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Algebraic Closures for Commutative Rings

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ABSTRACT

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This thesis is a study of the paper [18] by R. Raphael. Algebraic extensions of commutative semiprime rings are introduced and discussed leading to a characterization of algebraically closed regular rings. The equivalence of algebraically closed rings, totally integrally closed rings [8] and semiprime saturably closed rings [2] is established.

As applications, algebraic closures obtained in [18] for some rings of continuous functions and for some group rings are presented.

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CONTENTS

LIST OF SYMBOLS

$\mathbb{Z}$	ring of integers
$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
1	identity element of a ring
ϕ	empty set
$R \setminus I$	complement of I in R
$f\|S$	restriction of f to S
R/I	residue class ring
$A \times B$	Cartesian product of A and B.
$\langle r \rangle$	principal ideal generated by r
$R[x]$	ring of polynomials in x over R
rad R	prime radical of R
Rad R	Jacobson radical of R
Ker f	kernel of the mapping f
$\mathrm{Hom}_R(A, B)$	R-module homomorphisms from A to B
$Q(R)$	complete ring of quotients of R
$q^{-1}R$	page 13
$a \wedge b$	infimum of a and b
$a \vee b$	supremum of a and b
a'	complement of a
$B(R)$	set of idempotents of R
$\prod_{i \in I} R_i$	direct product of rings R_i
K^*	annihilator of K
IS	extension of I to S

Spec R	prime spectrum of R
V(A)	page 42
$\Omega(R)$	algebraic closure of R
R_S	localization of R at S
AG	page 75
ωH	page 76
C(X)	page 82
Z(f)	page 82
coz f	page 82
cl V	page 83
Hom D	page 83
$Q_R(X)$	page 82
$Q_C(X)$	page 87
$\varinjlim A_i$	page 89
$C[S]$	page 89
$V_o(X)$	page 90
$C^*(X)$	page 91
βX	page 91
G_δ	page 91
$G_o(X)$	page 91
$\overline{Q}_R(X)$	page 92
$\overline{Q}_C(X)$	page 92
L(V)	page 92
${}_RQ_L(X)$	page 92
${}_CQ_L$	page 92
C(X, F)	page 93
d(f)	page 107
$F_s(R)$	page 113

INTRODUCTION

An algebraic extension of a commutative ring R is defined in [18] to be a ring extension that is integral and essential over R. These extensions coincide in fields. Algebraic extensions of commutative semiprime rings were studied independently by E. Enochs [8] and R. Raphael [18], proceeding from different points of view. The resulting equivalent closures are unique up to isomorphism and have a universal property. The rings that are regular in the sense of von Neumann are of main interest in [18] where they are shown to play the same role with respect to semiprime rings as do fields with respect to integral domains.

A further investigation of algebraic extensions was undertaken by W. Borho [2], yielding a closure for arbitrary commutative rings that agrees with the algebraic closure whenever the ground ring is semiprime but is not unique in general.

In this thesis we study the paper [18] by R. Raphael. The totally integrally closed rings [8] due to Enochs are presented as are the saturably closed rings defined by Borho in [2]. To make this work self contained other results from commutative algebra are proved. We have taken the liberty to expand some of the proofs. *Lectures on Rings and Modules* [14] by J. Lambek was used as the main reference.

In what follows all rings are assumed to be commutative with 1 and all homomorphisms are 1-preserving.

CHAPTER 1

ESSENTIAL EXTENSIONS OF RINGS

1. Essential extensions.

1.1 Definition. Let R and S be rings and let R be a subring of S . If every non-zero ideal of S intersects R in a non-zero ideal we shall say that S is an essential extension of R , or that S is essential over R . An embedding of rings $m: R \to S$ will be called essential if S is essential over $m(R)$.

1.2 Proposition. Let R and S be rings and let R be a subring of S . Then the following statements are equivalent:

(1) S is an essential extension of R ;

(2) every non-zero principal ideal of S intersects R in a non-zero ideal;

(3) for any $s \neq 0$ in S there is a $t \in S$ such that $st \in R$ and $st \neq 0$;

(4) a ring homomorphism with domain S is a monomorphism if and only if its restriction to R is a monomorphism.

Proof. Clearly (1) implies (2). Suppose (2) holds and that sS is a non-zero principal ideal in S . By (2) $sS \cap R$ is a non-zero ideal of R , hence there is a $t \in S$ such that $st \in sS \cap R$ and $st \neq 0$. Therefore (2) implies (3).

Assume (3). Let f be a ring homomorphism with domain S and f' its restriction to R . Suppose f' is a monomorphism. If f has non-zero kernel there is an $s \in S$, $s \neq 0$, which is mapped by f into zero. By (3) there is a $t \in S$ such that $st \in R$ and

$st \neq 0$. But $f(st) = 0$, hence $f'(st) = 0$ and since f' is a monomorphism it follows that $st = 0$, a contradiction. Therefore f is a monomorphism. Conversely, if f is a monomorphism and r belongs to $Ker(f')$, then $f(r) = f'(r) = 0$, hence $r = 0$. Thus f' is a monomorphism and so (3) implies (4).

Finally assume (4). Let I be a non-zero ideal of S, and $h: S \to S/I$ the projection from S onto S/I. The restriction of h to R is the homomorphism $r \to r + I$ of R into S/I with kernel $R \cap I$. Since $Ker(h) = I \neq \langle 0 \rangle$ it follows from (4) that the kernel of the restriction is a non-zero ideal whence $R \cap I \neq \langle 0 \rangle$. Thus (4) $\Rightarrow$ (1) and the proof is complete.

1.3 Example. The essential extensions of a field are precisely its overfields.

Proof. Every homomorphism of a field into a non-zero ring is a monomorphism. Thus any overfield of a field F is essential over F by condition (4) above. On the other hand, if S is an essential ring extension of a field F, condition (4) implies that every homomorphism with domain S into a non-zero ring is a monomorphism. Hence S is a field.

1.4 Lemma (Transitivity). Let R, S and T be rings, $R \subseteq S \subseteq T$, let S be an essential extension of R, and let T be an essential extension of S. Then T is essential over R.

Proof. Let I be an ideal of T. If $I \neq \langle 0 \rangle$ then $I \cap S$ is a non-zero ideal of S. Now $I \cap R = I \cap (S \cap R) = (I \cap S) \cap R \neq \langle 0 \rangle$. Thus T is an essential extension of R.

2. Essential extensions of a semiprime ring.

1.5 Definition. The prime radical of a ring R, denoted rad R, is the intersection of all prime ideals of R. A ring is called semiprime if its prime radical is <0>.

1.6 Definition. An element r of a ring R is called nilpotent if $r^n = 0$ for some positive integer n.

1.7 Proposition. The prime radical of a ring R consists of all nilpotent elements of R.

Proof ([14]). Let R be a ring, $r \in R$. If r is nilpotent then $r^n = 0$ for some positive integer n. Consequently r^n belongs to each prime ideal P in R. It follows that $r \in P$ and therefore $r \in \text{rad } R$. Conversely suppose r is not nilpotent. Set $T = \{1, r, r^2, \ldots\}$, this is a multiplicatively closed set which does not contain 0. Let P be an ideal of R maximal with respect to the property that it does not meet T and let a and b be elements of R outside P; then P is properly contained in P + aR and in P + bR. By the maximality of P there are elements $r^m, r^n \in T$ such that $r^m \in P + aR$ and $r^n \in P + bR$, hence $r^{m+n} \in (P + aR)(P + bR) \subseteq P + abR$. Then $ab \notin P$, for otherwise $r^{m+n} \in P$, a contradiction. Hence P is prime and so there is a prime ideal of R which does not contain r. Thus r does not lie in the prime radical. ⌝

1.8 Corollary. A semiprime ring has no non-zero nilpotent elements.

1.9 Definition. Let I be an ideal of the ring R. The radical of I, denoted rad I, is the set of all elements of R having some positive power in I. It follows from the proof of (1.7) that rad I is the intersection of all prime ideals of R which contain I; it is therefore an ideal. An ideal I is said to be semiprime if $I =$ rad I.

1.10 Proposition. The radical of an ideal I is a semiprime ideal.

Proof ([16]). It is clear that rad $I \subseteq$ rad (rad I). Conversely let $r \in$ rad (rad I). Then $r^n \in$ rad I for some positive integer n, hence there is a positive integer m such that $r^{nm} \in I$. Accordingly $r \in$ rad I.

1.11 Proposition. Let R be a subring of the ring S. If R is semiprime, then S is essential over R if every non-zero semiprime ideal of S has non-zero intersection with R.

Proof. Let I be a non-zero ideal of S. Then rad $I \neq \langle 0 \rangle$ since it contains I, hence it is a non-zero semiprime ideal of S. If now rad I has non-zero intersection with R then there is a non-zero $r \in (\text{rad } I) \cap R$ and therefore $r^n \in I \cap R$ for some positive integer n. Furthermore $r^n \neq 0$ since r is a non-zero element of the semiprime ring R.

1.12 Lemma. Let R be a semiprime ring and let S be essential over R. Then S is semiprime.

Proof. Clearly $(\text{rad } S) \cap R = \text{rad } R$. Now rad $R = \langle 0 \rangle$, hence by the essentiality of S over R, rad $S = \langle 0 \rangle$. Thus S is semiprime.

3. The complete ring of quotients of a commutative ring.

The complete ring of quotients will be referred to many times over and is therefore presented in detail. The proofs for this are taken from [14], Section 2.3.

1.13 Definition. An ideal D in a ring R will be called *dense* if, for all $r \in R$, $rD = <0>$ implies $r = 0$. For example all non-zero ideals of the ring Z of integers are dense. On the other hand any ideal generated by a non-zero nilpotent element of a ring R is not dense.

1.14 Proposition. The following are properties of dense ideals:

(1) A ring R is dense.

(2) If D is a dense ideal and $D \subseteq D'$, then D' is dense.

(3) If D and D' are dense ideals, then so are DD' and $D \cap D'$.

(4) If $R \neq <0>$ then $<0>$ is not a dense ideal.

Proof. (1) For all $r \in R$, $rR = <0>$ implies $r = r \cdot 1 = 0$.

(2) If D' is not dense then $rD' = <0>$ for some $r \in R$. Therefore $rD = <0>$.

(3) Let D and D' be dense ideals and $r \in R$ such that $rDD' = <0>$. Then $rdD' = <0>$ for any $d \in D$ and therefore $rd = 0$. Thus $rD = <0>$ and hence $r = 0$. It follows that DD' is a dense ideal and since $DD' \subseteq D \cap D'$ the latter is also dense by (2).

(4) If $R \neq <0>$ there exists a non-zero r in R such that $r<0> = <0>$.

1.15 Definition. Let R be a ring. By a fraction is meant an element $f \in \mathrm{Hom}_R(D, R)$, the set of R-module homomorphisms of D to R, where D is any dense ideal. Thus f is a group homomorphism of D into R such that $f(dr) = (fd)r$ for any $d \in D$ and $r \in R$. The zero fraction and the identity fraction will be given so as to admit as domains the ring R itself, that is, $0,1 \in \mathrm{Hom}_R(R, R)$ are introduced by writing $0r = 0$, $1r = r$, for all $r \in R$.

Addition and multiplication of fractions $f_i \in \mathrm{Hom}_R(D_i, R)$, $i = 1,2$, are defined thus:

$f_1 + f_2 \in \mathrm{Hom}_R(D_1 \cap D_2, R)$, $(f_1 + f_2)d = f_1d + f_2d$,

$f_1f_2 \in \mathrm{Hom}_R(f_2^{-1}D_1, R)$, $(f_1f_2)d = f_1(f_2d)$.

Here $f_2^{-1}D_1 = \{r \in R \mid f_2r \in D_1\}$. This is a dense ideal since $f_2(D_2D_1) \subseteq D_1$, that is, $D_2D_1 \subseteq f_2^{-1}D_1$.

The set of fractions associated with a ring R thus form a system $(F,0,1,-,+,\cdot)$ where $(F,0,+)$ is an Abelian semigroup with identity element 0 and $(F,1,\cdot)$ is a semigroup with identity element 1. If f is an element of F having domain $D \neq R$, then $[f + (-f)]$ is not identically the 0 fraction since $[f + (-f)]$ has domain D and 0 has domain R. Thus f does not have an inverse relative to addition and therefore $(F,0,1,-,+,\cdot)$ is not a ring. It will be shown that an equivalence relation can be defined on F so that the set of all equivalence classes forms a ring.

1.16 Definition. Let C be the class of all systems sharing the set of operations and satisfying the set of identities of $(F,0,1,-,+,\cdot)$. An equivalence relation θ on a system F belonging to C is called a congruence relation if $f_1 \,\theta\, g_1$ and $f_2 \,\theta\, g_2$ imply $(-f_1) \,\theta\, (-g_1)$,

$(f_1 + f_2)\ \theta\ (g_1 + g_2)$, $(f_1 f_2)\ \theta\ (g_1 g_2)$ for any $f_1, f_2, g_1, g_2 \in F$. A congruence relation θ on F partitions F into a set F/θ of disjoint equivalence classes. The equivalence class of $f \in F$ is the subset $\theta f = \{f' \in F \mid f'\ \theta\ f\}$.

1.17 Proposition. If θ is a congruence relation on a system F belonging to C , then F/θ belongs to C .

Proof. Suppose θ is a congruence relation on F . Define $0 = \theta 0$, $1 = \theta 1$, $\theta f + \theta g = \theta(f + g)$, $\theta f \theta g = \theta(fg)$ and $-(\theta f) = \theta(-f) = \{f' \in F \mid f'\ \theta\ (-f)\}$.

It then follows from the definition of a congruence relation that the results of these operations depend only on the equivalence classes θf and θg and are independent of the choice of representatives f and g . Clearly θ preserves all operations defined on F , hence F/θ satisfies the set of identities that F satisfies. Thus $F/\theta \in C$.

1.18 Definition. Let D_1 and D_2 be dense ideals and $f_1 \in \mathrm{Hom}_R(D_1, R)$, $f_2 \in \mathrm{Hom}_R(D_2, R)$. By $f_1\ \theta\ f_2$ is meant that f_1 and f_2 agree on the intersection of their domains, that is $f_1 d = f_2 d$ for all $d \in D_1 \cap D_2$.

1.19 Lemma. $f_1\ \theta\ f_2$ if and only if f_1 and f_2 agree on some dense ideal.

Proof. If $f_1\ \theta\ f_2$ then f_1 and f_2 agree on the dense ideal $D_1 \cap D_2$. Conversely suppose f_1 and f_2 agree on the dense ideal D' . Let $d \in D_1 \cap D_2$ and $d' \in D'$. Then $(f_1 d)d' = f_1(dd') = f_2(dd') = (f_2 d)d'$. Hence $(f_1 d - f_2 d)D' = \langle 0 \rangle$ and therefore $f_1 d = f_2 d$ since D' is dense. Thus f_1 and f_2 agree on $D_1 \cap D_2$.

1.20 Lemma. θ is a congruence relation on the system $(F,0,1,-,+,\cdot)$.

Proof. It is clear that $f \theta f$ holds for any $f \in F$ and that $f_1 \theta f_2$ implies $f_2 \theta f_1$ for fractions f_1 and f_2 . Hence θ is reflexive and symmetric. To show θ is transitive suppose $f_1 \theta f_2$ and $f_2 \theta f_3$. Then f_1 agrees with f_2 on $D_1 \cap D_2$ and f_2 agrees with f_3 on $D_2 \cap D_3$, hence $f_1 d = f_3 d$ for any $d \in D_1 \cap D_2 \cap D_3$. This is a dense ideal consequently $f_1 \theta f_3$. Thus θ is an equivalence relation.

Now assume $f_1 \theta f_3$ and $f_2 \theta f_4$. Then $f_1 d = f_3 d$ on $D_1 \cap D_3$, hence $(-f_1)d = -(f_1 d) = -(f_3 d) = (-f_3)d$, therefore $(-f_1) \theta (-f_3)$. Furthermore $(f_1 + f_2) \in \mathrm{Hom}_R(D_1 \cap D_2, R)$ and $(f_3 + f_4) \in \mathrm{Hom}_R(D_3 \cap D_4, R)$ are both defined. Let d belong to $D_1 \cap D_2 \cap D_3 \cap D_4$, then $f_1 d = f_3 d$ and $f_2 d = f_4 d$ hence $(f_1 + f_2)d = f_1 d + f_2 d = f_3 d + f_4 d = (f_3 + f_4)d$ and therefore $(f_1 + f_2) \theta (f_3 + f_4)$. Finally $f_1 f_2 \in \mathrm{Hom}_R(f_2^{-1}D_1, R)$ and $f_3 f_4 \in \mathrm{Hom}_R(f_4^{-1}D_3, R)$ are both defined. Choose $d \in f_2^{-1}D_1 \cap f_4^{-1}D_3$, then $f_2 d \in D_1$, $f_4 d \in D_3$. Since $f_2 \theta f_4$ we have $f_2 d = f_4 d$ in $D_1 \cap D_3$ and from $f_1 \theta f_3$ it follows that $(f_1 f_2)d = f_1(f_2 d) = f_3(f_4 d) = (f_3 f_4)d$. Hence $f_1 f_2$ and $f_3 f_4$ agree on the intersection of their domains and so $(f_1 f_2) \theta (f_3 f_4)$. Thus θ is a congruence relation by (1.16).

1.21 Proposition. If R is a commutative ring, the system $(F,0,1,-,+,\cdot)/\theta = Q(R)$ is also a commutative ring. It extends R and will be called its complete ring of quotients.

Proof. By (1.17) any equation valid in F remains valid in F/θ . It therefore needs to be shown that the equivalence classes form a group under addition and that the distributive and commutative laws are satisfied. To this end suppose f is a fraction with domain D and $d \in D$. Then $[f + (-f)]d = fd - fd = 0d$, hence $[f + (-f)]$ agrees with 0 on D. Thus $\theta f + \theta(-f) = \theta[f + (-f)] = \theta 0$, that is, $\theta(-f)$ is the additive inverse of θf in $Q(R)$. Now let $f_i \in \mathrm{Hom}_R(D_i, R)$, $i = 1,2,3$, and let $d \in D_1 D_2 D_3$. Then $f_2 d \in D_1$, $f_3 d \in D_1$, and since $D_1 D_2 D_3 \subseteq D_2 \cap D_3$ we have $(f_2 + f_3)d = (f_2 d + f_3 d) \in D_1$. Then $[f_1(f_2 + f_3)]d = f_1[(f_2 + f_3)d] = f_1[f_2 d + f_3 d] = f_1(f_2 d) + f_1(f_3 d) = (f_1 f_2)d + (f_1 f_3)d$. Hence $\theta[f_1(f_2 + f_3)] = \theta[(f_1 f_2) + (f_1 f_3)]$. Also $f_1 f_2$ and $f_2 f_1$ agree on $D_1 D_2$. This follows from commutativity in R, for if

$\sum_{i=1}^{n} d_i d_i' \in D_1 D_2$, $(d_i \in D_1,\ d_i' \in D_2)$ then

$$\sum_{i=1}^{n} f_1[f_2(d_i d_i')] = \sum_{i=1}^{n} (f_1 d_i)(f_2 d_i') = \sum_{i=1}^{n} (f_2 d_i')(f_1 d_i) = \sum_{i=1}^{n} f_2[f_1(d_i d_i')] .$$

Finally, with every $r \in R$ one may associate the fraction $r/1$ with domain R which sends any $s \in R$ onto rs. Then the mapping $r \to \theta(r/1)$ of R into $Q(R)$ is a homomorphism since operations on fractions are preserved in $Q(R)$. It is actually a monomorphism, for $r/1 \;\theta\; 0/1$ only if $r/1$ and $0/1$ agree on some dense ideal D, that is, only when $rD = 0$, that is $r = 0$. The mapping $r \to \theta(r/1)$ will be called the <u>canonical monomorphism</u> of R into $Q(R)$. This completes the proof.

1.22 <u>Definition</u>. A fraction is called <u>irreducible</u> if it cannot be extended to a larger domain.

1.23 <u>Proposition</u>. Every equivalence class of fractions contains exactly one irreducible fraction, and this extends all fractions in the class.

Before proving this we state Zorn's lemma: If every simply ordered subset of a nonempty ordered set $(S, \leq)$ has an upper bound in S, then S has at least one maximal element m, maximal in the sense that $m \leq s$ implies $m = s$ for all $s \in S$. Recall that $(S, \leq)$ is an ordered set if $\leq$ satisfies the reflexive, transitive and anti-symmetric laws. An ordered set $(S, \leq)$ is called simply ordered if $a \leq b$ or $b \leq a$ for any two elements a and b in S.

<u>Proof</u>. Set S to be the set of fractions in an equivalence class and define a relation $\leq$ on S by $f_1 \leq f_2$ if $D_1 \subseteq D_2$. Consider a simply ordered family of fractions $\{f_i \mid i \in I\}$ in the equivalence class and let $D = \cup_{i \in I} D_i$. This is an ideal, for if $x, y \in D$ then $x \in D_i$ and $y \in D_j$, where $D_i \subseteq D$, $D_j \subseteq D$. Since $\{D_i \mid i \in I\}$ is simply ordered in R both x and y, and hence their sum, are in D_i or in D_j. Therefore $(x + y) \in D$. Define $f \in \mathrm{Hom}_R(D, R)$ by $fd = f_i d$ when $d \in D_i$. (If $d \in D_i \cap D_j$ then $f_i d = f_j d$ since $f_i \,\theta\, f_j$). Then f is an upper bound to the simply ordered family $\{f_i \mid i \in I\}$. By Zorn's lemma the equivalence class contains at least one irreducible fraction. Now suppose f_1 and f_2 with domains D_1 and D_2 respectively are both irreducible fractions in an equivalence class. Define $f \in \mathrm{Hom}_R(D_1 + D_2, R)$ by $f(d_1 + d_2) = f_1 d_1 + f_2 d_2$. One need only ~~verify~~ that f maps the zero element into zero, all other conditions for ~~f~~ to be a fraction being clearly satisfied. But if $d_1 = -d_2 \in D_1 \cap D_2$ then $f_1 d_1 = f_2 d_1 = f_2(-d_2) = -f_2 d_2$. Thus f is a fraction and so f_1 and f_2 have a common extension in the equivalence class, contrary to the assumption that they are irreducible.

1.24 Proposition. The following statements concerning the ring R are equivalent.

(1) Every irreducible fraction has domain R.

(2) For every fraction f there exists an element $s \in R$ such that $fd = sd$ for all $d \in D$, the domain of f.

(3) $Q(R) \cong R$ canonically

Under any of these conditions R will be called rationally complete.

Proof. (1) ⇒ (2). Suppose every irreducible fraction has domain R. Let f be any fraction, f' its irreducible extension. Since f' has domain R put $f'1 = s$. Then $fd = f'd = f'(1d) = (f'1)d = sd$ for all $d \in D$.

(2) ⇒ (3). Consider any element θf of $Q(R)$. By (2) there exists an element $s \in R$ such that $fd = sd = (s/1)d$ for all $d \in D$, the domain of f. Thus $f \, \theta \, s/1$ and therefore $\theta f = \theta(s/1)$, the canonical image of s in $Q(R)$. It follows that the canonical monomorphism of R into $Q(R)$ is surjective.

(3) ⇒ (1). Assume $Q(R) \cong R$ canonically. Let f be any irreducible fraction, then θf is the image of some $s \in R$ under the canonical isomorphism, that is $\theta f = \theta(s/1)$. Now $s/1$ is irreducible since it has domain R, hence by (1.23) $f = s/1$ and so f has domain R.

Following (1.21) R will now be identified with its canonical image in $Q(R)$. Thus we write $\theta(r/1) = r$.

1.25 Lemma. Let $q \in Q(R)$. Then $q^{-1}R = \{r \in R \mid qr \in R\}$ is a dense ideal.

Proof. Suppose $q = \theta f$, f a fraction with domain D and $d \in D$. Then the fractions $[f(d/1)]$ and $(fd)/1$ agree on R , for $[f(d/1)]$ has domain $\{r \in R \mid (d/1)r \in D\} = R =$ domain $(fd)/1$, and if $r \in R$ then $[f(d/1)]r = f(dr) = (fd)r = (fd/1)r$. It now follows that $qd = \theta f \theta(d/1) = \theta[f(d/1)] = \theta(fd/1) = fd$ for all $d \in D$. Hence $qD \subseteq R$ and therefore $D \subseteq q^{-1}R$. Thus $q^{-1}R$ contains a dense ideal.

1.26 Corollary. An element of $Q(R)$ which annihilates a dense ideal of R is zero.

Proof. Suppose q is a non-zero element of $Q(R)$ and $qD = \langle 0 \rangle$ for some dense ideal D of R. As shown in (1.25) there is an $r \in R$ such that $qr \in R$ and $qr \neq 0$. Therefore $qrD \neq \langle 0 \rangle$, a contradiction.

1.27 Proposition. $Q(R)$ is rationally complete.

Proof. Let ϕ be any fraction over $Q(R)$, K its domain. If k is a non-zero element in K then $\langle 0 \rangle \neq k(k^{-1}R) \subseteq K \cap R$. Set $D = \{r \in K \cap R \mid \phi r \in R\}$ and define $f \in Hom_R(D, R)$ by $fd = \phi d$. It will be shown that a) D is a dense ideal and b) for any $k \in K$, $\phi k = (\theta f)k$. The result will then follow by (1.24), condition 2.

