# Almost Periodic Differential Equations and a study of the Non-homogeneous Heat Equation

Mosleh Uddin Mazumder

A Thesis in The Department of Mathematics
and Statistics

Presented in Partial Fulfilment of the Requirements
for the Degree of Master of Science
at
**Concordia University**
**Montreal, Quebec, Canada**

January 1992

ISBN 0-315-80931-0

# ABSTRACT

## ALMOST PERIODIC DIFFERENTIAL EQUATIONS AND A STUDY OF THE NON-HOMOGENEOUS HEAT EQUATION

The Thesis is divided into four Chapters. The first chapter describes the preliminaries of the subject with a little background.

In chapter II, the basic works of Bohr and Neugebauer are reviewed and the theorem of Bohr and Neugebauer was re-established by using matrix theoretic tricks without getting deep into the subject. Favard's theorem based on the generalization of Bohr's work is revisited and the theorem is reproved by applying Bochner's almost periodicity criterion. Some observations are made on the theorems of Bohr-Neugebauer and Favard. An attempt is made to obtain almost periodic solutions for a system of two differential equations using the maximum principle and standard hypothesis.

In chapter III, the almost periodicity criteria for ordinary differential equations as found in recent literature are studied. Some aspects of the subject in the light of Stepanov's almost periodicity are simplified. A few relatively new results are discussed based on the use of Liapunov type of functions.

In chapter IV, an attempt is made to extend the work of Professor S. Zaidman on the almost periodic solution of non-homogeneous heat equation. The problem was suggested by my supervisor Dr. M. Zaki, Concordia University and I received helpful remarks from Professor S. Zaidman, University of Montreal for which I am thankful to him.

A few graphs have been included in the Appendix to show the difference between periodic and almost periodic functions.

# ACKNOWLEDGEMENT

The author wishes to express his deepest sense of gratitude and thankfulness to his supervisor Professor M. Zaki without whose sincere supervision, encouragement and guidance this work would never be possible.

He also thankfully acknowledges his indebtedness to the Department of Mathematics and Statistics, Concordia University, Montreal, Canada for supporting him with a Teaching Assistantship for a period of two years and four months. He is further thankful to Professors W. Byers and E. Cohen for their cooperation and assistance during the program.

# TABLE OF CONTENTS

*CHAPTER* I

# ALMOST PERIODIC FUNCTIONS

## 1. INTRODUCTION

In 1922, the Danish mathematician Harald Bohr (1887-1951) published the 1st formal paper on Almost Periodic Functions and since then he continued to publish a series of papers on this relatively new branch of mathematics. Starting from the 1st few papers, the works of Bohr attracted interest of quite a number of researchers who made significant contributions to the development of the Theory of Almost Periodic Functions. Among others, the works of A. Besicovitch, S. Bochner, N. N. Bogolyubov, J. Favard, J. von Neumann, V. V. Stepanov, H. Weyl and S. Zaidman are remarkable.

In recent years the theory of almost periodic functions has been developed in connection with problems of differential equations, stability theory, dynamical systems and many other disciplines.

There are a few equivalent definitions of almost periodic functions. We will use some of them according to our requirements.

DEFINITION (1.1). Let us take an arbitrary function $f(x) = \mu(x) + i\nu(x)$ continuous for $-\infty < x < +\infty$. The real number $\tau$ is called *Translation Number* of $f(x)$ corresponding to $\varepsilon > 0$, denoted by $\tau(\varepsilon)$ or $\tau_f(\varepsilon)$, whenever

$$|f(x+\tau) - f(x)| \leq \varepsilon, \text{ for } -\infty < x < +\infty.$$

A translation number of $f(x)$ corresponding to $\varepsilon$ corresponds a fortiori to every quantity $\varepsilon_1 > \varepsilon$ and together with $\tau$. Again $-\tau$ is also a translation number of $f(x)$ corresponding to $\varepsilon$.

Also

$$\tau(\varepsilon_1) + \tau(\varepsilon_2) = \tau(\varepsilon_1 + \varepsilon_2)$$

and

$$\tau(\varepsilon_1) - \tau(\varepsilon_2) = \tau(\varepsilon_1 + \varepsilon_2)$$

i.e. the sum or difference of translation numbers corresponding to $\varepsilon_1$ and $\varepsilon_2$ will at the same time be a translation number corresponding to $\varepsilon_1 + \varepsilon_2$.

DEFINITION (1.2). A subset $E$ of real numbers $\Re$ is called *Relatively Dense* if there are no arbitrary large gaps among the numbers of the set $E$. i.e. a subset $E$ of $\Re$ is called relatively dense if there exists a number $\delta > 0$ such that

$$[a, a+\delta] \cap E \neq \phi, \ \forall a \in \Re.$$

For instance, the numbers $np$ $(n = 0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots, p > 0)$ of an arithmetic progression are relatively dense and the set $\pm n^2$ $(n = 0, 1, 2, \ldots\ldots\ldots)$ is not relatively dense, since $(n+1)^2 - n^2 = 2n + 1 \to \infty$ as $n \to \infty$. Thus roughly speaking 'relatively dense' sets can be described as 'one which is just as dense as an arithmetic progression'.

DEFINITION (Bohr) (1.3). A function $f(x)$ continuous for $-\infty < x < +\infty$

is called *Almost Periodic*, if for every $\varepsilon > 0$, there exists a relatively dense set of translation numbers of $f(x)$ corresponding to $\varepsilon$.

i.e. to every $\varepsilon$, there exists a length $l(\varepsilon)$ such that each interval of length $l(\varepsilon)$ contains at least one translation number $\tau(\varepsilon)$.

NOTATION

For any bounded complex function $f(x)$ and $\varepsilon > 0$, we denote

$$T(f,\varepsilon) = \{\tau : |f(x+\tau) - f(x)| < \varepsilon, \forall x \in \Re\}$$

i.e. this $T(f,\varepsilon)$ is the set of translation numbers of $f(x)$ corresponding to $\varepsilon$. Using this notation Bohr's definition of almost periodic function is simplified as:

*"$f(x)$ is almost periodic if for every $\varepsilon > 0$, $T(f,\varepsilon)$ is relatively dense."*

Every periodic function is almost periodic while the converse is not true. If $f$ is periodic of period $T$, then all numbers of the form $nT$, where $n = \pm 1, \pm 2, \ldots\ldots\ldots\ldots$, are also periods of $f$ and so they are translation numbers of $f$ for any $\varepsilon > 0$. Thus $f$ is almost periodic as well. On the contrary, we observe that $f(t) = \cos t + \sin \sqrt{2}t$ is almost periodic but not periodic. A few graphs are given in the Appendix showing the differences between periodic and almost periodic functions.

Now we shall define almost periodic functions in metric spaces. Let $\Re$ be the real line, $X$ a complete metric space and $\rho = \rho(x_1, x_2)$ a metric on $X$.

DEFINITION (1.4). A number $\tau$ is called a translation number or an $\varepsilon$-almost period of $f : \Re \to X$ if

$$\sup_{t \in \Re} \rho(f(t+\tau), f(t)) \leq \varepsilon.$$

DEFINITION (1.5). A continuous function $f : \Re \to X$ is called *Almost Periodic* if it has a relatively dense set of translation numbers $\tau_\varepsilon$ for each $\varepsilon > 0$ i.e. if there is a number $l = l(\varepsilon) > 0$ such that each interval $[a, a+l] \subset \Re$ contains at least one number $\tau_\varepsilon$ satisfying definition (1.4).

Now we will define almost periodic functions using complex trigonometric polynomials.

The function $T(x) = \sum_{k=1}^{n} c_k e^{i\lambda_k x}$, where the coefficients $c_k$ are arbitrary complex quantities while the exponents $\lambda_k$ are arbitrary real quantities, is called a *Complex Trigonometric Polynomial* whose real and imaginary parts are real trigonometric polynomials.

DEFINITION (1.6) A complex valued function $f(x)$ defined for $-\infty < x < +\infty$ is called almost periodic, if for any $\varepsilon > 0$, $\exists$ a trigonometric polynomial $T_\varepsilon(x)$ such that

$$|f(x) - T_\varepsilon(x)| < \varepsilon, \text{ for } -\infty < x < +\infty.$$

From the above definitions it is clear that

(i) Any trigonometric polynomial is an almost periodic function.

(ii) Almost periodic functions are those functions defined on the real line which can be approximated uniformly by trigonometric polynomials.

(iii) From the theorem of approximation of periodic functions by trigonometric polynomials it follows that any periodic function is also almost periodic.

In 1927, S. Bochner defined almost periodic functions in a more useful and comprehensive way in terms of its applications to differential equations

and showed its equivalence to Bohr's definition.

DEFINITION (Bochner) (1.7). A function $f(x)$ is called *Almost Periodic* if from every sequence $\{s'_n\}$ one can extract a subsequence $\{s_n\}$ such that $\lim_{n\to\infty} f(x+s_n)$ exists uniformly on $\Re$.

NOTATIONS

For simplicity of the use of above definition, Bochner [1] introduced a few notations.

1. The sequence $\{s_n\}$ is denoted as $s_n$ i.e. braces are omitted. So $u_n \subset s_n$ means that $\{u_n\} \subset \{s_n\}$ i.e. $\{u_n\}$ is a subsequence of $\{s_n\}$.

2. Let $\lim_{n\to\infty} f(x+s_n) = g(x)$. This is denoted by $T_s f = g$ and is written only when the limit exists. Here the operator $T$ is known as *Translation Operator.*

3. $AP(C)$ denotes set of all almost periodic complex valued functions where $C$ designates complex numbers. Similarly when $\Re$ designates real numbers, $AP(R)$ denotes set of almost periodic real valued functions. $AP(E)$ will be intended to mean either one.

4. $H(f) = \{g : \exists s$ with $T_s f = g$ uniformly on $\Re\}$. This $H(f)$ is called the *Hull* of $f(x)$. If $f(x)$ is an almost periodic function such that $g \in H(f)$, we can easily show [2] that $H(g) = H(f)$.

## 2. ELEMENTARY PROPERTIES

Now we prove some of the simple properties of almost periodic functions. These are actually straight forward consequences of the definitions.

**PROPERTY I.** *An almost periodic function is bounded on the real line. i.e. there exists a constant $c$ such that $|f(x)| \leq c, \forall x \in \Re$.*

*PROOF.* An almost periodic function can be approximated by a trigonometric polynomial. Let $f(x)$ be an almost periodic function. Then for any $\varepsilon > 0$, there exists a trigonometric polynomial $T_\varepsilon(x)$ such that

$$|f(x) - T_\varepsilon(x)| < \varepsilon, \forall x \in \Re$$

Suppose $\varepsilon = 1/n$ and $|T_1(x)| \leq M$, where $M > 0$. Then we get

$$|f(x)| \leq |f(x) - T_1(x)| + |T_1(x)| \leq 1 + M, \forall x \in \Re$$

$\Rightarrow f(x)$ is bounded on $\Re$. $\diamond$

**PROPERTY II.** *An almost periodic function is uniformly continuous on the real line. i.e. given $\varepsilon > 0$, $\exists \delta(\varepsilon)$ such that*

$$|f(x_1) - f(x_2)| \leq \varepsilon \text{ for } |x_1 - x_2| \leq \delta.$$

*PROOF.* Since any trigonometric polynomial is a uniformly continuous function on the real line, for given $\varepsilon > 0$ we have a $\delta(\varepsilon)$ such that

$$|T_\varepsilon(x_1) - T_\varepsilon(x_2)| < \varepsilon/3, \text{ for } |x_1 - x_2| < \delta \tag{1}$$

Again from the definition of almost periodic function, we have

$$|f(x) - T_\varepsilon(x)| < \varepsilon/3, \forall x \in \Re. \tag{2}$$

$$\Rightarrow |f(x_1) - f(x_2)| \leq |f(x_1) - T_\varepsilon(x_1)| + |T_\varepsilon(x_1) - T_\varepsilon(x_2)| + |T_\varepsilon(x_2) - f(x_2)|.$$

Now using (1) and (2), we obtain

$$|f(x_1) - f(x_2)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

Thus $|f(x_1) - f(x_2)| < \varepsilon$ for $|x_1 - x_2| < \delta$.

$\Rightarrow f(x)$ is uniformly continuous on the real line. $\diamond$

**PROPERTY III.** *If $f(x)$ is almost periodic, $c$ is a complex number and $a$ is a real number, the functions $\overline{f}(x), cf(x), f(x+a)$ and $f(ax)$ are all almost periodic.*

*PROOF.* If $T(x)$ is a trigonometric polynomial, $\overline{T}(x)$, $cT(x)$, $T(x+a)$ and $f(ax)$ can be approximated by the corresponding trigonometric polynomials and therefore the proof follows directly from definition (1.6) of almost periodic function.

**PROPERTY IV.** *The sum $f(x) + g(x)$ of two almost periodic functions $f(x)$ and $g(x)$ is almost periodic.*

*PROOF* Since $f(x)$ and $g(x)$ are almost periodic, we have an $\varepsilon > 0$ and two trigonometric polynomials $T(x)$ and $S(x)$ such that

$$|f(x) - T(x)| < \varepsilon/2, \forall x \in \Re$$

and

$$|g(x) - S(x)| < \varepsilon/2, \forall x \in \Re$$
$$\Rightarrow |(f(x) + g(x)) - (T(x) + S(x))| \leq |f(x) - T(x)| + |g(x) - S(x)|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\Rightarrow f(x)+g(x)$ is almost periodic. $\diamond$

**PROPERTY V.** *The product $f(x)g(x)$ of two almost periodic functions $f(x)$ and $g(x)$ is almost periodic.*

*PROOF.* Let $\varepsilon$ be a real number such that $0 < \varepsilon < 1$. There exist two trigonometric polynomials $S_\varepsilon(x)$ and $T_\varepsilon(x)$ such that

$$|f(x) - S_\varepsilon(x)| < \frac{\varepsilon}{2(M+1)}$$

$$|g(x) - T_\varepsilon(x)| < \frac{\varepsilon}{2(M+1)} \tag{1}$$

where $M$ is a number for which $f(x) \leq M$ and $g(x) \leq M$ on the whole of real line.

$$\begin{aligned} \Rightarrow |S_\varepsilon(x)| &\leq |S_\varepsilon(x) - f(x)| + |f(x)| \\ &< M+1 \end{aligned} \tag{2}$$

But

$$|fg - S_\varepsilon T_\varepsilon| \leq |g|.|f - S_\varepsilon| + |S_\varepsilon||g - T_\varepsilon|$$

where applying (1) and (2), we obtain

$$|f(x)g(x) - S_\varepsilon(x)T_\varepsilon(x)| < \varepsilon, \ \forall x \in \Re$$

$\Rightarrow f(x)g(x)$ is almost periodic since $S_\varepsilon(x)T_\varepsilon(x)$ is a trigonometric polynomial. $\diamond$

**PROPERTY VI.** *The limit of a uniformly convergent sequence of almost periodic functions is an almost periodic function.*

*PROOF.* Let $f(x)$ be the limit of the sequence $f_n(x)$ of almost periodic functions uniformly convergent for $-\infty < x < +\infty$. Obviously $f(x)$ is continuous. Let $\varepsilon > 0$ be given and we choose $N = N(\varepsilon)$ such that

$$|f(x) - f_N(x)| \leq \varepsilon/3, \ \forall x \in \Re$$

Then every translation number $\tau = \tau_{f_N}(\varepsilon/3)$ is a translation number $\tau_f(\varepsilon)$. For, we have

$$\begin{aligned} |f(x+\tau) - f(x)| &\leq |f(x+\tau) - f_N(x+\tau)| + |f_N(x+\tau) - f_N(x)| + |f_N(x) - f(x)| \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon \end{aligned}$$

Since the translation numbers $\tau_{f_N}(\varepsilon/3)$ are relatively dense, the same is true for the translation numbers $\tau_f(\varepsilon)$

$\Rightarrow f(x)$ is almost periodic. $\diamond$

**PROPERTY VII.** *Let $\varphi(z_1, z_2, \ldots\ldots, z_n)$ be a uniformly continuous function of $(z_1, z_2, \ldots\ldots, z_n) \in \mathcal{M}$, where $\mathcal{M}$ is a set in the n-dimensional complex space. If $f_1(x), f_2(x), \ldots\ldots\ldots\ldots\ldots\ldots, f_n(x)$ are almost periodic functions such that $(f_1, f_2, \ldots\ldots\ldots\ldots\ldots\ldots, f_n) \in \mathcal{M}$ for any real $x$, then the function*

$$F(x) = \varphi(f_1, f_2, \ldots\ldots\ldots\ldots\ldots\ldots, f_n)$$

*is also almost periodic.*

*PROOF.* The functions $f_1(x), f_2(x), \ldots\ldots\ldots\ldots\ldots\ldots, f_n(x)$ are bounded on the real line $\Re$.

$\Rightarrow$ the set of points $(f_1, f_2, \ldots\ldots\ldots\ldots, f_n)$ in the n-dimensional complex space is bounded.

Thus without loss of generality, we can assume that $\mathcal{M}$ is bounded and closed. Then by Weierstrass approximation theorem, we conclude that for given $\varepsilon > 0$, $\exists$ a polynomial

$$P_\varepsilon(z_1, z_2, \ldots\ldots\ldots\ldots, z_n; \overline{z_1}, \overline{z_2}, \ldots\ldots\ldots\ldots, \overline{z_n})$$

such that

$$|\varphi(z_1, z_2, \ldots\ldots\ldots\ldots, z_n) - P_\varepsilon(z_1, z_2, \ldots\ldots\ldots\ldots, z_n; \overline{z_1}, \overline{z_2}, \ldots\ldots\ldots\ldots, \overline{z_n})| < \varepsilon.$$

for all $(z_1, z_2, \ldots\ldots\ldots\ldots, z_n) \in \mathcal{M}$

$$\Rightarrow |F(x) - P_\varepsilon(f_1(x), f_2(x), \ldots\ldots\ldots\ldots, f_n(x); \overline{f}_1(x), \overline{f}_2(x), \ldots\ldots\ldots\ldots, \overline{f}_n(x))| < \varepsilon, \forall x \in \Re$$

$\Rightarrow$ $F(x)$ can be approximated by almost periodic functions.

$\Rightarrow$ $F(x)$ is almost periodic. $\diamond$

**PROPERTY VIII.** *If $f(x)$ is almost periodic and $f'(x)$ is uniformly continuous on $\Re$, then $f'(x)$ is also almost periodic.*

*PROOF.* Here $f'(x)$ is uniformly continuous.

$\Rightarrow$ For given $\varepsilon > 0$, $\exists$ a $\delta = \delta(\varepsilon) > 0$ such that $|f'(x_1) - f'(x_2)| < \varepsilon$, when $|x_1 - x_2| < \delta$.

Suppose $1/n < \delta$. Then

$$|n(f(x + 1/n) - f'(x))| = |n \int_0^{1/n} (f'(x + s) - f'(x)) ds|$$

$$\leq n \int_0^{1/n} |f'(x + s) - f'(x)| \, dx$$

$$< \varepsilon.$$

So $n(f(x+1/n)-f(x)) \to f'(x)$ uniformly on $\Re$.

