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# A Sheaf-Theoretic Characterization of Commutative Hereditary Rings

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## ABSTRACT

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This thesis studies a characterization of commutative hereditary and semihereditary rings given by G. Bergman in [1]. Following R. Pierce in [12], a ring R is associated with a sheaf of rings over the topological space obtained from its ring of idempotents and studied in this context. The characterizations so obtained are consequently sheaf-theoretic in nature. It is shown that the commutative hereditary rings are the p.p. rings in which all the stalks are Dedekind domains, non-zero divisors are "almost" units, and the Boolean ring of idempotents of R is hereditary.

The ring associated with the one-point compactification of a discrete space is hereditary. The ring of continuous functions from a space whose associated ring is hereditary to a field with the discrete topology is hereditary.

An oft cited but seldom presented result of I. Kaplansky concerning the direct sum decomposition of projective modules over commutative semihereditary rings is proved in the appendix.

## DEDICATION

To Constance Tucker

## ACKNOWLEDGEMENTS

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# CONTENTS

# LIST OF SYMBOLS

| | |
|---|---|
| $1_R$ | the unit of R or the identity map on R |
| Rad(R) | Jacobson radical of R (p.3) |
| rad(R) | prime radical of R (p.4) |
| $A^*$ | annihilator of A (p.4) |
| $\mathfrak{X}_R$, Spec(R) | set of prime ideals of R (p.5) |
| $\Gamma(A)$ | the elements of $\mathfrak{X}_R$ not containing A (p.5) |
| B(R) | central idempotents of R (p.7) |
| $X_R$ | Spec(B(R)) (p.8) |
| $\bar{P}$ | RP, where $P \in X_R$ (p.9) |
| $R_P$ | the quotient ring $R/\bar{P}$ (p.13) |
| $\mathcal{R}$ | $\bigcup_{P \in X_R} R_P$ (p.10) |
| $\hat{r}$ | (p.10) |
| $r_P$ | image of r in $R_P$ (p.13) |
| $(X_R, \mathcal{R})$ | ringed space of R (p.14) |
| $C(X_R, \mathcal{R})$ | ring of sections of $\mathcal{R}$ (p.15) |
| S(r) | support of r (p.18) |
| $Ker(\varphi)$ | kernel of $\varphi$ (p.21) |
| $Im(\varphi)$ | image of $\varphi$ (p.21) |
| $RS^{-1}$ | ring of fractions of R with respect to S (p.33) |
| $(a)_r^*$ | right annihilator of a (p.37) |
| [R:A] | residual of A by R (p.43) |
| Hom(A,B) | the R-module homomorphisms from A to B (p.43) |
| $\mathbb{N}$ | the natural numbers |

| | | |
|---|---|---|
| 2 | the integers modulo 2 | (p.64) |
| $\mathcal{C}$ | the Cantor numbers | (p.64) |
| $\mathbf{Z}$ | ring of integers | |

# INTRODUCTION

A commutative ring R in which all ideals (respectively all finitely generated ideals) are projective as R-modules is called hereditary (respectively semihereditary). The hereditary (respectively semihereditary) integral domains are exactly the Dedekind domains (respectively the Prüfer domains).

This paper presents a sheaf theoretic characterization of commutative hereditary and semihereditary rings developed by George Bergman in [1]. The characterization depends ultimately on properties exhibited by the principal ideals of R and the Boolean ring of idempotents of R and the ring properties of the stalks associated with R.

Using [9] and [12] as models, we present two topologies connected with any ring R and demonstrate how they are used to associate a sheaf of rings to R. With the tools made available by this representation of R we retrace the path taken in [1]. Liberties have been taken in expanding proofs and changing the "flavour" of some (the different flavour coming from the use of the Pierce stalk of [12] rather than the direct limit of factor-rings in [1]).

Except when otherwise specified, R will represent a commutative ring with a unit different from zero.

# CHAPTER 1

# R AS A RINGED SPACE

## 1. *The space of prime ideals of a regular ring.*

Recall that a ring R, not necessarily commutative, is (*von Neumann*) *regular* if for every $a \in R$ there is some $b \in R$ such that $a = aba$. If R is regular then every $a \in R$ has a *quasi-inverse*: an element c such that $a = aca$, $c = cac$ (choose $c = bab$ where b is as above). If $a = aba$ then $(ab)^2 = abab = ab$ and $(ba)^2 = baba = ba$ so ab and ba are idempotents. If R commutes then quasi-inverses are unique.

Two immediate examples of regular rings are fields (inverses are quasi-inverses) and Boolean rings (each element its own quasi-inverse).

1.1 *Proposition* ([9], 2.2.3). If R is regular then

(1) Every non-unit of R is a zero-divisor

(2) Every prime ideal of R is maximal

(3) Every principal ideal of R is a direct summand.

*Proof.* (1) For $a \in R$, $a = a^2b \Rightarrow a(1 - ab) = 0$ for some $b \in R$. If a is not a unit then $1 - ab$ cannot be zero so a must be a zero-divisor.

(2) Let $P \subset R$ be a prime ideal. R/P is an integral domain so has no zero-divisors. Homomorphic images of regular rings are regular so every non-zero element of R/P

is a unit; hence $R/P$ is a field and $P$ must be maximal.

(3) For $a = a^2b$, $aR = eR$ where $e = ab$ is idempotent and $R = eR \oplus (1 - e)R$. □

1.2 *Lemma* ([9], 2.1.1). $M \subset R$ is a maximal ideal if and only if for every $r \notin M$, $1 - rx \in M$ for some $x \in R$.

*Proof.* ($\Rightarrow$) If $r \notin M$ then $M$ maximal implies that $M + rR = R$ so $(m + rx = 1) \Rightarrow (1 - rx \in M)$.

($\Leftarrow$) Let $a \notin M$ and consider $M + aR$. $(1 - ax) \in M$ for some $x$ so $1 - ax = m$ and $1 = m + ax$ is in $M + aR$. Therefore, $M + aR = R$ for any $a \notin M$ and $M$ is maximal. □

The intersection of all maximal ideals of $R$ is called the *radical* (or *Jacobson radical*) of $R$ and is denoted $\mathrm{Rad}(R)$.

1.3 *Lemma* ([9], 2.1.7). $\mathrm{Rad}(R) = \{r \in R \mid 1 - rx \text{ is a unit } \forall\, x \in R\}$.

*Proof.* If $r$ is in every maximal ideal then $1 - rx$ is not in any maximal ideal for any $x$; hence $1 - rx$ is a unit for all $x$ and $\mathrm{Rad}(R) \subseteq \{r \in R \mid 1 - rx \text{ is a unit } \forall\, x \in R\}$.

If $b \in \{r \in R \mid 1 - rx \text{ is a unit } \forall\, x \in R\}$ then $1 - bx$ is a unit for all $x$. Let $M$ be any maximal ideal and suppose $bx \notin M$. There is some $y$ such that $1 - (bx)y \in M$ so $1 - b(xy)$ is not a unit which is a contradiction unless $bx \in M\ \forall\, x \in R$ and we have $\{r \in R \mid 1 - rx \text{ is a unit } \forall\, x \in R\} \subseteq \mathrm{Rad}(R)$. □

1.4 *Proposition* ([9], 2.2.4). If $R$ is regular then $\mathrm{Rad}(R) = 0$.

*Proof.* If $a \in R$ then $a = a^2 b$ for some $b$ and $a(1 - ab) = 0$. If $a \neq 0$ then $1 - ab$ is a zero-divisor so by (1.3) $a \notin \mathrm{Rad}(R)$.$_\square$

The intersection of all prime ideals is called the *prime radical* and denoted $\mathrm{rad}(R)$. If $\mathrm{rad}(R) = 0$ then $R$ is *semiprime*. Since $\mathrm{rad}(R) \leq \mathrm{Rad}(R)$, regular rings are semiprime.

If $A \leq R$ is any subset then $A^{\vee} = \{r \in R | ra = 0 \ \forall\, a \in R\}$ is an ideal of $R$—the *annihilator* of $A$ in $R$. $J$ is an called an *annihilator ideal* if $J = A^*$ for some $A \leq R$.

1.5 *Lemma.* If $R$ is semiprime and $A$ an annihilator ideal of $R$ then $A = eR$ for some idempotent $e \in R$ if and only if $A + A^* = R$.

*Proof.* Suppose $A = eR$ for some idempotent $e$. It is easily checked that $(eR)^* = (1 - e)R$ and $R = eR \oplus (1 - e)R = A + A^*$.

Conversely, suppose $R = A + A^*$. When $R$ is semiprime, $A \cap A^* = 0$. To see this suppose $x$ is in the intersection; then $x \cdot x = x^2 = 0$ is nilpotent and lies in $\mathrm{rad}(R)$ ([9], 2.1.8) which is zero; thus $x$ is zero and $R = A \oplus A^*$.

We can write $1 = a + b$ for $a \in A$ and $b \in A^*$. Then $a = a(a + b) = a^2 + ab = a^2$ so $a$ is idempotent as is $b = 1 - a$.

Since $aR \leq A$ and $(1 - a)R \leq A^*$ and $R = aR \oplus (1 - a)R$ it follows that $A = aR$. $\square$

It can be shown ([9], 2.4.2) that in a semiprime ring the set of annihilator ideals forms a (complete) Boolean algebra when the infimum of two elements is taken to be their intersection and the complement of A is taken to be $A^*$.

Let $\mathfrak{X}_R$ be the collection of all prime ideals of R, often called the *spectrum* of R and denoted Spec(R). It is well known (see [9] for instance) that $\mathfrak{X}_R$ is a topology, often called the Stone-Zariski topology, when the open sets of $\mathfrak{X}_R$ are defined as those of the form

$$\Gamma(A) = \{P \in \mathfrak{X}_R | A \text{ is not contained in } P\}$$

where A is any subset of R.

$\Gamma(A) = \Gamma(\bigcap_{A \subset J} J)$ where J runs over all ideals of R containing A so it is enough to talk about ideals of R when specifying open sets of $\mathfrak{X}_R$. If $\{A_i\}^{i \in I}$ is any family of ideals then $\bigcup_I \Gamma(A_i) = \Gamma(\sum_I A_i)$ (where $\sum A_i$ is the ideal generated by the union of the $A_i$) and $\Gamma(A) \cap \Gamma(B) = \Gamma(AB)$ are open. Also, $\Gamma(0) = \emptyset$ and $\Gamma(R) = \mathfrak{X}_R$ are open. For any ideal A, $\Gamma(A) = \bigcup_{a \in A} \Gamma(a)$ so the sets $\{\Gamma(a)\}^{a \in R}$ form a basis for $\mathfrak{X}_R$.

1.6 *Proposition* ([9], 2.5.1). $\mathfrak{X}_R$ is a compact topological space.

*Proof*. Let $\bigcup_{i \in I} \Gamma(A_i)$ be an open cover of $\mathfrak{X}_R$, where the $A_i$'s are ideals of R. $\mathfrak{X}_R \subseteq \bigcup_I \Gamma(A_i) = \Gamma(\sum_I A_i) = \{P \in \mathfrak{X} | \sum_I A_i$ is not contained in P$\}$. Since the set of maximal ideals is contained in the set of prime ideals we know that $\sum A_i$ is in no maximal ideal of R. Therefore, $1 \in \sum_{f \in F} A_f$ for some finite $F \subset I$ and $R = \sum_F A_f$. It follows that $\bigcup_F [\Gamma(A_f)] = \Gamma(\sum_F A_f) = \Gamma(R) = \mathfrak{X}_R$ so $\bigcup_I \Gamma(A_i)$ contains a finite subcover.□

A set that is simultaneously open and closed will be called *clopen*.

1.7 *Proposition* ([3], p32). If R is regular then $\mathfrak{X}_R$ is an Hausdorff space with a basis of clopen sets.

*Proof* [3]. First notice that if e is an idempotent then $e(1 - e) = 0$ is in every $P \in \mathfrak{X}_R$. Therefore, $e \in P$ or $(1 - e) \in P$ by primality but never both (if both were in P then $e + (1 - e) = 1$ would be). Consequently, $\Gamma(e)$ and $\Gamma(1 - e)$ are disjoint open sets.

Let P and Q be distinct elements of $\mathfrak{X}_R$. Since both are maximal, P is not contained in Q and we can find an element $a \in P$, $a \notin Q$. Let b be such that $a = a^2b$, then $(a^2b \notin Q) \Rightarrow (e = ab \notin Q)$ by primality of Q. P contains e since $a \in P$ so $(1 - e) \notin P$. It follows that $Q \in \Gamma(e)$ and $P \in \Gamma(1 - e)$ are in disjoint open sets so $\mathfrak{X}_R$ is Hausdorff.

For the second part, Let $\mathcal{O}$ be any open set in $\mathfrak{X}_R$ and Q any prime ideal in $\mathcal{O}$. $\mathcal{O} = \Gamma(A)$ for some ideal $A \subset R$ and we can find an element $a = a^2b \in A$ that is not in Q by definition of $\Gamma(A)$. $Q \in \Gamma(a) = \Gamma(aR) = \Gamma(eR) = \Gamma(e)$ where $e = ab$ so $Q \in \Gamma(e) \subseteq \mathcal{O}$. The complement of $\Gamma(e)$ is $X_R - \Gamma(e) = \{P \in X_R | e \in P\} = \{P \in X_R | (1 - e) \notin P\} = \Gamma(1 - e)$ is open. Since the complement of an open set must be closed $\Gamma(e)$ is open and closed. Therefore, every element of $\mathcal{O}$ is contained in a clopen set in $\mathcal{O}$ and so $\mathcal{O}$ is the union of clopen sets.$_\square$

A topological space with a basis of clopen sets is called a *totally disconnected* space.

We summarize these facts in

1.8 *Theorem.* If R is a regular ring then $X_R$ is a compact, Hausdorff, totally disconnected topology.

Any space with these properties will be called a *Boolean space*.

## 2. *The sheaf of rings associated with R.*

[12] was the main reference in what follows.

Recall that a *Boolean ring* is a ring in which every element is an idempotent. It is immediate from the definition that Boolean rings are commutative and of characteristic two: $ef = fe$ and $e + e = 0$ for all ring elements e and f.

B(R) will denote the *Boolean algebra* of central idempotents of the ring R (see, for instance, [9]). B(R) is a Boolean ring contained in R sharing the same multiplication as R but with the addition: $e + f = e + f - 2ef$ where the operations on the right take place in R (so B(R) is not usually a subring of R).

For e,f in B(R), $eR \subseteq fR \Rightarrow e = ef$ since $(1 - f)$ annihilates eR. We also have that the ideal in R generated by e and f is $eR + fR = (e + f - ef)R$. To see this take $x \in (e + f - ef)R$; then $x = er + fr - efr = er + f(fr - er) \in eR + fR$ so $(e + f - ef)R \subseteq eR + fR$. In the other direction if $x = er_1 + fr_2$ then $er_1 = (e + f - ef)er_1$ and $fr_2 = (e + f - ef)fr_2$ are both elements of $(e + f - ef)R$ giving the opposite containment. In particular, if e and f are orthogonal, so that $ef = 0$, we have $eR + fR = (e + f)R$.

$X_R$ will be the collection of all prime ideals of B(R), Spec(B(R)). Since B(R) is a Boolean ring, hence regular, $X_R$ has all the properties described earlier for $\mathfrak{X}_R$.

1.9 *Proposition* $\mathbb{V} \subseteq X_R$ is clopen if and only if $\mathbb{V} = \Gamma(e)$ for some $e \in B(R)$.

*Proof*. Let $\mathbb{V}$ be clopen. By definition, $\mathbb{V} = \Gamma(A)$ for some ideal $A \subseteq B(R)$. $\mathbb{V}^c$, the complement of $\mathbb{V}$ in $X_R$, must also be open so $\mathbb{V}^c = \Gamma(J)$ for some ideal $J \subseteq B(R)$. $\Gamma(A + J) = \{P \in X_R | A + J \text{ is not contained in } P\}$. Suppose $A + J \subseteq P$ for some $P \in X_R$; then $J \subseteq P \Rightarrow P \notin \Gamma(J)$ so $P \in \Gamma(A)$, the complement of $\Gamma(J)$. But then A is not contained in P and for some $a \in A$, $a + 0 \notin P$ so $A + J$ is not contained in P which is a contradiction. It must be that $A + J$ is not in any $P \in X_R$ so $\Gamma(A + J) = X_R$ and $A + J = B(R)$.

To see that $A + J$ is direct let $x \in A \cap J$. $\Gamma(x) \subseteq \Gamma(A) \cap \Gamma(J) = \emptyset$ so $\Gamma(x) = \emptyset$ and $x$ must be zero. Therefore, $B(R) = A \oplus J$.

We claim that $J = A^*$, the annihilator of $A$. Clearly $J \subseteq A^*$. Suppose $x \in A^*$ but $x \notin J$. We know that $x = a + b$ for some $a \in A$, $b \in J$. For any $y \in A$: $0 = xy = (a + b)y = ay + by = ay = 0$ (since $J \subseteq A^*$) so $ay = 0$ for all $y \in A$. Specifically, $a^2 = 0$ so $a = 0$ since $B(R)$ is Boolean and $x = 0 + b \in J$, establishing the claim.

Therefore, if $V$ is clopen then $V = \Gamma(A)$ for an annihilator ideal $A$ and $B(R) = A \oplus A^*$. In any semiprime (hence regular) ring $R$ we have seen that this can only happen if $A = eR$ for some idempotent $e$ (lemma 1.5). Therefore, $\Gamma(A) = \Gamma[eB(R)] = \Gamma(e)$.

Conversely, if $V = \Gamma(e)$ for some $e \in B(R)$ then $V^c = \Gamma(1 - e)$ which is open.□

Therefore, the clopen sets in $X_R$ are precisely the basic open sets $\Gamma(e)$, $e \in B(R)$.

As mentioned in §1, the annihilator ideals of a semiprime ring, such as $B(R)$, form a Boolean algebra. In such rings $\Gamma$ is an isomorphism between the direct summands of $B(R)$, which are precisely the annihilator ideals $A$ such that $A + A^* = B(R)$, and the clopen sets of $X_R$ ([9], §2.5). In particular, $\Gamma[eB(R)] = \Gamma[fB(R)] \Leftrightarrow eB(R) = fB(R) \Leftrightarrow e = f$.

1.10 *Lemma* ([12], 1.6). If $R$ is any ring and $P \in X_R$ then $RP = \bar{P} = \{re \mid r \in R, e \in P\}$ is an ideal of $R$.

*Proof.* This is obvious except for closure under addition. Let re and sf be in $\bar{P}$; then $re + sf = (re + sf)(e + f - ef)$ and $e + f - ef \in P$. □

For a ring R and $P \in X_R$, $R_P$ will denote the quotient ring $R/\bar{P}$. Let $\mathcal{R}$ be the disjoint union $\mathcal{R} = \bigcup_{P \in X_R} R_P$ and define $\pi: \mathcal{R} \longrightarrow X_R$ by $\pi(r + \bar{P}) = P$. For $r \in R$, let $\hat{r}: X_R \longrightarrow \mathcal{R}$ be given by $\hat{r}(P) = r + \bar{P}$ and $\hat{r}[\Gamma(e)] = \{r + \bar{P} | P \in \Gamma(e)\}$ where $e \in B(R)$. Finally, let $\mathbb{B} = \{\hat{r}[\Gamma(e)] | r \in R, e \in B(R)\}$ and call the sets which are elements of $\mathbb{B}$, and all arbitrary unions of these sets, the open sets of $\mathcal{R}$.

1.11 *Theorem.* $\mathcal{R}$ is a topological space with basis $\mathbb{B}$.

*Proof.* $\mathcal{R}$ will be a topology with basis $\mathbb{B}$ if each $x \in \mathcal{R}$ is in at least one member of $\mathbb{B}$ and if x is in $B_\alpha \cap B_\beta$ then there exists $B_\gamma \subseteq B_\alpha \cap B_\beta$ such that $x \in B_\gamma$ where $B_\alpha$, $B_\beta$, $B_\gamma$ are in $\mathbb{B}$ ([8]).

Let $x = r + \bar{P}$ be any element of $\mathcal{R}$. Since $\Gamma(1) = X_R$ we always have $x \in \hat{r}[\Gamma(1)]$ so condition one is satisfied. (Also, since $\Gamma(0) = \emptyset$ we have that $\emptyset$ is open in $\mathcal{R}$.)

For the second condition we need

1.12 *Sublemma* ([12], 4.3). If $\hat{r}(P) = \hat{s}(P)$ for some $P \in X_R$ then there exists $e \in B(R)$ such that $P \in \Gamma(e)$ and $\hat{r}(M) = \hat{s}(M)$ for every $M \in \Gamma(e)$.