(a) Suppose $rD = \langle 0 \rangle$ for some $r \in R$. Let k be any element of K , then $\phi k \in Q(R)$. Put $D' = k^{-1}R \cap (\phi k)^{-1}R$. This is a dense ideal since both $k^{-1}R$ and $(\phi k)^{-1}R$ are dense by (1.25). Now $kD' \subseteq k(k^{-1}R) \subseteq K \cap R$ and $(\phi k)D' \subseteq \phi k[(\phi k)^{-1}R] \subseteq R$, therefore $\phi(kD') \subseteq R$ and so $kD' \subseteq D$. Thus $(rk)D' = r(kD') \subseteq rD = \langle 0 \rangle$, hence $rk = 0$, since D' is dense, and therefore $rK = 0$. But K is dense, hence $r = 0$.

(b) Let $k \in K$ and $d' \in D' = k^{-1}R \cap (\phi k)^{-1}R$. Because $kD' \subseteq D$ we have $\phi(kd') = f(kd')$. Then $(\phi k)d' = \phi(kd') = f(kd') = (\theta f)kd'$ as in (1.25). Therefore $\phi k - (\theta f)k$ annihilates the dense ideal D' of R and so $\phi k = (\theta f)k$ by (1.26).

1.28 Definition. Let S be a ring. A subgroup D of S may be called dense even if it is not an ideal of S, provided $sD = 0$ implies $s = 0$, for all $s \in S$. If R is a subring of S, then S is called a ring of quotients of R if and only if, for all $s \in S$, $s^{-1}R = \{r \in R \mid sr \in R\}$ is dense in S. Thus S is a ring of quotients of R if and only if, for all s and $t \in S$, $t \neq 0$ implies $t(s^{-1}R) \neq \langle 0 \rangle$.

1.29 Proposition. Let R be a subring of the ring S. Then the following statements are equivalent:

(1) S is a ring of quotients of R.

(2) For all $0 \neq s \in S$, $s^{-1}R$ is a dense ideal of R and $s(s^{-1}R) \neq \langle 0 \rangle$.

(3) There exists a monomorphism of S into $Q(R)$ which induces the canonical monomorphism of R into $Q(R)$.

Proof. (1) $\Rightarrow$ (2). Follows from (1.28).

(2) $\Rightarrow$ (3). Let $s \in S$ and define $\hat{s} \in \mathrm{Hom}_R(s^{-1}R, R)$ by $\hat{s}d = sd$ for all $d \in s^{-1}R$. By (2) $s^{-1}R$ is dense in R, hence $\hat{s}$ is a fraction. The mapping $s \to \theta\hat{s}$ is clearly a homomorphism which induces the canonical monomorphism $r \to \theta\hat{r} = \theta(r/1)$. Its kernel consists of

all $s \in S$ for which $\theta\hat{s} = \theta 0$, that is those s for which $s(s^{-1}R) = \langle 0 \rangle$, that is $s = 0$ by (2). Thus $s \to \theta\hat{s}$ is a monomorphism of S into $Q(R)$.

(3) ⇒ (1). By (3) one may assume $R \subseteq S \subseteq Q(R)$. Let $s = \theta f \in S$ and D the domain of f. Then $D \subseteq s^{-1}R$ as in (1.25). Suppose $t \in S$ and $t(s^{-1}R) = \langle 0 \rangle$. Then $tD = 0$. If $\theta f'$ is the image of t under the monomorphism of S into $Q(R)$ then $(\theta f')D = 0$ and $f' : D \to (\theta f')D$, defined by $f'd = (\theta f')d$, $d \in D$, is a fraction. Now $f'D = 0$, hence $f' \theta 0$ since D is dense, hence $t = \theta f' = 0$.

1.30 <u>Corollary</u>. If S is a ring of quotients of the ring R and D is a dense ideal of R then D is dense in S.

<u>Proof</u>. If $s = \theta f \in S$ and $sD = \langle 0 \rangle$, then $\theta f = \theta 0$ as was just shown in the proof of part (3) above.

1.31 <u>Definition</u>. Let S be an extension of a ring R and let $\sigma : R \to T$ be an embedding of rings. An embedding τ of S in T will be said to be <u>over σ</u> if the restriction of τ to R is equal to σ. We also say that τ extends σ. An embedding $\tau : S \to T$ is called <u>over R</u> if it induces the identity on R. By (1.29), $Q(R)$ contains a copy over R of any ring of quotients of R.

1.32 <u>Proposition</u>. Up to isomorphism over R, $Q(R)$ is the only rationally complete ring of quotients of a ring R.

<u>Proof</u>. Let S be any ring of quotients of R. In view of the last proposition one may write $R \subseteq S \subseteq Q(R)$. Let $q \in Q(R)$ and $D = \{s \in S \mid qs \in S\}$. This is a dense ideal of S since it contains

$\{r \in R \mid qr \in R\} = q^{-1}R$, which is dense in S by (1.30). Thus the mapping $d \to qd \in \mathrm{Hom}_S(D, S)$ is a fraction f . Now suppose S is rationally complete. By (1.24) there exists an $s \in S$ such that $qd = fd = sd$ for all $d \in D$, hence $(q-s)D = \langle 0 \rangle$ and so $(q-s)q^{-1}R = \langle 0 \rangle$. But $q^{-1}R$ is dense in $Q(R)$ by (1.26) hence $q = s$. Therefore $Q(R) = S$.

1.33 Example. Any ring of quotients of a commutative ring R is essential over R .

Proof. Let S be a ring of quotients of R . Then for all $0 \neq s$, $s(s^{-1}R) \neq \langle 0 \rangle$. Hence for any $s \neq 0$ in S there is an $r \in R$ such that $sr \in R$ and $sr \neq 0$. Therefore S is essential over R by (1.2), condition 3.

4. Baer rings.

1.34 Definition. A sum $\sum_{i \in I} K_i$ of subgroups K_i of an additive Abelian group will be called direct if $0 = \sum_{i \in I} k_i$, $k_i \in K_i$, implies $k_i = 0$ for all i .

1.35 Definition ([14]). An element e in a ring R is said to be idempotent if $e^2 = e$. If e is any idempotent in R then any element r of R can be written in the form $r = er + (1-e)r$, where $er \in eR$ and $(1-e)r \in (1-e)R$. Moreover, this is the only way in which r can be written as a sum of elements of the principal ideals eR and $(1-e)R$, since $r = ex + (1-e)y$ implies $er = e^2x + e(1-e)y = ex$ and $(1-e)r = (1-e)ex + (1-e)^2y = (1-e)y$. It follows that 0 cannot be written nontrivially as a sum of elements of eR and (1-e)R . Thus R = eR + (1-e)R is a direct sum. An ideal K of R is a direct summand if K = eR for some idempotent e of R .

1.36 Definition ([14]). Let K be a subset of a ring R . The annihilator of K , denoted K* , is an ideal of R consisting of all $r \in R$ such that rK = <0> . (K*)* will be written K** . The ideals of the form K* are called annihilator ideals; thus J is an annihilator ideal if and only if J = K* for some subset K of R . If K_1 and K_2 are subgroups of R then clearly $(K_1 + K_2)^* = K_1^* \cap K_2^*$. If $e = e^2 \in R$ then (eR)* = (1-e)R ; for if s is an element of R such that es = 0 , then s = es + (1-e)s = $(1-e)s \in (1-e)R$. The second inclusion is trivial.

1.37 Lemma ([14]). Let K and J be ideals in a ring R . Then

(1) $K \subseteq J \Rightarrow J^* \subseteq K^*$.

(2) $K \subseteq K^{**}$.

(3) $K^{***} = K^*$.

Proof. (1) Is clear.

(2) Since KK* = <0> , K is contained in the annihilator of K* .

(3) From (2) and (1) $K^{***} \subseteq K^*$. On the other hand $K^* \subseteq (K^*)^{**}$ by (2). It follows that J** = J whenever J is an annihilator ideal.

1.38 Lemma ([14]). If K is any ideal in a semiprime ring, then

(1) $K \cap K^* =$ <0> .

(2) K + K* is dense.

Proof. (1) $(K \cap K^*)^2 \subseteq K^*K = <0>$. Hence $K \cap K^* = <0>$ since R is semiprime.

(2) $(K + K^*)^* = K^* \cap K^{**} = <0>$ by part (1).

1.39 Definition. A ring is Baer if all its annihilator ideals are direct summands.

1.40 Proposition. A Baer ring is semiprime.

Proof. Let R be a Baer ring and suppose R is not semiprime. Then there exists an $x \in R$ and an integer $n > 1$ such that $x^n = 0$ and $x^{n-1} \neq 0$. Thus $x \in <x^{n-1}>^* = eR$ for some idempotent e in R , since the annihilator ideals in R are direct summands. Now $x \in eR$, $x^* = (1-e)R$, therefore $x = ex$ and so

$$0 = ex^{n-1} = (ex)x^{n-2} = x^{n-1} ,$$

a contradiction.

1.41 Proposition ([14]). If R is semiprime and rationally complete, then R is Baer.

Proof. Let K be an annihilator ideal in R . By (1.38) $K \cap K^* = <0>$, hence there is a mapping $f \in \mathrm{Hom}_R(K + K^*, R)$ defined by $f(a + b) = a$, where $a \in K$, $b \in K^*$. Now $K + K^*$ is dense, therefore f is a fraction, and since R is rationally complete there exists an element $e \in R$ such that $fd = ed$ for all $d \in K + K^*$. Thus $a = f(a + b) = e(a + b)$. Then $e^2(a + b) = e(a) = f(a) = a = e(a + b)$, hence $e^2 - e$ annihilates the dense ideal $K + K^*$ and so $e^2 = e$. Moreover $K = fK = eK \subseteq eR$ and since $eK^* = fK^* = <0>$ we also have $K^* \subseteq (eR)^* = (1-e)R$. Hence $eR = [(1-e)R]^* \subseteq K^{**} = K$.

Thus $K = eR$ is a direct summand. Accordingly R is Baer.

1.42 __Proposition__. If R is a semiprime ring, then R has an essential Baer extension.

__Proof__. From (1.21) we have the cannonical momomorphism of R into $Q(R)$. Moreover $Q(R)$ is rationally complete and is essential over R by (1.33). Since R is semiprime $Q(R)$ is semiprime by (1.12) and it follows from (1.41) that $Q(R)$ is Baer.

1.43 __Lemma__. Let R be a Baer ring and let S be essential over R. Then $S \setminus R = \{s \in S \mid s \notin R\}$ contains no idempotents.

__Proof__. By contradiction. Let $e = e^2 \in S \setminus R$. Then $e \neq 0$ and eS, $(1-e)S$ have non-zero intersection with R. eS is the annihilator of $(1-e)S$ in S. Therefore $(eS \cap R)[(1-e)S \cap R] = \langle 0 \rangle$, yielding $eS \cap R \subseteq [(1-e)S \cap R]^*$, the annihilator being taken in the ring R. Suppose now that $x \in R$ and that x is contained in the annihilator of $(1-e)S \cap R$. Then $x(1-e) = 0$, for if $x(1-e) \neq 0$ then by the essentiality of S over R there is a $y \in S$ such that $(1-e)xy$ is in $(1-e)S \cap R$ and is different from 0. But x annihilates $(1-e)S \cap R$ and so $x(1-e)xy = 0$. Therefore $[(1-e)xy]^2 = 0$ and hence $(1-e)xy = 0$ since R is semiprime, a contradiction. Thus $x(1-e) = 0$ and $x = ex \in eS \cap R$. Accordingly the annihilator of $(1-e)S \cap R$ in R is $eS \cap R$ and, because R is Baer, there is an idempotent $f \in R$ such that $eS \cap R = [(1-e)S \cap R]^* = fR$. It follows that $f(1-e) = 0$, hence $f = ef \in R$. Since $e \in S \setminus R$ assume $e \neq ef$. Then $e(1-f) \neq 0$, and by essentiality there exist $r \in R$ and $t \in S$ such that $0 \neq r = e(1-f)t$. Now $r \in eS \cap R = fR$ and $(fR)^* = (1-f)R$, hence $(1-f)r = 0$. Thus $r = fr = fe(1-f)t = 0$, a contradiction. Therefore $e = ef = f \in R$.

CHAPTER 2

REGULAR RINGS

1. Regular rings.

2.1 Definition. The Jacobson radical of a ring R, denoted Rad R, is the intersection of all maximal ideals of R.

2.2 Proposition. The Jacobson radical of R consists of all elements $r \in R$ such that $1 - rx$ is a unit for all $x \in R$.

Proof ([14], [1]). If the element r belongs to Rad R then for every maximal ideal M and for every element x, $rx \in M$. It follows that $1 - rx$ belongs to no maximal ideal, that is $1 - rx$ is a unit.

Conversely suppose there is a maximal ideal M which does not contain r. Then $rx + m = 1$ for some $x \in R$ and some $m \in M$. Hence $1 - rx \in M$ and is therefore not a unit.

2.3 Definition. A ring R is called semiprimitive if its Jacobson radical is $\langle 0 \rangle$. Thus R is semiprimitive if $r \neq 0$ implies $1 - rx$ is not a unit for some $x \in R$.

2.4 Definition. A ring R is regular in the sense of von Neumann if for every $r \in R$ there is at least one element $r' \in R$ such that $r = r^2 r'$. r' is called a quasi-inverse for r, after the case when R is a field.

2.5 Proposition. Every regular ring is semiprimitive.

Proof. Suppose R is a regular ring and $r \in \text{Rad } R$. Then $r(1 - rr') = 0$ for some $r' \in R$ and $1 - rr'$ is a unit. Therefore $r = 0$.

2.6 Proposition. Let R be a regular ring. Then

(1) Every non-unit is a zero-divisor.

(2) Every prime ideal is maximal.

(3) Every principal ideal is a direct summand.

(4) (von Neumann). Every finitely generated ideal is principal.

Proof ([14]). (1) For every $r \in R$ there is an $r' \in R$ such that $r(1 - rr') = 0$. If r is a non-zero-divisor then $rr' = 1$, hence r is a unit.

(2) Let P be any prime ideal, $r \notin P$. Then $r(1 - rr') = 0 \in P$, hence $1 - rr' \in P$ and therefore $1 \in rr' + P$. Thus r is invertible modulo P, that is R/P is a field. Accordingly P is a maximal ideal.

(3) If $r = rr'r$, then $rr' = (rr')^2$ is idempotent. Set $rr' = e$. Then $e \in rR$ and $r = er \in eR$. Thus $rR = eR$ is a direct summand.

(4) Consider the ideal $aR + bR$ $(a,b \in R)$. By (3) above $aR = eR$, where $e^2 = e$, and by (1.35) $bR \subseteq ebR + (1-e)bR$. Therefore $aR + bR = eR + (1-e)bR = eR + fR$, where $f^2 = f$ and $ef = 0$. Set $g = f(1-e)$. Then clearly $gf = f$, $g^2 = g$ and $eg = 0$. Moreover $g \in fR$ and $f \in gR$ hence $fR = gR$ and so $aR + bR = eR + gR$. Now $(e+g)R \subseteq eR + gR$. On the other hand $er + gr' =$ $e(er) + (eg)(r+r') + g(gr') = (e+g)(er + gr')$ for any $r,r' \in R$, hence $eR + gR \subseteq (e+g)R$ and therefore $aR + bR = eR + gR = (e+g)R$ is

a principal ideal. It follows that any finitely generated ideal in a regular ring is principal.

2.7 Lemma. Let R be a semiprime ring and let T be an essential extension of R. If S is a ring between R and T and if S is regular, then S is essential over R.

Proof. Let x be a non-zero element of S. For some $y \in S$, $x = x^2y$ and xy is a non-zero idempotent. By essentiality there exists $t \in T$ such that $xyt \in R$ and is non-zero. But $xyt = (xy)^2t = x[y(xyt)]$. Since $y(xyt)$ is in S, the proof is complete.

2.8 Definition. A ring R is π-regular if for each $r \in R$ there is an $x \in R$ and a positive integer n such that $r^n = (r^n)^2x$.

2.9 Definition. A prime ideal P of a ring R is a minimal prime ideal if it is minimal in the set of prime ideals of R ordered by inclusion.

2.10 Proposition ([4]). Every element of a minimal prime ideal is a zero-divisor.

Proof. Let x belong to a minimal prime ideal P and let $S = \{sx^k \mid s \in R\setminus P\}$, k a positive integer. Clearly S is a multiplicative subset of R. Suppose $0 \notin S$. Then, as in the proof of (1.7), there exists a prime ideal Q which is maximal in the set of ideals of R not meeting S. Moreover $Q \subseteq P$ for if $q \in Q \cap R\setminus P$ then $qx \in Q \cap S$ which is impossible. But $x \in P\setminus Q$, hence Q is properly contained in P, a contradiction. Therefore

x is a zero-divisor.

2.11 Lemma ([19]). Let R be a ring all whose prime ideals are maximal. Then R is π-regular.

Proof. If every prime ideal of R is maximal, then clearly every prime ideal of R is also a minimal prime ideal and this holds in every homomorphic image $\bar{R}$ of R. It then follows from (2.10) that the maximal ideals of any homomorphic image $\bar{R}$ of R consist solely of zero-divisors, hence each element of $\bar{R}$ is either a zero-divisor or a unit.

Now suppose R is not π-regular. Then there exists an $r \in R$ such that $r^{2n}x \neq r^n$ for any $x \in R$ and any positive integer n. If r is a unit there exists an $s \in R$ such that $rs = 1$, hence $r^n s^n = 1$ and therefore $r^{2n}s^n = r^n$, a contradiction. Thus assume r is a zero-divisor.

Let $\langle r\rangle^* \subseteq \langle r^2\rangle^* \subseteq \ldots \subseteq \langle r^n\rangle^* \subseteq \ldots$ be a chain of annihilator ideals and let their union be the ideal I. Then r is a non-zero-divisor modulo I, because if $rs \in I$ then $rs \in \langle r^n\rangle^*$ for some n, therefore $sr^{n+1} = 0$, that is $s \in \langle r^{n+1}\rangle^*$ and so $s \in I$. Moreover r is not invertible modulo I, for if $(rt - 1) \in I$, then $r^{m+1}t = r^m$ for a suitable m. But then $r^m = r^{m+1}t = r^{m+2}t^2 = \ldots = r^{2m}t^m$, a contradiction. Thus $\bar{r}$ is neither a zero-divisor nor a unit, again a contradiction. This completes the proof.

2.12 Corollary. A semiprime ring is regular if and only if each of its prime ideals is a maximal ideal.

Proof. A semiprime π-regular ring R is regular, for if $r \in R$ and $r^n(1 - r^n x) = 0$ for some $x \in R$, then $[r(1 - r^n x)]^n = 0$ whence $r(1 - r^n x) = 0$. Thus $r = r^2(r^{n-1}x)$. The opposite implication was proved in (2.6).

2. Integrally dependent rings.

2.13 Definition. Let R be a ring and let S be an over-ring of R. An element $x \in S$ is said to be integral over R (or integrally dependent on R), if there exists a finite set $\{r_0, \ldots, r_{n-1}\}$ of elements of R such that $x^n + r_{n-1}x^{n-1} + \ldots + r_0 = 0$, that is if x is a root of a monic polynomial with coefficients in R. This equation is called an equation of integral dependence satisfied by x over R.

2.14 Proposition. Let S be an over-ring of R and $x \in S$. Then the following are equivalent:

(1) x is integral over R.

(2) The ring $R[x]$ is a finitely generated R-module.

(3) The ring $R[x]$ is contained in a subring T of S which is a finitely generated R-module.

(4) There exists a finitely generated R-module M in S such that $xM \subseteq M$ and the annihilator of M in $R[x]$ is zero.

Proof ([1], [20]). (1) ⇒ (2). By induction on q. If x is integral over R, then $x^n = \sum_{i=0}^{n-1} r_i x^i$ for some positive integer n and $r_i \in R$, $i = 0,1,\ldots,n-1$, hence for any positive integer q we have $x^{n+q} = r_{n-1}x^{n+q-1} + \ldots + r_0 x^q$. By induction on q assume $x^{n+q} \in \sum_{i=0}^{n-1} Rx^i$, then, since x^n belongs to $\sum_{i=0}^{n-1} Rx^i$ so does x^{n+q+1} . Therefore $R[x]$ viewed as an R-module is generated by $1,x,\ldots,x^{n-1}$.

(2) ➡ (3). Take $T = R[x]$.

(3) ➡ (4). Take $M = T$. Then M is an $R[x]$-module, thus $xM \subseteq M$. Moreover $1 \in M$, hence if $y \in M^*$ for some $y \in R[x]$ then $y = 0$.

(4) ➡ (1). Suppose M is generated over R by $m_1,\ldots,m_n$. Since $xM \subseteq M$ we have $xm_i = \sum_{j=1}^{n} r_{ij} m_j$, $r_{ij} \in R$, $i = 1,\ldots,n$. This is a system of n linear homogeneous equations in the m_i . It can be written $\sum_{j=1}^{n} (\delta_{ij}x - r_{ij})m_j = 0$ where the δ_{ij} are the Kronecker symbols. Let $d \in R[x] = \det(\delta_{ij}x - r_{ij})$, $i,j = 1,\ldots,n$. If the matrix $(\delta_{ij}x - r_{ij})$ is multiplied by its adjoint this yields the matrix dI . It follows that $dm_i = 0$ for every i whence $dM = \langle 0\rangle$ and therefore $d = 0$, since $d \in R[x]$. Expanding the determinant then gives an equation that shows x is integral over R .

2.15 Lemma. Let S be a ring, R a subring of S and let $x_1,\ldots,x_n$ be elements of S , each integral over R . Then the ring $R_n = R[x_1,\ldots,x_n]$ is a finitely generated R-module.

Proof ([1]). By induction on n . When $n = 1$ the lemma is true by (2.14). By induction on n assume $R_{n-1} = R[x_1,\ldots,x_{n-1}]$ is a finitely generated R-module. Then $R_n = R[x_1,\ldots,x_n] = R_{n-1}[x_n]$ is a finitely generated R_{n-1}-module by the case $n = 1$ since x_n , being integral over R , is clearly integral over R_{n-1} . Now $R \subseteq R_{n-1} \subseteq R_n$; suppose $\{a_i \mid i \in I\}$ generate R_n over R_{n-1} and $\{b_j \mid j \in J\}$ generate R_{n-1} over R . Then R_n is finitely generated as an R-module by the mn products a_ib_j and the proof is complete.

2.16 Corollary. Let S be a ring, R a subring of S. Then the elements of S integral over R form a subring of S containing R.

Proof ([1], [20]). If x and y are elements of S integral over R, then $R[x,y]$ is a finitely generated R-module by (2.15) and it is a subring of S, hence by (2.14), part 3, $x-y$ and xy are integral over R. The elements of R are integral over R since each $r \in R$ is a root of the polynomial $X - r \in R[X]$.

2.17 Definition. Let S be a ring, R a subring of S. The ring of all elements of S which are integral over R is called the integral closure of R in S. When the integral closure of R in S is R itself then R is said to be integrally closed in S. If every element of S is integral over R then S is said to be integral over R.

2.18 Proposition (Transitivity). Let R, S and T be rings, $R \subset S \subset T$. Let S be an integral extension of R and let T be an integral extension of S. Then T is integral over R.

Proof ([20]). Let $x \in T$ and let $x^n + s_{n-1}x^{n-1} + \ldots + s_0 = 0$, $s_i \in S$, be an equation of integral dependence for x over S. Then the ring $S' = R[s_0,\ldots,s_{n-1}]$ is a finitely generated R-module by (2.15). Since x is integral over S' by the above equation of integral dependence, $S'[x]$ is a finitely generated S'-module. It follows from the proof of (2.15) that $S'[x]$ is a finitely generated R-module and therefore x is integral over R by part 3 of (2.14).

2.19 Corollary. Let S be a ring, R a subring of S and let T be the integral closure of R in S. Then T is integrally closed in S.

Proof ([1]). Let $s \in S$ be integral over T. By (2.18) s is integral over R, hence $s \in T$.

2.20 Proposition. Let $R \subseteq S$ be rings. S integral over R, and let I be an ideal of S. Then S/I is integral over $R/R \cap I$.

Proof ([20]). We recall that $R/R \cap I$ can be identified with the subring $R+I/I$ of S/I. Now let $\bar{s} \in S/I$, s a preimage of $\bar{s}$ in S, and let $P(X) \in R[X]$ be a monic polynomial with s as a root. For each coefficient r of $P(X)$ we have the cannonical map $r \to r + I$ of R into S/I, hence the image of $P(X)$ under the reduction homomorphism modulo I of $R[X]$ into $S/I[X]$ will be a monic polynomial with $\bar{s}$ as a root.

2.21 Proposition. Let $R \subseteq S$ be integral domains, S integral over R. Then R is a field if and only if S is a field.

Proof ([1]). Suppose R is a field and s a non-zero element of S. We recall that the polynomial ring $R[X]$ is a principal ideal domain whenever R is a field, hence the set of polynomials in $R[X]$ with s as a root form a principal ideal. Let $X^n + r_{n-1}X^{n-1} + \ldots + r_0$ $(r_i \in R)$ be its generator; that is, $s^n + r_{n-1}s^{n-1} + \ldots + r_0 = 0$ is an equation of integral dependence for s of smallest possible degree. Now $r_0 \neq 0$, for otherwise s is a zero-divisor, contrary to the assumption that S is an integral domain. Hence r_0 is invertible and therefore $[-r_0^{-1}(s^{n-1} + r_{n-1}s^{n-2} + \dots + r_1)]s = 1$. Hence s is invertible in S and so S is a field. Conversely let S be a field and y a nonzero element of R . Then y is invertible in S , hence y^{-1} is integral over R , and there is an equation of integral dependence $y^{-m} + r_1y^{-m+1} + \dots + r_m = 0$ $(r_i \in R)$. Multiplying through by y^m yields

$$1 = y[-(r_1 + r_2y + \dots + r_my^{m-1})] .$$

Hence y is invertible in R and therefore R is a field.

