the ε_{xz}. Recall also that this term results in a slanted wave propagation with respect to z axis. This will become more clear in the next section where the field distribution of symmetric layered waveguides in the equatorial configuration will be derived.

4.4. Symmetric Layered Waveguides

In the previous sections of this chapter the three configurations pertaining to the optic axis orientation in one of the planes of the device's coordinate system were examined. This section treats the symmetric boundary value problem illustrated in Fig. 4.9, showing a layered waveguide of normalized width 2ℓ, that consists of an anisotropic core and identical semi-infinite cladding regions. An analysis of the characteristic and dispersion equations will first be presented, followed by a discussion on cut-off and asymptotic (high frequency) behavior.

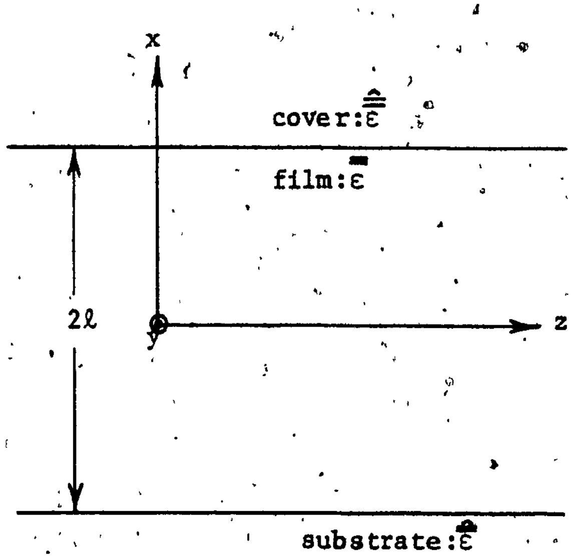

Fig. 4.9 A symmetric layered waveguide. The width of film is 2ℓ. The x,y,z and ℓ are normalized quantities.

Expressions are presented also for the internal and external power flow.

(1) Characteristic and Dispersion equations

The characteristic equation arises from the continuity of the transverse field components at the boundary. As a consequence of symmetry only one of the interfaces needs to be considered.

Consider an interface separating two media characterized by wave impedances $\overrightarrow{z}$ and $\overleftarrow{z}$, as shown in Fig. 4.10. These wave impedances provide the link between the transverse electric and magnetic field distributions at $x=\ell^{\pm}$. At the $x=\ell^{\pm}$ planes, the following equations are valid:

$$\begin{bmatrix} \eta_o^{-\frac{1}{2}}\overline{E}_\tau(\ell^-) \\ \sigma^T\eta_o^{\frac{1}{2}}\overline{H}_\tau(\ell^-) \end{bmatrix} = \begin{bmatrix} \overleftarrow{z} \\ I \end{bmatrix} \sigma^T\eta_o^{\frac{1}{2}}\,\overline{H}_\tau(\ell^-) \; ; \; \begin{bmatrix} \eta_o^{-\frac{1}{2}}\overline{E}_\tau(\ell^+) \\ \sigma^T\eta_o^{\frac{1}{2}}\overline{H}_\tau(\ell^+) \end{bmatrix} = \begin{bmatrix} \overrightarrow{z} \\ I \end{bmatrix} \sigma^T\eta_o^{\frac{1}{2}}\,\overline{H}_\tau(\ell^+) . \tag{4.4.1}$$

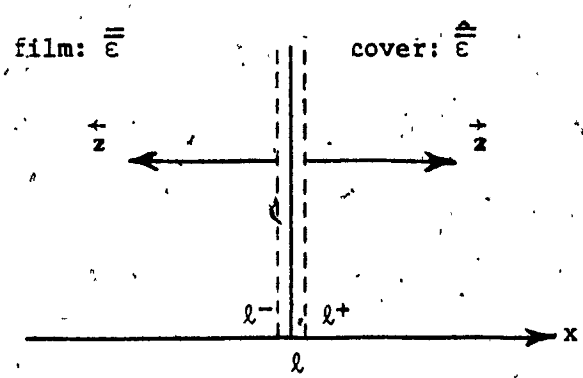

Fig. 4.10 Interface beween film and cover at $x=\ell$. The wave impedances $\overleftarrow{z}$ and $\overrightarrow{z}$ pertain to the corresponding media shown by the arrows.

Continuity of the transverse field distribution at the interface is assured by:

$$\begin{bmatrix} \overleftarrow{z} \\ I \end{bmatrix} \sigma^T\eta_o^{\frac{1}{2}}\,\overline{H}_\tau(\ell^-) = \begin{bmatrix} \overrightarrow{z} \\ I \end{bmatrix} \sigma^T\eta_o^{\frac{1}{2}}\,\overline{H}_\tau(\ell^+) . \tag{4.4.2}$$

Multiplying both sides with the left inverse of $\begin{bmatrix} \vec{z} \\ I \end{bmatrix}$ and transfering the remaining r.h.s. also to the left one has

$$\left\{ \tfrac{1}{2}\begin{bmatrix} \overleftrightarrow{y} & I \end{bmatrix} \begin{bmatrix} \overleftarrow{z} \\ I \end{bmatrix} - I \right\} \sigma^T \eta_o^{\frac{1}{2}} H_\tau(\ell) = 0$$

where the boundary condition, stipulating that $\bar{H}_\tau(\ell^+) = \bar{H}_\tau(\ell^-)$ has been considered. Thus the b.c. requires that

$$\mathrm{Det}[\tfrac{1}{2}\overleftrightarrow{y}\overleftarrow{z} - I] = \mathrm{Det}(\overleftarrow{z} - \vec{z}) = 0 \tag{4.4.3}$$

since $\overleftrightarrow{y}$ is nonsingular. This is the characteristic equation. Use of the notation

$$z = \frac{1}{D} \begin{bmatrix} z'_{11} & z'_{12} \\ z'_{21} & z'_{22} \end{bmatrix} \tag{4.4.4}$$

results in the open form of (4.4.3)

$$(\mathrm{Det}\vec{z} + \mathrm{Det}\overleftarrow{z})\vec{D}\overleftarrow{D} - (\vec{z}'_{11}\overleftarrow{z}'_{22} + \overleftarrow{z}'_{11}\vec{z}'_{22} - \vec{z}'_{12}\overleftarrow{z}'_{21} - \overleftarrow{z}'_{12}\vec{z}'_{21}) = 0 \tag{4.4.5}$$

Before substituting the z's of various cases, one has to examine the nature of the transverse effective guide index κ characterizing the film. This will be done by obtaining the characteristic equation with another method and comparing it with the resulting (4.4.5).

Another method to obtain the characteristic equation is found by assuming symmetric or antisymmetric field distributions in the film and subsequently satisfying the b.c.'s at $x=\ell$. The wave reflection introduced by the two interfaces of the film result in the existence of forward waves - characterized by κ_1,κ_3 with coefficients c_1,c_3, and backward waves in the film - characterized by κ_2,κ_4 with coefficients c_2,c_4. Coupling between the corresponding forward and backward modes is accounted by the expression of c_2 and c_4 in terms of c_1 and c_3. By choosing the c_2 and c_4 coefficients in (2.3.30) as $c_2=c_1$ and $c_4=c_3$, pure symmetric modes result, whereas by choosing $c_2=-c_1$ and $c_4=-c_3$, pure antisymmetric. Thus, substituting the appropriate c_2,c_4 into (2.3.30), the symmetric modes are characterized by :

$$\eta_o^{-\frac{1}{2}}E_y = c_1\cos\kappa_1 x + c_3\cos\kappa_3 x$$

and the antisymmetric by :

$$\eta_o^{-\frac{1}{2}}E_y = c_1\sin\kappa_1 x + c_3\sin\kappa_3 x,$$

where the coefficients c_i of $\cos\kappa_i x$ have been renamed to be $2c_i$ and the ones of $\sin\kappa_i x$ to be $-2jc_i$ $(i=1,3)$. The corresponding field expressions of these two distributions in the polar, longitudinal and equatorial configurations are shown in Tables 4.1-6. Since in the external region $(|x|\geq\ell)$ the fields must decay exponentially only $\hat{c}_1$ and $\hat{c}_3$ are nonzero and the corresponding effective transverse indices are purely imaginary, i.e., $\hat{\kappa}_{1,3}=-j\alpha_{1,3}$. Considering polar configuration, continuity of the field components of the

$\eta_o^{-\frac{1}{2}}E_x(x)$	$j\,\frac{\beta a}{\varepsilon_{xx}}\,(\kappa_1 h_1 c_1 \sin\kappa_1 x+\kappa_3 h_3 c_3 \sin\kappa_3 x)$
$\eta_o^{-\frac{1}{2}}E_y(x)$	$c_1\cos\kappa_1 x+c_3\cos\kappa_3 x$
$\eta_o^{-\frac{1}{2}}E_z(x)$	$h_1 c_1\cos\kappa_1 x + h_3 c_3\cos\kappa_3 x$
$\eta_o^{\frac{1}{2}}H_x(x)$	$-\beta(c_1\cos\kappa_1 x+c_3\cos\kappa_3 x)$
$-\eta_o^{\frac{1}{2}}H_y(x)$	$-ja(\kappa_1 h_1 c_1\sin\kappa_1 x+\kappa_3 h_3 c_3\sin\kappa_3 x)$
$\eta_o^{\frac{1}{2}}H_z(x)$	$-j(\kappa_1 c_1\sin\kappa_1 x+\kappa_3 c_3\sin\kappa_3 x)$
$\eta_o^{-\frac{1}{2}}E_x(x)$	$j\frac{x}{\lvert x\rvert}\;\frac{\beta\hat{a}}{\hat{\varepsilon}_{xx}}\{\alpha_1\hat{h}_1\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\alpha_3\hat{h}_3\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$\eta_o^{-\frac{1}{2}}E_y(x)$	$\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]$
$\eta_o^{-\frac{1}{2}}E_z(x)$	$\{\hat{h}_1\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\hat{h}_3\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$\eta_o^{\frac{1}{2}}H_x(x)$	$-\beta\{\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$-\eta_o^{\frac{1}{2}}H_y(x)$	$-j\frac{x}{\lvert x\rvert}\;\hat{a}\{\alpha_1\hat{h}_1\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\alpha_3\hat{h}_3\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$\eta_o^{\frac{1}{2}}H_z(x)$	$-j\frac{x}{\lvert x\rvert}\;\{\alpha_1\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\alpha_3\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$

Table 4.1 Symmetric field distribution in the polar configuration. The upper part of the table pertains to the film and the lower to the external regions. Note that E_y, E_z, H_x and E_x, H_y, H_z belong to groups of opposite parity.

$\eta_o^{-\frac{1}{2}}E_x(x)$	$-j\,\frac{\beta a}{\varepsilon_{xx}}\,(h_1\kappa_1c_1\cos\kappa_1x+h_3\kappa_3c_3\cos\kappa_3x)$
$\eta_o^{-\frac{1}{2}}E_y(x)$	$c_1\sin\kappa_1x+c_3\sin\kappa_3x$
$\eta_o^{-\frac{1}{2}}E_z(x)$	$h_1c_1\sin\kappa_1x+h_3c_3\sin\kappa_3x$
$\eta_o^{\frac{1}{2}}H_x(x)$	$-\beta(c_1\sin\kappa_1x+c_3\sin\kappa_3x)$
$-\eta_o^{\frac{1}{2}}H_y(x)$	$ja(h_1\kappa_1c_1\cos\kappa_1x+h_3\kappa_3c_3\cos\kappa_3x$
$\eta_o^{\frac{1}{2}}H_z(x)$	$j(\kappa_1c_1\cos\kappa_1x+\kappa_3c_3\cos\kappa_3x)$
$\eta_o^{-\frac{1}{2}}E_x(x)$	$j\,\frac{\beta\hat{a}}{\hat{\varepsilon}_{xx}}\,\{\alpha_1\hat{h}_1\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\alpha_3\hat{h}_3\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$\eta_o^{-\frac{1}{2}}E_y(x)$	$\frac{x}{\lvert x\rvert}\{\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$\eta_o^{-\frac{1}{2}}E_z(x)$	$\frac{x}{\lvert x\rvert}\{\hat{h}_1\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\hat{h}_3\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$\eta_o^{\frac{1}{2}}H_x(x)$	$-\beta\frac{x}{\lvert x\rvert}\{\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$-\eta_o^{\frac{1}{2}}H_y(x)$	$-j\hat{a}\{\alpha_1\hat{h}_1\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\alpha_3\hat{h}_3\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$\eta_o^{\frac{1}{2}}H_z(x)$	$-j\{\alpha_1\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\alpha_3\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$

Table 4.2 Antisymmetric field distribution in the polar configuration. The upper part of the table pertains to the film and the lower to the external regions. Note that E_y, E_z, H_x and E_x, H_y, H_z belong to groups of opposite parity.

$\eta_o^{-\frac{1}{2}}E_x(x)$	$\frac{1}{\beta}[(b\kappa_1^2-1)g_1c_1\cos\kappa_1x+(b\kappa_3^2-1)g_3c_3\cos\kappa_3x]$
$\eta_o^{-\frac{1}{2}}E_y(x)$	$c_1\cos\kappa_1x+c_3\cos\kappa_3x$
$\eta_o^{-\frac{1}{2}}E_z(x)$	$-jb(\kappa_1g_1c_1\sin\kappa_1x+\kappa_3g_3c_3\sin\kappa_3x)$
$\eta_o^{\frac{1}{2}}H_x(x)$	$-\beta(c_1\cos\kappa_1x+c_3\cos\kappa_3x)$
$-\eta_o^{\frac{1}{2}}H_y(x)$	$g_1c_1\cos\kappa_1x+g_3c_3\cos\kappa_3x$
$\eta_o^{\frac{1}{2}}H_z(x)$	$-j(\kappa_1c_1\sin\kappa_1x+\kappa_3c_3\sin\kappa_3x)$
$\eta_o^{-\frac{1}{2}}E_x(x)$	$-\frac{1}{\beta}\{(\hat{b}\alpha_1^2+1)\hat{g}_1\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+(\hat{b}\alpha_3^2+1)\hat{g}_3\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$\eta_o^{-\frac{1}{2}}E_y(x)$	$\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]$
$\eta_o^{-\frac{1}{2}}E_z(x)$	$-j\hat{b}\frac{x}{\lvert x\rvert}\{\hat{g}_1\alpha_1\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\hat{g}_3\alpha_3\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$\eta_o^{\frac{1}{2}}H_x(x)$	$-\beta\{\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$-\eta_o^{-\frac{1}{2}}H_y(x)$	$\hat{g}_1c_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\hat{g}_3\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]$
$\eta_o^{\frac{1}{2}}H_z(x)$	$-j\frac{x}{\lvert x\rvert}\{\alpha_1\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\alpha_3\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$

Table 4.3 Symmetric field distribution in the longitudinal configuration. The upper part of the table pertains to the film and the lower to the external regions. Note that E_x, E_y, H_x and H_y belong to the same parity group, whereas E_z and H_z to a group of opposite parity.

$\eta_o^{-\frac{1}{2}}E_x(x)$	$\frac{1}{\beta}\,[(b\kappa_1^2 - 1)g_1c_1\sin\kappa_1x+(b\kappa_3^2 - 1)g_3c_3\sin\kappa_3x]$
$\eta_o^{-\frac{1}{2}}E_y(x)$	$c_1\sin\kappa_1x+c_3\sin\kappa_3x$
$\eta_o^{-\frac{1}{2}}E_z(x)$	$jb(g_1\kappa_1c_1\cos\kappa_1x+g_3\kappa_3c_3\cos\kappa_3x)$
$\eta_o^{\frac{1}{2}}H_x(x)$	$-\beta(c_1\sin\kappa_1x+c_3\sin\kappa_3x)$
$-\eta_o^{\frac{1}{2}}H_y(x)$	$g_1c_1\sin\kappa_1x+g_3c_3\sin\kappa_3x$
$\eta_o^{\frac{1}{2}}H_z(x)$	$j(\kappa_1c_1\cos\kappa_1x+\kappa_3c_3\cos\kappa_3x)$
$\eta_o^{-\frac{1}{2}}E_x(x)$	$-\frac{1}{\beta}\frac{x}{\lvert x\rvert}\,\{(\hat{b}\alpha_1^2+1)\hat{g}_1\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+(\hat{b}\alpha_3^2+1)\hat{g}_3\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$\eta_o^{-\frac{1}{2}}E_y(x)$	$\frac{x}{\lvert x\rvert}\,\{\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$\eta_o^{-\frac{1}{2}}E_z(x)$	$-j\hat{b}\{\hat{g}_1\alpha_1\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\hat{g}_3\alpha_3\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$\eta_o^{\frac{1}{2}}H_x(x)$	$-\beta\,\frac{x}{\lvert x\rvert}\,\{\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$-\eta_o^{\frac{1}{2}}H_y(x)$	$\frac{x}{\lvert x\rvert}\,\{\hat{g}_1\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\hat{g}_3\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$
$\eta_o^{\frac{1}{2}}H_z(x)$	$-j\{\alpha_1\hat{c}_1\exp[-\alpha_1(\lvert x\rvert-\ell)]+\alpha_3\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]\}$

Table 4.4 Antisymmetric field distribution in the longitudinal configuration. The upper part of the table pertains to the film and the lower to the external regions. Note that E_x, E_y, H_x, H_y and E_z, H_z belong to groups of opposite parity.

$\eta_o^{-\frac{1}{2}} E_y$	$c_1 \cos\kappa_1 x$
$\eta_o^{\frac{1}{2}} H_x$	$-\beta c_1 \cos\kappa_1 x$
$\eta_o^{\frac{1}{2}} H_z$	$-j\kappa_1 c_1 \sin\kappa_1 x$
$\eta_o^{-\frac{1}{2}} E_y$	$\hat{c}_1 \exp[-\alpha_1(\lvert x\rvert-\ell)]$
$\eta_o^{\frac{1}{2}} H_x$	$-\beta\hat{c}_1 \exp[-\alpha_1(\lvert x\rvert-\ell)]$
$\eta_o^{\frac{1}{2}} H_z$	$-j\frac{x}{\lvert x\rvert}\alpha_1\hat{c}_1 \exp[-\alpha_1(\lvert x\rvert-\ell)]$

$\eta_o^{-\frac{1}{2}} E_y$	$c_1 \sin\kappa_1 x$
$\eta_o^{\frac{1}{2}} H_x$	$-\beta c_1 \sin\kappa_1 x$
$\eta_o^{\frac{1}{2}} H_z$	$j\kappa_1 c_1 \cos\kappa_1 x$
$\eta_o^{-\frac{1}{2}} E_y$	$\frac{x}{\lvert x\rvert}\ \hat{c}_1 \exp[-\alpha_1(\lvert x\rvert-\ell)]$
$\eta_o^{\frac{1}{2}} H_x$	$-\frac{x}{\lvert x\rvert}\ \beta\hat{c}_1 \exp[-\alpha_1(\lvert x\rvert-\ell)]$
$\eta_o^{\frac{1}{2}} H_z$	$-j\alpha_1\hat{c}_1 \exp[-\alpha_1(\lvert x\rvert-\ell)]$

Table 4.5 Field distribution of TE modes in the equatorial configuration. The upper part of the table pertains to symmetric modes and the lower to antisymmetric modes. The TE modes of an anisotropic symmetric layered waveguide are identical to the corresponding modes of the isotropic symmetric slab waveguide, given in Table 3.1.

$\eta_o^{\frac{1}{2}}E_x$	$-(\varepsilon_{xz}+\beta\kappa_3)c_3\exp(-j\kappa_3 x) - (\varepsilon_{xz}+\beta\kappa_4)c_4\exp(-j\kappa_4 x)$
$\eta_o^{-\frac{1}{2}}E_z$	$(\varepsilon_{xx}-\beta^2)[c_3\exp(-j\kappa_3 x)+c_4\exp(-j\kappa_4 x)]$
$-\eta_o^{\frac{1}{2}}H_y$	$(\varepsilon_{xx}\kappa_3+\beta\varepsilon_{xz})c_3\exp(-j\kappa_3 x)+(\varepsilon_{xx}\kappa_4+\beta\varepsilon_{xz})c_4\exp(-j\kappa_4 x)$
$\eta_o^{-\frac{1}{2}}E_x$	$-(\hat{\varepsilon}_{xz}-j\beta\alpha_3)\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]$
$\eta_o^{-\frac{1}{2}}E_z$	$(\hat{\varepsilon}_{xx}-\beta^2)\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]$
$-\eta_o^{\frac{1}{2}}H_y$	$(\beta\hat{\varepsilon}_{xz}-j\hat{\varepsilon}_{xx}\alpha_3)\hat{c}_3\exp[-\alpha_3(\lvert x\rvert-\ell)]$

Table 4.6 Field distribution of TM modes in the equatorial configuration. The lack of bidirectionality, when $\mathrm{Re}(\varepsilon_{xz}) \neq 0$, does not permit the existence of symmetric and antisymmetric modes.

symmetric field distribution at the boundary results in

$$c_1\cos\kappa_1\ell+c_3\cos\kappa_3\ell=\hat{c}_1+\hat{c}_3$$
$$-j\kappa_1c_1\sin\kappa_1\ell-j\kappa_3c_3\sin\kappa_3\ell=-j\alpha_1\hat{c}_1-j\alpha_3\hat{c}_3$$
$$h_1c_1\cos\kappa_1x+h_3c_3\cos\kappa_3x=\hat{h}_1\hat{c}_1+\hat{h}_3\hat{c}_3$$
$$-ja\kappa_1h_1c_1\sin\kappa_1\ell-ja\kappa_3h_3c_3\sin\kappa_3\ell=-j\hat{a}\hat{h}_1\alpha_1\hat{c}_1-j\hat{a}\hat{h}_3\alpha_3\hat{c}_3 \qquad (4.4.6)$$

These four equations can be written in matrix form. In order to have a nontrivial solution the determinant of the coefficient matrix must vanish, i.e.,

$$\mathrm{Det}\begin{bmatrix} 1 & 1 & 1 & 1 \\ j\kappa_1\tan\kappa_1\ell & j\kappa_3\tan\kappa_3\ell & j\alpha_1 & j\alpha_3 \\ h_1 & h_3 & \hat{h}_1 & \hat{h}_3 \\ ja\kappa_1h_1\tan\kappa_1\ell & ja\kappa_3h_3\tan\kappa_3\ell & j\hat{a}\alpha_1\hat{h}_1 & j\hat{a}\alpha_3\hat{h}_3 \end{bmatrix}=0 \qquad (4.4.7)$$

or in expanded form

$$-a(h_1-h_3)(\hat{h}_1-\hat{h}_3)\kappa_1\tan\kappa_1\ell\cdot\kappa_3\tan\kappa_3\ell$$
$$+[(ah_1-\hat{a}\hat{h}_1)(h_3-\hat{h}_3)\alpha_1+(ah_1-\hat{a}\hat{h}_3)(\hat{h}_1-h_3)\alpha_3]\kappa_1\tan\kappa_1\ell$$
$$+[(ah_3-\hat{a}\hat{h}_1)(\hat{h}_3-h_1)\alpha_1+(ah_3-\hat{a}\hat{h}_3)(h_1-\hat{h}_1)\alpha_3]\kappa_3\tan\kappa_3\ell$$
$$-\hat{a}(h_1-h_3)(\hat{h}_1-\hat{h}_3)\alpha_1\alpha_3=0. \qquad (4.4.8)$$

Returning now to the wave impedance formulation, referring to Fig. 4.8, the backward wave impedance $\overleftarrow{z}$ of the guide, in view of (4.1.15) and (4.1.16), is given by

$$\overleftarrow{z} = \frac{1}{a\kappa_1'\kappa_3'(h_1-h_3)} \begin{bmatrix} a(h_3\kappa_3'-h_1\kappa_1') & \kappa_1' - \kappa_3' \\ ah_1h_3(\kappa_3'-\kappa_1') & h_3\kappa_1' - h_1\kappa_3' \end{bmatrix} \tag{4.4.9}$$

and the forward $\overrightarrow{z}$ of the simi-infinite medium, by

$$\overrightarrow{z} = \frac{1}{j\hat{a}\alpha_1\alpha_3(\hat{h}_3-\hat{h}_1)} \begin{bmatrix} \hat{a}(\hat{h}_1\alpha_1-\hat{h}_3\alpha_3) & \alpha_3 - \alpha_1 \\ a\hat{h}_1\hat{h}_3(\alpha_1 - \alpha_3) & \hat{h}_1\alpha_3 - \hat{h}_3\alpha_1 \end{bmatrix} \tag{4.4.10}$$

where the primed κ_i's account for the effect of the definite thickness of the film [21]. Substituting $\overleftarrow{z}$ and $\overrightarrow{z}$ into the generalized characteristic equation (4.4.5) and arranging the terms in a similar form with (4.4.8), the following equation results

$$\begin{aligned} & a(h_1-h_3)(\hat{h}_1-\hat{h}_3)\kappa_1'\kappa_3' \\ & -j[(ah_1-\hat{a}\hat{h}_1)(h_3-\hat{h}_3)\alpha_1+(ah_1-\hat{a}\hat{h}_3)(\hat{h}_1-h_3)\alpha_3]\kappa_1' \\ & -j[(ah_3-\hat{a}\hat{h}_1)(\hat{h}_3-h_1)\alpha_1+(ah_3-\hat{a}\hat{h}_3)(h_1-\hat{h}_3)\alpha_3]\kappa_3' \\ & -\hat{a}(h_1-h_3)(\hat{h}_1-\hat{h}_3)\alpha_1\alpha_3 = 0 \end{aligned} \tag{4.4.11}$$

Equations (4.4.8) and (4.4.11) express the characteristic equation of a symmetric layered waveguide in the polar configuration. Since they have to be identical

$$\kappa_i' = j\kappa_i \tan\kappa_i \ell \quad ; \quad i=1,3 \tag{4.4.12}$$

must be satisfied. Working similarly as before, the same form of κ_i' results whenever the field distribution is of symmetric form, in either polar or longitudinal configuration. The corresponding expression of κ_i' for antisymmetric field distribution in the film is

$$\kappa_i' = j\kappa_i \cot(\kappa_i \ell) \quad ; \quad i=1,3 \qquad (4.4.13)$$

Taking now into consideration the 'primed' κ_i's in (4.4.7), the characteristic equation of a layered waveguide, where all three media are of polar configuration (PPP), is

$$\text{Det}\begin{bmatrix} 1 & 1 & 1 & 1 \\ \kappa_1' & \kappa_3' & j\alpha_1 & j\alpha_3 \\ h_1 & h_3 & \hat{h}_1 & \hat{h}_3 \\ ah_1\kappa_1' & ah_3\kappa_3' & j\hat{a}\hat{h}_1\alpha_1 & j\hat{a}\hat{h}_3\alpha_3 \end{bmatrix} = 0 \qquad (4.4.14)$$

where the left partition refers to the film and the right one to the external region. The corresponding equation for the longitudinal case (LLL), is

$$\text{Det}\begin{bmatrix} 1 & 1 & 1 & 1 \\ \kappa_1' & \kappa_3' & j\alpha_1 & j\alpha_3 \\ bg_1\kappa_1' & bg_3\kappa_3' & j\hat{b}\hat{g}_1\alpha_1 & j\hat{b}\hat{g}_3\alpha_3 \\ g_1 & g_3 & \hat{g}_1 & \hat{g}_3 \end{bmatrix} = 0 \qquad (4.4.15)$$

It is understood that the characteristic equation of mixed cases can be written from the composition of the correspondiing partitions of (4.4.14) and (4.4.15). It is worthnoting that the primes refer to the κ_i shown in the matrix forms and not to the ones of the h_i and g_i parameters. Recalling (4.4.11), the expanded form of a characteristic equation can be written

$$A\kappa'_1\kappa'_3 - jB\kappa'_1 - jC\kappa'_3 + D = 0 \qquad (4.4.16)$$

where A, B, C and D are functions of material parameters. The characteristic equations and the corresponding wave impedances of the four cases PPP, LPL, PLP and LLL are shown in Tables 4.7-10. Note the symmetry of the expanded forms of the characteristic equations. To get from a P(L) case the corresponding L(P) case, one has merely to perform the changes $a \leftrightarrow b$ and $h_i \leftrightarrow g_i$ for the internal distribution, or $\hat{a} \leftrightarrow \hat{b}$ and $\hat{h}_i \leftrightarrow \hat{g}_i$ for the corresponding external one.