*Proof.* $\hat{r}(P) = \hat{s}(P) \Leftrightarrow r + \bar{P} = s + \bar{P} \Leftrightarrow r - s \in \bar{P} \Leftrightarrow r - s = af$ for some $f \in P$. Set $e = 1 - f$ so $e \notin P$ and whenever $e \notin P$

we have $f \in P$, then $r - s \in \bar{M}$ for all $M \in \Gamma(e)$ (and $P \in \Gamma(e)$). This gives us that $\hat{r}(M) = \hat{s}(M)$ for every $M \in \Gamma(e)$.$_\square$

To continue with the proof of (1.11), let $x \in \hat{r}[\Gamma(e)] \cap \hat{s}[\Gamma(f)]$ so that $x = \hat{r}(P) = \hat{s}(P)$ for some P in $\Gamma(e) \cap \Gamma(f)$. By the preceeding, there is a $g \in B(R)$ such that $\Gamma(g)$ contains P and $\hat{r}(M) = \hat{s}(M)$ for every $M \in \Gamma(g)$. Therefore, $x = \hat{r}(P) = \hat{s}(P) \in \hat{r}[\Gamma(gef)] \subseteq \hat{r}[\Gamma(e)] \cap \hat{s}[\Gamma(f)]$ since P is in $\Gamma(g) \cap \Gamma(e) \cap \Gamma(f) = \Gamma(gef)$. This is the second statement.$_\square$

Using [12] (definition 3.1a) verbatim we state the following definition of a sheaf.

1.13 *Definition.* Let X be a topological space. Suppose that for each $x \in X$, a ring $R_x$ with zero $0_x$ and identity $1_x$ is given. Assume that $R_x \cap R_y = \emptyset$ for $x \neq y$. Let $\mathcal{R} = \bigcup_{x \in X} R_x$. Denote by $\pi$ the mapping of $\mathcal{R}$ to X defined by $\pi(r) = x$ if $r \in R_x$. Assume that a topology is imposed on $\mathcal{R}$ such that the following axioms are satisfied.

(1) If $r \in \mathcal{R}$, there exists open sets $\mathbb{U}$ in $\mathcal{R}$ with $r \in \mathbb{U}$ and $N \subseteq X$ such that $\pi$ maps $\mathbb{U}$ homeomorphically onto N.

(2) Let $\mathcal{R} + \mathcal{R}$ denote $\{(r,s) | \pi(r) = \pi(s)\}$, with the topology induced by the product topology in $\mathcal{R} \times \mathcal{R}$. Then the mapping $r \longrightarrow -r$ is continuous on $\mathcal{R}$ to $\mathcal{R}$ and the mappings $(r,s) \longrightarrow r + s$ and $(r,s) \longrightarrow rs$ are continuous on $\mathcal{R} + \mathcal{R}$ to $\mathcal{R}$.

(3) The mapping $x \longrightarrow 1_x$ is continuous on X to $\mathcal{R}$.

With these conditions $\mathcal{R}$ is called a *sheaf of rings over* X. The rings $R_x$ are called the *stalks* of the sheaf $\mathcal{R}$.

We will show that $\mathcal{R} = \bigcup_{P \in X_R} R_P$ is a sheaf and develop some of its properties.

1.14 *Proposition.* The mappings $\pi \colon \mathcal{R} \longrightarrow X_R$ and $\hat{r} \colon X_R \longrightarrow \mathcal{R}$ are continuous functions.

*Proof.* First note that for $P \in X_R$, $(\pi \circ \hat{r})(P) = \pi[\hat{r}(P)] = \pi(r + \bar{P}) = P$ so $\pi \circ \hat{r} = 1_{X_R}$, the identity map on $X_R$ for any $r \in R$. Also, $(\hat{r} \circ \pi)(r + \bar{P}) = \hat{r}[\pi(r + \bar{P})] = \hat{r}(P) = r + \bar{P}$ so $\hat{r} \circ \pi = 1_{\mathcal{R}}$, the identity map on $\mathcal{R}$ for any $r \in R$.

Let $V = \Gamma(e)$ be any basic open set in $X_R$. $\pi^{-1}(V) = \{r + \bar{P} \mid \pi(r + \bar{P}) \in \Gamma(e)\} = \{r + \bar{P} \mid P \in \Gamma(e)\} = \bigcup_{r \in R} \{\hat{r}[\Gamma(e)]\}$ is a union of basic open sets in $\mathcal{R}$, hence is open. Since any open set in $X_R$ is a union of sets of the type $\Gamma(e)$ it follows that inverse images of open sets are open so $\pi$ is continuous.

Consider a basic open set $\hat{s}[\Gamma(e)]$ in $\mathcal{R}$ with $P \in \Gamma(e)$. $\hat{r}^{-1}[\hat{s}(P)] = (\hat{r}^{-1} \circ \hat{s})(P) = (\pi \circ \hat{s})(P) = P$ (since $\pi \circ \hat{r} = 1_{X_R} \Rightarrow \pi = \hat{r}^{-1}$). Therefore, $\hat{r}^{-1}\{\hat{s}[\Gamma(e)]\} = \Gamma(e)$ which is open in $X_R$ and it follows that inverse images of any open sets are open so $\hat{r} \colon X_R \longrightarrow \mathcal{R}$ is continuous for each $r \in R$. $\square$

In particular, $\hat{1}(P) = 1 + \bar{P}$ is continuous on $X_R$ to $\mathcal{R}$ which is condition three of a sheaf.

For the first condition take $P \in \Gamma(e)$ and $r + \bar{P}$ in $\mathcal{R}$. Let $U = \hat{r}[\Gamma(e)]$. Then $\pi \circ \hat{r}(P) = P \ \forall\, P \in \Gamma(e)$ so $\pi[\hat{r}(\Gamma(e))] = \Gamma(e)$ and $\pi$ and $\hat{r}$ are continuous inverses of each other. So, for any basic open set $\mathcal{U}$ in $\mathcal{R}$, $\pi$ maps $\mathcal{U}$ homeomorphically onto the basic open set $\Gamma(e)$ in $X_R$.

The second condition remains and will be accomplished in steps. We introduce the notation $r_P$ for the image of r in $R_P = R/\bar{P}$; $r_P = r + \bar{P} = \hat{r}(P)$ and this will later supplant the $\hat{r}$ notation.

1.15 *Lemma*. The map $\varphi$: $\mathcal{R} \longrightarrow \mathcal{R}$ given by $r_P \longmapsto -r_P$ is continuous.

*Proof*. Let $\mathcal{U} = \hat{s}[\Gamma(e)]$, a basic open set in $\mathcal{R}$. $\varphi^{-1}(\mathcal{U}) = \{r_P \in \mathcal{R} | -r_P \in \mathcal{U}\} = \{r + \bar{P} | (-r) + \bar{P} = s + \bar{P}\}$ for some $P \in \Gamma(e)$. If $\varphi^{-1}(\mathcal{U}) \neq \emptyset$ (which is open) then $(-\hat{r})(P) = \hat{s}(P)$ for some $P \in \Gamma(e)$. It follows that there is some $f \in B(R)$ such that P is in $\Gamma(f)$ and $(-\hat{r})(Q) = \hat{s}(Q)\ \forall\ Q \in \Gamma(f)$ by (1.12). Either way $\varphi^{-1}(\mathcal{U})$ is open and $\varphi$ is continuous.□

$\mathcal{R} + \mathcal{R} = \{(r_P, s_P) | r,\ s \in R,\ P \in X_R\}$. Endow $\mathcal{R} \times \mathcal{R}$ with the product topology induced by the topology on $\mathcal{R}$. A basic open set in $\mathcal{R} \times \mathcal{R}$ will then be of the form $(\hat{r}[\Gamma(e)] \times \mathcal{R}) \cap (\mathcal{R} \times \hat{s}[\Gamma(f)]) = (\hat{r}[\Gamma(e)] \times \hat{s}[\Gamma(f)])$ where r, s are in R and e, f are in B(R). $\mathcal{R} + \mathcal{R}$ is topologized by taking as basic open those sets in $\mathcal{R} + \mathcal{R}$ that are intersections of basic open sets in $\mathcal{R} \times \mathcal{R}$ with $\mathcal{R} + \mathcal{R}$. Then $\mathcal{O}$ basic open in $\mathcal{R} + \mathcal{R}$ means $\mathcal{O} = \{(t_P, v_P) | t_P \in \hat{r}[\Gamma(e)],\ v_P \in \hat{s}[\Gamma(f)]\}$ for some t, v in R; e, f in B(R).

1.16 *Lemma*. The map $+ : (\mathcal{R} + \mathcal{R}) \longrightarrow \mathcal{R}$ given by $+(r_P, s_P) = r_P + s_P$ is continuous.

*Proof.* Let $\mathbf{V} = \hat{u}[\Gamma(e)]$ be a basic open set in $\mathcal{R}$. We want that $(+)^{-1}(\mathbf{V})$ is open in $\mathcal{R}$.

Consider some fixed pair $(r,s)$ in R such that $r_P + s_P = u_P$ for some $P \in \Gamma(e)$ so $(r \hat{+} s)(P) = \hat{u}(P)$ and $+(r_P,s_P) \in \mathbf{V}$. This equation must hold over a basic open set $\Gamma(f)$ containing P. Specifically, $(r\hat{+}s)(P) = \hat{u}(P)$ over $\Gamma(e) \cap \Gamma(f) = \Gamma(ef)$. This implies that $\{(r_P,s_P) | P \in \Gamma(ef)\} = \{\hat{r}[\Gamma(ef)] \times \hat{s}[\Gamma(ef)]\} \cap (\mathcal{R} + \mathcal{R})$ is contained in $(+)^{-1}(\mathbf{V})$ and is a basic open set in $\mathcal{R} + \mathcal{R}$. $(+)^{-1}(\mathbf{V})$ will consist of the union of all such sets as we run through pairs $(a,b)$ such that $a + b + \bar{P} = u + \bar{P}$ for some $P \in \Gamma(e)$. It follows that $(+)^{-1}(\mathbf{V})$ is open and that addition is a continuous operation on $\mathcal{R} + \mathcal{R}$ to $\mathcal{R}$.$_\square$

That the function $\times$: $\mathcal{R} + \mathcal{R} \longrightarrow \mathcal{R}$ given by $\times(r_P,s_P) = r_P s_P$ is continuous is shown by the same procedure so multiplication is also a continuous operation. With this we have the second condition and conclude that $\mathcal{R} = \bigcup_{P \in X_R} R_P$ is a sheaf of rings. Taken together, $(X_R,\mathcal{R})$ is called a *ringed space.*

We now turn to some of the consequences of $(X_R,\mathcal{R})$ being a ringed space.

1.17 *Definition.* A continuous map $\sigma$: $X_R \longrightarrow \mathcal{R}$ is called a *section* (of $\mathcal{R}$ over $X_R$) if $\pi[\sigma(P)] = P \; \forall \; P \in X_R$.

For example, we have seen that for any $r \in R$, $\hat{r}$: $X_R \longrightarrow \mathcal{R}$ is continuous. $\pi[\hat{r}(P)] = \pi(r+\bar{P}) = P \; \forall \; P \in X_R$ so

$\hat{r}$ is a section for every $r \in R$. Of paramount importance will be showing that *every* section of $\mathcal{R}$ over $X_R$ is of this form.

1.18 *Proposition.* Let $C(X_R,\mathcal{R}) = \{\hat{r} \mid r \in R\}$ and define $(r\hat{+}s)(P) = \hat{r}(P) + \hat{s}(P)$, $(r\hat{\cdot}s)(P) = \hat{r}(P)\cdot\hat{s}(P)$ for all $P \in X_R$, then $C(X_R,\mathcal{R})$ is a ring.

*Proof.* $C(X_R,\mathcal{R})$ has $\hat{1}$ and $\hat{0}$ as unit and zero respectively. It is straightforwaid to check that the map $P \longmapsto [\hat{r}(P),\hat{s}(P)]$ is continuous for any $r$, $s \in R$. By the second condition on sheaves then, $P \longmapsto [\hat{r}(P),\hat{s}(P)] \longmapsto \hat{r}(P) + \hat{s}(P)$ is a composition of continuous functions hence continuous so $(r\hat{+}s)(P) = \hat{r}(P) + \hat{s}(P)$ is continuous. Similarly, $(r\hat{\cdot}s) = \hat{r}\hat{s}$ is continuous.

Finally, $\pi[(r\hat{+}s)(P)] = P = \pi[r\hat{\cdot}s(P)]$ so if $\hat{r}$ and $\hat{s}$ are sections then so are $\hat{r} + \hat{s}$ and $\hat{r}\cdot\hat{s}$. Upon this it will follow that $C(X_R,\mathcal{R})$ is a ring.$_\square$

1.19 *Proposition.* If $f$ and $g$ are any sections of $\mathcal{R}$ that agree at $P_0 \in X_R$ then they agree on an open set containing $P_0$.

*Proof.* Suppose $f(P_0) = g(P_0)$. $f(P_0)$ is contained in some basic open set $\mathbb{U} \subseteq \mathcal{R}$ and we have seen that $\pi$ maps $\mathbb{U}$ homeomorphically onto a basic open set $\mathbb{U}' \subseteq X_R$. Set $\mathbb{V} = [f^{-1}(\mathbb{U})] \cap [g^{-1}(\mathbb{U})] \cap \mathbb{U}'$ which is an open set since $f$ and $g$ are continuous.

$P_0$ is in $\mathbb{V} \subseteq \mathbb{U}'$. If $P \in \mathbb{V}$ we have, by construction of

$\mathbf{V}$, that $f(P)$ and $g(P)$ are in $\mathbf{U}$. $\pi[f(P)] = \pi[g(P)] = P$ since f and g are sections and on any basic open set $\pi$ is an injection so it must be that $f(P) = g(P)$ for all $P \in \mathbf{V}$.□

Before we proceed to the main result of this section we establish the important *partition property* of $X_R$ in

1.20 *Theorem.* If $\{O_i\}^{i\in I}$ is any open covering of $X_R$ then there exists a finite collection of clopen sets $\{\Gamma(e_j)\}_{j=1}^{n}$ such that each $\Gamma(e_j) \subseteq O_i$ for some i and $\Gamma(e_j) \cap \Gamma(e_k) = \emptyset$ if $i \neq k$ and $X_R = \bigcup_{i=n}^{n} \Gamma(e_i)$.

*Proof.* Since $X_R$ is compact $\{O_i\}^I$ has a finite subcovering, say $X_R = \bigcup_{i=1}^{r} O_i$. Each of these is in turn a union of clopen sets: $X_R = \bigcup_{i=1}^{r} [\bigcup \Gamma(e_{O_i})]$ where $e_{O_i}$ is in B(R) and $\bigcup \Gamma(e_{O_i}) = O_i$. Again, by compactness, this contains a finite subcover $X_R = \bigcup_{i=1}^{k} \Gamma(e_i)$. Where each $\Gamma(e_j) \subseteq O_i$ for some i. Now, since this is a *finite* collection of clopen sets we can turn it into a disjoint union by a finite number of set operations involving finite intersections and complements each of which produce clopen sets, establishing the desired partition.

For instance, suppose $X_R = \Gamma(e_1) \cup \Gamma(e_2)$. We can write $X_R = [\Gamma(1 - e_1e_2) \cap \Gamma(e_1)] \cup \Gamma(e_1e_2) \cup [\Gamma(1 - e_1e_2) \cap \Gamma(e_2)]$ $= \Gamma(e_1 - e_1e_2) \cup \Gamma(e_1e_2) \cup \Gamma(e_2 - e_1e_2)$ and this is a disjoint union of clopen sets each one of which is contained in one of the original $O_i$.□

We apply the partition property at once to show that every section over $X_R$ is represented by a ring element.

1.21 *Theorem.* If $\sigma$ is any section of $\mathcal{R}$ over $X_R$ then $\sigma = \hat{r}$ for some $r \in R$.

*Proof.* Recall that if $e \in B(R)$ and $P \in X_R$ then either $e \in P$ or $(1 - e) \in P$ but never both. If $e \in P$ then $\hat{e}(P) = e + \bar{P} = \bar{P} = 0_P$. If $(1 - e) \in P$ then $\hat{e}(P) = e + \bar{P} = e + (1 - e) + \bar{P} = 1 + \bar{P} = 1_P$. So always $\hat{e}(P) = e_P = 1$ or $0$ for any $e \in X_R$.

Let $\sigma$ be any section and $P \in X_R$. We must have $\sigma(P) = \hat{r}(P)$ for some $r \in R$ so $\sigma$ and $\hat{r}$ must agree on a clopen set containing $P$, say $\Gamma(e)$, so $\sigma(M) = \hat{r}(M)\ \forall\ M \in \Gamma(e)$. Sets of this type cover $X_R$ and we can choose $\{e_i\}_{i=1}^{n}$, $e_i \in B(R)$, and $\{r_i\}_{i=1}^{n} \subset R$ such that $X_R = \bigcup_{i=1}^{n}\Gamma(e_i)$, $\Gamma(e_i) \cap \Gamma(e_j) = \emptyset$ if $i \neq j$ and $\sigma(P) = \hat{r}_i(P)\ \forall\ P \in \Gamma(e_i)$.

Set $s = \sum_{i=1}^{n} e_i r_i$ and take any $P \in X_R$. $P$ is in *exactly* one of the $\Gamma(e_i)$, say $\Gamma(e_j)$. For all $P \in \Gamma(e_j)$ we have $\hat{r}_j(P) = \sigma(P)$ and $\hat{e}_j(P) = 1$ since $e_j \notin P$. Therefore, $(e_j\hat{\ }r_j)(P) = \hat{e}_j(P)\hat{r}_j(P) = 1_P \cdot \sigma(P) = \sigma(P)$ on $\Gamma(e_j)$. Over $\Gamma(e_i)$, $i \neq j$, $\hat{e}_i(P) = 0$ and $\hat{r}_i(P) = \sigma(P)$ so $(e_i\hat{\ }r_i)(P) = \hat{e}_i(P)\hat{r}_i(P) = 0_P$. This gives $\hat{s}(P) = 0_P + \cdots + 0_P + \hat{r}_j(P) + 0_P + \cdots + 0_P = \sigma(P)$. □

With this result it is immediate that the map $\hat{\ }: R \longrightarrow C(X_R, \mathcal{R})$ given by $\hat{\ }(r) = \hat{r}$ is a ring epimorphism. To see that $\hat{\ }$ is an injection just suppose that $\hat{r}$ is the zero map: $[\hat{r}(P) = 0\ \forall\ P \in X_R] \Leftrightarrow [r + \bar{P} = \bar{P}\ \forall\ P \in X_R] \Rightarrow r = ae$ for some $a \in R$ and $e \in \bigcap_{P \in X_R} P$. But $B(R)$ is a regular ring and hence semiprime so $\bigcap_{P \in X_R} P = (0)$ and it must be that r = ae = 0. Therefore $[\hat{}(r) = \hat{r} = \hat{0}] \Rightarrow r = 0$ and ^ is an injection. The upshot is that R and $C(X_R, \mathcal{R})$ are isomorphic as rings and there is a *one-to-one* identification between the sections of $X_R$ over $\mathcal{R}$ and the elements of R.

This fact together with the partition property enjoyed by $X_R$ will allow study of R to take place in the context of the topologies on $X_R$ and $\mathcal{R}$ and the "local" behaviour of continuous functions (ring elements) can be "pieced together" to give results about R.

For any commutative ring R we will say that the equation a = b *holds* at $P \in X_R$ if $a_P = b_P$ (i.e. if $\hat{a}(P) = \hat{b}(P)$).

The *support* of $r \in R$, denoted S(r), will be the subset of $X_R$ over which $\hat{r}$ is non-zero, i.e. $S(r) = \{P \in X_R | r_P \neq 0_F\} = \{P \in X_R | r \notin \bar{P}\}$. In the case where $e \in B(R)$ then it is clear that S(e) is precisely $\Gamma(e)$.

1.22 *Proposition.* For a and b in R and e in B(R), $a_P = b_P$ for every P in $\Gamma(e)$ if and only if ae = be in R.

*Proof.* ($\Rightarrow$) $a_P = b_P \Leftrightarrow a + \bar{P} = b + \bar{P} \Leftrightarrow a - b \in \bar{P} \Leftrightarrow a - b = rf$ for some $f \in P$. Therefore, $[a_P = b_P \ \forall\, P \in \Gamma(e)] \Leftrightarrow [f \in P \ \forall\, P \in \Gamma(e)] \Leftrightarrow [\Gamma(e) \cap \Gamma(f) = \emptyset] \Leftrightarrow ef = 0$. So $a - b = rf \Rightarrow (a - b)e = rfe = 0$ and we have that ae = be.