Thus by Property VI, $f'(x)$ is almost periodic. $\diamond$

**PROPERTY (Bochner) IX.** *If $f(x)$ is almost periodic, then from every pair of sequences $u', v'$ there exists common subsequences $u \subset u'$, $v \subset v'$ such that*

$$T_{u+v}f = T_u T_v f$$

*pointwise.*

*PROOF.* By hypothesis, $f(x)$ is almost periodic.

So from given sequences $u', v'$ we can extract a subsequence $v'' \subset v'$ such that $T_{v''}f = g$ uniformly. This $g(x) \in AP(C)$. If $u'' \subset u'$ is common with $v''$, we can extract a subsequence $u''' \subset u''$ such that $T_{u'''}g = h$ uniformly.

Let $v''' \subset v''$ be common with $u'''$. Now we can find common subsequences $u \subset u''', v \subset v'''$ such that $T_{u+v}f = k$ uniformly.

Since $u \subset u''', v \subset v'''$, we find that

$$T_u g = h \text{ and } T_v f = g$$

uniformly. For sufficiently large $n$, given $\varepsilon > 0$, we have,

$$|k(x) - f(x+u_n+v_n)| < \varepsilon, \ \forall x \in \Re,$$

$$|g(x) - f(x+v_n)| < \varepsilon, \ \forall x \in \Re;$$

$$|h(x) - g(x+u_n)| < \varepsilon, \ \forall x \in \Re.$$

$$\Rightarrow |h(x) - k(x)| \leq |h(x) - g(x+u_n)| + |g(x+u_n) - f(x+u_n+v_n)| + |f(x+u_n+v_n) - k(x)|$$

$$< 3\varepsilon$$

Since $\varepsilon$ is arbitrary, we have

$$k(x) = h(x)$$

$$\Rightarrow T_{u+v}f = T_u g = T_u(T_v f)$$

$$= T_u T_v f. \diamond$$

NOTES

1. If $f(x)$ and $g(x)$ are two almost periodic functions with $|g(x)| > 0$, then $f(x)/g(x)$ is an almost periodic function.

2. If $f(x)$ and $g(x)$ are two almost periodic functions, then their difference $f(x) - g(x)$ is also almost periodic.

3. If $f(x)$ is an almost periodic function , then $(f(x))^2$ and $|f(x)|^2$ are also almost periodic.

4. Let $P(y_1, y_2, \ldots\ldots\ldots\ldots, y_n)$ be a polynomial and $f_1(x), f_2(x), \ldots\ldots\ldots\ldots, f_n(x)$ are almost periodic functions . Then $P(f_1, f_2, \ldots\ldots\ldots\ldots, f_n)$ is almost periodic.

5. The set of almost periodic functions is closed with respect to uniform convergence. In other words, any function that can be approximated uniformly by almost periodic functions with any accuracy, is almost periodic.

6. If one replaces the word "bounded" by "with values in a compact set", then the same results will follow.

CHAPTER II

# ALMOST PERIODIC SOLUTIONS OF O.D.E.

## 1. INTRODUCTION

It is Bohr, H. and Neugebauer, O. [3] who studied the almost periodicity problem of bounded solutions of O.D.E. for the first time. They considered the finite-dimensional linear system $x' = Ax + f(t)$ where $f(t) \in AP(E^n)$ and $A$, a constant matrix and proved that *any bounded solution of the system on the whole of* $\Re$ *is necessarily almost periodic*. The research direction initiated by Bohr and Neugebauer was an area of concentration for many researchers and consequently a great deal of results on the almost periodicity properties of solutions to Partial Differential Equations, to different other types of functional equations, or to Abstract Differential Equations, were obtained. In general, more sophisticated types of equations than the ordinary ones drew the attention of many of them. Nevertheless, the almost periodicity problem of bounded solutions for O.D.E. is still a challenge for many researchers in this field. In this chapter we shall review and develop some results on the almost periodicity problem of bounded solutions for O.D.E.

**THEOREM (2.1).** *Let the set of bounded solutions of a differential equation $x' = f(x,t)$ be non-empty. If $f$ is defined and bounded on $S \times \Re$ where $S$ is a sphere containing the origin and a bounded solution, then there is a solution of the equation with minimum norm.*

The set of bounded solutions of the equation $x' = f(x,t)$ is a convex set that does not contain $0$ if $f \neq 0$. Such sets should have an element close to $0$ i.e. with minimum norm. This theorem on the existence of minimum norm characterizing the bounded solution is found in the works of Favard and the proof in English is provided in Fink [2].

**THEOREM (Bohr) (2.2).** *Let $x' = f(t)$, $\forall t \in \Re$ where $f(t) \in AP(E^n)$. Then $x \in AP(E^n) \Leftrightarrow x(t)$ is bounded on the real line $\Re$.*

Doss [4] observed that $F(x) = \int_0^x f(t)\,dt$ for $f \in AP(C)$ has the property that $F(x+h)-F(x)$ is bounded and almost periodic for any $h$. This hypothesis makes sense without integration. So he poses the question: If $F$ is bounded and all differences $F(x+h)-F(x)$ are almost periodic , is $F$ almost periodic ? Bochner [1] studied a general first order system which includes the possibility of delays and pure difference equations. He showed that if one non-trivial difference is almost periodic and $F$ is bounded and uniformly continuous, then $F$ is almost periodic.

**THEOREM (2.3).** *The bounded solutions of $x' = Ax$, $A$ being a constant matrix, are precisely almost periodic.*

*PROOF.* The vector solutions of $x' = Ax$ for constant $A$ are

$$x(t) = e^{At}x_o = \sum_{k=1}^{n} P_k(t)e^{\lambda_k t}x_o$$

where $\lambda_k$ are eigenvalues of $A$ and $P_k$ are polynomials whose degrees do not exceed multiplicity of $\lambda_k$ [5]. Let $\lambda_k = \mu_k + i\nu_k$. Then

$$x(t) = \sum_{k=1}^{n} P_k e^{\mu_k t}x_o(\cos \nu_k t + i \sin \nu_k t)$$

where we observe that

(i) If $\mu_k \neq 0$, no solution except $0$ is bounded.

(ii) If $\mu_k = 0$, we get

$x = \sum P_k x_o(\cos \nu_k t + i \sin \nu_k t)$ which is a trigonometric polynomial.

$\Rightarrow x \in AP(E^n)$. $\diamond$

**THEOREM (2.4).** *Let*

$$y^{(n)} + a_1(t)y^{(n-1)} + \dots\dots\dots\dots + a_n(t)y = f(t) \tag{1}$$

*be a linear differential equation where the coefficients $a_i$ and $f$ are bounded on $\Re$. If $y$ is bounded on $\Re$, then $y', y'', \dots\dots\dots\dots, y^{(n)}$ are all bounded on $\Re$.*

Proof is provided in Fink [2].

## OBSERVATION

Changing of variables by substituting $x_1 = y, x_2 = y' \dots\dots\dots\dots, x_n = y^{(n-1)}$, the system (1) becomes

$$x' = A(t)x + g(t) \tag{2}$$

where $x = (x_1, x_2, \ldots\ldots\ldots\ldots, x_n)^T$, $g(t) = (0, 0, 0, \ldots\ldots\ldots, f(t))^T$ and $A(t)$ is the matrix

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_2 & -a_1 \end{pmatrix}.$$

If $f(t) = 0$, $g(t) = 0$ and then (2) becomes of the form

$$x' = Ax.$$

For constant $A$, the result is confirmed by theorem (2.3). By the hypothesis of (2.3), $x$ is bounded and so in the scalar case, $y, y', \ldots\ldots\ldots\ldots, y^{(n-1)}$ are all bounded. But reversing the procedure, we take $y$ as a bounded solution of the scalar equation, even, if one of the derivatives may be unbounded. In case of constant $A$ it may never happen, but certainly in the nonconstant case, it may happen. For instance, $y = \cos t^2$ is bounded while $y'$ is unbounded. It turns out that if $\cos t^2$ is a solution of a linear differential equation, then it must have unbounded coefficients.

## 2. GENERAL LINEAR EQUATION

We shall consider the general linear differential equation of the form

$$x' = Ax + f \tag{1}$$

where $A$ is a constant matrix and $f \in AP(E^n)$. A solution $x = (x_1, x_2, \ldots\ldots\ldots\ldots, x_n)$ of the system (1) will be called almost periodic if all its components are almost periodic functions of $t$. To find almost periodic solutions of the systems, such as (1), we must look for their bounded solutions. Since the systems of the form (1) have, in general, unbounded solutions, one cannot expect that all the solutions be almost periodic [6]. The following theorem, due to Bohr and Neugebauer, will show that the bounded solutions, if exist, are almost periodic.

**THEOREM (Bohr and Neugebauer) (2.5).** *A solution $x(t)$ of $x' = Ax + f$, where $A$ is a constant matrix and $f(t) \in AP(E^n)$, is almost periodic if and only if it is bounded.*

*PROOF.* Here we attempt to simplify the original proof of this theorem by applying Favard's minimum norm idea and Bochner's pointwise version of almost periodicity.

Let $x_0$ be a bounded solution of $x' = Ax + f$.

$\Rightarrow$ by theorem (2.1), there is a solution $x(f)$ with minimum norm.

By hypothesis, $f \in AP(E^n)$. So for some sequence $u'$, $T_{u'}f = g$ exists uniformly. Then $\exists\, u \subset u'$ such that $T_u x_o$ exists uniformly on compact sets and $T_u x_o$ is a bounded solution of $x' = Ax + T_{u'}f = Ax + g$

Thus we got a bounded solution of $x' = Ax + g$. Again by theorem (2.1), it has a solution $x(g)$ with minimum norm.

We need to prove that $x(g)$ is unique. Let $x_1$ and $x_2$ be two distinct solutions of $x' = Ax + g$ such that

$$||x_1(g)|| = ||x_2(g)|| = ||x(g)||.$$

Then $\frac{1}{2}(x_1 + x_2)$ is a solution of $x' = Ax + g$ and $\frac{1}{2}(x_1 - x_2)$ is a non-trivial solution of $x' = Ax$ so that

$$|\frac{1}{2}(x_1 - x_2)| \geq \rho > 0, \quad \forall t.$$

Now,

$$|\frac{1}{2}(x_1 + x_2)|^2 + |\frac{1}{2}(x_1 - x_2)|^2 = \frac{|x_1|^2 + |x_2|^2}{2} \leq ||x(g)||^2$$

$$\Rightarrow |\frac{1}{2}(x_1 + x_2)|^2 \leq ||x(g)||^2 - \rho^2$$

$$\Rightarrow |\frac{1}{2}(x_1 + x_2)|^2 < ||x(g)||^2$$

which contradicts the minimum property of $x(g)$. So $x(g)$ is unique.

We know that $T_v x(f) = x(T_v f)$ and $T_u T_v x(f) = T_{u+v} x(f)$ since both minimize the norm over bounded solutions of $x' = Ax + T_{u+v} f$. As any two solutions differ by some constant function, we may assume that

$$x_0(t) = x(f)(t) + z(t)$$

where $z(t)$ is a bounded solution of $x'(t) = Ax(t)$ which is almost periodic by theorem (2.3).

$\Rightarrow x_0 \in AP(E^n)$. $\diamond$

## OBSERVATIONS

In these observations, we shall use the matrix theoretical tricks to reach to the result of the above theorem.

1. We know that if $A \in C^{n \times n}$ and $\{\lambda_1, \lambda_2, \ldots\ldots, \lambda_n\}$ be the set of eigenvalues of $A$, then there exists a non-singular matrix $P \in C^{n \times n}$ such that

$$J(A) = P^{-1}AP = \begin{pmatrix} Q_1 & & & \\ & Q_2 & & \\ & & \ddots & \\ & & & Q_n \end{pmatrix}$$

where $Q_i$ is a Jordan block corresponding to the eigenvalue $\lambda_i$. The matrix $J(A)$ is referred to as the *Jordan Canonical Form* of A. Using the detailed procedure of matrix theory, we may transform the matrix $A$ into its Jordan cononical form. Then we look at a particular Jordan block to obtain

$$x' = \begin{pmatrix} \lambda & 1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda & 1 & \ldots & 0 & 0 \\ 0 & 0 & \lambda & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \lambda & 1 \\ 0 & 0 & 0 & \ldots & 0 & \lambda \end{pmatrix} x + \tilde{f}.$$

The last equation is of the form

$$x' = \lambda x + f$$

with $x$ a scalar. Proceeding up, the remaining equations are of the form

$$x' = \lambda x + g$$

where $x$ is a scalar and $g$ is a scalar almost periodic function.

Thus the theorem can be reconstructed by verifying the above sequence of scalar equations. Let us consider the equation

$$x' = \lambda x + f(t) \tag{3}$$

where $\lambda$ is a complex number and $f \in AP(E)$.

The general solution of (3) is given by

$$x(t) = e^{\lambda t}\left\{K + \int_0^t e^{-\lambda s} f(s)\, ds\right\} \tag{4}$$

where $K$ is the constant of integration.

Now we shall show that any bounded solution of such equation is almost periodic. Suppose $\lambda = a + ib$. Three different cases of the solution are coming up.

CASE I. When $a > 0$.

We see that $|e^{\lambda t}| = e^{at} \to +\infty$ as $t \to \infty$. But we are interested in bounded solutions. This is possible if

$$K + \int_0^t e^{-\lambda s} f(s) ds \to 0 \text{ as } t \to \infty,$$

i.e. when

$$K = -\int_0^\infty e^{-\lambda s} f(s) ds \tag{5}$$

which is a convergent improper integral, since

$$\begin{aligned} \left| -\int_0^\infty e^{-\lambda s} f(s) ds \right| &\leq |e^{-\lambda s} f(s)| \\ &\leq \sup |f(s)| e^{-as}, \ \forall s \geq 0. \end{aligned}$$

So from (4), we obtain

$$\begin{aligned} x(t) &= e^{\lambda t}\left\{ -\int_0^\infty e^{-\lambda s} f(s) ds + \int_0^t e^{-\lambda s} f(s) ds \right\} \\ &= e^{\lambda t}\left\{ -\int_t^\infty e^{-\lambda s} f(s) ds \right\} \\ &= -\int_t^\infty e^{\lambda(t-s)} f(s) ds \end{aligned} \tag{6}$$

$$\begin{aligned} \Rightarrow \ |x(t)| &\leq \sup |f(t)| . \left| \int_t^\infty e^{a(t-s)} ds \right| \\ &\leq \sup |f(t)| . \frac{1}{a} = \frac{M}{a} \end{aligned} \tag{7}$$

where $M = \sup |f(t)|, \forall t \in \Re$.

$\Rightarrow$ $x(t)$ is bounded on $\Re$.

We note that

$$|x(t+\tau) - x(t)| \leq \frac{1}{a} \sup |f(t+\tau) - f(t)|, \forall t \in \Re \tag{8}$$

This implies that any $\varepsilon a$-translation number of $f(t)$ is an $\varepsilon$-translation number of $x(t)$. So $x(t)$ is almost periodic.

CASE II. When $a < 0$.

By an analogous calculation, we get

$$x(t) = \int_{-\infty}^{t} e^{\lambda(t-s)} f(s) ds \tag{9}$$

with

$$|x(t)| \leq \frac{M}{|a|}.$$

As in (8), any $\varepsilon|a|$-translation number of $f(t)$ is an $\varepsilon$-translation number of $x(t)$ and $x(t)$ is almost periodic.

CASE III. When $a = 0$.

From equation (4), we have

$$x(t) = e^{ibt}\left\{K + \int_{0}^{t} e^{-ibs} f(s) ds\right\}. \tag{10}$$

$\Rightarrow$ $x(t)$ is bounded if and only if

$$\int_{0}^{t} e^{-ibs} f(s) ds$$

is bounded on $\Re$.

Since the function under integral sign is almost periodic and so bounded, $x(t)$ is bounded as well. Then by Bohr's theorem (2.2), it is almost periodic.

Thus applying all the three cases in all the scalar equations in succession, the theorem is re-established.

2. We note that for any matrix $A$, there exists a non-singular matrix $P$ of the same order such that $PAP^{-1}$ is a lower triangular matrix with the eigenvalues $\{\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n\}$ of $A$ on the diagonal, such as,

$$PAP^{-1} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ t_{21} & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ t_{n1} & t_{n2} & \ldots & \lambda_n \end{pmatrix}.$$

Substituting $x = P^{-1}y$ or $y = Px$ in equation (1), we obtain

$$y' = PAP^{-1}y + Pf.$$

$$\Rightarrow \quad y' = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ t_{21} & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ t_{n1} & t_{n2} & \ldots & \lambda_n \end{pmatrix} y + g \tag{11}$$

where $g = Pf$.

We note that $e^y \subset e^x$ by $y = Px$ and the reverse follows since $x = P^{-1}y$. Also

$$||x||.||P^{-1}||^{-1} \leq ||y|| \leq ||P||.||x||$$

so that the statements about norms on each equation can be transfered to the other. The constant equations of (11) are as follows:

$$
\begin{aligned}
y_1' &= \lambda_1 y_1 + g_1, \\
y_2' &= t_{21} y_1 + \lambda_2 y_2 + g_2, \\
&\dots \quad \dots \quad \dots \quad \dots \\
y_n' &= t_{n1} y_1 + t_{n2} y_2 + \dots + \lambda_n y_n + g_n.
\end{aligned} \tag{12}
$$

The first equation is of the form of the scalar equation (3) in the last observation, $\lambda$ being an eigenvalue of the matrix $A$, where we already found that $y$ is almost periodic. Coppel [7] proved an interesting result that $|\lambda - i\sigma_n| \geq \rho > 0$ for $\lambda$, an eigenvalue of $A$ and $\sigma_n \in e^f$. With this we get a unique almost periodic $y_1$ where $e^{y_1} = e^{g_1}$ and $||y_1|| \leq M\rho^{-1}||g_1||$.

The second equation in (12) is

$$y_2' = \lambda y_2 + (t_{21} y_1 + g_2) = \lambda y_2 + h_2$$

where $h_2 = g_2 + t_{21} y_1$ is almost periodic. So with the above hypothesis we get a unique almost periodic $y_2$ with $e^{y_2} = e^{h_2}$ and

$$
\begin{aligned}
||y_2|| &\leq M\rho^{-1}||h_2|| \\
&\leq M\rho^{-1}(||g_2|| + |t_{21}|.||y_1||) \\
&\leq M\rho^{-1}(||g|| + |t_{21}|.M\rho^{-1}||g||) \\
&\leq M\rho^{-1}||g||(1 + |t_{21}|.M\rho^{-1})
\end{aligned}
$$

Continuing in this procedure, it is possible to arrive at the almost periodic solution $y$ with $e^y \subset e^f$ and

$$||y|| \leq M\rho^{-1}||g||P$$

where $P$ is a polynomial of degree n with no constant term and the coefficients of $P$ are coming up from the elements of the matrix $PAP^{-1}$ and threfore depends on $A$ eventually.