The transverse effective guide indices κ_i, obtained in (4.1.5) and (4.2.5) are given by

$$2\kappa^2_{1,3} = (R_{34}R_{43}+R_{21}) \pm [(R_{34}R_{43}-R_{21})^2 + 4|R_{21}|^2 R_{34}]^{\frac{1}{2}} \qquad (4.4.17)$$

for the polar case and

$$2\kappa^2_{1,3} = (R_{34}R_{43}+R_{21}) \pm [(R_{34}R_{43}-R_{21})^2 + 4|R_{31}|^2 R_{43}]^{\frac{1}{2}} \qquad (4.4.18)$$

for the longitudinal. The corresponding α_i's are obtained from the same formulae by replacing κ_i with $-j\alpha_i x/|x|$ and R_{ij} by $\hat{R}_{ij}$. Substituting R_{ij} from (4.1.4) and (4.2.4), equations (4.4.17) and (4.4.18) take the form

$\mathrm{Det}\begin{bmatrix} 1 & 1 & 1 & 1 \\ \kappa_1' & \kappa_3' & j\alpha_1 & j\alpha_3 \\ h_1 & h_3 & \hat{h}_1 & \hat{h}_3 \\ ah_1\kappa_1' & ah_3\kappa_3' & j\hat{a}\hat{h}_1\alpha_1 & j\hat{a}\hat{h}_3\alpha_3 \end{bmatrix} = 0$	$\overleftarrow{z} = \frac{1}{a\kappa_1'\kappa_3'(h_1-h_3)}\begin{bmatrix} a(h_3\kappa_3'-h_1\kappa_1') & \kappa_1'-\kappa_3' \\ ah_1h_3(\kappa_3'-\kappa_1') & h_3\kappa_1'-h_1\kappa_3' \end{bmatrix}$ $\overrightarrow{z} = \frac{1}{j\hat{a}\alpha_1\alpha_3(\hat{h}_3-\hat{h}_1)}\begin{bmatrix} \hat{a}(\hat{h}_1\alpha_1-\hat{h}_3\alpha_3) & \alpha_3-\alpha_1 \\ \hat{a}\hat{h}_1\hat{h}_3(\alpha_3-\alpha_1) & h_1\alpha_3-h_3\alpha_1 \end{bmatrix}$
$\mathrm{Det}\,\overleftarrow{z} = 1/a\kappa_1'\kappa_3'$ $\mathrm{Det}\,\overrightarrow{z} = -1/\hat{a}\alpha_1\alpha_3$ Symmetric modes: $\kappa_i' = j\kappa_i\tan\kappa_i\ell$ Antisymmetric modes: $\kappa_i' = -j\kappa_i\cot\kappa_i\ell$	$A\kappa_1'\kappa_3' - jB\kappa_1' - jC\kappa_3' + D = 0$ $A = a(h_1-h_3)(\hat{h}_1-\hat{h}_3)$ $B = (ah_1-\hat{a}\hat{h}_1)(h_3-\hat{h}_3)\alpha_1+(ah_1-\hat{a}\hat{h}_3)(\hat{h}_1-h_3)\alpha_3$ $C = (ah_3-\hat{a}\hat{h}_1)(\hat{h}_3-h_1)\alpha_1+(ah_3-\hat{a}\hat{h}_3)(h_1-\hat{h}_1)\alpha_3$ $D = \hat{a}(h_1-h_3)(\hat{h}_3-\hat{h}_1)\alpha_1\alpha_3$

Table 4.7 Characteristic equation and wave impedances of a symmetric layered waveguide in the PPP case.

$\text{Det}\begin{bmatrix} 1 & 1 & 1 & 1 \\ \kappa_1' & \kappa_3' & j\alpha_1 & j\alpha_3 \\ h_1 & h_3 & j\hat{b}\hat{g}_1\alpha_1 & j\hat{b}\hat{g}_3\alpha_3 \\ ah_1\kappa_1' & ah_3\kappa_3' & \hat{g}_1 & \hat{g}_3 \end{bmatrix} = 0$	$\overleftarrow{z} = \frac{1}{a\kappa_1'\kappa_3'(h_1-h_3)}\begin{bmatrix} a(h_3\kappa_3'-h_1\kappa_1') & \kappa_1'-\kappa_3' \\ ah_1h_3(\kappa_3'-\kappa_1') & h_3\kappa_1'-h_1\kappa_3' \end{bmatrix}$ $\overrightarrow{z} = \frac{1}{j(\alpha_3\hat{g}_1-\alpha_1\hat{g}_3)}\begin{bmatrix} \hat{g}_3-\hat{g}_1 & j(\alpha_3-\alpha_1) \\ j\hat{g}_1\hat{g}_3\hat{b}(\alpha_3-\alpha_1) & \alpha_1\alpha_3\hat{b}(\hat{g}_1-\hat{g}_3) \end{bmatrix}$
$\text{Det}\ \overleftarrow{z} = 1/a\kappa_1'\kappa_3'$ $\text{Det}\ \overrightarrow{z} = \hat{b}(\alpha_1\hat{g}_1-\alpha_3\hat{g}_3)/(\alpha_3\hat{g}_1-\alpha_1\hat{g}_3)$ Symmetric modes: $\kappa_1' = j\kappa_1\tan\kappa_1\ell$ Antisymmetric modes: $\kappa_1' = -j\kappa_1\cot\kappa_1\ell$	$A\kappa_1'\kappa_3' - jB\kappa_1' - jC\kappa_3' + D = 0$ $A = ja\hat{b}(h_1-h_3)(\hat{g}_1\alpha_1-\hat{g}_3\alpha_3)$ $B = (ah_1h_3-\hat{b}\hat{g}_1\hat{g}_3)(\alpha_3-\alpha_1)+j(a\hat{b}h_1\alpha_1\alpha_3+h_3)(\hat{g}_1-\hat{g}_3)$ $C = (ah_1h_3-\hat{b}\hat{g}_1\hat{g}_3)(\alpha_1-\alpha_3)+j(a\hat{b}h_3\alpha_1\alpha_3+h_1)(\hat{g}_3-\hat{g}_1)$ $D = j(h_1-h_3)(\hat{g}_1\alpha_3-\hat{g}_3\alpha_1)$

Table 4.8 Characteristic equation and wave impedances of a symmetric layered waveguide in the LPL case.

$\mathrm{Det}\begin{bmatrix}1&1&1&1\\ \kappa_1'&\kappa_3'&j\alpha_1&j\alpha_3\\ bg_1\kappa_1'&bg_3\kappa_3'&\hat{h}_1&\hat{h}_3\\ g_1&g_3&j\hat{a}\hat{h}_1\alpha_1&j\hat{a}\hat{h}_3\alpha_3\end{bmatrix}=0$	$\overleftarrow{z}=\frac{1}{\kappa_1'g_3-\kappa_3'g_1}\begin{bmatrix}g_1-g_3&\kappa_1'-\kappa_3'\\ g_1g_3b(\kappa_1'-\kappa_3')&\kappa_1'\kappa_3'b(g_1-g_3)\end{bmatrix}$ $\overrightarrow{z}=\frac{1}{j\hat{a}\alpha_1\alpha_3(\hat{h}_3-\hat{h}_1)}\begin{bmatrix}\hat{a}(\hat{h}_1\alpha_1-\hat{h}_3\alpha_3)&\alpha_3-\alpha_1\\ \hat{a}\hat{h}_1\hat{h}_3(\alpha_1-\alpha_3)&\hat{h}_1\alpha_3-\hat{h}_3\alpha_1\end{bmatrix}$
$\mathrm{Det}\ \overleftarrow{z}=b(\kappa_3'g_3-\kappa_1'g_1)/(\kappa_1'g_3-\kappa_3'g_1)$ $\mathrm{Det}\ \overrightarrow{z}=-1/\hat{a}\alpha_1\alpha_3$ Symmetric modes: $\kappa_i'=j\kappa_i\tan\kappa_i\ell$ Antisymmetric modes: $\kappa_i'=-j\kappa_i\cot\kappa_i\ell$	$A\kappa_1'\kappa_3'-B\kappa_1'-C\kappa_3'+D=0$ $A=j\hat{a}b(g_1-g_3)(\hat{h}_1\alpha_1-\hat{h}_3\alpha_3)$ $B=(bg_1g_3-\hat{a}\hat{h}_1\hat{h}_3)(\alpha_3-\alpha_1)+j(\hat{a}bg_1\alpha_1\alpha_3+g_3)(\hat{h}_1-\hat{h}_3)$ $C=(bg_1g_3-\hat{a}\hat{h}_1\hat{h}_3)(\alpha_1-\alpha_3)+j(\hat{a}bg_3\alpha_1\alpha_3+g_1)(\hat{h}_3-\hat{h}_1)$ $D=j(g_1-g_3)(\hat{h}_1\alpha_3-\hat{h}_3\alpha_1)$

Table 4.9 Characteristic equation and wave impedances of a symmetric layered waveguide in the PLP case.

$$\mathrm{Det}\begin{bmatrix} 1 & 1 & 1 & 1 \\ \kappa_1' & \kappa_3' & j\alpha_1 & j\alpha_3 \\ bg_1\kappa_1' & bg_3\kappa_3' & j\hat{b}\hat{g}_1\alpha_1 & j\hat{b}\hat{g}_3\alpha_3 \\ g_1 & g_3 & \hat{g}_1 & \hat{g}_3 \end{bmatrix} = 0$$

$$\overleftarrow{z} = \frac{1}{\kappa_1' g_3 - \kappa_3' g_1} \begin{bmatrix} g_1 - g_3 & \kappa_1' - \kappa_3' \\ g_1 g_3 b(\kappa_1' - \kappa_3') & \kappa_1'\kappa_3' b(g_1 - g_3) \end{bmatrix}$$

$$\overrightarrow{z} = \frac{1}{j(\alpha_3 \hat{g}_1 - \alpha_1 \hat{g}_3)} \begin{bmatrix} \hat{g}_3 - \hat{g}_1 & j(\alpha_3 - \alpha_1) \\ j\hat{g}_1\hat{g}_3\hat{b}(\alpha_3 - \alpha_1) & \alpha_1\alpha_3\hat{b}(\hat{g}_1 - \hat{g}_3) \end{bmatrix}$$

$$\mathrm{Det}\ \overleftarrow{z} = b(\kappa_3' g_3 - \kappa_1' g_1)/(\kappa_1' g_3 - \kappa_3' g_1)$$

$$\mathrm{Det}\ \overrightarrow{z} = b(\alpha_1\hat{g}_1 - \alpha_3\hat{g}_3)/(\alpha_3\hat{g}_1 - \alpha_1\hat{g}_3)$$

Symmetric modes: $\kappa_i' = j\kappa_i \tan\kappa_i \ell$

Antisymmetric modes: $\kappa_i' = -j\kappa_i \cot\kappa_i \ell$

$$A\kappa_1'\kappa_3' - j\beta\kappa_1' - jC\kappa_3' + D = 0$$

$$A = b(g_1 - g_3)\ (\hat{g}_1 - \hat{g}_3)$$

$$B = [(bg_1 - \hat{b}\hat{g}_1)(g_3 - \hat{g}_3)\alpha_1 + (bg_1 - \hat{b}\hat{g}_3)(\hat{g}_1 - g_3)\alpha_3]$$

$$C = [(\hat{b}g_1 - bg_3)(g_1 - \hat{g}_3)\alpha_1 + (bg_3 - \hat{b}\hat{g}_3)(g_1 - \hat{g}_1)\alpha_3]$$

$$D = \hat{b}(g_1 - g_3)(\hat{g}_3 - \hat{g}_1)\alpha_1\alpha_3$$

Table 4.10 Characteristic equation and wave impedances of a symmetric layered waveguide in the LLL case.

$$2\kappa_{1,3}^2 = \varepsilon_{yy}+\varepsilon_{zz}-\beta^2(1+\varepsilon_{zz}/\varepsilon_{xx})\pm\left\{[\varepsilon_{yy}-\varepsilon_{zz}+\beta^2(\varepsilon_{zz}/\varepsilon_{xx}-1)]^2 + \right.$$
$$\left. + 4|\varepsilon\ |^2(1-\beta^2/\varepsilon_{xx})\right\}^{\frac{1}{2}} \tag{4.4.19}$$

in the polar and

$$2\kappa_{1,3}^2 = \varepsilon_{yy}+\varepsilon_{zz}-|\varepsilon_{xy}|^2/\varepsilon_{xx}-\beta^2(1+\varepsilon_{zz}/\varepsilon_{xx})\pm\left\{[\varepsilon_{zz}-\varepsilon_{yy}+|\varepsilon_{xy}|^2/\varepsilon_{xx} + \right.$$
$$\left. + \beta^2(1-\varepsilon_{zz}/\varepsilon_{xx})]^2+4\beta^2\varepsilon_{zz}|\varepsilon_{xy}|^2/\varepsilon_{xx}^2 \right\}^{\frac{1}{2}} \tag{4.4.20}$$

in the longitudinal configuration. In both cases κ_i is dependent on the square of the effective guide index β. By setting $\varepsilon_{ii}=\varepsilon$ $(i=x,y,z)$ and $\varepsilon_{ij}=0$ $(i,j=x,y,z;i\neq j)$, these equations reduce to (3.1.4) and (3.1.5) of the isotropic case.

The dispersion equation is obtained by substitution of the appropriate expression of κ^2 from (4.4.17) or (4.4.18) into the characteristic equation (4.4.16). The resulting equation relates the square of the effective guide index β to the (normalized) width 2ℓ, that is, the frequency, since $\ell = k_oL$ where $2L$ is the actual width. The expression is rather complicated, not as simple as the corresponding (3.1.11) of the isotropic case. The appropriate values of β^2 corresponding to propagating modes in a symmetric layered waveguide of specific width 2ℓ, are obtained numerically, by the use of computer.

(2) Cutoff Condition and Asymptotic Behaviour

The propagating modes are characterized by at least one real transverse guide index in the film and purely imaginary transverse indices in the semi-infinite media. Due to phase matching, the longitudinal guide index β is common to all three regions. Recalling from the corresponding isotropic case that a film in order to guide waves must be optically more dense than the external region, the values of β pertaining to guided waves must be restricted to lie between $n_{min}(\omega)$ and $n_{max}(\omega)$, where n_{min} is a refractive index characterizing the cladding and determining the cutoff condition, whereas n_{max} is a refractive index characterizing the core region, attained asymptotically as the frequency grows beyond all bounds.

Cutoff is reached when α_1 or α_3 becomes zero. To find this condition in the polar configuration, one has to set (4.4.19) to zero ($\hat{\kappa}=-j\alpha$) and obtain the solutions of β. These are

$$\beta_a^2=\hat{\varepsilon}_{xx} \quad \text{and} \quad \beta_b^2=\hat{\Delta}_{xx}/\hat{\varepsilon}_{zz} \tag{4.4.21}$$

If $\hat{\varepsilon}_{yy}>\hat{\varepsilon}_{xx}$ then $\alpha_3=0$ at $\beta=\beta_a$ while $\alpha_1=\pm j\sqrt{\hat{\varepsilon}_{yy}-\hat{\varepsilon}_{xx}}$. However, if $\hat{\varepsilon}_{yy}<\hat{\varepsilon}_{xx}$ then $\alpha_1=0$ at $\beta=\beta_a$ while $\alpha_3=\sqrt{\hat{\varepsilon}_{xx}-\hat{\varepsilon}_{yy}}$. Conversely, if $\hat{\varepsilon}_{zz}(1-\hat{\varepsilon}_{yy}/\hat{\varepsilon}_{xx})>|\varepsilon_{yz}|^2(1+\varepsilon_{zz}/\varepsilon_{xx})/\varepsilon_{zz}$ then $\alpha_3=0$ at $\beta=\beta_b$ while

$$\alpha_1=\pm j\sqrt{\varepsilon_{zz}(1-\varepsilon_{yy}/\varepsilon_{xx})+|\varepsilon_{yz}|^2(1+\varepsilon_{zz}/\varepsilon_{xx})/\varepsilon_{zz}}$$, whereas if

$\varepsilon_{zz}(1-\varepsilon_{yy}/\varepsilon_{xx})<|\varepsilon_{yz}|^2(1+\varepsilon_{zz}/\varepsilon_{xx})/\varepsilon_{zz}$ then $\alpha_1 = 0$ at $\beta = \beta_b$ and

$\alpha_3=\pm j\sqrt{\varepsilon_{zz}(1-\varepsilon_{yy}/\varepsilon_{xx})+|\varepsilon_{yz}|^2(1+\varepsilon_{zz}/\varepsilon_{xx})/\varepsilon_{zz}}$, where the plus-minus sign is to assure positive real and negative imaginary parts, respectively.

In high frequencies, the effective guide indices β of the film approach asymptotically the corresponding β_a or β_b defined accordingly with uncaretted ε_{ij}'s, whereas the transverse guide indices $\kappa_{1,3}$ are found from the corresponding $\alpha_{1,3}$ by changing $-j\alpha_i$ to κ_i and $\hat{\varepsilon}_{ij}$ to ε_{ij}. Note that the two cutoff conditions coalesce when $\beta_a = \beta_b$, i.e., when

$$\varepsilon_{yy}-\varepsilon_{xx} = |\varepsilon_{xy}|^2/\varepsilon_{zz} \qquad (4.4.22)$$

In the longitudinal case, the cutoff conditions are obtained by setting (4.4.20) to zero ($\kappa = -j\alpha$). The resulting solutions of β satisfy: $R_{21}R_{34} = |R_{31}|^2$, i.e.,

$$\beta^2_{c,d} = \tfrac{1}{2}\left\{\varepsilon_{xx}+\varepsilon_{yy}\pm\sqrt{(\varepsilon_{xx}-\varepsilon_{yy})^2+4|\varepsilon_{xy}|^2}\right\} \qquad (4.4.23)$$

where the plus (minus) refers to c(d). By defining

$$\beta^2_o = \left\{\varepsilon_{xx}(\varepsilon_{yy}+\varepsilon_{zz})-|\varepsilon_{xy}|^2\right\}/(\varepsilon_{xx}+\varepsilon_{zz}), \qquad (4.4.24)$$

if $\beta^2_{c,d} < \beta^2_o$ then $\alpha_3 = 0$ at $\beta^2 = \beta^2_{c,d}$ while

$\alpha_1 = \pm j\sqrt{(\hat{\varepsilon}_{yy}+\hat{\varepsilon}_{zz}-|\hat{\varepsilon}_{xz}|^2/\hat{\varepsilon}_{xx})-(1+\hat{\varepsilon}_{zz}/\hat{\varepsilon}_{xx})\beta^2_{c,d}}$. However, if

$\beta^2_{c,d} > \beta^2_o$, then $\alpha_1 = 0$ at $\beta^2 = \beta^2_{c,d}$ while

$\alpha_3 = \sqrt{(1+\hat{\varepsilon}_{zz}/\hat{\varepsilon}_{xx})\beta^2_{c,d}-(\hat{\varepsilon}_{yy}+\hat{\varepsilon}_{zz}-|\hat{\varepsilon}_{xy}|^2/\hat{\varepsilon}_{xx})}$, where the ± sign is to assure positive real and negative imaginary parts, respectively. The corresponding high frequency asymptotic behaviour is obtained by changing the $\hat{\varepsilon}_{ij}$ to ε_{ij} and $-j\alpha_i$ to κ_i.

To illustrate these situations the following example is chosen. Consider a symmetric layered waveguide in the polar configuration characterized by:

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} 2.60 & 0. & 0. \\ 0. & 2.64 & 0.001 \\ 0. & 0.001 & 2.62 \end{bmatrix} \quad \text{and} \quad \hat{\bar{\bar{\varepsilon}}} = \begin{bmatrix} 1.96 & 0. & 0. \\ 0. & 1.92 & 0.01 \\ 0. & 0.01 & 1.90 \end{bmatrix} \tag{4.4.25}$$

Since $\hat{\varepsilon}_{yy} < \hat{\varepsilon}_{xx}$, $\alpha_1 = 0$ at $\beta = \beta_a = 1.4$ while $\alpha_3 = 0.2$. Noting that $\hat{\varepsilon}_{zz}(1-\hat{\varepsilon}_{yy}/\hat{\varepsilon}_{xx}) = 0.0388$ and $|\hat{\varepsilon}_{yz}|^2(1+\hat{\varepsilon}_{zz}/\hat{\varepsilon}_{xx})/\hat{\varepsilon}_{zz} = 0.0001$, the alternate guide indices are $\alpha_1 = j0.1972$ and $\alpha_3 = 0$ at $\beta = \beta_b = 1.3856$. Similarly in high frequencies, since $\varepsilon_{yy} > \varepsilon_{xx}$: $\kappa_1 = 0.2$, $\kappa_3 = 0$ at $\beta = \beta_a = 1.6125$ and since $\varepsilon_{zz}(1-\varepsilon_{yy}/\varepsilon_{xx}) = -0.0403$ and $|\varepsilon_{yz}|^2(1+\varepsilon_{zz}/\varepsilon_{xx})/\varepsilon_{zz} = 8\text{x}10^{-7}$:

$\kappa_1 = 0$ and $\kappa_3 = j0.0063$ at $\beta = \beta_b = 1.6248$. This example is illustrated in the $\beta^2 = \beta^2(\ell)$ diagram in Fig. 4.11

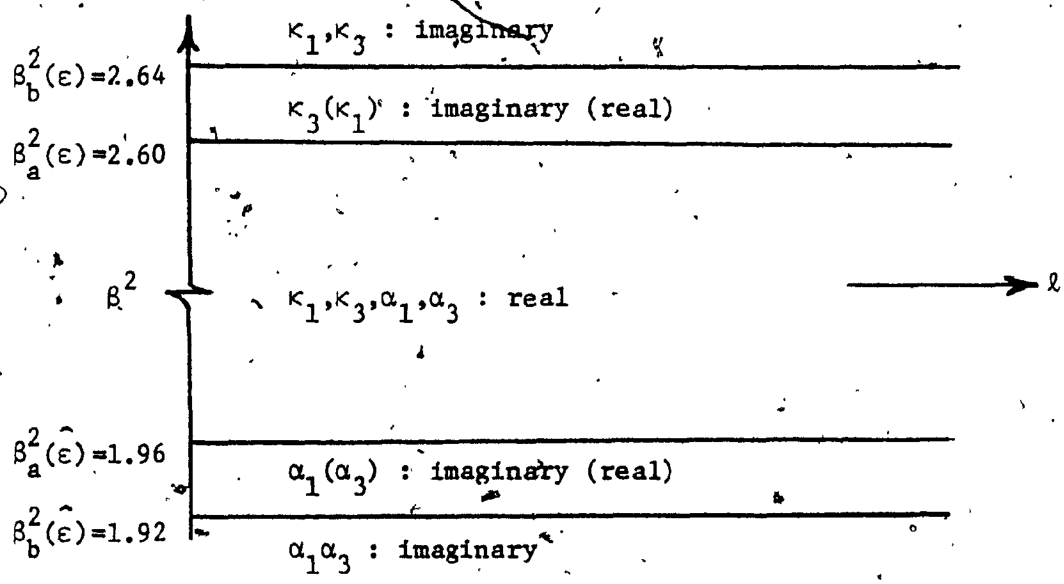

Fig. 4.11 Cutoff conditions and high frequency confinement of a PPP symmetric layered waveguide, characterized by the $\overline{\overline{\varepsilon}}$ and $\hat{\hat{\varepsilon}}$ given in (4.4.25).