($\Leftarrow$) If ae = be for $e \in B(R)$ and $P \in \Gamma(e)$ then $(ae)_P = a_P e_P = a_P 1_P = a_P$. Similarly, $(be)_P = b_P$. □

1.23 *Corollary.* If $a_P = b_P \ \forall \ P \in X_R$ then $a = b$ in R.

*Proof.* $a_P = b_P$ holds on $X_R = \Gamma(R) = \Gamma(1)$ so $a_P = b_P \Rightarrow a \cdot 1 = b \cdot 1$.□

1.24 *Lemma.* If I is an ideal of R and $I_P$ its image in $R_P$ and $I_P = R_P \ \forall \ P \in \Gamma(e)$ then $Ie = eR$.

*Proof.* $\Gamma(e)$, being a clopen subset of a Boolean space, is itself a Boolean space and inherits the partition property.

At each $P \in \Gamma(e)$ there is an $i^{(P)} \in I$ such that $i_P^{(P)} = 1_P$ and this must be true over a clopen set $\Gamma(f^{(P)}) \subset \Gamma(e)$ and so $i^{(P)}f^{(P)} = f^{(P)}$. The sets $\{\Gamma(f^{(P)})\}_{P \in X_R}$ are an open cover of $\Gamma(e)$ and by the partition property we can write $\Gamma(e) = \bigcup_{j=1}^{n} \Gamma(f_j)$ where $\Gamma(f_i) \cap \Gamma(f_j) = \Gamma(f_i f_j) = \varnothing$ and $\sum_{j=1}^{n} f_j = e$. Over each $\Gamma(f_i)$ there is an $i^{(P_i)} \in I$ such that $i^{(P_i)} f_i = f_i$. Piecing these together we have $\sum_{j=1}^{n} i^{(P_j)} f_j = \sum_{j=1}^{n} f_j = e$. It follows that $e \in I$ so $eR \subset eI$. Since $eI \subset eR$ we have $Ie = eR$.□

Before leaving this section, we add that an almost identical development can be given for an R-*module* over $X_R$ with essentially the same results (see [12], §§1-4). If A is an R-module then the stalk at $P \in X_R$ is $A_P = A/\bar{P}A$ where $\bar{P}$ is as already described; $X_R$ and $\mathcal{S} = \bigcup_{P \in X_R} A_P$ are topologized in a manner completely analogous to the preceeding. In particular, an equation holding at a point in $X_R$ must hold on an open set containing that point.

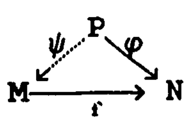
## CHAPTER 2

## PROJECTIVE MODULES AND BOOLEAN RINGS

### 1. *Projective modules.*

We will recall some results about R-modules. Since the rings we are working over are all commutative the distinction between left and right R-modules is unimportant. When dealing with an R-module, homomorphism will mean *R-module homomorphism*. First, the well known

2.1 *Definition.* The R-module P is *projective* if whenever there is a surjective homomorphism $f: M \longrightarrow N$ for any R-modules M and N and $\varphi: P \longrightarrow N$ is any homomorphism then there exists $\psi: P \longrightarrow M$ making the diagram commutative:




i.e. $\varphi = f \circ \psi$.

Loosely, an R-module P is projective if P is a direct summand of every R-module that maps onto it.

A sequence of R-modules

$\cdots \longrightarrow L \xrightarrow{\varphi_{n-1}} M \xrightarrow{\varphi_n} P \xrightarrow{\varphi_{n+1}} \cdots$ is *exact* if $\mathrm{Im}(\varphi_{n-1}) = \mathrm{Ker}(\varphi_n)$. An exact sequence of the form

$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$ (*) is called *short exact*. Clearly, (*) is exact if and only if g is surjective and f is injective.

The exact sequence $M \xrightarrow{g} P \longrightarrow 0$ is *split exact*, or *splits*, if there exists a homomorphism $g': P \longrightarrow M$ satisfying $g \circ g' = 1_P$. The exact sequence $0 \longrightarrow L \xrightarrow{f} M$

splits if there is $f'\colon M \longrightarrow L$ satisfying $f'\circ f = 1_L$. If the sequence $M \longrightarrow P \longrightarrow 0$ above splits then $M \cong \mathrm{Im}(g') \oplus \mathrm{Ker}(g) \cong P \oplus \mathrm{Ker}(g)$.

The following equivalences about the (left) R-module P are well known and presented without proof (see, for example, [6] or [10]).

2.2 *Theorem.* If L, M, P are any R-modules then the following are equivalent

(1) P is projective.

(2) P is a direct summand of a free module.

(3) Every short exact sequence $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$ is such that $\mathrm{Im}(f) = \mathrm{Ker}(g)$ is a direct summand of M (equivalently, every short exact sequence $M \longrightarrow P \longrightarrow 0$ splits).

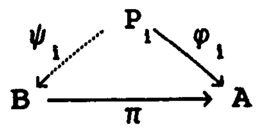
2.3 *Proposition* ([9], 4.3). If $\{P_i\}_{i\in I}$ is a family of R-modules then $P = \bigoplus_I P_i$ is projective if and only if each $P_i$ is projective.

*Proof.* ($\Rightarrow$) Let $\varphi\colon P \longrightarrow A$ be any homomorphism and $\pi\colon B \longrightarrow A$ an epimorphism. For each $i \in I$ let $K_i\colon P_i \longrightarrow P$ be the canonical injection, $\rho_i\colon P \longrightarrow P_i$ the canonical projection of P onto $P_i$ ($\rho_i \circ K_i = 1_{P_i}$) and $\varphi_i \in \mathrm{Hom}(P_i, A)$.

For each $i \in I$, $\varphi_i' = \varphi_i \circ \rho_i$ is a homomorphism of $P \longrightarrow A$. P is projective so for each i there is a $\psi^{(i)}\colon P \longrightarrow B$ such that $\pi\circ\psi^{(i)} = \varphi_i'$. Set $\psi_i = \psi^{(i)}\circ K_i$ so $\psi_i\colon P_i \longrightarrow B$. We have then $\pi\circ\psi_i = \pi\circ\psi^{(i)}\circ K_i = \varphi_i'\circ K_i =$

$\varphi_i \circ \rho_i \circ K_i = \varphi_i \circ 1_{P_i} = \varphi_i$

i.e.




is commutative for each i.

($\Leftarrow$) Suppose each $P_i$ is projective. Let $\varphi: P \longrightarrow A$ be any homomorphism and $\pi: B \longrightarrow A$ an epimorphism. Since $P_i$ projects, there exists $\psi_i: P_i \longrightarrow B$ satisfying $\pi \circ \psi_i = \varphi \circ K_i$, $K_i$ as above.

Now, $\{\psi_i\}_I$ is a family of homomorphisms $P_i \longrightarrow B$ and P is the direct sum of the $P_i$'s so there exists a unique homomorphism $\psi: P \longrightarrow B$ such that $\psi \circ K_i = \psi_i$ for each $i \in I$ ([9], 4.1). This implies that $\pi \circ (\psi \circ K_i) = \pi \circ \psi_i = \varphi \circ K_i$ and $(\pi \circ \psi) \circ K_i = \varphi \circ K_i$ implies $\pi \circ \psi = \varphi$ since $K_i$ is injective. This establishes the projectivity of P.$_\square$

We focus now on Boolean rings, develop some of their properties and examine when an ideal in a Boolean ring is projective. B will denote a Boolean ring in what remains of this chapter.

2.4 *Lemma*. Finitely generated ideals in B are principal.

*Proof*. It suffices to consider the case of an ideal generated by two elements.

For e and f in B, $er + fs = (e + f + ef)er + (e + f + ef)fs$ so $eB + fB \subseteq (e + f + ef)B$. Clearly, $(e + f + ef)B \subseteq eB + fB$ so $eB + fB = (e + f + ef)B$.$_\square$

[Lemma (2.4) is true for any *regular* ring R (using one-sided ideals if R does not commute): $xR + yR = (e + f)R$ where e and f are orthogonal idempotents (above we have that $(e + f)$ and $ef$ are orthogonal).]

2.5 *Lemma*. If $U \subseteq X_B$ is an open set then the collection J of elements of B whose support lie in U is an ideal.

*Proof*. Recall that the support of $e \in B$ is $\Gamma(e) = \{P \in X_B | \hat{e}(P) \neq 0\} = \{P \in X_B | e \notin P\}$. $\Gamma(0) = \emptyset \in U$ so $0 \in J$ and J is non-empty. Let e, f be elements of J. $\Gamma(e)$ and $\Gamma(f)$ contained in U implies that $\Gamma(e) \cup \Gamma(f) = \Gamma(e + f + ef) \subseteq U$. To see that $\Gamma(e + f) \subseteq \Gamma(e + f + ef)$ consider $Q \in \Gamma(e + f)$, so $e + f \notin Q$. If $e + f + ef \in Q$ then $(e + f)(e + f + ef) = e + f$ is in Q which it is not. Therefore, $e + f + ef \notin Q$ so $Q \in \Gamma(e + f + ef)$ and we have $\Gamma(e + f) \subseteq \Gamma(e + f + ef) \subseteq U$ so $e + f$ is in J.

$\Gamma(e) = \Gamma(-e)$ and $\Gamma(e) = \Gamma(eB)$ so it follows that J is an ideal of B.□

To each open set U in $X_B$ we can, in consequence, associate an ideal of B. In the other direction, to any ideal J in B we associate the open set $U = \Gamma(J) = \bigcup_{j \in J} \Gamma(j)$, the union of the supports of the elements of J. If $A \subseteq B$ generates J then $U = \Gamma(J) = \Gamma(\sum_{a \in A} aB) = \bigcup_A \Gamma(a)$. On the other hand, if J is the ideal associated with $U = \bigcup_A \Gamma(a)$ then $[x \in J] \Rightarrow [\Gamma(x) \subseteq \Gamma(a)$ for some $a] \Rightarrow x \in aB$. Therefore, $A \subseteq B$ generates J if and only if the union of the supports of the elements of A is U.

2.6 *Proposition.* Let $\{B_i\}^I$ be a family of ideals in B, then $\sum_I B_i$ is a direct sum

(1) if and only if $B_iB_j = 0 \ \forall \ i \neq j$ in I

(2) if and only if the associated open sets in $X_B$ are disjoint.

*Proof.* (1) $\sum_I B_i$ is direct if $B_i \cap \sum_{i \neq j} B_j = (0)$. Consider $B_iB_j$ for any $i \neq j$: $B_iB_j \subseteq B_i \cap B_j \subseteq B_i \cap \sum_{i \neq j} B_j = 0$.

Conversely, assume $B_iB_j = 0$ for $i \neq j$ and take $x \in B_i \cap \sum_{i \neq j} B_j$. Then $x = b_i \in B_i$ and $x = \sum_{i \neq j} b_j$ for $b_j \in B_j$. Therefore, $[b_i = \sum_{i \neq j} b_j] \Rightarrow b_i^2 = b_i = \sum_{i \neq j} b_ib_j = \sum 0 = 0$ since $B_iB_j = 0$ for $i \neq j$.

(2) Suppose $\sum B_i$ is direct and consider the ideals $B_i$, $B_j$ and their associated open sets $\mathfrak{U}_i$, $\mathfrak{U}_j$ in $X_B$.

Let $P \in \mathfrak{U}_i \cap \mathfrak{U}_j$. $B_i$ is not contained in P so there is a $b_i \in B_i$ with $b_i \notin P$. Similarly, there exists $b_j \in B_j$ with $b_j \notin P$ so $P \in \Gamma(b_i) \cap \Gamma(b_j) = \Gamma(b_ib_j) = \Gamma(0) = \emptyset$ ($B_iB_j = 0$ by (1)) so $\mathfrak{U}_i \cap \mathfrak{U}_j = \emptyset$.

Conversely, assume $\mathfrak{U}_i \cap \mathfrak{U}_j = \emptyset \ \forall \ i \neq j$ and consider $b_i \in B_i$, $b_j \in B_j$. $\Gamma(b_i) \cap \Gamma(b_j) = \emptyset = \Gamma(b_ib_j)$ so $b_ib_j$ is in every $P \in X_B$. Therefore, $[b_ib_j \in \bigcap_{P \in X_B} P = (0)] \Rightarrow [b_ib_j = 0] \Rightarrow [B_iB_j = 0$ for $i \neq j] \Rightarrow \sum B_i$ is direct.$\square$

2.7 *Corollary.* For any $e \in B$, $B = eB \oplus (1-e)B$.

*Proof.* If $b \in B$ then $b = eb + (1 - e)b$ so $B = eB + (1 - e)B$ and clearly $(eB)\cdot((1 - e)B) = 0$.$\square$

## 2. *Projective ideals in Boolean Rings.*

Recall that a ring R is right *hereditary* (respectively *semihereditary*, *$\alpha$-hereditary*) if every right ideal (resp. every finitely generated right ideal, every right ideal generated by $\leq \alpha$ elements) is projective as a right R-module, where $\alpha$ is any cardinal. (Since our rings are commutative, and ideals are two-sided, the qualifier right (or left) ideal is irrelevant).

By (2.7) every principal ideal of a Boolean ring B is a direct summand of the free B-module B so every principal ideal is projective. Since finitely generated ideals in B are principal, Boolean rings are semihereditary.

2.8 *Proposition.* If $I \subseteq B$ is an ideal and $\mathbb{U} \subseteq X_B$ its associated open set then I has an orthogonal system of generators if and only if $\mathbb{U}$ is a disjoint union of clopen sets.

*Proof.* ($\Rightarrow$) Let $E = \{e_j\}^{j \in J}$ be an orthogonal generating set for I. $\mathbb{U} = \bigcup_J \Gamma(e_j)$ and $\Gamma(e_i) \cap \Gamma(e_j) = \Gamma(e_i e_j) = \Gamma(0) = \emptyset$ if $i \neq j$.

($\Leftarrow$) Let $S(I) = \mathbb{U} = \dot{\bigcup_J} \Gamma(e_j)$ where $\Gamma(e_j) \cap \Gamma(e_k) = \emptyset$ so $e_j e_k = 0$ if $j \neq k$ and set $I' = \bigoplus_J e_j B$ which also has $\mathbb{U}$ for support. If $x \in I$ then $S(x) = \Gamma(x) \subseteq \mathbb{U}$ so $\Gamma(x) \subseteq \bigcup_{J'} \Gamma(e_j) = \Gamma(\sum_{J'} e_j)$ for a subset $J' \subseteq J$. $\Gamma(x) = \Gamma(x) \cap \Gamma(\sum_{J'} e_j) =$

$\Gamma(x\sum_{J'} e_j)$ so $x = x\sum_{J'} e_j$ and $xB \subseteq (\sum_{J'} e_j)\cdot B = \sum_{J'} e_j B$ so $x \in I'$. Similarly, $I' \subseteq I$ and the two coincide.$_\square$

2.9 *Proposition*. If $I \subseteq B$ is an ideal then $I$ has an orthogonal set of generators if and only if $I$ is projective.

Before proceeding we state without proof (see appendix) the following

2.10 *Lemma* ([7], theorem 3). If R is a commutative semi-hereditary ring and P a projective R-module then P is a direct sum of modules each of which is isomorphic to a finitely generated ideal in R.

*Proof* (of 2.9). ($\Rightarrow$) By hypothesis $I = \sum_{j\in J} e_j B$ where $e_i e_j = 0$ for all $i \neq j$ in $J$ so $I$ is the direct sum of the $e_j B$'s. Therefore, $I$ is the direct sum of projective ideals.

($\Leftarrow$) If $I \subseteq B$ is projective then, since $B$ is semihereditary, (2.10) gives that $I$ is a direct sum of finitely generated ideals. Finitely generated ideals in Boolean rings (in fact in all regular rings) are principal so $I$ is a direct sum of principal ideals. Pairwise products of ideals in such a sum must be zero so it follows that the generators of $I$ are orthogonal.$_\square$

To summarize we state

2.11 *Theorem* ([1], lemma 1.1). For the ideal $I \subseteq B$ and $U \subseteq X_B$ its associated open set, the following are equivalent:

(1) I is projective.

(2) I has an orthogonal family of generators.

(3) U is a disjoint union of clopen sets.

2.12 *Corollary.* Every countably generated ideal of B is projective.

*Proof* ([1]). Let I be generated by $A = \{a_1, a_2, a_3, \ldots\} \subseteq B$. Define A′ as follows: $a_1' = a_1$, $a_2' = a_2 + a_1a_2$, $a_3' = a_3 + a_1a_3 + a_2a_3 + a_1a_2a_3$, $a_4' = a_4 + a_1a_4 + a_2a_4 + a_3a_4 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4 + a_1a_2a_3a_4$, and so on. Direct calculation will confirm that $a_i'a_j' = 0$ for every $i \neq j$ (each term in the product will be repeated an even number of times and B has characteristic two) so A′ is orthogonal.

If I′ is the ideal genertated by A′ then clearly $I' \subseteq I$ since each $a_i'$ is a finite sum of elements of I.

Direct calculation reveals: $a_1 = a_1'$, $a_2 = a_2' + a_2a_1'$, $a_3 = a_3' + a_3(a_1' + a_2')$, ... , $a_n = a_n' + a_n(a_1' + \cdots + a_{n-1}')$, .... Therefore, each $a_i$ is a finite sum of elements in I′ so $I \subseteq I'$ and $I = I'$ has an orthogonal generating set.□

Thus, every ideal in a Boolean ring generated by $\leq \aleph_0$ elements is projective so Boolean rings may be called $\aleph_0$*-hereditary.* In particular, if B is a countable Boolean ring then B is hereditary.

The remainder of this section is devoted to demonstrating that if $B \subseteq B'$ are Boolean rings then ideals in B that generate (i.e. extend to) projective ideals in B′ are themselves projective.

To this end, let $I \subseteq B$ be such that $IB'$ is projective. By (2.11), $IB'$ must have an orthogonal generating set, $S \subseteq B'$, so $IB' = \oplus_{s \in S} sB'$. Following Bergman, call $I_0 \subseteq I$ an *S-I-ideal* if $I_0$ is an ideal in $I$ such that $I_0B'$ is generated as an ideal of $B'$ by a subset of $S$, say $S_0$ (for example, (0) and $I$ are S-I-ideals). Let $I\backslash I_0$ denote the annihilator of $I_0$ in $I$. We will show that if $I_0$ is an S-I-ideal (with $I_0B' = S_0B'$, $S_0 \subseteq S$) then so is $I\backslash I_0$, with $(I\backslash I_0)B'$ generated by $S\backslash S_0$ and $I = I_0 \oplus (I\backslash I_0)$.

We have $IB' = \oplus_S sB' = (\oplus_{S\backslash S_0} sB') \oplus (\oplus_{S_0} sB')$. Let $\rho$ be the canonical projection of $IB'$ onto the direct summand $S_0B' = I_0B'$.

2.14 *Lemma*. With notation as above, if $I_0$ is an S-I-ideal then $\rho(I) \subseteq I_0$.

*Proof*. The proof will show that $\rho$ applied to $x \in I$ can be accomplished by multiplying $x$ by some $a \in I_0$, so $\rho(x) = ax \in I_0$.

Take $x \in I$. In $B'$, $x$ can be written as a finite sum involving $S$. For notational convenience, suppose $x = s_1b_1 + \cdots + s_nb_n$. If none of the $s_i$'s are in $S_0$ then $\rho(x) = 0 = 0x \in I_0$. Suppose some of the $s_i$'s are in $S_0$. For convenience suppose that $s_1,\ldots,s_k$ are in $S_0$ with $k \leq n$ and write $x = s_1b_1 +\cdots+ s_kb_k + s'_{k+1}b_{k+1} +\cdots+ s_n'b_n$ where the $'$ indicates $s_j' \in S\backslash S_0$. Each of the $s_i$, being in $I_0B'$, can be written as a finite sum of the form $\sum i_jb_j'$, $i_j \in I_0$. Collect all of the $i_j$'s for each of the $s_1,\ldots,s_k$ and let

$A \subseteq I_0$ be the ideal generated by them. Being finitely generated, $A = aB$ for some $a \in I_0$ and in $B'$ the ideal $AB' = aB' \subset S_0B'$ and contains $\{s_1, \ldots, s_k\}$.