2.22 <u>Corollary</u>. Let $R \subseteq S$ be rings, S integral over R , and let Q be a prime ideal of S . Then Q is a maximal ideal of S if and only if $Q \cap R$ is a maximal ideal of R .

<u>Proof</u> ([1], [16]). If Q is a prime ideal of S then $Q \cap R$ is a prime ideal in R . For if x and y are elements of R such that $xy \in Q \cap R$ and $y \notin Q \cap R$ then $xy \in Q$ and $y \notin Q$. It follows that $x \in Q$ and hence $x \in Q \cap R$. Now S/Q is integral over $R/Q \cap R$ by (2.20) and both these rings are integral domains. By (2.21) S/Q is a field if and only if $R/Q \cap R$ is a field, that is, Q is maximal if and only if $Q \cap R$ is maximal.

2.23 <u>Lemma</u>. Let R be a regular ring and let S be an over-ring of R which is semiprime and is integrally dependent on R . Then S is a regular ring.

<u>Proof</u>. Suppose Q is a prime ideal of S and $P = Q \cap R$. Then P is a prime ideal of R , as was shown in (2.22). Because R is regular it is a maximal ideal, hence Q is a maximal ideal

by Corollary 2.22 and since S is semiprime it follows from (2.12) that S is regular.

2.24 Corollary. If R is a regular ring and S is an over-ring of R which is both integral and essential over R, then S is regular.

Proof. A regular ring is semiprime by (2.5) and an essential extension of a semiprime ring is again semiprime by (1.12). Thus S is semiprime and integrally dependent on R, hence regular by the result just proved.

3. The Boolean ring of idempotents of a ring.

The Boolean algebra of idempotents of a regular ring has a bearing upon the structure of the ring and that of its extensions. We shall therefore review Boolean algebras in some detail. The reference for this is [14, Section 1.1].

2.25 Definition. An ordered set (or partially ordered set) is a system $(S, \leq)$, where S is a set and $\leq$ is a relation on S satisfying the reflexive, transitive and symmetric laws:

i) $a \leq a$ ii) $(a \leq b$ and $b \leq c) \Rightarrow a \leq c$

iii) $(a \leq b$ and $b \leq a) \Rightarrow a = b$, for all $a,b,c \in S$.

A semilattice is an ordered set in which any two elements a and b have a greatest lower bound or inf $a \wedge b$. Thus a semilattice is a system $(S, \leq, \wedge)$ where $(S, \leq)$ is an ordered set and $\wedge$ is a binary operation satisfying the law:

$$c \leq a \wedge b \iff (c \leq a \text{ and } c \leq b).$$

2.26 Proposition. If $(S, \leq, \wedge)$ is a semilattice, the system $(S, \wedge)$ is a semigroup satisfying the idempotent and commutative laws: $a \wedge a = a$, $a \wedge b = b \wedge a$. Conversely any semigroup $(S, \cdot)$ which satisfies the idempotent and commutative laws is a semilattice relative to a suitable definition of $\leq$.

Proof. Let $(S, \leq, \wedge)$ be a semilattice. Then $\wedge$ is associative since for any $s \in S$ $s \leq (a \wedge b) \wedge c \Leftrightarrow (s \leq a$ and $s \leq b$ and $s \leq c) \Leftrightarrow s \leq a \wedge (b \wedge c)$. This shows that $(a \wedge b) \wedge c \leq a \wedge (b \wedge c)$ and $a \wedge (b \wedge c) \leq (a \wedge b) \wedge c$, hence $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and so $(S, \wedge)$ is a semigroup. Now $a \leq a \Rightarrow (a \leq a$ and $a \leq a) \Rightarrow a \leq a \wedge a \Rightarrow (a \leq a \wedge a$ and $a \wedge a \leq a) \Rightarrow a = a \wedge a$. Also $[(a \wedge b \leq b$ and $a \wedge b \leq a)$ and $(b \wedge a \leq a$ and $b \wedge a \leq b)] \Rightarrow [(a \wedge b \leq b \wedge a)$ and $(b \wedge a \leq a \wedge b)] \Rightarrow a \wedge b = b \wedge a$. Furthermore $a \leq b \Leftrightarrow a \wedge b = a$, for clearly $a \wedge b = a \Rightarrow a \leq b$, and on the other hand $a \leq b \Rightarrow (a \leq a$ and $a \leq b) \Rightarrow a \leq a \wedge b \Rightarrow (a \leq a \wedge b$ and $a \wedge b \leq a) \Rightarrow a \wedge b = a$. Hence if $(S, \cdot)$ is a semigroup which satisfies the idempotent and commutative laws one defines a relation $\leq$ on S by $a \leq b \Leftrightarrow ab = a$. We have

i) $aa = a \Rightarrow a \leq a$ (reflexive law);

ii) $(a \leq b$ and $b \leq c) \Rightarrow (ab = a$ and $bc = b) \Rightarrow$
$(a = ab = a(bc) = (ab)c = ac) \Rightarrow a \leq c$ (transitive law);

iii) $(a \leq b$ and $b \leq a) \Rightarrow (ab = a$ and $ba = b) \Rightarrow (a = ab = ba = b)$
$\Rightarrow a = b$ (antisymmetric law).

Finally $[ab = (aa)b = a(ab)$ and $ab = a(bb) = (ab)b] \Leftrightarrow (ab \leq a$ and $ab \leq b)$ and for any $s \in S$, $s \leq ab \Leftrightarrow [s = s(ab) = (sb)a = (sa)b] \Leftrightarrow [s = (sb)(aa) = sa$ and $s = (sa)(bb) = sb] \Leftrightarrow s \leq a$ and $s \leq b$.

Thus a and b have an inf, ab, and so the elements of $(S, \cdot)$ form a semilattice.

2.27 Definition. A lattice is a system $(S, \leq, \wedge, \vee)$ in which any two elements a and b have an inf $a \wedge b$ and a least upper bound or sup $a \vee b$, where $\vee$ is the binary operation satisfying the law

$$a \vee b \leq c \Longleftrightarrow (a \leq c \text{ and } b \leq c) .$$

A lattice with 0 and 1 is a lattice with elements so designated such that $0 \leq a$, $a \leq 1$, for each $a \in S$. An element a' is called a complement of a if $a \wedge a' = 0$, $a \vee a' = 1$. If every element of S has a complement, the lattice is called complemented.

A lattice is called distributive if identically

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) ,$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) .$$

2.28 Remark. In a distributive lattice the complement of an element a, if it exists, is uniquely determined. For if a' and a^* are both complements of a then

$$a^* = a^* \wedge 1 = a^* \wedge (a \vee a') = (a^* \wedge a) \vee (a^* \wedge a') =$$

$$0 \vee (a^* \wedge a') = (a' \wedge a) \vee (a' \wedge a^*) = a' \wedge (a \vee a^*) = a' \wedge 1 = a' .$$

2.29 Definition. Let $(S, \leq)$ be an ordered set. The element s of S is called an upper bound of the subset T of S if $t \leq s$ for all $t \in T$; it is called a sup or least upper bound of T if $s \leq s'$ for every upper bound s' of T. If both s and s' are least upper bounds of T then $s \leq s'$ and $s' \leq s$, hence $s = s'$, therefore the sup of a subset T of S is uniquely determined, if it exists at all. Lower bound and inf of a subset are defined dually.

An ordered set $(S, \leq)$ is called a complete lattice if every subset of S has both an inf and a sup. It suffices to assume an inf for each subset of S ; for the sup of any subset may then be defined as the inf of all its upper bounds. In particular, the sup of the empty set is inf S .

2.30 Definition. A Boolean ring is a ring which satisfies the idempotent law $aa = a$. A Boolean algebra is a system $(S, 0, ', \wedge)$ where $(S, \wedge)$ is a semilattice, 0 is an element of S and $'$ is a unary operation such that

$$a \wedge b' = 0 \Leftrightarrow a \wedge b = a \quad (\text{i.e., } a \leq b) .$$

2.31 Lemma ([14], p.4). In any Boolean algebra $a'' = (a')' = a$.

The following two propositions show that complemented distributive lattices, Boolean algebras and Boolean rings are all the same objects.

2.32 Proposition ([14], p.4). A Boolean algebra becomes a complemented distributive lattice by defining $a \vee b = (a' \wedge b')'$, $1 = 0'$. Conversely, any complemented distributive lattice is a Boolean algebra in which these equations are provable identities.

2.33 Corollary. If $(S, 0, ', \wedge)$ is a Boolean algebra, then so is $(S, 1, ', \vee)$.

Proof. $a \vee b = b \Leftrightarrow a \leq b \Leftrightarrow a \wedge b = a \Leftrightarrow a' \vee b = (a \wedge b')' = 0' = 1$.

2.34 Proposition. A Boolean algebra $(S, 0, ', \wedge)$ can be turned into a Boolean ring $(S, 0, 1, -, +, \cdot)$ by defining i) $a \cdot b = a \wedge b$

ii) $1 = 0'$ iii) $--a = a$ iv) $a + b = (a \wedge b') \vee (b \wedge a')$, where $a \vee b = (a' \wedge b')'$ as in (2.32). Conversely any Boolean ring can be regarded as a Boolean algebra with $a' = 1 - a$, and the above definitions of 1, −, and + then become provable identities.

Proof. If $(S, 0, ', \wedge)$ is a Boolean algebra then $(S, \wedge)$ is a semilattice, hence by (2.26) $(S, \cdot)$ is a semigroup satisfying the commutative and idempotent laws. Also $a \wedge 0 = 0 \Longrightarrow a \wedge 0'' = 0 \Longrightarrow a \wedge 0' = a$ and so $(S, 1, \cdot)$ is a semigroup with identity element $1 = 0'$. The operation + is clearly commutative and from $a + b = (a \wedge b') \vee (b \wedge a')$ we have $(a + b)' = (a' \vee b) \wedge (b' \vee a) = [(a' \vee b) \wedge b'] \vee [(a' \vee b) \wedge a] = (a' \wedge b') \vee (b \wedge a)$. Then

$(a + b) + c = [[(a \wedge b') \vee (b \wedge a')] \wedge c'] \vee [c \wedge [(a' \wedge b') \vee (b \wedge a)]]$

$= (a \wedge b' \wedge c') \vee (b \wedge a' \wedge c') \vee (c \wedge a' \wedge b') \vee (c \wedge b \wedge a)$

$= (a \wedge b' \wedge c') \vee (a \wedge c \wedge b) \vee (b \wedge c' \wedge a') \vee (c \wedge b' \wedge a')$

$= [a \wedge [(b' \wedge c') \vee (c \wedge b)]] \vee [[(b \wedge c') \vee (c \wedge b')] \wedge a']$

$= a + (b + c)$. That is, + is associative. Moreover

$a + 0 = (a \wedge 0') \vee (0 \wedge a') = a$, and $a + a = (a \wedge a') \vee (a' \wedge a) = 0$; Therefore 0 is the identity element relative to addition and $a = -a$, the additive inverse of a . Thus $(S, 0, -, +)$ is an Abelian group.

Now $ac + bc = [(a \wedge c) \wedge (b \wedge c)'] \vee [(b \wedge c) \wedge (a \wedge c)']$

$= [(a \wedge c) \wedge (b' \vee c')] \vee [(b \wedge c) \wedge (a' \vee c')]$

$= (a \wedge c \wedge b') \vee (a \wedge c \wedge c') \vee (b \wedge c \wedge a') \vee (b \wedge c \wedge c')$

$= (a \wedge b' \wedge c) \vee (b \wedge a' \wedge c) = [(a \wedge b') \vee (b \wedge a')] \wedge c = (a + b)c$.

Therefore the system $(S, 0, 1, -, +, \cdot)$ satisfies both distributive laws and so it is a Boolean ring.

Conversely suppose $(S, 0, 1, -, +, \cdot)$ is a Boolean ring. Then $(S, \cdot)$ is a semilattice by (2.26) and 0 is an element of S. Now define $a' = 1 - a$, then $a(1 - b) = 0 \Longleftrightarrow a = ab$, hence $(S, 0, ', \cdot)$ is a Boolean algebra.

It can now be seen that

1) $1 = 1 - 0 = 0'$.

2) $(-a) \wedge a' = (-a)(1 - a) = -a + a^2 = 0$ and

$(-a) \vee a' = [(-a)' \wedge a]' = 1 - [(1 - (-a))a] = 1 - (a + a^2)$

$= 1 - (a + a) = 1$. Therefore $-a = a'' = a$.

3) $a + b = 1 - [1 - a(1 - b)][1 - b(1 - a)] =$

$[(a \wedge b')' \wedge (b \wedge a')']' = (a \wedge b') \vee (b \wedge a')$.

2.35 <u>Proposition</u>. The set of idempotents of a ring R form a Boolean ring.

<u>Proof</u> ([14]). Let R be a ring and $B(R)$ the set of idempotents of R. If e and $f \in B(R)$ then clearly $ef \in B(R)$. Hence $(B(R), \cdot)$ is a semigroup satisfying the commutative and idempotent laws, where $\cdot$ denotes multiplication in R. By (2.26) it is a semilattice in which $e \leq f \Longleftrightarrow ef = e$. Moreover $0 \in B(R)$ and if $e \in B(R)$ then so is $1 - e$. Setting $e' = 1 - e$ yields $ef' = 0 \Longleftrightarrow e(1-f) = 0 \Longleftrightarrow ef = e$. Thus $(B(R), 0, ', \cdot)$ is a Boolean algebra by (2.30), and it follows from (2.34) that it can be turned into a Boolean ring.

As shown above, multiplication in $B(R)$ coincides with that of R, but addition differs in general, for if e and $f \in B(R)$, their sum in $B(R)$ is $ef' \vee fe' = [(ef')'(fe')']' =$

$1 - [1 - e(1 - f)][1 - f(1 - e)] = e + f - 2ef$. Here $+$ denotes addition in R.

We recall that an over-ring S of a ring R is a ring of quotients of R if and only if, for all $s \in S$, $s^{-1}R = \{r \in R | sr \in R\}$ is dense in S. These rings will now be applied to Boolean algebras.

2.36 Lemma ([14], p.46, Exercise 5). If R is a semiprime ring and S an over-ring of R, then S is a ring of quotients of R if $s(s^{-1}R) \neq \langle 0 \rangle$ for all non-zero elements s of S.

Proof. Let $s \neq 0$ be an element of S, $s^{-1}R = \{r \in R \mid sr \in R\}$ and $s(s^{-1}R) \neq \langle 0 \rangle$. By (1.29), part 2, S will be a ring of quotients of R if $s^{-1}R$ is dense in R. Suppose $a(s^{-1}R) = \langle 0 \rangle$ for some $a \in R$. Then $a[s(s^{-1}R)] = as(s^{-1}R) = \langle 0 \rangle$. If $as \in R$ then $a \in s^{-1}R \cap (s^{-1}R)^* = \langle 0 \rangle$ and the proof is complete. Assume $a \neq 0$. We have $as \notin R$, hence $as \in S \setminus R$, therefore $as[(as)^{-1}R] \neq \langle 0 \rangle$, that is there is an $r \in R$ such that $r(as) = (ra)s \in R$ and $(ra)s \neq 0$. Thus $ra \in s^{-1}R \cap (s^{-1}R)^* = \langle 0 \rangle$ and so $ras = 0$, as contradiction.

2.37 Proposition. Let R be a regular ring, and let S be a regular over-ring of R. Then S is an essential extension of R if and only if $B(S)$ is a ring of quotients of $B(R)$.

Proof. Assume S is essential over R and let $e \in B(S)$. By essentiality there exist $s \in S$ and $0 \neq r \in R$ such that $es = r$. Since R is regular there is an $r' \in R$ such that $r = r^2r'$ and $rr' \in B(R)$. Let $rr' = f$. Then clearly $f \neq 0$ and $f = rr' = (es)r' = e(esr') = ef$. Hence $ef \in B(R)$ and $ef \neq 0$. It now follows from (2.36) that $B(S)$ is a ring of quotients of $B(R)$.

Conversely suppose $B(S)$ is a ring of quotients of $B(R)$. Let s be a non-zero element of S, s' a quasi-inverse for s. Then as above $ss' \neq 0$ and $ss' \in B(S)$. Because $B(S)$ is a ring of quotients of $B(R)$ it is essential over $B(R)$. Hence there is an $e \in B(S)$ and $f \in B(R)$, $f \neq 0$, such that $f = (ss')e = s(s'e)$. Accordingly S is essential over R.

2.38 <u>Definition</u>. A Boolean algebra is called <u>complete</u> if it is a complete lattice.

2.39 <u>Lemma</u> ([14]). The annihilator ideals in a semiprime ring form a complete Boolean algebra.

<u>Proof</u>. Clearly the annihilator ideals in a semiprime ring R form a semilattice when ordered by inclusion and with intersection as inf, since $\cap_{i\in I} K_i^* = (\sum_{i\in I} K_i)^*$, where K_i, $i \in I$ are subsets of R. Now suppose J and K are annihilator ideals. Then $J \cap K^* = \langle 0 \rangle \Leftrightarrow J \subseteq K$; for if $J \subseteq K$ then $J \cap K^* \subseteq K \cap K^* = \langle 0 \rangle$ and conversely, if $J \cap K^* = \langle 0 \rangle$ then $JK^* = \langle 0 \rangle$, hence $J \subseteq K^{**} = K$. It follows that with $*$ as complementation the annihilator ideals in R form a Boolean algebra. It is complete by (2.29).

2.40 <u>Lemma</u>. A regular ring is Baer if and only if its Boolean ring of idempotents is complete.

<u>Proof</u>. In a regular ring principal ideals are direct summands. Hence if r is an element of a regular ring R and $rR = eR$, $e \in B(R)$, then $r^* = (1 - e)R$. Thus the annihilators of individual elements are also direct summands. By (2.39) the annihilator ideals in R form a complete Boolean algebra. Therefore R will be Baer if and

only if any intersection of direct summands is again a direct summand, that is, if and only if $B(R)$ is a complete Boolean algebra.

2.41 <u>Corollary</u>. Let R be a regular Baer ring and let S be a regular essential extension of R. Then S is Baer.

<u>Proof</u>. By (1.43) $S \setminus R$ contains no idempotents. Therefore $B(S) = B(R)$, a complete Boolean algebra.

4. <u>Extended and contracted ideals</u>.

2.42 <u>Definition</u>. Let R be a ring, S an over-ring of R and let J be an ideal of S. The ideal $J \cap R$ is called the <u>contraction</u> of J to R. An ideal of R is contracted (with respect to S) if it is the contraction of an ideal of S. If I is an ideal of R, then IS is its <u>extension</u> to S and an ideal of S is extended if it is the extension of an ideal of R. An extension IS consists of all elements of S of the form $a_1 s_1 + \ldots + a_n s_n$, where n is a positive integer, $a_j \in I$, $s_j \in S$, $j = 1,2,\ldots,n$. Thus IS is an ideal of S and $I \subseteq IS \cap R$. The set of extended ideals in S is closed under sum and product. On the other hand $(J \cap R)S \subseteq J$ and the set of contracted ideals in R is closed under intersection, radical formation and quotient formation.

2.43 <u>Lemma</u>. Let $R \subseteq S$ be rings, J an ideal of S and I an ideal of R. Denote by E the set of all ideals in S extended with respect to R and C the set of all ideals in R contracted with respect to S. Then $J \to J \cap R$ and $I \to IS$ are 1-1 and are inverse mappings of E onto C and C onto E.

Proof ([20]). We have $I \subseteq IS \cap R$, $(J \cap R)S \subseteq J$. Therefore $J \cap R \subseteq [(J \cap R)S \cap R] \subseteq J \cap R$ and $(IS \cap R)S \subseteq IS \subseteq (IS \cap R)S$. Thus $J \cap R = (J \cap R)S \cap R$ and $IS = (IS \cap R)S$. Now if I is a contracted ideal there is an ideal J in S such that $I = J \cap R = (J \cap R)S \cap R = IS \cap R$, that is I is the contraction of its extension. Similarly when J is an extended ideal in S there is an ideal I in R such that $J = IS = (IS \cap R)S = (J \cap R)S$ and therefore J is the extension of its contraction.

2.44 Lemma. Let R be a regular ring and let S be an over-ring of R. Then any ideal of R is the contraction of an ideal of S.

Proof. A factor ring $\overline{R}$ of a regular ring R is clearly regular hence by (2.5) the zero ideal of $\overline{R}$ is the intersection of all the maximal ideals. Thus any ideal of a regular ring is the intersection of the maximal ideals containing it. Since the contracted ideals of R are closed under intersection it suffices to establish the result for maximal ideals. Let M be a maximal ideal in R and assume $MS = S$. Then for suitable $m_i \in M$ and $s_i \in S$, $i = 1,2,\ldots,k$, $1 = m_1 s_1 + \ldots + m_k s_k$. Thus $S = 1S \subseteq (m_1R)S + \ldots + (m_kR)S = (m_1R + \ldots + m_kR)S \subseteq MS = S$. Now $S = (m_1R + \ldots + m_kR)S$. Since R is regular, by (2.6) there exists an $e \in M$ such that $e^2 = e$ and $m_1R + \ldots + m_kR = eR$. Thus $S = (eR)S = eRS = eS$. Hence $1 = es$ for some $s \in S$ and therefore $1 - e = (1 - e)1 = (1 - e)es = 0$. Thus $1 = e \in M$ which contradicts the fact that M is a proper ideal of R. Therefore MS is proper in S and so $MS \cap R$ is a proper ideal in R. Since $M \subseteq MS \cap R$ and M is maximal, $M = MS \cap R$.

2.45 Proposition. Let R be a regular ring and let S be a regular essential extension of R. Then all ideals of S are extensions of ideals of R if and only if $B(R) = B(S)$.

Proof. If $B(R) = B(S)$ and $x \in J$ (J an arbitrary ideal in S), then by regularity of S, $x = x(xy)$ for some $y \in S$. Hence $x = ex$, where $e = xy$ is an idempotent in the ideal J, hence in $J \cap R$. Thus $x \in (J \cap R)S$. Therefore $J \subseteq (J \cap R)S$. The opposite inclusion is clear and so J is the extension of $J \cap R$.

Suppose $B(R) \neq B(S)$ but that the ideal generated by $e \in B(S) \setminus B(R)$ is the extension of an ideal A of R, so that $eS = AS$. Then as in (2.44) $eS = (a_1 R + \ldots + a_n R)S$, $a_i \in A$, $i = 1,2,\ldots,n$. Then there is an idempotent $f \in R$ such that $eS = fRS = fS$. This implies that $e = ef = f$, contradicting the fact that f is in R and e is not. Thus e generates an ideal that is not extended with respect to R.

2.46 Remark. If R is regular and Baer and S is a regular essential extension of R, we have by (1.43) that $B(R) = B(S)$, hence by (2.44) and (2.45) that all ideals of R are contracted with respect to S and that all ideals of S are extended with respect to R. It then follows from (2.43) that the operations of contraction and extension define an isomorphism between the lattice of ideals of R and the lattice of ideals of S.

2.47 Proposition ([14]). Let R be a ring and $Q(R)$ its complete ring of quotients. Then $Q(R)$ is regular if and only if R is semiprime.

Proof. If $Q(R)$ is regular then it is semiprime by (2.5). Then R is semiprime since $\text{rad } R = [\text{rad } Q(R)] \cap R = \langle 0 \rangle$. Now suppose R is semiprime. $Q(R)$ will be regular if for each fraction f over R there is a fraction f' such that $\theta f = \theta(ff'f)$. Let f be a fraction with domain D and kernel $K \subset D$. Then $\langle 0 \rangle \neq K^*D \subseteq D \cap K^*$ and the restriction of f to $D \cap K^*$ is a monomorphism since it has kernel $D \cap K^* \cap K = \langle 0 \rangle$. Let $E = f(D \cap K^*)$. Then $E \cap E^* = \langle 0 \rangle$, thus define $f' \in \text{Hom}_R(E + E^*, R)$ by putting $f'(fd) = d$ for all $fd \in E$ and $f'r = 0$ for all $r \in E^*$. Then $f'[f(D \cap K^*)] = D \cap K^*$ hence $ff'fd = fd$ when $d \in D \cap K^*$. Now $f[K + (D \cap K^*)] = fK + f(D \cap K^*) = ff'f[K + (D \cap K^*)]$ and, because $K \subset D$, $K + (D \cap K^*) = D \cap (K + K^*)$ by the modular law. This is an intersection of two dense ideals, hence dense. Thus $ff'f - f$ annihilates a dense ideal and therefore $ff'f \; \theta \; f$.

2.48 Proposition. Let R be a regular Baer ring and let S be essential over R. Then the minimal prime ideals of S are precisely the ideals which are extensions of maximal ideals in R.

Proof. Consider the embeddings $R \to S \to Q(S)$ where $Q(S)$ is the complete ring of quotients of S. By (1.4) $Q(S)$ is essential over R, and by (1.12) and (2.47) $Q(S)$ is regular. Let M be a maximal ideal in R and $MQ(S)$ its extension in $Q(S)$. By (2.46) $MQ(S)$ is a maximal ideal in $Q(S)$. Suppose that $x \in MQ(S) \cap S$; then $xQ(S) = eQ(S) \subseteq MQ(S)$ for some idempotent $e \in BQ(S) = B(R)$, since R is Baer. Thus $e \in MQ(S) \cap R = M$ and because $xQ(S) = eQ(S)$, $x = ex \in MS$. Therefore $MQ(S) \cap S \subseteq MS$ which implies that $MQ(S) \cap S = MS$, the second inclusion being trivial. Furthermore

MS is a prime ideal in S as was shown in (2.22). Now suppose that P is a prime ideal in S contained in MS. Then $P \cap R \subseteq MS \cap R = M$. But $P \cap R$ is a prime ideal in R, a regular ring, and so it is a maximal ideal and $P \cap R = M$. Therefore $P = PS \supseteq (P \cap R)S = MS \supseteq P$. Therefore MS is a minimal prime ideal of S.