Thus the theorem is reverified.

**COROLLARY (2.6).** *If $|\lambda - i\sigma_n| \geq \rho > 0$ for all eigenvalues $\lambda$ of $A$ and $\sigma_n \in e^f$, then there is an almost periodic solution of the equation (1).*

*PROOF.* Using $|\lambda - i\sigma_n| \geq \rho > 0$, we already obtained an almost periodic solution of the equation $y' = PAP^{-1}y + g$ in the preceeding discussion. Now simply by substituting $x = P^{-1}y$, we can obtain the solution $x(f)$. Here the only thing we need is to prove the uniqueness of the solution. Suppose we have two solutions $x_1$ and $x_2$ of the equation (1) with exponents in $e^f$. Let

$$x_1 \simeq \sum_{n=1}^{\infty} a_n e^{i\sigma_n t}, \quad \sigma_n \in e^f$$

and

$$x_2 \simeq \sum_{n=1}^{\infty} b_n e^{i\sigma_n t}, \quad \sigma_n \in e^f.$$

$$\Rightarrow \quad x_1 - x_2 \simeq \sum_{n=1}^{\infty} (a_n - b_n) e^{i\sigma_n t}$$

$$\Rightarrow \quad (x_1 - x_2)' \simeq \sum_{n=1}^{\infty} i\sigma_n (a_n - b_n) e^{i\sigma_n t}$$

Thus from $(x_1 - x_2)' = A(x_1 - x_2)$, we have $(i\sigma_n)(a_n - b_n) = A(a_n - b_n)$.

$\Rightarrow$ Since $i\sigma_n$ is not an eigenvalue of $A$, we get $a_n - b_n = 0$, $\forall \sigma_n \in \sigma^f$. So $x_1 = x_2$. $\diamond$

**COROLLARY (2.7).** *If $f(t) \in AP(E^n)$ and the matrix $A$ has no eigenvalues with real part zero, then the system (1) admits a unique almost periodic solution. If $x = (x_1, x_2, \ldots \ \ldots \ \ldots, x_n)$ is the almost periodic solution and $M = \max_i\{\sup|f_i(t)|\}$, $t \in \Re$, then $|x_i(t)| \leq KM$, where $K$ is a positive constant depending only on the matrix $A$.*

*PROOF.* Suppose $A$ is not a triangular matrix. Then we introduce some unknown functions by substituting $x = Pz$, the matrix $P$ is so chosen that $P^{-1}AP$ is triangular. Now substituting for $x$ in (1), we get

$$Pz' = APz + f$$

$$\Rightarrow z' = P^{-1}APz + \tilde{f}, \ \ \tilde{f} = P^{-1}f$$

$$\Rightarrow \begin{pmatrix} z_1' \\ z_2' \\ \vdots \\ z_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & t_{12} & \ldots & t_{1n} \\ 0 & \lambda_2 & \ldots & t_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} + \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \\ \tilde{f}_n \end{pmatrix}. \tag{13}$$

It is clear that if $x$ is a bounded solution of (1), then the corresponding solution $z$ of (13) is also bounded and conversely. In the previous observation we already discussed the cases where the real part of the eigenvalues of $A$ are $>$ or $< 0$. So all we have to do here is to prove the uniqueness and existence of an almost periodic solution only to the system (13). It will suffice to show that the system (13) has a unique bounded solution on the whole of $\Re$ where $\lambda_i \neq 0, i = 1, 2, \ldots\ldots, n$. In accordance with equations (7) and (9), from the last component equation of (13), we have

$$z_n = -\int_t^{\infty} e^{\lambda_n(t-s)} \tilde{f}_n(s) ds, \ \ Re(\lambda_n) > 0 \tag{14}$$

and

$$z_n = \int_{-\infty}^{t} e^{\lambda_n(t-s)} \tilde{f}_n(s)ds, \; Re(\lambda_n) < 0. \tag{15}$$

Substituting this values of $z_n(t)$ in the 2nd component equation of (13) from the last, we get an equation for $z_{n-1}(t)$ in the form of equation (3). Since $Re(\lambda_{n-1}) \neq 0$, we can find a unique solution for $z_{n-1}$ and so on. Hence the 1st part of the Corollary.

Let $\mu > 0$ where

$$|Re(\lambda_i)| \geq \mu, \;\; i = 1, 2, \ldots\ldots, n. \tag{16}$$

From (14) or (15), it follows that

$$|z_n(t)| \leq \frac{M}{\mu} = K_n M. \tag{17}$$

From (13), we have

$$z'_{n-1} = \lambda_{n-1} z_{n-1} + b_{n-1,n} z_n + \tilde{f}_{n-1}.$$

Exressing $z_{n-1}(t)$ by a formula of the form (14) or (15) and estimating the integral, we get

$$|z_{n-1}(t)| \leq \frac{1}{\mu}\left\{|b_{n-1,n}|\frac{M}{\mu} + M\right\} = K_{n-1} M.$$

Continuing in the same fashion, we obtain

$$|z_i(t)| \leq K_i M, \; i = 1, 2, \ldots\ldots, n. \tag{18}$$

This inequality is for the system (13) according to the hypothesis of the Corollary. Now by substitution $z = P^{-1}x$, a similar inequality is obvious in terms of $x_i$.

The constant $K$ depends on the transformation $x = Pz$, i.e. on the matrix $P$ for which $P^{-1}AP$ is an upper triangular matrix. So eventually $K$ depends only on $A$. In no way $K$ depends on $f(t)$.

Hence the Corollary. $\diamond$

## 3. FAVARD'S GENERALISATION

In the general linear equation $x' = Ax + f$, the matrix $A$ was assumed to be a constant matrix. Favard attempted to generalise to the case when $A$ is an *Almost Periodic Matrix* whose elements are individually almost periodic functions. So he considered the linear almost periodic system

$$x' = A(t)x + f(t) \tag{1}$$

and its associated homogeneous equation

$$x' = A(t)x \tag{2}$$

where $A(t)$ is an almost periodic matrix and $f(t)$ is an almost periodic vector, each with complex valued components.

If, for some sequence $u$, $T_u A = B$ and $T_u f = g$ i.e. if $B \in H(A)$ and $g \in H(f)$, then the equation

$$x' = B(t)x + g(t) \tag{3}$$

is known as the equation in the *Non-homogeneous Hull* of (1) and the equation

$$x' = B(t)x \tag{4}$$

is known as the equation in the *Homogeneous Hull* of (1). Now let us prove Favard's theorem using Bochner's criterion.

**THEOREM (Favard ) (2.8).** *Suppose that all bounded non-trivial solutions of (4) satisfy*

$$\inf_{t \in \mathbb{R}} |x(t)| > 0$$

*and there is a bounded solution of (1). Then the equation (1) has an almost periodic solution.*

*PROOF.* First, if the set of bounded solutions of (3) is non-empty, then this set is a convex set which has a unique element with minimum norm. This is an argument using parallelogram identity. For two distinct minimizing solutions $x$ and $y$, we have

$$|\frac{x+y}{2}|^2 + |\frac{x-y}{2}|^2 = \frac{1}{2}\left\{|x|^2 + |y|^2\right\} \tag{5}$$

Since $\frac{1}{2}(x-y)$ is a bounded solution of equation (4),

$$|\frac{x-y}{2}|^2 > \delta > 0.$$

Now we shall use the notations $(A,f)$ and $(B,g)$ to mean equations (1) and (3). Let $x(B,g)$ be the minimum norm solution of $(B,g)$ in the hull of $(A,f)$. If $T_u(A,f) \to (B,g)$, then by taking subsequences if necessary, $T_u x(A,f) = y$ is a solution of (3) and

$$||y|| \leq ||x(A,f)||. \tag{6}$$

Repeating the argument with the sequence $-u$, we get $T_{-u}y$ is a solution of (1) and $||T_{-u}y|| \leq ||y|| \leq ||x(A,f)||$. By uniqueness $T_{-u}y = x(A,f)$. It follows that $T_u x(A,f) = x(B,g)$. i.e. the least norm solutions are translate of each other.

Thus $T_s T_u x(A,f)$ and $T_{s+u}x(A,f)$ are both translates of a least norm solution and solutions of the same equation.

By uniqueness they are the same and $x(A,f)$ satisfies Bocher's criterion of almost periodicity. Hence $x(A,f)$ is almost periodic. $\diamond$

**LEMMA (2.9).** *If $A(t)$ is an almost periodic matrix and $x$ is an almost periodic solution to $x' = A(t)x$, then $\inf_{t\in\Re} |x(t)| > 0$ or otherwise $x(t) = 0$.*

*PROOF.* If $\inf |x(t)| = 0$, let $|x(u'_n)| \to 0$. Then we find a subsequence $u \subset u'$ such that $T_u A = B, T_{-u} B = A, T_u x = y$ and $T_{-u} y = x$ exist uniformly, by Bochner's criterion. Then $y' = By$ and $y(0) = 0$.

Thus $y = 0$ and $x = T_{-u} y = 0$. $\diamond$

**COROLLARY (Bochner) (2.10).** *If every bounded solution of $x' = B(t)x$ is almost periodic, then all bounded solutions of $x' = A(t)x + f(t)$ are almost periodic.*

*PROOF.* According to the above Lemma, the hypothesis of the Corollary implies the hypothesis of Theorem (2.8). So we get an almost periodic solution, $\phi$ (say). But if $\psi$ be another bounded solution, then $\phi = \psi + \chi$, where $\chi$ is a solution of (2) and so is almost periodic. $\diamond$

**COROLLARY (2.11).** *If $A(t)$ is periodic, then any bounded solution of $x' = A(t)x + f(t)$ is almost periodic.*

*PROOF.* By Floquet Theory, there is a periodic matrix $P$ so that $P, P'$ and $P^{-1}$ are bounded. Multiplying throughout by $P$, the given equation gives

$$Px' = PAx + Pf$$

$$\Rightarrow (Px)' = (PA + P')x + Pf$$

By applying the transformation $Px = y$, we get

$$y' = (PAP^{-1} + P'P^{-1})y + Pf$$

$$\Rightarrow y' = Ry + Pf \tag{7}$$

where $R = PAP^{-1} + P'P^{-1}$ is a constant matrix.

Similarly equation (2), the associated homogemeous equation, is transformed into

$$y' = Ry \tag{8}$$

Now boundedness conditions on $x$ carry over to the same ones for (7) and (8). So all bounded solutions of (8) are almost periodic and the hypothesis of Corollary (2.10) are satisfied by the system (7). So $y$, the solution of (7) is almost periodic.

So $x = P^{-1}y$ is almost periodic, since $P^{-1}$ is periodic. $\diamond$

## OBSERVATIONS

1. From Theorems (2.5) and (2.8), it has now become clear that if $A$ is a constant matrix, there is a bounded solution of $x' = Ax + f$ for every bounded $f$ which is almost periodic; whereas if $A$ is non-constant and almost periodic, then in order to get almost periodic solutions to $x' = Ax + f$, we need estimates for the fundamental solutions of the associated homogeneous equation $x' = Ax$ and the equation $x' = B(t)x$ in the homogeneous hull of $x' = Ax + f$. Now we want to relate it with Exponential Dichotomy [8].

The equation $x' = A(t)x$ is said to satisfy an *Exponential Dichotomy* if there exists a projection $P$ and positive constants $\mu_1, \mu_2, \chi_1, \chi_2$ so that

$$|X(t)PX^{-1}(s)| \leq \chi_1 e^{-\mu_1(t-s)}, \quad t \geq s$$

$$|X(t)(I-P)X^{-1}(s)| \leq \chi_2 e^{-\mu_2(t-s)}, \quad t \leq s \tag{9}$$

for a fundamental matrix solution $X(t)$.

We notice that the function

$$\phi(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)f(s)ds + \int_{t}^{\infty} X(t)(I-P)X^{-1}(s)f(s)ds$$

is formally a solution to the system (1) [See [2], page 126]. We have

$$\begin{aligned}
||\phi|| &\leq \left\{\int_{-\infty}^{t} |X(t)PX^{-1}(s)|ds + \int_{t}^{\infty} |X(t)(I-P)X^{-1}(s)|ds\right\}||f|| \\
&\leq \left\{\int_{-\infty}^{t} \chi_1 e^{-\mu_1(t-s)}ds + \int_{t}^{\infty} \chi_2 e^{\mu_2(t-s)}ds\right\}||f|| \\
&\leq \left\{\frac{\chi_1}{\mu_1} + \frac{\chi_2}{\mu_2}\right\}||f||.
\end{aligned}$$

Since the homogeneous equation has no non-trivial solutions, $x$ is the unique bounded solution. Moreover, every equation in the hull has a unique bounded solution. Thus from given sequences $u'$, $v'$ , we can find common subsequences $u \subset u', v \subset v'$ so that $T_{u+v}A = T_uT_vA$; $T_{u+v}f = T_uT_vf$; $y = T_{u+v}\phi$ and $z = T_uT_v\phi$ exist uniformly on compact sets. Since both are bounded solutions of the same equation we have $y = z$ and therefore $T_{u+v}\phi = T_uT_v\phi$. So $\phi$ is almost periodic.

Thus if the homogeneous equation $x' = A(t)x$ satisfies the exponential dechotomy (9) and $f$ is an almost periodic function, then the system (1) admits a unique almost periodic solution $\phi$.

2. From Floquet theory, if $A$ is a periodic matrix, then

$$X(t) = Q(t)e^{Bt} \tag{10}$$

where $Q(t)$ is periodic and $B$ is another constant matrix. So it is obvious that the equation $x' = A(t)x$ has an exponential dichotomy only when the matrix $B$ has no eigenvalues with real part zero, i.e. $B$ has all its eigenvalues off the imaginary axis. Similar is the case with the equation $x' = Bx$.

Substituting $x = Q(t)y$ in $x' = A(t)x$ with $y' = By$, we have

$$Q' = AQ - QB \tag{11}$$

If $Q$ is a constant matrix, then (11) implies that $A$ and $B$ are similar matrices.

Thus exponential dichotomy is, in fact, equivalent to the eigenvalues of $A$ being off the imaginary axis if $A$ is constant. In deed, if all eigenvalues have negative real parts, then solutions decay to $0$ exponentially as $t \to \infty$; while the situation for non-constant coefficients is quite different. Let us take the following matrix

$$A(t) = \begin{pmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\cos t \sin t \\ -1 - \frac{3}{2}\cos t \sin t & -1 + \frac{3}{2}\sin^2 t \end{pmatrix}$$

whose eigenvalues are the constants $-\frac{1}{4} \pm \frac{1}{4}\sqrt{7}i$. Although $(-e^{t/2}\cos t, e^{t/2}\sin t)$ is a solution whose norm approaches to $\infty$ as $t \to \infty$.

## 4. SYSTEM OF TWO DIFFERENTIAL EQUATIONS

In this Section we shall study the almost periodic solutions for a system of two differential equations. Let us consider the following system of two equations

$$x' = f(x,y,t)$$

$$y' = g(x,y,t) \tag{1}$$

If the functions $f,g \in AP(C^3)$ are almost periodic in $t$ uniformly for $(x,y) \in K$, where $K$ is compact subset of $\Re^2$, then for each sequence $u'_n$, there is a subsequence $u_n$ such that there exist

$$\tilde{f}(x,y,t) = \lim f(x,y,t+u_n),$$

$$\tilde{g}(x,y,t) = \lim g(x,y,t+u_n) \tag{2}$$

and these limits exist uniformly for $t \in \Re, (x,y) \in K$, where $K$ is compact in $\Re^2$.

The Hull of $(f,g)$, denoted by $H(f,g)$, is the collection of all pairs $(\tilde{f},\tilde{g})$ such that there is a sequence $u_n$ with the limits (2) existing uniformly for all $(x,y,t) \in K \times \Re$.

DEFINITION (2.1). The equation

$$x'' = F(x,x',t) \tag{3}$$

is said to satisfy the *Maximum Principle* if for arbitrary $a < b$ and solutions $x_1(t), x_2(t)$, the inequalities

$$x_2(a) - x_1(a) \leq M$$

$$x_2(b) - x_1(b) \leq M \tag{4}$$

and

$$0 \leq x_2(t) - x_1(t) \text{ on } [a,b]$$

imply that $x_2(t) - x_1(t) \leq M$ on $[a,b]$.

The notion of maximum principle for the system of two differential equations as in system (1) may be defined with respect to one of the components of the solutions [9]. The system (1) satisfies the maximum principle *with respect to* $x$ if for any arbitrary $a < b$ and the solutions $\{(x_1, y_1), (x_2, y_2)\}$, the inequalities

$$x_2(a) - x_1(a) \leq M,$$

$$x_2(b) - x_1(b) \leq M$$

and

$$0 \leq x_2(t) - x_1(t) \text{ on } [a,b]$$

imply that $x_2(t) - x_1(t) \leq M$ on $[a,b]$.

**LEMMA (2.12).** *Suppose that $f, g \in AP(C^3)$ are almost periodic in $t$ uniformly for $(x,y) \in K$, a compact subset of $\Re^2$. If each system of equations in the hull $H(f,g)$ of system (1) has a unique solution in $\Re^2$ with values in $K$, then these solutions are almost periodic.*

*PROOF.* Let $(x,y) \in K$ be the solution of the system

$$x' = f(x,y,t)$$

$$y' = g(x, y, t)$$

Since $f, g$ are almost periodic, for given sequences $u', v'$, we can find common subsequences $u \subset u', v \subset v'$ so that $T_{u+v}f = T_u T_v f$ and so that $T_u T_v x$ and $T_{u+v}x$ exist uniformly on compact sets. Then $T_u T_v x$ and $T_{u+v}x$ are solutions in $K$ of the same system of equations in $H(f, g)$. Thus $T_u T_v x = T_{u+v}x$ and therefore $x$ is almost periodic.

Similarly, we can show that $T_r T_s y = T_{r+s}y$ and so $y$ is almost periodic. Thus $(x, y)$ is almost periodic. $\diamond$

DEFINITION (2.2). The system (1) is said to satisfy *Standard Hypothesis* if there exists a compact subset $K$ in $\Re^2$ such that (i) the functions $f, g$ are almost periodic in $t$ uniformly for $(x, y) \in K$ and (ii) each system in the Hull has a unique solution of initial value problem in $K$.

**THEOREM (2.13).** *Let the system (1) satisfies the Standard Hypothesis on $K$ and that each equation in the hull satisfies the maximum principle with respect to $x$. If there is a bounded solution $(x, y)$ on $\Re^2$ with values in $K$, then the system has an almost periodic solution.*

*PROOF.* Let us define the functional

$$\lambda(x) = \sup_{t \in R} x(t) - \inf_{t \in R} x(t)$$

By Theorem (2.1) and according to the Note 6 of Chapter I, we have a solution that minimizes $\lambda(x)$. Now we define $K$ as $K = K_1 \times K_2$ where $K_1$ is the range of the 1st component of the minimizing solution. We claim that no other minimizing solution is in $K$.