For some $c_i \in B'$ and $s_i \in S_0$, $a = c_1s_1 + \cdots + c_ks_k + c_{k+1}s_{k+1} + \cdots + c_{k+m}s_{k+m}$ in terms of the direct sum $S_0B'$ where $c_j \neq 0 \; \forall \; j \leq k$. Multiplying a successively by the orthogonal $s_i$'s gives $as_i = c_is_i$ for each i so $a = as_1 + \cdots + as_k + \cdots + as_{k+m}$. Each of these $s_i$ are in $aB'$ so $s_i = ab' \Rightarrow as_i = ab'$ so $as_i = s_i$. Therefore, $a = \sum_{i=1}^{k+m} s_i$ and

$ax = (s_1 + \cdots + s_{k+m})(s_1b_1 + \cdots + s_kb_k + s'_{k+1}b_{k+1} + \cdots + s'_nb_n)$

$= s_1b_1 + \cdots + s_kb_k$ by the orthogonality of the $s_i$'s. Therefore, multiplication by a annihilates the component of x "outside" $S_0$ and leaves the rest unchanged, precisely the action of $\rho$; hence $\rho(x) = ax \in I_o$.□

2.15 *Lemma*. With notation as above, $I = I_0 \oplus I\backslash I_0$.

*Proof*. For $x \in I$ write x as it appears in $IB' = SB'$, say $x = s_1b_1' + \cdots + s_nb_n'$. By successive mulitiplications by the $s_i$'s we obtain $x = x(s_1 + \cdots + s_n)$. For notational convenience suppose $s_1, \ldots, s_k$ are in $S_0$. Write $x = x(s_1 + \cdots + s_k) + x(s_{k+1} + \cdots + s_n)$. From the earlier discussion we know we can find $a \in I_0$ such that $ax = x(s_1 + \cdots + s_k)$ so $x = xa + x(s_{k+1} + \cdots + s_n)$ and $xa \in I_0$; hence $x(s_{k+1} + \cdots + s_n) = x - xa \in I$. Clearly $x(s_{k+1} + \cdots + s_n)a = 0 \;\; \forall \; a \in I_0$ (just write a as it appears in $I_0B' = S_0B'$). So $x(s_{k+1} + \cdots + s_n) \in I\backslash I_0$ and $I = I_0 + (I\backslash I_0)$ and since $(I_0)(I\backslash I_0) = 0$ the sum is direct.□

Now, if $A \oplus C$ is a direct sum in $B$ we have $(A \oplus C)B' = AB' + CB'$ and since $AC = 0$, $(AB')(CB') = 0$ so $(A \oplus C)B' = AB' \oplus CB'$ is direct. It follows that $IB' = I_0B' \oplus (I\backslash I_0)B' = S_0B' \oplus (S\backslash S_0)B'$ and $IB' = SB' = S_0B' \oplus (S\backslash S_0)B'$. So, up to isomorphism, $(I\backslash I_0)B' = (S\backslash S_0)B'$.

We conclude that if $I_0 \leq I$ is an S-I-ideal then so is $I\backslash I_0$; $I = I_0 \oplus I\backslash I_0$ and $(I\backslash I_0)B' = (S\backslash S_0)B'$.

2.16 *Lemma*. If $I_0$ is a non-zero S-I-ideal ($I_0B' = S_0B'$, $S_0 \leq S$) then $I_0$ contains a non-zero countably generated S-I-ideal.

*Proof* ([1]). Let $J^{(0)} = aB$ where a is a non-zero element of $I_0$ so $J^{(0)} \leq I_0$ is finitely generated. Pick $K^{(0)} \leq S_0$ such that $J^{(0)}B' \leq K^{(0)}B'$ and $K^{(0)}$ is finite (possible since $a \in S_0B'$ is a finite sum involving $S_0$). From the earlier discussion we can find $a_2 \in I_0$ such that $(a_2)B' \supsetneq K^{(0)}$.

Set $J^{(1)} = (a,a_2)B \leq I_0$ and pick $K^{(1)} \leq S_0$ finite such that $K^{(0)} \leq K^{(1)}$ (so $K^{(0)}B' \leq K^{(1)}B'$) and $J^{(1)}B' \leq K^{(1)}B'$. Continue this process and let $J = \bigcup_{i=0}^{\infty} J^{(i)} = (a,a_2,a_3,\ldots)B$, and $K = \bigcup_{i=0}^{\infty} K^{(i)} \leq S_0$. By this construction, $JB' = KB'$ is a countably generated S-I-ideal contained in $I_0$. □

Let $\mathbb{T}$ be the collection of all families of pairwise disjoint, countably generated S-I-ideals. $\{(0)\}$ is such a family so $\mathbb{T} \neq \emptyset$. $\mathbb{T}$ is partially ordered by set inclusion and if $\{T_i\}$ is a chain in $\mathbb{T}$ then $T = \bigcup T_i$ is an upper bound of $\{T_i\}$ in $\mathbb{T}$. Zorn's Lemma then guarantees maximal elements in $\mathbb{T}$.

Let T be maximal in $\mathbb{T}$ and set $I_0 = \bigoplus_{J \in T} J$, clearly an S-I-ideal. Suppose $I_0 \subsetneq I$; then $I \backslash I_0$ is a non-zero S-I-ideal and contains a countably generated S-I-ideal disjoint form each $J \in T$, contradicting the maximality of T. Therefore, $I = I_0$ is a sum of countably generated ideals so, by (2.12), a sum of projective ideals. To summarize and expand, we have

2.17 *Theorem* ([1], 1.2). If $B \leq B'$ are Boolean rings and $I \leq B$ an ideal that generates a projective ideal $IB'$ in $B'$, then I is projective. Any subring of an hereditary ($\alpha$-hereditary) Boolean ring is hereditary ($\alpha$-hereditary). Any quotient of an $\alpha$-hereditary Boolean ring is $\alpha$-hereditary.

*Proof*. The first statement has been demonstrated. The second is immediate since any ideal (resp. $\alpha$-generated ideal) in $B \subset B'$ generates a projective ideal in $B'$ when $B'$ is hereditary (resp. $\alpha$-hereditary).

For the third statement, let B be $\alpha$-hereditary and $I \leq B$ an ideal and $\bar{B} = B/I$. Let $\bar{J} = \bar{S}\bar{B}$ be the ideal generated by $\bar{S}$ where $\bar{S}$ has $\leq \alpha$ elements and $J = SB$ its preimage in B (so S is a preimage of $\bar{S}$). J is projective so we can take S to be an orthogonal set so $\bar{S}$ is also orthogonal. Then, by (2.11), $\bar{B}$ is $\alpha$-hereditary.$_\square$

## CHAPTER 3

## HEREDITARY AND SEMIHEREDITARY RINGS

### 1. *The ring of fractions of $R/\bar{P}$.*

In what remains, the ring of fractions of the commutative ring R with respect to its set of non-zero divisors—the *total ring of fractions* of R—will be of some importance. In particular, we will be interested in the total ring of fractions of $R_p = R/\bar{P}$ (P will always denote a prime ideal of B(R)) and will therefore formally present these rings.

Let S be the multiplicative set of non-zero divisors of R, then $K = RS^{-1}$ will denote the ring of fractions of R with respect to S. $KP \subseteq K$ will be the ideal $KP = \{ke \mid k \in K,\ e \in P\}$ where $P \in X_R$. That this is an ideal is the same proof as lemma 1.10. We will show that if S is the set of all non-zero divisors of R then the ring of fractions of $R_p$ with respect to $(S)_p$ is $R_p(S)_p^{-1} = K/KP$.

Recall ([11]) that K is the ring of fractions of R with respect to the multiplicative set S if and only if there exists a ring homomorphism $f:R \longrightarrow K$ such that:

(1) f(s) is a unit in K for all $s \in S$

(2) if $g:R \longrightarrow M$ is a ring homomorphism such that g(s) is a unit in M $\forall\ s \in S$ then there exists a unique homomorphism $h:K \longrightarrow M$ such that $g = h \circ f$.

Also recall that if $f:R \longrightarrow R'$ is a ring homomorphism and $I \subseteq R$ and $J \subseteq R'$ are ideals satisfying $f(I) \subseteq J$ then $f$ induces the ring homomorphism $f': R/I \longrightarrow R'/J$ defined by $f'(r + I) = f(r) + J$.

3.1 *Proposition.* If $\bar{P} = RP$ is a prime ideal of R then KP is a prime ideal of K.

*Proof.* Note that if $s \in S$ then $s \notin \bar{P}$ for any $P \in X_R$ since if $s = te$ for some $e \in B(R)$ then $s(1 - e) = 0$ which contradicts that s is a non-zero divisor.

Let $(r/s) \in KP$, so $(r/s) = (u/t)e$ where s,t are in S and $e \in P$. Then $rt = (su)e$ is in $\bar{P}$. But t cannot be in $\bar{P}$ so $r \in \bar{P}$ by primality. It follows that $(r/s) \in KP \Leftrightarrow r \in \bar{P}$, or, by contraposition, $(r/s) \notin KP \Leftrightarrow r \notin \bar{P}$.

Suppose $(r/s)(u/t) \in KP$ but $(u/t) \notin KP$. $(ru)/(st) = (x/y)f$ for some $y \in S$ and $f \in P$ so $ru = st(x/y)f$ is in KP. Now, $ru/1 \in KP \Leftrightarrow ru \in \bar{P}$ so $r \in \bar{P}$ since $u \notin \bar{P}$. Therefore, r/s is in KP and KP must be prime.□

As an aside, it should be noted that the full hypothesis is necessary—the primality of $P \in X_R$ does not guarantee primality of $\bar{P}$ in general. To see this, let R = $\{(a,b) \mid a \equiv b \bmod(c),\ c \geq 2\}$ where a,b,c are integers. B(R) = $\{(0,0), (1,1)\}$ and has only one proper prime ideal, the zero ideal. So, if P is the zero ideal of B(R) then $\bar{P}$ is the zero ideal of R which is not prime and in $K = RS^{-1} = \mathbb{Q} \times \mathbb{Q}$, KP is not prime. Notice that R is not p.p. (to be defined presently) since, for instance, the ideal generated by (c,0) in R is not projective. (This example, in another context, is found in [1].)

3.2 *Proposition.* If $\bar{P}$ is a prime ideal then $S_p$, the image of S in $R_p$, is a multiplicative set of non-zero divisors.

*Proof.* For any $s \in S$, $s \notin \bar{P}$ for any P so $s_p \neq 0_p$ and $0 \notin S_p$. If $s_p r_p = 0$ then $sr \in \bar{P}$ gives $r \in \bar{P}$ by primality of $\bar{P}$; hence $r_p = 0$ and $s_p$ is a non-zero divisor.

Clearly, $S_p$ is closed under multiplication.$_\square$

Define $f:R \longrightarrow K$ by $f(r) = r/1$ and suppose that $g:R \longrightarrow M$ is any homomorphism such that g(s) is a unit in M $\forall\ s \in S$. Then $h:K \longrightarrow M$ defined by $h(r/s) = g(r)[g(s)]^{-1}$ is the unique homomorphism such that $g = h \circ f$ whose existence is guaranteed by the fact that K is the ring of fractions of R with respect to S.

The homomorphism f induces the homomorphism $f':R/\bar{P} \longrightarrow K/KP$ given by $f'(r_p) = [f(r)]_{KP}$ (where the KP subscript denotes the image of f(r) in K/KP). Let $s_p = s + \bar{P}$ be any element of $S_p$; then $f'(s_p) = (s/1)_{KP}$ and has inverse $(1/s)_{KP}$ in K/KP ($(1/s) \notin KP$ since $1 \notin \bar{P}$). Therefore, f′ satisfies the first condition of a ring of fractions.

Suppose that $g:R/\bar{P} \longrightarrow M$ is a homomorphism such that $g(s_p)$ is a unit in M for every $s_p$ in $S_p$. Define $h:K/KP \longrightarrow M$ by $h[(r/s)_{KP}] = g(r_p)[g(s_p)]^{-1}$. That h is a homomorphism is an easy check. To see that it is well-defined suppose $(r/s)_{KP} = (u/v)_{KP}$. We have

$h[(r/s)_{KP}] - h[(u/v)_{KP}] = h[((rv - us)/(sv))_{KP}] =$ $g[(rv - us)_P]\cdot[g((sv)_P)]^{-1}$. However, $rv - us$ is in $\bar{P}$ so $g[(rv - us)_P] = g(0) = 0$ and it follows that h does not depend on the coset representative (r/s).

A straightforward check reveals $h\circ f' = g$. If h′ is another homomorphism with the same properties as h then $h'\circ f' = g = h\circ f'$. Uniqueness will follow from the fact that when S contains no zero divisors of R then f is an injection ($\mathrm{Ker}(f) = \{r \in R | sr = 0$ for some $s \in S\}$ since $f(r) = r/1 = (r/1)\cdot(s/s)$ for any $s \in S$), forcing f′ to be an injection. Thus the conditions are fulfilled and K/KP is the ring of fractions of $R_P$ with respect to $S_P$. We have shown

3.3 *Proposition.* If R is a commutative ring, S its set of non-zero divisors, $P \in X_R$, and $K = RS^{-1}$ its total ring of fractions, then the ring of fractions of $R/\bar{P} = R_P$ is $K/KP = K_P = R_P(S^{-1})_P$.

2. *p.p. and regular rings.*

3.4 *Definition.* R is (right, left) *p.p.* if every principal (right, left) ideal is projective as a (right, left) R-module.

3.5 *Proposition.* For R not necessarily commutative, the right ideal aR is projective if and only if the kernel of the left multiplication map is a direct summand of R.

(In the following two proofs we are considering R and aR as right R-modules.)

*Proof.* ($\Rightarrow$) Define $\mu : R \longrightarrow aR$ by $\mu(r) = ar$, the left multiplication map. $\mu$ is surjective and has $Ker(\mu) = \{a\}_r^*$, the right annihilator of a. If aR is projective then the sequence $R \xrightarrow{\mu} aR \longrightarrow 0$ is split exact with splitting homomorphism $\rho : aR \longrightarrow R$, $\mu \circ \rho = 1_{aR}$. Now, any $r \in R$ can be written as $r = \rho(\mu(r)) + [r - \rho(\mu(r))]$ and $r - \rho(\mu(r))$ is in $Ker(\mu)$ for all $r \in R$ so $R = Im(\rho) + Ker(\mu)$. If $r \in Im(\rho) \cap Ker(\mu)$ then $r = \rho(x)$ and $0 = \mu(r) = \mu(\rho(x)) = 1(x) = x$ so $r = 0$ and the sum is direct. We have then that $R = Im(\rho) \oplus Ker(\mu)$ and $Ker(\mu)$ is a direct summand of R. Since $Im(\rho) \cong aR$ and $Ker(\mu) = \{a\}_r^*$ we can also write $R \cong aR \oplus \{a\}_r^*$.

($\Leftarrow$) Suppose $R = A \oplus Ker(\mu) = A \oplus \{a\}_r^*$; then $A \cong R/\{a\}_r^* \cong aR$. Therefore, aR is isomorphic to a direct summand of the projective module R.$_\square$

3.6 *Proposition.* For R not necessarily commutative, aR is projective if and only if $\{a\}_r^* = \{e\}_r^*$ where $e^2 = e$ is an idempotent in R.

*Proof.* ($\Rightarrow$) With the notation of the last result, $R = Im(\rho) \oplus \{a\}_r^*$ so $1 = x + y$ for some $x \in Im(\rho)$ and $y \in \{a\}_r^*$. Since $y(x + y) = (x + y)y$, x and y commute. Therefore $1 = 1^2 = x^2 + xy + yx + y^2 = x^2 + y(2x + y) = x + y$ where $x^2 \in Im(\rho)$ and $y(2x + y) \in \{a\}_r^*$. By unique representation in the direct sum we see that $x^2 = x$ is an idempotent in R (similarly, $y^2 = y$). Also, $a = a\cdot 1 = ax + ay$ and, since $y \in (a)_r^*$, we have that $a = ax$.

The claim is that $(a)_r^* = (x)_r^*$. Clearly, $(x)_r^* \subseteq (a)_r^*$ since if $xt = 0$ then $at = axt = 0$. Let $t \in (a)_r^*$ so $at = 0$. We have $xt \in \mathrm{Im}(\rho)$ since $x \in \mathrm{Im}(\rho)$. However, $\mu(xt) = axt = at = 0$ so $xt \in \mathrm{Ker}(\mu)$. Therefore, $xt \in \mathrm{Im}(\rho) \cap \mathrm{Ker}(\mu) = (0)$ so $xt = 0$ and $(a)_r^* \subseteq (x)_r^*$, establishing the claim.

($\Leftarrow$) Suppose $(a)_r^* = (e)_r^*$ where $e^2 = e$. Then $R = eR \oplus (1 - e)R$ and $(1 - e)R = (e)_r^* = (a)_r^*$. Therefore, $aR \simeq R/(a)_r^* = R/(1 - e)R \simeq eR$ so $aR$ is isomorphic to a direct summand of a projective module.$_\square$

3.7 *Proposition*. If R is commutative then the idempotent of (3.6) is unique.

*Proof*. Let $(a)^* = (e)^*$ where $e^2 = e$. Suppose we also have $(f)^* = (a)^*$ where $f^2 = f$. It must be that $f(1 - e) = 0$ and that $e(1 - f) = 0$ and together these say that $e = f$.$_\square$

In the proof of (3.6) is contained the fact that if $(a)^* = (e)^*$ for some idempotent e then $a = ae$. We will call e the idempotent associated with a. A *commutative* p.p. ring R, then, is one in which every $a \in R$ can be uniquely associated with an idempotent e: $a = ae$, $(a)^* = (e)^*$.

3.8 *Lemma*. If R is a regular ring, not necessarily commutative, then R is (right and left) p.p..

*Proof*. If a is any element of R then $a = aba$ for some $b \in$

R. (ab) and (ba) are both idempotents and it is a simple check that $\{a\}_r^* = \{ba\}_r^*$ and $\{a\}_l^* = \{ab\}_l^*$.□

3.9 *Lemma*. If R is p.p. and S is the set of all non-zero divisors of R then $B(R) = B(RS^{-1})$.

*Proof*. $B(R) \leq B(RS^{-1})$ always. For $r \neq 0$ and $s \in S$ suppose that $(r/s)^2 = r/s$ in $RS^{-1}$. Then $sr^2 = s^2r \Rightarrow sr(r - s) = 0$. Since s is a non-zero divisor we must have $r(r - s) = 0$. If r is a non-zero divisor then $r = s$ and $r/s = 1$.

If r is a zero divisor then R p.p. $\Rightarrow r = re$ where $\{r\}^* = \{e\}^* = (1 - e)R$. Then $[r(r - s) = 0] \Rightarrow [r - s \in (1 - e)R] \Rightarrow [r - s = t - te] \Rightarrow re = se$. Therefore $r/s = re/s = se/s = e \in R$, so $B(RS^{-1}) \leq B(R)$.□

3.10 *Theorem* ([1], 3.1). Let R be commutative and S the set of non-zero divisors of R and $RS^{-1}$ the ring of fractions of R with respect to S, then the following are equivalent:

1) R is p.p.

2) $B(R) = B(RS^{-1})$, the support of $a \in R$ is clopen and $R_P$ is an integral domain for every $P \in X_R$.

3) The stalk of $RS^{-1}$ is a field at each $P \in X_R$.

4) $RS^{-1}$ is regular and $B(RS^{-1}) = B(R)$.

*Proof*. (1⇒2) The first statement is lemma 3.9.

Let $a_P$ denote the image of a in $R_P = R/\bar{P}$. $S(a) = \{P \in X_R | a_P \neq 0_P\} = \{P \in X_R \mid a \notin \bar{P}\}$. If $a \in B(R)$ then $S(a) = \Gamma(a)$ which is clopen so we assume that $a \notin B(R)$. In this case we can write $a = ae$ where e is the idempotent associated with a; then $[a_P = ae + \bar{P} \neq 0] \Leftrightarrow [ae \notin \bar{P}] \Leftrightarrow [e \notin P] \Leftrightarrow P \in \Gamma(e)$ which is clopen.

For the third statement, suppose $a_P$ and $b_P$ are non-zero in $R_P$. This means that $a = ae$, $b = bf$ where $e, f \notin P$. $[a_Pb_P = (ab)_P = 0] \Leftrightarrow [abef \in \bar{P}] \Leftrightarrow [ef \in P]$ and $ef \notin P$ by primality of P. It follows that $R_P$ is an integral domain. (We have then, for free, that "R p.p. $\Rightarrow \bar{P}$ is prime.")