Now let L be a minimal prime ideal in S. Then $L = LS \supseteq (L \cap R)S$. $L \cap R$ is a prime ideal in R, thus a maximal ideal, hence $(L \cap R)S$ is a minimal prime ideal in S, as was just shown. Therefore $L = (L \cap R)S$ and L is of the claimed form. This completes the proof.

2.49 Remark. It is interesting to observe that the ring S, as described above, has the property that each prime ideal of S contains a unique minimal prime ideal.

2.50 Definition. A ring is Bézout if all its finitely generated ideals are principal.

2.51 Proposition. Let R be a regular Baer ring and let S be an essential extension of R. Then the following are equivalent:

(1) S is regular ;

(2) S is Bézout and all non-zero divisors in S are units.

Proof. (1) $\Rightarrow$ (2). This is true by (2.6), parts 1 and 4.

(2) $\Rightarrow$ (1). By (2.48) we can write an arbitrary minimal prime ideal of S in the form MS, where M [illegible] maximal ideal of R. Let N be an ideal of S such that $N \supseteq MS$. Let $Q(S)$ be the complete ring of quotients of S. Then $NQ(S) \supseteq MSQ(S) = MQ(S)$. But by (2.46) $MQ(S)$ is a maximal ideal in $Q(S)$, since $Q(S)$ is a regular

essential extension extension of R . Therefore $NQ(S) = Q(S)$ or $NQ(S) = MQ(S)$.

Suppose that $NQ(S) = Q(S)$. Then for suitable $n_i \in N$, $q_i \in Q(S)$, $i = 1,2,\ldots,k$, we have $n_1q_1 + \ldots + n_kq_k = 1$. Thus $Q(S) = (n_1S + \ldots + n_kS)Q(S)$, as in (2.44). Since S is Bézout, there exists $x \in S$ such that $xS = n_1S + \ldots + n_kS$.

Clearly $x \in N$, and $Q(S) = (xS)Q(S) = xQ(S)$. Thus $xq = 1$ for some $q \in Q(S)$, hence x is a unit in the ring $Q(S)$. Clearly it is not at the same time a zero-divisor in S , and so it must be a unit in S . But $x \in N$; thus $N = S$.

Suppose that $NQ(S) = MQ(S)$. Then $N = N \cap S \subset NQ(S) \cap S = MQ(S) \cap S = MS$ as in (2.48). Therefore $N = MS$. Thus the minimal prime ideals of S are maximal ideals, and so all prime ideals of S are maximal. S is also semiprime by (1.12), and hence it is regular by (2.12).

5. The space of prime ideals of a regular ring.

2.52 Proposition. Let R be a ring and let Π be the set of all prime ideals of R . Then Π may be made into a topological space by taking as open sets all sets of the form $V(A) = \{P \in \Pi \mid A \not\subset P\}$, where A is any subset of R . One notes that $V(A) = \bigcup_{a \in A} V(a)$ thus the sets $V(a)$ form a basis of the open sets of Π .

Proof ([14]). It is easy to see that for any subset A of R , $VA = V(A')$, where A' is the intersection of all prime ideals of Π containing A , hence an ideal. Now let $\{A_i\}_{i \in I}$ be a family of subsets of R . Then

(1) $\cup_{i\in I} V(A_i) = \{P \in \Pi \mid A_i \not\subset P \text{ for some } i \in I\}$

$= \{P \in \Pi \mid A_i' \not\subset P \text{ for some } i \in I\}$

$= \{P \in \Pi \mid \sum_{i\in I} A_i' \not\subset P\}$

$= V(\sum_{i\in I} A_i')$.

(2) $V(A_i) \cap V(A_j) = \{P \in \Pi \mid A_i \not\subset P \text{ and } A_j \not\subset P\}$

$= \{P \in \Pi \mid A_i A_j \not\subset P\}$

$= V(A_i A_j)$.

(3) $V(R) = \{P \in \Pi \mid R \not\subset P\} = \Pi$

$V(\{0\}) = \{P \in \Pi \mid \{0\} \not\subset P\} = \phi$.

Thus Π with the family of all sets $V(a) = \{P \in \Pi \mid a \notin P\}$ as a basis for the open sets becomes a topological space. This space is called the <u>prime spectrum</u> of R and is written Spec R . Its topology is attributed by some authors to Stone and by others to Zariski.

2.53 <u>Proposition</u>. If Spec R contains all maximal ideals, then Spec R is compact.

<u>Proof</u> ([14]). Suppose Spec $R = \bigcup_{i\in I} V(A_i) = V(\sum_{i\in I} A_i)$. Then $\sum_{i\in I} A_i$ is contained in no maximal ideal and thus contains 1. Therefore there is a finite subset F of I so that $1 \in \sum_{i\in F} A_i$, hence Spec $R = V(\sum_{i\in F} A_i) = \bigcup_{i\in F} V(A_i)$.

2.54 <u>Definition</u>. The union of all open sets contained in a subset E of a topological space is called the <u>interior</u> of E . Thus E is open if and only if the interior of E is E .

2.55 <u>Proposition</u>. For any subset E of Spec R , $V(\bigcap_{P \in E} P)$ is the interior of the complement of E .

<u>Proof</u> ([14]). $P' \in V(\bigcap_{P \in E} P) \Leftrightarrow (\bigcap_{P \in E} P) \not\subseteq P'$

$\Leftrightarrow$ there exists $r \in R$ such that $r \in P$ for all $P \in E$ and $r \notin P'$.

$\Leftrightarrow$ there exists $r \in R$ such that $P' \in V(r)$ and $P \notin V(r)$ for all $P \in E$.

Thus there is a basic open set $V(r)$ containing P' and not meeting E , that is P' belongs to the interior of the complement of E .

2.56 <u>Proposition</u>. Let R be semiprime. Then for any subset A of R , $\bigcap_{P \in V(A)} P$, is the annihilator A^* of A .

<u>Proof</u> ([14]). Suppose $r \in \bigcap_{P \in V(A)} P$. Then r belongs to all prime ideals of Spec R not containing A and therefore rA belongs to each prime ideal of R . Thus $rA = \langle 0 \rangle$ since R is semiprime.

2.57 <u>Proposition</u>. If R is semiprime then the closed-open subsets of Spec R are precisely those of the form $V(e) = \{P \in \text{Spec } R \mid e \notin P\}$, e an idempotent of R .

Proof. Suppose the subset E of Spec R is both closed and open in Spec R . Since E is open there is a subset A of R such that $E = V(A) = \{P \in \text{Spec } R \mid A \not\subseteq P\}$. Since $V(A)$ is also closed, Spec $R \setminus V(A)$ is open and thus coincides with its interior. Then by (2.55) and (2.56) $\text{Spec } R \setminus V(A) = V(\bigcap_{P \in V(A)} P) = V(A^*)$ and so $V(R) = \text{Spec } R = V(A) \cup V(A^*) = V(A + A^*)$. Now there are elements $e \in A$, $f \in A^*$ with $e + f = 1$, and because $ef = 0$ we have $e^2 = e$ and $a = ea$ for each $a \in A$. Hence $V(A) = V(eA) = V(e) \cap V(A) \subseteq V(e)$ and therefore $V(A) = V(e)$. The opposite implication is obvious.

2.58 Definition. A space X is a Hausdorff space if and only if whenever x and y are distinct points of X , there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.

A subset Y of X is connected if and only if the only subsets of Y which are both open and closed in Y are Y and the void set.

A space X is totally disconnected if and only if the only nonempty connected subsets of X are the one-point sets.

A compact totally disconnected space has a base of open-and-closed sets ([10], p.247).

2.59 Proposition ([12], p.32). If R is a regular ring, then Spec R is a compact, Hausdorff, totally disconnected space.

2.60 Definition. A compact space X is called projective when the following property holds: If $f: X \to Z$ is continuous and $g: Y \to Z$ is continuous and onto, then there exists a continuous map $h: X \to Y$ such that $f = gh$.

2.61 <u>Definition</u> ([11], p.484). A space X is <u>extremally disconnected</u> if the closure of every open set in X is open. In the category of compact spaces and continuous maps, the projective Hausdorff spaces are precisely the extremally disconnected spaces.

2.62 <u>Lemma</u>. If S is a regular Baer ring then $\operatorname{Spec} S$ is extremally disconnected.

<u>Proof</u>. Let $V(A) = \{M \in \operatorname{Spec} S \mid A \not\subset M\}$ be an open set in $\operatorname{Spec} S$. Then the intersection of all the maximal ideals in $V(A)$ is A^* by (2.56) and, since S is Baer, there is an idempotent $e \in S$ such that $A^* = eS$. Now the closure of $V(A)$ is $\{N \in \operatorname{Spec} S \mid \cap V(A) \subset N\} = \{N \in \operatorname{Spec} S \mid eS \subset N\} = \{N \in \operatorname{Spec} S \mid 1-e \notin N\}$. This is an open set, hence $\operatorname{Spec} S$ is extremally disconnected and is therefore projective by (2.61).

2.63 <u>Definition</u> ([11], p.486). Let X be a compact Hausdorff space. The <u>projective cover</u> of X is a compact Hausdorff extremally disconnected space $G(X)$, together with a continuous mapping of $G(X)$ onto X having the property that no closed subset of $G(X)$ maps onto X.

2.64 <u>Proposition</u>. Let R be regular and let S be a regular Baer ring which is essential over R ($Q(R)$ for example). Then $(\operatorname{Spec} S, f)$ is the projective cover of $\operatorname{Spec} R$, where f is defined by contracting the prime ideals of S to those of R.

Proof. Since R and S are regular, both spaces in question are compact and Hausdorff and Spec S is extremally disconnected by (2.62). Furthermore f is continuous (see for example [1], p. 13, Exercise 21), and (2.44) shows that it is onto. Now let C be a proper closed subset of Spec S. By the definition of the Stone topology, $C = \{M_i\}_{i \in I}$, where $\{M_i\}$ is the set of all maximal ideals of S containing the ideal $J = \cap_i M_i$. J is a non-zero ideal, as otherwise we would have C = Spec S, contradicting the fact that it is proper. Then $f(C) = \{M_i \cap R\}$, the closed set in Spec R defined by the ideal $J \cap R$. Since S is essential over R, $J \cap R \neq \langle 0 \rangle$. Thus there exists a maximal ideal in R which does not contain $J \cap R$ and f(C) is proper.

CHAPTER 3

ALGEBRAIC EXTENSIONS

1. Algebraic extensions.

3.1 Definition. Let R be a ring and let S be an over-ring of R. We shall call S an algebraic extension of R if it is both essential and integral over R. The following examples show that the two properties are independent.

3.2 Example. If F is a field and x is an indeterminate, then $F(x)$ is essential but not integral over R.

Proof. The quotient field $F(x)$ of the polynomial ring $F[x]$ is essential over F by (1.3) but $F(x)$ is well known to be a transcendental field extension of F, hence is not integral over F.

3.3 Example. The complete Boolean algebra on a two-element set is integral but not essential over the copy of the two-element field which it contains.

Proof. By (2.34) the complete Boolean algebra on the set of subsets of a two-element set $\{a, b\}$ may be regarded as a Boolean ring S with $0 = \phi$ and $1 = \{a, b\}$. We recall that a Boolean ring is a field if and only if it has exactly two elements: 0 and 1. Thus S is not a field and it follows from (1.3) that it is not essential over the copy of the two-element field which it contains. It is an integral extension since $s^2 - s = 0$ for each $s \in S$.

3.4 Proposition. Let R be a semiprime ring and let S be an over-ring of R. Then S is algebraic over R if and only if for each $s \in S$, $s \neq 0$, there exist $r_i \in R$, $i = 0,1,\ldots,n-1$, $r_0 \neq 0$, such that $s^n + r_{n-1}s^{n-1} + \ldots + r_1 s + r_0 = 0$.

Proof. Let S be algebraic over R and let $s \in S$, $s \neq 0$. Since S is integral over R we have

$$(1) \qquad s^n + r_{n-1}s^{n-1} + \ldots + r_1 s + r_0 = 0$$

for some $r_i \in R$, $i = 0,1,\ldots,n-1$. If $r_0 \neq 0$, then the proof is complete.

Suppose that $r_0 = 0$. Since S is essential over R, there exists $t \in S$ such that $st = a \in R$, $a \neq 0$. If $a^i r_i = 0$ for $i = 0,1,\ldots,n-1$ then multiplication of (1) by t^n yields:

$$0 = t^n s^n + t r_{n-1} a^{n-1} + t^2 r_{n-2} a^{n-2} + \ldots + t^{n-1} r_1 a = a^n .$$

Because R is semiprime this implies that $a = 0$, a contradiction. Hence there exists a positive integer $m < n$ such that $a^m r_m \neq 0$ and $a^i r_i = 0$ for all $i < m$. Thus $a^m r_i = 0$ for all $i < m$.

Multiplication of (1) by t^m yields:

$$(2) \qquad a^m s^{n-m} + a^m r_{n-1} s^{n-m-1} + \ldots + a^m r_{m+1} s + a^m r_m = 0 .$$

It is easy to see that addition of equations (1) and (2) yields an equation of the desired form.

Conversely suppose that the condition holds. Clearly S is integral over R. Let $s \in S$, $s \neq 0$. Then for appropriate $r_i \in R$, $i = 0,1,\ldots,n-1$, one has

$$0 \neq r_0 = s(r_1 + r_2 s + \dots + r_{n-1}s^{n-2} + s^{n-1}) \in sS \cap R$$

showing that S is essential over R .

3.5 Proposition. Let R be a regular ring and let S be a regular essential extension of R . The following are equivalent:

(1) S is an algebraic extension of R ;

(2) all between rings of R and S are regular;

(3) $R[s]$ is a regular ring for all $s \in S$;

(4) $R[u]$ is a regular ring, u any unit of S ;

(5) all units of S are integral over R .

Proof. Clearly (2) ⇒ (3) ⇒ (4).

(1) ⇒ (2). Since S is regular, it is semiprime and so are all the between rings of R and S . Moreover, the latter are integral over R , since S is. Now (2) follows from (2.23).

(4) ⇒ (5). Let u be a unit in S . $R[u^{-1}]$ is regular, and so its non-zero-divisors are units by (2.6). Therefore u^{-1} is a unit in $R[u^{-1}]$. Thus there exist $r_i \in R$, $i = 0,1,\dots,n$ such that

$$1 = u^{-1}(r_n u^{-n} + \dots + r_1 u^{-1} + r_0) =$$

$$r_n u^{-(n+1)} + \dots + r_1 u^{-2} + r_0 u^{-1} .$$

Transposition and multiplication of both sides of the equation by u^{n+1} yield

$$u^{n+1} - r_0 u^n - \dots - r_{n-1}u - r_n = 0$$

which is an equation of integral dependence over R for u .

(5) ⇒ (1). It is pointed out ([14], p.36, Exercise 4) that in a commutative regular ring the quasi-inverse of an element may be chosen to be a unit. (If s' is a quasi-inverse for s then $s = ss's = s(s's)^2 = s^2s'ss' + s^2 - s\,ss's = s^2(s'ss' + 1 - s's) = s^2u$, where $u^{-1} = s + 1 - ss'$). If $s \in S$ and u is both a quasi-inverse for s and a unit, then $su = e$, an idempotent, and $s = eu^{-1}$; that is every element of S is the product of an idempotent and a unit. Clearly the idempotents of S are integral over R; thus if the units are integral as well, S is integral over R by (2.16).

3.6 __Lemma__ (Transitivity). Let R, S and T be rings, $R \subseteq S \subseteq T$. Suppose that S is an algebraic extension of R and that T is an algebraic extension of S. Then T is an algebraic extension of R.

__Proof__. The transitivity of essentiality was pointed out in (1.4) and the transitivity of integral dependence in (2.18).

3.7 __Lemma__. Let R be a semiprime ring. Then R has a Baer algebraic extension.

__Proof__. From (1.42) we know that R can be embedded into $Q(R)$, a Baer ring. Let T be the integral closure of R in $Q(R)$. If $t = \theta f$ is a non-zero element of T, then as in (1.25), $t^{-1}R = \{r \in R \mid tr \in R\}$ contains D, the domain of f, and there is a $d \in D$ such that td is a non-zero element of R; hence T is essential and integral over R. Moreover $B(T) = B(Q(R))$, for if $e = e^2 \in B(Q(R))$ then e is integral over R, that is $e \in B(T)$. Now let t be a zero-divisor in T. Since $Q(R)$ is Baer, there exists $e \in B(T)$ such that $eQ(R) = (t)^*$ in $Q(R)$. Therefore $eT = eQ(R) \cap T \subseteq (t)^*$. On the other hand if $st = 0$, for some $s \in T$, then $s \in eQ(R) \cap T = eT$. Therefore $eT = (t)^*$ in T. This shows that the annihilators of elements are direct summands. But $B(T)$ is complete by (2.40) and so all annihilators in T are direct summands, and T is Baer.

2. Algebraically closed rings.

3.8 Definition. Let R be a ring. Then R is algebraically closed if it has no proper algebraic extensions. In terms of maps this means that if $m: R \to S$ is a monomorphism such that S is algebraic over $m(R)$, then m is onto. For example, an algebraically closed field is an algebraically closed ring since it possesses no proper integral extensions.

3.9 Proposition. Let R be a regular ring. Then the following are equivalent:

(1) R is algebraically closed;

(2) R is Baer and every monic polynomial equation over R has a root in R;

(3) R is Baer and all the factor fields of R are algebraically closed.

Proof. (1) $\Rightarrow$ (2). If R is not Baer, then the embedding given in (3.7) is proper and algebraic; hence R is not algebraically closed. Now suppose that $f(x) = x^n + r_{n-1}x^{n-1} + \ldots + r_1x + r_0$ is a monic polynomial over R with the property that no element of R is a zero of it. One notes that $r_0 \neq 0$, as otherwise $f(0) = 0$. A root will now be "adjoined". Embed R into $R[x]$, where x is an indeterminate and consider the ideal $I = \langle f(x) \rangle$. Then $I \cap R = \langle 0 \rangle$. For a typical element of I is of the form $f(x) \cdot g(x)$, where $g(x)$ is a polynomial in x over R. If the highest non-zero coefficient appearing in g is b, as the coefficient of x^m say, then $f(x) \cdot g(x)$ is a polynomial with $1 \cdot b$ as the coefficient of x^{m+n}. Because 1 is a non-zero-divisor and x is an indeterminate such a polynomial is not in R.

In the family of ideals of $R[x]$ which contains I and have trivial intersection with R, every simply ordered subset of ideals is bounded above by its union. Hence, in view of Zorn's lemma, there exists an ideal J, which is maximal in the family. Let p be the projection from $R[x]$ to $R[x]/J = S$, say. Then

$$R \to R[x] \xrightarrow{p} S$$

and $p|R$ is a monomorphism since $J \cap R = \langle 0 \rangle$. By the choice of J every non-zero ideal of S intersects $p|R$ in a non-zero ideal; thus S is essential over $p|R$. Also since J contains I we have $\bar{x}^n + \bar{r}_{n-1}\bar{x}^{n-1} + \ldots + \bar{r}_1\bar{x} + \bar{r}_0 = 0$ and it follows that $p(x)$ is integral over the image of R in S. But $p(x)$ and $p(R)$ generate the ring S. Therefore S is integral and essential over $p(R)$, that is $p|R$ is an algebraic embedding of R into S. Furthermore, the embedding is proper since $p(x)$ satisfies the equation $f(x) = 0$, as does no element of R. Thus there is a proper algebraic extension of R, a ring which is given by assumption to be algebraically closed; a contradiction. Therefore all monic polynomials over R have zeros in R .

(2) → (3). The field R/M will be algebraically closed if it has a root for every monic polynomial equation over itself. Choose such a monic over R/M and lift each of the coefficients back to one of its preimages under the cannonical map from R to R/M . Then the preimage of the first coefficient can be taken to be the 1 in R . By (2), the resulting monic over R has a root in R . Then the image of this root under the cannonical map above is a root for the monic over R/M . Since the question of being Baer is not at issue, this part of the proof is complete.

(3) → (1). Before proving this we make some preliminary remarks. If a ring S is regular then there is a homeomorphism of Spec S onto Spec B(S) defined by $M \to M \cap B(S)$ ([12], p.91). For $M \in$ Spec S , $M = (M \cap B(S))S = \{es \mid e \in M \cap B(S), s \in S\}$. By (2.57) the closed-open subsets of Spec S are those of the form $V(e) = \{M \in \text{Spec } S \mid e \notin M\}$, e an idempotent of S . Let $\{V(e_i)\}$, $i \in I$, be a cover of Spec S by closed-open sets. Because Spec S is a compact space there exists a finite subfamily $V(e_1), \ldots, V(e_n)$ that cover Spec S and which are, in general, not pairwise disjoint. A disjoint partition of Spec S can then be found in the following way: Since Spec S = $\bigcup_{i=1}^{n} V(e_i)$ we write $1 = \prod_{i=1}^{n} (e_i + (1-e_i))$. Expanding the right hand side yields $1 = \sum_{j=1}^{m} f_j$, where $m \leq 2^n - 1$, that is there are at most $2^n - 1$ non-zero terms in the expansion since $\prod_{i=1}^{n} (1 - e_i) = 0$ when Spec S = $\bigcup_{i=1}^{n} V(e_i)$. Each f_j , $j = 1,2,\ldots,m$, is a product of n factors, the kth factor being either e_k or $1 - e_k$. It is clear that $\{f_j\}$, $j = 1,2,\ldots,m$ are orthogonal

idempotents and therefore $V(f_i) \cap V(f_j) = \phi$ if $i \neq j$. Also, for each $j \leq m$ there is an $i \leq n$ such that $V(f_j) \subset V(e_i)$. Finally, every maximal ideal M of S belongs to a subset $V(f_j)$ for some $j \leq m$ as otherwise $1 \in M$. Therefore $\bigcup_{j=1}^{m} V(f_j)$ is a disjoint cover of Spec S by closed-open sets.

Now assume (3) and suppose there is an algebraic embedding of R into S. We may write $R \subset S$. Then $B(S) = B(R)$ by (1.43) and S is a regular Baer ring by (2.24) and (2.41).

Since S is integral over R we have, from (2.20) and (1.3), an algebraic embedding $R/M \cap R \to S/M$ at each maximal ideal of S and it follows from (3) that each embedding is onto.

Let $s \in S$ be an arbitrary element of S and suppose that M is a maximal ideal of S. In view of the isomorphism between S/M and $R/M \cap R$ there exists an element $r_M \in R$ such that $s + M = r_M + M$ in S/M. It is shown in ([17], p.16) that there is then an $e \in B(S)$ with $M \in V(e)$ such that $s + N = r_M + N$ at all $N \in V(e)$. For if $s + M = r_M + M$ then $s - r_M \in M$. Thus $s - r_M = gt$ where $g \in M \cap B(S)$ and $t \in S$. Let $e = 1 - g$. Then $M \in V(e)$. If $N \in V(e)$ then $g \in N$ so that $s - r_M \in N$. Hence $s + N = r_M + N$. Thus for any maximal ideal M of S there is an $r_M \in R$ and a closed-open set $V(e_M)$ containing M so that r_M and s agree on $V(e_M)$. Clearly $\{V(e_M) \mid M \in \text{Spec } S\}$ is a cover of Spec S. By compactness there exists a finite subfamily $V(e_{M_1}), \ldots, V(e_{M_n})$ that cover Spec S and elements $r_{M_1}, \ldots, r_{M_n}$ in R so that r_{M_i} and s agree on $V(e_{M_i})$, $i = 1,2,\ldots,n$. As in the remark above we then have a cover of Spec S by pairwise disjoint closed-open sets $V(f_1), \ldots, V(f_m)$, $m \leq 2^n - 1$, and for each $j \leq m$ there is an

$i \leq n$ such that $V(f_j) \subset V(e_{M_i})$. There is therefore for each $j \leq m$ a suitable element r_{M_i} to be chosen from the subfamily $r_{M_1}, \ldots, r_{M_n}$ so that r_{M_i} and s agree on $V(f_j)$. For $j = 1,2,\ldots,m$ we shall write $r_j = r_{M_i}$ to denote the element r_{M_i} so chosen. It is clear that $r_1, r_2, \ldots, r_m$ need not all be distinct.

Now set $r = \sum_{j=1}^{m} r_j f_j$. Then $r \in R$ since $B(S) = B(R)$. Furthermore $rf_j = r_j f_j$ since $f_1, f_2, \ldots, f_m$ are orthogonal idempotents. If M is a maximal ideal of S we have $M \in V(f_j)$ for some $j \leq m$, hence $s - r_j \in M$ and so $(s - r_j)f_j \in M$. But $(s - r_j)f_j = sf_j - r_j f_j = sf_j - rf_j = (s - r)f_j$, therefore $s - r \in M$. Thus $s - r$ belongs to $\cap\text{Spec } S = \langle 0 \rangle$ and so the algebraic embedding of R into S is onto and R is algebraically closed.

3.10 <u>Corollary</u>. If R is regular and algebraically closed and I is an ideal of R , then R/I is algebraically closed if and only if it is Baer.