(3) Power Flow

Of interest is the fractional z-directed power flow in the core and in the cladding. A criterion for zero power-flow density in the $\pm x$ direction is also needed.

The power propagating in the $\pm x$ direction, is characterized by the power flow density S_x, where

$$S_x = \tfrac{1}{2}\mathrm{Re}\left\{(\eta_o^{-\frac{1}{2}} E_y)(\eta_o^{\frac{1}{2}} H_z)^* - (\eta_o^{-\frac{1}{2}} E_z)(\eta_o^{\frac{1}{2}} H_y)^*\right\}. \qquad (4.4.26)$$

Taking into consideration that S_x in the external region is

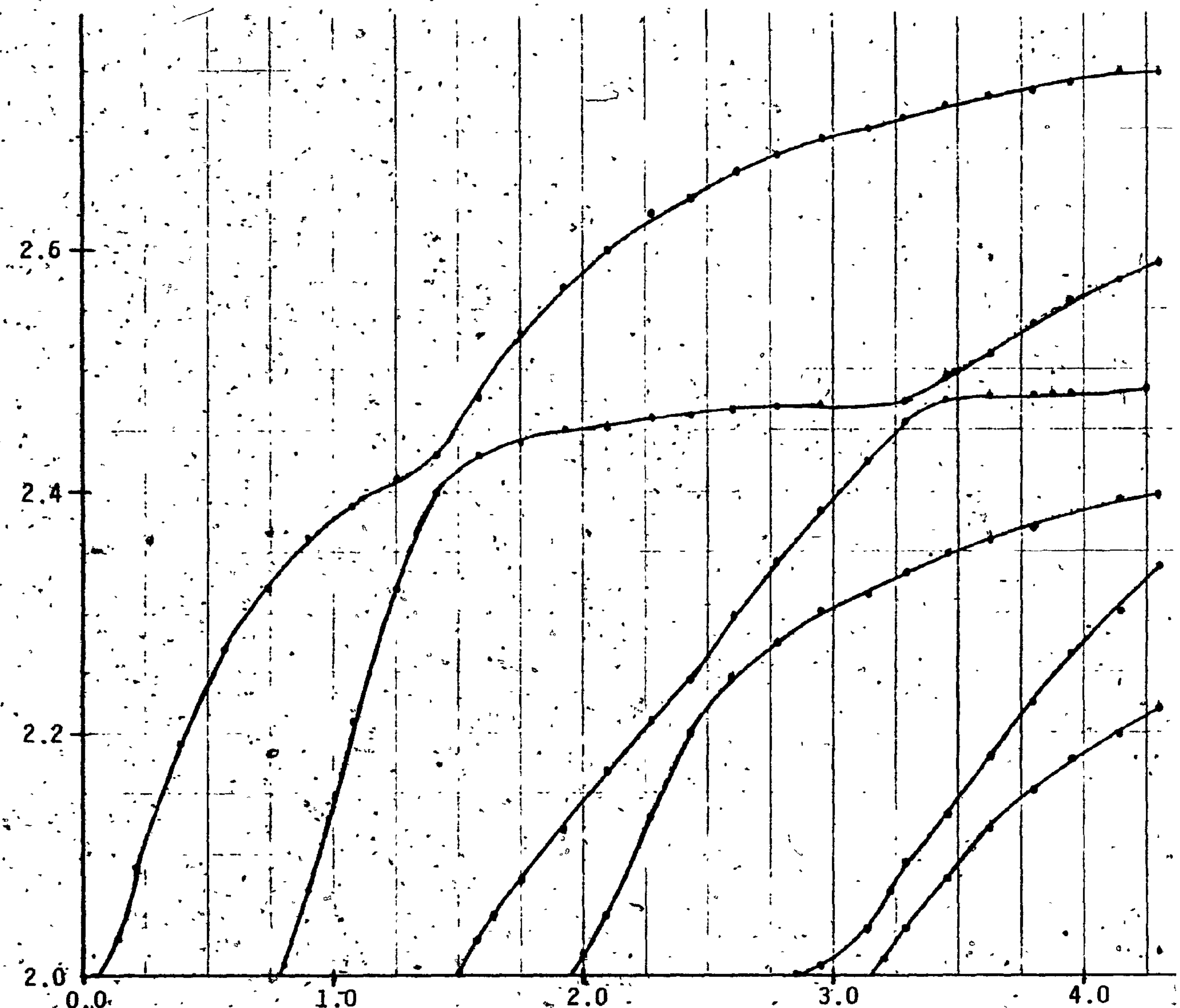

Fig. Modes found using the program LAYER for a PPP waveguide assuming a symmetric field distribution. The permittivity matrices used are:

$$\bar{\bar{\epsilon}} = \begin{matrix} 2.8 & 0.0 & 0.0 \\ 0.0 & 2.5 & 0.1 \\ 0.0 & 0.1 & 2.6 \end{matrix}$$

and

$$\bar{\bar{\epsilon}} = \begin{matrix} 1.8 & 0.0 & 0.0 \\ 0.0 & 2.0 & j0.07 \\ 0.0 & -j0.07 & 1.7 \end{matrix}$$

It is seen that the modes are not equispaced as in the isotropic waveguide and are mutually coupled.

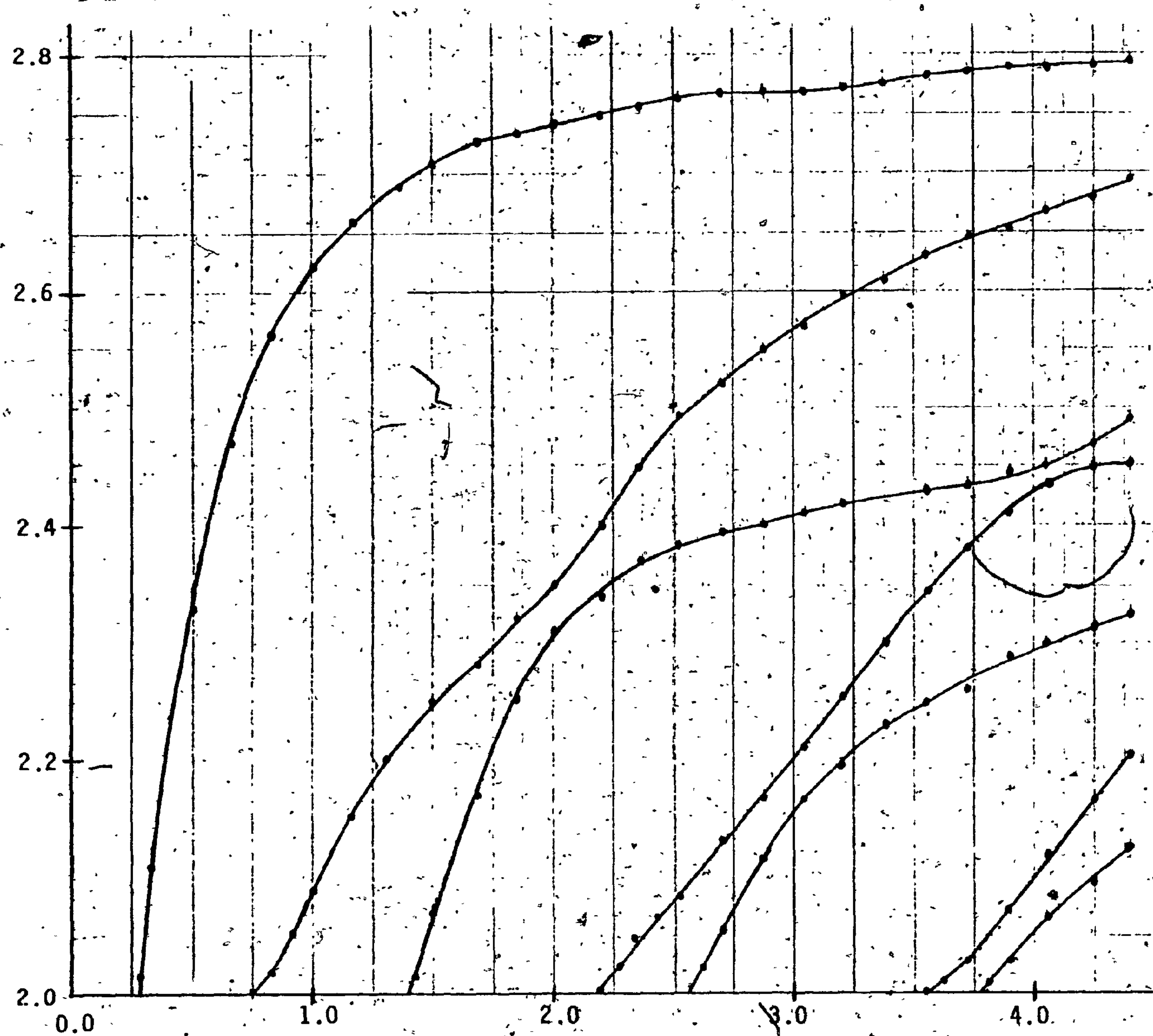

Fig. Modes found using the program LAYER for a PPP waveguide assuming a antisymmetric field distribution. The permittivity matrices used are the same as in the previous example.

the same in symmetric and antisymmetric modes, substitution of the corresponding field components from Table 4.1 or 4.2 into (3.1.13) yields

$$S_x = j\tfrac{1}{2}\,\frac{x}{|x|}\Big\{\alpha_1(1+\hat{a}|\hat{h}_1|^2)|\hat{e}_1|^2\exp[-2\alpha_1(|x|-\ell)] + \alpha_3(1+\hat{a}|\hat{h}_3|^2)|\hat{e}_3|^2\exp[-2\alpha_3(|x|-\ell)]\Big\}, \qquad (4.4.27)$$

whereas in the longitudinal, from Table 4.3 or 4.4

$$S_x = j\tfrac{1}{2}\,\frac{x}{|x|}\Big\{\alpha_1(1-\hat{b}|\hat{g}_1|^2)|\hat{e}_1|^2\exp[-2\alpha_1(|x|-\ell)] + \alpha_3(1-\hat{b}|\hat{g}_3|^2)|\hat{e}_3|^2\exp[-2\alpha_3(|x|-\ell)]\Big\}. \qquad (4.4.28)$$

The fact that both expressions of S_x are imaginary when α_1 and α_3 are both real proves that power propagates along the z direction only, that is, the field distribution corresponds to a propagating mode. In the event that α_1 and/or α_3 is imaginary, one can no longer speak of energy confinement, leakage occurs in the lateral direction; the waveguide does not support a propagating mode.

The z-directed power flow per unit width in the y-direction, is given by:

$$P = \frac{1}{2k_o}\int_{-\infty}^{\infty}\Big\{(\eta_o^{\frac{1}{2}}E_x)(\eta_o^{-\frac{1}{2}}H_y)^* - (\eta_o^{-\frac{1}{2}}E_y)(\eta_o^{\frac{1}{2}}H_x)^*\Big\}dx \qquad (4.4.29)$$

In the symmetric PPP case, for example, P assumes the form

$$k_o P = \tfrac{1}{2}\beta\ell |c_1|^2 \left\{ \frac{a^2}{\varepsilon_{xx}} |\kappa_1|^2 |h_1|^2 \cdot \left| 1 - \frac{\sin 2\kappa_1 \ell}{2\kappa_1 \ell} \right| + \left(1 + \frac{\sin 2\kappa_1 \ell}{2\kappa_1 \ell}\right) \right\}$$

$$+ \tfrac{1}{2}\beta\ell |c_3|^2 \left\{ \frac{a^2}{\varepsilon_{xx}} |\kappa_3|^2 |h_3|^2 \cdot \left| 1 - \frac{\sin 2\kappa_3 \ell}{2\kappa_3 \ell} \right| + \left(1 + \frac{\sin 2\kappa_3 \ell}{2\kappa_3 \ell}\right) \right\}$$

$$- \frac{ab}{\varepsilon_{xx}} \mathrm{Re} \left\{ \kappa_1 \kappa_3^* c_1 c_3^* \left[\frac{\sin(\kappa_1 - \kappa_3^*)\ell}{\kappa_1 - \kappa_3^*} - \frac{\sin(\kappa_1 + \kappa_3^*)\ell}{\kappa_1 + \kappa_3^*} \right] \right\}$$

$$+ \beta \mathrm{Re} \left\{ c_1 c_3^* \left[\frac{\sin(\kappa_1 - \kappa_3^*)\ell}{\kappa_1 - \kappa_3^*} + \frac{\sin(\kappa_1 + \kappa_3^*)\ell}{\kappa_1 + \kappa_3^*} \right] \right\}$$

$$+ \tfrac{1}{2} \frac{\beta}{\alpha_1} |\hat{c}_1|^2 \left(1 + \frac{\hat{a}^2}{\hat{\varepsilon}_{xx}} \alpha_1^2 |\hat{h}_1|^2\right)$$

$$+ \tfrac{1}{2} \frac{\beta}{\alpha_3} |\hat{c}_3|^2 \left(1 + \frac{\hat{a}^2}{\hat{\varepsilon}_{xx}} \alpha_3^2 |\hat{h}_3|^2\right)$$

$$+ \frac{2\beta}{\alpha_1 + \alpha_3} \mathrm{Re}(c_1 c_3^*) \left(1 - \frac{\hat{a}}{\hat{\varepsilon}_{xx}} \alpha_1 \alpha_3\right) \qquad (4.4.30)$$

where the first 4 terms pertain to power flow in the film and the last 3 to power flow in the external region. Note that taking $c_3 = \hat{c}_3 = h_1 = h_3 = \hat{h}_1 = \hat{h}_3 = 0$ and considering that at the boundary: $\hat{c}_1 = c_1 \cos\kappa_1 \ell$, (4.4.30) becomes identical to (3.1.17) of the corresponding isotropic case. The form of $k_o P$ in the rest of the cases, is given in Appendix .

In all cases, k_oP is real provided α_1 and α_3 are both real. Note that some of the terms in the axial power flow expressions are imaginary. Closer investigation reveals that in a lossless waveguide in polar or longitudinal configuration the axial power flow is real and the imaginary terms vanish. This can be shown either from the principle of energy conservation or from a detailed evaluation of c_3 in terms of c_1.

In this section the forms of the effective transverse guide indices of layered media were given and the characteristic equations of waveguides comprised of a film and identical semi-infinite regions, supporting symmetric-antisymmetric field distributions in the polar and longitudinal configurations, were derived. The cutoff conditions and high frequency confinements were derived and depicted in the β^2-ℓ diagram. Finally, it was shown that the modes being supported by such media, are propagating modes along the axial z-direction.

4.5 Optical Transmission Characteristics of Liquid-Crystal Twist-Cells

Liquid-crystal twist-cells are structures made of a uniaxial anisotropic liquid crystal layered between two parallel conducting transparent electrodes. At equilibrium, the liquid crystal molecules lie parallel to the electrode surfaces and the ones adjacent to the two surfaces are normal to each other [27]. Upon the application of an electric field normal to the surfaces, the molecules away from them tend to realign with the field. The crystal's molecular orientation is described by the twist angle δ, which in the

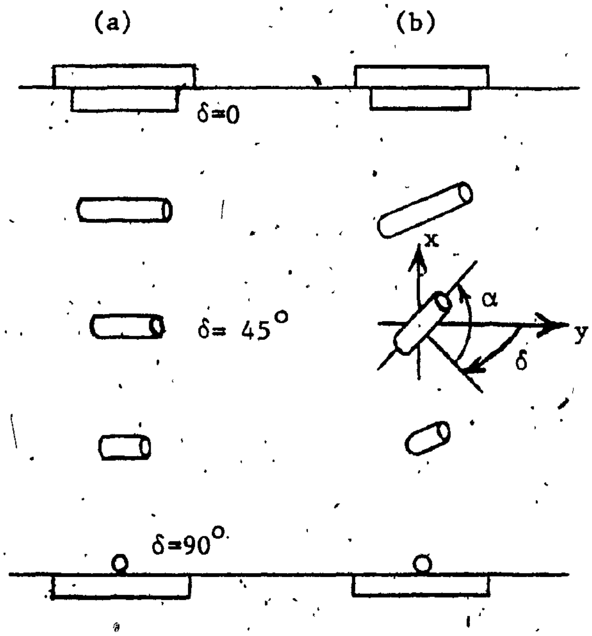

Fig.4.12 Molecular orientation in a liquid-crystal twist-cell: (a) at equilibrium and (b) with an applied field in the transverse direction x. Twist angle δ refers to the rotation of the molecule in the y-z plane whereas tilt angle to its spatial rotation with respect to y-z.

equilibrium state varies linearly with x from 0° to 90° and the tilt angle α, which depends on the strength of the field.

Considering light propagation confined along the transverse direction x, the effective guide indices β and γ take zero values. The (relative) permittivity tensor along the molecular coordinate system is

$$\bar{\bar{\varepsilon}}_p = \begin{bmatrix} \varepsilon_2 & 0 & 0 \\ 0 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_2 \end{bmatrix} \qquad (4.5.1)$$

and the coordinate rotation matrix A, corresponding to

$\theta = \alpha, \ \phi + 3\pi/2 = \delta$ and $\psi = 0$,

$$A = \begin{bmatrix} -\sin\delta & -\cos\alpha\cos\delta & \sin\alpha\cos\delta \\ \cos\delta & \cos\alpha\sin\delta & -\sin\alpha\sin\delta \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \quad (4.5.2)$$

The resulting rotated ε_p is

$$\overline{\overline{\varepsilon}} = \begin{bmatrix} \varepsilon_1\cos^2\alpha\cos^2\delta+\varepsilon_2(1-\cos^2\alpha\cos^2\delta) & (\varepsilon_1-\varepsilon_2)\cos^2\alpha\sin\delta\cos\delta & (\varepsilon_1-\varepsilon_2)\sin\alpha\cos\alpha\cos\delta \\ & \varepsilon_1\cos^2\alpha\sin^2\delta+\varepsilon_2(1-\cos^2\alpha\sin^2\delta) & (\varepsilon_1-\varepsilon_2)\sin\alpha\cos\alpha\sin\delta \\ \text{sym} & & \varepsilon_1\sin^2\alpha+\varepsilon_2\cos^2\alpha \end{bmatrix} \quad (4.5.3)$$

Coupling of the transverse field components due to the off-diagonal elements of ε is expressed by

$$\frac{d}{dx}\,\overline{g}(x) = -jR\overline{g}(x) \quad (4.5.4)$$

where the coupling matrix R is

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \Delta_{zz}/\varepsilon_{xx} & 0 & -\Delta_{yz}/\varepsilon_{xx} & 0 \\ 0 & 0 & 0 & 1 \\ -\Delta_{yz}/\varepsilon_{xx} & 0 & \Delta_{yy}/\varepsilon_{xx} & 0 \end{bmatrix} \quad (4.5.5)$$

The corresponding characteristic equation is quadratic in κ^2 and its roots give the effective guide indices κ_i, viz.,

$$\kappa_{1,3}^2 = \left\{\Delta_{yy} + \Delta_{zz} \pm\sqrt{(\Delta_{yy} - \Delta_{zz})^2 + 4\Delta_{yz}^2}\right\}/2\varepsilon_{xx} \tag{4.5.6}$$

where subscripts 1 and 3 pertain to the upper and lower sign respectively. Finally, the form of the modal matrix is

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \kappa_1 & -\kappa_1 & \kappa_3 & -\kappa_3 \\ A_1 & A_1 & A_3 & A_3 \\ \kappa_1 A_1 & -\kappa_1 A_1 & \kappa_3 A_3 & -\kappa_3 A_3 \end{bmatrix} \tag{4.5.7}$$

where $A_{1,3} = (\Delta_{zz} - \varepsilon_{xx}\kappa_{1,3}^2)/\Delta_{yz}$.

Note that the liquid crystal is a nonuniform medium, in which $\bar{\bar{\varepsilon}}$ (and therefore κ_i,U) is a function of x. The field vector $\bar{g}(x)$ can be written as a linear combination of the eigenvectors $\bar{u}_i(x)$. Hence,

$$\bar{g}(x) = U(x)\bar{a}(x) \tag{4.5.8}$$

where $\bar{a}(x)$ is a proper coefficient-vector. Substituting (4.5.8) into (4.5.4), the following differential equation for $\bar{a}$ obtains

$$\dot{\overline{a}} = -j(U^{-1}\cdot R\cdot U - jU^{-1}\cdot\dot{U})\overline{a} \qquad (4.5.9)$$

where the dot abbreviates differentiation with respect to x. A second transformation via a diagonal matrix $\Lambda(x)$ is introduced

$$\overline{a}(x) = \Lambda(x)\overline{c}(x) \qquad (4.5.10)$$

where the elements of the diagonal matrix are defined to be:

$$\Lambda_{ii}(x) = \exp[-j\int_0^x \kappa_i(\lambda)\,d\lambda]; \quad i = 1 \text{ to } 4. \qquad (4.5.11)$$

Define now T(x), the transfer matrix of $\overline{c}(x)$:

$$\overline{c}(x) = T(x)\overline{c}(0), \qquad (4.5.12)$$

then successive back substitution shows that

$$\overline{g}(\ell) = U(\ell)\cdot\Lambda(\ell)T(\ell)U^{-1}(0)\overline{g}(0) \qquad (4.5.13)$$

where the only unknown matrix is $T(\ell)$. The transfer matrix can be found either numerically or by use of the modal method.

The numerical solution of T [28] is obtained from the solution of a differential equation of $\overline{c}(x)$. The latter, substituting (4.5.10) into (4.5.9), is found to have the form

$$\dot{\bar{c}} = -\Lambda^{-1}\cdot U^{-1}\cdot \dot{U}\Lambda\bar{c} \qquad (4.5.14)$$

After solving this for $c(\ell)$, the form of $T(\ell)$ is deduced from equation (4.5.12) for a known excitation $\bar{c}(0)$.

Another way to find T, is to subdivide the crystal cell in a number of N sublayers, where each one is aseumed to be uniform. The general expression of $\bar{g}(\ell)$ then follows from (3.3.3), where the overall transfer impedance matrix results from the product of the corresponding $G_i(x = \frac{\ell}{N})$ to each individual sublayer, matrices.

Summarizing, field analysis of the nonuniform liquid-crystal twist-cell was outlined and the corresponding formalism, adapted from the one derived earlier, was presented. It was seen that in a nonuniform medium, the discrete propagation factor matrix Λ_g is replaced by Λ of (4.5.11), and the transfer properties of the field coefficients are characterized by a matrix T which is not directly related to Λ or U. Finally, it was shown that the transmission characteristics in a nonuniform medium can be obtained either from the numerical solution of a differential equation satisfied by the field coefficients $\bar{c}(x)$ or from the subdivision of the nonuniform medium in a sufficient number of sub-layers and subsequent application of the field analysis presented earlier for uniform media.

4.6 Reflection Matrix at an Interface of a Biaxial Crystal and Determination of its Complex Dielectric Tensor

The reflection matrix Γ was introduced in the study of electric field reflection at an interface of a multilayered crystal structure. In this section, reflection of light waves impinging normally at the interface of a bulk biaxial crystal will be studied and the general expression of Γ matrix will be derived with respect to the complex permittivity parameters.

Consider an interface of a biaxial crystal lying on the y-z plane, as shown in Fig. 4.13. The permittivity tensor of the crystal is assumed to be in its most general form:

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \qquad (4.6.1)$$

Normal incidence of light waves on the y-z interface, will excite waves in the crystals whose wavefront is parallel to the interface. In other words, $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$ and $\beta = \gamma = 0$.

Thus the wave equation (2.3.30b) reduces to:

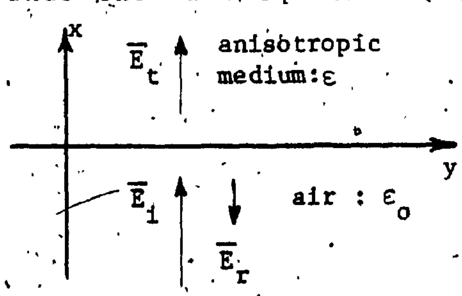

Fig. 4.13 Coordinate orientation of the interface between the crystal and the surrounding air. Light waves impinge normally from the air on the interface.

$$\begin{bmatrix} \varepsilon_{xy} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy}-\kappa^2 & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}-\kappa^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0. \qquad (4.6.2)$$

To find the form of the effective guide indices κ_i, one has first to write the coupling matrix R_g, which recalling (2.3.17) with $\beta = 0$, is

$$\underset{\sim}{R}_g = \frac{1}{\varepsilon_{xx}} \begin{bmatrix} 0 & \varepsilon_{xx} & 0 & 0 \\ \Delta_{zz} & 0 & -\Delta_{zy} & 0 \\ 0 & 0 & 0 & \varepsilon_{xx} \\ -\Delta_{yz} & 0 & \Delta_{yy} & 0 \end{bmatrix} \qquad (4.6.3)$$

and then substitute the corresponding entries of R_g into the characteristic equation (2.3.21). Hence

$$\kappa^4 + b\kappa^2 + d = 0 \qquad (4.6.4a)$$

where

$$b = -\frac{1}{\varepsilon_{xx}} (\Delta_{yy} + \Delta_{zz})$$

$$d = \frac{1}{\varepsilon_{xx}^2} (\Delta_{yy}\Delta_{zz} - |\Delta_{yz}|^2) \qquad (4.6.4b)$$

The quartic form of the characteristic equation (4.6.4a) assures wave bidirectionality, with $\kappa_{1,3}$ the guide indices of the forward travelling waves, given by:

$$\kappa_{1,3}^2 = \frac{1}{2\varepsilon_{xx}} \left[\Delta_{yy}+\Delta_{zz} \pm \sqrt{(\Delta_{yy}-\Delta_{zz})^2+4|\Delta_{yz}|^2} \right] \tag{4.6.5}$$

Note that, since the root in (4.6.5) is always real, κ_1 and κ_3 are always purely real or imaginary.