($2 \Rightarrow 3$) Since $X_R = X_{(RS^{-1})}$, $RS^{-1}$ is made into a ringed space by setting the stalk at P as $K_P = K/KP$ (and we have that $K_P = R_P(S^{-1})_P$). To show that $K_P$ is a field we must demonstrate that any non-zero $a_P \in R_P$ is a unit in $R_P(S^{-1})_P$, which amounts to showing that there exists $a' \in S$ such that $(a')_P = a_P$.

By hypothesis, $S(a) = \Gamma(e)$ for some $e \in B(R)$. $a_P \neq 0 \Rightarrow a = re$ for some $e \notin P$. Set $a' = a + (1 - e)$. For all $Q \in \Gamma(e)$ we have $(a')_Q = a_Q \neq 0$ (since $1 - e \in Q$ and $a = re$, $e \notin Q$) so $a'$ is a preimage in R of $a_P$. For all $Q \in \Gamma(1 - e)$ we have $(a')_Q = a_Q + (1 - e)_Q = (re)_Q + 1_Q - e_Q = 1_Q$ since $e \in Q$. Therefore $(a')_Q$ is non-zero for all $Q \in X_R$.

Suppose $a'b = 0$ for some $b \neq 0$ in R, then $[(a'b)_Q = (a')_Qb_Q = a_Qb_Q = 0] \Rightarrow b_Q = 0$ for all $Q \in \Gamma(e)$ since $R_Q$ is an integral domain and we have $b \in \bar{Q}$ for all $Q \in \Gamma(e)$. For $Q \in \Gamma(1 - e)$, $(a'b)_Q = 1_Qb_Q = 0$ so $b \in \bar{Q}$ for all $Q \in \Gamma(1 - e)$. Therefore, $b \in \bar{P}$ for all $P \in X_R$ so $b = ce$ for some $c \in R$ and $[e \in \bigcap_{P \in X_R} P = (0)] \Rightarrow b = c0 = 0$ which is a contradiction. It must be that $a'$ is a non-zero divisor, i.e. $a' \in S$. It follows that any non-zero element of $R_P$ has an inverse in

$R_P(S^{-1})_P$ and so the stalk of $RS^{-1}$ at P is a field.

(3⇒4) Let a be any element in $RS^{-1}$. For each P choose $b_P^{(P)} \in R$ such that $b_P^{(P)} = (a_P)^{-1}$ if $a_P \neq 0_P$ and $b_P^{(P)} = 0$ if $a_P = 0$, then $a_P = a_P^2 b_P^{(P)}$ at P. Since an equation holding at a point must hold on a basic open set, $a_P = a_P^2 b_P^{(P)}$ over say, $\Gamma(e_P)$.

As P ranges over $X_R$, the sets $\{\Gamma(e_P)\}$ give an open cover of $X_R$ and by the partition property we can choose a finite cover of sets $\{\Gamma(e_i)\}_{i=1}^{n}$ such that each $\Gamma(e_i) \subseteq \Gamma(e_P)$ for some P and $\Gamma(e_i) \cap \Gamma(e_j) = \emptyset$ if $i \neq j$ and $\bigcup \Gamma(e_i) = X_R$.

Since $X_R = \bigcup_{i=1}^{n} \Gamma(e_i)$ is a disjoint union the $e_i$'s are orthogonal and $\bigcup_{i=1}^{n} \Gamma(e_i) = \bigcup \Gamma(e_i R) = \Gamma[(e_1 + \cdots + e_n)R] = \Gamma(\sum e_i)$. Therefore, $[\sum e_i \notin P \ \forall P \in X_R] \Rightarrow$ $[1 - \sum e_i \in P \ \forall P \in X_R] \Rightarrow 1 - \sum e_i = 0$ so $\sum e_i = 1$.

Over $\Gamma(e_i)$ we have $a_{P_i} = a_{P_i}^2 b_{P_i}^{(P_i)} \ \forall P_i \in \Gamma(e_i)$ so in R we have $ae_i = a^2 b^{(P_i)} e_i$. Let $b = \sum_{i=1}^{n} b^{(P_i)} e_i$, then $aba = \sum_{i=1}^{n} a^2 b^{(P_i)} e_i = \sum_{i=1}^{n} ae_i = a \sum_{i=1}^{n} e_i = a \cdot 1 = a$ and b is a quasi-inverse of a.

For the second part, let $k \in RS^{-1}$ be idempotent; then $[(k_P)^2 = k_P] \Rightarrow k_P(k_P - 1_P) = 0_P$ and since $(RS^{-1})_P$ is a field, $k_P = 1$ or 0 at each $P \in X_R$.

By the partition property we can find a partition $X_R = \bigcup_{i=1}^{n} e^{(P_i)}$ where over $\Gamma(e^{(P_i)})$ we have $k_{P_i} = (\delta_i)_{P_i}$, $\delta_i =$ 0 or 1, and $ke^{(P_i)} = \delta_i e^{(P_i)}$. These piece together to yield

$\sum_{i=1}^{n} ke^{(P_i)} = k\sum_{i=1}^{n} e^{(P_i)} = k = \sum_{i=1}^{n} \delta_i e^{(P_i)} \in B(R)$ since $\delta_i$ is 0 or 1 and $e^{(P_i)} \in B(R)$.

(4⇒1) $RS^{-1}$ is regular, hence p.p., and $B(RS^{-1}) = B(R)$. Let a be any non-unit in R and write $RS^{-1} = aRS^{-1} \oplus \{a\}^*$.

In $RS^{-1}$ we have $1 = as + s'$ for some $s \in RS^{-1}$, $s' \in \{a\}^*$ and $e = as$ is the idempotent of $RS^{-1}$ associated with a. But, since e is in B(R), $\{a\}^* = \{e\}^*$. To see this let $x \in \{a\}^*$, so $xe = xas = 0$ and $\{a\}^* \subseteq \{e\}^*$. If $x \in \{e\}^*$ then $xa = xae = 0a = 0$ so $\{e\}^* \subseteq \{a\}^*$ and the two are equal. Therefore, the annihilator of a in R is equal to the annihilator of an idempotent in R so aR is projective and R is p.p. .□

3.11 *Corollary* ([1], 3.2). For the commutative ring R the following are equivalent:

(1) R is (von Neumann) regular.

(2) R is p.p. and non-zero divisors are units.

(3) The stalk of R at each $P \in X_R$ is a field.

*Proof*. (1⇒2) "Regular ⇒ p.p." is lemma 3.8. .

Let a be a non-zero divisor. Then $a = a^2b$ for some $b \in R$ so $a(1 - ab) = 0$ and since a is a non-zero divisor it must be that $ab = 1$.

(2⇒3) If every non-zero divisor of R is a unit then $RS^{-1} = R$ so $R_P$ is a field by the last theorem.

(3⇒1) $R_P$ is a field at each $P \in X_R$ so for each $a \in R$, $a_P$ has a quasi-inverse (i.e. $a_P^{-1}$) on a clopen set containing P. Following the same strategy as in "3 ⇒ 4" of the last result we can piece together these clopen sets to find a quasi-inverse of a in R. □

## 3. *Commutative semihereditary rings.*

3.12 *Definition* ([10], 6.5). R is a *Prüfer domain* if R is an integral domain in which every non-zero finitely generated ideal of R is invertible.

The aim of this section is to show that the commutative semihereditary rings are precisely the p.p. rings with stalk at each point of $X_R$ a Prüfer domain. We first confirm that the commutative semihereditary *domains* are exactly the Prüfer domains.

Let $K = RS^{-1}$, the total ring of fractions of R, so S is the set of all non-zero divisors of R.

Recall that A is a *fractional ideal* of $R \le K$ if A is an R-module such that $dA \le R$ for some non-zero divisor $d \in R$. If $AB = R$ for some fractional ideal B then A is *invertible* and we write $B = A^{-1}$. If A is invertible then $A^{-1} = [R:A] = \{k \in K | kA \le R\}$. It is easily verified that A is invertible if and only if $1_R \in A[R:A]$ and that A invertible implies that A is finitely generated as an R-module (this follows from $AB = R \Rightarrow 1 = a_1b_1 + \cdots + a_nb_n$ for a finite sum with $a_i \in A$, $b_i \in B$).

3.13 *Lemma* ([9], p86, ex. 1). An R-module P is projective if and only if $\exists \{p_i\}^{i\in I} \subseteq P$ and $\varphi_{p_i} \in \mathrm{Hom}(P,R)$ for each $i \in I$ such that $\varphi_{p_i}(p) = 0$ for all but finitely many $i \in I$ and for any $p \in P$ we have $p = \sum_I \varphi_{p_i}(p)p_i$.

*Proof.* ($\Leftarrow$) Let F be the free R-module on I. Define $\varphi: F \longrightarrow P$ by setting $\varphi(i) = p_i$ and extending linearly so $\varphi$ is an R-epimorphism and $F \xrightarrow{\varphi} P \longrightarrow 0^{(*)}$ is an exact sequence. Define $\psi: P \longrightarrow F$ by $\psi(p) = \sum_I \varphi_{p_i}(p)i$ (this is a finite sum) where $p = \sum_I \varphi_{p_i}(p)p_i$. $\psi$ is an R-homomorphism and $\varphi\circ\psi = 1_P$ so (*) is split exact and $\psi$ is an injection. Therefore, $\mathrm{Im}(\psi) \simeq P$ and $F \simeq P \oplus K$ and we have P as a direct summand of a free module.

($\Rightarrow$) If P is projective there exists a free module F such that $F \simeq P \oplus Q$. Let $J \subseteq F$ be a basis for F and $\varphi: F \longrightarrow P$ a projection of F onto P and $S = \{\varphi(j) \mid j \in J\} = \{p_j\} \subseteq P$.

$F \xrightarrow{\varphi} P \longrightarrow 0$ is split exact since P is projective so there is a map $\psi: P \longrightarrow F$ such that $\varphi\circ\psi = 1_P$. Take $f \in F$ so $f = \sum_J r_j j$, a finite sum, and define $\rho_j: F \longrightarrow R$ by $\rho_j(f) = r_j$ for each $j \in J$ and define $f_j: P \longrightarrow R$ by $f_j(p) = (\rho_j\circ\psi)(p)$ for each $j \in J$. It follows that $f_j(p) = 0$ for all but finitely many $f_j$ and that for any $p \in P$, $p = \sum_J f_j(p)p_j$. □

3.14 *Proposition.* Let R be an integral domain and $A \subseteq R$ a non-zero ideal, then A is invertible if and only if A is projective as an R-module.

*Proof.* ($\Rightarrow$) If $A \subseteq R$ is invertible with inverse $B$ we can write $1 = a_1b_1 + \cdots + a_nb_n$, a finite sum in $AB$. Then, for any $a \in A$ we have $a = \sum_{i=1}^{n} a_i(ab_i)$ since $R$ commutes and $ab_i \in R$ for all $i$. Fix $S = \{a_i\} \subseteq A$ and define $\varphi_{a_i}: A \longrightarrow R$ by $\varphi_{a_i}(a) = ab_i$. Then $\varphi_{a_i} \in \mathrm{Hom}(A,R)$ for each $i$ and fulfills the criteria of (3.13), giving that $A$ is projective.

($\Leftarrow$) If $A$ is projective then there exists $\{a_i\}^I \subseteq A$ such that for each $i \in I$ there is a $\varphi_i \in \mathrm{Hom}(A,R)$, $\varphi_i(a) = 0$ for all but finitely many $i$ and $a = \sum_I \varphi_i(a)a_i$. Let $p,q, \in A$, $r \in R$ and $i \in I$ with $p,q,r$ non-zero. Then $\varphi_i(rpq) = rp\varphi_i(q) = rq\varphi_i(p)$ and since $R$ is an integral domain we can cancel and we have that in $K$, the quotient field of $R$, $\varphi_i(p)/p = \varphi_i(q)/q$. Set $q_i = \varphi_i(p)/p$. Since there are only finitely many non-zero $\varphi_i(p)$ and $\varphi_i(p)/p = q_i$ for all non-zero $p \in A$, the set $Q = \{q_i\}^I$ is finite, say $Q = \{q_1,\ldots,q_m\}$. For $a \in A$, $\varphi_i(a)/a = q_i$ so $\varphi_i(a) = aq_i$ and $a = \sum \varphi_i(a)a_i$. Therefore, in $K$, $1 = a/a = \sum_I (\varphi_i(a)/a)a_i = \sum_{i=1}^{m} q_ia_i$. So, for any $a$ we can write $a = a\sum q_ia_i = \sum a(\varphi_i(a)/a)a_i = \sum_{i=1}^{m} \varphi_i(a)a_i \in \sum_{i=1}^{m} Ra_i$ so $A \subseteq \sum Ra_i$. The opposite containment is immediate and it follows that $\{a_1,\ldots,a_m\}$ generates $A$.

Let $Q'$ be the $R$-module generated by $Q$. We can always find $r \in R$ such that $rQ' \subseteq R$ since each $q_i \in K$, so $Q'$ is a fractional ideal of $K$. Let $x = aq$ ($a \in A$, $q \in Q'$); then $q = \sum_{i=1}^{m} r_iq_i = \sum r_i\varphi_i(a)/a = (1/a)\sum r_i\varphi_i(a)$ so $x = aq =$

$\sum r_i\varphi_i(a) \in R$ and $AQ' \subseteq R$. In the other direction $r = r\sum q_i a_i = \sum (ra_i)q_i$ with $ra_i \in A$ and $q_i \in Q'$ so $R \subseteq AQ'$ and we have that $Q' = A^{-1}$ so A is invertible.□

(As an aside we notice that "invertible ⇒ projective" requires only that R be commutative.)

This then characterizes the Prüfer domains as the integral domains in which finitely generated ideals are projective—the semihereditary commutative domains. This will reduce to a special case of theorem 3.21.

A *Dedekind domain* is an integral domain in which the non-zero ideals are invertible. The foregoing argument above also demonstrates that R is a Dedekind domain if and only if R is a commutative hereditary domain.

Before proceeding to the general case where R is any commutative ring we need

3.15 *Lemma* ([10], 6.6). If R is an integral domain in which ideals generated by two elements are invertible then R is a Prüfer domain.

*Proof* [10]. Let $C = (c_1,\ldots,c_n)R$ be any finitely generated ideal in R. If C is invertible then C is Prüfer. We proceed by induction on the number of generators. We have the result for n = 1 and 2; suppose it holds for ideals with less than n generators. Let $A = (c_1,\ldots,c_{n-1})R$, $B = (c_2,\ldots,c_n)R$, $D = (c_1,c_n)R$, $E = (c_1RA^{-1}D^{-1} + c_nRB^{-1}D^{-1})$. A,B,D are invertible by the induction hypothesis.

$CE = [A + C_nR](c_1)R\cdot A^{-1}D^{-1} + [c_1R + B](c_n)R\cdot B^{-1}D^{-1}$
(since $C = A + c_nR = c_1R + B$)
$= (c_1)R\cdot D^{-1} + (c_nc_1)R\cdot A^{-1}D^{-1} + (c_1c_n)R\cdot B^{-1}D^{-1} + (c_n)R\cdot D^{-1}$
$= (c_1)R\cdot D^{-1}[R + (c_n)R\cdot B^{-1}] + (c_n)R\cdot D^{-1}[R + (c_1)R\cdot A^{-1}]$
$= (c_1)R\cdot D^{-1} + (c_n)R\cdot D^{-1} = D^{-1}[(c_1)R + (c_n)R]$ (since $(c_n)R \subseteq B \Rightarrow (c_n)R\cdot B^{-1} \subseteq R$ and similarly, $(c_1)R\cdot A \subseteq R$)
$= D^{-1}D = R$. So $E = C^{-1}$. □

3.16 *Proposition*. If R is a commutative semihereditary ring then R is p.p. and the stalk $R_P$ is a Prüfer domain for each $P \in X_R$.

*Proof*. That R is p.p. is immediate. For the second statement, consider an ideal of $R_P$ generated by two elements, say $I = a_PR_P + b_PR_P$ $(a,b \in R)$. We first show that I is projective. For this we will consider, in this instance only, a sheaf of R-*modules* over $X_R$. The R-module in question will be the free R-module on two generators: $F = uR \oplus vR$. The stalk at each $P \in X_R$ will be $F/\bar{P}F$. In R, $aR + bR$ is projective so $(aR + bR) \oplus C \simeq F$ for some R-module C. At P we have $[(aR + bR) \oplus C]_P \simeq (uR \oplus vR)_P$ so $(a_PR_P + b_PR_P) + C_P \simeq u_PR_P + v_PR_P$. We need that the right hand side is free and that $a_PR_P + b_PR_P$ is a direct summand. To this end suppose that $u_PR_P + v_PR_P$ is not free. Then there is some $r_P \neq 0_P$ such that $u_Pr_P + v_Ps_P = 0_P$ and this is true on a clopen set $\Gamma(e)$ containing P. This says that $ure = vse = 0$ in F so $re = se = 0$ since F is free. But then $[(re)_P = r_Pe_P = r_P1_P = 0_P] \Rightarrow r_P = 0_P$ which is a contradiction unless $F_P$ is free.

On the left hand side suppose $a_P r_P' + b_P s_P + c_P t_P = 0_P$ and $(a_P r_P + b_P s_P) \neq 0$ or $c_P t_P \neq 0$. There exists a clopen set $\Gamma(f)$ containing P over which this holds so $arf + bsf + ctf = 0$ in $(aR + bR) \oplus C$. By unique representation in a direct sum, $arf + bsf = 0$ and $ctf = 0$. But then $(ar + bs)_P = 0$ and $(ct)_P = 0$ and again we have a contradiction unless the left hand side is a direct sum.

It follows that $(aR + bR)_P \oplus C_P \simeq u_P R_P \oplus v_P R_P \simeq R/\bar{P} \oplus R/\bar{P}$ and that $(aR + bR)_P \simeq a_P R_P + b_P R_P$ is a projective ideal in an integral domain ($R_P$ is a domain by (3.10)) and by (3.14) must be invertible. By (3.15) then, $R_P$ is a Prüfer domain.$\square$

3.17 *Lemma*. Let R be a commutative p.p. ring and $I \leq R$ a finitely generated ideal. If $\mathfrak{U}$ is the open set in $X_R$ associated with I then $\mathfrak{U}$ is clopen.

*Proof*. Let $I = (a_1,\ldots,a_n)R$. The support of $a_i$ is $\Gamma(e_i)$ where $e_i$ is the idempotent associated with $a_i$. For any $r \in R$, $S(a_i r) = \Gamma(e_i) \cap \Gamma(f) \leq S(a_i) = \Gamma(e_i)$ $\forall$ $r \in R$ where $r = rf$. Therefore, $S(a_i R) = \bigcup S(a_i r) \leq S(a_i) = \Gamma(e_i)$ and $\mathfrak{U} = S(I) \leq \bigcup_{i=1}^{n} S(a_i R) \leq \bigcup_{i=1}^{n} \Gamma(e_i) = \Gamma(e)$ where $eB(R)$ is the ideal of $B(R)$ generated by $e_1,\ldots,e_n$. That $\Gamma(e) \leq \mathfrak{U}$ is clear.$\square$

3.18 *Lemma*. Let I be a fractional ideal of $R \leq K = RS^{-1}$ and suppose that for some $k \in K$, $k_P \in I_P$ $\forall$ $P \in X_R$ (where $k_P$, $I_P$ denote the images of k, I in $K_P = R_P(S^{-1})_P$), then $k \in I$.

*Proof.* $k_P \in I_P \Rightarrow k_P = i_P^{(P)}$ for some $i^{(P)} \in I$ and this holds on a basic set $\Gamma(e^{(P)})$ so $ke^{(P)} = i^{(P)}e^{(P)}$ in K. The sets $\{e^{(P)}\}^{P\in X_R}$ cover $X_R$ and by the partition property we can pick $\{\Gamma(e_j)\}_{j=1}^{n}$ such that for each j, $\Gamma(e_j) \subseteq \Gamma(e^{(P)})$ for some P and $\Gamma(e_i) \cap \Gamma(e_j) = \emptyset$ for $i \neq j$ and $X_R = \bigcup_{i=1}^{n}\Gamma(e_i)$ (so $e_ie_j = 0$ and $\sum e_i = 1$). Over $\Gamma(e_j)$ we have $k_{P_j} = i_{P_j}^{(P_j)} \Rightarrow$ $ke_j = i^{(P_j)}e_j \in I$ since I is an R-module. And $\sum_{j=1}^{n} i^{(P_j)}e_j =$ $\sum ke_j = k(\sum e_j) = k\cdot 1 \in I.$ □

3.19 *Proposition.* If R is a p.p. ring in which $R_P$ is a Prüfer domain $\forall\, P \in X_R$ and $I \subseteq R$ is a finitely generated ideal with all of $X_R$ for support then I is invertible.