<u>Proof</u>. Clearly R/I is regular. The factor fields of R/I are of the form $(R/I)/(M/I) \cong R/M$, M a maximal ideal of R containing I . Thus the factor fields of R/I are algebraically closed whenever R is. The result now follows from the equivalence of (1) and (3) in (3.9).

3.11 <u>Proposition</u>. A product of algebraically closed regular rings is also algebraically closed.

<u>Proof</u>. Let $R = \Pi R_i$, where $\{R_i\}_{i \in I}$ is a family of algebraically closed regular rings. It is easily seen that R is regular. Also $B(R) \cong \Pi B(R_i)$, for if $e \in B(R)$ we have $e: I \to \cup_{i \in I} R_i$ and

$e(i) = e^2(i) = e(i)^2 \in B(R_i)$ for all $i \in I$. Thus $B(R)$ is a product of complete Boolean algebras. Furthermore by (1.33) and (1.43) $B(R_i) = Q(B(R_i))$ for each $i \in I$, and therefore $B(R) \cong \Pi Q(B(R_i)) \cong Q(\Pi B(R_i))$ by ([14], p.41, Proposition 8). Hence $B(R)$ is rationally complete (see 1.24) and is a complete Boolean algebra by (1.41) and so R is Baer by (2.40). To demonstrate a root for a monic polynomial one notes that the polynomial gives a monic over each R_i under the projection $R[x] \to R_i[x]$ onto the ith component. Thus one can solve locally to arrive at the sought root.

3. The algebraic closure of a semiprime ring.

The existence of an algebraic extension for a semiprime ring was proved in (3.7). It will further be shown that, given a semiprime ring R, there exists an algebraic extension of R which is algebraically closed. These algebraically closed rings coincide with the totally integrally closed rings of Enochs [8] which will now be introduced.

3.12 Definition ([8]). A ring D is said to be totally integrally closed if for any ring homomorphism $\sigma: B \to D$ and any integral extension C of B there is a homomorphism $C \to D$ extending σ.

3.13 Proposition. If A is the direct product of a family of rings $\{A_i \mid i \in I\}$ with projections $\pi_i: A \to A_i$ then for every ring B and for every family of homomorphisms $\phi_i: B \to A_i$ there exists a unique homomorphism of rings $\phi: B \to A$ such that $\pi_i \cdot \phi = \phi_i$.

Proof ([14]). We recall that if $a \in A$, that is $a: I \to \bigcup_{i\in I} A_i$ with $a(i) \in A_i$ for all $i \in I$, then the projection $\pi_i: A \to A_i$ is defined by $\pi_i a = a(i)$. Now let $\phi_i: B \to A_i$ be a family of ring homomorphisms and define $\phi: B \to A$ by $(\phi b)(i) = \phi_i b$. It is clear that ϕ is a ring homomorphism and that $(\pi_i \circ \phi)b = \pi_i(\phi b) = (\phi b)(i) = \phi_i b$. Thus $\pi_i\phi = \phi_i$. If also $\pi_i \circ \psi = \phi_i$ then $(\psi b)(i) = \pi_i(\psi b) = (\pi_i \circ \psi)b = \phi_i b = (\phi b)(i)$, hence $\psi = \phi$.

3.14 Proposition. If $\{A_i\}_{i\in I}$ is a family of rings, then $A = \Pi_{i\in I} A_i$ is totally integrally closed if and only if each ring A_i is totally integrally closed.

Proof. Assume A is totally integrally closed. Let B be a ring and for each $i \in I$ let $\phi_i: B \to A_i$ be a ring homomorphism. Then by (3.13) there exists a unique homomorphism of rings $\phi: B \to A$ such that $\pi_i\phi = \phi_i$ where $\pi_i: A \to A_i$ is the projection onto the ith coordinate. If C is an integral extension of B and $\kappa: B \to C$ a monomorphism, then there is a homomorphism $\psi: C \to A$ extending ϕ, since A is totally integrally closed. Thus $\psi\kappa = \phi$. Therefore $\phi_i = \pi_i\phi = \pi_i\psi\kappa$ and so $\pi_i\psi$ is a homomorphism $C \to A_i$ extending ϕ_i and A_i is totally integrally closed.

Conversely, as in ([14], p.82, Proposition 3), assume each ring A_i is totally integrally closed and suppose $\phi: B \to A$ is a ring homomorphism. There is a family of projections $\pi_i: A \to A_i$, $i \in I$, hence there exists for each $i \in I$ a homomorphism $\pi_i\phi: B \to A_i$. Let $\kappa: B \to C$ be an embedding of B into an integral extension C.

Since the A_i are totally integrally closed there is for each $i \in I$ a homomorphism $\psi_i: C \to A_i$ extending $\pi_i\phi$ and by (3.13) $\phi: B \to A$ is now the unique homomorphism so that $\pi_i\phi = \psi_i\kappa$. Furthermore there exists a unique homomorphism $\psi: C \to A$ with the property that $\pi_i\psi = \psi_i$. Therefore $\pi_i\psi\kappa = \psi_i\kappa = \pi_i\phi$ and by the uniqueness of ϕ we have $\psi\kappa = \phi$. Thus ψ extends ϕ and A is totally integrally closed.

3.15. Proposition. If A is a subring of a ring D and A is a retract of D (i.e. there is a homomorphism $r: D \to A$ with $r|A = 1_A$) , then if D is totally integrally closed, A is totally integrally closed.

Proof. Suppose A is a subring of a totally integrally closed ring D and $r: D \to A$ is a retraction. Let $\sigma: B \to A$ be a ring homomorphism and let C be an integral extension of B . Since D is totally integrally closed there is a homomorphism $\phi: C \to D$ extending $(r|A)^{-1}\sigma: B \to D$ and therefore $r|A\,\phi: C \to A$ extends σ .

Before showing further results on totally integrally closed rings it will be convenient to recall some properties of the localization of a ring at a multiplicative set and the particular case of passing from an integral domain to its quotient field.

3.16 Definition. Let R be a ring and let S be a multiplicative set of R , that is S is multiplicatively closed and 1 belongs to S . We also assume that $0 \notin S$. A relation θ will be defined on $R \times S$ by $(r, s)\ \theta\ (r', s') \iff (rs' - sr')t = 0$ for some $t \in S$.

This is easily seen to be an equivalence relation. Denote by r/s the equivalence class of (r, s) and by R_S the set of equivalence classes. It is trivially verified that R_S can then be made into a ring with addition and multiplication given by the rule $r/s + r'/s' = (rs' + r's)/ss'$, $(r/s)(r'/s') = rr'/ss'$, having unit element $1/1$ and $0/1$ the identity element under addition. There is also a ring homomorphism $h: R \to R_S$ defined by $h(r) = r/1$ for each $r \in R$, with the property that every element of $h(S)$ is invertible in R_S. One notes that h is injective whenever S consists of non-zero-divisors of R. For $r/1 = 0/1$ only if there is an $s \in S$ such that $sr = 0$ and it follows that $r = 0$ if s is a non-zero-divisor. The process of passing from $R \to R_S$ will be called the <u>localization of R at the multiplicative set S</u>.

3.17 <u>Proposition</u> ([1], p.37). Let R, R_S and $h: R \to R_S$ be as in (3.16) and let $f: R \to T$ be a ring homomorphism such that $f(s)$ is a unit in T for all $s \in S$. Then there exists a unique ring homomorphism $g: R_S \to T$ such that $f = gh$.

3.18 <u>Proposition</u>. Let $R \subseteq T$ be rings, T integral over R. If S is a multiplicative set of R then T_S is integral over R_S.

<u>Proof</u> ([1]). Clearly S is a multiplicative set of T. Now suppose t/s is an element of T_S, $t \in T$, $s \in S$. Then t satisfies an equation of integral dependence $t^n + \ldots + r_1 t + r_0 = 0$, $r_i \in R$, $i = 0,1,\ldots,n-1$. Therefore $(t/s)^n + (r_{n-1}/s)(t/s)^{n-1} + \ldots + r_0/s^n = 0$.

3.19 Definition. Let A be an integral domain and let $S = A - \{0\}$. Then S is a multiplicative set and A_S is then a field called the quotient field of A.

3.20 Proposition. Let $A \subseteq B$ be integral domains, B integral over A. If $S = A - \{0\}$, then B_S is a quotient field of B.

Proof. It follows from (3.18) and (2.21) that B_S is a field. If $T = B - \{0\}$ then by (3.17) there exists a homomorphism $g\colon B_T \to B_S$ extending $f\colon B \to B_S$. Now g is a homomorphism of fields, thus a monomorphism and since $S \subseteq T$ we have $B_S \subseteq B_T$. Hence g is onto and therefore B_S is a quotient field of B.

3.21 Proposition ([8]). An integral domain A is totally integrally closed if and only if it is integrally closed in an algebraic closure Ω of its quotient field.

Proof. Suppose A is integrally closed in Ω. Let $\sigma\colon B \to A$ be a ring homomorphism and let C be an integral extension of B. Then $\operatorname{Ker}\sigma = P$ is a prime ideal of B because A is an integral domain, and since C is integral over B the Lying-over theorem (Cohen and Seidenberg) establishes the existence of a prime ideal Q of C such that $Q \cap B = P$. Put $\overline{C} = C/Q$ and $\overline{B} = B/P$. It then follows from (2.20) that $\overline{C}$ is an integral extension of $\overline{B}$. Since A may be identified with its image in Ω the injective homomorphism $\overline{B} \to A$ induced by σ gives rise to a homomorphism $\phi\colon \overline{B} \to \Omega$ with $\phi(\overline{B}) = \sigma(B)$. Let L be the quotient field of $\overline{B}$. By (3.18) and (3.20) there exists a quotient field M of $\overline{C}$ such that M is an integral extension of L, hence M is an algebraic field extension of L.

Now $\phi: \overline{B} \to \Omega$ can be extended to $\phi': L \to \Omega$ by (3.17) and because Ω is an algebraically closed field there is in turn an extension of ϕ' to an embedding ψ' of M in Ω. By restriction we have $\psi'|\overline{C} = \psi: \overline{C} \to \Omega$, where $\psi|\overline{B} = \phi: \overline{B} \to \Omega$. Therefore there is a homomorphism $\tau: C \to \Omega$ with $\tau(C) = \psi(\overline{C})$ and $\tau|B = \sigma$. It follows from the fact that C is an integral extension of B that the elements of $\tau(C)$ are integral over $\tau(B) = \sigma(B) \subseteq A$ and hence belong to A since A is integrally closed in Ω. Thus $\tau(C) \subseteq A$ and so $\sigma: B \to A$ has an extension $C \to A$.

Conversely assume A is totally integrally closed. If B is the integral closure of A in Ω then the identity mapping on A has an extension $\phi: B \to A$ and $\operatorname{Ker} \phi \cap A = \langle 0 \rangle$. But then $\operatorname{Ker} \phi = \langle 0 \rangle$. For if b is a non-zero element of B contained in $\operatorname{Ker} \phi$ there is an equation of integral dependence of smallest possible degree $b^n + \ldots + a_1 b + a_0 = 0$, $a_i \in A$, $i = 0,1,\ldots,n-1$. Because B is an integral domain a_0 is different from zero, as otherwise there would exist an equation of integral dependence having degree $n - 1$. Hence $a_0 = b^n + \ldots + a_1 b \in \operatorname{Ker} \phi$ and therefore $b = 0$. Accordingly ϕ is an isomorphism and $B = A$.

3.22 <u>Theorem</u> ([8]). A ring A is a subring of a totally integrally closed ring if and only if A is semiprime.

<u>Proof</u>. We first recall that a ring A is a subdirect product of a family of rings $\{S_i \mid i \in I\}$ if there is a monomorphism $\kappa: A \to S = \prod_{i\in I} S_i$ such that $\pi_i \cdot \kappa$ is onto for all $i \in I$, where $\pi_i: S \to S_i$ cannonically; if and only if $S_i \cong A/J_i$, J_i an ideal of

A, and $\cap_{i\in I} J_i = \langle 0\rangle$. (See [14], p.30). Now suppose A is semiprime and let $\{P_i \mid i \in I\}$ be the set of all prime ideals of A. Then A is a subdirect product of the rings A/P_i and so there is a monomorphism $\kappa: A \to \Pi_{i\in I} A/P_i$. Because each A/P_i is an integral domain it is a subring of a totally integrally closed ring by (3.21), hence $\Pi_{i\in I} A/P_i$ is a subring of a totally integrally closed ring by (3.14) and therefore A is a subring of a totally integrally closed ring.

The proof of the opposite implication is due to Borho and Weber ([3]). Assume A is not semiprime. There exists a non-zero element s of A with $s^2 = 0$. Let T be its annihilator s^* and consider the free A/T-module M with a base $\{x_i\}_{i\in I}$, where I is any set. M becomes an A-module by defining $am = \bar{a}m$ for all $a \in A$, $m \in M$, and one then forms the direct product $A \times M$ of the A-modules A and M. Now let multiplication be given in $A \times M$ so that $x_i^2 = s$ for all $i \in I$ and $x_i x_j = 0$ if $i \neq j$, that is $(0, \bar{a}x_i)(0, \bar{b}x_i) = (abs, 0)$, $a,b \in A$. To verify that this operation is well defined suppose $\bar{a} = \bar{a}'$ and $\bar{b} = \bar{b}'$. Then $a - a' \in T$ and $b - b' \in T$, therefore $(a - a')b + a'(b - b') = ab - a'b' \in T$; thus $s(ab - a'b') = 0$ whence $(sab, 0) = (sa'b', 0)$. The product in $A \times M$ of elements $(a, \sum_{i=1}^{n} \bar{a}_i x_i)$, $(b, \sum_{j=1}^{m} \bar{b}_j x_j)$ is then $(ab + s \sum_{i=j} a_i b_j,\ a \sum_{j=1}^{m} \bar{b}_j x_j + b \sum_{i=1}^{n} \bar{a}_i x_i)$. It is easily seen that with multiplication so defined and unit element $(1, 0)$, $A \times M$ becomes a ring. It extends A under the embedding $a \to (a, 0)$ for each $a \in A$ and furthermore the embedding is algebraic. For if $\omega = (a, \sum_{i=1}^{n} \bar{a}_i x_i)$ is an element of $A \times M$ such that $\sum_{i=1}^{n} \bar{a}_i x_i \neq 0$

then $\omega^2 - 2a\omega + a^2 - s \sum_{i=1}^{n} a_i^2 = 0$. Thus each element of $A \times M$ satisfies an equation of integral dependence over A . To see that $A \times M$ is essential over the image of A that it contains we first assume $a \in A \backslash T$. Then $(s, 0)(a, \sum_{i=1}^{n} \bar{a}_i x_i) = (sa, 0)$ and $0 \neq sa \in A$. If a belongs to T then $\overline{ax}_i = 0$, $i = 1,2,\ldots,n$, but since $\sum_{i=1}^{n} \bar{a}_i x_i \neq 0$ there is an a_i , $i \leq n$, which does not belong to T . Hence $a_i s$ is a non-zero element of A and $(a, \sum_{i=1}^{n} \bar{a}_i x_i)(0, \bar{1}x_i) = (a_i s, 0) \neq 0$.

Now suppose A is a subring of a totally integrally closed ring D . Let I be a set having cardinality strictly greater than that of D and let $\{x_i\}_{i \in I}$ be the basis for the free A-module M . Clearly $\text{card}(I) \leq \text{card}(A \times M)$. There is a monomorphism $\sigma: A \to D$ and because $A \times M$ is an integral extension of A there is a homomorphism of rings $\phi: A \times M \to D$ extending σ . Moreover $\text{Ker}\,\phi \cap A = \langle 0 \rangle$, hence $\text{Ker}\,\phi = \langle 0 \rangle$ by essentiality and therefore ϕ is an embedding. But

$$\text{card}(D) < \text{card}(I) \leq \text{card}(A \times M) ,$$

a contradiction. Thus A lies in no totally integrally closed ring.

3.23 Proposition. If a semiprime ring A is a subring of a totally integrally closed ring D , then the integral closure C of A in D is totally integrally closed.

Proof. Let $\sigma: R \to C$ be any ring homomorphism and let S be an integral extension of R . Since $C \subseteq D$ there is a homomorphism of rings $\phi: S \to D$ extending σ , consequently $\phi(S)$ is integral over $\phi(R) = \sigma(R) \subseteq C$. But C is integrally closed in D hence $\phi(S) \subseteq C$ and so C is totally integrally closed.

3.24 Theorem ([8]). If A is a semiprime ring there is a totally integrally closed algebraic extension A' of A. If A'' is any other such extension then any homomorphism $A' \to A''$ over A is an isomorphism.

Proof. Let A be a semiprime ring contained in a totally integrally closed ring B. In view of (3.23) we shall assume that B is integral over A. If also B is essential over A it is a totally integrally closed algebraic extension of A and the first part of the theorem is proved. Thus suppose B is not essential over A and let $\{A_\alpha\}$ be the family of rings such that $A \subseteq A_\alpha \subset B$ and $A \to A_\alpha$ is essential. Then $\{A_\alpha\}$ is not empty because it contains A. Let $\{C_\gamma\}$ be a simply ordered subset of $\{A_\alpha\}$ and let $C = \cup C_\gamma$. As in the proof of (1.23) C is a ring and furthermore C is an essential extension of A. For if $x \in C$, $x \neq 0$, then $x \in C_\gamma$ for some γ, hence there is a $y \in C_\gamma \subseteq C$ such that $xy \in A$ and $xy \neq 0$. Therefore $A \subseteq C \subset B$ and C is an upper bound for $\{C_\gamma\}$. By Zorn's lemma one now has an essential extension A' of A in B such that if A'' is essential over A', $A'' \subset B$, then $A' = A''$. Clearly B is an integral extension of A'. Let J be an ideal of B such that $A' \cap J = \langle 0\rangle$ and J is maximal with respect to this property. Then $A' \cong A' + J/J \subseteq B/J$. Moreover B/J is an integral extension of A' by (2.20) and is essential over the copy of A' it contains by the maximality of J. Because B is totally integrally closed the embedding $A' \to B$ can be extended to a homomorphism $\phi: B/J \to B$ with $\text{Ker}\,\phi \cap A' = \langle 0\rangle$. Consequently $\text{Ker}\,\phi = \langle 0\rangle$ by essentiality. One thus assumes $A' \subseteq B/J \subset B$ and by the choice of A' this implies that $B/J = A' = A' + J/J$. It follows that $B - A' \subseteq J$, that is $B \subseteq A' + J$, hence $B = A' + J$. We recall that also $A' \cap J = \langle 0 \rangle$, therefore the projection $p: B \to B/J = A'$ will map the elements of $B \setminus A'$ into zero while $p|A' = 1_{A'}$. Thus p is a retraction and by (3.15) A' is totally integrally closed.

If now A' and A'' are both totally integrally closed algebraic extensions of A let $\sigma: A' \to A''$ be a homomorphism over A. Then $\text{Ker}\,\sigma \cap A = \langle 0 \rangle$ hence $\text{Ker}\,\sigma = \langle 0 \rangle$ by essentiality and therefore there is an isomorphism of A' with $\sigma(A')$; thus $\sigma(A')$ is a totally integrally closed extension of A. Since A'' is integral over A it is integral over $\sigma(A')$, consequently there is a homomorphism $r: A'' \to \sigma(A')$ over $\sigma(A')$. Since $\text{Ker}\,r \cap \sigma(A') = \langle 0 \rangle$ we have $\text{Ker}\,r \cap A = \langle 0 \rangle$ and so $\text{Ker}\,r = \langle 0 \rangle$. Accordingly $\sigma(A') = A''$.

3.25 <u>Corollary</u>. A semiprime ring is totally integrally closed if and only if it is algebraically closed.

<u>Proof</u>. Let A be a semiprime ring. As was just shown in (3.24) A has a totally integrally closed algebraic extension. If now A is algebraically closed then any such extension will be isomorphic with A by (3.8) and therefore A is totally integrally closed. Conversely assume A is totally integrally closed. By (3.7) there exists an algebraic extension D of A and so we have a homomorphism $\phi: D \to A$ extending the identity map on A. This is obviously a monomorphism. It follows that the algebraic embedding $A \to D$ is onto and A is algebraically closed by (3.8).

3.26 Corollary. If R is a semiprime ring there exists an algebraic extension $\Omega(R)$ which is algebraically closed. Furthermore $\Omega(R)$ is unique up to isomorphism over R and contains a copy over R of every algebraic extension of R. We call $\Omega(R)$ the algebraic closure of R. It is clear from (2.24) that the algebraic closure of a regular ring is regular.

3.27 Proposition. Let R be a semiprime ring. Then $\Omega(R)$ can be realized as the integral closure of R in the algebraic closure of its complete ring of quotients.

Proof. If R is semiprime and $Q(R)$ its complete ring of quotients there exists an algebraic closure $\Omega(Q(R))$ of $Q(R)$ which is essential over R by transitivity. Let T be the integral closure of R in $\Omega(Q(R))$ and let $t \in T$, $t \neq 0$. Since T is integral over R one has an equation of integral dependence

$$(1) \qquad t^n + r_{n-1}t^{n-1} + \dots + r_1 t + r_0 = 0 \, , \quad r_i \in R \, .$$

The proof will be complete if $r_0 \neq 0$, for then T is algebraic over R by (3.4). Suppose $r_0 = 0$. Since $\Omega(Q(R))$ is essential over R there exists $\omega \in \Omega(Q(R))$ such that $t\omega = a \in R$, $a \neq 0$. As in the proof of (3.4) there exists a positive integer $m < n$ such that $a^m r_m \neq 0$ and $a^m r_i = 0$ for all $i < m$, hence multiplication of (1) by ω^m yields:

$$a^m t^{n-m} + a^m r_{n-1} t^{n-m-1} + \dots + a^m r_{m+1} t + a^m r_m = 0 \, .$$

Therefore $0 \neq -a^m r_m = t(a^m t^{n-m-1} + a^m r_{n-1} t^{n-m-2} + \dots + a^m r_{m+1}) \in tT \cap R$, showing that T is essential over R . Thus the integral closure of R in $\Omega(Q(R))$ is an algebraic extension of R which is algebraically closed by (3.23).

3.28 Proposition. Ω commutes with finite Cartesian products.

Proof. Let $R = \prod_{i=1}^{n} R_i$ be a semiprime ring and $\Omega(R)$ its algebraic closure. By (3.26) there is an algebraic closure $\Omega(R_i)$ for each of the rings R_i , $i \leq n$, and it follows from (3.14) that $\Pi\Omega(R_i)$ is algebraically closed. Let $s = (s_1, s_2, \ldots, s_n)$ be a non-zero element of $\prod_{i=1}^{n}\Omega(R_i)$. For each $i = 1,2,\ldots,n$ s_i satisfies the equational condition of (3.4) with respect to the corresponding R_i since $\Omega(R_i)$ is algebraic over R_i . Suppose m is the degree of the equation having highest degree. Then multiplication of each equation of lesser degree by a suitable power of the respective s_j will yield monic equations of degree m and under componentwise operations one obtains an equation $s^m + r_{m-1}s^{m-1} + \ldots + r_0 = 0$, where $r_j \in \prod_{i=1}^{n} R_i$, $j = 0,1,\ldots,m-1$, and $r_0 \neq 0$ because $r_0(i)$ is a non zero element of R_i for some $i \leq n$. Hence $\prod_{i=1}^{n} \Omega(R_i)$ is algebraic over $\prod_{i=1}^{n} R_i$ by (3.4). But $\prod_{i=1}^{n} \Omega(R_i)$ is algebraically closed and therefore there is, by (3.24), an isomorphism over R of $\prod_{i=1}^{n} \Omega(R_i)$ onto $\Omega(R)$.

3.29 Proposition. Let R be a semiprime ring and let $\Omega(R)$ be its algebraic closure; then $\Omega(R)$ is Baer.

Proof. If $\Omega(R)$ were not Baer, then by (3.7) it would have an algebraic extension which, being Baer, would be a proper extension.

4. Localizations of an algebraically closed regular ring.

3.30 Lemma. Let R be a regular ring and let I be an ideal in R. Then idempotents can be lifted modulo I; that is, an element of R/I is an idempotent if and only if it is the image under factoring by I, of an idempotent of R.

Proof. The images of idempotents are again idempotents. Conversely, let $\bar{x} = \bar{x}^2$ in R/I. Then $x^2 - x \in I$. By regularity there exists $y \in R$ such that $x = x^2y = xe$ where $e = e^2 = xy$. Since $x^2 - x$ is in I, so is $y(x^2 - x) = x - e$. Thus the idempotent e is mapped onto $\bar{x}$ in R/I.

3.31 Lemma. Let R be a regular ring and let $B(R)$ be its ring of idempotents. Then:

(1) If I is any ideal of R, then the ring of idempotents of R/I is isomorphic to $B(R)/I \cap B(R)$.

(2) Any ideal of $B(R)$ is extended by an ideal of R.

Proof. (1) From (2.35) multiplication in $B(R)$ coincides with that in R, and if $e \in B(R)$, $f \in B(R)$, their sum in $B(R)$ is $ef' \vee fe'$, which becomes $e + f - 2ef$ when $+$ denotes addition in R. Let h be the projection from R onto R/I. It is easily seen that $h|B(R) \to B(R/I)$ is a ring homomorphism. Furthermore, its kernel is $I \cap B(R)$ and (3.30) ensures that it is onto.

(2) Let J be any ideal in $B(R)$. Let $I = \{er \mid e \in J, r \in R\}$. It is shown in ([17], p.7, Lemma 1.6) that if e_1r_1 and $e_2r_2 \in I$, then $e_1r_1 + e_2r_2 = (e_1 + e_2 - e_1e_2)(e_1r_1 + e_2r_2)$

$= (e_1'e_2')'(e_1r_1 + e_2r_2) = (e_1 \vee e_2)(e_1r_1 + e_2r_2)$.