Assume $(x_1, y_1)$ and $(x_2, y_2)$ are two distinct minimum solutions in $K$. We shall abort it by using maximum principle.

Let $u(t) = x_2(t) - x_1(t)$ and suppose $u(t_o) \geq \varepsilon > 0$. i.e., $x_2(t_o) - x_1(t_o) > 0$. Now $u'(t)$ may be $>, <, = 0$. If $u'(t_o) > 0$, then $u(t) > u(t_o) > 0$ for some $t > t_o$. If there is $t_1 > t_o$ such that $u(t_1) < u(t_o)$ and $u(t) \geq 0$ on $[t_o, t_1]$, then it contradicts the maximum principle. Thus $u(t) \geq \varepsilon$ on $[t_o, \infty)$.

Similarly, if $u'(t_o) < 0$, then we have $u(t) \geq \varepsilon$ on $(-\infty, t_o]$.

If $u'(t) = 0$, then we get both. That is, $u(t) \geq \varepsilon$ on both $(-\infty, t_o]$ and $[t_o, \infty)$.

So in all cases $u(t) \geq \varepsilon$ on a half-line. If $(x_1, y_1)$ is a minimizing solution, then

$$\sup_{t \in \Re^+} x_1(t) = \sup_{t \in \Re^-} x_1(t)$$

so that

$$\sup_{t \in \Re} x_1(t) = \sup_{t \in \Re} \{x_2(t) - u(t)\} < \sup_{t \in \Re} \{x_2(t) - \varepsilon\}.$$

$\Rightarrow$ Both $x_1, x_2$ cannot be in $K_1$.

So by Lemma (2.12), the system has an almost periodic solution. $\diamond$

NOTE

If the system (1) satisfies the maximum principle with respect to $x$ on $t_o \leq t < \infty$ (or on $-\infty < t \leq t_o$) and $u(t) = x_2(t) - x_1(t)$ is increasing ( or decreasing), the result of the Theorem is also true.

**LEMMA (2.14).** *Suppose the system (1) has unique solution of initial value problem when $f, g \in AP(C^3)$. If (i) $f$ is increasing in $y$ for fixed $t, x$ and non-decreasing in $x$ for fixed $t, y$, (ii) $g$ is increasing in $x$ for fixed $t, y$ and non-increasing in $y$ for fixed $t, x$, then the system (1) satisfies the maximum principle with respect to $y$ or, for some $t_o \in \Re$, satisfies the maximum principle on $[t_o, \infty)$ with respect to $x$.*

*PROOF.* Suppose $\{(x_1, y_1), (x_2, y_2)\}$ be solutions on $\Re$ of the system (1) and let

$$u(t) = x_2(t) - x_1(t)$$

and

$$v(t) = y_2(t) - y_1(t).$$

If there is $t_o \in \Re$ such that $v(t_o) > 0$ and $v'(t_o) = 0$, then

$$u(t_o) = x_2(t_o) - x_1(t_o) \geq 0$$

and from (1), we get

$$u'(t_o) = x_2'(t_o) - x_1'(t_o) = f(t_o, x_2(t_o), y_2(t_o)) - f(t_o, x_1(t_o), y_1(t_o)) > 0.$$

Hence $v(t) > 0, u(t) > 0$ and $u'(t) > 0$ on some interval $(t_o, t_1)$. If there is $t_1 < t_o$ such that $v(t_1) = 0$ and $u(t_1) = 0$, a contradiction with $v'(t_1) = 0$ is obtained. If there is $t_1 > t_o$ such that $u(t_1) = 0$ and $v(t_1) > 0$, a contradiction with $u'(t_1) > 0$ is obtained. Thus $u'(t) > 0$ on $[t_o, \infty)$ and system (1) satisfies the maximum principle on $[t_o, \infty)$ with respect to $x$ and $x_2(t) - x_1(t)$ is increasing.

If $v'(t) \neq 0$ when $v(t) > 0$, then (1) satisfies the maximum principle with respect to $y$. $\diamond$

**THEOREM (2.15).** *If the Lemma (2.14) holds for the system (1) and $f, g$ are almost periodic in*

*$t$ uniformly for $(x, y) \in K$, $K$ is compact in $\Re^2$ and there is a bounded solution on $\Re^2$, then there exists an almost periodic solution.*

*PROOF.* By hypothesis of the theorem, the system (1) satisfies the maximum principle with respect to $y$ or for some $t_o \in \Re$ the maximum principle is satisfied on $[t_o, \infty)$ with respect to $x$. Then according to the Theorem (2.13) and the Note, there exists an almost periodic solution for the system. $\diamond$

NOTE

The almost periodic solutions for systems of differential equations is actually itself a vast field of study. Some interesting results were found in the works of Seifert [10] which could be extended in great details.

CHAPTER III

# SOME ALMOST PERIODICITY CRITERIA

## 1. INTRODUCTION

In this chapter we will basically review some almost periodicity creteria for O.D.E. as found in literatures of the recent time [11], [12]. In [12], some results were obtained based on the use of some Liapunov's type of functions. The new feature of the results consists of certain differential inequalities that seem to be particularly adequate when the unknown function is bounded on the entire real line. The conditions imposed to the almost periodic system for obtaining the almost periodicity of the bounded solution lead immediately to the uniqueness of such a solution.

The method applied in deriving a few criteria of almost periodicity of bounded solutions of O.D.E. is the comparison method, i.e. the simultaneous use of Liapunov's functions and differential inequalities. Since the almost periodic functions (Bohr) are bounded on the real axis, the differential inequalities involved will usually hold on the entire real line. No initial conditions will be required, except for the case when the inequalities are

restructed to a half-axis.

**LEMMA (3.1).** *Let $x(t)$ be a differentiable map from $\Re$ into $(0, \infty)$ such that*

$$x'(t) \leq \omega(x(t)), \ \forall t \in \Re \tag{1}$$

*with $\omega$ continuous from $(0, \infty)$ into $\Re$ satisfying the condition $\omega(x) < 0$ when $x > M > 0$. If $x(t)$ is a bounded solution of (1) on the real line, then*

$$x(t) \leq M, \ \forall t \in \Re. \tag{2}$$

*PROOF.* Since $x(t)$ is a bounded solution of (1) on $\Re$, there are two distinct cases for consideration.

First, when $x(t)$ attains its maximum value at a point $t_o \in \Re$. Then by the principles of maxima or minima of $x(t)$, we have $x'(t_o) = 0$ and therefore $x(t_o)$ is such that $\omega(x(t_o)) \geq 0$ by (1). So obviously $x(t) \leq x(t_o) = M$.

Secondly, there is a sequence $\{t_n\}$ with $t_n \to \infty$ (or $t_n \to -\infty$) such that

$$\lim_{n \to \infty} x(t_n) = x_{max}.$$

If $x(t) \to x_{max}$ as $t \to \infty$, then we can assume, without loss of generality, that $x'(t_n) \geq 0$ for sufficiently large $n$. Indeed, the contrary case would mean $x'(t)$ is negative for $t \geq T$, which is impossible. Therefore, for such a sequence $\{t_n\}$ we obtain $\omega(x(t_n)) \geq 0$ by (1) for sufficiently large $n$, which obviously implies

$$\omega(x_{max}) \geq 0 \text{ and } x_{max} \leq M.$$

$$\Rightarrow \ x(t) \leq M, \ t \in \Re.$$

In case $x(t)$ does not tend to $x_{max}$ as $t \to \infty$, there is a number $x_o, 0 \leq x_o < x_{max}$, such that for a conveniently chosen sequence $\{\bar{t}\}, \bar{t} \to \infty$, we have

$$\lim_{n \to \infty} x(\bar{t}_n) = x_o.$$

In this case, the sequence $\{t_n\}$ can be chosen in such a manner that $\{x(t_n)\}$ represents a sequence of local maxima for $x(t)$ and therefore again by the principles of maxima or minima of $x(t)$, we have $x'(t_n) = 0$ for any $n$. Again we obtain $\omega(x(t_n)) \geq 0$, from which we get $\omega(x_{max}) \geq 0$. Hence $x_{max} \leq M$.

When the sequence $\{t_n\}$, on which $x(t)$ tends to $x_{max}$, is such that $t_n \to -\infty$, similar arguments hold true. For the case $x(t) \to x_{max}$ as $t \to -\infty$, it is useful to notice the existence of a sequence $\{t_n\}$, such that $x'(t_n) \to 0$.

Hence the Lemma. $\diamond$

**LEMMA (3.2).** *Let $x(t)$ be a differentiable map from $\Re$ into $(0, \infty)$ such that*

$$x'(t) \geq \omega(x(t)), \ \forall t \in \Re$$

*with $\omega$ continuous from $(0, \infty)$ into $\Re$ satisfying the condition $\omega(x) > 0$ when $x > M > 0$. If $x(t)$ is a bounded solution of (1) on the real line, then*

$$x(t) \leq M, \ \forall t \in \Re.$$

*PROOF.* Actaully this is the dual of Lemma (3.1). It is obtained when $t$ is changed into $-t$ in the proof of Lemma (3.1).

**LEMMA (3.3).** *Let $x(t)$ be continuously differentiable of the second order from $\Re$ into $(0, \infty)$ such that*

$$x''(t) \geq \omega(x(t)), \forall t \in \Re \tag{3}$$

*with $\omega$ continuous from $(0, \infty)$ into $\Re$ satisfying*

$$\omega(x) > 0 \text{ for } x > M > 0. \tag{3a}$$

*If $x(t)$ is a bounded solution of (3) on $\Re$, then*

$$x(t) \leq M, \forall t \in \Re \tag{3b}$$

*PROOF.* Since $x(t)$ is a bounded solution of (3) on $\Re$, we have two distinct possibilities.

First, when $\sup x(t)$ is attained at a certain point $\tilde{t} \in \Re$. In such case, by the principles of maxima or minima of $x(t)$, we have $x''(\tilde{t}) \leq 0$ and therefore (3) implies $\omega(x(\tilde{t})) < 0$. So according to (3a), the only possibility, we have, is $x(\tilde{t}) \leq M$. Hence (3b) holds true in this case.

Secondly, we assume that there is no point $\tilde{t} \in \Re$ such that

$$x(\tilde{t}) = \sup_{t \in \Re} x(t).$$

In this case, either of the following two situations must occur:

(i) Either we can find a sequence $\{t_m\}$, $t_m \to \infty$, such that

$$x(t_m) \to X = \sup_{t \in \Re} x(t)$$

or, (ii) a sequence $\{t'_m\}, t'_m \to -\infty$, such that

$$x(t'_m) \to X \text{ as } m \to \infty.$$

Since changing $t$ by $-t$ does not affect (3), we check only the case when $X = \lim x(t_m)$, $t_m \to +\infty$. Again, we face two distinct situations.

(a) when $\lim x(t) = X$ as $t \to +\infty$, and

(b) when there exists another sequence $\{s_m\}, s_m \to +\infty$, such that $\lim x(s_m) = X_o < X$, as $m \to \infty$.

If the situation in (a) occurs and $x(t) < X$ for any $t \in \Re$, then we can find a sequence $\{t_m\}$, $t_m \to +\infty$, such that

$$x''(t_m) \leq 0, \; m = 1, 2, 3, \ldots \ldots .$$

Actually if we assume $x''(t) > 0$ for $t \geq T$, then $t \to x(t)$ is a convex map. From the boundedness of $x(t)$ we easily obtain $x(t) \to X$, in contradiction with the fact that $x(t) < X, \forall t$. Hence

$$\omega(x(t_m)) \leq 0, \; m = 1, 2, 3, \ldots \ldots$$

which implies $\omega(X) \leq 0$. According to 3(a), this is possible only when $x(t) \leq X = M$.

If the situation (b) occurs, then from $x(t_m) \to X$ and $x(s_m) \to X_o < X$, as $m \to \infty$, we easily find that a new sequence $\{\bar{s}_m\}, \bar{s}_m \to +\infty$, must exist, with the property $x(\bar{s}_m) \to X$, as $m \to \infty$, and such that $x(\bar{s}_m)$ is a local maximum for $x(t)$. At such a point we shall have $x''(\bar{s}_m) \leq 0$, and therefore

$$\omega(x(\bar{s}_m)) \leq 0, \; m = 1, 2, 3, \ldots \ldots .$$

Thus $\omega(X) \leq 0$ and again by (3a), this is possible only when

$$x(t) \leq M.$$

Hence the Lemma. $\diamond$

NOTE

The above Lemmas are playing very important roles in determining the almost periodicity of bounded solutions to the system of equations which are admitting almost periodicity properties in Bohr's sense. But there are some difficulties in establishing the almost periodicity of the system when it happens that the right sides of some of the equations are almost periodic in a weaker sense. To overcome these situations Corduneanu [13], used some differential inequalities involving functions which are not necessarily bounded on $\Re$ in order to obtain boundedness and estimates for their solutions. He considered the differential inequality

$$x'(t) \leq -kx(t) + f(t)\lambda(x(t)), \forall t \in \Re, \tag{4}$$

or equivalently

$$x'(t) \geq kx(t) - f(t)\lambda(x(t)), \forall t \in \Re, \tag{4a}$$

where $k > 0$ is a constant, $f$ is a locally integrable map from $\Re$ into $(0, \infty)$ such that

$$||f||_M = \sup \int_t^{t+1} f(s)ds < \infty, \forall t \in \Re \tag{5}$$

assuming that $\lambda(r)$ is a map of $(0, \infty)$ into itself, continuous and nondecreasing such that $\lambda(r) = 0$ implies $r = 0$, while

$$\mu(r) = \frac{r}{\lambda(r)}, r > 0, \mu(0) = 0, \tag{6}$$

is strictly increasing for $r > 0$, and continuous and approaches to infinity with $r$.

**LEMMA (3.4).** *Let $x(t)$ be a differentiable map from $\Re$ into $(0, \infty)$ such that the inequality (4) holds true, where $k$, $f$ and $\lambda$ satisfy the conditions (5) and (6). If $x(t)$ is a bounded solution of (4) on $\Re$, then*

$$\sup_{t\in\Re} x(t) \leq \mu^{-1}(K\|f\|_M), \tag{7}$$

*where $K$ is a constant depending on $k$ only.*

*PROOF.* Multiplying both sides of (4) by $e^{kt}$ and then integrating from $t_o$ to $t$ $(t_o < t)$, we obtain

$$\left[e^{kt}x(t)\right]_{t_o}^{t} \leq \int_{t_o}^{t} \lambda(x(s))f(s)e^{ks}ds$$

$$\Rightarrow x(t) \leq x(t_o)e^{-k(t-t_o)} + \int_{t_o}^{t} \lambda(x(s))f(s)e^{-k(t-s)}ds. \tag{8}$$

Since $x(t)$ is bounded, for fixed $t$ and when $t_o \to -\infty$, the inequality (8) gives the following

$$x(t) \leq \int_{-\infty}^{t} \lambda(x(t))f(s)e^{-k(t-s)}ds, \tag{9}$$

Taking supremum with respect to $t \in \Re$ on both sides of (9) and denoting $m = \sup x(t)$, $t \in \Re$, and then dividing both sides by $\lambda(m)$, we get

$$\mu(m) \leq K\|f\|_M,$$

$$\Rightarrow \sup_{t\in\Re} x(t) \leq \mu^{-1}(K\|f\|_M). \tag{10}$$

Hence the Lemma. $\diamond$

**LEMMA (3.5).** *Let $x(t)$ be a map from $\Re$ into $(0, \infty)$, twice differentiable and such that*

$$x''(t) \geq k^2 x(t) - f(t)\lambda(x(t)), \forall t \in \Re. \tag{11}$$

*If $x(t)$ is bounded on $\Re$ and $k$, $f$ and $\lambda$ satisfy the conditions stated in Lemma (3.4), then $x(t)$ satisfies*

$$\sup_{t \in \Re} x(t) \leq \mu^{-1}(K\|f\|_M), \tag{12}$$

*with $K > 0$ depending on $k$ only.*

*PROOF.* Rewriting the inequality in (11) in the form

$$(x' - kx)' + k(x' - kx) \geq -f(t)\lambda(x(t)), \; t \in \Re, \tag{13}$$

Now multiplying both sides by $e^{kt}$ and then integrating from $t_o$ to $t$, $(t > t_o)$, we get

$$\left[(x' - kx)e^{kt}\right]_{t_o}^{t} \geq -\int_{t_o}^{t} \lambda(x(s))f(s)e^{ks}ds$$

$$\Rightarrow \{x'(t) - kx(t)\}e^{kt} \geq -\{x'(t_o) - kx(t_o)\}e^{kt_o} - \int_{t_o}^{t} \lambda(x(s))f(s)e^{ks}ds. \tag{14}$$

The boundedness of $x(t)$ implies the existence of a sequence $\{t_n\}$ such that when $t_n \to -\infty$, we get $x'(t_n) \to 0$. If $t_o \to -\infty$ on such a sequence in (14), we obtain the inequality

$$x'(t) - kx(t) \geq -\int_{-\infty}^{t} \lambda(x(s))f(s)e^{-k(t-s)}ds, \; \forall t \in \Re. \tag{15}$$

Now taking supremum with respect to $t$ on both sides and denoting $\sup x(t) = m$, $t \in \Re$ as in Lemma (3.4), the inequality (15) gives

$$m \leq \lambda(m) K ||f||_M$$

Hence the Lemma. $\diamond$

## 2. STEPANOV'S ALMOST PERIODICITY

Before we go into details of the discussion of almost periodicity criteria, we need a little background of the difference between the almost periodicity conceptions according to Bohr and Stepanov. The work of Stepanov was mainly on the almost periodicity of functions with values in Banach spaces. Naturally there is always a difference.

Let $X$ be a complex Banach space with the norm topology; $x, y, \ldots\ldots$ are elements from $X$ of the norm $\|x\|, \|y\|, \ldots\ldots$; $\Re$ is the set of real numbers; $f$, or $f : \Re \to X$, or $t \to f(t)$, or $x = f(t)$, where $t \in \Re$, is a function defined on the set of real numbers $\Re$ with values in the Banach space $X$.

DEFINITION (3.1). If $f(t)$ is a function defined on $\Re$ with values in $X$ and $h$ is a fixed real number, then the function $f_h : \Re \to X$ defined by

$$f_h(t) = f(t+h), \quad \forall t \in \Re \tag{1}$$

is called the *h-Translate of f* i.e. of the function $f(t)$.

DEFINITION (3.2). A function $T : \Re \to X$ defined by

$$T(t) = \sum_{k=1}^{n} c_k e^{i\lambda_k t}, \forall t \in \Re \tag{2}$$

where $\lambda_k$ are real numbers, $c_k$ are elements from $X$ and $i$ is the imaginary unit, is called *Trigonometric Polynomial* with values in $X$.