*Proof.* Since R is p.p., $K_P = R_P(S^{-1})_P$ is the *field* of fractions of $R_P$. We show that $([R:I]_K)_P = [R_P:I_P]_{K_P}$ and the result will follow from (3.18).

Let $a_P \in ([R:I]_K)_P$, so $a_P = a + KP$ for some $a \in K$ with the property that $aI \subseteq R$, then $(aI)_P = a_PI_P \subseteq R_P$ giving $([R:I]_K)_P \subseteq [R_P:I_P]_{K_P}$.

For the opposite containment, suppose for some $k \in K$ we have $k_PI_P \subseteq R_P$ so $k_P \in [R_P:I_P]_{K_P}$. Say $I = (a_1,\ldots,a_n)R$. For each $a_i$, $k_P(a_i)_P = r_P$ for some r and this must hold on a basic set, say $\Gamma(f_i)$, containing P. So $ka_if_i = rf_i \Rightarrow$ $(ka_i - r)f_i = 0$ and $ka_i - r \in \{f_i\}^*$. Then $ka_i - r =$ $(1 - f_i)r'$ and $ka_i = r + (1 - f_i)r' \in R$. Since there are only finitely many $a_i$ it must be that $kI \subseteq R$ over a basic open set containing P, call it $\Gamma(g)$, say $\Gamma(g) = \bigcap_{i=1}^{n} \Gamma(f_i)$. So, for any i, $(ki)_P = r_P$ holds $\forall$ $P \in \Gamma(g)$ and $kig = rg$ in R. This implies that $kg \in [R:I]_K$ and $(kg)_P \in ([R:I]_K)_P$. But, $g \in B(R)$ so $g_P = 1_P$ and $(kg)_P = k_P$, so $[R_P:I_P]_{K_P} \subseteq ([R:I]_K)_P$, establishing the claim.

$R_P$ is a Prüfer domain at each $P \in X_R$ so $I_P$, being the image of a finitely genetrated ideal, hence itself finitely generated, is invertible and $(I_P)^{-1} = [R_P:I_P]_{K_P}$. Therefore $1_P \in I_P[R_P:I_P]_{K_P} = (I[R:I]_K)_P$ $\forall$ $P \in X_R$ so by (3.18), $1 \in I[R:I]_K$ so $[R:I]_K = I^{-1}$ and I is invertible.□

3.20 *Proposition.* If R is a p.p. ring in which $R_P$ is a Prüfer domain $\forall$ $P \in X_R$ then R is semihereditary.

*Proof.* Let I be any finitely generated ideal of R. By (3.17) the support of I is a clopen set, say $\Gamma(e)$. We can write $X_R = \Gamma(e) \cup \Gamma(1 - e)$, a disjoint union, and thus obtain a decomposition of R; $R = eR \oplus (1 - e)R = R' \oplus R''$ into the factor rings $R/(1 - e)R$, $R/eR$. $I = I' \oplus (0)$ for an ideal $I' \subseteq R'$ (otherwise I would have non-zero support in $\Gamma(1 - e)$). This reduces us to the case of a finitely generated ideal, I′, having all of $X_{R'}$ for support and we saw in (3.19) that such an ideal is invertible and hence projective. Since (0) is projective we have written I as a sum of projectives so I is projective and R is semihereditary.□

We summarize these facts in

3.21 *Theorem* ([1], 4.1). A commutative ring R is semihereditary if and only if:

(a) R is p.p. and

(b) $R_P$ is a Prüfer domain for every $P \in X_R$.

(If R is, in addition, an integral domain, then B(R) = {0,1} and $X_R$ consists solely of (0) and R so $R_P$ = (0) or R for all $P \in X_R$. (3.21) is then the statement that the commutative semihereditary domains are the Prüfer domains.)

## 4. *Commutative hereditary rings.*

A simple criterion for a ring to be Noetherian is presented in

3.22 *Lemma*. If R is a ring in which countably generated ideals are finitely generated then R is Noetherian.

*Proof*. We will demonstrate the contrapositive. Assume R is not Noetherian, so there exists an infinite, strictly increasing, chain of ideals $I_1 \subset I_2 \subset I_3 \subset \cdots$ . Define $J_1 = a_1R$, $a_1 \in I_1$; $J_2 = a_2R$, $a_2 \in I_2 - I_1$; ... ; $J_n = a_nR$, $a_n \in I_n - I_{n-1}$ and so on. The chain $J_1 \subset J_2 \subset J_3 \subset \cdots$ is strictly increasing and $J = \bigcup_{i=1}^{\infty} J_i = (a_1, a_2, a_3, \ldots)R$ is countably generated but not finitely generated and this gives us the contrapositive statement.$\square$

We now display an association between the ideals of R and those of B(R) by considering the supports of ideals. Recall that the support of $I \leq R$ is the open set $U \leq X_R$ such that $U = \bigcup_{i \in I} S(i)$, the union of the supports of the elements of I.

3.23 *Lemma*. If I, J are ideals in the p.p. ring R then $I \cap J = (0)$ if and only if the supports of I and J are disjoint.

*Proof*. ($\Rightarrow$) Suppose $I \cap J = (0)$ but $S(I) \cap S(J) \neq \emptyset$. We can then find some $i \in I$, $j \in J$ and $P \in X_R$ such that $i_P$ and $j_P$ are non-zero. Since $R_P$ is an integral domain when R is p.p. (by (3.10)), $(ij)_P$ is non-zero so $ij \notin \bar{P}$ and, in particular, $ij \neq 0$. But this is a contradiction since $ij \in I \cap J = (0)$.

($\Leftarrow$) Let the supports be disjoint and suppose there is a non-zero $x \in I \cap J$. Since $[x_P = 0 \ \forall \ P \in X_R] \Rightarrow x = 0$, there must be some $P \in X_R$ such that $x_P \neq 0$. So $P \in S(I) \cap S(J) = \emptyset$ which is a contradiction.$\square$

Let $U = S(I)$ and $B = \{e \in B(R) | \Gamma(e) \leq U\}$. B is easily verified as an ideal of B(R) with the same support as I. If $I = \bigoplus_{j \in J} I_j$ is a direct sum of subideals $I_j \leq I$ (J can be infinite) then to each ideal $I_j$ we associate the ideal $B_j = \{f \in B(R) | \Gamma(f) \leq S(I_j)\}$. If $e \in B_j$ and $f \in B_k$ then $\Gamma(e) \cap \Gamma(f) = \emptyset$ since $I_j \cap I_k = (0)$, so $ef = 0$. This gives that $B_j B_k = (0) \ \forall \ j \neq k$ and the sum of the $B_j$'s is direct.

So I is associated with $B = \oplus_{j\in J} B_j$, the direct sum of ideals associated with the $I_j$'s.

Suppose R is p.p. and $I \subseteq R$ is generated by $A \subseteq R$. $S(A) \subseteq S(I)$ since $A \subseteq I$. To see that $S(I) \subseteq S(A)$ consider $i = r_1a_1 + \cdots + r_na_n \in A\cdot R = I$. If $P \in S(i)$ then $r_1a_1 + \cdots + r_na_n \notin \bar{P}$. If $P \notin S(r_ja_j)$ for some j then $i_P = 0$ so $P \in S(r_ja_j)$ for some j and $r_ja_j \notin \bar{P} \Rightarrow a_j \notin \bar{P}$ by primality of $\bar{P}$. Therefore $P \in S(a) \subseteq S(A)$ and it follows that $S(I) = S(A)$. So, a subset of R and the ideal it generates have the same support in $X_R$.

3.24 *Lemma*. If R is a p.p. ring and $I \subseteq R$ is finitely generated then so is $B \subseteq B(R)$, where B is the ideal associated with I.

*Proof*. It suffices to consider $I = (a,b)R$. R is p.p. so any single element has clopen support and from the remarks following (3.23) we know that $S(I) = S(a) \cup S(b) = \Gamma(e) \cup \Gamma(f)$ for some $e,f \in B(R)$. $S(B) = \Gamma(e) \cup \Gamma(f)$ and the ideal in B(R) with this support is $(e,f)B(R) = (e + f - ef)B(R)$.□

3.25 *Lemma*. If R is p.p. and $a \in R$ then

(1) a is a non-zero divisor if and only if $S(a) = X_R$

(2) Any ideal containing the non-zero divisor a cannot be an infinite direct sum of subideals.

*Proof*. For the first statement, if a is a non-zero divisor then $a \notin \bar{P}\ \forall\ P \in X_R$ since otherwise $a = ae$ for some non-trivial $e \in B(R)$ and $a(1 - e) = 0$ then implies that a is a zero divisor. Therefore $S(a) = X_R$.

In the other direction, suppose an element a has $X_R$ for support but is a zero divisor. We can find $b \neq 0$ such that $ab = 0$ so $a_P b_P = 0_P \; \forall \; P \in X_R$. Since a is not in any $\bar{P}$ ($a_P \neq 0 \; \forall \; P \in X_R$), b is in every $\bar{P}$ by primality. This means that $b = be$ where $e \in \bigcap_{P \in X_R} P = (0)$. Therefore $b = 0$ which is a contradiction unless a is a non-zero divisor.

For the second statement let $I \leq R$ be an ideal and $a \in I$ a non-zero divisor, so $S(a) = S(I) = X_R$. The ideal associated with I is all of B(R) so if $I = \bigoplus_J I_j$ where J is infinite then we have a corresponding decomposition of B(R): $B(R) = \bigoplus_J B_j$ (where $B_j$ is associated with $I_j$). It follows that $X_R = \bigcup_J S(B_j)$ and $S(B_j) \cap S(B_k) = \varnothing$ and we have found an infinite open cover of $X_R$ which admits no finite subcover $\{\Gamma(e_i)\}$, where $\Gamma(e_i) \leq S(B_j)$ for some j, as guaranteed by the partition property of $X_R$. It cannot be that I is an infinite direct sum.$_\square$

3.26 *Lemma.* If R is a p.p. ring and $I \leq R$ has $X_R$ for support then I contains a non-zero divisor.

*Proof.* By hypothesis, $X_R = \bigcup_{i \in I} S(i)$ and by compactness we can select a finite subcover: $X_R = \bigcup_{j=1}^{n} S(i_j e_j)$ such that each $i_j = i_j e_j$ for some $e_j \in B(R)$.

$X_R = \bigcup_{j=1}^{n} S(i_j e_j) = \bigcup_{j=1}^{n} \Gamma(e_j)$. By the partition property we can write $X_R = \bigcup_{k=1}^{m} \Gamma(f_k)$ where this is a disjoint union and each $\Gamma(f_k) \leq \Gamma(e_j)$ for some j. Let $i' = \sum_{k=1}^{m} i'_k f_k \in I$, where $i'_k = i_j$ if $\Gamma(f_k) \leq \Gamma(e_j)$ (so the $i'_k$ are not necessarily distinct). $S(i') = S(\sum i'_k f_k) = \bigcup \Gamma(f_k) = X_R$ since the union is disjoint and by (3.25) i′ must be a non-zero divisor.□

3.27 *Lemma*. Let R be a p.p. ring in which $R_P$ is Noetherian $\forall\ P \in X_R$ and any non-zero divisor $a \in R$ is such that $a_P$ is invertible for all but finitely many $P \in X_R$. Then any ideal of R that has $X_R$ for support is finitely generated.

*Proof*. Let $I \subset R$ be an ideal with $X_R$ for support. By (3.26) I contains a non-zero divisor a so, by hypothesis, $a_P$ is a unit for all but a finite set, $\mathbb{P} = \{P_1,\dots,P_2\}$, of prime ideals in $X_R$. At each $P_i$, $R_{P_i}$ is Noetherian so $I_{P_i}$ is finitely generated by, say, $\{b_1^{(i)},\dots,b_{m_i}^{(i)}\}_{P_i}$. It turns out that I is generated by the finite collection $\{a,\ b_1^{(1)},\dots,b_{m_1}^{(1)},b_1^{(2)},\dots,b_{m_2}^{(2)},\dots b_{m_n}^{(n)}\}$.

At each $Q \in X_R - \mathbb{P}$, $a_Q$ is a unit so $I_Q = R_Q = \langle a,\dots,b_{m_n}^{(n)}\rangle_Q$. At each $P_i \in \mathbb{P}$, $I_{P_i}$ is generated by $\{b_1^{(i)},\dots,b_{m_i}^{(i)}\}_{P_i}$, and the images of the "left-overs" are in $I_{P_i}$ so $I_{P_i} = \langle a,\dots,b_{m_n}^{(n)}\rangle_{P_i}$ for each i.

The result follows if $I_P = J_P$ for all $P \in X_R$ implies that I = J. To see that it does, consider some $x \in I$. Then $x_P = j_P$ for some $j \in J$ at each $P \in X_R$. Over some finite cover, $\{\Gamma(e_i)\}_{i=1}^{n}$, we have $xe_i = j_ie_i$ for each i and these add together to yield $x = \sum_{i=1}^{n} j_ie_i \in J$ so $I \subset J$. Similarly,

$J \subset I$ and we have that $I = J$.$\square$

3.28 *Proposition* ([1], 4.2). For a p.p. ring R and $\alpha$ an infinite cardinal, the following are equivalent:

(1) Every ideal $I \leq R$ generated by $\leq \alpha$ elements is a direct sum of finitely generated subideals.

(2) a: The stalk at P, $R_P$, is Noetherian $\forall\ P \in X_R$.

b: For any non-zero divisor $a \in R$, $a_P$ is invertible for all but finitely many $P \in X_R$.

c: B(R) is $\alpha$-hereditary.

*Proof.* (1⇒2a) Let $I_P$ be any non-zero countably generated ideal of $R_P$ and $I \leq R$ its countably generated preimage in R. By hypothesis, $I = \oplus_J I_j$ where $I_j \leq I$ is finitely generated. If $P \in S(I)$ then $P \in S(I_j)$ for precisely one of the $I_j$ since their supports must, by (3.23), be pairwise disjoint. Therefore, $(I_k)_P = (0)_P\ \forall\ k \neq j$ and $I_P = (I_j)_P$ which is finitely generated. This gives us that countably generated ideals are finitely generated and so $R_P$ is Noetherian at each $P \in X_R$ by (3.22).

(1⇒2b) Let $a \in R$ be a non-zero divisor. If $a \in I$ for some ideal $I \leq R$ then I cannot be an infinite direct sum of ideals by (3.25). The hypothesis then forces I to be finitely generated. By way of contradiction, we assume $a_P$ is not invertible at infinitely many $P \in X_R$ and construct a non-finitely generated ideal containing a.

Let $\mathbb{P} = \{P_1, P_2, P_3, \ldots\}$ be an infinite collection of distinct primes in $X_R$ such that $a_{P_i}$ is not invertible for

$i = 1,2,3,\ldots.$ Since $X_R$ is Hausdorff we can separate $P_1$ and $P_2$ by a partition; say $P_1 \in \Gamma(e_1)$, $P_2 \in \Gamma(1 - e_1)$. One of these, possibly both, contains infinitely many elements of $\mathbb{P}$, say $\Gamma(1 - e_1)$ does. Suppose $P_3 \in \Gamma(1 - e_1)$ and separate $P_2$ and $P_3$ by a partition: $P_2 \in \Gamma(f)$, $P_3 \in \Gamma(1 - f)$.

$\Gamma(1 - e_1) = \Gamma[f(1 - e_1)] \cup \Gamma[(1 - f)(1 - e_1)]$ is a partition of $\Gamma(1 - e_1)$ and $P_2 \in \Gamma(e_2)$, $P_3 \in \Gamma(1 - e_1 - e_2)$ where $e_2 = f(1 - e_1)$ (so $e_1$ and $e_2$ are orthogonal). One of these must contain infinitely many elements of $\mathbb{P}$, say $\Gamma(1 - e_1 - e_2)$. Take a point in here, say $P_4$, and separate it from $P_3$ by a partition and "shrink" this to a partition of $\Gamma(1 - e_1 - e_2)$ and continue the process. In this manner is obtained a sequence of idempotents $\{f_1,f_2,f_3,\ldots\}$ with $f_1 = e_1$, $f_2 = e_1 + e_2$, $f_3 = e_1 + e_2 + e_3$, etc., such that $\Gamma(\sum_{i=1}^{n} e_i) = \bigcup_{i=1}^{n}\Gamma(e_i)$, $e_ie_j = 0$ if $i \neq j$ and $\Gamma(f_1) \subsetneq \Gamma(f_2) \subsetneq \Gamma(f_3) \subsetneq \cdots$ and $\mathbb{P}$ is not contained in any $\Gamma(f_i)$. This gives a chain of ideals: $aR$, $aR + f_1R$, $aR + f_1R + f_2R$, .... Suppose the chain becomes stationary at some $n \geq 1$. Then $aR + \sum_{i=1}^{n} f_iR = aR + \sum_{i=1}^{n+1} f_iR$ so $f_{n+1}R \subseteq aR + \sum_{i=1}^{n} f_iR$ and $f_{n+1} = ar + \sum_{i=1}^{n} f_ir_i$ for suitable ring elements. By construction, $\Gamma(f_n) \subsetneq \Gamma(f_{n+1})$ so $\exists\, P \in \Gamma(f_{n+1})$ which is not in any $\Gamma(f_j)$, $j \leq n$. Therefore, at $P$ we have $[(f_{n+1})_P = (ar)_P + (\sum f_ir_i)_P]$ $\Rightarrow 1_P = a_Pr_P$ since $(f_i)_P = 0\ \forall\ i \leq n$ and $(f_{n+1})_P = 1_P$ since $P \in \Gamma(f_{n+1})$ and $f_{n+1} \in B(R)$. This says that $a$ is a unit at $P$ which is a contradiction, so the chain is strictly increasing and $I = \bigcup (aR, aR + f_1R, \ldots) = (a,f_1,f_2,\ldots)R$ is not finitely generated yet contains a which is a contradiction unless $\mathbf{P}$ is finite.

(Under this hypothesis then, p.p. rings are rings in which non-zero divisors are "almost" units in the sense that a non-zero divisor is almost everywhere invertible.)

($1\Rightarrow 2c$) Let $B \subseteq B(R)$ be an ideal generated by $\leq \alpha$ elements, say $B = G\cdot B(R)$. Let $I = G\cdot R$ be the ideal in R generated by G. $S(I) = S(G) = S(B)$ so I and B are associated. By hypothesis, $I = \oplus_J I_j$, a direct sum of finitely generated subideals. This gives us the decomposition $B = \oplus_J B_j$, where $B_j$ is associated with $I_j$ and is finitely generated, hence principal. Therefore, B is a direct sum of principal ideals and principal ideals are projective in Boolean rings, hence B is projective and B(R) is $\alpha$-hereditary.

($2\Rightarrow 1$) Let I be an ideal of R generated by $\leq \alpha$ elements, say $I = G\cdot R$, and let $B \subseteq B(R)$ be its associated ideal. Let $e_g$ be a non-zero element of B(R) such that $\Gamma(e_g) = S(g)$ for each $g \in G$, then $S[\{e_g\}^{g\in G}] = S(I)$ and B is generated by these elements which are $\leq \alpha$ in number. B(R) is $\alpha$-hereditary so B is projective and has an orthogonal basis by (2.11) so $B = \oplus_{j\in J} e_j B(R)$. Therefore I also decomposes as a direct sum of ideals: $I = \oplus_{j\in J} I_j$ where $I_j$ is associated with $e_j B(R)$ and has clopen support.

If an ideal with clopen support in a p.p. ring is finitely generated then we are done. Suppose J is an ideal with $S(J) = \Gamma(e)$. Let $J' = J \oplus (1 - e)R$ (the sum is direct since J and (1 - e)R have disjoint supports). J′ contains a non-zero divisor by (3.26) and $a_P$ is invertible for all but finitely many $P \in X_R$ by hypothesis. Therefore (3.27) applies and J′ is finitely generated so J must follow suit.

We have then that I is a direct sum of finitely generated ideals.□

3.29 *Corollary* ([1], 4.3). For a p.p. ring R, the following are equivalent:

(1) Countably generated ideals are direct sums of finitely generated ideals.

(2) R satisfies a and b of (3.28).

(3) Ideals containing a non-zero divisor are finitely generated.

(4) Ideals of R with clopen support in $X_R$ are finitely generated.

*Proof.* (1⇒2) Take $\alpha \leq \aleph_0$ in the preceding result.

(2⇒3) This is contained in "2⇒1" of (3.28).