Therefore $e_1r_1 + e_2r_2 \in I$ and so I is an ideal of R. If $er \in I \cap B(R)$ then $e \in J$, hence $e(er) = er \in J$. Thus $J = I \cap B(R)$.

3.32 <u>Definition</u>. Let R be a ring. An R-module M is called <u>irreducible</u> if it has exactly two submodules. That is, $M \neq 0$ and M has no proper submodules. If J is an ideal of R then J is irreducible as an R-module if and only if J is a minimal non-zero ideal.

An R-module M is said to be <u>completely reducible</u> if it is isomorphic to a direct sum of irreducible modules. The ring R will be called completely reducible if it is completely reducible as an R-module. (Such a ring is also called "semisimple").

3.33 <u>Proposition</u> ([14], p.65). A commutative ring R is completely reducible if and only if it is isomorphic to a finite direct product of fields.

3.34 <u>Proposition</u> ([14], p.68). A ring R is completely reducible if and only if it is Noetherian and regular.

3.35 <u>Corollary</u>. A completely reducible ring R is regular Baer.

<u>Proof</u>. Since R is Noetherian every ideal is finitely generated. It then follows from (2.6) that every ideal is a direct summand.

3.36 <u>Lemma</u>. Let R be a regular Baer ring with ring of idempotents $B(R)$. Then R is completely reducible if and only if $B(R)$ is finite.

Proof. If R is completely reducible then it is isomorphic to a direct product of fields $\prod_{i=1}^{n} F_i$ having idempotents $e: \{1,2,\ldots,n\} \to \bigcup_{i=1}^{n} F_i$ such that $e(i) \in F_i$, $i = 1,2,\ldots,n$, and $e(i) = 0$ or $e(i) = 1$. Therefore B(R) is finite.

The converse can be established by induction on the order of B(R). If $|B(R)| = 2$ then R is a field. If $|B(R)| = n > 2$, let $e \in B(R)\setminus\{0,1\}$. Then eR is a ring with identity element e. It is not a subring of R since $e \in B(R)\setminus\{0,1\}$. Now $R \cong eR \times (1-e)R$ by (1.35) and $|B(eR)| < n$, $|B(1-e)R| < n$. Thus R is the Cartesian product of finitely many fields.

3.37 Theorem ([7], p.456, Theorem 4.3). A Boolean algebra has the property that all of its quotient algebras are complete if and only if it is finite.

3.38 Proposition. If R is semiprime and rationally complete then all quotient rings are rationally complete if and only if R is completely reducible.

Proof. If R is semiprime and rationally complete then it is Baer by (1.41) and regular by (1.24) and (2.47). Suppose that R is not completely reducible. By (3.36) B(R) is infinite. By (3.37) there exists an ideal J in B(R) such that B(R)/J is not complete. By (3.31), part 2, J is extended by an ideal say I of R, and by (3.31), part 1, the ring of idempotents of R/I is not complete. Thus R/I is not rationally complete. Thus if all quotient objects of R are to be complete, then B(R) must be finite and R must be completely reducible. The opposite implication is straightforward.

3.39 <u>Proposition</u>. If R is regular and algebraically closed then all quotient objects of R are algebraically closed if and only if R is completely reducible.

<u>Proof</u>. A regular algebraically closed ring is Baer by (3.9). The quotient rings of R are always regular and their factor fields are algebraically closed as shown in the proof of (3.10). Thus the only property at issue is that of being Baer and this argument proceeds as in (3.38).

3.40 <u>Definition</u>. We recall from (3.16) that R_S is an extension of a ring R provided S is a multiplicative set of R consisting of non-zero-divisors. Now assume that S is a multiplicative set in R with $0 \notin S$ and that S contains a divisor of zero. As shown in ([20], p.221) a localization $R \to R_S$ is then defined thus: Consider a homomorphism f of R into a ring T such that $f(s)$ is a unit for every $s \in S$. If x is an element of R such that $sx = 0$ for some $s \in S$ then $0 = f(xs) = f(x)f(s)$, and since $f(s)$ is a unit in T this implies $f(x) = 0$. Thus the kernel of f must contain the set I of all elements x in R for which there exists an element s in S such that $sx = 0$. The set I is a proper ideal of R. For if $x_1, x_2 \in I$ then $s_1x_1 = 0$, $s_2x_2 = 0$, $s_1, s_2 \in S$, whence $s_1s_2(x_1 + x_2) = 0$. Furthermore $1 \notin I$ as otherwise $s = 1 \cdot s = 0$ for some $s \in S$. Now the cannonical image $\overline{S} = S + I/I$ of S in R/I is obviously closed under multiplication and if $\overline{x}\,\overline{s} = 0$, $\overline{x} \in R/I$, $\overline{s} \in \overline{S}$, then $xs \in I$, $xss' = 0$ for a suitable $s' \in S$, and since $ss' \in S$ this implies that $x \in I$ and $\overline{x} = 0$. Therefore $\overline{S}$ is a multiplicative set containing no zero-divisors. As in (3.16) one has a ring $(R/I)_{\overline{S}}$ which will be called the localization of R at the multiplicative set S and denoted R_S. We shall write r/s to denote the equivalence class of $(\overline{r}, \overline{s})$ in R_S. The homomorphism $h: R \to R_S$ will be given by $h = \psi\phi$, where ϕ is the cannonical homomorphism of R onto R/I and ψ is the monomorphism of R/I into R_S defined by $\overline{r} \to r/1$. The kernel I of h is then the set of all elements x in R for which there exists s in S such that $xs = 0$. Furthermore, following (3.16), every element of h(S) is invertible in R_S. We note that the monomorphism $R/I \to R_S$ will be onto whenever $\overline{S}$ consists of units of R/I.

3.41 Lemma. Let R be a regular ring and let I be an ideal of R. Then I is the kernel of a localization of R, and the localization is R/I.

Proof. Since R is regular $I = \cap_i M_i$ where $\{M_i\}$ is the family of maximal ideals of R containing I. The set $S = \cap_i (R - M_i)$ is multiplicative and I is the kernel of the localization with respect to S. For if r is in the kernel, $rs = 0$ for some $s \in S$. Since s is in no M_i, r is in each M_i, and therefore in I. Conversely, if $r \in I$, then by the regularity of R $rR = eR$ for some idempotent e of R. Since e is in each M_i, $1 - e$ is in each $R - M_i$, i.e. in S. Since $r(1-e) = 0$, one concludes that r is in the kernel of the localization with respect to S. The localization R_S will be R/I because the image of S in R/I consists of non-zero-divisors as was just shown in (3.40). Therefore the elements of $\overline{S}$ are units by (2.6) and so the monomorphism $R/I \to R_S$ is onto.

3.42 <u>Proposition</u>. Let R be an algebraically closed regular ring. Then every localization of R is algebraically closed if and only if R is completely reducible, i.e. if and only if R is a finite Cartesian product of algebraically closed fields.

<u>Proof</u>. (3.39) and (3.41).

3.43 <u>Remark</u>. If $R = \Pi_{\aleph_0} F$, F an algebraically closed field, then R is an algebraically closed regular ring by (3.11). Thus every monic polynomial over R has a root in R and the field images of R are algebraically closed. But R is not completely reducible, hence there exists a localization R/J of R that is not algebraically closed. R/J has a root for every polynomial equation over itself, as was shown in the proof of (3.9), and furthermore its quotient fields are algebraically closed because they lie among those of R. Therefore R/J is not Baer and it is now clear that the demands that the ring be Baer in (3.9) are not superfluous.

CHAPTER 4

APPLICATIONS TO GROUP RINGS

1. Group Rings.

4.1 Definition. Given a multiplicative group G and a ring A, which may be not commutative, the group ring $R = AG$ consists of all functions $r: G \to A$ with finite support. The support of r is $\{g \in G \mid r(g) \neq 0\}$. R is endowed with ring operations by defining

$0(g) = 0$,

$1(g) = 1$ if $g = 1$, $= 0$ if $g \neq 1$,

$(-r)g = -r(g)$,

$(r + r')g = r(g) + r'(g)$,

$(rr')g = \Sigma_{g=hh'}\, r(h)r'(h')$.

It is easy to see that these operations satisfy the associative and distributive laws and so R is in fact a ring. It contains A as a subring under the identification of $a \in A$ with a function $a: G \to A$ given by

$a(h) = 0$ if $h \neq 1$, $h \in G$,

$a(1) = a$.

Similarly for any $g \in G$ put

$g(h) = 0$ if $h \neq g$, $h \in G$,

$g(g) = 1$.

Then G may be identified with a subsemigroup of the multiplicative semigroup of R and R has unit element $1 = 1_G$. It follows that for any $r \in R$

$$r = \Sigma_{g\in G}\, r(g)g = \Sigma_{g\in G}\, gr(g).$$

Thus R is a commutative ring if A is commutative and G is an Abelian group. Furthermore R is a free A-module generated by the set of elements of G. For if $\sum_{i=1}^{n} r(g_i)g_i = 0$ then

$$\left[\sum_{i=1}^{n} r(g_i)g_i\right](g_j) = \sum_{i=1}^{n} [r(g_i)(1) \cdot g_i(g_j)] = 0 , \quad j = 1,2,\ldots,n ,$$

and since $g_i(g_j) = 0$ for all $j \neq i$ we have

$$r(g_i)(1) \cdot g_i(g_i) = r(g_i)(1) \cdot 1 = r(g_i) = 0 , \quad i = 1,2,\ldots,n .$$

There is also a mapping $\delta: R \to A$ defined by $\delta(\Sigma\, r(g)g) = \Sigma\, r(g)$. This is clearly a ring homomorphism of R onto A.

4.2 Lemma ([6], p.651). Let $R = AG$. There is a mapping ω from the lattice of subgroups of G to the lattice of right ideals of R. For any subgroup H of G, ωH is defined to be the right ideal generated by the set $\{1 - h \mid h \in H\}$. If H is a normal subgroup ωH is an ideal. In fact it is the kernel of the homomorphism $AG \to A(G/H)$ given by $\sum_{i=1}^{n} r(g_i)g_i \to \sum_{i=1}^{n} r(g_i)g_iH$.

4.3 Corollary. If $R = AG$ then $R/\omega G \cong A$.

Proof. $R/\omega G \cong A(G/G) \cong A$.

4.4 Definition. Let A be a ring and let $n \in \mathbb{Z}$. Then n is said to be invertible in A if $n \cdot 1_A$ is a unit of A.

4.5 <u>Theorem</u> ([6], p.660). The group ring $R = AG$ is regular if and only if

(1) A is regular,

(2) every finitely generated subgroup of G is finite,

(3) the order of any finite subgroup of G is invertible in A.

4.6 <u>Proposition</u> ([6], p.659). If $R = AG$ then $g \in G$ has finite order if and only if $1 - g$ is a zero-divisor. When g has order n the right annihilator of $1 - g$ is $(1 + g + \ldots + g^{n-1})R$.

4.7 <u>Theorem</u> ([6], p.667). Let A be commutative. Then $R = AG$ is semiprime if and only if A is semiprime and the order of any finite normal subgroup of G is a non-zero-divisor in A.

It will henceforward be assumed that $R = AG$ is a commutative ring.

4.8 <u>Proposition</u>. If $R = AG$ is Baer, then A is Baer and the orders of the elements of G are invertible in A.

<u>Proof</u>. Let R be Baer and let J be an ideal of A. Then $JR = \{r \in R \mid \text{for all } g \in G, r(g) \in J\}$ is an ideal of R, consequently $(JR)^* = eR$, e an idempotent of R. If I is the annihilator of J in A then $I = \delta(e)A$. For $IR \subset (JR)^* = eR$ and therefore $I = IA = \delta(I)\delta(R) = \delta(IR) \subseteq \delta(eR) = \delta(e)A$. On the other hand we have $eJR = \langle 0 \rangle$, hence $\delta(eJR) = \delta(e)J = \langle 0 \rangle$ and so $\delta(e) \in I$. Furthermore $I = \delta(e)A$ is a direct summand of A because $\delta(e) = \delta(e^2) = [\delta(e)]^2$. Thus A is Baer.

In view of (4.3) the image of $\delta(e)A$ in $R/\omega G$ is a direct summand of $R/\omega G$ whose preimage in R is eR. Therefore $\omega G \subset eR$ and since this implies that $(1 - e)R \subset (\omega G)^*$ it follows from (4.6) that the elements of G have finite order. To see that the order of $g \in G$ is invertible in A suppose $g^n = 1$. If $(1 - g)^* = fR$, f an idempotent in R, then $(1 + g + \ldots + g^{n-1})R = fR$ by (4.6), hence $(n \cdot 1)A = \delta(f)A$ and by (4.7) $1 - \delta(f) = 0$. Accordingly $(n \cdot 1)A = A$.

2. Algebraically closed group rings.

4.9 Lemma. Let $R = AG$, R, A both regular. Then the quotient fields of R are algebraically closed if and only if the quotient fields of A are algebraically closed.

Proof. By (4.3) $A \cong R/\omega G$ and therefore the maximal ideals of A are the ideals $M/\omega G$, M a maximal ideal of R containing ωG. If now R/M is algebraically closed then so is $(R/\omega G)/(M/\omega G) \cong R/M$.

Conversely assume the quotient fields of A are algebraically closed. We recall from (2.44) that every maximal ideal of A is the contraction of an ideal of R, therefore it is the contraction of a maximal ideal of R. Let M be a maximal ideal in R and consider the field embedding $\bar{A} = A/M \cap A \to R/M = \bar{R}$. Let g be any element of G. By (4.5) $g^n = 1$ for some positive integer n and so $\bar{g}$ in $\bar{R}$ is integral over $\bar{A}$. Thus $\bar{R}$ is integral over $\bar{A}$ and by (1.3) the field embedding is algebraic. Since $\bar{A}$ is algebraically closed the embedding is an isomorphism by (3.8).

4.10 <u>Proposition</u>. Let $R = AG$ be regular. Then if R is algebraically closed, so is A.

<u>Proof</u>. 4.5, 4.8, 4.9, 3.9.

4.11 <u>Lemma</u>. Let G be a finite group, and $R = AG$, a regular group ring, where A is regular, algebraically closed and of prime characteristic. Then R is algebraically closed.

<u>Proof</u>. By (4.9) and (3.9), it suffices to show that R is Baer. $Q(R) = Q(A)G$ by ([5], 3.6). Let e be an idempotent in $Q(R)$, say

$$e = \sum_{i=1}^{n} q_i g_i \ , \quad q_i \in Q(A) \ , \quad g_i \in G \ , \quad i = 1,2,\dots, n \ .$$

Let p be the characteristic of A. By the binomial theorem $e = e^{p^\alpha} = \sum_{i=1}^{n} q_i^{p^\alpha} g_i^{p^\alpha}$, α any positive integer, because the binomial coefficients $\binom{p^\alpha}{k}$ $0 < k < p^\alpha$ are multiples of p. Thus exponentiation by p^α acts as a permutation on the support of e.

Let g_i be an element in the support of e. By the above remarks

$$0 \neq e(g_i^{p^\alpha}) = e^{p^\alpha}(g_i^{p^\alpha}) = [e(g_i)]^{p^\alpha} = [e(g_i)]^{p\alpha}[e(g_i)]^{p^\alpha - p\alpha}$$

and it follows that for any positive integer α $e(g_i^{p\alpha}) \neq 0$ and $e(g_i^{p^\alpha - p\alpha}) \neq 0$. Now the set $\{g_i^{pj} \mid j = 0,1,2,\dots\}$ lies in the (finite) support of e and therefore there exist positive integers m, n, $m > n$, such that $g_i^{pm} = g_i^{pn}$. Reading off its coefficients in the equation $e^{pm} = e^{pn}$ yields the equation $q_i^{pm} = q_i^{pn}$, since $Q(A)G$ is a free $Q(A)$-module. Thus q_i is integral over A. But

A is algebraically closed and so by (1.33) it coincides with its integral closure in $Q(A)$. Thus AG contains all idempotents of $Q(A)G$ and is consequently Baer by (1.41) and (2.40)

4.12 <u>Proposition</u>. Partial converse to (4.10). Let $R = AG$ be a regular group ring. Then R is algebraically closed provided that G is finite and A is algebraically closed and of non-zero characteristic.

<u>Proof</u>. A is regular and so it has no non-trivial nilpotent elements; in particular the characteristic of A must be square-free, say $n = \prod_{i=1}^{m} p_i$. It follows from (2.6) that there exists for each p_i an idempotent $1 - e_i \in A$ such that $(1 \cdot p_i)A = (1 - e_i)A$ and if $A_i = \{a \in A \mid p_i a = 0\}$ it is easy to see that $A_i = e_i A$. Thus, A_i is a ring with unit element e_i and since p_i is prime it is of characteristic p_i. Furthermore A_i is a regular ring. For if $a \in A_i$ there is $c \in A$ such that $a = a^2 c$. If $c^2 a = d$ then $a = a^2 c = (a^2 c)ac = a^2 d$ hence $d \in A_i$ is a quasi-inverse for a.

Because A_i is of characteristic p_i the rings $A_1, A_2, \ldots, A_m$ are pairwise disjoint, hence $e_1, e_2, \ldots, e_m$ are orthogonal idempotents and therefore

$$1 - (e_1 + e_2 + \ldots + e_m) = (1 - e_1)(1 - e_2) \ldots (1 - e_m) = 0 .$$

By (1.35) $A = A_1 + A_2 + \ldots + A_m$ is a direct sum, consequently there is an isomorphism of A onto the finite direct product of the rings A_i. Since the ideals of $\prod A_i$ are direct products of ideals of the rings A_i it follows from the fact that A is Baer that each

A_i is Baer. Then by (1.35) $A_i = e_iA \cong A/(1-e_i)A$ whence it is algebraically closed by 3.10. Now $R = AG \cong \prod A_iG$. By (4.11) each A_iG is algebraically closed. By (3.11) R is algebraically closed.

CHAPTER 5

APPLICATIONS TO RINGS OF CONTINUOUS FUNCTIONS

1. The complete ring of quotients of C(X).

5.1 Definition. The set of all continuous real-valued functions on a topological space X is denoted by $C(X)$. A ring structure is given to $C(X)$ by defining

$$(f+g)(x) = f(x) + g(x) , \quad (fg)(x) = f(x)g(x) ,$$
$$(-f)(x) = -f(x) .$$

The unit element of $C(X)$ is the constant function 1 whose constant value is the real number 1 and the zero element is the constant function 0. The multiplicative inverse f^{-1}, when it exists, is characterized by the formula $f^{-1}(x) = \frac{1}{f(x)}$. It is obvious that with operations thus defined $C(X)$ is a commutative semiprime ring. The complete ring of quotients of $C(X)$, denoted $Q_R(X)$, is then a regular ring by 2.47.

Following ([10], 3.1) it will be assumed that the space X is completely regular, i.e. it is a Hausdorff space such that whenever F is a closed set and x is a point in its complement, there exists a function $f \in C(X)$ such that $f(x) = 1$ and $f(F) = \{0\}$.

5.2 Definition. The zero-set of $f \in C(X)$ is the set $Z(f) = \{x \in X \mid f(x) = 0\}$. Every zero-set is closed because it is the preimage of a point in $\mathbb{R}$. The complement of a zero-set is the cozero-set of f, denoted $\operatorname{coz} f$.

For any ideal I in C(X)

$Z(I) = \cap_{f\in I} Z(f)$,

$\text{coz } I = \cup_{f\in I} \text{coz } f$.

5.3 <u>Definition</u>. The <u>closure</u> of a subset V of a topological space X is the intersection of the members of the family of all closed sets containing V . The closure of V in X will be denoted cl V . A set V is <u>dense</u> in X if and only if the closure of V is X .

5.4 <u>Lemma</u>. If V is dense in X then the ring homomorphism $f \to f|V$ from C(X) into C(V) is a monomorphism.

<u>Proof</u>. Let $0 \neq f \in C(X)$. Then $Z(f) \not\supset V$ since Z(f) is closed and V is dense in X . Thus $f|V \neq 0$. One may then write $C(X) \subset C(V)$.

An ideal I in a ring R may, of course, be regarded as an R-module. The set of all R-module homomorphisms from I into R will now be denoted Hom I ; it is well known that Hom I is itself an R-module. We recall from (1.13) that an ideal D of R is said to be (rationally) dense if its only annihilator in R is <0> .

5.5 <u>Lemma</u> ([9]). If D and D' are dense ideals with $D \supset D'$ then the restriction homomorphism $\phi \to \phi|D'$ from Hom D into Hom D' is a monomorphism of R-modules.

<u>Proof</u>. If $0 \neq \phi \in \text{Hom } D$, then $\phi(d) \neq 0$ from some $d \in D$; since D' is dense, there exists $d' \in D'$ such that $0 \neq \phi(d) \cdot d' = \phi(dd')$; therefore $\phi|D' \neq 0$. One may then write $\text{Hom } D \subset \text{Hom } D'$.

5.6 Theorem ([9]). An ideal D in $C(X)$ is (rationally) dense if and only if $\operatorname{coz} D$ is (topologically) dense in X.

Proof. D is dense if and only if for all $g \in C(X)$, $gD = \langle 0 \rangle$ implies $g = 0$; if and only if for all $g \in C(X)$, $Z(g) \supset \operatorname{coz} D$ implies $g = 0$. If now $\operatorname{cl}(\operatorname{coz} D) = F \neq X$ there is an $x \in X - F$ and by complete regularity there exists $g \in C(X)$ such that $g(x) \neq 0$ and $g(F) = \{0\}$. Then $Z(g) \supset F \supset \operatorname{coz} D$ and so $g = 0$, a contradiction. Thus $\operatorname{coz} D$ is dense.

Conversely suppose $F = X$. If $Z(g) \supset \operatorname{coz} D$ for any $g \in C(X)$ then clearly $Z(g) = X$ hence $g = 0$.

5.7 Lemma ([9]). Every open set U in X is of the form $\operatorname{coz} I$ for some ideal I in $C(X)$.

Proof. Define $I = \{f \in C(X) \mid \operatorname{coz} f \subset U\}$. Then $0 \in I$ since $\operatorname{coz} 0 = \phi$. If $f, g \in I$ and $h \in C(X)$ then

$\operatorname{coz} f + g = \{x \in X \mid f(x) + g(x) \neq 0\} \subseteq \operatorname{coz} f \cup \operatorname{coz} g \subset U$ and

$\operatorname{coz} hf = \{x \in X \mid h(x)f(x) \neq 0\} = \operatorname{coz} h \cap \operatorname{coz} f \subset U$. Thus I is an ideal and $\operatorname{coz} I \subset U$.

Conversely assume $y \in U$. Because U is open it is a neighbourhood of y and by complete regularity there exists $f \in C(X)$ such that $f(y) \neq 0$ and $f(X - U) = \{0\}$. Then $f \in I$ and

$y \in \operatorname{coz} f \subseteq \cup_{f \in I} \operatorname{coz} f = \operatorname{coz} I$.

5.8 Corollary. The dense open sets of X are precisely the sets $\operatorname{coz} D$, D a dense ideal of $C(X)$.

Proof. (5.6), (5.7).

5.9 Proposition ([9]). If V is a dense open set in X then $C(V)$ is a ring of quotients of $C(X)$.

Proof. $C(V) \supset C(X)$ by (5.4). Now consider any $h \in C(V)$ and let $v \in V$. By complete regularity there exists $f \in C(X)$ that vanishes on $X - V$ but not at v , hence $v \in \text{coz } f$. If $h \neq 0$ choose $y \in \text{coz } h$, then $v \in \text{coz } h \cap \text{coz } f = \text{coz } hf$ and $hf \neq 0$. Furthermore $Z(hf) = Z(h) \cup Z(f) \supset Z(f) \supset X - V$, therefore $hf \in C(X)$. Thus $h(h^{-1}C(X)) \neq 0$ and, since $C(X)$ is semiprime, $C(V)$ is a ring of quotients of $C(X)$ by (2.36).

5.10 Definition. A filter F in a set X is a family of non-empty subsets of X such that

(1) The intersection of two members of F belongs to F .

(2) If $F_1 \in F$ and $F_1 \subset F_2$ then $F_2 \in F$. In view of (5.6) and (1.14) the family of dense open sets of a completely regular space X forms a filter.

5.11 Proposition. $Q_R(X)$ can be realized as the set of all continuous real-valued functions defined on dense open sets of X , modulo the relation which identifies functions that agree on the intersection of their domains.

Proof. Let V be a dense open subset of X and let $f \in C(V)$. Then f may be identified with the mapping $d \to fd$, $d \in f^{-1}C(X)$; as such it belongs to $\text{Hom } f^{-1}C(X)$. Now suppose V_1 and V_2 are dense open sets, $f_1 \in C(V_1)$ and $f_2 \in C(V_2)$. Then by (5.10) $C(V_1 \cap V_2)$ is a ring of quotients of $C(X)$ and there are restriction monomorphisms $f_i \to f_i \mid V_1 \cap V_2$, $i = 1, 2$; thus

$f_1 + (-f_2) \in C(V_1 \cap V_2)$. If f_1 and f_2 agree on $V_1 \cap V_2$ then $f_1 d = f_2 d$ for all $d \in (f_1 + (-f_2))^{-1} C(X)$ and by (1.19) we have $\theta f_1 = \theta f_2$.