DEFINITION (3.3). A continuous function $f : \Re \to X$ on $\Re$ is called *Almost Periodic*, if for any number $\varepsilon > 0$, we can find a number $l(\varepsilon) > 0$ such that any interval on the real line of length $l(\varepsilon)$ contains at least one point of abscissa $\tau$ with the property that

$$||f(t+\tau) - f(t)|| < \varepsilon, \ \forall t \in \Re. \tag{3}$$

The number $\tau$ for which the above inequality holds true is called $\varepsilon$-translation number of the function $f$. The above property says that for every $\varepsilon > 0$, the function $f$ has a set of $\varepsilon$-translation numbers which is relatively dense in $\Re$.

DEFINITION (3.4). A continuous function $f : \Re \to X$ is called *Normal* if any set of translates of $f$ has a subsequence uniformly convergent on $\Re$ in the sense of the norm.

DEFINITION (3.5). A function $f : \Re \to X$ is said to possess the *Approximation Property*, if for any number $\varepsilon > 0$, we can determine a trigonometric polynomial $T_\varepsilon$ with values in $X$ such that

$$||f(t) - T_\varepsilon(t)|| < \varepsilon, \ \forall t \in \Re. \tag{4}$$

A function with the approximation property is obviously continuous. The approximation property is equivalent to the fact that there exists at least one sequence of trigonometric polynomials $T_n$ with values in $X$, uniformly convergent on $\Re$ to $f$.

Now we shall state (without proofs) a few essenial properties of Almost

Periodicity with values in Banach spaces from standard texts.

PROPERTY I. An almost periodic function with values in Banach space $X$ is bounded in $X$ i.e. bounded in the norm.

PROPERTY II. An almost periodic function is uniformly continuous on the real line $\Re$.

PROPERTY III. If $f$ is almost periodic with values in $X$, then $\lambda f$ where $\lambda$ is a complex number and any translate $f_h$ are almost periodic functions. The numerical function $t \to \|f(t)\|$ is also almost periodic.

PROPERTY IV. If $f_n$ is a sequence of almost periodic functions with values in $X$ and if

$$\lim_{n \to \infty} f_n(t) = f(t)$$

uniformly on $\Re$ in the sense of convergence in the norm, then $f$ is almost periodic.

PROPERTY V. The set of values of an almost periodic function with values in $X$ is relatively compact in $X$.

PROPERTY VI. The necessary and sufficient condition for a continuous function $f : \Re \to X$ to be almost periodic is that it is normal.

PROPERTY VII. The sum of two almost periodic functions with values in $X$ is an almost periodic function.

PROPERTY VIII. A function $f : \Re \to X$ with approximation property is almost periodic.

DEFINITION (3.6). Let us suppose that $X = L_p[0, 1], p \geq 1$ and the functions to be considered may take complex values. As known, $L_p[0, 1]$ is a Banach space and in the case $p = 2$ it is a Hilbert Space. Let $f(x,t)$ be a function defined for all $x \in [0, 1]$ and for any $t \in \Re$ such that $f(x,t) \in L_p[0,1]$ for any $t \in \Re$. Then $f(x,t)$ is said to be *Almost Periodic in the mean of order p*, $p \geq 1$, if the function $t \to f(x,t) \in L_p[0,1]$ is almost periodic.

In other words, $f(x,t)$ is almost periodic in the mean of order $p$, if there exist some functions $\delta(\varepsilon,t) > 0$ and $l(\varepsilon) > 0$, defined for $\varepsilon > 0$ and $t \in \Re$ with the following properties:

(1). The inequality $|t_1 - t| < \delta$ implies

$$\int_0^1 |f(x,t_1) - f(x,t)|^p \, dx < \varepsilon \tag{5}$$

(2). Any interval of length $l$ on the real line contains at least a point $\tau$ such that

$$\int_0^1 |f(x,t+\tau) - f(x,t)|^p \, dx < \varepsilon^p, \ t \in \Re. \tag{6}$$

The general properties of almost periodicity with values in Banach space $X$ are applied to this special case as well and we shall obtain the following results:

(i). If $f(x,t)$ is an almost periodic function in the mean of order $p$, then there exists a number $M_f > 0$ such that

$$\left\{\int_0^1 |f(x,t)|^p\,dx\right\}^{1/p} < M_f,\ \forall t \in \Re. \tag{7}$$

(ii). If $f(x,t)$ is an almost periodic function in the mean of order $p$, then for any $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that

$$|t_1 - t_2| < \delta,\ t_1, t_2 \in \Re$$

implies

$$\int_0^1 |f(x,t_1) - f(x,t_2)|^p\,dx < \varepsilon^p. \tag{8}$$

(iii). If $f(x,t)$ is almost periodic in the mean of order $p$, then from any sequence $\{f(x,t+h_n)\}$ we may extract a subsequence which converges in the mean of order $p$, uniformly with respect to $t \in \Re$.

(iv). The sum of two almost periodic functions of order $p$ is a function of the same kind.

Conversely, if $f(x,t) \in L_p[0,1]$ for $t \in \Re$ and if the condition (1) is satisfied i.e. if $f(x,t)$ is continuous in the mean of order $p$, then the normality in the sense of the convergence in the mean of order $p$, uniformly with respect to $t \in \Re$ implies the almost periodicity in the mean of order $p$ for the function $f(x,t)$.

Now we are able to discuss little bit on the notion of almost periodic functions in the Stepanov sense. Let $f(t)$ be a numerical function defined almost everywhere in $\Re$ such that $f \in L_p[a,b]$ for any bounded interval $[a,b] \in \Re$.

DEFINITION (3.7). The function $f(t)$ is said to be *Almost Periodic in the sense of Stepanov or $S^p$-almost periodic* if for any $\varepsilon > 0$, there is a number $l(\varepsilon) > 0$ such that

any interval of length $l(\varepsilon)$ of the real line contains at least one point $\tau$ for which

$$\sup_{x\in\Re}\left\{\int_x^{x+1} |f(t+\tau)-f(t)|^p\,dt\right\}^{1/p} < \varepsilon. \tag{9}$$

The same class of functions is obtained replacing (9) by

$$\sup_{x\in\Re}\left\{\frac{1}{\alpha}\int_x^{x+\alpha} |f(t+\tau)-f(t)|^p\,dt\right\}^{1/p} < \varepsilon \tag{10}$$

where $\alpha$ is an arbitrary positive number [14].

Let now $f(t)$ be an $S^p$-almost periodic function. Consider the function of two variables $\phi(x,t) \equiv f(x+t)$, defined for $0 \le x \le 1$ and $t \in \Re$. From (9), it follows that

$$\left\{\int_0^1 |\phi(x,t+\tau)-\phi(x,t)|^p\,dx\right\}^{1/p} < \varepsilon, \quad t \in \Re. \tag{11}$$

Since

$$\lim_{h\to 0}\int_0^1 |\phi(x,t+h)-\phi(x,t)|^p dx = 0, \tag{12}$$

it means that $\phi(x,t)$ is an almost periodic function in the mean of order $p$.

Properties (i), (ii) and (iii) stated above for almost periodic functions also hold for $S^p$-almost periodic functions. The following characterisation shows that an $S^p$-almost periodic function can be reduced to an almost periodic function in the Bohr sense.

**THEOREM (3.1).** *If an $S^p$-almost periodic function ($p \ge 1$) is uniformly continuous on the real*

*line, then it is almost periodic in the Bohr sense.*

*PROOF.* From Holder's inequality, we have

$$\int_x^{x+1} |f(x+\tau) - f(x)|dx \leq \left\{ \int_x^{x+1} |f(x+\tau) - f(x)|^p dx \right\}^{1/p}. \tag{13}$$

This inequality shows that it suffices to consider the case of $p = 1$ only. We consider

$$\phi_h(x) = \frac{1}{h} \int_0^h f(x+t)dt, \quad h > 0. \tag{14}$$

Using the above, we can reach to

$$|\phi_h(x+\tau) - \phi_h(x)| \leq \frac{1}{h} \int_x^{x+h} |f(t+\tau) - f(t)|dt. \tag{15}$$

Therefore $\phi_h(x)$ is continuous and almost periodic in the Bohr sense provided that $f(t)$ is considered to be $S^1$-almost periodic.

It is now observed that

$$|\phi_h(x) - f(x)| = \frac{1}{h} \left| \int_0^h \{f(x+t) - f(x)\}dt \right|$$

$$\leq \frac{1}{h} \int_0^h |f(x+t) - f(x)|dt. \tag{16}$$

where $0 \leq t \leq h$. The uniform continuity of the function $f(x)$ leads to

$$|f(x+t) - f(x)| < \varepsilon, \text{ if } h < \delta(\varepsilon). \tag{17}$$

Therefore, we obtain

$$|f_h(x) - f(x)| < \varepsilon, \quad \forall x \in \Re \tag{18}$$

provided $h < \delta(\varepsilon)$ which shows that $f(x)$ is the limit in the sense of uniform convergence on the whole line of almost periodic functions in the Bohr sense.

Therefore according to Property VI of Chapter I, $f(x)$ is almost periodic in the Bohr sense. $\diamond$

NOTE

This theorem shows that Stepanov's definition of almost periodicity is more general than that given by Bohr.

## 3. RESULTS ON ALMOST PERIODICITY

Let us consider the following systems of differential equations

$$x' = f(x,t), t \in \Re \tag{1}$$

and

$$x'' = f(x,t), t \in \Re \tag{2}$$

where $f(x,t)$ is almost periodic in $t$, uniformly with respect to $x$ in any compact subset of $R^n$. The almost periodicity of $f(x,t)$ will mean either Bohr's almost periodicity or that of Stepanov; whereas we will always mean the almost periodicity of the solutions to be in Bohr's sense.

We will be required a few assumptions in obtaining the almost periodicity criteria which are as follows:

(i). The map $f$ from $R^n \times R$ into $R^n$ is continuous and such that there exist two functions $\phi$ and $\psi$ from $(0, \infty)$ into itself with

$$< f(x,t) - f(y,t),\ x - y > \geq\ -\phi(|x-y|) - \psi(|x-y|). \tag{3}$$

(ii). The functions $\phi$ and $\psi$ satisfy

$$\phi(r) = o(r) \text{ as } r \to 0 \tag{4}$$

and

$$\liminf \frac{1}{r}\{\phi(r) - \psi(r)\} > 0 \text{ as } r \to \infty, \tag{5}$$

while the greatest root $r(\varepsilon)$, say, of the equation

$$\phi(r) - \psi(r) - \varepsilon r = 0,\ \varepsilon > 0, \tag{6}$$

is such that

$$\lim r(\varepsilon) = 0 \text{ as } \varepsilon \to 0. \tag{7}$$

(iii). The map $t \to f(x,t)$ from $\Re$ into $R^n \times R$, is almost periodic in Bohr's sense, uniformly with respect to the second argument in any bounded set of $R^n$.

Now we are in a position to state the following results.

**THEOREM (3.2).** *Let $x(t)$ be a bounded solution on $\Re$ of the system (1) in which $f$ satisfies the assumptions (i), (ii) and (iii). Then $x(t)$ is Bohr almost periodic.*

*PROOF.* Let us denote that

$$y(t) = |x(t+\tau) - x(t)|^2, \ \forall t \in \Re. \tag{8}$$

where $\tau \in \Re$ is a translation number. Differentiating (8) with respect to $t$ and using (1), we get

$$y' = 2\sqrt{y}\,|f(x,t+\tau) - f(x,t)|$$

Taking supremum on both sides, using (6) first and then subtracting $2\varepsilon\sqrt{y}$ from the right side, we obtain

$$y' \ \geq \ 2\{\phi(\sqrt{y}) - \psi(\sqrt{y}) - \varepsilon\sqrt{y}\}, \ \forall t \in \Re. \tag{9}$$

where the number $\varepsilon$ is defined by

$$\varepsilon = \sup\{|f(x(t),t+\tau) - f(x(t),t)|, \ \forall t \in \Re\}, \tag{10}$$

and is certainly finite because of the assumption (iii) on $f(x,t)$. Of course, we can choose $\varepsilon$ as small as we want whenever $\tau$ is chosen among the $\varepsilon$-almost periods of the almost periodic function $f(x,t)$.

Now according to (5), we have $\omega(y) \equiv \phi(\sqrt{y}) - \psi(\sqrt{y}) - \varepsilon\sqrt{y} > 0$ when $\sqrt{y}$ is large enough. Again by (4) and the condition $\psi(\sqrt{y}) \geq 0$, we see that $\phi(\sqrt{y}) - \psi(\sqrt{y}) - \varepsilon\sqrt{y} < 0$ for small values of $\sqrt{y}$. Let $y(\varepsilon)$ be the root of $\phi(\sqrt{y}) - \psi(\sqrt{y}) - \varepsilon\sqrt{y} = 0$ and by (7) obviously $\sqrt{y} > y(\varepsilon) > 0$. Therefore, by applying Lemma (3.2) and taking into account the boundedness of $y(t)$ on $\Re$, equation (9) implies

$$\sqrt{y} < y(\varepsilon)$$

$$\Rightarrow |x(t+\tau) - x(t)| < y(\varepsilon), \forall t \in \Re \tag{11}$$

So by (7), $x(t)$ is almost periodic. $\diamond$

**THEOREM (3.3).** *Let $x(t)$ be a map from $\Re$ into $R^n$, twice differentiable, bounded on $\Re$ and satisfying the system (2). We assume that $f(x,t)$ verifies the assumptions (i), (ii) and (iii) stated above. Then $x(t)$ is Bohr almost periodic.*

*PROOF.* In this proof, we will follow the same strategy as that of the Theorem (3.2). So as before, we consider

$$y(t) = |x(t+\tau) - x(t)|^2$$

By using (2), we obtain

$$y'' = 2\sqrt{y}|f(x,t+\tau) - f(x,t)| + \frac{1}{2y}.y'$$

Taking supremum on both sides, we get

$$y'' = 2\sqrt{y}\varepsilon + \frac{1}{2y}.y' \qquad \text{[By (10)]}$$

$$\geq 2\sqrt{y}\varepsilon$$

$$\geq 2\sqrt{y}\left\{\frac{\phi(\sqrt{y})-\psi(\sqrt{y})}{\sqrt{y}}\right\} \quad \text{[By (6) ]}$$

$$\geq 2\{\phi(\sqrt{y})-\psi(\sqrt{y})-\varepsilon\sqrt{y}\},\ \forall t \in \Re \tag{12}$$

According to (5), we have

$$\omega(y) \equiv \phi(\sqrt{y})-\psi(\sqrt{y})-\varepsilon\sqrt{y} > 0$$

when $y$ is large enough. If $y(\varepsilon)$ be the root of $\phi(\sqrt{y})-\psi(\sqrt{y})-\varepsilon\sqrt{y}=0$, by (7) it is obvious that $\sqrt{y} > y(\varepsilon) > 0$. So now applying Lemma (3.3) to the inequality (12), we obtain

$$\{y(t)\}^{\frac{1}{2}} = |x(t+\tau)-x(t)| \ < \ y(\varepsilon),\ \forall t \in \Re. \tag{13}$$

According to (7), (13) implies that $x(t)$ is almost periodic. $\diamond$

NOTE

If we choose $\psi(r)$ to be identically zero and $\phi(r)=mr^2$, i.e. condition (3) takes the form of a monotonicity condition for $f(x,t)$,

$$< f(x,t)-f(y,t), x-y > \geq \ m|x-y|^2,\ m>0. \tag{14}$$

As we have stated that when the right hand sides of the system of equations are weakly almost periodic, such as in the sense of Stepanov, we will face an awkward situation. In order to deal with such cases we will now

apply Lemmas (3.4) and (3.5); nevertheless, we are assuming the continuity of $f(x,t)$ in order to secure the existence of continuously differentiable solutions.

**THEOREM (3.4).** *Consider the system (1) under the following assumptions:*

*(i) The map $f(x,t)$ from $R \times R^n$ into $R^n$ is continuous and such that assumption (iii) holds true.*

*(ii) The map $t \to f(.,t)$, from $\Re$ into $R^n$ is almost periodic in Stepanov's sense with $p = 1$, uniformly with respect to the second argument in any compact set of $R^n$.*

*If $x(t)$ is a bounded solution of (1) on $\Re$, then $x(t)$ is Bohr almost periodic. Moreover, the almost periodic solution of (1) is unique.*

*PROOF.* Let $\tau \in \Re$ be a fixed number and consider again the function $y(t)$ defined by (8) where $x(t)$ is the bounded solution of (1) on $\Re$ whose existence is guaranteed by our hypotheses. Following again the strategy as that in the proof of Theorem (3.2), we obtain the following inequality

$$y'(t) \geq 2\{my(t) - g(t)\sqrt{y(t)}\}, \; t \in \Re, \tag{15}$$

where $g(t)$ is defined as

$$g(t) = |f(x(t), t+\tau) - f(x(t), t)|, \; t \in \Re. \tag{16}$$

Now applying the Lemma (3.4) on (15), we get

$$y(t) \leq K|g|_M$$

$$\Rightarrow \{y(t)\}^{\frac{1}{2}} = |x(t+\varepsilon) - x(t)| \leq K|g|_M, \tag{17}$$

where the constant $K$ depends only on $m$. From (16), it is obvious that

$$|g|_M = \sup \int_t^{t+1} |f(x(s), s+\tau) - f(x(s), s)| ds, \ t \in \Re, \tag{18}$$

can be done arbitrarily small, provided the translation number $\tau$ is chosen among the almost periods of $f$. Consequently, any $K^{-1}\varepsilon$-almost period of $f$ is an $\varepsilon$-almost period for the solution $x(t)$. Thus $x(t)$ is almost periodic.

Now we look for the uniqueness of the almost periodic solution to (1) using the Lemma (3.4) again. If we take into account the fact that $y(t) = |x(t) - \tilde{x}(t)|^2$ where $x(t)$ and $\tilde{x}(t)$ stand for two bounded solutions of (1), verifies the inequality

$$y \geq 2my - \varepsilon,$$

for every positive $\varepsilon$. $\diamond$

**THEOREM (3.5).** *Consider the differential system (2) subject to the assumptions (i) and (ii) of Theorem (3.4). If $x(t)$ is a bounded solution of (2) on $\Re$, then $x(t)$ is Bohr almost periodic and the solution is unique.*

*PROOF.* The proof of this theorem will be in the same line as that in Theorem (3.3). We cosider the same function

$$y(t) = |x(t+\tau) - x(t)|^2, \ \forall t \in \Re$$

Simple calculation gives

$$y'' \geq 2\{my(t) - g(t)\sqrt{y(t)} \tag{19}$$

where $g(t)$ is given by (16). Since $y(t)$ is obviously bounded on $\Re$, from (19) and Lemma (3.5), we obtain

$$\sqrt{y(t)} = |x(t+\tau) - x(t)| \leq K||g||_M, \ \forall t \in \Re \tag{20}$$

with $K$ depending on $m$ only. If we consider the almost periodicity of $f(x,t)$ in the sense of Stepanov, the inequality (20) implies the almost periodicity of $x(t)$ in the sense of Bohr.