With κ_i known, one can now expand the cofactors of the first row of the matrix in (4.6.2) to obtain the polarization vector $\bar{p}_i$, which along with the corresponding $\bar{q}_i$ are given by:

$$\bar{p}_i = \begin{bmatrix} (\varepsilon_{yy}-\kappa_i^2)(\varepsilon_{zz}-\kappa_i^2)-\varepsilon_{yz}\varepsilon_{zy} \\ \varepsilon_{yz}\varepsilon_{zx}-\varepsilon_{yx}(\varepsilon_{zz}-\kappa_i^2) \\ \varepsilon_{yx}\varepsilon_{zy}-\varepsilon_{zx}(\varepsilon_{yy}-\kappa_i^2) \end{bmatrix}, \quad \bar{q} = \begin{bmatrix} 0 \\ -\kappa_i p_{iz} \\ \kappa_i p_{iy} \end{bmatrix} \quad ; \; i=1,3. \tag{4.6.6}$$

The transverse electromagnetic field distribution in the crystal thus is:

$$\begin{aligned} \eta_o^{-\frac{1}{2}} E_y &= c_1 p_{1y}\exp(-j\kappa_1 x) + c_3 p_{3y}\exp(-j\kappa_3 x) \\ \eta_o^{-\frac{1}{2}} E_z &= c_1 p_{1z}\exp(-j\kappa_1 x) + c_3 p_{3z}\exp(-j\kappa_3 x) \\ \eta_o^{\frac{1}{2}} H_y &= -c_1\kappa_1 p_{1z}\exp(-j\kappa_1 x) - c_3\kappa_3 p_{3z}\exp(-j\kappa_3 x) \\ \eta_o^{\frac{1}{2}} H_z &= c_1\kappa_1 p_{1y}\exp(-j\kappa_1 x) + c_3\kappa_3 p_{3y}\exp(-j\kappa_3 x) \end{aligned} \tag{4.6.7}$$

Assuming now an incident light wave polarized parallel to the y axis with field amplitude e_o, the superimposed incident and reflected waves in the air, are:

$$\eta_o^{-\frac{1}{2}} E_y = e_o \exp(-jx) + e_1 \exp(jx)$$
$$\eta_o^{-\frac{1}{2}} E_z = e_2 \exp(jx)$$
$$\eta_o^{\frac{1}{2}} H_y = e_2 \exp(jx)$$
$$\eta_o^{\frac{1}{2}} H_z = e_o \exp(-jx) - e_1 \exp(jx), \qquad (4.6.8)$$

where e_o and e_1, e_2 are the incident and reflected amplitudes, respectively. Continuity of the transverse field components - given at (4.6.7) and (4.6.8), at the interface, yields:

$$c_1 p_{1y} + c_3 p_{3y} = e_o + e_1$$
$$c_1 p_{1z} + c_3 p_{3z} = e_2$$
$$c_1 \kappa_1 p_{1z} + c_3 \kappa_3 p_{3z} = -e_2$$
$$c_1 \kappa_1 p_{1y} + c_3 \kappa_3 p_{3y} = e_o - e_1 \qquad (4.6.9)$$

The electric field reflection coefficient matrix Γ, defined in (3.5.7) from the equation

$$\eta_o^{-\frac{1}{2}} E_{\tau r} = \Gamma(\eta_o^{-\frac{1}{2}} E_{\tau i}) \qquad (4.6.10)$$

can be found by first evaluating from (4.6.9).

$$\Gamma_{yy} = \frac{e_1}{e_o} \quad , \quad \Gamma_{zy} = \frac{e_2}{e_o} \tag{4.6.11}$$

and then interchanging y and z to obtain Γ_{yz} and Γ_{zz}.

$$\Gamma = \frac{2}{p_{1y}p_{3z}-p_{3y}p_{1z}} \begin{bmatrix} p_{1y}p_{3z}m_1-p_{3y}p_{1z}m_3 & p_{1y}p_{3y}(m_1-m_3) \\ p_{1z}p_{3z}(m_1-m_3) & p_{3y}p_{1z}m_1-p_{1y}p_{3z}m_3 \end{bmatrix} \tag{4.6.12}$$

where

$$m_i = 1/(1+\kappa_i) \quad ; \quad i = 1,3 \tag{4.6.13}$$

Equation (4.6.12) gives the reflection coefficient matrix Γ, at the interface between an isotropic medium and a biaxial crystal, for normal incidence of electromagnetic waves propagating in the isotropic medium. Note that Γ_{jk} (j,k=y,z) is the reflection coefficient of a k-polarized incident wave which is reflected as j-polarized.

Although the form of Γ was derived for normal-incidence at the y-z interface, by proper change of the subscripts, similar formulae to (4.6.12) can be written for reflection at the z-x and x-y interfaces. Thus, the complex Γ_{yy}, Γ_{zz} at y-z, Γ_{zz}, Γ_{xx} at z-x and Γ_{xx}, Γ_{yy} at x-y interfaces can theoretically be expressed with respect to $\bar{\bar{\varepsilon}}$. By taking into account their amplitude and phase, twelve equations for the real and imaginary parts of ε_{ij} can be obtained, whose solution yields $\bar{\bar{\varepsilon}}$ (assuming it is symmetric or c.c.). Note that this is rather awkward task and time consuming. Instead, it has been proposed that one can use dispersion analysis to determine $\bar{\bar{\varepsilon}}$ from the six transverse reflectances $|\Gamma_{ii}|^2$ ($i = x,y,z$) at the three crystal's interfaces [29]. However, this is out of the purpose of the present thesis and will not be dealt here.

Summarizing, the complex reflection coefficient matrix at the three interfaces of a bulk crystal (assumed to have a cubic shape), was found. Theoretical expression of reflectances were derived with respect to the permittivity tensor parameters, for arbitrary directions of the crystal's coordinate system. Finally, it was suggested that the reflectances can be considered as sample measurements of $\bar{\bar{\varepsilon}}$ and can be used for its measurement.

CHAPTER 5

CONCLUSION

The main objective of this thesis has been a formalism of mode analysis in birefringent media leading to the network representation of layered anisotropic structures.

To facilitate understanding of wave propagation in birefringent media, first the classical formulation of the macroscopic response of an anisotropic medium to an electromagnetic wave excitation has been reviewed. The electro-optic effect, optical activity and Faraday rotation have also been encompassed to form a generalized treatment of birefringence. Maxwell's equations have been expressed in a 4x4 matrix form, by suppressing two unused components of the field. The eigenvalues of the resulting coupling (system) matrix provide the four transverse guide indices. The electromagnetic field distribution in each layer was shown to be a linear combination of the polarization vectors - found from the eigenvectors of the above coupling matrix.

In Chapter 3, under the assumption of wave bidirectionality, the forward/backward wave impedance has been defined and discussed. The bidirectional form of the propagating modes led to a four-port network representation of the anisotropic layer. Laws governing losslessness, reciprocity and symmetry have been discussed and expressed in terms of the layer transfer matrix and the system coupling matrix. The new concept of semireciprocity, realized by connecting in series a reciprocal and an antireciprocal four-port, has also been examined. Reflection and transmission at an interface between two dissimilar anisotropic media has been analyzed in terms of the field, wave and coefficient vectors.

Expressions have also been given for the reflection coefficient matrix of a stack of layers. The characteristic equation of a layered structure terminated by mode-converting two ports has been presented using the transverse resonance condition.

In Chapter 4, three bidirectional configurations have been analyzed. Symmetric and antisymmetric field distributions were obtained. Mixed distributions comprised of symmetric and antisymmetric terms have been found to violate the boundary conditions. The polar and longitudinal configurations were found to be complementary, in a sense that, the effect of a real off-diagonal term in the polar permittivity tensor is similar to the effect of an imaginary off-diagonal term in the longitudinal case. TE-TM mode coupling characterizing the polar and longitudinal configurations does not apply to the equatorial case, where pure TE and TM modes are valid solutions. In the latter case, the TE modes are identical to those in the isotropic slab waveguide, whereas the TM modes exhibit a peculiar behaviour.

Wave propagation in symmetric layered waveguides comprised of anisotropic media of similar and dissimilar configurations has also been analyzed. Two equivalent methods for obtaining the characteristic equation were presented, one utilizing the secular equation obtained from the boundary conditions, the other, the impedance concept. The input impedance to a terminated anisotropic layer has been found to show an analogous behaviour to that of the isotropic case. The polar and longitudinal configurations have been analyzed using a characteristic parameter (h and g, respectively), which enables one to treat them in a unified manner. This is particularly useful in numerical work and in programming.

The propagating modes are characterized by at least one real transverse guide index in the film and purely imaginary indices in the cladding. The cutoff conditions, which result when the imaginary part of the cladding's guide index vanishes, have been determined together with the asymptotic high frequency behaviour of the film's axial guide index. Finally, the power carried by the propagating modes has been examined and power formulae have been derived for the symmetric waveguides considered.

In summary, a detailed matrix analysis of wave propagation in birefringent layered media has been presented. Transfer and reflection matrices have been developed for an interface. The characteristic equation, power flow, cutoff conditions and the equivalent network has been obtained for the three layer symmetric waveguide fabricated using anisotropic media in the longitudinal and/or polar configuration. The analysis has been restricted to lossless media and transversally bidirectional configuration. Suggested further work would deal with wave propagation in asymmetric multi-layered waveguides, as well as with layered structures utilizing lossy, biaxial media.

REFERENCES

1. Felsen B. L., Marcuvitz N. : "Radiation and Scattering of Waves", Prentice-Hall, Inc., New York, 1973, pp. 79-80.

2. Yariv A. : "Quantum Electronics", John Wiley & Sons, New York, Second Edition, 1975, pp. 84-85.

3. Born M., Wolf E. : "Principles of Optics", Pergamon Press, Sixth Edition, 1980, pp. 673-676.

4. Ramaswamy V. : "Propagation in Asymmetrical Anisotropic Film Waveguides", Applied Optics, Vol. 13, No. 6, June 1974, pp. 1363-1371.

5. Ramaswamy V. : "Ray Model of Energy and Power Flow in Anisotropic Film Waveguides", J. O. S. A., Vol. 64, No. 10, October 1974, pp. 1313-1320.

6. Yamamoto S., Koyamada Y., Makimoto T. : "Normal-Mode Analysis of Anisotropic and Gyrotropic Thin-Film Waveguides for Integrated Optics", J. Appl. Phys., Vol. 43, No. 12, December 1972, pp. 5090-5097.

7. Marcuse D. : "Theory of Dielectric Optical Waveguides", Academic Press, New York, 1974, pp. 5-7.

8. Auld B. A. : "Acoustic Fields and Waves in Solids", John Wiley & Sons, New York, 1973, Vol. I, pp. 250-256.

9. Nye J. F. : "Physical Properties of Crystals", Oxford University Press, 1957, pp. 260-267.

10. Portigal D. L., Burstein E. : "Acoustical Activity and other First Order Spatial Dispersion Effects in Crystals", Phys. Rev., Vol. 170, No. 3, 1968, pp. 673-678.

11. Landau L. D., Lifschitz E. M. : "Statistical Physics", Pergamon Press, Ltd., Oxford, 1958, p. 104.

12. Sommerfeld A. : "Optics", Academic Press, New York, 1973, Vol. I, pp. 250-256.

13. Nussbaum A., Phillips R. A. : "Contemporary Optics for Scientists and Engineers", Prentice-Hall, Inc., New Jersey, 1976, pp. 384-387.

14. Tabor W. J. : "Magneto-optic Materials", in Laser Handbook, Arecchi F. T. Schulz-Dubois E. O., Eds., North-Holland Publ. Co., Amsterdam, 1972, p. 1011.

15. Yariv A., Lotspeich J. F. : "Coupled-Mode Analysis of Light Propagation in Optically Active Crystals", J. O. S. A., Vol. 72, No. 2, February 1982, pp. 273-277.

16. Goldstein H. : "Classical Mechanics", Addison-Wesley Publ. Co., Inc., Reading, Mass., 1965, Sec. 4.4.

17. Marcuse D. : "Light Transmission Optics", Van Nostrand Reinhold Co., New York, 1972, Sec. 8.3.

18. Schwelb O. : "The Transverse Resonance Method and its Applications", Internal Technical Report, Dept. of Electrical Engineering, Concordia University, 1980.

19. Tien P. K., Ulrich R. : "Theory of Prism-Film Coupler and Thin-Film Light Guides", J. O. S. A., Vol. 60, No. 10, October 1970, pp. 1325-1337.

20. Seshadri S. R. : "Fundamentals of Transmission Lines and Electromagnetic Fields", Addison-Wesley Publ. Co., Reading, Mass., 1971, Sec. 2.11.

21. Schwelb O. : "Network Representation and Transverse Resonance for Layered Anisotropic Dielectric Waveguides", M. T. T., Vol. 30, No. 6, June 1982, pp. 899-905.

22. Noble B., Daniel J. W. : "Applied Linear Algebra" Prentice-Hall, Inc., New Jersey, second edition, 1977, Sec. 8.4.

23. Schwelb O., Antepyan R. : "Conservation Laws for Integrated Circuit Four Ports", submitted for publication.

24. Harrington R. F., Villeneuve A. T. : "Reciprocal Relations for Gyrotropic Media", Trans. IEEE, Vol. MTT-6, No. 3, July 1958, pp 308-310.

25. Marcuse D. : "Modes of a Symmetric Slab Optical Waveguide in Birefringent Media - Part I : Optical Axis Not in Plane of Slab", IEEE Journal of Q. E., Vol. QE-14, No. 10, October 1978, pp. 736-741.

26. Burns W. K., Warner J. : "Mode Dispersion in Uniaxial Optical Waveguides", J. O. S. A., Vol. 64, No. 4, April 1974, pp. 441-446.

27. Berreman D. W. : "Dynamics of liquid-crystal twist-cells", Appl. Phys. Let., Vol. 25, No. 1, July 1974, pp. 12-15.

28. Gagnon R. J. : "Liquid-Crystal Twist-Cell Optics", J. O. S. A., Vol. 71, No. 3, March 1981, pp. 348-353.

29. Emslie A. G., Aronson J. R. : "Determination of the Complex Dielectric Tensor of Triclinic Crystals : Theory", J. O. S. A., Vol. 73, No. 7, July 1983, pp. 916-919.

30. Wang S., Shah M., Crow J. : "Studies of the Use of Gyrotropic and Anisotropic Materials for Mode Conversion in Thin-Film Optical-Waveguide Applications", J. Appl. Phys., Vol. 43, No. 4, April 1972, pp. 1861-1875.

31. Wang S., Crow J. D., Wong S. L., Shah M. : "Eigenmode Analysis of Wave Propagation in Optical Waveguides Deposited on Gyrotropic and Anisotropic Substrates", J. Appl. Phys., Vol. 44, No. 7, July 1973, pp. 3232-3239.

32. Schesser J., Eichmann G. : "Propagation of Plane Waves in Biaxially Anisotropic Layered Media", J. O. S. A., Vol. 62, No. 6, June 1972, pp. 786-791.

33. Berreman D. W. : "Optics in Stratified and Anisotropic Media : 4x4 Matrix Formulation", J. O. S. A., Vol. 62, No. 4, April 1972, pp. 502-509.

34. Kishioka K. Rokushima K. : "Unified Analysis of Guided and Radiation Modes in Anisotropic Slab Waveguide", Electr. and Com. in Japan, Vol. 61-B, No. 6, 1978, pp. 62-71.

35. Yeh P. : "Electromagnetic Propagation in Birefringent Layered Media", J. O. S. A., Vol. 69, No. 5, May 1979, pp. 742-756.

36. Adair J.E.:"Coupled-mode analysis of nonuniform coupled transmission lines", Ph.D. Thesis 1968, University of Michigan, Ann Arbor, Michigan.

APPENDIX

I. Power flow in symmetric layered waveguides.

II. Fortran program WAVE.

III. Fortran program LAYER.

APPENDIX I

Power Flow In Symmetric Layered Waveguides

The z-directed power flow P per unit width in the y-direction, defined by

$$k_oP = \tfrac{1}{2}\int_{-\infty}^{\infty}\left\{(\eta_o^{-\frac{1}{2}}E_x)(\eta_o^{\frac{1}{2}}H_y^*) - (\eta_o^{-\frac{1}{2}}E_y)(\eta_o^{\frac{1}{2}}H_x^*)\right\}dx,$$

is formulated below for layered symmetric waveguides, supporting symmetric/antisymmetric field distributions in the polar and longitudinal configurations. The power formulae of the core pertain to a film of normalized width 2ℓ ($\ell = k_oL$) and the ones of cladding to the total power of substrate and cover. Summation of the proper formulae of film and cladding results in the z-directed power flow per unit width of the corresponding waveguiding structure. The imaginary terms appearing in the power formulae of the longitudinal configuration must be zero in the guided wave region because there can be no reactive z-directed power flow in a lossless waveguide.

1. Film

(a) Polar configuration with symmetric modes

$$k_oP = \tfrac{1}{2}\beta\ell|c_1|^2\cdot\left\{|\kappa_1|^2\cdot|h_1|^2\frac{a^2}{\varepsilon_{xx}}\left|1-\frac{\sin 2\kappa_1\ell}{2\kappa_1\ell}\right| + (1+\frac{\sin 2\kappa_1\ell}{2\kappa_1\ell})\right\}$$

$$+\tfrac{1}{2}\beta\ell|c_3|^2\left\{|\kappa_3|^2\cdot|h_3|^2\frac{a^2}{\varepsilon_{xx}}\left|1-\frac{\sin 2\kappa_3\ell}{2\kappa_3\ell}\right| + (1+\frac{\sin 2\kappa_3\ell}{2\kappa_3\ell})\right\}$$

$$-\frac{\beta a}{\varepsilon_{xx}}\cdot\mathrm{Re}\left\{\kappa_1\kappa_3^*c_1c_3^*\left[\frac{\sin(\kappa_1-\kappa_3^*)}{\kappa_1-\kappa_3^*} - \frac{\sin(\kappa_1+\kappa_3^*)\ell}{\kappa_1+\kappa_3^*}\right]\right\}$$

$$+ \beta \mathrm{Re}\left\{c_1 c_3^* \left[\frac{\sin(\kappa_1-\kappa_3^*)\ell}{\kappa_1-\kappa_3^*} + \frac{\sin(\kappa_1+\kappa_3^*)\ell}{\kappa_1+\kappa_3^*}\right]\right\}$$

(b) Polar configuration with antisymmetric modes

$$k_o P = \tfrac{1}{2}\beta\ell|c_1|^2 \cdot \left\{|\kappa_1|^2 \cdot |h_1|^2 \frac{a^2}{\varepsilon_{xx}} \left(1 + \frac{\sin 2\kappa_1\ell}{2\kappa_1\ell}\right) + \left|1 - \frac{\sin 2\kappa_1\ell}{2\kappa_1\ell}\right|\right\}$$

$$+ \tfrac{1}{2}\beta\ell|c_3|^2 \cdot \left\{|\kappa_3|^2 \cdot |h_3|^2 \frac{a^2}{\varepsilon_{xx}} \left(1 + \frac{\sin 2\kappa_3\ell}{2\kappa_3\ell}\right) + \left|1 - \frac{\sin 2\kappa_3\ell}{2\kappa_3\ell}\right|\right\}$$

$$- \frac{\beta a}{\varepsilon_{xx}} \mathrm{Re}\left\{\kappa_1\kappa_3^* c_1 c_3^* \left[\frac{\sin(\kappa_1-\kappa_3^*)\ell}{\kappa_1-\kappa_3^*} + \frac{\sin(\kappa_1+\kappa_3^*)\ell}{\kappa_1+\kappa_3^*}\right]\right\}$$

$$+ \beta \mathrm{Re}\left\{c_1 c_3^* \left[\frac{\sin(\kappa_1-\kappa_3^*)\ell}{\kappa_1-\kappa_3^*} - \frac{\sin(\kappa_1+\kappa_3^*)\ell}{\kappa_1+\kappa_3^*}\right]\right\}$$

(c) Longitudinal configuration with symmetric modes

$$k_o P = \tfrac{1}{2}\beta\ell|c_1|^2 \left(1 + \frac{\sin 2\kappa_1\ell}{2\kappa_1\ell}\right) \left\{1 + \frac{|g_1|^2}{\beta^2} (1 - b\kappa_1^2)\right\}$$

$$+ \tfrac{1}{2}\beta\ell|c_3|^2 \left(1 + \frac{\sin 2\kappa_3\ell}{2\kappa_3\ell}\right) \left\{1 + \frac{|g_3|^2}{\beta^2} (1 - b\kappa_3^2)\right\}$$

$$+ \frac{1}{\beta}\left(\beta^2 + \kappa_1^2 + \kappa_3^2 - \frac{2}{b}\right) \mathrm{Re}\left\{c_1 c_3^* \left[\frac{\sin(\kappa_1-\kappa_3^*)\ell}{\kappa_1-\kappa_3^*} + \frac{\sin(\kappa_1+\kappa_3^*)\ell}{\kappa_1+\kappa_3^*}\right]\right\}$$

$$+ j\frac{1}{\beta}(\kappa_1^2 - \kappa_3^2) \mathrm{Im}\left\{c_1 c_3^* \left[\frac{\sin(\kappa_1-\kappa_3^*)\ell}{\kappa_1-\kappa_3^*} + \frac{\sin(\kappa_1+\kappa_3^*)\ell}{\kappa_1+\kappa_3^*}\right]\right\}$$

(d) Longitudinal configuration with antisymmetric modes

$$k_o P = \tfrac{1}{2}\beta\ell|c_1|^2 \cdot \left|1 - \frac{\sin 2\kappa_1\ell}{2\kappa_1\ell}\right| \left\{1 + \frac{|g_1|^2}{\beta^2} (1 - b\kappa_1^2)\right\}$$

$$+ \tfrac{1}{2}\beta\ell|c_3|^2 \cdot \left|1 - \frac{\sin 2\kappa_3\ell}{2\kappa_3\ell}\right| \cdot \left\{1 + \frac{|g_3|^2}{\beta^2}(1 - b\,\kappa_3^2)\right\}$$

$$+ \frac{1}{\beta}(\beta^2 + \kappa_1^2 + \kappa_3^{*2} - \frac{2}{b})\,\mathrm{Re}\left\{c_1 c_3^* \left[\frac{\sin(\kappa_1 - \kappa_3^*)\ell}{\kappa_1 - \kappa_3^*} - \frac{\sin(\kappa_1 + \kappa_3^*)\ell}{\kappa_1 + \kappa_3^*}\right]\right\}$$

$$+ j\,\frac{(\kappa_1^2 - \kappa_3^2)}{\beta}\,\mathrm{Im}\left\{c_1 c_3^* \left[\frac{\sin(\kappa_1 - \kappa_3^*)\ell}{\kappa_1 - \kappa_3^*} - \frac{\sin(\kappa_1 + \kappa_3^*)\ell}{\kappa_1 + \kappa_3^*}\right]\right\}$$

2. Cladding

(a) Polar configuration

$$k_o P = \frac{\beta}{2\alpha_1} |\hat{c}_1|^2 \left\{1 + \alpha_1^2 |\hat{h}_1|^2 \frac{\hat{a}^2}{\hat{\varepsilon}_{xx}}\right\}$$

$$+ \frac{\beta}{2\alpha_3} |\hat{c}_3|^2 \left\{1 + \alpha_3^2 |\hat{h}_3| \frac{\hat{a}^2}{\hat{\varepsilon}_{xx}}\right\}$$

$$+ \frac{2\beta}{\alpha_1 + \alpha_3}\,\mathrm{Re}(\hat{c}_1 \hat{c}_3^*)(1 - \alpha_1 \alpha_3 \frac{\hat{a}}{\hat{\varepsilon}_{xx}})$$

(b) Longitudinal configuration

$$k_o P = \frac{\beta}{2\alpha_1} |\hat{c}_1|^2 \cdot \left\{1 + \frac{|\hat{g}_1|^2}{\beta^2}(1 + \hat{b}\,\alpha_1^2)\right\}$$

$$+ \frac{\beta}{2\alpha_3} |\hat{c}_3|^2 \cdot \left\{1 + \frac{|\hat{g}_3|^2}{\beta^2}(1 + \hat{b}\,\alpha_3^2)\right\}$$

$$+ \frac{1}{\beta(\alpha_1 + \alpha_3)}(2\beta^2 - \alpha_1^2 - \alpha_3^2 - \frac{2}{b})\,\mathrm{Re}(\hat{c}_1 \hat{c}_3^*)$$

$$+ j\,\frac{\alpha_3 - \alpha_1}{\beta}\,\mathrm{Im}(\hat{c}_1 \hat{c}_3^*)$$

APPENDIX II

Fortran Program WAVE

Program WAVE computes the propagation and transfer characteristics of an unconfined anisotropic medium in the polar and longitudinal configurations. The program works with the formulae derived in this thesis and is mostly self explanatory.