Conversely, as in ([9], 2.5) assume D is a dense ideal of C(X) and let $\phi \in \text{Hom } D$ be given. For $x \in \text{coz } D$, choose $d \in D$ for which $d(x) \neq 0$ and define

$$g(x) = \frac{\phi(d)(x)}{d(x)} .$$

Since $\phi(d)d' = \phi(d')d$ it follows that $\frac{\phi(d)}{d} = \frac{\phi(d')}{d'}$ if $\text{coz } d = \text{coz } d'$, therefore this definition is independent of d . Furthermore $\phi(d) \in C(X) \subseteq \check{C}(\text{coz } D)$ by (5.6) and (5.4). Thus for each $x \in \text{coz } D$ there is a continuous function that agrees with g on a neighbourhood coz d of x and so g is continuous on its domain coz D . Now consider any $d \in D$. For each $x \in \text{coz } D$ a suitable d' can be found so that $\phi(d)(x) = \frac{\phi(d')(x)}{d'(x)} d(x) = g(x)d(x)$, hence $\phi(d) = g \cdot d$ and therefore $\theta\phi = \theta g$.

5.12 <u>Definition</u> ([10]). A totally ordered field F is said to be <u>real-closed</u> if it satisfies the following conditions which are known to be equivalent:

(1) Every positive element is a square, and every polynomial over F of odd degree has a zero in F .

(2) $F(\sqrt{-1})$ is algebraically closed.

(3) F has no proper algebraic extension to a totally ordered field.

5.13 **Lemma**. The factor fields of $Q_R(X)$ are real-closed.

Proof. It is established in ([10], 5.5) that the residue class rings $C(X)/P$ are totally ordered whenever P is a prime ideal in $C(X)$; in particular, if $f \in C(X)$ is positive on some zero-set of P then $\overline{f} > 0$, $\overline{f} \in C(X)/P$. Therefore the factor fields of $Q_R(X)$ are totally ordered. For if $\overline{q} \in Q_R(X)/M$, q a preimage in $Q_R(X)$, let q denote the representative function defined on a dense open set V in X . Then $\overline{q} \in C(V)/M \cap C(V)$ and if q is positive on some zero-set of the prime ideal $M \cap C(V)$ then $\overline{q} > 0$.

To see that the factor fields of $Q_R(X)$ are real-closed one observes that the proof of the theorem for the ring $C(X)$, given in ([10], 13.4), goes over entirely; the coefficients $q_1, q_2, \ldots, q_n$ in a polynomial of odd degree $P(\lambda) = \lambda^n + q_1\lambda^{n-1} + \ldots + q_n$ over $Q_R(X)$ may be viewed as representative functions defined on dense open sets $V_1, V_2, \ldots, V_n$ of X. Since $V = \bigcap_n V_k$ is a dense open set in X by (5.10) one now argues on V .

5.14 **Theorem**. Let $Q_C(X)$ denote the ring of all complex-valued functions defined on dense open subsets of X , modulo the relation which identifies functions that agree on the intersection of their domains. Then $\Omega(Q_R(X)) = Q_C(X)$.

Proof. The natural embedding in algebraic. Take $f \in Q_C(X)$, $f \neq 0$. Let a and b denote the purely real and purely complex parts of f ; if p_1, p_2 are the projections from $\mathbb{C}$ to $\mathbb{R}$ then $a = p_1 f$, $b = p_2 f$, are each the composition of continuous maps, therefore they are continuous real-valued functions on the domain of f . Now

$f = a + ib$ and $f^2 - 2af + (a^2 + b^2) = 0$ identically on the domain of f. This is a monic equation in f with coefficients from $Q_R(X)$; furthermore, the absolute term in not zero because f is not the zero function.

It follows from (2.24) and (2.41) that $Q_C(X)$ is a regular Baer ring. Therefore by (2.46) there is a one-one correspondence between the maximal ideals of $Q_R(X)$ and those of $Q_C(X)$ and so the factor fields of $Q_C(X)$ are algebraic over the real-closed factor fields of $Q_R(X)$ by (1.3) and (2.20). Since they contain the image of the function which has constant value i they are algebraically closed by (5.12) and the result follows from (3.9).

2. <u>The Dedekind completion of $Q_R(X)$.</u>

5.15 <u>Definition</u> ([15]). A partially ordered set I is called a <u>directed</u> set if given $i,j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$. Assume I is directed. Let A be a category and $\{A_i\}$ a family of objects in A indexed by I. For each pair i,j in I such that $i \leq j$ let $f_{ij}: A_i \to A_j$ be a morphism satisfying the conditions:

(1) f_{ii} is the identity morphism of A_i for all $i \in I$;

(2) $f_{ik} = f_{jk} \cdot f_{ij}$ whenever $i \leq j \leq k$.

Then the objects A_i and morphisms f_{ij} are said to form a direct system (A_i, f_{ij}).

Let B be the category whose objects are the pairs $(A, (f_i))$, where $A \in Ob(A)$ and (f_i) is a family of morphisms $f_i: A_i \to A$, $i \in I$, such that $f_i = f_j \cdot f_{ij}$ whenever $i \leq j$. A <u>direct limit</u>

for the family $\{f_{ij}\}$, written $\lim\limits_{\to} A_i$, is an object $(A, (f_i))$ in $\mathcal{B}$ such that for every object in $\mathcal{B}$ there exists a unique morphism of $(A, (f_i))$ into this object.

5.16 Proposition. Let $\mathcal{S}(X)$ be a filter base consisting of dense subsets of a completely regular space X. Then $\mathcal{S}$ is a directed set indexing the family $\{C(S)\}_{S\in\mathcal{S}}$ and $\lim\limits_{\to S\in\mathcal{S}} C(S)$ may be realized as the set of all continuous real-valued functions defined on subsets of $\mathcal{S}$, modulo the relation which identifies functions that agree on the intersection of their domains.

Proof. $\mathcal{S}$ is closed under finite intersection, consequently $(\mathcal{S}, \supseteq)$ is a directed set. Furthermore the restriction homomorphisms $\phi_{SS'}: f \to f|S'$, when $f \in C(S)$ and $S \supseteq S'$, are one-one by (5.4) and therefore satisfy conditions (1) and (2) above. Now denote by $C[\mathcal{S}]$ the ring of equivalence classes with respect to the relation on $\bigcup_{S\in\mathcal{S}} C(S)$ that identifies functions which agree on the intersection of their domains, and let $\phi_S: C(S) \to C[\mathcal{S}]$ be defined as follows: $\phi_S(f_S) = f$ whenever f_S is a representative function for f. Clearly $\{\phi_S\}_{S\in\mathcal{S}}$ is a family of ring homomorphisms such that $\phi_S = \phi_{S'} \cdot \phi_{SS'}$ whenever $S \supseteq S'$.

It remains to show that $C[\mathcal{S}] = \lim\limits_{\to S\in\mathcal{S}} C(S)$. Thus suppose $(T, \{\psi_S\})$ is a pair consisting of a ring T and a family of homomorphisms $\psi_S: C(S) \to T$, $S \in \mathcal{S}$, such that $\psi_S = \psi_{S'} \cdot \phi_{SS'}$ whenever $S \supseteq S'$. Define $\psi: C[\mathcal{S}] \to T$ by $\psi(f) = \psi_S(f_S)$, where f_S is any representative function for f. Then it is easily seen that ψ is a homomorphism of rings. If also $\psi' \cdot \phi_S = \psi_S$ then $\psi(f) = \psi \cdot \phi_S(f_S) = \psi_S(f_S) = \psi' \cdot \phi_S(f_S) = \psi'(f)$, hence $\psi' = \psi$.

5.17 <u>Corollary</u>. Let $V_o(X)$ denote the filter base of all dense open subsets of X. Then $C[V_o(X)] = \lim_{\to V \in V_o} C(V) = Q_R(X)$.

5.18 <u>Remark</u>. As shown in ([10], 1.2), $C(X)$ is a lattice ordered ring under the partial order given by:

$$f \leq g \text{ if and only if } f(x) \leq g(x) \text{ for all } x \in X .$$

To define an ordering on $C[S]$ suppose $f,g \in C[S]$ and let f_1, g_1 be any of their representative functions defined respectively on dense subsets S_1, T_1 in S. By $f \leq g$ will then be meant that $f_1 \leq g_1$ on $S_1 \cap T_1$. If also $f_2 \in C(S_2)$, $g_2 \in C(T_2)$ are representative functions for f and g then clearly $f_2 \leq g_2$ on $S = S_1 \cap S_2 \cap T_1 \cap T_2$, therefore the function $(g_2 - f_2) \wedge 0 = (g_2 - f_2) - |g_2 - f_2| \in C(S_2 \cap T_2)$ agrees with the constant function 0 on S, and because S is a dense subset of X this means that $(g_2 - f_2) \wedge 0 = 0$. Thus the ordering is well defined and so each ring $C[S]$ is a lattice with respect to the pointwise definition of order.

5.19 <u>Definition</u> ([9]). Let $C = C[S]$ be as in (5.16). If every nonvoid subset with an upper bound in C has a supremum in C, then C is <u>Dedekind - complete</u>. If C is Dedekind - complete, $B \subset C$, and every element of C is the supremum of some subset of B, then C is the <u>Dedekind completion</u> of B.

5.20 Definition. The subring of bounded functions in $C(X)$ is denoted by $C^*(X)$. It is shown in ([9], Theorem 2.3) that $C(X)$ is a ring of quotients of $C^*(X)$ and that therefore $Q_R(X)$ is the maximal ring of quotients of $C^*(X)$.

5.21 Definition. A compactification of a completely regular space X is a compact space in which X is dense. Any completely regular space X has a compactification βX called the Stone - Čech compactification of X in which X is C^*-embedded, that is every function in $C^*(X)$ can be extended to one in $C(\beta X)$. Moreover βX is completely regular and Hausdorff (See [10], 3.14 and 6.5).

5.22 Lemma. $Q_R(\beta X) = Q_R(X)$.

Proof. We recall that a continuous real-valued function on a compact space is bounded, thus $C(\beta X) = C^*(\beta X)$. Since X is dense in βX it follows from (5.4) that the embedding of $C^*(X)$ into $C(\beta X)$ is onto, hence $C^*(\beta X) = C^*(X)$ and therefore $Q_R(\beta X) = Q_R(X)$ by (5.20).

5.23 Definition. A subset of a space X is a G_δ if it is a countable intersection of open sets.

By the Baire category theorem a countable intersection of dense open sets of a compact Hausdorff space X is dense in X, therefore the family $G_o(\beta X)$ of all dense G_δ's in βX is closed under countable intersection and any finite intersection of G_δ's contains a member of G_o. Thus $G_o(\beta X)$ is a filter base of dense subsets of βX and by (5.16) $C[G_o(\beta X)]$ is the ring of all continuous real-valued functions defined on dense G_δ's of βX modulo the usual relation. As is proved in ([9], 4.6 and 4.9) $C[G_o(\beta X)]$ is the ring

$\overline{Q}_R(X)$ presented in ([9], 4.1) and $\overline{Q}_R(X)$ is the Dedekind completion of $Q_R(X)$. We denote by $\overline{Q}_C(X)$, the ring of all complex-valued continuous functions defined by the same filter, modulo the same relation.

5.24 Theorem. $\Omega(\overline{Q}_R(X)) = \overline{Q}_C(X)$.

Proof. The fact that $\overline{Q}_C(X)$ has algebraically closed factor fields and that the embedding is algebraic proceeds as in (5.14). $\overline{Q}_C(X)$ is Baer because $\overline{Q}_R(X)$ is rationally complete ([9], 4.8) and therefore Baer by (1.41).

3. Locally constant functions in $Q_R(X)$.

5.25 Definition ([9]). Let $V \in \mathcal{V}_o(X)$. The function $f \in C(V)$ is locally constant provided that $\{x \mid f(x) = r\}$ is open in V for each $r \in \mathbb{R}$. Define $L(V)$ to be the ring of locally constant functions in $C(V)$, and ${}_RQ_L(X) = \varinjlim_{V\in\mathcal{V}_o(X)} L(V)$, those functions in $Q_R(X)$ which are locally constant on their domain of definition.

5.26 Theorem. $\Omega[{}_RQ_L(X)] = {}_CQ_L(X)$.

Proof. It is easily seen that the composition of a locally constant complex-valued function with either projection to the reals is still locally constant. Thus the natural embedding is algebraic as in (5.14). Also, since ${}_RQ_L(X)$ is rationally complete ([9], 4.3), both ${}_RQ_L(X)$ and ${}_CQ_L(X)$ are regular Baer rings.

Now a monic polynomial over ${}_CQ_L(X)$ is one over $Q_C(X)$, therefore it has a root in $Q_C(X)$ since this ring is algebraically closed by

(5.14). In fact the root lies in ${}_{C}Q_{L}(X)$. For suppose $g \in Q_{C}(X)$ satisfies $g^{n} + f_{n-1}g^{n-1} + \ldots + f_{1}g + f_{0} = 0$, $f_{i} \in {}_{C}Q_{L}(X)$, and D is a dense open set in X , common to the domains of definition of the $n + 1$ functions appearing in the equation. We use the same letters to denote members of Q and their representative functions defined on D . Assume $g(d) = z$, z some complex number. Then z is a root of the equation

$$(1) \quad x^{n} + f_{n-1}(d)x^{n-1} + \ldots + f_{1}(d)x + f_{0}(d) = 0 .$$

Since each f_{i} is locally constant, there exist U_{i} , $i = 0,1,\ldots,n-1$, open neighbourhoods of d in D such that f_{i} is fixed on U_{i} . Let $U = \cap_{i} U_{i}$. Then each f_{i} is fixed on the open set U and it follows that each element of g(U) is a root of the equation (1), that is g assumes on U only values among the finitely many roots of (1). Because $\mathbf{C}$ is Hausdorff there exists an open set W in $\mathbf{C}$ containing z and excluding all other roots of (1), hence g is constant on the open neighbourhood $U \cap g^{-1}(W)$ of d and so $g \in L(D)$. Thus all monics over ${}_{C}Q_{L}(X)$ have roots in ${}_{C}Q_{L}(X)$ and by (3.9) ${}_{C}Q_{L}(X)$ is algebraically closed.

4. Rings of functions into a finite field.

5.27 Definition. The ring of continuous functions from a topological space X into a finite field F , topologically discrete, is denoted by C(X, F).

We apply the proof of ([10] , Theorem 3.9) to show that compact Hausdorff totally disconnected spaces suffice for the study of functions to F , just as completely regular spaces suffice for the study of real-valued functions.

5.28 <u>Proposition</u>. For every topological space X, there exists a compact Hausdorff totally disconnected space Y, such that the mapping $g \to g \circ \tau$ is an isomorphism of $C(Y, F)$ onto $C(X, F)$.

<u>Proof</u>. Let X be a topological space and let Y be the set of all equivalence classes with respect to the relation on X given by $x \equiv x'$ if $f(x) = f(x')$ for every $f \in C(X, F)$. We define a mapping τ of X onto Y thus: τx is the equivalence class that contains x.

With each $f \in C(X, F)$ associate a function $g: Y \to F$ such that $g(y)$ is the common value of $f(x)$ at every point $x \in y$. Then $f = g \circ \tau$. Let C' denote the family of all such functions, that is, $g \in C'$ if and only if $g \circ \tau \in C(X, F)$. Now endowing Y with the weak topology induced by C', we find that every function in C' is continuous on Y, hence $C' \subset C(Y, F)$. Furthermore $\tau: X \to Y$ is continuous by ([10], 3.8).

Since F is finite and topologically discrete the weak topology generated by C' has a subbase consisting of finitely many closed-and-open sets. Therefore Y is compact and totally disconnected ([10], 16.17). It is Hausdorff because if y and y' are distinct points of Y there evidently exists $g \in C'$ such that $g(y) \neq g(y')$.

Finally consider any function $h \in C(Y, F)$. Since τ is continuous, $h \circ \tau$ is continuous on X. This implies that $h \in C'$, consequently $C' \supset C(Y, F)$. Thus $C' = C(Y, F)$; and it is clear that the mapping $g \to g \circ \tau$ is an isomorphism.

5.29 <u>Proposition</u>. Let X be compact, Hausdorff, and totally disconnected. Then $Q(C(X, F)) \cong C(G(X), F)$, where $G(X)$ is the projective cover of X , due to Gleason.

<u>Proof</u>. From (2.63) there is a surjection $t: G(X) \to X$ with the property that t maps any proper closed subset of $G(X)$ onto a proper subset of X . As well, $G(X)$ is compact, T_2 and extremally disconnected. It is clear that t induces a ring monomorphism $t^*: C(X, F) \to C(G(X), F)$ defined by $t^*(g) = g \circ t$.

By ([17], p. 104, 24.2), $C(G(X), F)$ is self injective (see [14], p. 46, exercise 6), therefore it is rationally complete by (1.24). One claims that $C(G(X), F)$ is a ring of quotients of $C(X, F)$. Take $f \in C(G(X), F)$, $f \neq 0$, then f defines a partition of $G(X)$ into disjoint clopen sets $A_1, A_2, \ldots, A_n$, where the A_i are the inverse images under f of the different elements of F . Since f is non-zero, we assume, without loss of generality, that $f(A_1) = d \neq 0$ in F . The set $B = \cup_{i=2}^{n} A_i$ is a proper closed set in $G(X)$. Thus $t(B)$ is a proper closed set in X . Let $D = X \setminus t(B)$. Then D is open and because X has a base of clopen sets it contains a non-void clopen set, say C . Clearly $t^{-1}(C) \subset A_1$. Consider the function $h \in C(X, F)$ defined as follows: $h(C) = 1$, and $h(X \setminus C) = 0$. Since each $x \in C$ is the image $t(y)$ of some $y \in t^{-1}(C)$ we have $h(x) = h(ty) = h \circ t(y) = t^*(h)(y)$, thus $t^*(h) \in C(G(X), F)$ is 1 on the clopen set $t^{-1}(C)$ in A_1, and zero elsewhere. Thus $ft^*(h)$ is the function which is d on $t^{-1}(C)$. But $ft^*(h)(t^{-1}(C)) = d \cdot h(t\, t^{-1}(C)) = dh \circ t(t^{-1}(C)) = t^*(dh)(t^{-1}(C))$. Therefore $ft^*(h) = t^*(dh) \neq 0$.

5.30 Remark. $Q(C(X, F))$ is an algebraic extension of $C(X, F)$. For if n is the order of F then $r^n(x) = r(x)^n = r(x)$, hence $r^n = r$ for all $r \in C(X, F)$. If $q \in Q(C(X, F))$, then by (1.25) there is a dense ideal D in $C(X, F)$ such that $qD \subset C(X, F)$. If $d \in D$, then $q^n d = q^n d^n = (qd)^n = qd$. Therefore $(q^n - q)D = \langle 0 \rangle$ and $q^n - q = 0$, an equation of integral dependence.

Thus by (3.27) one can restrict the study of the algebraic closure of $C(X, F)$ to the case where X is extremally disconnected.

5.31 Theorem. Let X be compact, Hausdorff, and extremally disconnected. Then $\Omega C(X, F) = C(X, \Omega F)$, where $\Omega(F)$ is given the discrete topology.

Proof. It is clear that $C(X, \Omega F)$, under pointwise addition and multiplication, is a commutative semiprime ring with 1 the constant function whose constant value is the identity element of $\Omega(F)$. As well, $C(X, \Omega F)$ extends $C(X, F)$. To see that $C(X, \Omega F)$ is regular we first observe that because X is compact there is for each $f \in C(X, \Omega F)$ a finite cover of X by disjoint clopen sets $f^{\leftarrow}(a_1), f^{\leftarrow}(a_2), \ldots, f^{\leftarrow}(a_n)$, the preimages of points $a_1, a_2, \ldots, a_n \in \Omega(F)$. This implies that f has a quasi-inverse $g \in C(X, \Omega F)$ defined thus: $g(x) = 0$ for $x \in f^{\leftarrow}(0)$ and $g(x) = \frac{1}{f(x)}$ for $x \in f^{\leftarrow}(a)$, $a \neq 0$.

Now $C(X, \Omega F)$ is essential over $C(X, F)$. For if $f \neq 0$ then fg is defined on a finite partition of X and is 0- or 1- valued. Since $f \neq 0$ it follows that $0 \neq fg \in C(X, F)$ establishing essentiality.

$C(X, \Omega F)$ is integral over $C(X, F)$. Consider an arbitrary finite (clopen) partition of X , say $X = \cup_{i=1}^{n} A_i$. Let k_i be the function defined as follows: $k_i(A_i) = x_i$, an arbitrary element of $\Omega(F)$, $k_i(X \setminus A_i) = 0$. Since the element x_i satisfies an integral equation over F, it follows that k_i does as well. (As coefficients in the equation for k_i choose functions from $C(X, F)$, defined so as to take on A_i the appropriate value in F and 0 elsewhere). But the sum of integrally dependent elements is again integrally dependent, and so the function which has arbitrary values of $\Omega(F)$ assigned to the elements of an arbitrary finite partition of X is integral over $C(X, F)$. The set of these functions is precisely $C(X, \Omega F)$.

Since $C(X, F)$ is self-injective it is Baer by (1.41) and therefore $C(X, \Omega F)$ is Baer by (1.43). One claims that every monic equation over $C(X, \Omega F)$ has a root in $C(X, \Omega F)$. Let

$$x^n + x^{n-1}g_{n-1} + \dots + xg_1 + g_0 = 0 , \quad g_i \in C(X, \Omega F) ,$$

be such a monic equation for which one seeks a root. Each g_i is constant on the elements of a finite clopen partition of X. Let Π be the common refinement of all these partitions. Π is clearly, itself, a finite clopen partition, say $X = \cup_j D_j$. Since each g_i is constant on each D_j, and since $\Omega(F)$ is an algebraically closed field, it follows that there is a root, say y_j in $\Omega(F)$ for the equation

$$\sum_{i=0}^{n} x^i g_i(D_j) = 0 .$$

The function $y: X \to \Omega(F)$ which has value y_j on D_j is in $C(X, \Omega F)$, and it satisfies the equation in question. By (3.9) $C(X, \Omega F)$ is algebraically closed and the proof of the theorem is complete.

5. The algebraic closure of a Boolean ring.

5.32 Definition. A topological space is said to be a Boolean space if it is compact Hausdorff and totally disconnected. A Boolean space is complete if the closure of every open set is open, that is, if it is extremally disconnected.

The algebra of all clopen sets in a Boolean space X is called the dual algebra of X .

5.33 Theorem ([13], p.92). The dual algebra A of a Boolean space X is complete if and only if X is complete.

5.34 Definition. Following [13] the two-element field will be denoted by 2 . It is shown in ([13], Section 17 and Lemma 2, Section 18) that the family of all functions from an arbitrary topological space into the topologically discrete space 2 is a Boolean space and so is the family of all homomorphisms from an arbitrary Boolean algebra to the Boolean algebra 2.

The set X of all 2-valued homomorphisms on a Boolean algebra A is called the dual space of A .

5.35 Lemma ([13], p.77). For every non-zero element a of every Boolean algebra A there is a 2-valued homomorphism x on A such that x(a) = 1 .

Proof. Since a Boolean ring is semiprimitive the conclusion can be rephrased as follows: there exists a maximal ideal M in A such that $a \notin M$.

5.36 Theorem (Stone representation theorem, [13], p.78). The second dual of every Boolean algebra A is isomorphic to A. More explicitly, if B is the dual algebra of the dual space X of A, and if $f(a) = \{x \in X \mid x(a) = 1\}$ for each a in A, then f is an isomorphism from A onto B.

5.37 Corollary. If A is a complete Boolean ring then Spec A is an extremally disconnected Boolean space by (2.59) and (2.62), and it follows from (5.33) that the Boolean algebra B of all sets $V(a) = \{M \in \text{Spec } A \mid a \notin M\}$ (see 2.57) is complete. We have $V(a \vee b) = V(a) \cup V(b)$, $V(a \wedge b) = V(a) \cap V(b)$ and $V(a') = B \setminus V(a)$. Furthermore V is onto by (2.57) and one-one by (5.35). It is therefore a Boolean isomorphism from A onto B and by (5.36) this implies that A is isomorphic to the second dual C of B. One notes that every function in C is continuous on Spec A because it is continuous on $\setminus V(a)$ and on $B \setminus V(a)$.

5.38 Remark. In view of Stone duality, any complete Boolean algebra can be represented as the ring of continuous functions from its spectrum to the two-element field. Thus (5.31) contains as a special case a representation for the algebraic closure of a complete Boolean ring. Since the complete ring of quotients of a Boolean ring is a Boolean ring ([14], p.44), the embedding of any Boolean ring into its complete ring of quotients is an algebraic embedding into a complete Boolean ring; and it follows from (3.6) that this disposes of the algebraic closure of all Boolean rings.

CHAPTER 6

SATURABLY CLOSED RINGS

The saturable closure of a commutative ring R was defined by W. Borho ([2]). As is $\Omega(R)$, the saturable closure of R is a ring essential and integral over R, coinciding in fields with its algebraic closure.

It should be pointed out that extensions of the ring R are viewed in [2] as R-algebras. A homomorphism of extensions of R thus induces the identity on R and any homomorphism of essential extensions is then a monomorphism by (1.2).

1. Essential F-extensions.

6.1 Definition ([2]). Let R be a ring, $F \subset R[x]$ a subset of monic polynomials and let $f \in F$. A ring S extending R will be called an f-extension of R if S is generated as an R-algebra by roots of f. We say a ring S is an F-extension of R whenever S is generated by f-extensions S_f, $f \in F$.