(3⇒4) Let $S(I) = \Gamma(e)$, then $I' = I \oplus (1-e)R$ contains a non-zero divisor and is finitely generated.

(4⇒1) B(R) is $\aleph_0$-hereditary so countably generated ideals of R can be associated with countably generated, hence projective, ideals of B(R) yielding a direct sum decomposition of ideals of R with clopen support.□

We now have all the tools at hand to characterize commutative hereditary rings in

3.30 *Theorem* ([1], 4.4). R is hereditary if and only if

(a) R is p.p.

(b) The stalk of R at every $P \in X_R$ is a Dedekind domain.

(c) For any non-zero divisor $a \in R$ and $P \in X_R$, $a_P$ is invertible on all but finitely many of the stalks $R_P$.

(d) B(R) is hereditary.

*Proof.* ($\Rightarrow$) (a) is trivial. For (b), any ideal $I \subseteq R$ is projective so by (2.10) I is a direct sum of finitely generated ideals and $R_P$ is Noetherian for every $P \in X_R$ by (3.28). By (3.21), each $R_P$ is also Prüfer so every ideal of $R_P$ is finitely generated and hence invertible.

(c) is true since R satisfies (1) of the last corollary. For (d), any ideal $B \subseteq B(R)$ can be paired with an ideal of R which, since R is hereditary, decomposes as a direct sum of finitely generated ideals, yielding a decomposition of B into finitely generated, hence projective, ideals.

($\Leftarrow$) Let $I \subseteq R$ be any ideal and $B \subseteq B(R)$ its associated ideal. B(R) hereditary implies B is projective so B can be written as a direct sum of principal ideals (by (2.11)). This gives a decomposition of I into ideals with clopen support and by (3.29) each of these is finitely generated. R p.p. and $R_P$ Prüfer for all P implies R is semihereditary; therefore each of these ideals is projective so I is projective. $\square$

An immediate generalization is that R is $\alpha$-hereditary if and only if a, b, and c above are satisfied and d becomes "B(R) is $\alpha$-hereditary."

3.31 *Corollary* ([1], 4.5). If R is commutative then the following are equivalent:

(1) R is $\aleph_0$-hereditary.

(2) R satisfies a, b and c of (3.30).

(3) R is p.p. and every ideal of R containing a non-zero divisor is finitely generated and projective.

(4) R is p.p. and any ideal of R with clopen support is finitely generated and projective.

*Proof*. (1⇒2) R p.p. is trivial. R semihereditary implies $R_P$ Prüfer for all P by (3.21) and R satisfies condition one of (3.28) by (2.10) and any Noetherian Prüfer domain is Dedekind, giving b and c.

(2⇒3) Under the hypothesis, R is p.p. and satisfies condition two of (3.29) so every ideal containing a non-zero divisor is finitely generated. (3.21) is also satisfied so R is semihereditary and finitely generated ideals are projective.

(3⇒4) Ideals with clopen support are finitely generated by (3.29). Suppose $I \subseteq R$ has $S(I) = \Gamma(e)$ and consider $I' = I \oplus (1 - e)R$ which has $X_R$ for support, hence contains a non-zero divisor. Then $I'$ is projective by hypothesis so I must be also.

(4⇒1) Let $I \subseteq R$ be countably generated. Under the hypothesis, (3.29) says that I is a direct sum of finitely generated ideals. Each of these is then associated with a finitely generated, hence principal, ideal in B(R). Therefore the support of each summand of I is clopen so I is a sum of projectives and R is $\aleph_0$-hereditary.□

3.32 *Corollary* ([1], 4.6). A commutative regular ring R is ($\alpha$-)hereditary if and only if B(R) is ($\alpha$-)hereditary. Any regular subring of an ($\alpha$-)hereditary commutative ring R is ($\alpha$-)hereditary.

*Proof.* By theorem 3.30, B(R) is ($\alpha$-)hereditary when R is. Conversely, by (3.11), R regular implies R p.p. and every non-zero divisor is a unit and $R_P$ is a field (hence a Dedekind domain) for all $P \in X_R$ so if B(R) is ($\alpha$-)hereditary then so is R by (3.30).

For the second statement suppose $R' \subseteq R$ is a subring where R is commutative and ($\alpha$-)hereditary. As Boolean rings, $B(R') \subseteq B(R)$ and by (2.17) any subring of an ($\alpha$-)hereditary Boolean ring is ($\alpha$-)hereditary. By the first part then, R′ is ($\alpha$-)hereditary.□

# CHAPTER 4

## EXAMPLES

This chapter will detail three examples of hereditary rings cited by Bergman in [1]. Because they are set-theoretic in nature it will often be more convenient to discuss Boolean algebras rather than Boolean rings. These are, of course, equivalent notions. If B is a Boolean algebra with operations $\wedge$, $\vee$, $'$ (meet, join and complementation, which in our examples will be set-theoretic intersection, union and complement) then B is a Boolean ring with addition $P + Q = (P' \wedge Q) \vee (P \wedge Q')$ (symmetric difference), multiplication $P \cdot Q = P \wedge Q$ and $P' = 1 + P$.

The simplest Boolean algebra is the integers modulo 2 and this will be denoted simply as 2.

The Boolean ring associated with a topological space X is obtained from the Boolean algebra of subsets of X. It is the algebra of all clopen sets of X. ([4])

1. *The ring associated with the Cantor set.*

Let $\mathscr{C}$ be the Cantor middle third set (all numbers of the form $\sum_{i=1}^{\infty} (a_i)/(3^i)$ where $a_i$ is zero or two) and $X = \prod_{i \in \mathbb{N}} 2^{(i)}$, the direct product of countably many copies of 2. If $f_i: X \longrightarrow 2^{(i)}$ is the projection of X onto the ith copy of 2 (the ith "coordinate" of X) then $f(x) = \sum_{i=1}^{\infty} (2f_i(x))/(3^i)$ is a one-to-one identification of X and $\mathfrak{C}$.

We endow X with the product topology induced by the discrete topology on 2: Let $O_i = \{x \in X | f_i(x) = \delta\}$ where $\delta$ is zero or one (then $O_i$ is the set of elements of X with ith coordinate fixed at $\delta$ and all others free). The collection of all finite intersections of such sets then forms a basis for the topology on X. The complement of such a basic open set is a finite union of such sets and hence is also open. In other words, X has a basis of clopen sets. That X is Hausdorff is immediate. X is compact by Tychonoff's theorem: the product of any number of compact spaces is compact in the product topology. X is therefore a Boolean space.

(In this example we need only countably many copies of 2 but this is no restriction—the space associated with uncountably many copies of 2 is likewise Boolean. (Halmos calls such spaces *Cantor spaces*.))

The collection of all clopen sets of X, the *field of clopen sets*, is the Boolean algebra (ring) associated with $\mathfrak{C}$, call it B.

Let $R = \mathbb{Z}[x_1, x_2, x_3, \ldots]$, the polynomial ring over $\mathbb{Z}$ in $\aleph_0$ commuting indeterminates. An easy counting argument shows that R is a countable ring. Let I be the ideal of R generated by two and all $x_i^2 - x_i$. Then $R/I \cong \bar{R} = (\mathbb{Z}/2\mathbb{Z})[\bar{x}_1, \bar{x}_2, \ldots]$ where $(\bar{x}_i)^2 = \bar{x}_i$. $\bar{R}$ is therefore a countable Boolean ring, the free Boolean ring on countably many generators.

We now associate B and $\bar{R}$. To the clopen set consisting of all elements with ith coordinate fixed at one associate $x_i$ in $\bar{R}$ (the bar over x will be dropped for notational convenience, remembering that $x_i^2 = x_i$). To the clopen set consisting of all elements with jth coordinate zero associate $(1 + x_j)$ (which is the compliment of the clopen set associated with $x_j$). Products in B (which are intersections in the algebra B) become products of the associated elements of $\bar{R}$. Sums in B (which are symmetric differences) are then the sums of the corresponding associates in $\bar{R}$. The null set (the zero of B) is then realized in $\bar{R}$ as, for instance, $x_i + x_i = 0$. X (the unit of B) can be realized in $\bar{R}$ as $x_i + (1 + x_i) = 1$, the symmetric difference of a clopen set and its compliment.

The above association is an isomorphism of rings. B is isomorphic to $\bar{R}$, the free Boolean ring on $\aleph_0$ commuting generators, which is countable. Therefore, B is a countable Boolean ring and it follows from the corollary to theorem 2.11 that B is hereditary (countable Boolean rings are hereditary).

To obtain an example of a Boolean ring that is not hereditary we can follow the same strategy as above only this time considering the free Boolean ring on uncountably many generators. The associated Boolean space is the Cantor space $X = 2^S$ where S is uncountable, constructed as for $X = 2^I$ where I is countably infinite.

To show that the Boolean ring B associated with $X = 2^S$ (which is isomorphic to the free Boolean ring on S) is non-hereditary we demonstrate that in such a space it is impossible to have an uncountable collection of non-empty, disjoint, clopen sets, and so B cannot fulfill the conditions of (2.11).

Since X is a compact topological group there exists a Haar measure, $\mu$, on X. That is, there is a Borel measure, $\mu$, such that $\mu(U) > 0$ for all non-empty open sets U in X and $\mu(X) < \infty$ (since X is compact). ([5], §58)

Let $\mathbb{D}$ be any class of non-empty, mutually disjoint, clopen sets in X. Set $D_n = \{D \in \mathbb{D} | \mu(D) > 1/n\}$ for n = 1,2,3,.... It is easy to see that $D_n$ is a finite collection for each n and it follows (since $\mu(D) > 0 \ \forall \ D \in \mathbb{D}$) that $\mathbb{D}$ can contain, at most, countably many sets.

Let $I \subset B$ be any ideal that cannot be generated by less than uncountably many generators. (For instance, an ideal generated by an uncountable subset of the free generators of B.) If I were projective then it would, by (2.11), have an uncountable set of generators whose associated open sets were disjoint and clopen and such a collection, we have just seen, is impossible. Therefore, I cannot be projective and B is not hereditary.

2. *The one-point compactification of a discrete space.*

Let X be the one-point compactification of an infinite discrete space with compactifying point $\infty$ (so $X - \{\infty\}$ is an infinite collection of points endowed with the discrete topology). X is topologized by calling every subset of $X - \{\infty\}$ open and declaring a set containing $\infty$ open if and only if it is cofinite (has a finite compliment). It is easily verified that $A \subseteq X$ is clopen if and only if A is a finite subset of $X - \{\infty\}$ or A is a cofinite subset of X containing $\infty$.

So topologized, it is immediate that X is a Boolean space. The Boolean algebra (ring) associated with X, call it B, is the algebra of clopen sets which, in this case, is isomorphic to the finite-cofinite algebra of subsets of $X - \{\infty\}$.

The thrust of what follows is that X so topologized is homeomorphic to $X_B$ topologized as usual (since B is Boolean, $X_B$ and the spectrum of B coincide).

Consider the spectrum of B and recall that in Boolean algebras maximal ideals and prime ideals are the same. For a singleton $\{a\}$ in B, $a \neq \infty$, define $M_a = \{e \in B | \{a\}$ is not contained in $e\}$. $M_a$ is easily verified as a maximal ideal of B (the relevant characterization being that an ideal M in a Boolean algebra B is maximal if and only if for every $p \in B$ either $p \in M$ or $p' \in M$ but never both). Also immediate is that if $\{a\}$ and $\{b\}$ are distinct then $M_a \neq M_b$. Set $M_\infty = \{$all finite subsets of B$\}$ which is also a maximal ideal.

If M is any maximal ideal of B distinct from $M_\infty$ then it must fail to contain a singleton $\{a\}$ and it will follow that

$M = M_a$. Therefore all the maximal ideals of B have been found and we know its spectrum.

Define f: $X \longrightarrow X_B$ by $f(x) = M_x$ (which includes $f(\infty) = M_\infty$). f is one-to-one and onto and a straightforward check shows that f and $f^{-1}$ are continuous. Therefore, X and $X_B$ are homeomorphic and any statement about open sets can be made with equal weight in either space. Let U be any open set in X. If $\infty \in U$ then U is cofinite and hence clopen. If $\infty \notin U$ then we can express U as the disjoint union of its singletons which are clopen. Therefore, the support of any ideal of B is either a clopen set or the disjoint union of clopen sets and hence is projective by (2.11). Consequently, B is an hereditary ring.

3. *A ring of* [illegible]*ctions.*

As abo[illegible] X be the one-point compactification of a discrete space [illegible] B its associated Boolean algebra (ring). Let F be any field and R the ring of functions from X to F continuous with respect to the discrete topology on F. We will show that R is an hereditary regular ring.

With the discrete topology, the elements of R are particularly simple. To see this let f: $X \longrightarrow F$ be continuous. Since F has the discrete topology, the sets $f^{-1}(k)$ are open for each $k \in F$ and $\bigcup_{k\in F}\{f^{-1}(k)\}$ is an open cover of X. X is a Boolean space and has the partition property so there exists a finite subcover of disjoint clopen sets $\{e_i\}_{i=1}^{n}$ where each $e_i \subseteq f^{-1}(k_i)$ for some $k_i \in F$ (so $f(x) = k_i$ if $\{x\} \subseteq e_i$). For each clopen set $e \subseteq X$

define $f_e: X \longrightarrow F$ by $f_e(x) = 1$ if $\{x\} \subseteq e$ and $f_e(x) = 0$ otherwise. Clearly, each $f_e$ is an idempotent element of R. It is easily confirmed that f above is then given by $f = k_1 f_{e_1} + k_2 f_{e_2} + \cdots + k_n f_{e_n}$. In other words, f continuous implies that $f(x) \in \{k_1,\ldots,k_n\}$ for some finite subset of F depending on f. Conversely, If $f: X \longrightarrow F$ has this form then f is continuous and we have found all the elements of R.

It is clear from this representation of $f \in R$ that the idempotent functions (the elements of B(R)) are precisely the functions f such that $f = f_e$ for some clopen set e of X and these are the continuous functions from X into 2.

Now consider $X_R$, the collection of maximal ideals of B(R). For $x \in X$, let $K_x$ be the kernel of the evaluation homomorphism (i.e. the kernel of $E_x$: $B(R) \longrightarrow 2$ given by $E_x(f) = f(x)$) so $K_x = \{f_e \in B(R) | f_e(x) = 0\}$. These are maximal ideals for every $x \in X$ and so $x \longmapsto K_x$ is a map from X to $X_R$. If $x \neq y$ in X then $K_x \neq K_y$ (for instance, let $e = X - \{x\}$ which is clopen, then $f_e \in K_x$ but $f_e \notin K_y$) so the map is one-to-one.

To see that every element of $X_R$ can be so obtained, suppose M is a maximal ideal of B(R) and that $M \neq K_x$ for any $x \in X$. Then, at each $x \in X$, there exists an $f^{(x)} \in M$ such that $f^{(x)}(x) = 1$. By definition, $f^{(x)} = 1$ on a clopen set $e^{(x)}$ and zero off of $e^{(x)}$. The sets $\{e^{(x)}\}$ cover X so we can find a finite disjoint subcover $\{e^{(x_i)}\}_{i=1}^{n}$ where $f^{(x_i)} = 1$ on each $e^{(x_i)}$ and zero elsewhere. It follows that

$f = f^{(x_1)} + \cdots + f^{(x_n)}$ is in M and this is the unit of B(R) so M = B(R) if $M \neq K_x$ for some $x \in X$. The map $x \longmapsto K_x$ is therefore one-to-one and onto.

The basic clopen sets of $X_R$ are all of the form $U = \Gamma(f_e) = \{K_x \in X_R | f_e \notin K_x\} = [K_x \in X_R | \{x\} \subseteq e]$. The inverse image of U is e which is clopen so the map is continuous. We can conclude that X and $X_R$ are homeomorphic. So, if $I \subseteq B(R)$ is an ideal of B(R) then its support in $X_R$ is an open set which, as we saw in the last example, must be clopen or the disjoint union of clopens. Therefore B(R) is hereditary. If it turns out that R is regular then (3.32) will yield that R is hereditary.

R will be regular, by (3.11), if the stalk at each point of $X_R$ is a field. For $K_x \in X_R$ the stalk is $R/RK_x = R/\bar{K}_x$. Consider $g \in \bar{K}_x$ so $g = f \cdot f_e$ for some $f \in R$ and $f_e \in K_x$. We have $g(x) = f(x)f_e(x) = f(x) \cdot 0 = 0$ so g is in the kernel of the evaluation homomorphism $\hat{E}_x\colon R \longrightarrow F$. Therefore, $\bar{K}_x$ is contained in this maximal ideal. In the other direction, assume g is in $Ker(\hat{E}_x)$ so $g \in R$ and $g(x) = 0$. There are elements $k_i$ in F such that $g = k_1 f_{e_1} + \cdots + k_n f_{e_n}$ for disjoint clopen sets $e_i$ that cover X. Since $x \in e_j$ for precisely one of the $e_j$ we have $0 = g(x) = k_j \cdot f_{e_j}(x) = k_j$ so $g = \sum_{i \neq j} k_i f_{e_i}$ and {x} is not contained in $e_i$ for each i. If we let $e = \sum_{i \neq j} e_i$ then $g = g \cdot f_e$ and $f_e \in K_x$. Therefore, $Ker(\hat{E}_x) \subseteq \bar{K}_x$ and the two are equal giving that $\bar{K}_x$ is maximal and the stalk at $K_x$ is consequently a field and R is regular. (Since the evaluation homomorphism is onto, each stalk is a copy of F.)

Less can be said if the structure on F is weakened. If we suppose that F is an integral domain that is not a field then R is neither hereditary nor $\aleph_0$-hereditary. To see this let $k \neq 0$ be a non-unit of F. The constant function $f(x) = k$ is in R and the image of f is not invertible on any of the stalks of $X_R$ so (c) of (3.30) fails.

If F is a Prüfer domain which is not field then each stalk is Prüfer (a copy of F). That R is p.p. can be shown by considering the annihilator of $f \in R$. By (3.21) then, R will be semihereditary.

(The last two assertions assume that X is infinite. If X is finite then R will be hereditary.)

Finally, it should be mentioned that if X is *any* Boolean space whose associated Boolean ring is hereditary then an exactly parallel argument produces the same conclusions about the ring of functions R. X was chosen to be the one-point-compactification of a discrete space only for concreteness.

# APPENDIX

In this appendix we prove the result of I. Kaplansky cited in the proof of (2.9): projective modules over commutative semihereditary rings are direct sums of modules, each isomorphic to a finitely generated ideal.

The proof is complex and rests on four main facts. These will be listed first as lemmas and used to demonstrate the main result before turning to their proofs.

In the appendix, R is commutative only if so stated. When R is not commutative, R-module means right R-module.

A.1 *Lemma* ([7], theorem 1). If an R-module M is a direct sum of countably generated R-modules then any direct summand of M is itself a direct sum of countably generated R-modules.

A.2 *Lemma* ([7], lemma 1). Let M be a countably generated R-module such that any direct summand N of M has the following property: any element of N can be embedded in a free (resp. finitely generated) direct summand of N; then M is free (resp. a direct sum of finitely generated modules).

A.3 *Lemma* ([7], lemma 3). Let R be a commutative semihereditary ring and P a projective R-module, then any element of P can be embedded in a finitely generated direct summand of P.

A.4 *Lemma* ([2], p.14). If R is right semihereditary then

*Proof of lemma* A.1 ([7], [11] p10). Let $M = \oplus_{\Lambda} M_{\lambda}$ where each $M_{\lambda}$ is countably generated and suppose that $M = P \oplus Q$. We show that P is a direct sum of countably generated modules. This will be done by constructing, via transfinite induction, a well-ordered family of submodules, $\{S_{\alpha}\}$, of M with the following properties:

(1) If $\beta < \alpha$ then $S_{\beta} \subset S_{\alpha}$

(2) $M = \bigcup_{\alpha} S_{\alpha}$

(3) If $\alpha$ is a limiting ordinal then $S_{\alpha} = \bigcup_{\beta<\alpha} S_{\beta}$

(4) $S_{\alpha+1}/S_{\alpha}$ is countably generated

(5) $S_{\alpha} = P_{\alpha} \oplus Q_{\alpha}$ where $P_{\alpha} = P \cap S_{\alpha}$, $Q_{\alpha} = Q \cap S_{\alpha}$

(6) $S_{\alpha}$ is a direct sum of the $M_{\lambda}$'s for a suitable subset of $\Lambda$.