Given the permittivity matrix (ERP/ERL in polar/longitudinal configuration), the axial effective guide index β (BETA) and a normalized transverse distance ℓ (L), it computes the:

1) coupling matrix R_g (RGP/RGL),
2) effective transverse guide indices κ_i (KP/KL),
3) modal matrices U and V (UP, UL/VP, VL) formed by right and left normalized eigenvectors of the coupling matrix R_g,
4) forward/backward wave impedances z_w (ZWP, ZWPB/ZWL, ZWLB),
5) propagation-factor matrix Λ_g (LGP/LGL),
6) impedance-transfer matrix G (GP/GL) and its principal 2x2 partitions J and F (JFP, JBP/FFL, FBL) in the forward/backward direction and finally the
7) reflection matrix Γ at an interface with an isotropic medium characterized by ε_s (ES).

It performs tests for losslessness, reciprocity, antireciprocity, semi-reciprocity, bilateral/transversal symmetry using the impedance transfer matrix. It also includes an orthogonality test between the right and left modal matrices U and V. To compute the inverse of 2×2 and 4×4 matrices, WAVE calls the LEQ2C IMSL subroutine, whereas to evaluate the product of three matrices, it calls subroutine PRODU. This subroutine, using adjustable arrays, can evaluate the product of 2×2 and 4×4 matrices. Finally, the print-out is adjustable through control-flags (M) provided in the output section of the main program.

```
      PROGRAM WAVE(INPUT,DEFAUL,OUTPUT,TAPE1=INPUT,TAPE5=DEFAUL)
C
      COMPLEX ERL(3,3),ERP(3,3),RGL(4,4),RGP(4,4)
      COMPLEX   KL(4), KP(4),DUMMY(4,4)
      COMPLEX UL(4,4),UP(4,4),QLZ(4),PLZ(4),QLY(4),QPZ(4),PPZ(4),QPY(4)
      COMPLEX LRL(4,4),LRP(4,4),PFL(2,2),PFP(2,2),PBL(2,2),PBP(2,2)
      COMPLEX QFL(2,2),QFP(2,2),QBL(2,2),QBP(2,2),LFL(2,2),LBL(2,2)
      COMPLEX LFP(2,2),LBP(2,2),P23(4,4),KS
      COMPLEX QFLINV(2,2),QFPINV(2,2),QBLINV(2,2),QBPINV(2,2)
      COMPLEX PFLINV(2,2),PFPINV(2,2),PBLINV(2,2),PBPINV(2,2)
      COMPLEX  FFL(2,2), FFP(2,2), FBL(2,2), FBP(2,2)
      COMPLEX  JFL(2,2), JFP(2,2), JBL(2,2), JBP(2,2)
      COMPLEX ZWL(2,2),ZWP(2,2),ZWLB(2,2),ZWPB(2,2)
      REAL L,LGIVEN,ES,WK(4)
      COMPLEX ULTIL(4,4),UPTIL(4,4)
      COMPLEX ID(2,2),EXPOL(4),EXPOP(4)
      COMPLEX GL(4,4),GP(4,4),LGL(4,4),LGP(4,4),ULTILI(4,4),UPTILI(4,4)
      COMPLEX LGLTIL(4,4),LGPTIL(4,4)
      COMPLEX H1L(2,2),H2L(2,2),H3L(2,2),H4L(2,2)
      COMPLEX H1P(2,2),H2P(2,2),H3P(2,2),H4P(2,2)
      COMPLEX K1L(2,2),K2L(2,2),K3L(2,2),K4L(2,2)
      COMPLEX K1P(2,2),K2P(2,2),K3P(2,2),K4P(2,2)
      COMPLEX IDEN(4,4),MODALL(4,4),MODALP(4,4),WA(24)
      COMPLEX LGL1(4,4),LGP1(4,4),LGL1TL(4,4),LGP1TL(4,4)
      COMPLEX GL1(4,4),GP1(4,4),GLT(4,4),GPT(4,4)
      COMPLEX GLREC(4,4),GPREC(4,4),DUM1(2,2),DUM2(2,2),DUM3(2,2)
      COMPLEX GLOSSL(4,4),GLOSSP(4,4),GLHER(4,4),GPHER(4,4)
      COMPLEX KLCC(4),KPCC(4),QLZCC(4),GAMAL(2,2),GAMAP(2,2)
      COMPLEX PLZCC(4),QLYCC(4),VL(4,4),VP(4,4),QPZCC(4),PPZCC(4)
      COMPLEX QPYCC(4),VUL(4,4),VUP(4,4)
      COMPLEX VL1(4),VL2(4),VL3(4),VL4(4),VP1(4),VP2(4),VP3(4),VP4(4)
      COMPLEX VLHERM(4,4),VPHERM(4,4),Z1PCC,Z2PCC,DUM4(2,2),JJ
      COMPLEX MATRIX1(2,2),MATRIX2(2,2),MATRIX3(2,2),MATRIX4(2,2)
      COMPLEX YFL(2,2),YFP(2,2),YF1(2,2),YB1(2,2)
      COMPLEX KSAI1(4,4),KSAI2(4,4),KSAI3(4,4),SIGMA1(4,4)
      COMPLEX GLBILA(4,4),GPBILA(4,4),TRIAL(4,4),GLSEMI(4,4)
      COMPLEX GPSEMI(4,4),GLANTI(4,4),GPANTI(4,4),SIGMA2(4,4)
       COMPLEX GLTRAN(4,4),GPTRAN(4,4)
C
C     INITIALIZATION
C     ==============
C
      BETA=1.483189
      PI=4*ATAN(1.0)
      L=0.8*PI
      SMALL=1.E-6
      ES=1.0
      LGIVEN=L/PI
      BETA2=BETA**2
      KS=ES-BETA2
      KS=CSQRT(KS)
C
      DO 2 I=1,2
      DO 2 J=1,2
      ID(I,J)=CMPLX(0.,0.)
      ID(I,I)=CMPLX(1.,0.)
2     CONTINUE
      DO 4 I=1,4
      DO 4 J=1,4
      KSAI1(I,J)=KSAI2(I,J)=KSAI3(I,J)=SIGMA1(I,J)=CMPLX(0.,0.)
      P23(I,J)=IDEN(I,J)=SIGMA2(I,J)=CMPLX(0.,0.)
      IDEN(I,I)=CMPLX(1.,0.)
4     CONTINUE
      P23(1,1)=P23(2,3)=P23(3,2)=P23(4,4)=CMPLX(1.,0.)
      KSAI1(1,1)=KSAI1(2,2)=KSAI2(1,3)=KSAI2(2,4)=CMPLX(1.,0.)
      KSAI2(3,1)=KSAI2(4,2)=SIGMA1(1,1)=SIGMA1(3,3)=CMPLX(1.,0.)
      KSAI3(1,3)=KSAI3(2,4)=CMPLX(0.,-1.)
      KSAI3(3,1)=KSAI3(4,2)=CMPLX(0.,1.)
      KSAI1(3,3)=KSAI1(4,4)=SIGMA1(2,2)=SIGMA1(4,4)=CMPLX(-1.,0.)
      SIGMA2(1,2)=SIGMA2(2,1)=SIGMA2(3,4)=SIGMA2(4,3)=CMPLX(1.,0.)
C
C     COMPUTATION OF THE RELATIVE PERMITTIVITY MATRICES
C     =================================================
C
      CALL PERMIT(ERL,ERP)
      CALL EIGP(KP,BETA,ERP,RGP)
      CALL EIGL(KL,BETA,ERL,RGL)
C
C     COMPUTATION OF THE MODAL MATRIX U (RIGHT EIGENVECTORS)
C     ======================================================
C
      CALL MODAL(RGL,KL,QLZ,PLZ,QLY,UL,PFL,PBL,QFL,QBL)
      CALL MODAL(RGP,KP,QPZ,PPZ,QPY,UP,PFP,PBP,QFP,QBP)
C
C     COMPUTATION OF THE MODAL MATRIX V (LEFT EIGENVECTORS)
C     =====================================================
C
```

```
      DO 6 I=1,4
      KLCC(I)=CONJG(KL(I))
 6    KPCC(I)=CONJG(KP(I))
      CALL MODAL(RGL,KLCC,QLZCC,PLZCC,QLYCC,VL,DUM1,DUM2,DUM3,DUM4)
      CALL MODAL(RGP,KPCC,QPZCC,PPZCC,QPYCC,VP,DUM1,DUM2,DUM3,DUM4)
      DO 8 I=1,4
      VL1(I)=VL(1,I)
      VL2(I)=VL(2,I)
      VL3(I)=VL(3,I)
      VL4(I)=VL(4,I)
      VP1(I)=VP(1,I)
      VP2(I)=VP(2,I)
      VP3(I)=VP(3,I)
 8    VP4(I)=VP(4,I)
      DO 10 J=1,4
      VL(1,J)=VL2(J)
      VL(2,J)=VL1(J)
      VL(3,J)=VL4(J)
      VL(4,J)=VL3(J)
      VP(1,J)=VP2(J)
      VP(2,J)=VP1(J)
      VP(3,J)=VP4(J)
 10   VP(4,J)=VP3(J)
C
C     ORTHOGONALITY TEST BETWEEN U AND V
C     ******************************
C
      DO 12 I=1,4
      DO 12 J=1,4
      VLHERM(I,J)=CONJG(VL(J,I))
 12   VPHERM(I,J)=CONJG(VP(J,I))
      CALL PRODU(IDEN,VLHERM,UL,VUL,4,DUMMY)
      CALL PRODU(IDEN,VPHERM,UP,VUP,4,DUMMY)
      IORTHOP=IORTHOL=0
      DO 13 I=1,4
      DO 13 J=1,4
      IF (I.NE.J) THEN
      REP=ABS(REAL(VUP(I,J)))
      REL=ABS(REAL(VUL(I,J)))
      AIMP=ABS(AIMAG(VUP(I,J)))
      AIML=ABS(AIMAG(VUL(I,J)))
      IF ((REP.GT.SMALL).OR.(AIMP.GT.SMALL)) IORTHOP=1
      IF ((REL.GT.SMALL).OR.(AIML.GT.SMALL)) IORTHOL=1
      END IF
 13   CONTINUE
C
C     COMPUTATION OF THE EIGENVALUE MATRIX LR
C     ***********************************
C
      DO 14  I=1,4
      DO 14  J=1,4
      LRL(I,J)=0.
      LRP(I,J)=0.
      LRL(I,I)=KL(I)
 14   LRP(I,I)=KP(I)
C
C     COMPUTATION OF THE WAVE IMPEDANCE IN THE FORWARD AND BACKWARD CASES
C     ***************************************************************
C
      DO 15 I=1,2
      DO 15 J=1,2
      MATRIX1(I,J)=QFL(I,J)
      MATRIX2(I,J)=QFP(I,J)
      MATRIX3(I,J)=QBL(I,J)
      MATRIX4(I,J)=QBP(I,J)
      QFLINV(I,J)=ID(I,J)
      QFPINV(I,J)=ID(I,J)
      QBLINV(I,J)=ID(I,J)
 15   QBPINV(I,J)=ID(I,J)
      CALL LEQ2C(MATRIX1,2,2,QFLINV,2,2,0,WA,WK,IER)
      CALL LEQ2C(MATRIX2,2,2,QFPINV,2,2,0,WA,WK,IER)
      CALL LEQ2C(MATRIX3,2,2,QBLINV,2,2,0,WA,WK,IER)
      CALL LEQ2C(MATRIX4,2,2,QBPINV,2,2,0,WA,WK,IER)
      CALL PRODU(PFL,QFLINV,ID,ZWL,2,DUMMY)
      CALL PRODU(PFP,QFPINV,ID,ZWP,2,DUMMY)
      CALL PRODU(PBL,QBLINV,ID,ZWLB,2,DUMMY)
      CALL PRODU(PBP,QBPINV,ID,ZWPB,2,DUMMY)
C
C     COMPUTATION OF THE LF AND LB  2X2 MATRICES
C     **************************************
C
      JJ=CMPLX(0.,-1.)
      DO 16 I=1,4
      EXPOL(I)=CEXP(JJ*L*KL(I))
 16   EXPOP(I)=CEXP(JJ*L*KP(I))
      DO 18 I=1,2
```

```
      DO 18 J=1,2
      LFL(I,J)=CMPLX(0.,0.)
      LBL(I,J)=CMPLX(0.,0.)
      LFP(I,J)=CMPLX(0.,0.)
18    LBP(I,J)=CMPLX(0.,0.)
      LFL(1,1)=EXPOL(1)
      LFL(2,2)=EXPOL(3)
      LBL(1,1)=EXPOL(2)
      LBL(2,2)=EXPOL(4)
      LFP(1,1)=EXPOP(1)
      LFP(2,2)=EXPOP(3)
      LBP(1,1)=EXPOP(2)
      LBP(2,2)=EXPOP(4)
C
C     COMPUTATION OF THE LG PROPAGATION-FACTOR MATRIX
C     ***********************************************
C
      DO 20  I=1,4
      DO 20  J=1,4
      LGL(I,J)=CMPLX(0.,0.)
      LGP(I,J)=CMPLX(0.,0.)
      LGL(I,I)=EXPOL(I)
20    LGP(I,I)=EXPOP(I)
C
C     EVALUATION OF THE F AND J MATRICES
C     **********************************
C
      DO 21 I=1,2
      DO 21 J=1,2
      MATRIX1(I,J)=PFL(I,J)
      MATRIX2(I,J)=PFP(I,J)
      MATRIX3(I,J)=PBL(I,J)
      MATRIX4(I,J)=PBP(I,J)
      PFLINV(I,J)=ID(I,J)
      PFPINV(I,J)=ID(I,J)
      PBLINV(I,J)=ID(I,J)
21    PBPINV(I,J)=ID(I,J)
      CALL LEQ2C(MATRIX1,2,2,PFLINV,2,2,0,WA,WK,IER)
      CALL LEQ2C(MATRIX2,2,2,PFPINV,2,2,0,WA,WK,IER)
      CALL LEQ2C(MATRIX3,2,2,PBLINV,2,2,0,WA,WK,IER)
      CALL LEQ2C(MATRIX4,2,2,PBPINV,2,2,0,WA,WK,IER)
      CALL PRODU(QFL,LFL,QFLINV,FFL,2,DUMMY)
      CALL PRODU(QFP,LFP,QFPINV,FFP,2,DUMMY)
      CALL PRODU(QBL,LBL,QBLINV,FBL,2,DUMMY)
      CALL PRODU(QBP,LBP,QBPINV,FBP,2,DUMMY)
      CALL PRODU(PFL,LFL,PFLINV,JFL,2,DUMMY)
      CALL PRODU(PFP,LFP,PFPINV,JFP,2,DUMMY)
      CALL PRODU(PBL,LBL,PBLINV,JBL,2,DUMMY)
      CALL PRODU(PBP,LBP,PBPINV,JBP,2,DUMMY)
C
C     COMPUTATION OF THE INVERSE OF THE MODAL MATRIX U-TILDE
C     ******************************************************
C
      CALL PRODU(P23,UL,P23,ULTIL,4,DUMMY)
      CALL PRODU(P23,UP,P23,UPTIL,4,DUMMY)
      DO 22 I=1,4
      DO 22 J=1,4
      MODALL(I,J)=ULTIL(I,J)
      MODALP(I,J)=UPTIL(I,J)
      ULTILI(I,J)=CMPLX(0.,0.)
      UPTILI(I,J)=CMPLX(0.,0.)
      ULTILI(I,I)=CMPLX(1.,0.)
22    UPTILI(I,I)=CMPLX(1.,0.)
      CALL LEQ2C(MODALL,4,4,ULTILI,4,4,0,WA,WK,IER)
      CALL LEQ2C(MODALP,4,4,UPTILI,4,4,0,WA,WK,IER)
C
C     COMPUTATION OF THE G(L) MATRIX (TILDE)
C     **************************************
C
      CALL PRODU(P23,LGL,P23,LGLTIL,4,DUMMY)
      CALL PRODU(P23,LGP,P23,LGPTIL,4,DUMMY)
      CALL PRODU(ULTIL,LGLTIL,ULTILI,GL,4,DUMMY)
      CALL PRODU(UPTIL,LGPTIL,UPTILI,GP,4,DUMMY)
C
C     COMPUTATION OF THE LG(-L) MATRIX
C     ********************************
C
      DO 24  I=1,4
      DO 24  J=1,4
      LGL1(I,J)=0.
      LGP1(I,J)=0.
      LGL1(I,I)=1./LGL(I,I)
24    LGP1(I,I)=1./LGP(I,I)
C
C     COMPUTATION OF THE G(-L) MATRIX (TILDE)
C     ***************************************
C
```

```
      CALL PRODU(P23,LGL1,P23,LGL1TL,4,DUMMY)
      CALL PRODU(P23,LGP1,P23,LGP1TL,4,DUMMY)
      CALL PRODU(ULTIL,LGL1TL,ULTILI,GL1,4,DUMMY)
      CALL PRODU(UPTIL,LGP1TL,UPTILI,GP1,4,DUMMY)
C
C     RECIPROCITY TEST
C     ***************
C
      IF (G.NE.0.) GO TO 27
      DO 26 I=1,4
      DO 26 J=1,4
      GLT(I,J)=GL(J,I)
26    GPT(I,J)=GP(J,I)
      CALL PRODU(KSAI3,GLT,KSAI3,GLREC,4,DUMMY)
      CALL PRODU(KSAI3,GPT,KSAI3,GPREC,4,DUMMY)
C
C     ANTIRECIPROCITY TEST
C     ********************
C
      CALL PRODU(KSAI2,GLT,KSAI2,GLANTI,4,DUMMY)
      CALL PRODU(KSAI2,GPT,KSAI2,GPANTI,4,DUMMY)
C
C     COMPUTATION OF THE H1,H2,H3,H4 AND K1,K2,K3,K4 MATRICES
C     *******************************************************
C
27    CALL H(QFL,QBL,QLZ,QLY,H1L,H2L,H3L,H4L)
      CALL H(QFP,QBP,QPZ,QPY,H1P,H2P,H3P,H4P)
      CALL KI(PFL,PBL,PLZ,K1L,K2L,K3L,K4L)
      CALL KI(PFP,PBP,PPZ,K1P,K2P,K3P,K4P)
C
C     LOSSLESSNESS TEST
C     *****************
C
      DO 28 I=1,4
      DO 28 J=1,4
      GLOSSL(I,J)=GLOSSP(I,J)=CMPLX(0.,0.)
      GPHER(I,J)=CONJG(GP(J,I))
      GLHER(I,J)=CONJG(GL(J,I))
28    CONTINUE
      CALL PRODU(KSAI2,GLHER,KSAI2,GLOSSL,4,DUMMY)
      CALL PRODU(KSAI2,GPHER,KSAI2,GLOSSP,4,DUMMY)
C
C     BILATERAL SYMMETRY
C     ******************
C
      CALL PRODU(KSAI1,GL,KSAI1,GLBILA,4,DUMMY)
      CALL PRODU(KSAI1,GP,KSAI1,GPBILA,4,DUMMY)
C
C     TRANSVERSAL SYMMETRY
C     ********************
C
      CALL PRODU(SIGMA2,GL,SIGMA2,GLTRAN,4,DUMMY)
      CALL PRODU(SIGMA2,GP,SIGMA2,GPTRAN,4,DUMMY)
C
C     SEMIRECIPROCITY
C     ***************
C
      CALL PRODU(SIGMA1,GLT,SIGMA1,TRIAL,4,DUMMY)
      CALL PRODU(KSAI3,TRIAL,KSAI3,GLSEMI,4,DUMMY)
      CALL PRODU(SIGMA1,GPT,SIGMA1,TRIAL,4,DUMMY)
      CALL PRODU(KSAI3,TRIAL,KSAI3,GPSEMI,4,DUMMY)
C
      ILOSSP=IRECP=IANTIP=ISEMIP=IBILAP=ITRANP=0
      ILOSSL=IRECL=IANTIL=ISEMIL=IBILAL=ITRANL=0
      DO 29 I=1,4
      DO 29 J=1,4
      A=ABS(REAL(GLOSSP(I,J)-GP1(I,J)))
      B=ABS(AIMAG(GLOSSP(I,J)-GP1(I,J)))
      IF((A.GT.SMALL).OR.(B.GT.SMALL)) ILOSSP=1
      A=ABS(REAL(GLOSSL(I,J)-GL1(I,J)))
      B=ABS(AIMAG(GLOSSL(I,J)-GL1(I,J)))
      IF((A.GT.SMALL).OR.(B.GT.SMALL)) ILOSSL=1
      A=ABS(REAL(GPREC(I,J)-GP1(I,J)))
      B=ABS(AIMAG(GPREC(I,J)-GP1(I,J)))
      IF((A.GT.SMALL).OR.(B.GT.SMALL)) IRECP=1
      A=ABS(REAL(GLREC(I,J)-GL1(I,J)))
      B=ABS(AIMAG(GLREC(I,J)-GL1(I,J)))
      IF((A.GT.SMALL).OR.(A.GT.SMALL)) IRECL=1
      A=ABS(REAL(GPANTI(I,J)-GP1(I,J)))
      B=ABS(AIMAG(GPANTI(I,J)-GP1(I,J)))
      IF((A.GT.SMALL).OR.(B.GT.SMALL)) IANTIP=1
      A=ABS(REAL(GLANTI(I,J)-GL1(I,J)))
      B=ABS(AIMAG(GLANTI(I,J)-GL1(I,J)))
      IF((A.GT.SMALL).OR.(B.GT.SMALL)) IANTIL=1
      A=ABS(REAL(GPSEMI(I,J)-GP1(I,J)))
      B=ABS(AIMAG(GPSEMI(I,J)-GP1(I,J)))
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```
      IF((A.GT.SMALL).OR.(B.GT.SMALL)) ISEMIP=1
      A=ABS(REAL(GLSEMI(I,J)-GL1(I,J)))
      B=ABS(AIMAG(GLSEMI(I,J)-GL1(I,J)))
      IF((A.GT.SMALL).OR.(B.GT.SMALL)) ISEMIL=1
      A=ABS(REAL(GPBILA(I,J)-GP1(I,J)))
      B=ABS(AIMAG(GPBILA(I,J)-GP1(I,J)))
      IF((A.GT.SMALL).OR.(B.GT.SMALL)) IBILAP=1
      A=ABS(REAL(GLBILA(I,J)-GL1(I,J)))
      B=ABS(AIMAG(GLBILA(I,J)-GL1(I,J)))
      IF((A.GT.SMALL).OR.(B.GT.SMALL)) IBILAL=1
      A=ABS(REAL(GPTRAN(I,J)-GP(I,J)))
      B=ABS(AIMAG(GPTRAN(I,J)-GP(I,J)))
      IF((A.GT.SMALL).OR.(B.GT.SMALL)) ITRANP=1
      A=ABS(REAL(GLTRAN(I,J)-GL(I,J)))
      B=ABS(AIMAG(GLTRAN(I,J)-GL(I,J)))
      IF((A.GT.SMALL).OR.(B.GT.SMALL)) ITRANL=1
29    CONTINUE
C
C
C     COMPUTATION OF THE REFLECTION MATRIX GAMMA
C     ==========================================
C
      CALL PRODU(QFL,PFLINV,ID,YFL,2,DUMMY)
      CALL PRODU(QFP,PFPINV,ID,YFP,2,DUMMY)
      DO 30 I=1,2
      DO 30 J=1,2
      YF1(I,J)=CMPLX(0.,0.)
30    YB1(I,J)=CMPLX(0.,0.)
      YF1(1,1)=KS
      YF1(2,2)=1/KS
      YB1(1,1)=-YF1(1,1)
      YB1(2,2)=-YF1(2,2)
      CALL GAMMA(YFL,YF1,YB1,GAMAL)
      CALL GAMMA(YFP,YF1,YB1,GAMAP)
C
C************************** PRINT-OUT  STATEMENTS ************************
C
C                           ************************
C
      M=0
      IF (M.EQ.0) GO TO 500
      GO TO 501
C
500   WRITE(5,190)  LGIVEN
190   FORMAT('1'/10X,'NORMALIZED WIDTH :L=',F6.3,'* PI S'/10X,
     1'=================')
      WRITE(5,191) BETA
191   FORMAT(/10X,'NORMALIZED PROPAGATION CONSTANT :'
     1,3X,'BETA=',F10.8/10X,'==============================')
      WRITE(5,192)
192   FORMAT(/10X,'THE MEDIUM IS :')
      IF (ILOSSP.EQ.0) WRITE(5,193)
      IF (IRECP.EQ.0)  WRITE(5,194)
      IF (IANTIP.EQ.0) WRITE(5,195)
      IF (ISEMIP.EQ.0) WRITE(5,196)
      IF (IBILAP.EQ.0) WRITE(5,197)
      IF (ITRANP.EQ.0) WRITE(5,198)
      IF (ILOSSL.EQ.0) WRITE(5,199)
      IF (IRECL.EQ.0)  WRITE(5,200)
      IF (IANTIL.EQ.0) WRITE(5,201)
      IF (ISEMIL.EQ.0) WRITE(5,202)
      IF (IBILAL.EQ.0) WRITE(5,203)
      IF (ITRANL.EQ.0) WRITE(5,204)
193   FORMAT(25X,'LOSSLESS                 (POLAR)')
194   FORMAT(25X,'RECIPROCAL               (POLAR)')
195   FORMAT(25X,'ANTIRECIPROCAL           (POLAR)')
196   FORMAT(25X,'SEMIRECIPROCAL           (POLAR)')
197   FORMAT(25X,'BILATERALLY SYMMETRIC    (POLAR)')
198   FORMAT(25X,'TRANSVERSALLY SYMMETRIC  (POLAR)')
199   FORMAT(25X,'LOSSLESS                 (LONGITUDINAL)')
200   FORMAT(25X,'RECIPROCAL               (LONGITUDINAL)')