To prove the uniqueness of $f(x,t)$, let us take two different solutions $x(t)$ and $\tilde{x}(t)$ of (2). Then

$$y(t) = x(t) - \tilde{x}(t)$$

satisfies the inequality

$$y'' = 2my, \quad \forall t \in \Re \tag{21}$$

which is strengthened if $\varepsilon > 0$ is subtracted from the right side of (21). $\diamond$

CHAPTER IV

# THE NON-HOMOGENEOUS HEAT EQUATION

## 1. INTRODUCTION

This work is based on the works of S. Zaidman [15] and [16]. In [15], Zaidman considered the following non-homogeneous heat equation

$$u_t(x,t) = \sum_{i=1}^{m} u_{x_i x_i}(x,t) + f(x,t), \quad t \in \Re \tag{1}$$

where $x = (x_1, x_2, x_3, \dots \dots \dots, x_m) \in R^m$, $u(x,t)$ and $f(x,t)$ are functions from $(-\infty, \infty)$ to the complete space $L^2(R^m)$ and $f(x,t)$ is almost periodic from $R$ into $L^2(R^m)$.

The function $u(x,t)$ is said to be weakly continuously differentiable if the scalar function

$$\int_{R^m} u(x,t)\phi(x)dx, \quad \forall \phi(x) \in L^2(R^m)$$

is continuously differentiable with respect to the variable $t$.

The function $u(x,t)$ is a weak solution of the equation (1) if for any $\phi(x) \in S(R^m)$, Schwartz space of infinitely differentiable rapidly decreasing functions [17], the following relation holds

$$\frac{d}{dt}\int_{R^m} u(x,t)\phi(x)dx = \int_{R^m} u(x,t)\Big\{\sum_{i=1}^{m}\phi_{x_i x_i}(x)\Big\}dx + \int_{R^m} f(x,t)\phi(x)dx. \tag{2}$$

Again the function $u(x,t)$ is said to satisfy the initial condition $u(x,t_o) = u(x) \in L^2(R^m)$ if, for any $\phi(x) \in L^2(R^m)$,

$$\lim_{t\to t_o}\int_{R^m} u(x,t)\phi(x)dx = \int_{R^m} u(x)\phi(x)dx. \tag{3}$$

Applying Fourier-Plancherel transformation in $L^2(R^m)$ and $S(R^m)$, it is possible to find an equation which is equivalent to (2) viz:

$$\frac{d}{dt}\int_{R^m} U(s,t)\Phi(s)ds = \int_{R^m} U(s,t)\{-s^2\}\Phi(s)ds + \int_{R^m} F(s,t)\Phi(s)ds, \tag{4}$$

where $U(s,t)$, $\Phi(s)$ and $F(s,t)$ are the Fourier transforms of $u(x,t)$ $\phi(x)$ and $f(x,t)$ respectively and $s^2 = s_1^2 + s_2^2 + \dots\dots + s_m^2$, $U(s,t)$ is weakly continuously differentiable and $F(s,t)$ is weakly continuous.

A solution $U(s,t)$ of (4) will be said to satisfy the initial condition $U(s,t_o) = U(s) \in L^2(R^m)$ if, for every $\Phi(s) \in L^2(R^m)$,

$$\lim_{t\to t_o}\int_{R^m} U(s,t)\Phi(s)ds = \int_{R^m} U(s)\Phi(s)ds. \tag{5}$$

Then Zaidman [15] proved the following theorem.

**THEOREM (4.1).** *Equation (4) for initial condition (5) has a unique $L^2$-bounded solution given by*

$$U(s,t) = e^{-s^2(t-t_o)}U(s) + \int_{t_o}^{t} e^{-s^2(t-\tau)}F(s,\tau)d\tau \tag{6}$$

*where $t \in [a,b]$, an interval within the domain of definition of $F(s,\tau)$.*

## 2. THE HEAT EQUATION

Now we are going to develop the works of Zaidman in the following way. Let us consider the following non-homogeneous heat equation

$$\frac{\partial U(x,t)}{\partial t} = \sum_{k=1}^{m} \frac{\partial^2 U(x,t)}{\partial x_k^2} + \sum_{k=1}^{m} a_k \frac{\partial U}{\partial x_k} + F(x,t) \tag{1}$$

where $a_k$'s are real constants, $x = (x_1, x_2, x_3, \dots \dots \dots, x_m) \in R^m$, $U(x,t)$ and $F(x,t)$ are functions from $(-\infty, \infty)$ to the complete space $L^2(R^m)$ and $F(x,t)$ is almost periodic from $R$ into $L^2(R^m)$.

DEFINITION (4.1). The function $U(x,t)$ is said to be *Weakly Continuously Differentiable* if the scalar function

$$\int_{R^m} U(x,t)\phi(x)dx, \quad \forall \phi(x) \in L^2(R^m)$$

is continuously differentiable with respect to the time variable $t$.

DEFINITION (4.2). The function $U(x,t)$ is said to be a *Weak Solution* of the equation (1) if, for any $\Phi(x) \in S(R^m)$, the following relation holds

$$\frac{d}{dt}\int_{R^m} U(x,t)\Phi(x)dx = \int_{R^m} U(x,t)\Big\{\sum_{k=1}^{m} \frac{\partial^2}{\partial x^2} + \sum_{k=1}^{m} a_k \frac{\partial}{\partial x_k}\Big\}\Phi(x)dx + \int_{R^m} F(x,t)\Phi(x)dx. \tag{2}$$

DEFINITION (4.3). The function $U(x,t)$ satisfies the *Initial Condition* $U(x,t_o) = U(x) \in L^2(R^m)$ if, for any $\Phi(x) \in L^2(R^m)$,

$$\lim_{t \to t_o} \int_{R^m} U(x,t)\Phi(x)dx = \int_{R^m} U(x)\Phi(x)dx.$$

From the properties of Fourier-Plancherel transformations in $L^2(R^m)$ and $S(R^m)$, it is possible to find an equation which is equivalent to (2) viz:

$$\frac{d}{dt}\int_{R^m} u(s,t)\phi(s)ds = \int_{R^m} u(s,t)\left\{-s^2 + i\sum_1^m a_k s_k\right\}\phi(s)ds + \int_{R^m} f(s,t)\phi(s)ds,\ i = \sqrt{-1} \quad (3)$$

where $u(s,t)$, $\phi(s)$ and $f(s,t)$ are the Fourier transforms of $U(x,t)$, $\Phi(x)$ and $F(x,t)$ respectively, $s = (s_1, s_2, \dots \dots \dots, s_m)$, $s^2 = s_1^2 + s_2^2 + \dots \dots + s_m^2$, $u(s,t)$ is weakly continuously differentiable and $f(s,t)$ is weakly continuous. Substituting

$$P(s) = -s^2 + i\sum_1^m a_k s_k$$

in (3), we get

$$\frac{d}{dt}\int_{R^m} u(s,t)\phi(s)ds = \int_{R^m} u(s,t)P(s)\phi(s)ds + \int_{R^m} f(s,t)\phi(s)ds. \quad (4)$$

Analogously, a solution $u(s,t)$ of (4) is said to satisfy the initial condition $u(s,t_o) = u(s) \in L^2(R^m)$ if, for every $\phi(s) \in L^2(R^m)$,

$$\lim_{t\to t_o}\int_{R^m} u(s,t)\phi(s)ds = \int_{R^m} u(s)\phi(s)ds. \quad (5)$$

Now we are going to prove the following theorem.

**THEOREM (4.2).** *If $f(s,t)$ is almost periodic from $R$ to $L^2(R^m)$, then any solution $u(s,t)$ of the equation (4) which is defined and $L^2$-bounded on $R$ is almost periodic.*

*PROOF.* Let us take the characteristic functions

$$\chi_p(s) = \begin{cases} 1, & if\ s \in E_p \\ 0, & if\ s \notin E_p \end{cases} \quad (7)$$

for $p = 1, 2, \dots \dots$ where $E_p = \left\{s : \frac{1}{p+1} \le |s| < \frac{1}{p}\right\}$.

We now claim that the function

$$u_p(s,t) = \chi_p(s)u(s,t), \tag{8}$$

for each $p$ is a solution of (4) for the known term

$$f_p(s,t) = \chi_p(s)f(s,t) \tag{9}$$

in place of $f(s,t)$ in (4). First, we prove that $u_p(s,t)$ is $L^2$-bounded. We have

$$\int_{R^m} |u_p(s,t)|^2 ds = \int_{E_p} |u(s,t)|^2 ds$$

$$\leq \int_{R^m} |u(s,t)|^2 ds$$

$$\leq k$$

since $u(s,t)$ is $L^2$-bounded. Hence $u_p(s,t)$ is $L^2$-bounded.

Secondly, we prove that $u_p(s,t)$ is also weakly continuously differentiable. Consider

$$\frac{d}{dt}\int_{R^m} u_p(s,t)\phi(s)ds = \frac{d}{dt}\int_{E_p} u(s,t)\chi_p(s)\phi(s)ds, \phi(s) \in L^2(R^m)$$

$$= \frac{d}{dt}\int_{E_p} u(s,t)\psi(s)ds$$

where $\psi(s) = \chi_p(s)\phi(s)$ and $\psi(s) \in L^2(R^m)$. Since $u(s,t)$ is weakly continuously differentiable, the scalar function

$$\int_{E_p} u(s,t)\psi(s)ds$$

is continuously differentiable and therefore $u_p(s,t)$ is weakly continuously differentiable.

If $\phi(s) \in S(R^m)$, then $P(s)\phi(s) \in L^2(R^m)$ because of the rapidly decreasing character of $\phi(s)$. Hence both the integrals

$$\int_{R^m} u(s,t)P(s)\chi_p(s)\phi(s)ds, \quad \phi(s) \in S(R^m) \tag{10}$$

and

$$\int_{R^m} f(s,t)\chi_p(s)\phi(s)ds \tag{11}$$

exist. Remembering that $S(R^m)$ is dense in $L^2(R^m)$, we can find a sequence $\{\phi_n(s)\}$ in $S(R^m)$ such that

$$\lim_{n\to\infty} \phi_n(s) = \chi_p(s)\phi(s), \tag{12}$$

for any chosen $\phi(s)$ in $S(R^m)$, the limit being taken in $L^2(R^m)$. As a convergent sequence in $L^2(R^m)$, $\{\phi_n(s)\}_1^\infty$ is $L^2$-bounded.

Now

$$|P(s)| = |-s^2 + i\sum_1^m a_k s_k|$$

$$= \sqrt{s^4 + (\sum_1^m a_k s_k)^2}$$

$$= \sqrt{s^4 + |\sum_1^m a_k s_k|^2}$$

But $|\sum_1^m a_k s_k| \le \sum_1^m |a_k|.|s_k| \le ma|s|$ where $a = \max\{|a_1|, \dots\dots\dots, |a_m|\}$. For $s \in E_p$, we have

$$\frac{1}{p+1} \leq |s| < \frac{1}{p}$$

Therefore when $s \in E_p$, we get

$$|P(s)| \leq \sqrt{\frac{1}{p^4} + \frac{m^2a^2}{p^2}}$$

$$= \frac{\sqrt{1 + m^2a^2p^2}}{p^2}$$

$$= \frac{\sqrt{1 + cp^2}}{p^2} \tag{13}$$

where $c = m^2a^2 \geq 0$.

Using (13), we can show now that

$$\lim_{n\to\infty} P(s)\phi_n(s) = \chi_p(s)\phi(s)P(s) \quad \in \quad L^2(R^m) \tag{14}$$

We observe that

$$||P(s)\phi_n(s) - \chi_p(s)\phi(s)P(s)|| \leq \max|P(s)|.||\phi_n(s) - \chi_p(s)\phi(s)||$$

$$\leq \frac{\sqrt{1 + cp^2}}{p^2}||\phi_n(s) - \chi_p(s)\phi(s)||_{L^2}$$

since the effective range of integration will be $E_p$. Again, for each $p$,

$$\lim_{n\to\infty} ||\phi_n(s) - \chi_p(s)\phi(s)||_{L^2} = 0$$

and therefore, we have,

$$||P(s)\phi_n(s) - \chi_p(s)\phi(s)P(s)|| \to 0 \tag{15}$$

Also $\phi_n(s) \in S(R^m)$ for each n, and $u(s,t)$ is a solution of (4). So

$$\frac{d}{dt}\int_{R^m} u(s,t)\phi_n(s)ds = \int_{R^m} u(s,t)P(s)\phi_n(s)ds + \int_{R^m} f(s,t)\phi_n(s)ds \tag{16}$$

holds for each $n = 1,2,3,\ldots\ldots\ldots$. But

$$\sup_{t\in\Re} ||u(s,t)||_{L^2} < \infty \tag{17}$$

by hypothesis. So from (15), we have

$$||u(s,t)P(s)\phi_n(s) - u(s,t)\chi_p(s)\phi(s)P(s)||_{L^2} \leq ||u(s,t)||_{L^2}.||P(s)\phi_n(s) - \chi_p(s)\phi(s)P(s)|| \to 0, \tag{18}$$

in fact uniformly as $n \to \infty$. Moreover,

$$||u(s,t)P(s)\phi_n(s)||_{L^2} \leq ||u(s,t)||_{L^2}.||P(s)\phi_n(s)||_{L^2} < \infty$$

because $\{P(s)\phi_n(s)\}$ as a convergent sequence in $L^2(R^m)$ must be bounded in norm.

Again $u(s,t)\{P(s)\phi_n(s)\}$ is in $L^2(R^m)$ because both $u(s,t)$ and $P(s)\phi_n(s)$, for each $n$, are in $L^2(R^m)$. Hence from Lebesque's theorem, we obtain

$$\lim_{n\to\infty}\int_{R^m} u(s,t)P(s)\phi_n(s)ds = \int_{R^m} u(s,t)\chi_p(s)P(s)\phi(s)ds \tag{19}$$

Now,

$$||f(s,t)\phi_n(s) - f(s,t)\chi_p(s)\phi(s)||_{L^2} \leq ||f(s,t)||_{L^2}.||\phi_n(s) - \chi_p(s)\phi(s)||$$

$$\leq k||\phi_n(s) - \chi_p\phi(s)||_{L^2} \to 0 \tag{20}$$

from (12) as $n \to \infty$, in fact uniformly. Also

$$||f(s,t)\phi_n(s)||_{L^2} \leq ||f(s,t)||_{L^2}.||\phi_n(s)||_{L^2} \leq \kappa < \infty$$

since $||\phi_n(s)||_{L^2}$ is a convergent sequence in $L^2(R^m)$ and $f(s,t)$ is almost periodic, both are bounded. So

$$f(s,t)\phi_n(s) \in L^2(R^m).$$

Hence again we can pass the limit under the integral sign from Lebesgue's theorem:

$$\lim_{n\to\infty} \int_{R^m} f(s,t)\phi_n(s)ds = \int_{R^m} f(s,t)\chi_p(s)\phi(s)ds \tag{21}$$

Putting (19) and (21) in (16), it follows that

$$\lim_{n\to\infty} \frac{d}{dt} \int_{R^m} u(s,t)\phi_n(s)ds = \int_{R^m} u(s,t)\lambda_p(s)P(s)\phi(s)ds + \int_{R^m} f(s,t)\chi_p(s)\phi(s)ds \tag{22}$$

Because of uniform convergence in equations (18) and (20), the convergence in equations (19) and (21) is uniform. Hence we can pass the limit under the differentiation sign on the left in equation (21) and

$$\lim_{n\to\infty} \frac{d}{dt} \int_{R^m} u(s,t)\phi_n(s)ds = \frac{d}{dt}\left\{\lim_{n\to\infty} \int_{R^m} u(s,t)\phi_n(s)ds\right\}$$

$$= \frac{d}{dt} \int_{R^m} \left\{\lim_{n\to\infty} u(s,t)\phi_n(s)\right\}ds$$

$$= \frac{d}{dt} \int_{R^m} u(s,t)\chi_p(s)\phi(s)ds.$$

Hence from equation (22), we have

$$\frac{d}{dt} \int_{R^m} u(s,t)\chi_p(s)\phi(s)ds = \int_{R^m} u(s,t)\chi_p(s)P(s)\phi(s)ds + \int_{R^m} f(s,t)\chi_p(s)\phi(s)ds \tag{23}$$

for arbitrary $\phi \in S(R^m)$ and each $p = 1,2,3,\ldots\ldots$. This exactly means $u_p(s,t)$ is a solution of equation (4) when the known term is $f_p(s,t) = \chi_p(s)f(s,t)$ in place of $f(s,t)$ in equation (4).

Define

$$\overline{u}(s,t) = \begin{cases} u(s,t), & 0 \le |s| < 1 \\ 0, & |s| \ge 1 \end{cases} \tag{24}$$

and

$$\overline{f}(s,t) = \begin{cases} f(s,t), & 0 \le |s| < 1 \\ 0, & |s| \ge 1 \end{cases} \tag{24'}$$

For reasons similar to those for $u_p(s,t)$, we have $\overline{u}(s,t)$ a solution of equation (4) when the known term $f(s,t)$ in equation (4) is replaced by $\overline{f}(s,t)$. Using Zaidman's theorem (4.1), the solution for the equation (4) is given by

$$u(s,t) = e^{P(s)(t-t_o)}u(s) + \int_{-t_o}^{t} e^{P(s)(t-u)}f(s,u)du \tag{25}$$

Let us define

$$v_p(s,t) = \lim_{R\to\infty} \int_{-R}^{t} e^{P(s)(t-u)}f_p(s,u)du \tag{26}$$

where the integral is taken in the sense of Bochner and the limit is taken strongly in $L^2(R^m)$. Put

$$G_p(s,u) = e^{P(s)(t-u)}f_p(s,u) \tag{27}$$

Using the definition of norm, we have

$$||G_p(s,u)||_{L^2} = \left\{ \int_{R^m} |e^{P(s)(t-u)} f_p(s,u)|^2 \, ds \right\}^{\frac{1}{2}}$$

$$= \left\{ \int_{E_p} |e^{(-s^2+i\sum_1^m a_k s_k)(t-u)}|^2 . |f_p(s,u)|^2 ds \right\}^{\frac{1}{2}}$$

$$= \left\{ \int_{E_p} e^{-2s^2(t-u)} |f_p(s,u)|^2 ds \right\}^{\frac{1}{2}}$$

$$\leq e^{-\frac{t-u}{(p+1)^2}} \left\{ \int_{R^m} |f_p(s,u)|^2 ds \right\}^{\frac{1}{2}}$$

$$\leq \sup_{u \in \Re} ||f_p(s,u)||_{L^2} . e^{-\frac{(t-u)}{(p+1)^2}}$$

$$= M_p e^{-\frac{(t-u)}{(p+1)^2}}$$

where

$$M_p = \sup_{u \in \Re} ||f_p(s,u)||_{L^2}$$

The function $e^{-\frac{(t-u)}{(p+1)^2}}$ is integrable on $-\infty < u < t$. In fact,

$$\int_{-\infty}^{t} e^{-\frac{(t-u)}{(p+1)^2}} du = (p+1)^2. \tag{28}$$

Hence it follows that $G_p(s,u)$ is Bochner integrable on $(-\infty, t)$ and

$$|| \int_{-\infty}^{t} e^{P(s)(t-u)} f_p(s,u) du ||_{L^2} \leq \int_{-\infty}^{t} ||e^{P(s)(t-u)} f_p(s,u)||_{L^2} du$$

$$\leq \int_{-\infty}^{t} M_p e^{-\frac{(t-u)}{(p+1)^2}} du$$

$$= M_p(p+1)^2 \tag{29}$$

and this is true for any real $t$. Hence

$$\lim_{R\to\infty} \int_{-R}^{t} e^{P(s)(t-u)} f_p(s,u)du$$

exists strongly in $L^2(R^m)$ and $v_p(s,t)$ in (26) is well defined.