Assume first that such a family is at hand. A quick check yields $S_{\alpha+1}/S_{\alpha} = P_{\alpha+1}/P_{\alpha} \oplus Q_{\alpha+1}/Q_{\alpha}$ and hence $P_{\alpha+1}/P_{\alpha}$ is a homomorphic image of a countably generated module and so is likewise countably generated. Further, $P_{\alpha}$ is a direct summand of $S_{\alpha}$ which is a direct summand of M so $P_{\alpha}$ is a direct summand of M. Since $P_{\alpha} \subset P_{\alpha+1}$ which is also a direct summand of M we know that $P_{\alpha}$ is a direct summand of $P_{\alpha+1}$ by (A.6). Therefore, $P_{\alpha+1} = P_{\alpha} \oplus P'_{\alpha+1}$ and $P'_{\alpha+1}$ is countably generated. We claim that for any $\alpha$, $P_{\alpha} = \oplus_{\beta\leq\alpha} P'_{\beta}$. This is shown by transfinite induction. It is clearly true for the least $P_{\alpha}$: $P_0 = (0)$. Assume the claim for all $P_{\beta}$ with $\beta < \alpha$. If $\alpha$ is a limiting ordinal then set $P'_{\alpha} = (0)$:

$$P_{\alpha} = P \cap S_{\alpha} = P \cap (\bigcup_{\beta<\alpha} S_{\beta}) = \bigcup_{\beta<\alpha} (P \cap S_{\beta}) = \bigcup_{\beta<\alpha} P_{\beta}$$

$$= \bigcup_{\beta<\alpha} [ \bigoplus_{\gamma<\beta} P'_\gamma ] = \bigoplus_{\beta<\alpha} P'_\beta = \bigoplus_{\beta<\alpha} P'_\beta \oplus P'_\alpha = \bigoplus_{\beta\leq\alpha} P'_\alpha$$

since $P'_\alpha = (0)$.

If $\alpha$ is not a limit ordinal then $\alpha = \beta + 1$ for some $\beta$. Then $P_{\beta+1} = P_\beta \oplus P'_{\beta+1} = ( \bigoplus_{\gamma\leq\beta} P'_\gamma) \oplus P'_{\beta+1}$ (by the induction)

$= \bigoplus_{\gamma\leq\beta+1} P'_\gamma$ and this establishes the claim.

Now, $P = P \cap S = P \cap (\bigcup_\alpha S_\alpha) = \bigcup_\alpha (P \cap S_\alpha)$

$= \bigcup_\alpha P_\alpha = \bigcup_\alpha ( \bigoplus_{\beta\leq\alpha} P'_\alpha) = \bigoplus_\alpha P'_\alpha$ and each $P'_\alpha$ is countably generated as desired.

To complete the proof we construct the well-ordered family $\{S_\alpha\}$. Let $S_0 = (0)$ be the least element. For any ordinal $\alpha$ assume that we have $S_\beta$ for all $\beta < \alpha$. If $\alpha$ is a limiting ordinal then set $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$.

If $\alpha$ is not a limiting ordinal then $\alpha$ has the form $\alpha = \beta + 1$. Let $Q_1$ be any of the $M_\lambda$'s not contained in $S_{\beta+1}$ (if there is no such $M_\lambda$ then the induction stops: $S_\beta = S$). $Q_1$ is countably generated by, say $\{x_{11}, x_{12}, x_{13}, \ldots\}$, and this will be the first row of an infinite matrix. Write $x_{11} = p_{11} + q_{11}$ where $p_{11} \in P$, $q_{11} \in Q$. Let $Q_2$ be the direct sum of the finitely many $M_\lambda$'s needed to express $p_{11}$ and $q_{11}$ in the sum $M = \bigoplus_\Lambda M_\lambda$. $Q_2$ is countably generated by, say $\{x_{21}, x_{22}, x_{23}, \ldots\}$, and let this be row two of the matrix. Next decompose $x_{12}$: $x_{12} = p_{12} + q_{12}$. Let $Q_3$ be the direct sum of $M_\lambda$'s needed to express $p_{12}$ and $q_{12}$ in $M = \bigoplus_\Lambda M_\lambda$. $Q_3 = \langle x_{31}, x_{32}, X_{33}, \ldots \rangle$ and let these generators be the third row of the matrix. Continue this process, pursuing the elements of the matrix along successive diagonals, that is, in the order $x_{11}$, $x_{12}$, $x_{21}$, $x_{13}$, $x_{22}$, $x_{31}$, $x_{14}$, .... The matrix thus produced has countably many elements $x_{ij}$. Let $<x_{ij}>$ be the R-module generated by them and set $S_{\beta+1} = S_\beta + <x_{ij}>$.

That $\{S_\alpha\}$ so constructed has all the properties required is immediate except possibly property (5). We will check this one and leave the rest.

We want $S_\alpha = P_\alpha \oplus Q_\alpha$. That the sum is direct is immediate from the definition of $P_\alpha$ and $Q_\alpha$. Also immediate is that $P_\alpha + Q_\alpha \subset S_\alpha$ for all $\alpha$ and we need only demonstrate the opposite containment.

Suppose first that $\alpha$ is a limiting ordinal. $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ so $x \in S_\alpha$ implies $x \in S_\beta$ for some $\beta < \alpha$. Therefore, $x \in P_\beta \oplus Q_\beta$ by induction and $P_\beta = P \cap S_\beta \subset P \cap S_\alpha = P_\alpha$ since $S_\beta \subset S_\alpha$. Similarly, $Q_\beta \subset Q_\alpha$ so $S_\alpha \subset P_\alpha + Q_\alpha$ and we have the case for $\alpha$ a limiting ordinal.

Suppose $\alpha = \beta + 1$. $S_{\beta+1} = S_\beta + <x_{ij}> =$ $(P_\beta \oplus Q_\beta) + <x_{ij}>$ by the induction. If $x \in S_{\beta+1}$ then $x = p + q + \sum r_{ij}x_{ij}$ where $p \in P_\beta \subset P_{\beta+1}$, $q \in Q_\beta \subset Q_{\beta+1}$ and $\sum r_{ij}x_{ij}$ is a finite sum involving the generators $x_{ij}$. Consider any one of the $x_{ij}$. We can write $x_{ij} = p_{ij} + q_{ij}$ for $p_{ij} \in P$, $q_{ij} \in Q$. By the construction of $S_{\beta+1}$ there is a module $Q_k \subset S_{\beta+1}$ which is the direct sum of the $M_\lambda$'s needed to express $p_{ij}$, $q_{ij}$ in $M = \bigoplus_\Lambda M_\lambda$ and $Q_k$ has generators $\{x_{k1}, x_{k2}, x_{k3}, \ldots\}$ which are included in $<x_{ij}>$. Therefore, $x_{ij} = p_{ij} + q_{ij}$ where $p_{ij}$ and $q_{ij}$ can each be rewritten as a finite R-linear combination of generators $x_{i'j'}$ contained in $\langle x_{ij} \rangle$. That is, $x_{ij} = p_{ij} + q_{ij}$ where $p_{ij} \in P \cap S_{\beta+1}$, $q_{ij} \in Q \cap S_{\beta+1}$ and the statement follows from this.$_\square$

*Proof of lemma* A.2. Let $\{x_1, x_2, x_3, \ldots\}$ be a generating set for M and let $N_1$ be any direct summand of M containing $x_1$: $M = N_1 \oplus N_1'$. By hypothesis, there is a free module $F_1$ such that $N_1 = F_1 \oplus N_1''$ and $x_1 \in F_1$ and we can write $M = F_1 \oplus (N_1' \oplus N_1'')$. If $F_1$ contains all $\{x_1, x_2, \ldots\}$ then $M = F_1$ (and $N_1' \oplus N_1'' = 0$) and we can stop. Otherwise, consider the "next" generator of M that is not in $F_1$. Without loss of generality, suppose that $x_2 \notin F_1$. Write $x_2$ in its $F_1$, $N_1'$, $N_1''$ components: $x_2 = f_1 + n_1' + n_1''$. There exists a free module $F_2$ such that $N_1' = F_2 \oplus N_2$ and $n_1' \in F_2$ and a free module $F_3$ such that $N_1'' = F_3 \oplus N_3$ and $n_1'' \in F_3$. Therefore, $M = F \oplus N_2 \oplus N_3$ where $F = F_1 \oplus F_2 \oplus F_3$ is free and $x_1R + x_2R \subset F$. If F contains all $\{x_1, x_2, x_3, \ldots\}$ then it follows that $M = F$ is free (and $N_2 \oplus N_3 = 0$). If not, consider the "next" generator not contained in F and repeat the process.

After, at most, countably many steps we will have written M as a sum of free modules.

The proof of the parenthetic statement is parallel.$_\square$

*Proof of lemma* A.3 ([7]). Let P be projective over R commutative and semihereditary and let x be any element of P. There exists a free module F with a basis $\{u_i\}^I$ such that $F \simeq P \oplus Q$. Let $S = \{y \in F | by = cx$ for some $b, c \in R$ and b a non-zero divisor}. S is a submodule of F containing x. It is also contained in P. To see this let $s \in S$ and write $s = p + q$ where $p \in P$, $q \in Q$. There is a non-zero divisor $b \in R$ such that $bs = cx$ so we can write $bs = bp + bq = cx$ and $cx \in P$. This implies that $bq = 0$ and so $q = 0$ (write q in terms of F components and recall that b is a non-zero divisor). Hence, $s = p \in P$ and S is a submodule of P containing x. We will show that S is a finitely generated direct summand of F, and therefore of P, yielding the result.

Suppose $x = a_1u_1 + \cdots + a_nu_n$ in F. Let $G = Ru_1 \oplus \cdots \oplus Ru_n$, the free submodule generated by $u_1, \ldots, u_n$. We first claim that $S \leq G$. For this let $y \in S$ so $by = cx$ for a non-zero divisor $b \in R$ and we have $by = ca_1u_1 + \cdots + ca_nu_n$. In F, $y = \sum_{i=1}^{r} d_iu_i$ where $r \geq n$. Hence: $by = bd_1u_1 + \cdots + bd_nu_n + \cdots + bd_ru_r = \sum_{i=1}^{n} ca_iu_i$ and this implies that $bd_k = 0 \;\forall\; k > n$ so $d_k = 0 \;\forall\; k > n$ since b is a non-zero divisor. It follows that $y \in G$, establishing the claim.

We will now proceed with the proof under the assumption that at least one of the R coefficients of x is a non-zero divisor in R and show, after the fact, that this is a reasonable assumption. So, without loss of generality, assume that $a_1$ is a non-zero divisor.

Let $y = \sum_{i=1}^{n} d_iu_i$ be any element of S. We know $by = cx$ so $by = \sum_{i=1}^{n} bd_iu_i = \sum_{i=1}^{n} ca_iu_i$. Because $a_1$ is a non-zero divisor $ca_1 \neq 0$, so $bd_1 \neq 0$ either and this implies that

$d_1 \neq 0$ since b is a non-zero divisor. Thus, every element of S has a non-zero coefficient of $u_1$.

We now claim that the factor module $\bar{G} = G/S$ is projective. Let $v_i = \bar{u}_i = u_i + S$ and set $\bar{H} = Rv_2 + \cdots + Rv_n$. $\bar{H}$ is a free submodule of $\bar{G}$ since if $\sum_{i=2}^{n} r_i v_i = 0$ then $\sum_{i=2}^{n} r_i u_i$ is in S but any non-zero element of S must have a non-zero coefficient of $u_1$ so it must be that $r_2 = \cdots = r_n = 0$ and $v_2, \ldots, v_n$ is a basis for $\bar{H}$.

If $t \in F$ and $a_1 t \in S$ then $t \in S$ because if $b(a_1 t) = cx$ where b is a non-zero divisor then $(ba_1)t = cx$ and $ba_1$ is a non-zero divisor. Define $\mu : \bar{G} \longrightarrow \bar{G}$ by $\mu(\bar{g}) = a_1\bar{g}$. If $\bar{g} = \sum_{i=1}^{n} r_i v_i$ then $a_1\bar{g} = 0 \Leftrightarrow a_1 \sum_{i=1}^{n} r_i u_i \in S \Leftrightarrow \sum_{i=1}^{n} r_i u_i \in S$ and so $\mu$ is an injection. Actually, $\mathrm{Im}(\mu) \subseteq \bar{H}$. To see this suppose that $\sum_{i=1}^{n} a_i u_i \in S$ so $\sum_{i=1}^{n} a_i v_i = 0$ in $\bar{G}$, then $a_1 v_1 = -a_2 v_2 - \cdots - a_n v_n \in \bar{H}$. Therefore, if $\bar{g} = \sum_{i=1}^{n} r_i v_i$ then $\mu(\bar{g}) = \sum_{i=1}^{n} r_i a_1 v_i \in \bar{H}$ since $r_i a_1 v_i \in \bar{H}$. This means that $\mu(\bar{G})$, an isomorphic copy of $\bar{G}$, is a finitely generated submodule of $\bar{H}$ which is free and (A.4) states that $\mu(\bar{G})$ is then a sum of projectives so $\bar{G} = G/S$ is projective as claimed.

Since G/S is projective, the exact sequence $0 \longrightarrow S \xrightarrow{\psi} G \xrightarrow{\pi} G/S$ splits (where $\psi$ is the inclusion map and $\pi$ is the canonical map) and we can state that $G \simeq S \oplus G/S$. G is finitely generated so S is and we have S as a finitely generated direct summand of F. $S \subseteq P$ which is also a direct summand of F so S is a direct summand of P by (A.6) and the result is proved.□

The assumption that $a_1$ is a non-zero divisor in the above must now be justified. To this end suppose that $a_1$ is in fact a zero-divisor. Since R is p.p., $(a_1)^* = (e_1)^* = (1 - e_1)R$ for some $e_1 \in B(R)$ and $R = e_1R \oplus (1 - e)R \simeq a_1R \oplus (1 - e_1)R$. In the ring $e_1R$, $a_1 = a_1e_1$ is a non-zero divisor. With notation as above, $e_1F$ is a free $(e_1R)$-module with basis $(e_1u_i)^I$ and $e_1F \simeq e_1P \oplus e_1Q$ and $e_1x = \sum_{i=1}^{n} a_ie_1u_i$ is in $e_1P$. We have the result for $e_1x \in e_1P$ so there exists a finitely generated module $A \subset e_1P$ such that $e_1P \simeq A \oplus B$ and $e_1x$ is embedded in A.

$(1 - e_1)x = \sum_{i=1}^{n} (1 - e_1)a_iu_i = \sum_{i=2}^{n} (1 - e_1)a_iu_i$ since $(1 - e_1) \in (a_1)^*$. If $(1 - e_1)$ also annihilates $a_2, \ldots, a_n$ then $(1 - e_1)x = 0$ and $e_1x = x$. This would give $x \in A$ and $P \simeq e_1P \oplus (1 - e_1)P \simeq A \oplus B \oplus (1 - e_1)P$ and we can stop.

On the other hand, suppose, without loss of generality, that $(1 - e_1)a_2 \neq 0$. Then $((1 - e_1)a_2)^* = (e_2)^* = (1 - e_2)R$ for some $e_2 \in B(R)$. $[e_1 \in ((1 - e_1)a_2)^*] \Rightarrow [e_1R \leq (1 - e_2)R] \Rightarrow [e_1 = e_1(1 - e_2)] \Rightarrow e_1e_2 = 0$. Then $e_2(1 - e_1)x = \sum_{i=2}^{n} e_2(1 - e_1)a_iu_i \in e_2P$ and $(1 - e_1)a_2 = e_2(1 - e_1)a_2$ is a non-zero divisor in $e_2R$. The foregoing argument applies to $e_2(1 - e_1)x$ to give that $e_2P \simeq A' \oplus B'$ where $A'$ is finitely generated and $e_2(1 - e_1)x \in A'$.

$(1 - e_2)(1 - e_1)x = \sum_{i=2}^{n} (1 - e_2)(1 - e_1)a_iu_i = \sum_{i=3}^{n} (1 - e_2)(1 - e_1)a_iu_i$. If $(1 - e_2)$ annihilates $a_3, \ldots, a_n$

then $(1 - e_2)(1 - e_1)x = (1 - e_1 - e_2)x = 0$ and $x = e_1x + e_2x = e_1x + (1 - e_1)e_2x$ with $e_1x \in e_1P$ and $(1 - e_1)e_2x \in e_2P$. $P \simeq e_1P \oplus e_2P \oplus (1 - e_1 - e_2)P \simeq A \oplus B \oplus A' \oplus B' \oplus (1 - e_1 - e_2)P$ and we have x embedded in $A \oplus A'$, a finitely generated direct summand of P. If $(1 - e_2)$ does not annihilate $a_3,\dots,a_n$ we need only repeat the process a finite number of times to achieve the result and thus the assumption that $a_1$ is a non-zero divisor is a reasonable one.

*Proof of Lemma* A.4 ([2]). Let F be free over R with basis $(x_i)^I$ and $A \subset F$ a finitely generated submodule. A is contained in a finite free summand of F, say $A \subset x_1R \oplus\cdots\oplus x_nR = F'$. Let B be the submodule consisting of all elements in A that can be written in terms of $x_1,\dots,x_{n-1}$. Then any $a \in A$ has a unique representation $a = x_n\lambda + b$; $b \in B$, $\lambda \in R$.

Define $\varphi: A \longrightarrow R$ by $\varphi(a) = \lambda$. $\varphi$ maps A onto $I = \varphi(A)$, a right ideal of R

We proceed by induction on n, the number of generators of F'.

I is a finitely generated R-submodule of R, being the homomorphic image of a finitely generated module and, since R is right semihereditary, I is projective. It follows that the exact sequence $0 \longrightarrow B \xrightarrow{\psi} A \xrightarrow{\varphi} I \longrightarrow 0$, where $\psi$ is the inclusion map, is split exact. This implies that $A \simeq B \oplus I$.

If $n = 1$ then $B = 0$ and we have the result. Assume its truth for less than n generators. $A \simeq B \oplus I \Rightarrow B \simeq A/I$ so B is finitely generated since A is. We have B, then, as a finitely generated submodule of a free R-module on (n - 1) generators and the induction applies to B. Therefore, $B \simeq B_1 \oplus \cdots \oplus B_r$ where $B_j \simeq I_j$, a finitely generated right ideal of R. This gives us that $A \simeq B_1 \oplus \cdots \oplus B_r \oplus I$.□

With these proofs the purpose of the appendix is accomplished. Out of interest we present some further results contained in [7].

If R is an integral domain then R is p.p. since any principal ideal is, as an R-module, an isomorphic copy of R. An integral domain in which finitely generated ideals are principal is, in consequence, semihereditary. Therefore, as a corollary to theorem A.5 we have

A.7 *Corollary* ([7]). If R is an integral domain in which finitely generated ideals are principal then any projective R-module is free.

*Proof*. If P is projective then, since R is semihereditary, $P \simeq \bigoplus_I P_i$ where each $P_i$ is isomorphic to a finitely generated ideal of R, say $P_i \simeq J_i$, by (A.5). $J_i = a_iR$ for some $a_i \in R$ so $P \simeq \bigoplus_I a_iR \simeq \bigoplus_I R^{(1)}$.□

A.8 *Lemma* ([7], lemma 4). If P is a projective module over a regular ring R, R not necessarily commutative, then any finitely generated submodule of P is a direct summand of P.

*Proof.* If P is projective then there exists a free module F such that $F \simeq P \oplus Q$. Let $A \subset P$ be finitely generated. Lemma A.4 applies to A: $A \simeq \bigoplus_{i=1}^{n} B_i$ where each $B_i \simeq J_i$, a finitely generated right ideal of R. R is regular so $J_i \simeq e_iR$ for some idempotent $e_i$ and we have $A \simeq \oplus\, e_iR$;
$R \simeq e_iR \oplus (1 - e_i)R$. F is free so F is isomorphic to the direct sum of copies of R:

$$F \simeq \bigoplus_I R^{(I)} \text{ for some index set I.}$$
$$\simeq [e_1R \oplus (1 - e_1)R] \oplus\cdots\oplus [e_nR \oplus (1 - e_n)R] \oplus [\bigoplus_{I'} R^{(I')}]$$
$$\simeq [e_1R \oplus\cdots\oplus e_nR] \oplus \cdots \quad \simeq A \oplus \cdots$$

and A is a direct summand of F. Since P is also a direct sumand of F and $A \subset P$ we have A as a direct summand of P. □

A.9 *Theorem* ([7], theorem 4). Let R be a regular ring, not necessarily commutative, and P a projective right R-module; then P is a direct sum of right R-modules each of which is isomorphic to a principal right ideal.

*Sketch of proof.* Apply lemmas (A.1) and (A.8). □

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