201   FORMAT(25X,'ANTIRECIPROCAL           (LONGITUDINAL)')
202   FORMAT(25X,'SEMIRECIPROCAL           (LONGITUDINAL)')
203   FORMAT(25X,'BILATERALLY SYMMETRIC    (LONGITUDINAL)')
204   FORMAT(25X,'TRANSVERSALLY SYMMETRIC  (LONGITUDINAL)')
C
C
501   M=2
      IF (M.EQ.2) GO TO 502
      GO TO 503
C
502   WRITE(5,205)
205   FORMAT(/10X,'RELATIVE PERMITTIVITY MATRIX'/10X,
     1      '============================'/)
      WRITE(5,206)
206   FORMAT(/25X,'LONGITUDINAL CASE'/25X,'================='/)
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      WRITE(5,207) ((ERL(I,J),J=1,3),I=1,3)
207   FORMAT((10X,'*',69X,'*'/10X,'*',3(3X,'(',F8.4,',',F8.4,
     1       ')'),3X,'*'/10X,'*',69X,'*' ))
      WRITE(5,208)
208   FORMAT(/25X,'POLAR CASE'/25X,'=========='/)
      WRITE(5,207) ((ERP(I,J),J=1,3),I=1,3)
C
503   M=4
      IF (M.EQ.4) GO TO 504
      GO TO 505
C
504   WRITE(5,209)
209   FORMAT(/10X,'COUPLING MATRIX'/10X,'==============='/)
      WRITE(5,206)
      WRITE(5,210)  ((RGL(I,J),J=1,4),I=1,4)
210   FORMAT((10X,'*',113X,'*'/10X,'*',4(1X,'(',F12.5,',',F12.5,
     $        ')'),1X,'*'/10X,'*',113X,'*'))
C
505   M=6
      IF (M.EQ.6) GO TO 506
      GO TO 507
C
506   WRITE(5,208)
      WRITE(5,210)  ((RGP(I,J),J=1,4),I=1,4)
C
507   M=8
      IF (M.EQ.8) GO TO 508
      GO TO 509
C
508   WRITE(5,211)
211   FORMAT(/10X,'EIGENVALUES OF THE COUPLING MATRIX'/
     110X,'=================================='/)
      WRITE(5,206)
      WRITE(5,212)  (EL(I), I=1,4)
212   FORMAT(/10X,4(1X,'(',1X,F10.7,1X,',',1X,F10.7,1X,')')/)
C
509   M=10
      IF (M.EQ.10) GO TO 510
      GO TO 511
C
510   WRITE(5,208)
      WRITE(5,212)  (EP(I),I=1,4)
C
511   M=12
      IF (M.EQ.12) GO TO 512
      GO TO 513
C
512   WRITE(5,213)
213   FORMAT(/10X,'THE MODAL MATRIX U'/10X,'=================='/)
      WRITE(5,206)
      WRITE(5,210)    ((UL(I,J),J=1,4),I=1,4)
      WRITE(5,208)
      WRITE(5,210) ((UP(I,J),J=1,4),I=1,4)
C
513   M=14
      IF (M.EQ.14) GO TO 514
      GO TO 515
C
514   WRITE(5,214)
214   FORMAT(/10X,'THE MODAL MATRIX V'/10X,'=================='/)
      WRITE(5,206)
      WRITE(5,210) ((VL(I,J),J=1,4),I=1,4)
      WRITE(5,208)
      WRITE(5,210) ((VP(I,J),J=1,4),I=1,4)
      IF (IORTHOP.EQ.1) WRITE(5,215)
      IF (IORTHOL.EQ.1) WRITE(5,227)
215   FORMAT(/10X,'U AND V ARE NOT ORTHOGONAL IN THE POLAR CASE')
227   FORMAT(/10X,'U AND V ARE NOT ORTHOGONAL IN THE LONGITUDINALCASE')
C
515   M=16
      IF (M.EQ.16) GO TO 516
      GO TO 517
C
516   WRITE(5,216)
216   FORMAT(/10X,'THE PF, PB, QF, QB  2X2 SUBMATRICES OF THE PARTITIONE
     1D MATRIX U'/10X,
     2'=================================================================='/)
      WRITE(5,206)
      WRITE(5,217)  ((PFL(I,J),J=1,2),I=1,2)
217   FORMAT(//(25X,'*',78X,'*'/25X,'*',2(2X,'(',F16.7,',',F16.7,')'
     1       ,2X),'*'/25X,'*',78X,'*'))
      WRITE(5,217)  ((PBL(I,J),J=1,2),I=1,2)
      WRITE(5,217)  ((QFL(I,J),J=1,2),I=1,2)
      WRITE(5,217)  ((QBL(I,J),J=1,2),I=1,2)
      WRITE(5,208)
      WRITE(5,217)  ((PFP(I,J),J=1,2),I=1,2)
```

```
      WRITE(5,217)  ((PBP(I,J),J=1,2),I=1,2)
      WRITE(5,217)  ((QFP(I,J),J=1,2),I=1,2)
      WRITE(5,217)  ((QBP(K,L),L=1,2),K=1,2)
C
517   M=18
      IF (M.EQ.18) GO TO 518
      GO TO 519
C
518   WRITE(5,218)
218   FORMAT(/10X,'FORWARD WAVE IMPEDANCE'/10X,'======================'/)
      WRITE(5,206)
      WRITE(5,217)  ((ZWL(I,J),J=1,2),I=1,2)
      WRITE(5,208)
      WRITE(5,217) ((ZWP(I,J),J=1,2),I=1,2)
      WRITE(5,219)
219   FORMAT(/10X,'BACKWARD WAVE IMPEDANCE'/10X,'======================
     1'/)
      WRITE(5,206)
      WRITE(5,217)  ((ZWLB(I,J),J=1,2),I=1,2)
      WRITE(5,208)
      WRITE(5,217)  ((ZWPB(I,J),J=1,2),I=1,2)
C
519   M=20
      IF (M.EQ.20) GO TO 520
      GO TO 521
C
520   WRITE(5,220)
220   FORMAT(/10X,'THE LF AND LB PROPAGATION-FACTOR MATRICES'/10X,
     1'========================================='/)
      WRITE(5,206)
      WRITE(5,217)   ((LFL(I,J),J=1,2),I=1,2)
      WRITE(5,217)    ((LBL(I,J),J=1,2),I=1,2)
      WRITE(5,208)
      WRITE(5,217)     ((LFP(I,J),J=1,2),I=1,2)
      WRITE(5,217)     ((LBP(I,J),J=1,2),I=1,2)
C
521   M=22
      IF (M.EQ.22) GO TO 522
      GO TO 523
C
522   WRITE(5,221)
221   FORMAT(/10X,'FF AND FB MATRICES'/10X,'=================='/)
      WRITE(5,206)
      WRITE(5,217)    ((FFL(I,J),J=1,2),I=1,2)
      WRITE(5,217)    ((FBL(I,J),J=1,2),I=1,2)
      WRITE(5,208)
      WRITE(5,217)    ((FFP(I,J),J=1,2),I=1,2)
      WRITE(5,217)    ((FBP(I,J),J=1,2),I=1,2)
      WRITE(5,222)
222   FORMAT(/10X,'JF AND JB MATRICES'/10X,'=================='/)
      WRITE(5,206)
      WRITE(5,217)    ((JFL(I,J),J=1,2),I=1,2)
      WRITE(5,217)    ((JBL(I,J),J=1,2),I=1,2)
      WRITE(5,208)
      WRITE(5,217)    ((JFP(I,J),J=1,2),I=1,2)
      WRITE(5,217)    ((JBP(I,J),J=1,2),I=1,2)
C
523   M=24
      IF (M.EQ.24) GO TO 524
      GO TO 525
C
524   WRITE(5,223)
223   FORMAT(/10X,'LAYER TRANSFER MATRIX G(L)'/10X,'====================
     1======'/)
      WRITE(5,206)
      WRITE(5,210) ((GL(I,J),J=1,4),I=1,4)
      WRITE(5,208)
      WRITE(5,210) ((GP(I,J),J=1,4),I=1,4)
C
525   M=26
      IF (M.EQ.26) GO TO 526
      GO TO 527
C
526   WRITE(5,224)
224   FORMAT(/10X,'THE G(-L) MATRIX'/10X,'================'/)
      WRITE(5,206)
      WRITE(5,210)  ((GL1(I,J),J=1,4),I=1,4)
      WRITE(5,208)
      WRITE(5,210)  ((GP1(I,J),J=1,4),I=1,4)
C
527   M=28
      IF (M.EQ.28) GO TO 528
      GO TO 529
C
528   WRITE(5,225)
225   FORMAT(/10X,'THE H1,H2,H3 AND H4 MATRICES'/
```

```
     110X,'*************************'/)
      WRITE(5,206)
      WRITE(5,217) ((H1L(I,J),J=1,2),I=1,2)
      WRITE(5,217) ((H2L(I,J),J=1,2),I=1,2)
      WRITE(5,217) ((H3L(I,J),J=1,2),I=1,2)
      WRITE(5,217) ((H4L(I,J),J=1,2),I=1,2)
      WRITE(5,208)
      WRITE(5,217) ((H1P(I,J),J=1,2),I=1,2)
      WRITE(5,217) ((H2P(I,J),J=1,2),I=1,2)
      WRITE(5,217) ((H3P(I,J),J=1,2),I=1,2)
      WRITE(5,217) ((H4P(I,J),J=1,2),I=1,2)
C
529   M=31
      IF (M.EQ.31) GO TO 530
      GO TO 531
C
530   WRITE(5,226)
226   FORMAT(/10X,'THE K1,K2,K3 AND K4 MATRICES'/
     1       10X,'****************************'/)
      WRITE(5,206)
      WRITE(5,217) ((K1L(I,J),J=1,2),I=1,2)
      WRITE(5,217) ((K2L(I,J),J=1,2),I=1,2)
      WRITE(5,217) ((K3L(I,J),J=1,2),I=1,2)
      WRITE(5,217) ((K4L(I,J),J=1,2),I=1,2)
      WRITE(5,208)
      WRITE(5,217) ((K1P(I,J),J=1,2),I=1,2)
      WRITE(5,217) ((K2P(I,J),J=1,2),I=1,2)
      WRITE(5,217) ((K3P(I,J),J=1,2),I=1,2)
      WRITE(5,217) ((K4P(I,J),J=1,2),I=1,2)
C
531   M=32
      IF (M.EQ.32) GO TO 532
      GO TO 533
C
532   WRITE(5,230)
230   FORMAT(/10X,'THE REFLECTION MATRIX GAMA'/10X,
     1'**************************'/)
      WRITE(5,206)
      WRITE(5,217)  ((GAMAL(I,J),J=1,2),I=1,2)
      WRITE(5,208)
      WRITE(5,217)  ((GAMAP(I,J),J=1,2),I=1,2)
C
533   STOP
      END
      SUBROUTINE PERMIT(EL,EP)
C***********************************************************************
C     THIS SUBROUTINE INITIALIZES THE VALUES OF THE PERMITTIVITY MATRICES,
C     IN THE LONGITUDINAL AND POLAR CASES
C***********************************************************************
      COMMON /AAA/ L
      REAL L
      COMPLEX  EL(3,3),EP(3,3),EHERMP(3,3),EHERML(3,3),SUM1,SUM2
C
      DO 1 I=1,3
      DO 1 J=1,3
1     EL(I,J)=EP(I,J)=CMPLX(0.,0.)
C
C  LONGITUDINAL CASE
      EL(1,1)=CMPLX(2.4,0.0)
      EL(1,2)=CMPLX(0.0,0.1)
      EL(2,1)=CONJG(EL(1,2))
      EL(2,2)=CMPLX(2.45,0.0)
      EL(3,3)=CMPLX(2.5,0.0)
C
C  POLAR CASE
      EP(1,1)=CMPLX(1.8,0.0)
      EP(2,2)=CMPLX(2.0,0.0)
      EP(2,3)=CMPLX(0.01,0.0)
      EP(3,2)=CONJG(EP(2,3))
      EP(3,3)=CMPLX(1.7,0.0)
C
      DO 2 II=1,3
      SUM1=SUM2=CMPLX(0.,0.)
      DO 2 JJ=1,3
      EHERML(II,JJ)=SUM1+EL(II,JJ)*CONJG(EL(II,JJ))
      EHERMP(II,JJ)=SUM2+EP(II,JJ)*CONJG(EP(II,JJ))
2     CONTINUE
      DO 3 II=1,3
      DO 3 JJ=1,3
      IF (AIMAG(EHERML(II,JJ)).NE.0.) THEN
      WRITE(5,11)
11    FORMAT(/'THE FOLLOWING PERMITTIVITY MATRIX IS NOT HERMITEAN')
      WRITE(5,22) ((EL(I,J),J=1,3),I=1,3)
22    FORMAT((10X,'*',69X,'*'/10X,'*',3(3X,'(',F8.5,',',F8.4,
     *       ')'),3X,'*'/10X,'*',69X,'*'))
      END IF
```

```
      IF (AIMAG(EHERMP(II,JJ)).NE.0.) THEN
      WRITE(5,11)
      WRITE(5,22) ((EP(I,J),J=1,3),I=1,3)
      END IF
3     CONTINUE
C
      RETURN
      END
      SUBROUTINE PCOUPLI(R,E,BETA)
C=========================================================================
C     PCOUPLI EVALUATES THE COUPLING MATRIX R IN THE POLAR CASE
C=========================================================================
      COMPLEX E(3,3),R(4,4)
      DO 1 I=1,4
      DO 1 J=1,4
1     R(I,J)=CMPLX(0.,0.)
      R(1,2)=CMPLX(1.,0.)
      R(2,1)=E(2,2)-BETA**2
      R(2,3)=E(2,3)
      R(3,4)=1-BETA**2/E(1,1)
      R(4,1)=E(3,2)
      R(4,3)=E(3,3)
      RETURN
      END
      SUBROUTINE LCOUPLI(R,E,BETA)
C=========================================================================
C     LCOUPLI EVALUATES THE COUPLING MATRIX R IN THE LONGITUDINAL CASE
C=========================================================================
      COMPLEX E(3,3),R(4,4),DZZ
C
      DO 1 I=1,4
      DO 1 J=1,4
1     R(I,J)=CMPLX(0.,0.)
C
      DZZ=E(1,1)*E(2,2)-E(1,2)*E(2,1)
      R(1,2)=CMPLX(1.,0.)
      R(2,1)=DZZ/E(1,1)-BETA**2
      R(2,4)=-BETA*E(2,1)/E(1,1)
      R(3,1)=-BETA*E(1,2)/E(1,1)
      R(3,4)=1-BETA**2/E(1,1)
      R(4,3)=E(3,3)
C
      RETURN
      END
      SUBROUTINE EIGP(K,BETA,E,R)
C=========================================================================
C     EIGP EVALUATES THE EIGENVALUES KAPPA IN THE POLAR CASE
C=========================================================================
      COMPLEX K(4),A,B,R(4,4),E(3,3),DUMMY
C
      CALL PCOUPLI(R,E,BETA)
      A=(R(3,4)*R(4,3)+R(2,1))/2
      B=A**2-(R(2,1)*R(4,3)-CABS(R(4,1))**2)*R(3,4)
      K(1)=A+CSQRT(B)
      K(3)=A-CSQRT(B)
      K(1)=CSQRT(K(1))
      K(2)=-K(1)
      K(3)=CSQRT(K(3))
      K(4)=-K(3)
C
      AIM1=AIMAG(K(1))
      AIM3=AIMAG(K(3))
      IF (ABS(AIM1).LT.ABS(AIM3)) THEN
      DUMMY=K(1)
      K(1)=K(3)
      K(2)=-K(1)
      K(3)=DUMMY
      K(4)=-K(3)
      END IF
      RETURN
      END
      SUBROUTINE EIGL(K,BETA,E,R)
C=========================================================================
C     EIGL EVALUATES THE EIGENVALUES KAPPA IN THE LONGITUDINAL CASE
C=========================================================================
      COMPLEX K(4),R(4,4),A,B,E(3,3),DUMMY
C
      CALL LCOUPLI(R,E,BETA)
      A=(R(3,4)*R(4,3)+R(2,1))/2
      B=A**2-(R(2,1)*R(3,4)-CABS(R(3,1))**2)*R(4,3)
      K(1)=A+CSQRT(B)
      K(1)=CSQRT(K(1))
      K(2)=-K(1)
      K(3)=A-CSQRT(B)
      K(3)=CSQRT(K(3))
      K(4)=-K(3)
```

```
C
      AIM1=AIMAG(K(1))
      AIM3=AIMAG(K(3))
      IF (ABS(AIM1).LT.ABS(AIM3)) THEN
      DUMMY=K(1)
      K(1)=K(3)
      K(2)=-K(1)
      K(3)=DUMMY
      K(4)=-K(3)
      END IF
      RETURN
      END
      SUBROUTINE MODAL(R,K,QZ,PZ,QY,U,PF,PB,QF,QB)
      COMPLEX PF(2,2),PB(2,2),QF(2,2),QB(2,2),R33CC
      COMPLEX RNUME1(4),RNUME2(4),RDENO(4)
      COMPLEX QZ(4), QY(4), PZ(4), R(4,4), K(4), U(4,4)
C
      R33CC=CONJG(R(3,3))
      DO 100  I=1,4
           RNUME1(I)=R(3,1)*(R33CC-K(I))-R(3,4)*R(4,1)
           RDENO(I)=R(4,3)*R(3,4)-(R33CC-K(I))*(R(3,3)-K(I))
              RNUME2(I)=R(4,1)*(R(3,3)-K(I))-R(4,3)*R(3,1)
           QZ(I)=K(I)
           PZ(I)=(RNUME1(I))/(RDENO(I))
           QY(I)=-(RNUME2(I))/(RDENO(I))
      U(1,I)=CMPLX(1.,0.)
      U(2,I)=QZ(I)
      U(3,I)=PZ(I)
      U(4,I)=-QY(I)
100   CONTINUE
C
      PF(1,1)=U(1,1)
      PF(1,2)=U(1,3)
      PF(2,1)=U(3,1)
      PF(2,2)=U(3,3)
      PB(1,1)=U(1,2)
      PB(1,2)=U(1,4)
      PB(2,1)=U(3,2)
      PB(2,2)=U(3,4)
      QF(1,1)=U(2,1)
      QF(1,2)=U(2,3)
      QF(2,1)=U(4,1)
      QF(2,2)=U(4,3)
      QB(1,1)=U(2,2)
      QB(1,2)=U(2,4)
      QB(2,1)=U(4,2)
      QB(2,2)=U(4,4)
      RETURN
      END
      SUBROUTINE PRODU(M1,M2,M3,PRO,N,PRO1)
C
C     THIS SUBROUTINE EVALUATES THE PRODUCT OF THREE NXN MATRICES
C     M1,M2 AND M3 IN THE FOLLOWING MANNER:  PRO=M1*M2*M3
C     ***************************************************
C
      COMPLEX M1(N,N),M2(N,N),M3(N,N),PRO1(N,N),PRO(N,N)
      DO 210  I=1,N
      DO 210  J=1,N
         PRO1(I,J)=0.
         PRO(I,J)=0.
210   CONTINUE
C
C
      DO 211  I=1,N
      DO 211  J=1,N
      DO 211  K=1,N
         PRO1(I,J)=PRO1(I,J)+M1(I,K)*M2(K,J)
211   CONTINUE
C
C
      DO 212  I=1,N
      DO 212  J=1,N
      DO 212  K=1,N
         PRO(I,J)=PRO(I,J)+PRO1(I,K)*M3(K,J)
212   CONTINUE
C
      RETURN
      END
      SUBROUTINE H(QF,QB,QZ,QY,H1,H2,H3,H4)
C
C     THIS SUBROUTINE EVALUATES THE 4 2X2 H MATRICES
C     **********************************************
C
      COMPLEX QF(2,2),QB(2,2),H1(2,2),H2(2,2),H3(2,2),H4(2,2)
      COMPLEX DETQF,DETQB,QZ(4),QY(4)
      COMPLEX H11(2,2),H22(2,2),H33(2,2),H44(2,2)
```

```
C
      DETQF=QZ(3)*QY(1)-QZ(1)*QY(3)
      DETQB=QY(2)*QZ(4)-QZ(2)*QY(4)
C
      H11(1,1)=-QZ(1)*QY(3)
      H11(1,2)=-QZ(1)*QZ(3)
      H11(2,1)=QY(1)*QY(3)
      H11(2,2)=QY(1)*QZ(3)
C
      H33(1,1)=QY(1)*QZ(3)
      H33(1,2)=QZ(3)*QZ(1)
      H33(2,1)=-QY(1)*QY(3)
      H33(2,2)=-QY(3)*QZ(1)
C
      H22(1,1)=-QZ(2)*QY(4)
      H22(1,2)=-QZ(2)*QZ(4)
      H22(2,1)=QY(2)*QY(4)
      H22(2,2)=QY(2)*QZ(4)
C
      H44(1,1)=QY(2)*QZ(4)
      H44(1,2)=QZ(2)*QZ(4)
      H44(2,1)=-QY(2)*QY(4)
      H44(2,2)=-QZ(2)*QY(4)
C
      DO 100 I=1,2
      DO 100 J=1,2
      H1(I,J)=H11(I,J)/DETQF
      H3(I,J)=H33(I,J)/DETQF
      H2(I,J)=H22(I,J)/DETQB
      H4(I,J)=H44(I,J)/DETQB
100   CONTINUE
C
      RETURN
      END
      SUBROUTINE KI(PF,PB,PZ,K1,K2,K3,K4)
C
C     THIS SUBROUTINE EVALUATES THE 4  2X2  K  MATRICES
C     ==============================================
C
      COMPLEX PF(2,2),PB(2,2),PZ(4),K1(2,2),K2(2,2),K3(2,2),K4(2,2)
      COMPLEX DETPF,DETPB,K11(2,2),K22(2,2),K33(2,2),K44(2,2)
C
      DETPF=PZ(3)-PZ(1)
      DETPB=PZ(4)-PZ(2)
C
      K11(1,1)=PZ(3)
      K11(1,2)=-1.
      K11(2,1)=PZ(3)*PZ(1)
      K11(2,2)=-PZ(1)
C
      K33(1,1)=-PZ(1)
      K33(1,2)=1.
      K33(2,1)=-PZ(1)*PZ(3)
      K33(2,2)=PZ(3)
C
      K22(1,1)=PZ(4)
      K22(1,2)=-1.
      K22(2,1)=PZ(2)*PZ(4)
      K22(2,2)=-PZ(2)
C
      K44(1,1)=-PZ(2)
      K44(1,2)=1.
      K44(2,1)=-PZ(2)*PZ(4)
      K44(2,2)=PZ(4)
C
      DO 100 I=1,2
      DO 100 J=1,2
      K1(I,J)=K11(I,J)/DETPF
      K3(I,J)=K33(I,J)/DETPF
      K2(I,J)=K22(I,J)/DETPB
      K4(I,J)=K44(I,J)/DETPB
100   CONTINUE
C
      RETURN
      END
      SUBROUTINE GAMMA(YF,YF1,YB1,GAMA)
C
C     THIS SUBROUTINE EVALUATES THE REFLECTION-COEFFICIENT MATRIX GAMMA
C     LOOKING TOWARDS THE RIGHT, AT THE INTERFACE BETWEEN AN ISOTROPIC
C     MEDIUM, CHARACTERIZED BY FORWARD(BACKWARD) ADMITTANCE YF1(YB1) AND
C     AN ANISOTROPIC MEDIUM, CHARACTERIZED BY WAVE-ADMITTANCE YF.
C
      REAL WK(4)
      COMPLEX WA(24),YINV(2,2),MATRIX(2,2),GAMA(2,2),YF(2,2),YF1(2,2)
      COMPLEX YB1(2,2),Y2(2,2),ID(2,2),DUMMY(2,2)
C
```

```
C
      DO 1 I=1,2
      DO 1 J=1,2
      MATRIX(I,J)=YF(I,J)-YB1(I,J)
      ID(I,J)=CMPLX(0.,0.)
      ID(I,I)=CMPLX(1.,0.)
      YINV(I,J)=CMPLX(0.,0.)
      YINV(I,I)=CMPLX(1.,0.)
    1 Y2(I,J)=YF1(I,J)-YF(I,J)
      CALL LEQ2C(MATRIX,2,2,YINV,2,2,0,WA,WK,IER)
      CALL PRODU(YINV,Y2,ID,GAMA,2,DUMMY)
      RETURN
      END
```