PROPERTIES OF $v_p(s,t)$. We shall now investigate a few properties of $v_p(s,t)$. First let us show that $v_p(s,t)$ is almost periodic. It is continuous (from its integral representation) and uniformly bounded, from (29) $f_p(s,t)$ is almost periodic from $R$ to $L^2(R^m)$. Let $\varepsilon > 0$ be given and $\tau_o$ be the $\varepsilon$-tranlation number for $f(s,t)$ which is almost periodic by hypothesis. Then

$$||f_p(s,t+\tau_o) - f_p(s,t)||_{L^2} = ||\lambda_p(s)f(s,t+\tau_o) - \lambda_p(s)f(s,t)||_{L^2}$$

$$\leq \max_{s\in R^m} |\lambda_p(s)|.||f(s,t+\tau_o) - f(s,t)||_{L^2}$$

$$\leq 1.\varepsilon = \varepsilon \tag{30}$$

For a given $\varepsilon > 0$, choose $\varepsilon/(p+1)^2$ translation number for the almost periodic function $f_p(s,t)$. Then

$$v_p(s,t+\tau_o) - v_p(s,t) = \int_{-\infty}^{t} e^{P(s)(t-u)}\{f_p(s,u+\tau_o) - f_p(s,u)\}du \tag{31}$$

as can be seen by an easy change of variable.

$$||v_p(s,t+\tau_o) - v_p(s,t)||_{L^2} \leq \int_{-\infty}^{t} ||e^{P(s)(t-u)}\{f_p(s,u+\tau_o) - f_p(s,u)\}||_{L^2}du$$

$$\leq \sup_{u \in \Re} ||f_p(s, u + \tau_o) - f_p(s, u)||_{L^2} . \int_{-\infty}^{t} e^{-\frac{(t-u)}{(p+1)^2}} du$$

$$\leq \frac{\varepsilon}{(p+1)^2} . (p+1)^2$$

$$= \varepsilon \tag{32}$$

for every $t$ in $R$, which proves the almost periodicity of $v_p(s,t)$.

Now we show that $v_p(s,t)$ as a function from $R$ to $L^2(R^m)$ for each $p$, is strongly differentiable and

$$\frac{d}{dt} v_p(s,t) = P(s) v_p(s,t) + f_p(s,t)$$

We have

$$\frac{1}{h}\{v_p(s, t+h) - v_p(s,t)\}$$

$$= \int_{-\infty}^{t} \frac{1}{h} \left[ e^{P(s)(t+h-u)} - e^{P(s)(t-u)} \right] f_p(s,u) du + \frac{1}{h} \int_{t}^{t+h} e^{P(s)(t+h-u)} f_p(s,u) du \tag{33}$$

$$= I_1 + I_2, \text{ (say).}$$

We are going to show that

$$\lim_{h \to 0} ||I_1 - P(s) v_p(s,t)||_{L^2} = 0 \tag{34}$$

and

$$\lim_{h \to 0} ||I_2 - f_p(s,t)||_{L^2} = 0 \tag{35}$$

We have

$$\|I_2 - f_p(s,t)\|_{L^2} = \|\frac{1}{h}\int_t^{t+h}\left\{e^{P(s)(t+h-u)}f_p(s,u) - f_p(s,t)\right\}du\|_{L^2}$$

$$\leq \frac{1}{h}\int_t^{t+h}\|e^{P(s)(t+h-u)}\{f_p(s,u) - f_p(s,t)\}\|_{L^2}du$$

$$+ \frac{1}{h}\int_t^{t+h}\|\left\{e^{P(s)(t+h-u)} - 1\right\}f_p(s,t)\|_{L^2}du$$

$$= I_2' + I_2'', \text{ (say).}$$

Again, we have

$$I_2' \leq \frac{1}{h}\max|e^{P(s)(t+h-u)}|\int_t^{t+h}\|f_p(s,u) - f_p(s,t)\|_{L^2}du \to 0 \tag{36}$$

as $h \to 0$, because $f_p(s,t)$ is strongly continuous in $L^2(R^m)$ and $u \to t$ when $h \to 0$.

[NOTE

We will use the following principle: For $I_2''$, we consider the function $e^{P(s)(t+h-u)}$ as a function in $h$, say, $F(h)$ and note that $t \leq u \leq t+h$ so that when $h = 0$, $u = t$

$$\frac{F(h) - F(0)}{h} = F'(\theta h).]$$

Now for $I_2''$, we have,

$$I_2'' = \frac{1}{h}\int_t^{t+h}\|\left\{e^{P(s)(t-u+h)} - 1\right\}f_p(s,t)\|_{L^2}du$$

$$= \frac{1}{h}\int_t^{t+h}\left\{\int_{E_p}\left|e^{P(s)(t-u+h)}-1\right|^2.|f_p(s,t)|^2ds\right\}^{\frac{1}{2}}du$$

$$= \int_t^{t+h}\left\{\int_{E_p}\left|\frac{e^{P(s)(t-u+h)}-1}{h}\right|^2.|f_p(s,t)|^2ds\right\}^{\frac{1}{2}}du$$

$$= \int_t^{t+h}\left\{\int_{E_p}\left|P(s)e^{P(s)(t-u+\theta h)}\right|^2.|f_p(s,t)|^2ds\right\}^{\frac{1}{2}}du,\ 0\le\theta\le 1$$

$$= \int_t^{t+h}\left\{\int_{E_p}|P(s)|^2e^{-2s^2(t-u+\theta h)}|f_p(s,t)|^2ds\right\}^{\frac{1}{2}}du$$

$$\le \max_{s\in E_p}|P(s)|e^{-s^2(t-u+\theta h)}\left\{\int_{R^m}|f_p(s,t)|^2ds\right\}^{\frac{1}{2}}\int_t^{t+h}du$$

$$\le \frac{\sqrt{1+cp^2}}{p^2}.M_p.h \ \to\ 0 \text{ as } h\to 0 \tag{37}$$

Hence

$$\lim_{h\to 0}\{I_2'+I_2''\}=0, \text{ and so } \lim_{h\to 0}||I_2-f_p(s,t)||=0$$

Now

$$||I_1-P(s)v_p(s,t)||_{L^2}$$

$$= ||\int_{-\infty}^t\frac{1}{h}\{e^{P(s)(t-u+h)}-e^{P(s)(t-u)}\}f_p(s,u)du-P(s)\int_{-\infty}^t e^{P(s)(t-u)}f_p(s,u)du||_{L^2}$$

$$\le \int_{-\infty}^t\left[\int_{E_p}\left|\frac{1}{h}\left\{e^{P(s)(t-u+h)}-e^{P(s)(t-u)}\right\}-P(s)e^{P(s)(t-u)}\right|^2.|f_p(s,u)|^2ds\right]^{\frac{1}{2}}du$$

$$= \int_{-\infty}^t\left\{\int_{E_p}\left|P(s)\left(e^{P(s)(t-u+\theta h)}-e^{P(s)(t-u)}\right)\right|^2.|f_p(s,u)|^2ds\right\}^{\frac{1}{2}}du \tag{38}$$

Then

$$\left\{\int_{E_p}\left|P(s)\left(e^{P(s)(t-u+\theta h)}-e^{P(s)(t-u)}\right)\right|^2.|f_p(s,u)|^2ds\right\}^{\frac{1}{2}}$$

$$\leq \frac{\sqrt{1+cp^2}}{p^2}\left\{e^{-\frac{t-u+\theta h}{(p+1)^2}}-e^{-\frac{t-u}{(p+1)^2}}\right\}.M_p \tag{39}$$

and both functions inside the bracket are integrable over $-\infty < u < t$. Hence equation (38) exists and therefore we can pass the limit under $\int_{-\infty}^{t}$ in equation (38). The integrand in (39) is also bounded uniformly and (39) exists for all positive values of $h$. Hence we can pass the limit under the integral sign. Again

$$e^{P(s)(t-u+\theta h)}-e^{P(s)(t-u)} \to 0$$

for every $s$. Therefore from Lebesgue's Theorem

$$\lim_{h\to 0}\int_{E_p}\left|P(s)\left\{e^{P(s)(t-u+\theta h)}-e^{P(s)(t-u)}\right\}\right|^2.|f_p(s,u)|^2ds=0$$

and this is true for any value of $u$ in $-\infty < u < t$. Therefore

$$\int_{-\infty}^{t}\left[\int_{E_p}\left|P(s)\left\{e^{P(s)(t-u+\theta h)}-e^{P(s)(t-u)}\right\}\right|^2.|f_p(s,u)|^2ds\right]^{\frac{1}{2}}du=0$$

whose existence has been established before. Because of the presence of $f_p(s,t)$ in the definition of $v_p(s,t)$, the effective variation of $s$ is in $E_p$.

We can show that $P(s)v_p(s,t)$, for each $p$, is an almost periodic function from $R$ to $L^2(R^m)$. For a given $\varepsilon > 0$, choose a $p^2\varepsilon/\sqrt{1+cp^2}$ translation number for the almost periodic function $v_p(s,t)$. Then

$$||P(s)v_p(s,t+\tau_o)-P(s)v_p(s,t)||_{L^2} \leq \max_{s\in E_p}|P(s)|.||v_p(s,t+\tau_o)-v_p(s,t)||_{L^2}$$

$$< \frac{\sqrt{1+cp^2}}{p^2} \cdot \frac{p^2\varepsilon}{\sqrt{1+cp^2}}$$

$$= \varepsilon \tag{40}$$

Thus from equations (30) and (40), it follows that the function

$$\frac{d}{dt} v_p(s,t) = P(s)v_p(s,t) + f_p(s,t) \tag{41}$$

is almost periodic as the sum of two almost periodic functions, which incidentally shows that $v_p(s,t)$ is strongly continuously differentiable and hence certainly weakly continuously differentiable. Therefore, multiplying both sides of (41) by $\phi(s) \in S(R^m)$ and then integrating, we get

$$\int_{R^m} \frac{d}{dt} v_p(s,t)\phi(s)ds = \int_{R^m} v_p(s,t)P(s)\phi(s)ds + \int_{R^m} f_p(s,t)\phi(s)ds$$

But

$$\frac{d}{dt}\int_{R^m} v_p(s,t)\phi(s)ds = \int_{R^m} \frac{d}{dt}\{v_p(s,t)\phi(s)\}ds$$

because as shown above, the strong derivative of $v_p(s,t)$ exists continuously and hence the weak derivative also must exist continuously and the two should be equal. Therefore we have

$$\frac{d}{dt}\int_{R^m} v_p(s,t)\phi(s)ds = \int_{R^m} v_p(s,t)P(s)\phi(s)ds + \int_{R^m} f_p(s,t)\phi(s)ds. \tag{42}$$

Thus $v_p(s,t)$ is an $L^2$-bounded solution of equation (4) when the known term $f(s,t)$ is taken to be $f_p(s,t)$. The same thing was shown in (23) for $u_p(s,t)$. Hence from the uniqueness part of Zaidman's Theorem (4.1), we have

$$u_p(s,t) = v_p(s,t) \text{ for each } p = 1,2,\ldots\ldots. \tag{43}$$

Now

$$\sum_{p=1}^{\infty} u_p(s,t) = \sum_{p=1}^{\infty} u(s,t)\chi_p(s)$$

$$= u(s,t)\sum_{p=1}^{\infty}\chi_p(s) \text{ for any } t \in R$$

$$= u(s,t)\chi_c(s), \text{ say}$$

$$= \overline{u}(s,t) \text{ as defined in (24)}$$

where

$$\chi_c(s) = \begin{cases} 1, & 0 \le |s| < 1 \\ 0, & |s| \ge 1. \end{cases}$$

Moreover,

$$\sum_{p=1}^{\infty} ||u_p(s,t)||_{L^2}^2 = \sum_{p=1}^{\infty}\int_{E_p} |u(s,t)|^2 . |\chi_p(s)|^2 ds$$

$$= \sum_{p=1}^{\infty}\int_{E_p} |u(s,t)|^2 ds \tag{44}$$

$$= \int_{\cup E_p} |u(s,t)|^2 ds \quad \text{, since } E_p\text{'s are disjoint [18]}$$

$$= \int_{0\le|s|<1} |u(s,t)|^2 ds \quad \text{, since } \cup E_p = [0,1)$$

$$= \int_{R^m} |u(s,t)\chi_c(s)|^2 ds = ||\overline{u}(s,t)||_{L^2}^2$$

$$\le ||u(s,t)||_{L^2}^2 \le \kappa. \tag{45}$$

We will now show that the convergence in

$$\sum_{1}^{\infty} u_p(s,t) = \overline{u}(s,t)$$

and

$$\sum_{1}^{\infty} ||u_p(s,t)||_{L^2}^2 = ||\overline{u}(s,t)||_{L^2}^2$$

is uniform for $t$ in $(-\infty, \infty)$.

First let us estimate the volume of a shell formed by two concentric spheres with radii $\frac{1}{p}$ and $\frac{1}{p+1}$ in our space $R^m$.

The volume of a sphere of radius $r$ in $R^m$ is $V = C_m r^m$ where

$$C_m = \frac{\pi^{\frac{m}{2}}}{(m/2)!}$$

if $m$ is even and

$$C_m = \frac{2^m \pi^{\frac{m-1}{2}} \left(\frac{m}{2} - \frac{1}{2}\right)}{m!}$$

if $n$ is odd. We need to evaluate

$$\int_{\frac{1}{p+1} \le s < \frac{1}{p}} ds = C_m \left\{ \frac{1}{p^m} - \frac{1}{(p+1)^m} \right\}$$

$$= C_m \left\{ \frac{(p+1)^m - p^m}{p^m (p+1)^m} \right\}$$

$$= \frac{C_m [{}^m C_1 p^{m-1} + {}^m C_2 p^{m-2} + \dots \dots + 1]}{p^m (p+1)^m}$$

$$< \frac{C_m 2^m p^{m-1}}{p^m (p+1)^m}$$

[because $p \geq 1$ and so $p^{m-1} \geq p^{m-2} \geq \ldots\ldots$ and ${}^mC_1 + {}^mC_2 + \ldots\ldots < 2^m$.]

$$< \frac{C_m 2^m p^{m-1}}{p^{2m}}$$

$$= \frac{c}{p^{m+1}} \tag{46}$$

where $c$ is some positive constant depending on the dimension of the space. Again for every $t \in \Re$, we have

$$\|u_p(s,t)\|_{L^2}^2 = \|u(s,t)\chi_p(s)\|_{L^2}^2$$

$$\leq \|u(s,t)\|_{L^2}^2 \|\cdot\| \chi_p(s)\|_{L^2}^2$$

$$\leq \kappa \|\chi_p(s)\|_{L^2}^2, \text{ since } u(x,t) \text{ is } L^2\text{-bounded} \tag{47}$$

$$= \kappa \int_{R^m} |\chi_p(s)|^2 ds$$

$$= \kappa \int_{E_p} ds$$

$$\leq \frac{\kappa c}{p^{m+1}}, \quad \text{[by (46)]}$$

Hence

$$\|u_p(s,t)\|_{L^2}^2 \leq \frac{\kappa c}{p^{m+1}} \tag{48}$$

for all $t$ in $(-\infty, \infty)$ and $\sum_1^\infty \frac{\kappa c}{p^{m+1}}$ is convergent and therefore $\sum_1^\infty \|u_p(s,t)\|^2$ is

uniformly convergent for $t$ in $(-\infty, \infty)$, that is

$$\lim_{n\to\infty} \sum_{p=1}^{n} ||u_p(s,t)||^2 = ||\overline{u}(s,t)||^2 \tag{49}$$

uniformly in $t$.

Moreover, it can be seen easily that

$$||\sum_{1}^{n} u_p(s,t)||^2_{L^2} = \sum_{1}^{n} ||u_p(s,t)||^2_{L^2}$$

for every $n$ and $t$. Hence from equation (49),

$$\lim_{n\to\infty} ||\sum_{p=1}^{n} u_p(s,t)||^2_{L^2} = ||\overline{u}(s,t)||^2_{L^2}$$

uniformly in $t$.

We have already established that $\sum_1^\infty u_p(s,t) = \overline{u}(s,t)$ in the strong topology of $L^2(R^m)$. Hence

$$\lim_{n\to\infty} ||\sum_{p=1}^{n} u_p(s,t)|| = ||\overline{u}(s,t)||_{L^2}$$

uniformly in $t$.

Since each $u_p(s,t) = v_p(s,t)$ is almost periodic, $\overline{u}(s,t)$ is almost periodic.

Let us now consider the function

$$\tilde{u}(s,t) = u(s,t) - \overline{u}(s,t)$$

$$= \begin{cases} u(s,t), & |s| \geq 1 \\ 0, & 0 \leq |s| < 1, \end{cases} \quad \text{by (24).}$$

Our intention is to show that $\tilde{u}(s,t)$ is also almost periodic and that will prove the almost periodicity of $u(s,t)$.

Take the characteristic functions

$$\tilde{\chi}_p(s) = \begin{cases} 1, & s \in \xi_p \\ 0, & s \notin \xi_p \end{cases} \tag{51}$$

where

$$\xi_p = \{s : p \leq |s| < p+1\},\ p = 1,2,3,\ldots\ldots. \tag{52}$$

From arguments similar to (7) - (23), it can be seen that the functions

$$\tilde{u}_p(s,t) = \tilde{\chi}_p(s)u(s,t) \tag{53}$$

are the solutions of (4) when the known term $f(s,t)$ is replaced by

$$\tilde{f}_p(s,t) = \tilde{\chi}_p(s,t)f(s,t). \tag{54}$$

Consider the function

$$\tilde{V}_p(s,t) = \int_{-\infty}^{t} e^{P(s)(t-u)}\ \tilde{f}_p(s,u)du$$

The existence of the integral follows the same way as before in (42). But this time

$$||\tilde{V}_p(s,t)||_{L^2} \leq \frac{1}{p^2} \sup_{t\in\Re} ||f(s,t)||_{L^2} \tag{55}$$

Almost periodicity and strong differentiability of $\tilde{V}_p(s,t)$ is obtained by arguments similar to (33) - (41). The only alteration we have to make is that

$$|P(s)| \leq (p+1)\sqrt{c+(p+1)^2}$$

where $c$ is the non-negative constant in equation (13) in place of

$$|P(s)| \leq \frac{\sqrt{1+cp^2}}{p^2}$$

which we took in our earlier arguments. Similarly we have $\tilde{u}_p(s,t) = \tilde{v}_p(s,t)$ and $\sum_1^\infty \tilde{u}_p(s,t) = \tilde{u}(s,t)$ in the strong topology of $L^2(R^m)$, the convergence being uniform in $t$, which proves the almost periodicity of $\tilde{u}(s,t)$. Since

$$u(s,t) = \tilde{u}(s,t) + \overline{u}(s,t),$$

the sum of two almost periodic functions, our theorem is completely proved.