The class of F-extensions of a ring R is closed under direct limits (see 5.15). For suppose (S, ϕ_i) is the direct limit of a family $\{S_i\}$, $i \in I$ of F-extensions of R indexed by a directed set I. We recall that S is an R-algebra each of whose elements can be written in the form $\phi_i(s_i)$ for some $i \in I$ and some $s_i \in S_i$. Since each S_i is an F-extension so is S.

6.2 Lemma. A direct limit of essential extensions of a ring R is essential over R.

Proof. Let $\{S_i\}$ $i \in I$ be a family of essential extensions of R indexed by a directed set I and let (S, ϕ_i) be its direct limit. Any non-zero $s \in S$ can then be written as $s = \phi_i(s_i)$ for some $i \in I$ and for some non-zero $s_i \in S_i$. Now S_i is essential over R and so there is a $t_i \in S_i$ such that $0 \neq s_i t_i = r \in R$. Therefore $\phi_i(s_i)\phi_i(t_i) = \phi_i(r) = \phi_i(1)r = r$.

6.3 Lemma ([2]). Let R be a ring and let S be any extension of R. Then there is a homomorphism $\sigma: S \to T$ onto an essential extension T of R.

Proof. Suppose S is not essential over R. As in the proof of (3.24) there exists an ideal I of S that is maximal with respect to the property $I \cap R = \langle 0 \rangle$, hence $R \cong I + R/I \subset S/I = T$. Now $\sigma: S \to T$ is onto and it follows from the maximality of I that T is essential over R.

6.4 Definition ([2]). Let R be a ring and let $F \subseteq R[x]$ be a subset of monic polynomials. By an F-split extension of R will be meant a ring S extending R such that each f in F splits into linear factors in S. An F-split F-extension of R will be called an F-splitting ring.

6.5 __Lemma__ ([2]). If $F \subset R[x]$ is a subset of monic polynomials then there exists an F-splitting ring that is essential over R.

__Proof__. Let $f(x) = x^n + r_{n-1}x^{n-1} + \ldots + r_0$ be a monic polynomial in F. As shown in the proof of (3.9) there is an embedding $R \to R[x]/\langle f(x)\rangle$ such that $R[x]/\langle f(x)\rangle$ contains a root of r, thus assume there exists an extension $R[\alpha]$ of R generated over R by a root of f. Then $x - \alpha$ divides $f(x)$, hence $f(x) = (x-\alpha)g(x)$ where $g(x) \in R[\alpha][x]$ is a monic polynomial and any root of $g(x)$ is a root of $f(x)$. Since no monic polynomial is a zero-divisor it follows that repeated applications of the division algorithm yield an extension $R[\alpha_1,\ldots,\alpha_n]$ of R generated over R by roots of f such that f splits in $R[\alpha_1,\ldots,\alpha_n]$ into linear factors $(x-\alpha_1),\ldots,(x-\alpha_n)$. Therefore there is for any $f \in F$ an f-splitting ring S_f.

For each finite subset G of F let S_G denote the tensor product over R of the S_f for $f \in G$. If G' is another finite subset of F and $G \subseteq G'$ then there are R-algebra homomorphisms $S_f \to S_G$ and $S_f \to S_{G'}$ for $f \in G$, consequently there is an R-algebra homomorphism $\phi_{GG'}: S_G \to S_{G'}$ (See [15], p.420). It is easily seen that the finite subsets of F form a directed set indexing the family $\{S_G\}$ and that $(S_G, \phi_{GG'})$ is a direct system (5.15). Let S denote its direct limit. Then S is an F-splitting ring because each S_G is a G-splitting ring and the result follows from (6.3).

6.6 __Definition__. Let R be a ring, let $F \subset R[x]$ be a subset of monic polynomials and denote by K the class of essential F-extensions of R. An extension $S \in K$ will be called __K-maximal__ if from $T \in K$ and

$S \subseteq T$ follows $S = T$. A K-maximal extension of R is not unique in general as will now be shown. We consider the following cases:

(1) There exists a cardinal number γ such that card $S \leq \gamma$ for all $S \in K$.

(2) Any pair $S_1 \in K$, $S_2 \in K$, can be embedded into some $T \in K$.

6.7 <u>Theorem</u> ([2]). (a) If K is a class of essential F-extensions of R satisfying condition (1) then for each $S \in K$ there is a $T \in K$ such that T is K-maximal and $S \subseteq T$.

(b) A ring R has a unique K-maximal extension (up to isomorphism) if and only if conditions (1) and (2) are both satisfied.

<u>Proof</u>. (a) Assume there is a cardinal number γ such that card $S \leq \gamma$ for all $S \in K$. Let M be a set with card $M = \gamma$ and let $R[X_M]$ be the ring of polynomials over R whose indeterminates are in 1-1 correspondence with the elements of M. Then $R[X_M]$ is the free R-algebra generated by the set of indeterminates X_M and therefore card $R[X_M] = \gamma$ (when M is an infinite set). We recall that the free R-algebra $R[X_M]$ has the property that for any mapping $\phi: X_M \to S$ into an R-algebra S there exists a homomorphism $\psi: R[X_M] \to S$ which extends ϕ. Since each $S \in K$ is an R-algebra with card $S \leq \gamma$ define $\phi: X_M \to S$ by mapping the elements of X_M to the generators of S, then each $S \in K$ is a homomorphic image of $R[X_M]$. If now $\mathcal{T}(S)$ is the isomorphism type of S then $\mathcal{T} = \{\mathcal{T}(S)\}$, $S \in K$, is a set because card $\hat{\mathcal{T}} \leq$ card $H \leq 2^{\gamma}$, where H is the set of homomorphic images of $R[X_M]$.

Let K_M be the set obtained by choosing from each $\mathcal{T}(S)$ in $\mathcal{T}$ a representative element $S \in \mathcal{T}(S)$. K_M can be partially ordered if by $S_1 \leq S_2$ is meant that $S_1 \to S_2$ is a monomorphism over R , and it follows from (6.1)' and (6.2) that under this ordering every simply ordered subset in K_M has an upper bound in K . But if $T \in K$ is an upper bound for a chain in K_M we have card $T \leq \gamma$ and so T can be viewed as the representative element of $\mathcal{T}(T)$ in K_M . Thus K_M contains a maximal element T' by Zorn's lemma and it is clear from the construction of K_M that T' is maximal in K .

(b) If R has a unique K-maximal extension $\overline{R}$ suppose card $\overline{R} = \gamma$. Then card $S \leq \gamma$ for all $S \in K$ and each S can be embedded into $\overline{R}$. Conversely assume both conditions are satisfied. Then there exists a K-maximal $S \in K$. If also S' is K-maximal there are embeddings $S \to T$, $S' \to T$, for some $T \in K$, therefore $S = T = S'$.

2. F-saturated extensions.

6.8 Lemma ([2]). Let R be a ring and let $F \subset R[x]$ be a subset of monic polynomials. The following conditions are equivalent:

(1) If S is an essential F-extension of R then S = R .

(2) For any essential F-extension S of R there is a retraction of S onto R .

Under these conditions R is called F-saturated.

Proof. The retraction $S \to R$ is a monomorphism by (1.2).

6.9 Corollary. An F-saturated ring is F-splitting.

Proof. (6.8) and (6.5).

6.10 <u>Theorem</u> ([2]). Let R be a ring, $d \in R$, $0 \neq a \in R$, $a^2 = da = 0$, $g(x) = x^2 - dx$. Then there is no g-saturated extension of R. More precisely, for any R-algebra S extending R there is a proper essential extension T of S generated by two roots of g.

<u>Proof</u>. Let S be any R-algebra extending R and let $F = S1 \times Su \times Sv$ be a free S-module generated by the three elements $1, u, v$. With multiplication given as $u^2 = v^2 = 0$, $uv = vu = a$, F becomes a commutative S-algebra whose unit element is the unit element of S. We identify S with $S1$. F is not associative because $a \neq 0$ entails $0 \neq au = (vu)u \in Su$, but $0 = u^2v \in Sv$. Denote by A the annihilator a^* of a in S, then $a, d \in A$. Furthermore Au and Av are ideals of F. For $Auv = Aa = 0$ implies that $FAu \subseteq Au$, other conditions being trivially satisfied, and similarly $FAv \subseteq Av$. It will be shown that $F/(Au + Av) = T$ is an extension of S having the sought properties.

Let $h: F \to T$ be the cannonical S-algebra homomorphism. Since $a \in A$ we have $a\bar{u} = 0$ hence $\bar{u}^2\bar{v} = 0 = \bar{u}a = \bar{u}(\bar{u}\bar{v})$ from which follows that multiplication in T is associative.

Now $h|S$ is an embedding of rings because $(Au + Av) \cap S = 0$ and therefore S may be identified with $\bar{S}$. The embedding is proper because $\bar{u}, \bar{v} \notin S$. As well, T is a g-extension of S since $g(\bar{u}) = \bar{u}^2 - d\bar{u} = -d\bar{u} \in A\bar{u} = 0$ and similarly $g(\bar{v}) = 0$.

Finally T is essential over S. Let B be a non-zero ideal of T and let $0 \neq b = s + s_1\bar{u} + s_2\bar{v} \in B$, where $s, s_1, s_2 \in S$.

It suffices to verify that $B \cap S \neq \langle 0\rangle$ in the following three distinct cases:

(1) $s, s_1, s_2 \in A$. Then $b = s \in B \cap S$.

(2) $s \notin A$. Then $sa \neq 0$ and $ba = sa \in B \cap S$.

(3) $s \in A$ and one of s_1 or $s_2 \notin A$. Assume $s_1 \notin A$. Then
$b\bar{v} = s\bar{v} + s_1\bar{u}\bar{v} + s_2\bar{v}^2 = s_1\bar{u}\bar{v} = s_1a \neq 0$ and $b\bar{v} = s_1a \in B \cap S$.

Therefore $B \cap S \neq \langle 0\rangle$ and T is an essential extension of S .

6.11 Definition Let R be a ring. A polynomial in $R[x_1,\ldots,x_n]$ which is unchanged by any permutation of the indeterminates $x_1,\ldots,x_n$ is called a symmetric polynomial of $x_1,\ldots,x_n$. The elementary symmetric polynomials of $x_1,\ldots,x_n$ are defined thus:

$$\sigma_1 = x_1 + x_2 + \ldots + x_n ,$$
$$\sigma_2 = x_1x_2 + x_1x_3 + \ldots + x_2x_3 + \ldots + x_{n-1}x_n ,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\sigma_n = x_1x_2 \ldots x_n .$$

If x is a variable over $R[x_1,\ldots,x_n]$ the elementary symmetric polynomials of $x_1,\ldots,x_n$ are the coefficients σ_i of the powers of x in the polynomial

$$(x-x_1)(x-x_2) \ldots (x-x_n) = x^n - \sigma_1x^{n-1} + \sigma_2x^{n-2} - \ldots + (-1)^n\sigma_n .$$

It is shown in ([15], Theorem 11, p.133) that a symmetric polynomial in $R[x_1,\ldots,x_n]$ can be written as a polynomial $g(\sigma_1,\ldots,\sigma_n)$.

6.12 Definition ([15]). The discriminant of a polynomial $(x-x_1) \dots (x-x_n) = x^n - \sigma_1 x^{n-1} + \dots + (-1)^n \sigma_n$ is the expression $\prod_{i<j} (x_i - x_j)^2$. This is a symmetric polynomial in $x_1,\dots,x_n$ because $\prod_{i<j} (x_i - x_j)$ is mapped to $\pm \prod_{i<j} (x_i - x_j)$ by any permutation of the x_i. It may therefore be viewed as a polynomial $g(\sigma_1,\dots,\sigma_n)$ in the elementary symmetric functions.

One notes that the discriminant of an arbitrary monic polynomial $f(x) = x^n + r_1 x^{n-1} + \dots + r_n = (x-t_1) \dots (x-t_n)$ is uniquely determined. For suppose $f(x)$ can also be written as a product of linear factors $(x-s_1) \dots (x-s_n)$. Then $(-1)^i \sigma_i(t_1,\dots,t_n) = r_i = (-1)^i \sigma_i(s_1,\dots,s_n)$ for $i = 1,2,\dots,n$ and therefore $\prod_{i<j} (t_i - t_j)^2$ and $\prod_{i<j} (s_i - s_j)^2$ contain exactly the same terms when written as polynomials in $\sigma_1,\dots,\sigma_n$. We write $d(f)$ to denote the discriminant of a monic polynomial f.

6.13 Corollary. If $f(x) = x^n + r_1 x^{n-1} + \dots + r_n$ is a monic polynomial over a ring R then $d(f)$ is an element of R.

Proof. By (6.5) there exists an extension T of R such that f splits in T into linear factors $(x-t_1) \dots (x-t_n)$. Therefore $d(f)$ can be written as a polynomial in $r_1,\dots,r_n$ where $r_i = (-1)^i \sigma_i(t_1,\dots,t_n)$.

6.14 Lemma ([2]). Let R be a ring that has no non-trivial idempotents and let $f \in R[x]$ be a monic polynomial whose discriminant is invertible in R. If $f(x)$ can be written as $(x-\alpha_1) \dots (x-\alpha_n)$, $\alpha_i \in R$, then $\alpha_1,\dots,\alpha_n$ are the only roots of f in R.

<u>Proof</u>. If $f(x) = (x-\alpha_1) \ldots (x-\alpha_n)$ has an invertible discriminant $d(f) = \prod_{i<j} (\alpha_i - \alpha_j)^2$ then the values taken by the derivative $f'(x)$ at $\alpha_1,\ldots,\alpha_n$ are also invertible, for clearly $f'(\alpha_i) = \prod_{\substack{j=1 \\ j\neq i}}^{n} \alpha_i - \alpha_j$ divides $d(f)$.

Now suppose $\beta = \alpha_i + \delta_i$, $i = 1,2,\ldots,n$, is a root of f . Expanding the terms of $f(\alpha_i + \delta_i)$ yields

(1) $$0 = f(\alpha_i + \delta_i) - f(\alpha_i) = \delta_i f'(\alpha_i) + \delta_i^2 r_i \ , \quad r_i \in R \ .$$

Put $s_i = (f'(\alpha_i))^{-1}$. Upon multiplication of (1) by $r_i s_i^2$ we obtain

$$0 = \delta_i r_i s_i + (\delta_i r_i s_i)^2$$

therefore $e_i = \delta_i r_i s_i$ is idempotent and this means that $e_i = 0$ or $e_i = 1$. If $e_i = 1$, $i = 1,2,\ldots,n$, then all δ_i are invertible, consequently $\beta - \alpha_i = \delta_i$ belongs to no maximal ideal. Then $\overline{\beta} \neq \overline{\alpha}_i$ in any factor field $\overline{R} = R/M$ which is impossible because $(\overline{\beta} - \overline{\alpha}_1) \ldots (\overline{\beta} - \overline{\alpha}_n) = \overline{0}$.

Thus $e_i = \delta_i r_i s_i = 0$ for some i , hence $\delta_i r_i = 0$ since s_i is a unit, and it follows from (1) that $\delta_i f'(\alpha_i) = 0$. But $f'(\alpha_i)$ is a unit, therefore $\delta_i = 0$ and so $\beta = \alpha_i$.

6.15 <u>Theorem</u> ([2]). Let R be a ring, $F \subset R[x]$ a subset of monic polynomials. The following statements are equivalent:

(1) R has an F-saturated extension.

(2) R has an f-saturated extension for each $f \in F$.

(3) No non-zero nilpotent element of R annihilates the discriminant of any $f \in F$.

(4) Each essential extension R' of R may be embedded into a product $S = \prod_{i \in J} S_i$ of rings S_i having no non-trivial idempotents. The set $W(f)$ of roots of f in S has the form

$$W(f) = \{\beta \in S \mid \beta = \alpha_1 e_1 + \ldots + \alpha_n e_n \ , \ e_1,\ldots,e_n \text{ orthogonal idempotents of } S\} ,$$

where $\alpha_1,\ldots,\alpha_n$ are roots of f in S such that f can be written as $f(x) = (x-\alpha_1) \ldots (x-\alpha_n)$.

(5) There exists a cardinal number $\gamma = \gamma(R, F)$ such that card $R' \leq \gamma$ for any essential F-extension R' of R .

Proof. (1) ⇒ (2) is clear.

(2) ⇒ (3). By contradiction. Let $f \in F$, let $d = d(f)$ be the discriminant of f and suppose $cd = \bar{0}$ for some $c \in \text{rad } R$, $c \neq 0$. We may assume that $c^2 = 0$. If S is an arbitrary extension of R the result will follow once it is shown that there is a proper essential f-extension of S .

By (6.5) there exists an essential f-extension S' of S which is also an f-splitting ring. If $S' \neq S$ the proof is complete. Assume therefore that $S' = S$. Let $f(x) = (x-\alpha_1) \ldots (x-\alpha_n)$, $\alpha_i \in S$, $i = 1,2,\ldots,n$, and let $J = \{(k, i) \mid 1 \leq k, i \leq n, k \neq i\}$. We show that there is an $a \in S$ such that $a \neq 0$, $a^2 = 0$ and $a(\alpha_i - \alpha_j) = 0$ for some $(i,j) \in J$. These conditions are satisfied by c in case $c(\alpha_k - \alpha_i) = 0$ for all $(k,i) \in J$. Otherwise there exists a subset $M \subset J$ maximal with respect to the property

$$c \prod_{(k,i)\in M} (\alpha_k - \alpha_i) = a \neq 0 ,$$

where $J \setminus M$ is non-empty because

$$cd = c \prod_{(k,i)\in J} (\alpha_k - \alpha_i) = 0$$

and therefore, in view of the maximality of M, we have a pair (i,j) in $J \setminus M$ such that

$$a(\alpha_i - \alpha_j) = 0 \text{ , with } a \neq 0 \text{ , } a^2 = 0 \text{ , } a \in S \text{ .}$$

Put $\delta = \alpha_i - \alpha_j$ and let $g(x) = x^2 - \delta x$. Then
$g(x - \alpha_j) = (x - \alpha_j)^2 - \delta(x - \alpha_j) = (x - \alpha_j)(x - \alpha_j - \delta) = (x - \alpha_j)(x - \alpha_i)$ divides f, consequently any $g(x - \alpha_j)$-extension of S is also an f-extension. As follows from (6.10) S has a proper essential extension generated by roots of $g(x - \alpha_j)$ and this is now a proper essential f-extension of S.

(3) ⇒ (4). Assume (3) and let R' be an essential extension of R. We first show that there is an embedding of rings $\theta\colon R' \to S = \prod_{i\in j} S_i$ into a product of rings $S_i = Se_i$ (e_i the unit element of S_i) containing no non-trivial idempotents, such that the discriminant $d(f)$ of each $f \in F$ is invertible in any S_i that is not an integral domain.

Put $N = \text{rad } R$, $\dot{N} = N \setminus 0$ and let $D = \{r \in R \mid r\dot{N} \subseteq \dot{N}\}$. This is a multiplicatively closed set, for if $r_1\dot{N} \subseteq \dot{N}$ and $r_2\dot{N} \subseteq \dot{N}$ then $r_1r_2\dot{N} = r_1(r_2\dot{N}) \subseteq r_1\dot{N} \subseteq \dot{N}$. By (3) we have $d(F) = \{d(f) \mid f \in F\} \subseteq D$. Let R'_D be the localization of R' at the multiplicative set D (3.40). The kernel of the cannonical homomorphism $\delta\colon R' \to R'_D$ is then the set of all $r \in R'$ for which there exists d in D such that $rd = 0$, therefore

$\text{Ker}\ \delta \cap \text{rad}\ R = \langle 0 \rangle$. If $\{M_i\}_{i \in I}$ is the family of maximal ideals of R'_D then the mapping $R'_D \to \prod_{i \in I} (R'_D)_{M_i} = A$ is injective by ([4], p.88, Corollary 2). The rings $(R'_D)_{M_i}$ are well known to be local (a ring is local if it has a unique maximal ideal), thus there is a homomorphism

$$\alpha \colon R' \to A = \prod_{i \in I} A_i$$

into a product of local rings A_i such that $\text{Ker}\ \alpha \cap \text{rad}\ R = \langle 0 \rangle$. As well, there is a homomorphism

$$\beta \colon R' \to \prod_{P \in \text{Spec}\ R'} R'/P = B$$

whose kernel is clearly $\text{rad}\ R'$. Therefore there is a further homomorphism

$$\theta = \alpha \times \beta \colon R' \to A \times B = S$$

which is an embedding of rings because $R \cap \text{Ker}\ \theta = (R \cap \text{Ker}\ \beta) \cap \text{Ker}\ \alpha = \text{rad}\ R \cap \text{Ker}\ \alpha = 0$, consequently $\text{Ker}\ \theta = \langle 0 \rangle$ by essentiality. Now the components of $S = \prod_{i \in J} S_i$ are either local rings or integral domains. Since all non-units of a local ring are contained in its unique maximal ideal such rings have no non-trivial idempotents and this obviously holds for integral domains. Furthermore the discriminant of each $f \in F$ is a unit of R'_D since $d(F) \subseteq D$ and it follows that all $d(f)$, $f \in F$, are invertible in each of the local rings among the S_i . Thus $\theta \colon R' \to S$ is the sought embedding.

It remains to describe the roots of $f \in F$ in S. By (6.5) R' has an essential F-splitting extension which is essential over R by transitivity. One may therefore assume that f splits in R' into linear factors. Now suppose f can be written as

$$f(x) = (x - \alpha_1) \ldots (x - \alpha_n), \quad \alpha_i \in S,$$

and identify R' with its image in $S = \prod_{i \in J} Se_i$, where e_i is the identity element of $S_i = Se_i$. Then the projections of $\alpha_1, \ldots, \alpha_n$ onto the ith coordinate yield a polynomial

$$e_i f(x) = f(x_i) = (x_i - e_i\alpha_1) \ldots (x_i - e_i\alpha_n), \quad x_i = e_i x_i,$$

and it follows from (6.14) that $e_i\alpha_1, \ldots, e_i\alpha_n$ are the only roots of f in Se_i. Thus $\beta \in S$ is a root of f precisely when $e_i\beta$ is a root of $e_i f$ for all $i \in J$, therefore any root of f in S has the form

$$\beta = \prod_{i \in J} e_i \alpha_{j(i)}, \text{ where } j: J \to \{1,2,\ldots,n\}.$$

Let $e_{(k)} \in S$ be defined by $e_{(k)}(i) = e_i$ if $e_i\beta = e_i\alpha_k$, $= 0$ if $e_i\beta \neq e_i\alpha_k$, $k = 1,2,\ldots,n$. Then $e_{(1)}, \ldots, e_{(n)}$ are orthogonal idempotents of S, $e_{(1)} + \ldots + e_{(n)} = 1$ and $\beta = e_{(1)}\alpha_1 + \ldots + e_{(n)}\alpha_n$.

(4) ⇒ (5). Assume R' is an essential F-extension of R and for $f \in F$ denote by $W_{R'}(f)$ the set of roots of f in R'. By (4) there is an embedding $\theta = \prod_{i \in J} \theta_i : R' \to S = \prod_{i \in J} S_i$ into a product of rings S_i having no non-trivial idempotents, hence

$$\text{card } W_{R'}(f) \leq \text{card } W(f) \leq n^{\text{card } J}$$

where $W(f)$ is the set of roots of f in S and $n = \deg f$. Because R' is generated over R by roots of the $f \in F$ the result will follow once it is shown that J may be chosen as $J = \dot{R} = R\setminus 0$.

Now $\operatorname{Ker}\theta = \bigcap_{i\in J} \operatorname{Ker}\theta_i = \langle 0\rangle$ and so there is for each $r \in \dot{R}$ an $i(r) \in J$ such that $r \notin \operatorname{Ker}\theta_{i(r)}$. Therefore the mapping

$$\theta' = \prod_{r\in\dot{R}} \theta_{i(r)} : R' \to S' = \prod_{r\in\dot{R}} S_{i(r)}$$

satisfies $R \cap \operatorname{Ker}\theta' = R \cap (\bigcap_{r\in\dot{R}} \operatorname{Ker}\theta_{i(r)}) = \langle 0\rangle$. Since R' is an essential extension of R we have an embedding of R' into a ring S' with the properties stated in (4), thus $\dot{R}$ can be chosen in place of the indexing set J and the proof is complete.

(5) ⇒ (1). By (6.7) R has a maximal essential F-extension. This is an F-saturated extension.

6.16 Definition ([2]). Let R be a ring and let $f \in R[x]$ be a monic polynomial. We say f is saturable (over R) if R has an f-saturated extension. The family of all saturable polynomials in $R[x]$ will be denoted by $F_s(R)$. It follows from (6.15) that $F_s(R)$ consists of all monic polynomials in $R[x]$ whenever R is a semiprime ring.

6.17 Definition ([2]). A saturable closure of a ring R is a maximal essential $F_s(R)$ extension. A ring is called saturably closed if it is F_s-saturated.

6.18 Proposition. Let R be a semiprime ring. The following statements are equivalent:

(1) R is saturably closed.

(2) R is algebraically closed.

Proof. Assume R is a saturably closed semiprime ring and suppose T is a ring algebraic over R. Let t be an element of T. Then t satisfies an equation of integral dependence $f(x) = x^n + r_1x^{n-1} + \ldots + r_n = 0$, $r_i \in R$, whence $R[t]$ is an f-extension of R. It is an essential f-extension of R because T is algebraic over R. But f belongs to $F_s(R)$ since R is semiprime, therefore by (6.8) the embedding $R \to R[t]$ is onto and it follows from (3.8) that R is algebraically closed.

The opposite implication is clear, an essential F_s-extension of a ring R being algebraic over R.

6.19 Corollary. The saturable closure of a semiprime ring R is unique up to isomorphism over R.

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