APPENDIX III

Fortran Program LAYER

Program LAYER solves numerically the dispersion equation of layered symmetric waveguides. The film and the cladding can be either of polar or longitudinal configuration.

For a particular waveguiding structure, LAYER varies the β and ℓ within prescribed ranges (DO loops) and computes the values of the characteristic equations, given in Tables 4.7-10, using the proper transverse guide indices of (4.4.19) and (4.4.20). Upon the detection of a change of sign between two successive values, it refines the step of β, under the same normalized width ℓ, to obtain a desired accuracy of the 'zero' of the dispersion equation. Due to the form of κ_i' ($\kappa_i' = j\kappa_i \tan\kappa_i\ell$; $\kappa_i' = -j\kappa_i \cot\kappa_i\ell$) special care is taken to find and abort the poles of the dispersion equation. The output consists of the values of $\beta, \kappa_1, \kappa_3, \alpha_1, \alpha_3$ and ℓ at which the dispersion equation attains close to zero values within predefined tolerance. It also plots the $\beta^2 = f(\ell)$ diagram of the 'zeros' of the dispersion equation and Re, Im $(\kappa_i, \alpha_i) = f(\beta^2)$ diagrams.

Program LAYER consists of:

<u>Main Program:</u> The input is distinguished as in-program and interactive input. The in-program input data are the entries of the permittivity tensors of the film and cladding, assigned values in subroutine PERMIT. The interactive input data are provided by the user via the terminal. Terminal print-outs explaining the input variables and free-format read statements make the interactive input simple and easy. The latter input consists of initialization of $\beta_o(\ell_o)$ = the initial value of $\beta(\ell)$ DO

loop, $\beta_{incr}(\ell_{incr})$ - the step of this loop, β_{min}/β_{max} - the ranges of β^2 to be used in the plots, a self-explanatory code for the waveguide configuration to be treated (e.g. PLP with κ_1 of symmetric form) and ℓ_{max} - the number of iterations of this DO loop. Although all the waveguide configurations are listed, using code-flags only one of these is examined. The computation of the cutoff axial guide indices, treated in Section 4.4(2), is followed by the nested DO loops of β and ℓ, and the print-out statements. In the DO loops, the dispersion equation is evaluated calling the pertinent Function and each value is compared in sign with the previous one. Change of sign results in the call of Subroutine SEARCH which refines the step of β to obtain the 'zero' of the dispersion equation. These 'zeros' are recorded along with the corresponding parameters and are finally printed out.

The arrays K1REC, K3REC, A1REC, A3REC, BET, SUM, YPLOT, YPLOT1, XAXIS, MAX and LREC which correspond to recorded values used in the plots, have been dimensioned so as to be able to accommodate 50 zeros of the characteristic equation. If more zeros are anticipated the dimensionality of these arrays should be augmented accordingly.

<u>Subroutine PERMIT</u>: This initializes the permittivity tensors of the film and cladding regions. It is called in the main program and in the Functions evaluating the value of the dispersion equations.

<u>Subroutines PCOUPLI/LCOUPLI</u>: These are used to evaluate the coupling matrices R_g in the polar/longitudinal configurations in the film and cladding.

<u>Subroutines EIGP/EIGL</u>: These are used to evaluate the transverse guide indices κ_1 and α_1 (eigenvalues of R_g).

Functions DPPP, DLPL, DPLP, DLLL: These functions evaluate the $h_i/\hat{h}_i$, $g_i/\tilde{g}_i$ parameters and the symmetric/antisymmetric guide indices κ_i'. Subsequently, they compute the value of the dispersion equation by calling the following Subroutines.

Subroutines PPP, LPL, PLP, LLL: These subroutines are called by the corresponding Functions to compute the dispersion equation. The form of the programmed characteristic equation is not the same as in Tables 4.7-10 but is a modified equivalent, where the total sum (determinant) is in real form instead of complex. This saves considerable amount of computer time.

Subroutines SYMMET/ASYMET: These are called by the Functions to compute the κ_i', h_i and g_i parameters. A code-flag from the Main Program regulates the use of the symmetric or antisymmetric form of the κ_i'.

Subroutine SEARCH: This Subroutine is called from the Main Program to refine the step of β in the case the value of the characteristic equation changes sign. It calls the IMSL Subroutine ZBRENT to find the 'zero' within a prescribed accuracy. Specifically, a 'zero' is accepted as the actual root when: (a) two successive β's resulting in values of opposite sign differ less that EPS (e.g. 10^{-7}), (b) two successive values of the dispersion equation differ by NSIG, or (c) after a number MAX of evaluations of the dispersion equation, the algorithm fails to obtain a refined β of the (a) or (b) cases.

Subroutine POLE: This is called by SEARCH to find and abort the poles of the dispersion equation.

Subroutine PLOT: This is called in the Main Program to plot the 'zeros' in the $\beta^2 = f(\ell)$ diagram using the IMSL subroutine USPLO. It is separate from the main program due to the advantage of adjustable arrays (of magnitude equal to the number of 'zeros') since the number of 'zeros' is not a priori known.