NOTE

This work may be extended for the case when $U(x,t)$ is strongly continuously differentiable and also for the case of an almost automorphic solution.

# APPENDIX

## GRAPHS SHOWING DIFFERENCE BETWEEN PERIODIC AND ALMOST PERIODIC FUNCTIONS

PLOT OF $f(t) = \cos(t) + \sin(t)$

PERIODIC

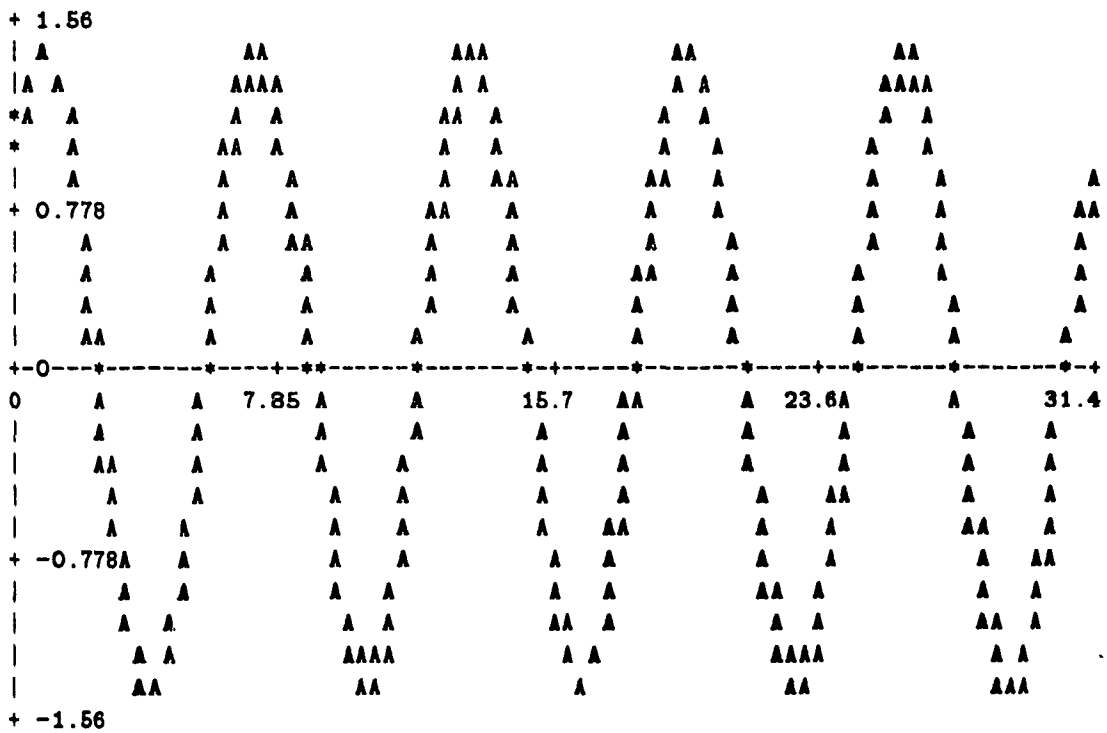


Fig.1

PLOT OF $f(t) = \cos(t) + \sin(2t)$

PERIODIC

```
+ 1.94
|AA              A              AA              AA              A
|AA             AAA             AAA             AA            A A
|AAA            A A             A A           A A             A A
*  A            A A            AA A            A  A            A A
*  A            A  A           A  A            A  A            A A
+ 0.968        A   A           A  A            A  A          A  AA           A
|  A           A   A           A  AA         A   A           A   A           A
|  A           A   A           A   A         A   A           A   A          AA
|   A     AA   A   A     AA   A    A     AA   A   A     AAA   A   A     AA   A
|   A    AAA   A   A     A A  A    A     AA   A    A    A A  A    A     AAA  A
+-0-*----*--*-*----+-----*-*--*----*---+*--*-*-----*----*-+*-*----*----*--*--*+
0   A    A  A A  7.85   A   AAA    A 15.7  AAA     A    23.6 A    A    A  A31.4
|   A    A   AA     A   A   AA     A    A   AA     A   A   AAA     A   A   AA
|   A   A          A   A          AA  AA         A   A           A   A
|   A   A          A   A           A  A          A   A           A   A
|    A  A          A   A           A  A          A   A           A  A
+ -0.968A          A  A            A  A           A A            A  A
|    A A            A A            A  A           A A            A  A
|    A A            A A            AAA             A A             A A
|    AAA            AAA             AA             A A             AA
|     A              A              AA              A              AA
+ -1.94
```

Fig. 2

PLOT OF $f(t) = \cos(t) + \sin(\sqrt{2}t)$

ALMOST PERIODIC BUT NOT PERIODIC

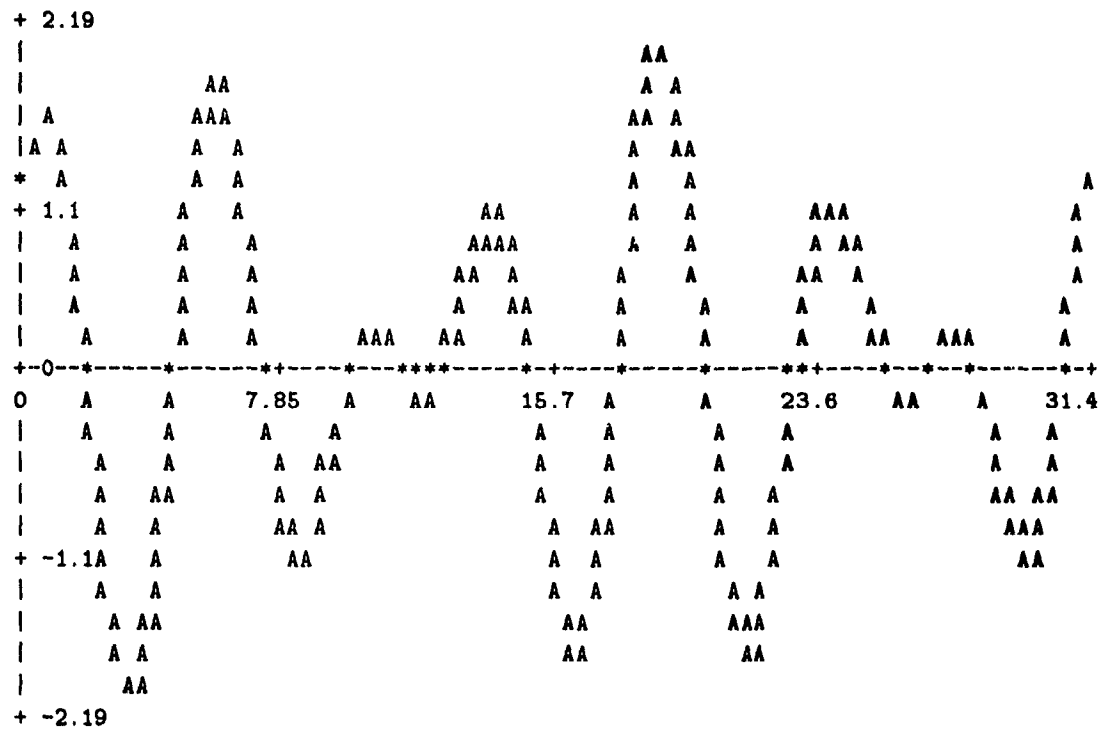


Fig. 3

PLOT OF $f(t) = \cos(2t) + \sin(t)$

PERIODIC

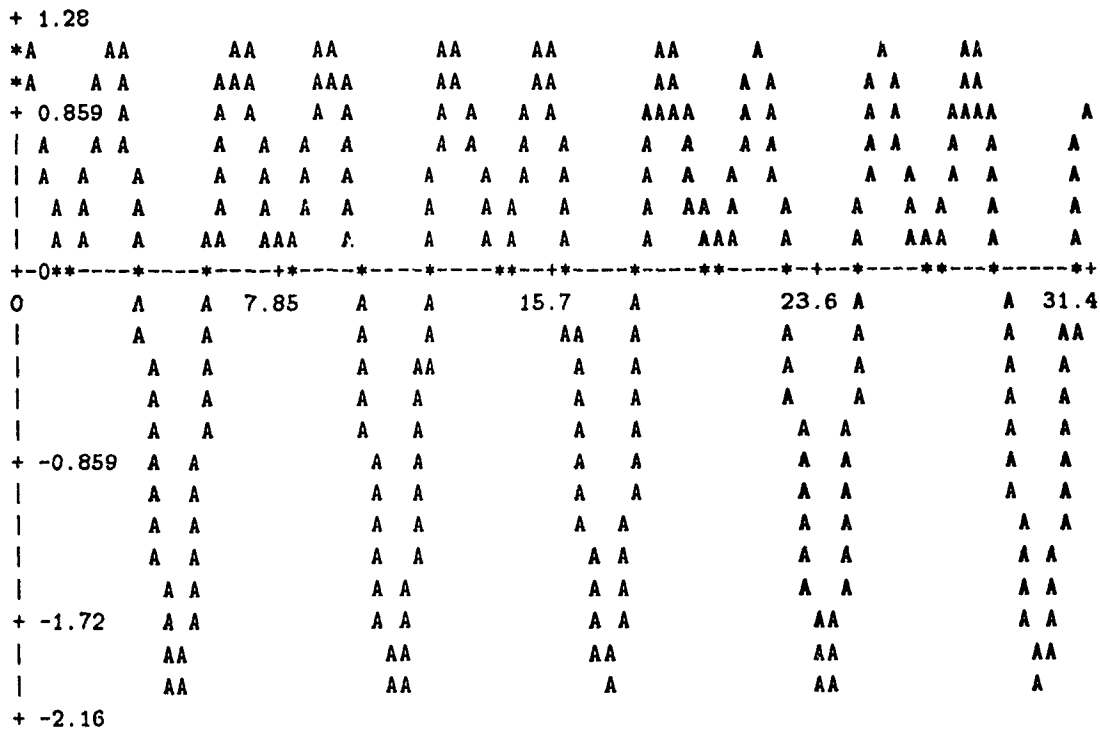


Fig. 4

PLOT OF $f(t) = \cos(\sqrt{2}t) + \sin(\sqrt{2}t)$

PERIODIC

```
+ 1.56
|AA          AA          AA          AA          AA          AA          AA
|AA          AA          AA          AA          AA          AA          AA
* A         A AA        A  A        AAAA        A  A        AA A        A  A         A
*  A        A  A        A  A        A  A        A  A        A  A        A  A        A
|  A        A  A        A  A        A  A        A  A        A  A        A  A        A
+ 0.778     A  A        A  A        A  A        A  A        A  A        A  A        A
|  A       A   A       A    A      A    A       A  A       A   A       A   A       A
|   A      A    A      A    A      A    A      A    A      A    A      A    A      A
|   A      A    A      A    A      A    A      A    A      A    A      A    A     A
|   A      A    A      A    A      A    A      A    A      A    A      A    A     A
+-0-*-----*----*---+-*----*-----*----*-+---*----*-----*---+*-----*----*-----*-+
0   A    A      A 7.85      A     A     15.7  A    A      A 23.6     A    AA   31.4
|    A   A      AA   A      A    A      A   A      A   A      A    A      A    A
|    A   A       A   A       A   A      A   A      A   A      A   A      A   A
|    A   A       A   A       A   A      A   A      A   A      A   A      A   A
!    A   A       A   A       A   A      A   A      A   A      A   A      A   A
+ -0.778A        A  A        A  A       A   A      A   A      A   A      A   A
|    AA A         A A        AA A        A A        A A        A AA       A A
|     A A         A A         A A        A A        A A        A A        A A
|     A A         AAA         A A        A A        A A        A A        AAA
|     AA          AA          AA          A          A          AA         AA
+ -1.56
```

Fig. 5

PLOT OF $f(t) = \cos(\sqrt{2}t) + \sin(t)$

ALMOST PERIODIC BUT NOT PERIODIC

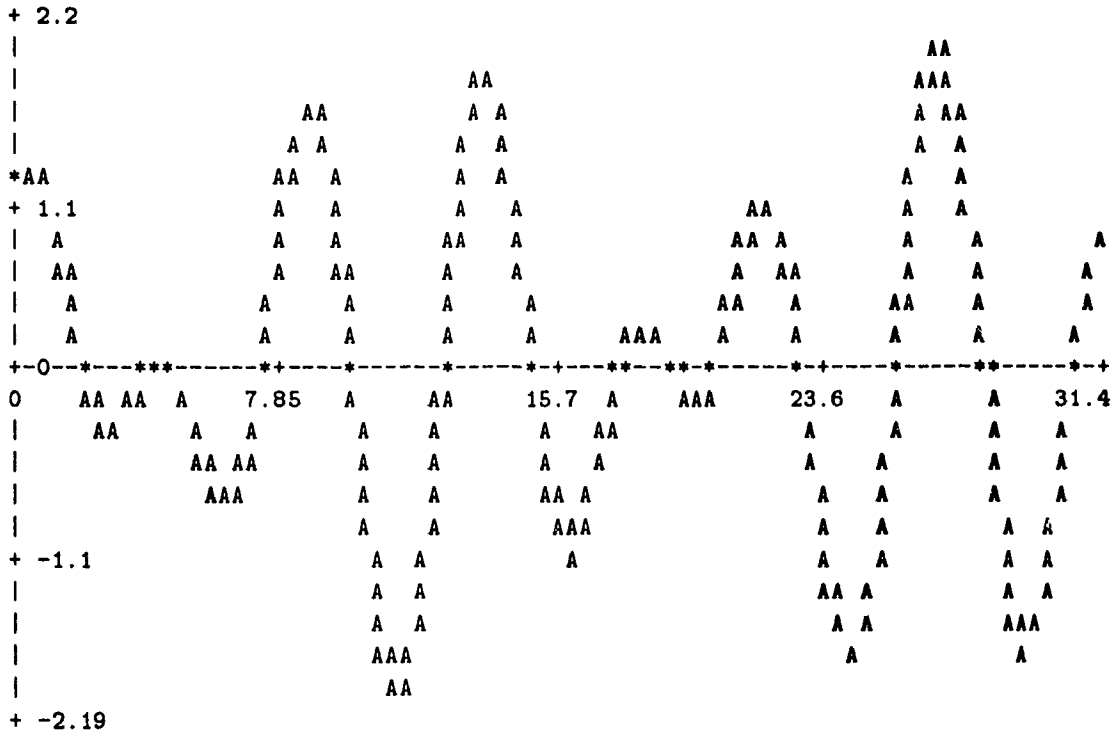


Fig. 6

PLOT OF $f(t) = \cos(\sqrt{3}t) + \sin(t)$

ALMOST PERIODIC BUT NOT PERIODIC

```
+ 2.14
|                 A                AA
|                 AA               AAA
|                A A               A A
|                A  A             A  A               AA        AA
+A1.09           A  A             A  A              AAA        A A
* A              A  A             A   A             A AA       A A
| A     A       AA  A             A   A            A   A      A  AA
|  A   AAA      A    A           A    A            A   A      A   A
|  A  AA A      A    A           A    A      AAA   A   A      A   A   AAA
|   A A   A     A    A           A    A     AA AA A    AA     A    A  A AA
+-0-***---*-----*--+-**---***----*-----+----*---*-*-----*-+--*-----*-*---*----+
0         A    A 7.85 A  AA A   A    15.7   A    A      23.6 A      AA   A 31.4
|          A   A      A  A  A   A      A    A           A    A           AA
|          A   A      A A    A  A      A   A            A    A            A   A
|          A   A       AA    AAAA      A   A             A   A            A   A
+ -1.09    A   A              AA        A  A             A  A             A  A
|           A A                         A AA             A  A              A A
|           A A                         A A              A  A              A A
|           A A                         AA               A  A              AAA
|            AA                                           AA
|                                                         AA
+ -2.2
```

Fig. 7

# REFERENCES


1. Bochner, S., A new approach to almost periodicity, Proc. Nat. Acad. Sci., U.S.A. 48 (1962), pp.2039-2043.

2. Fink, A. M., Almost periodic differential equations, Lecture Notes in Math, Springer-Verlag, 1974, Berlin, Heidelberg, New York.

3. Bohr, H. and Neugebauer, O., Uber lineare differential-glechungen mit konstanten , Nachr. Ges. Wiss. Gottingen, Math-Phys. Klasse, 1926.

4. Doss, R., On bounded functions with almost periodic differences, Proc. Am. Math. Soc., 12 (1961), pp.488-489.

5. Berman, A., Neumann, M., and Stern, R. J., Non-negative Matrices in Dynamic Systems, John Wiley & Sons, N.Y., 1989.

6. Jiang, D. P., Almost periodic solutions of some first order differential equations, Annals of Differential Equations, 2(1), 1986, pp.11-27.

7. Coppel, W. A., Almost periodic properties of partial differential equations, Ann. Mat. Pura Appl. 76(1967), pp.27-50.

8. Lin, Z. S., Theory of linear systems, Annals of Differential Equations, 6(2), 1990, pp.153-175.

9. Vornicescu, N., Almost periodic solutions for a system of differential equations, Seminar on Mathematical Analysis, Preprint No. 7, "Babes-Bolyai" University, Romania, 1988, pp.45-50.

10. Seifert, G., Almost periodic solutions for a certain class of almost periodic systems, Proc. Am. Math. Soc., Vol. 84, No. 1, 1982, pp. 47-51.

11. Haraux, A., A simple almost periodicity criterion and applications, Journal of Differential Equations, Vol. 66, 1987, pp.51-61.

12. Corduneanu, C., Some almost periodic criteria for ordinary differential equations, Libertas Mathematica, Vol. 3, 1983, pp.21-43.

13. Corduneanu, C., Two qualitative inequalities, Journal of Differential Equations, Vol. 64, No. 1, 1986, pp. 16-25.

14. Rao, A. S., On the Stepanov almost periodic solution of an abstract differential equation, Indiana University Mathematics Journal, Vol. 23, No. 3, 1973, pp.205-208.

15. Zaidman, S., Note presentate da soci, Lincei - Rend. Sc. Fis. Mat. e Nat., Vol.XXXI, 1961, pp.362-368.

16. Zaidman, S., A partial differential equation with almost periodic right hand side, Ann. Sc. Math. Que, Vol. III, No. 2, 1979, pp.273-285.

17. Schwartz, L., Les equations d'evolution liees au produit de composition, Ann. Inst. Fourier, 1950, pp.19-49.

18. E. Hille and R. S. Phillips, Functional Analysis and Semi-groups, A.M.S., Colloquim Publications, 31, 1957.