Subroutine RECORD: This is called in the Main Program to record all the 'zeros' (β) found along with the corresponding β^2, κ_1, κ_3, α_1, α_3, value of the dispersion equation, number of iterations used by SEARCH to obtain that value and ℓ.

```
      PROGRAM LAYER(INPUT,DEFAUL,OUTPUT,TAPE6=DEFAUL)
      COMMON /AAA/ L
      COMMON /BBB/ PI
      COMMON /CCC/ ISA,ICODE
      PARAMETER (IMAX=101,LMAX=150)
      COMPLEX KL(2),KP(2),KL2(2),KP2(2)
      COMPLEX K1REC(LMAX),K3REC(LMAX)
      REAL A1REC(LMAX),A3REC(LMAX)
      REAL BET(LMAX),SUM(LMAX),YPLOT1(LMAX),XAXIS(LMAX),RANGE(4)
      INTEGER MAX(LMAX)
      COMPLEX EP(3,3),EL(3,3),EPEXT(3,3),ELEXT(3,3)
      REAL BINCR,IMKP1,IMKP3,IMKL1,IMKL3,YPLOT(IMAX,1),LINC,LINCR
      REAL L,LPLSU1,LPLSU2,LLLSU1,LLLSU2,LLMIN,LREC(LMAX)
      REAL YKP(IMAX,2),YKL(IMAX,2),YAP(IMAX,2),YAL(IMAX,2),B2PL(IMAX)
      CHARACTER DUMMY*10,IPOLAR*10,ILONG*17,IBETA2*7,ICHAR*2
      CHARACTER*18 IKTI,IATI
      EXTERNAL DPPP,DPLP,DLPL,DLLL
      DATA I1/1/,IPOLAR/'POLAR CASE'/,ILONG/'LONGITUDINAL CASE'/
      DATA ICHAR/'13'/,IBETA2/'BETA**2'/,IKTI/'MAGNITUDE OF KAPPA'/
      DATA IATI/'MAGNITUDE OF ALPHA'/
C
C==================================================================
C=====INPUT FROM THE TERMINAL=======================================
C==================================================================
      PRINT *,'ASSIGN VALUE FOR BZERO (FOR THE BETA LOOP)'
      READ *,B0
      PRINT *,'ASSIGN VALUE FOR BINCR (BFINAL=B0+101*BINCR)'
      READ *,BINCR
      PRINT *,'ASSIGN VALUE FOR BMIN AND BMAX (PLOT. RANGES OF B **2)'
      READ *,BMIN,BMAX
      PRINT *,'ASSIGN VALUE FOR THE TWO-DIGIT INTEGER ICODE, WHERE:'
      PRINT *,'FIRST DIGIT =1(PPP), 2(LPL), 3(PLP), 4(LLL)'
      PRINT *,'SECOND DIGIT=1(SYMMETRIC MODES),2(ANTISYMMETRIC MODES)'
      READ *, ICODE
      PRINT*,'ASSIGN VALUE TO LLMIN, WHERE LLMIN*PI=MIN OF L'
      READ *,  LLMIN
      PRINT *,'ASSIGN VALUE TO LINCR, AS FRACTIONS OF PI'
      READ *,  LINCR
      PRINT *,' ASSIGN VALUE TO LLMAX: LLMAX-1=# OF LINCR*PI  '
      READ *, LLMAX
C==================================================================
C=====INITIALIZATION===============================================
C==================================================================
      DO 2 I=1,LMAX
2     BET(I)=LREC(I)=0.0
      RANGE(1)=BMIN
      RANGE(2)=BMAX
      RANGE(3)=0.0
      RANGE(4)=1.0
      ICASE=INT(ICODE/10.)
      ISA=INT(ICODE-10*ICASE)
      PI=4*ATAN(1.0)
      LINC=LINCR
      LINCR=LINCR*PI
      BFINAL=B0+(IMAX-1)*BINCR
      XLMAX=LLMIN+(LLMAX-1)*LINC
      CALL PERMIT(EL,1)
      CALL PERMIT(EP,2)
      CALL PERMIT(ELEXT,3)
      CALL PERMIT(EPEXT,4)
C==================================================================
C=====CUTOFF AXIAL GUIDE INDICES===================================
C=====POLAR CASE-HIGH FREQUENCY====================================
      IF ((ICASE.EQ.1).OR.(ICASE.EQ.2)) THEN
      BAP2=REAL(EP(1,1))
      BBP2=REAL((EP(2,2)*EP(3,3)-EP(2,3)*EP(3,2))/EP(3,3))
      BEP=REAL(EP(2,2)-EP(1,1))
      BCP2=REAL(EP(3,3)*(1.-EP(2,2)/EP(1,1))+(CABS(EP(2,3))**2/EP(3,3))*
     #      (1.+EP(3,3)/EP(1,r)))
      BDP2=REAL(EP(3,3)*(1.-EP(2,2)/EP(1,1))-(CABS(EP(2,3))**2/EP(3,3))*
     #      (1.+EP(3,3)/EP(1,1)))
      END IF
C2===POLAR CASE-CUTOFF=============================================
      IF ((ICASE.EQ.1).OR.(ICASE.EQ.3)) THEN
      BAP2EX=REAL(EPEXT(1,1))
      BBP2EX=REAL((EPEXT(2,2)*EPEXT(3,3)-EPEXT(2,3)*EPEXT(3,2))/EPEXT(3,
     #       3))
      BEPEX=REAL(EPEXT(2,2)-EPEXT(1,1))
      BCP2EX=REAL(EPEXT(3,3)*(1.-EPEXT(2,2)/EPEXT(1,1))+(CABS(EPEXT(2,3)
     #       )**2/EPEXT(3,3))*(1.+EPEXT(3,3)/EPEXT(1,1)))
      BDP2EX=REAL(EPEXT(3,3)*(1.-EPEXT(2,2)/EPEXT(1,1))-(CABS(EPEXT(2,3)
     #       )**2/EPEXT(3,3))*(1.+EPEXT(3,3)/EPEXT(1,1)))
      END IF
C3===LONGITUDINAL CASE-HIGH FREQUENCY==============================
      IF ((ICASE.EQ.3).OR.(ICASE.EQ.4)) THEN
      BCL2=REAL(EL(1,1)+EL(2,2)+CSQRT((EL(1,1)-EL(2,2))**2+4*CABS(EL(1
     #     ,2))**2))/2
```

```
      BDL2=REAL(EL(1,1)+EL(2,2)-CSQRT((EL(1,1)-EL(2,2))**2+4*CABS(EL(1,
     #     2))**2))/2
      B1L2=REAL(EL(2,2)+EL(3,3)-(CABS(EL(1,2))**2)/EL(1,1)-(1+EL(3,3)/
     #     EL(1,1))*BCL2)
      B2L2=REAL(EL(2,2)+EL(3,3)-(CABS(EL(1,2))**2)/EL(1,1)-(1+EL(3,3)/
     #     EL(1,1))*BDL2)
      END IF
C4===LONGITUDINAL CASE-CUTOFF============================================
      IF ((ICASE.EQ.2).OR.(ICASE.EQ.4)) THEN
      BCL2EX=REAL(ELEXT(1,1)+ELEXT(2,2)+CSQRT((ELEXT(1,1)-ELEXT(2,2))**
     #       2+4*CABS(ELEXT(1,2))**2))/2
      BDL2EX=REAL(ELEXT(1,1)+ELEXT(2,2)-CSQRT((ELEXT(1,1)-ELEXT(2,2))**
     #       2+4*CABS(ELEXT(1,2))**2))/2
      B1L2EX=REAL(ELEXT(2,2)+ELEXT(3,3)-(CABS(ELEXT(1,2))**2)/ELEXT(1,1)
     #       -(1.+ELEXT(3,3)/ELEXT(1,1))*BCL2EX)
      B2L2EX=REAL(ELEXT(2,2)+ELEXT(3,3)-(CABS(ELEXT(1,2))**2)/ELEXT(1,1)
     #       -(1.+ELEXT(3,3)/ELEXT(1,1))*BDL2EX)
      END IF
C=======================================================================
      DO 10 LL=1,LLMAX
      L=LLMIN*PI+(LL-1)*LINCR
      INDEX=1
C
      DO 10 I=1,IMAX
      BETA=B0+(I-1)*BINCR
      CALL EIGP(KP,BETA,EP,1)
      CALL EIGP(KP2,BETA,EPEXT,2)
      CALL EIGL(KL,BETA,EL,1)
      CALL EIGL(KL2,BETA,ELEXT,2)
      REKL21=REAL(KL2(1))
      REKL23=REAL(KL2(2))
      REKP21=REAL(KP2(1))
      REKP23=REAL(KP2(2))
      IMKL1=AIMAG(KL(1))
      IMKL3=AIMAG(KL(2))
      IMKP1=AIMAG(KP(1))
      IMKP3=AIMAG(KP(2))
C
      YKL(I,1)=CABS(KL(1))
      YKL(I,2)=CABS(KL(2))
      YKP(I,1)=CABS(KP(1))
      YKP(I,2)=CABS(KP(2))
      YAL(I,1)=CABS(KL2(1))
      YAL(I,2)=CABS(KL2(2))
      YAP(I,1)=CABS(KP2(1))
      YAP(I,2)=CABS(KP2(2))
      B2PL(I)=BETA**2
C=======================================================================
C==1==POLAR/POLAR/POLAR=================================================
C=======================================================================
      IF (ICASE.EQ.1) THEN
      IF ((IMKP1.NE.0.).AND.(IMKP3.NE.0.)) GO TO 10
      IF ((REKP21.NE.0.).OR.(REKP23.NE.0.)) GO TO 10
      IF (INDEX.NE.1) GO TO 15
      PPPSU1=DPPP(BETA)
      INDEX=INDEX+1
      GO TO 10
C======================
15    PPPSU2=DPPP(BETA)
      PROD=PPPSU1*PPPSU2
      IF (PROD.LT.0.) THEN
      BE=BETA
      CALL SEARCH(BE,DPPP,BINCR,BETA1,M1,PPPSU1)
      IF (BETA1.NE.0.) THEN
      BET(I1)=BETA1
      CALL EIGP(KP,BET(I1),EP,1)
      CALL EIGP(KP2,BET(I1),EPEXT,2)
      CALL RECORD(BET(I1),M1,MAX(I1),KP,K1REC(I1),
     #     K3REC(I1),KP2,A1REC(I1),A3REC(I1),SUM(I1),DPPP)
      LREC(I1)=L/PI
      I1=I1+1
      END IF
      PPPSU1=PPPSU2
      ELSE IF (PROD.EQ.0.) THEN
      BET(I1)=BETA
      CALL RECORD(BET(I1),0,MAX(I1),KP,K1REC(I1),
     #     K3REC(I1),KP2,A1REC(I1),A3REC(I1),SUM,DPPP)
      LREC(I1)=L/PI
      I1=I1+1
      PPPSU1=-PPPSU1
      ELSE IF (PROD.GT.0.) THEN
      PPPSU1=PPPSU2
      END IF
      END IF
C======================================================================
C=== LONG./POLAR/LONG. =================================================
```

```
C=========================================================================
      IF (ICASE.EQ.2) THEN
      IF ((REKL21.NE.0.).OR.(REKL23.NE.0.)) GO TO 10
      IF (INDEX.NE.1) GO TO 25
      LPLSU1=DLPL(BETA)
      INDEX=INDEX+1
      GO TO 10
C=======================
25    LPLSU2=DLPL(BETA)
      PROD=LPLSU1*LPLSU2
      IF (PROD.LT.0.) THEN
      BE=BETA
      CALL SEARCH(BE,DLPL,BINCR,BETA1,M1,LPLSU1)
      IF (BETA1.NE.0.) THEN
      BET(I1)=BETA1
      CALL EIGP(KP,BET(I1),EP,1)
      CALL EIGL(KL2,BET(I1),ELEXT,2)
      CALL RECORD(BET(I1),M1,MAX(I1),KP,K1REC(I1),
     #     K3REC(I1),KL2,A1REC(I1),A3REC(I1),SUM(I1),DLPL)
      LREC(I1)=L/PI
      I1=I1+1
      END IF
      LPLSU1=LPLSU2
      ELSE IF (PROD.EQ.0.) THEN
      BET(I1)=BETA
      CALL RECORD(BET(I1),0,MAX(I1),KP,K1REC(I1),
     #     K3REC(I1),KL2,A1REC(I1),A3REC(I1),SUM(I1),DLPL)
      LREC(I1)=L/PI
      I1=I1+1
      LPLSU1=-LPLSU1
      ELSE IF (PROD.GT.0.) THEN
      LPLSU1=LPLSU2
      END IF
      END IF
C=========================================================================
C=====POLAR/LONG./POLAR===================================================
C=========================================================================
      IF (ICASE.EQ.3) THEN
      IF ((IMKL1.NE.0.).AND.(IMKL3.NE.0.)) GO TO 10
      IF ((REKP21.NE.0.0).OR.(REKP23.NE.0.0)) GO TO 10
      IF (INDEX.NE.1) GO TO 35
      PLPSU1=DPLP(BETA)
      INDEX=INDEX+1
      GO TO 10
C=======================
35    PLPSU2=DPLP(BETA)
      PROD=PLPSU1*PLPSU2
      IF (PROD.LT.0.) THEN
      BE=BETA
      CALL SEARCH(BE,DPLP,BINCR,BETA1,M1,PLPSU1)
      IF (BETA1.NE.0.) THEN
      BET(I1)=BETA1
      CALL EIGL(KL,BET(I1),EL,1)
      CALL EIGP(KP2,BET(I1),EPEXT,2)
      CALL RECORD(BET(I1),M1,MAX(I1),KL,K1REC(I1),
     #     K3REC(I1),KP2,A1REC(I1),A3REC(I1),SUM(I1),DPLP)
      LREC(I1)=L/PI
      I1=I1+1
      END IF
      PLPSU1=PLPSU2
      ELSE IF (PROD.EQ.0.) THEN
      BET(I1)=BETA
      CALL RECORD(BET(I1),0,MAX(I1),KL,K1REC(I1),
     #     K3REC(I1),KP2,A1REC(I1),A3REC(I1),SUM(I1),DPLP)
      LREC(I1)=L/PI
      I1=I1+1
      PLPSU1=-PLPSU1
      ELSE IF (PROD.GT.0.) THEN
      PLPSU1=PLPSU2
      END IF
      END IF
C=========================================================================
C=====LONG./LONG./LONG.===================================================
C=========================================================================
      IF (ICASE.EQ.4) THEN
      IF ((REKL21.NE.0.0).OR.(REKL23.NE.0.0)) GO TO 10
      IF (INDEX.NE.1) GO TO 45
      LLLSU1=DLLL(BETA)
      INDEX=INDEX+1
      GO TO 10
C=======================
45    LLLSU2=DLLL(BETA)
      PROD=LLLSU1*LLLSU2
      IF (PROD.LT.0.) THEN
      BE=BETA
      CALL SEARCH(BE,DLLL,BINCR,BETA1,M1,LLLSU1)
```

```
      IF (BETA1.NE.0.) THEN
      BET(I1)=BETA1
      CALL EIGL(KL,BET(I1),EL,1)
      CALL EIGL(KL2,BET(I1),ELEXT,2)
      CALL RECORD(BET(I1),M1,MAX(I1),KL,K1REC(I1),
     #     K3REC(I1),KL2,A1REC(I1),A3REC(I1),SUM(I1),DLLL)
      LREC(I1)=L/PI
      I1=I1+1
      END IF
      LLLSU1=LLLSU2
      ELSE IF (PROD.EQ.0.) THEN
      BET(I1)=BETA
      CALL RECORD(BET(I1),0,MAX(I1),KL,K1REC(I1),
     #     K3REC(I1),KL2,A1REC(I1),A3REC(I1),SUM(I1),DLLL)
      LREC(I1)=L/PI
      I1=I1+1
      LLLSU1=-LLLSU1
      ELSE IF (PROD.GT.0.) THEN
      LLLSU1=LLLSU2
      END IF
      END IF
10    CONTINUE
C=========================================================================
C==== PRINT-OUT STATEMENTS ===============================================
C=========================================================================
      WRITE(6,80) BO,BINCR,BFINAL
80    FORMAT('1'//10X,'BMIN=',F6.3,6X,'BINCR=',F7.4,6X,'BMAX=',F6.3/)
      WRITE(6,81) LLMIN,LINC,XLMAX
81    FORMAT(/10X,'LMIN/PI=',F6.3,3X,'LINCR/PI=',F6.3,3X,'LMAX/PI='
     #       ,F6.3/)
      WRITE(6,82)
82    FORMAT(/10X,'PERMITTIVITY MATRIX OF LAYER'/10X,
     #'============================'/)
      IF ((ICASE.EQ.3).OR.(ICASE.EQ.4)) THEN
      WRITE(6,103)
103   FORMAT(7X,'LONGITUDINAL CASE'/)
      WRITE(6,104) ((EL(I,J),J=1,3),I=1,3)
104   FORMAT((10X,'*',67X,'*'/10X,'*',3(1X,'(',F9.6,',',F9.6,
     #       ')'),1X,'*'/10X,'*',67X,'*'))
      WRITE(6,84) BCL2,BDL2,B1L2,B2L2
84    FORMAT(/10X,'B C**2=',F10.7,6X,'B D**2=',F10.7,6X,'B01**2=',
     #       F10.7,6X,'B02**2=',F10.7/)
      END IF
      IF ((ICASE.EQ.1).OR.(ICASE.EQ.2)) THEN
      WRITE(6,105)
105   FORMAT(7X,'POLAR CASE'/)
      WRITE(6,104) ((EP(I,J),J=1,3),I=1,3)
      WRITE(6,85) BAP2,BBP2,BCP2,BDP2,BEP
85    FORMAT(/10X,'B A**2=',F10.7,6X,'B B**2=',F10.7,6X,'B C**2=',
     #       F10.7,6X,'B D**2=',F10.7,6X,'EYY-EXX=',F10.7/)
      END IF
      WRITE(6,83)
83    FORMAT(/10X,'PERMITTIVITY MATRIX OF SUBSTRATE'/10X,
     #'================================'/)
      IF ((ICASE.EQ.2).OR.(ICASE.EQ.4)) THEN
      WRITE(6,103)
      WRITE(6,104) ((ELEXT(I,J),J=1,3),I=1,3)
      WRITE(6,84) BCL2EX,BDL2EX,B1L2EX,B2L2EX
      END IF
      IF ((ICASE.EQ.1).OR.(ICASE.EQ.3)) THEN
      WRITE(6,105)
      WRITE(6,104) ((EPEXT(I,J),J=1,3),I=1,3)
      WRITE(6,85) BAP2EX,BBP2EX,BCP2EX,BDP2EX,BEPEX
      END IF
C
      IF (ICASE.EQ.1) WRITE(6,95)
      IF (ICASE.EQ.2) WRITE(6,98)
      IF (ICASE.EQ.3) WRITE(6,99)
      IF (ICASE.EQ.4) WRITE(6,100)
C
      IF (ISA.EQ.1) WRITE(6,86)
      IF (ISA.EQ.2) WRITE(6,89)
C
      WRITE(6,96)
      I1=I1-1
      DO 200 M=1,I1
      B2=BET(M)**2
      WRITE(6,97) BET(M),B2,K1REC(M),K3REC(M),A1REC(M),A3REC(M),
     #            SUM(M),MAX(M),LREC(M)
200   CONTINUE
C
95    FORMAT(//10X,'POL/POL/POL CASE'/)
98    FORMAT(//10X,'LONG/POL/LONG CASE'/)
99    FORMAT(//10X,'POL/LONG/POL CASE'/)
100   FORMAT(//10X,'LONG/LONG/LONG CASE'/)
86    FORMAT(/10X,'SYMMETRIC DISTRIBUTION'/)
```

```
89    FORMAT(/10X,'ANTISYMMETRIC DISTRIBUTION'/)
96    FORMAT(5X,'BETA',5X,'BETA**2',10X,'KAPPA-1',20X,'KAPPA-3',11X,
     +'ALPHA-1',4X,'ALPHA-3',6X,'DETERM',4X,'MAXFN',3X,'L/PI'/5X,
     2'====',5X,'=======',10X,'=======',20X,'=======',11X,'=======',4X,
     #'=======',6X,'======',4X,'=====',3X,'====')
97    FORMAT(1X,'* ',F8.6,' * ',F8.6,' *',2('(',1X,F8.6,1X,',',1X,F8.6,
     #' ) * '),2(F8.6,1X,'*',1X),E10.3,' * ',I3,' * ',F6.4,' *')
C=======================================================================
C==== PLOT THE BETA**2 VERSUS NORMALIZED LENGTH ========================
C=======================================================================
      DO 3 I=1,I1
      XAXIS(I)=YPLOT1(I)=0.0
3     CONTINUE
      DO 4 I=1,I1
      XAXIS(I)=LREC(I)
      YPLOT1(I)=BET(I)**2
4     CONTINUE
C
      CALL PLOT(XAXIS,YPLOT,YPLOT1,I1,LLMIN,XLMAX,BMIN,BMAX)
      STEPB=(BMAX-BMIN)/10
      STEPL=(XLMAX-LLMIN)/10
      WRITE(6,106) BMIN,BMAX,STEPB,LLMIN,XLMAX,STEPL
106   FORMAT(///10X,'BMIN=',F6.4,'   BMAX=',F6.4,'   STEP=',F6.4,
     #'  LMIN=',F7.4,'   LMAX=',F7.4,'   STEP=',F7.4)
C=======================================================================
C==== PLOT THE K, A'S VERSUS BETA**2 ===================================
C=======================================================================
      CALL USPLO(B2PL,YKP,100,100,2,1,IPOLAR,10,IBETA2,7,IKTI,18,
     #           RANGE,ICHAR,1,IER)
      CALL USPLO(B2PL,YAP,100,100,2,1,IPOLAR,10,IBETA2,7,IATI,18,
     #           RANGE,ICHAR,1,IER)
      CALL USPLO(B2PL,YKL,100,100,2,1,ILONG,17,IBETA2,7,IKTI,18,
     #           RANGE,ICHAR,1,IER)
      CALL USPLO(B2PL,YAL,100,100,2,1,ILONG,17,IBETA2,7,IATI,18,
     #           RANGE,ICHAR,1,IER)
C
      STOP
      END
      SUBROUTINE RECORD(BETA,M,MNEW,K,K1,K3,KEX,A1,A3,D,SUM)
C=====================================================
C      RECORD RECORDS THE 'ZEROS' OF THE DISPERSION
C     EQUATION ALONG WITH THE CORRESPONDING PARAMETERS
C=====================================================
      COMMON /AAA/ L
      EXTERNAL SUM
      COMPLEX K(2),K1,K3,KEX(2)
      REAL L
C
      K1=K3=CMPLX(0.0,0.0)
      A1=A3=0.0
      MNEW=M
      K1=K(1)
      K3=K(2)
      A1=AIMAG(KEX(1))
      A3=AIMAG(KEX(2))
      D=SUM(BETA)
C
      RETURN
      END
      SUBROUTINE PERMIT(E,N)
C=======================================================================
C     PERMIT INITIALIZES THE VALUES OF THE PERMITTIVITY MATRICES IN THE
C     LAYERED AND SEMI-INFINITE MEDIA, IN THE LONGITUDINAL AND POLAR CASES
C=======================================================================
      COMMON /AAA/ L
      REAL L
      COMPLEX E(3,3),EHERM(3,3),SUM
C
      DO 10 J=1,3
      DO 10 K=1,3
10    E(J,K)=CMPLX(0.,0.)
C LONGITUDINAL INTERNAL
      IF (N.EQ.1) THEN
      E(1,1)=CMPLX(2.8,0.0)
      E(1,2)=CMPLX(0.1,0.0)
      E(2,1)=CONJG(E(1,2))
      E(2,2)=CMPLX(2.5,0.0)
      E(3,3)=CMPLX(2.6,0.0)
C  POLAR INTERNAL
      ELSE IF (N.EQ.2) THEN
      E(1,1)=CMPLX(2.8,0.0)
      E(2,2)=CMPLX(2.5,0.0)
      E(2,3)=CMPLX(0.1,0.0)
      E(3,2)=CONJG(E(2,3))
      E(3,3)=CMPLX(2.6,0.0)
C  LONGITUDINAL EXTERNAL
```

```
      ELSE IF (N.EQ.3) THEN
      E(1,1)=CMPLX(1.8,0.0)
      E(1,2)=CMPLX(0.0,0.01)
      E(2,1)=CONJG(E(1,2))
      E(2,2)=CMPLX(2.0,0.0)
      E(3,3)=CMPLX(1.7,0.0)
C  POLAR EXTERNAL
      ELSE IF (N.EQ.4) THEN
      E(1,1)=CMPLX(1.8,0.0)
      E(2,2)=CMPLX(2.0,0.0)
      E(2,3)=CMPLX(0.0,0.01)
      E(3,2)=CONJG(E(2,3))
      E(3,3)=CMPLX(1.7,0.0)
      END IF
C
      DO 20 II=1,3
      DO 20 JJ=1,3
20    EHERM(II,JJ)=CONJG(E(II,JJ))
C
      IF ((ICASE.EQ.1).OR.(ICASE.EQ.2)) THEN
      IF ((E(1,2).NE.ZERO).OR.(E(1,3).NE.ZERO)) THEN
      WRITE(6,33)
      WRITE(6,22) ((E(I,J),J=1,3),I=1,3)
      STOP
      END IF
      ELSE IF ((ICASE.EQ.3).OR.(ICASE.EQ.4)) THEN
      IF ((E(1,3).NE.ZERO).OR.(E(2,3).NE.ZERO)) THEN
      WRITE(6,34)
      WRITE(6,22) ((E(I,J),J=1,3),I=1,3)
      END IF
      END IF
      IF ((ICASE.EQ.1).OR.(ICASE.EQ.3)) THEN
      IF ((E(1,2).NE.ZERO).OR.(E(1,3).NE.ZERO)) THEN
      WRITE(6,33)
      WRITE(6,22) ((E(I,J),J=1,3),I=1,3)
      STOP
      END IF
      ELSE IF ((ICASE.EQ.2).OR.(ICASE.EQ.4)) THEN
      IF ((E(1,3).NE.ZERO).OR.(E(2,3).NE.ZERO)) THEN
      WRITE(6,34)
      WRITE(6,22) ((E(I,J),J=1,3),I=1,3)
      STOP
      END IF
      END IF
      DO 30 II=1,3
      DO 30 JJ=1,3
      IF (EHERM(II,JJ).NE.E(II,JJ)) THEN
      IF (N.EQ.1) THEN
      WRITE(6,11)
      WRITE(6,22) ((E(I,J),J=1,3),I=1,3)
      STOP
      END IF
      IF (N.EQ.2) THEN
      WRITE(6,12)
      WRITE(6,22) ((E(I,J),J=1,3),I=1,3)
      STOP
      END IF
      END IF
30    CONTINUE
11    FORMAT('1','THE PERMITTIVITY MATRIX OF THE FILM IS NOT HERMITEAN')
12    FORMAT('1','THE PERMITTIVITY MATRIX OF THE CLADDING IS NOT HERITEA
     #N')
22    FORMAT((10X,'*',3(1X,F[illegible]',',F6.3,' *')))
33    FORMAT('1'/,'THE FOLLOWING PERM. MATRIX IS NOT OF THE POLAR FORM')
34    FORMAT('1'/,'THE FOLLOWING PERM. MATRIX IS NOT OF THE LONG. FORM')
      RETURN
      END
      SUBROUTINE LCOUPLI(R,E,BETA)
C*************************************************************************
C      LCOUPLI EVALUATES THE COUPLING MATRIX R IN THE LONGITUDINAL CASE
C*************************************************************************
      COMPLEX R(4,4),E(3,3),DZZ
C
      DO 10 I=1,4
      DO 10 J=1,4
10    R(I,J)=CMPLX(0.,0.)
      DZZ=E(1,1)*E(2,2)-E(1,2)*E(2,1)
      R(1,2)=CMPLX(1.,0.)
      R(2,1)=DZZ/E(1,1)-BETA**2
      R(2,4)=-BETA*E(2,1)/E(1,1)
      R(3,1)=-BETA*E(1,2)/E(1,1)
      R(3,4)=1-BETA**2/E(1,1)
      R(4,3)=E(3,3)
C
      RETURN
      END
```

```
      SUBROUTINE PCOUPLI(R,E,BETA)
C=========================================================================
C      PCOUPLI EVALUATES THE COUPLING MATRIX R IN THE POLAR CASE
C=========================================================================
      COMPLEX R(4,4),E(3,3)
C
      DO 10 I=1,4
      DO 10 J=1,4
10    R(I,J)=CMPLX(0.,0.)
      R(1,2)=CMPLX(1.,0.)
      R(2,1)=E(2,2)-BETA**2
      R(2,3)=E(2,3)
      R(3,4)=1.0-BETA**2/E(1,1)
      R(4,1)=E(3,2)
      R(4,3)=E(3,3)
C
      RETURN
      END
      SUBROUTINE EIGL(K,BETA,E,M)
C=================================================================
C     EIGL EVALUATES THE EIGENVALUES KAPPA IN THE LONGITUDINAL CASE
C=================================================================
      COMPLEX K(2),R(4,4),A,B,E(3,3),DUMMY
C
      CALL LCOUPLI(R,E,BETA)
      A=R(3,4)*R(4,3)+R(2,1)
      B=(R(3,4)*R(4,3)-R(2,1))**2 + 4*R(4,3)*CABS(R(3,1))**2
      K(1)=(A+CSQRT(B))/2
      K(1)=CSQRT(K(1))
      K(2)=(A-CSQRT(B))/2
      K(2)=CSQRT(K(2))
C
      IF (M.EQ.1) RETURN
      AIM1=AIMAG(K(1))
      AIM3=AIMAG(K(2))
      IF (ABS(AIM1).LT.ABS(AIM3)) THEN
      DUMMY=K(1)
      K(1)=K(2)
      K(2)=DUMMY
      END IF
C
      RETURN
      END
      SUBROUTINE EIGP(K,BETA,E,M)
C=================================================================
C     EIGP EVALUATES THE EIGENVALUES KAPPA IN THE POLAR CASE
C=================================================================
      COMPLEX K(2),A,B,R(4,4),E(3,3),DUMMY
C
      CALL PCOUPLI(R,E,BETA)
      A=R(3,4)*R(4,3)+R(2,1)
      B=(R(3,4)*R(4,3)-R(2,1))**2+4*R(3,4)*R(4,1)*CONJG(R(4,1))
      K(1)=(A+CSQRT(B))/2
      K(2)=(A-CSQRT(B))/2
      K(1)=CSQRT(K(1))
      K(2)=CSQRT(K(2))
C
      IF (M.EQ.1) RETURN
      AIM1=AIMAG(K(1))
      AIM3=AIMAG(K(2))
      IF (ABS(AIM1).LT.ABS(AIM3)) THEN
      DUMMY=K(1)
      K(1)=K(2)
      K(2)=DUMMY
      END IF
C
      RETURN
      END
      FUNCTION DPPP(BETA)
C=========================================================================
C     DPPP EVALUATES IN THE POL/POL/POL CASE THE VALUE OF THE
C     CHARACTERISTIC EQUATION
C=========================================================================
      COMMON /AAA/ L
      COMMON /CCC/ ISA,ICODE
      COMPLEX K(2),K2(2),E(3,3),E2(3,3)
      REAL L,K1,K3
C
      CALL PERMIT(E,2)
      CALL PERMIT(E2,4)
      CALL EIGP(K,BETA,E,1)
      CALL EIGP(K2,BETA,E2,2)
      IF (ISA.EQ.1) CALL SYMMET(K(1),L,K1,H1,G1,E,BETA)
      IF (ISA.EQ.2) CALL ASYMET(K(1),L,K1,H1,G1,E,BETA)
      IF (ISA.EQ.1) CALL SYMMET(K(2),L,K3,H3,G3,E,BETA)
      IF (ISA.EQ.2) CALL ASYMET(K(2),L,K3,H3,G3,E,BETA)
```

```
      A1=AIMAG(K2(1))
      A3=AIMAG(K2(2))
      E2XX=REAL(E2(1,1))
      E2ZZ=REAL(E2(3,3))
      H12=-(E2XX-BETA**2)/(E2XX*A1**2+E2ZZ*(E2XX-BETA**2))
      H32=-(E2XX-BETA**2)/(E2XX*A3**2+E2ZZ*(E2XX-BETA**2))
      CALL PPP(K1,K3,H1,H3,H12,H32,A1,A3,E,E2,VALUE)
      DPPP=VALUE
      RETURN
      END
      FUNCTION DLPL(BETA)
C=================================================================
C     DLPLSY IS THE CORRESPONDING DPPP IN THE LONG/POL/LONG CASE
C=================================================================
      COMMON /AAA/ L
      COMMON /CCC/ ISA,ICODE
      COMPLEX K(2),K2(2),E(3,3),E2(3,3)
      REAL L,K1,K3
C
      CALL PERMIT(E,2)
      H1=3850
      CALL PERMIT(E2,3)
      CALL EIGP(K,BETA,E,1)
      CALL EIGL(K2,BETA,E2,2)
      IF (ISA.EQ.1) CALL SYMMET(K(1),L,K1,H1,G1,E,BETA)
      IF (ISA.EQ.2) CALL ASYMET(K(1),L,K1,H1,G1,E,BETA)
      IF (ISA.EQ.1) CALL SYMMET(K(2),L,K3,H3,G3,E,BETA)
      IF (ISA.EQ.2) CALL ASYMET(K(2),L,K3,H3,G3,E,BETA)
      A1=AIMAG(K2(1))
      A3=AIMAG(K2(2))
      E2XX=REAL(E2(1,1))
      E2ZZ=REAL(E2(3,3))
      G12=BETA*E2ZZ/(E2ZZ*(E2XX-BETA**2)+E2XX*A1**2)
      G32=BETA*E2ZZ/(E2ZZ*(E2XX-BETA**2)+E2XX*A3**2)
      CALL LPL(K1,K3,H1,H3,G12,G32,A1,A3,E,E2,VALUE)
      DLPL=VALUE
      RETURN
      END
      FUNCTION DPLP(BETA)
C=================================================================
C     DPLPSY IS THE CORRESPONDING DPPP IN THE POL/LONG/POL CASE
C=================================================================
      COMMON /AAA/ L
      COMMON /CCC/ ISA,ICODE
      COMPLEX K(2),K2(2),E(3,3),E2(3,3)
      REAL L,K1,K3
C
      CALL PERMIT(E,1)
      CALL PERMIT(E2,4)
      CALL EIGL(K,BETA,E,1)
      CALL EIGP(K2,BETA,E2,2)
      IF (ISA.EQ.1) CALL SYMMET(K(1),L,K1,H1,G1,E,BETA)
      IF (ISA.EQ.2) CALL ASYMET(K(1),L,K1,H1,G1,E,BETA)
      IF (ISA.EQ.1) CALL SYMMET(K(2),L,K3,H3,G3,E,BETA)
      IF (ISA.EQ.2) CALL ASYMET(K(2),L,K3,H3,G3,E,BETA)
      A1=AIMAG(K2(1))
      A3=AIMAG(K2(2))
      E2XX=REAL(E2(1,1))
      E2ZZ=REAL(E2(3,3))
      H12=-(E2XX-BETA**2)/(E2XX*A1**2+E2ZZ*(E2XX-BETA**2))
      H32=-(E2XX-BETA**2)/(E2XX*A3**2+E2ZZ*(E2XX-BETA**2))
      CALL PLP(K1,K3,G1,G3,H12,H32,A1,A3,E,E2,VALUE)
      DPLP=VALUE
C
      RETURN
      END
      FUNCTION DLLL(BETA)
C=================================================================
C     DLLLSY IS THE CORRESPONDING DPPP IN THE LONG/LONG/LONG CASE
C=================================================================
      COMMON /AAA/ L
      COMMON /CCC/ ISA,ICODE
      COMPLEX K(2),K2(2),E(3,3),E2(3,3)
      REAL L,K1,K3
C
      CALL PERMIT(E,1)
      CALL PERMIT(E2,3)
      CALL EIGL(K,BETA,E,1)
      CALL EIGL(K2,BETA,E2,2)
      IF (ISA.EQ.1) CALL SYMMET(K(1),L,K1,H1,G1,E,BETA)
      IF (ISA.EQ.2) CALL ASYMET(K(1),L,K1,H1,G1,E,BETA)
      IF (ISA.EQ.1) CALL SYMMET(K(2),L,K3,H3,G3,E,BETA)
      IF (ISA.EQ.2) CALL ASYMET(K(2),L,K3,H3,G3,E,BETA)
      A1=AIMAG(K2(1))
      A3=AIMAG(K2(2))
      E2XX=REAL(E2(1,1))
```

```
      E2ZZ=REAL(E2(3,3))
      G12=BETA*E2ZZ/(E2ZZ*(E2XX-BETA**2)+E2XX*A1**2)
      G32=BETA*E2ZZ/(E2ZZ*(E2XX-BETA**2)+E2XX*A3**2)
      CALL LLL(K1,K3,G1,G3,G12,G32,A1,A3,E,E2,VALUE)
      DLLL=VALUE
      RETURN
      END
      SUBROUTINE SYMMET(K,L,KPR,H,G,E,BETA)
C=========================================================================
C     SYMMET EVALUATES THE K1 AND K3 OF THE LAYERED MEDIUM FROM THE GENERALLY
C     COMPLEX K(1) AND K(3) PROPAGATION CONSTANTS, IN THE SYMMETRIC CASE
C=========================================================================
      COMMON /BBB/ PI
      REAL L,KPR,L1
      COMPLEX E(3,3),K
C
      L1=L/PI
      EXX=REAL(E(1,1))
      EZZ=REAL(E(3,3))
      B2X=1-BETA**2/EXX
      REK=REAL(K)
      AIMK=AIMAG(K)
      IF((AIMK.NE.0.).AND.(REK.NE.0.)) WRITE(6,11)K,BETA
 11   FORMAT(1X,'CMPLX K IN SYMMET: K=(',F6.3,',',F6.3,')',F6.3)
      IF (REK.EQ.0.) THEN
      KPR=-AIMK*TANH(AIMK*L)
      H=-B2X/(AIMK**2+EZZ*B2X)
      G=BETA*EZZ/(EXX*(EZZ*B2X+AIMK**2))
      ELSE IF (AIMK.EQ.0.) THEN
      KPR=REK*TAN(REK*L)
      H=B2X/(REK**2-EZZ*B2X)
      G=BETA*EZZ/(EXX*(EZZ*B2X-REK**2))
      END IF
      RETURN
      END
      SUBROUTINE ASYMET(K,L,KPR,H,G,E,BETA)
C=========================================================================
C     ASYMET IS THE CORRESPONDING SYMMET IN THE ANTISYMMETRIC CASE
C=========================================================================
      COMMON /BBB/ PI
      REAL L,KPR,L1
      COMPLEX E(3,3),K
C
      L1=L/PI
      EE=1.E-7
C
      EXX=REAL(E(1,1))
      EZZ=REAL(E(3,3))
      B2X=1.-BETA**2/EXX
      REK=REAL(K)
      AIMK=AIMAG(K)
      IF((AIMK.NE.0.).AND.(REK.NE.0.)) WRITE(6,11) K,BETA
 11   FORMAT(1X,' COMPLEX K IN ASYMET: K=(',F6.3,',',F6.3,')',
     #        ' BETA=',F6.4/)
      IF (REK.EQ.0.) THEN
      KPR=-AIMK/TANH(AIMK*L)
      H=-B2X/(AIMK**2+EZZ*B2X)
      G=BETA*EZZ/(EXX*(EZZ*B2X+AIMK**2))
      ELSE IF (AIMK.EQ.0.) THEN
      KPR=-REK/TAN(REK*L)
      H=B2X/(REK**2-EZZ*B2X)
      G=BETA*EZZ/(EXX*(EZZ*B2X-REK**2))
      END IF
      RETURN
      END
      SUBROUTINE PPP(K1,K3,H1,H3,H12,H32,A1,A3,E,E2,VALUE)
C=========================================================================
C     PPP EVALUATES THE VALUE OF THE CHARACTERISTIC EQUATION IN THE
C     POL/POL/POL CASE
C=========================================================================
      REAL K1,K3
      COMPLEX E(3,3),E2(3,3)
C
      VALUE=0.
      EYZ2=CABS(E2(2,3))**2
      PPP1=K1*K3*(1/H1-1/H3)*(H12-H32)*EYZ2
      PPP2=A1*A3*CABS(E(2,3))**2*(H1-H3)*(1/H12-1/H32)
      PPP3=CABS(E(2,3))**2*(H1*K3-H3*K1)*(A1/H32-A3/H12)
      PPP4=(K1/H3-K3/H1)*(A3*H12-A1*H32)*EYZ2
      PPP5=2*REAL(E(2,3)*CONJG(E2(2,3)))*(K1-K3)*(A1-A3)
      VALUE=PPP1+PPP2+PPP3+PPP4+PPP5
C
      RETURN
      END
      SUBROUTINE LPL(K1,K3,H1,H3,G12,G32,A1,A3,E,E2,VALUE)
C=========================================================
```

```
C     LPL EVALUATES THE VALUE OF THE CHARACTERISTIC EQUATION IN
C     THE LONG/POL/LONG CASE
C*****************************************************************
      COMPLEX E(3,3),E2(3,3)
      REAL K1,K3,LPL1,LPL2,LPL3,LPL4,LPL5
C
      EZZ=REAL(E2(3,3))
      VALUE=0.
      LPL1=K1*K3*(1/H1-1/H3)*(A1*G12-A3*G32)/EZZ
      LPL2=A1*A3*(K1/H3-K3/H1)*(G12-G32)/EZZ
      LPL3=CABS(E(2,3))**2*EZZ*(H3-H1)*(A1/G12-A3/G32)/CABS(E2(1,2))**2
      LPL4=CABS(E(2,3))**2*EZZ*(H1*K3-H3*K1)*(1/G12-1/G32)/
     #     CABS(E2(1,2))**2
      LPL5=2*AIMAG(E(2,3)/E2(2,1))*(K1-K3)*(A1-A3)
      VALUE=LPL1+LPL2+LPL3+LPL4+LPL5
C
      RETURN
      END
      SUBROUTINE PLP(K1,K3,G1,G3,H12,H32,A1,A3,E,E2,VALUE)
C*****************************************************************
C     PLP EVALUATES THE VALUE OF THE CHARACTERISTIC EQUATION
C     IN THE POL/LONG/POL CASE
C*****************************************************************
      REAL K1,K3
      COMPLEX E(3,3),E2(3,3)
C
      EZZ=REAL(E(3,3))
      VALUE=0.
      PLP1=K1*K3*(G1-G3)*(A1/H32-A3/H12)/EZZ
      PLP2=A1*A3*(G1*K1-G3*K3)*(1/H12-1/H32)/EZZ
      PLP3=CABS(E2(2,3))**2*EZZ*(K1/G1-K3/G3)*(H32-H12)/CABS(E(1,2))**2
      PLP4=CABS(E2(2,3))**2*EZZ*(1/G1-1/G3)*(A3*H12-A1*H32)/
     #     CABS(E(1,2))**2
      PLP5=2*AIMAG(E2(2,3)/E(2,1))*(K1-K3)*(A1-A3)
      VALUE=PLP1+PLP2+PLP3+PLP4+PLP5
C
      RETURN
      END
      SUBROUTINE LLL(K1,K3,G1,G3,G12,G32,A1,A3,E,E2,VALUE)
C*******************************************************************
C     LLL EVALUATES THE VALUE OF THE CHARACTERISTIC EQUATION
C     IN THE LONG/LONG/LONG CASE
C*******************************************************************
      REAL LLL1,LLL2,LLL3,LLL4,LLL5,K1,K3
      COMPLEX E(3,3),E2(3,3)
C
      E2ZZ=REAL(E2(3,3))
      EZZ=REAL(E(3,3))
      VALUE=0.
      LLL1=K1*K3*E2ZZ*(G1-G3)*(1/G12-1/G32)/EZZ
      LLL2=A1*A3*EZZ*CABS(E2(1,2))**2*(1/G1-1/G3)*(G12-G32)/
     #     (CABS(E(1,2))**2*E2ZZ)
      LLL3=E2ZZ*(G3*K3-G1*K1)*(A1/G12-A3/G32)/EZZ
      LLL4=CABS(E2(1,2))**2*EZZ*(K3/G3-K1/G1)*(A1*G12-A3*G32)/
     #     (CABS(E(1,2))**2*E2ZZ)
      LLL5=2*REAL(E2(1,2)/E(1,2))*(K1-K3)*(A1-A3)
      VALUE=LLL1+LLL2+LLL3+LLL4+LLL5
C
      RETURN
      END
      SUBROUTINE POLE(B2,DET1,ROOT,BETA,BINCR,DETERM)
C**************************************************************************
C     POLE EXAMINES WHETHER THE 'ROOT' FOUND BY ZBRENT IS THE ACTUAL ROOT
C     OR A POLE. IN THE CASE OF A POLE, IT SETS THE RECORDED ROOT TO ZERO VALUE
C**************************************************************************
      EXTERNAL DETERM
C
      BETA1=BETA-BINCR
      BETA2=BETA
C
      IF ((B2.EQ.BETA1).OR.(B2.EQ.BETA2)) THEN
      ROOT=0.
      RETURN
      END IF
      IF (BETA.EQ.0.) THEN
      ROOT=0.
      RETURN
      END IF
C
      D2=DETERM(B2)
      IF (ABS(D2).GT.ABS(DET1)) ROOT=0.
C
      RETURN
      END
      SUBROUTINE SEARCH(BETA,DETERM,BINC,BE,MAX,DET1)
C
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      COMMON /AAA/ L
      COMMON /BBB/ PI
      REAL L
      EXTERNAL DETERM
      DATA EPS/1.E-7/,NSIG/7/
      BET1=BETA-BINC
      BET2=BETA
      MAX=100
      CALL ZBRENT(DETERM,EPS,NSIG,BET1,BET2,MAX,IER)
      BE=BET2
      CALL POLE(BET2,DET1,BE,BETA,BINC,DETERM)
      RETURN
      END
      SUBROUTINE PLOT(X,Y,Y1,II,IXMIN,IXMAX,IYMIN,IYMAX)
C
      REAL X(II),Y(II,1),RANGE(4),IYMIN,IYMAX,IXMIN,IXMAX,Y1(II)
      CHARACTER ITITLE*36,IXLABL*20,IYLABL*7,ICHAR*1
C
      DO 5 I=1,II
      Y(I,1)=0.0
5     CONTINUE
      DO 10 I=1,II
10    Y(I,1)=Y1(I)
      ICHAR='X'
      ITITLE='PLOT OF BETA**2 VS NORMALIZED LENGTH'
      IXLABL='NORMALIZED LENGTH'
      IYLABL='BETA**2'
      RANGE(1)=IXMIN
      RANGE(2)=IXMAX
      RANGE(3)=IYMIN
      RANGE(4)=IYMAX
C
      CALL USPLO(X,Y,II,II,1,1,ITITLE,36,IXLABL,20,IYLABL,7,
     #           RANGE,ICHAR,1,IER)
      RETURN
      END
```