AN APPLICATION OF FUNCTIONAL ANALYSIS TO MARKOV'S INEQUALITY FOR THE DERIVATIVE OF POLYNOMIALS

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ABSTRACT

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Let $P(x)$ be a polynomial of degree n and $|P(x)| \leq 1$ on $[0,1]$. The problem we discuss is, how large $|P'(\xi)|$ can be for a given real number ξ. In 1889, A.A. Markov considered this problem and established a result known as: Markov's Theorem: <u>If</u> $P(x)$ <u>is a polynomial of degree</u> n <u>such that</u> $|P(x)| \leq 1$ <u>on</u> $[0,1]$, <u>then</u>

$$|P'(x)| \leq 2n^2$$

<u>on</u> $[0,1]$.

Later, in 1912 S.N. Bernstein observed that the estimate in Markov's Theorem can be considerably improved, if we restrict ourselves to the open interval $(0,1)$. He proved: Bernstein's Theorem: <u>If</u> $P(x)$ <u>is a polynomial of degree</u> n <u>and</u> $|P(x)| \leq 1$ <u>on</u> $[0,1]$ <u>then</u>

$$|P'(x)| \leq \frac{n}{\sqrt{x(1-x)}}$$

<u>on</u> $(0,1)$.

The problem proposed by Markov was studied by E.V. Voronovskaja. In 1956 she established, by the use of the methods of Functional Analysis, a result we call Markov-Voronovakaja Theorem: <u>If</u> $P(x)$ <u>is a polynomial of degree</u> n <u>with</u> $|P(x)| \leq 1$ <u>then</u>

$$|P'(\xi)| \leq \begin{cases} |T_n'(\xi)| & \text{for } \xi \in E_T \\ |Z_n(\xi,\sigma_\xi)| & \text{for } \xi \in E_Z \end{cases}$$

where $T_n(x)$ denotes the Cebysev polynomial and $Z_n(x,\sigma_x)$ the Zolotarev polynomials: E_T and E_Z are sets where they are respectively extremal.

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INTRODUCTION

In the year 1889 A.A. Markov published a paper "On a question posed by D.J. Mendelyeff": There, he proves Markov Theorem: If $P(x)$ is a polynomial of degree n such that $|P(x)| \leq 1$ on $[0,1]$, then on $[0,1]$

$$|P'(x)| \leq 2n^2 .$$

Twenty-three years later, in 1912 S.N. Bernstein proved the following result known as Bernstein Theorem: If $P(x)$ is a polynomial of degree n and $|P(x)| \leq 1$ on $[0,1]$, then on $(0,1)$

$$|P'(x)| \leq \frac{n}{\sqrt{x(1-x)}} .$$

In 1956 E.V. Voronovskaja extended Markov's and Bernstein's results. She proved the theorem (we call) Markov-Voronovskaja Theorem: If $P(x)$ is a polynomial of degree n with $|P(x)| \leq 1$ on $[0,1]$ then

$$|P'(\xi)| \leq \begin{cases} |T_n'(\xi)| & \text{for } \xi \in E_T \\ |Z_n'(\xi,\sigma_\xi)| & \text{for } \xi \in E_Z \end{cases} ,$$

where $T_n(x)$ denotes the Čebyšev polynomial of degree n, and E_T is called the Čebyšev interval; $Z_n(x,\sigma_x)$ represents the Zolotorev polynomial and E_Z is called the Zolotorev interval.

In fact, A.A. Markov in his 1889 paper considered the following problem: For a fixed point $\xi \in R$ how large $|P'(\xi)|$ can be when $|P(x)| \leq 1$ on $[0,1]$. He discussed this problem in detail but did not complete the study. By use of the methods of Functional Analysis, Voronovskaja answered this problem.

The aim of this work is to give a complete and comprehensive presentation of Voronovskaja's solution to Markov's problem.

Voronovskaja has succeeded in creating a unified method called "The Functional Method", for studying and solving certain problem of Čebyšev type which could not be solved by classical methods. The work of Voronovskaja is not well known and not yet well explained anywhere. Only her monograph written in 1963 and translated from Russian by R.P. Boas in 1970 give her technique. In [2] R.P. Boas discussed the importance of this technique, for further discussion see also [6].

The first chapter of our work deals with A.A. Markov's paper. Since there exists no published English translation of this paper, we feel that it is of considerable interest to see what Markov has done. We take this occasion to thank Mrs. Tanya Khalil for translating Markov's paper from Russian into English, for the purpose of this thesis.

The second chapter is concerned with S.N. Bernstein's original results. It is not hard to prove Markov's theorem from Bernstein's theorem see [4, p.137], and there are several easy proofs of the Bernstein theorem.

This might be one of the reasons that not much attention was given to the original work of Markov. However, it is still interesting to see the original proof due to Bernstein, and compare it with the proof due to Markov. We observe how close these two proofs are.

The third and fourth chapters deal with the work of Voronovskaja. She considered the space of all polynomials of degree $\leq n$ on $[0,1]$, and thought of $P'(\xi)$ as the derivative functional F_ξ acting on $P(x)$ i.e. $F_\xi(P) = P'(\xi)$. The problem is then to find the norm $N(\xi)$ of the functional F_ξ and the extremal polynomial $Q(x)$ for F_ξ i.e. $N(\xi) = \|F_\xi\| = F_\xi(Q)$, $\max_{[0,1]} |Q(x)| = 1$. We present this study in two parts. The first part, which is Chapter III, deals with the case when Čebyšev polynomial is extremal. In this chapter we have also included the general theory for the existence of the extremal polynomial for arbitrary functional. By the use of Riesz Representation Theorem, we determine for what intervals E_T the extremal polynomials are the Čebyšev polynomials $\pm T_n(x)$. In fact there are n disjoint intervals E_T where $\pm T_n(x)$ are extremal and the norm $N(\xi), \xi \in E_T$, increases from one Čebyšev interval E_T to the next for all $E_T \subset [\frac{1}{2}, 1]$.

In the complement of the Čebyšev intervals, called the Zolotorev intervals the derivative functional F_ξ is served by the extremal polynomial called the Zolotorev polynomials. The study of the class of Zolotorev polynomials is quite involving. Most of Chapter IV deals with this topic. We have tried to keep to a minimum the study of this class of extremal polynomials and present only those results needed for our study.

CHAPTER I

MARKOV'S THEOREM

<u>1.1 Introduction</u> Let Π_n denote the family of all polynomials $P(x)$ of degree $\leq n$. Concerning the derivative of a polynomial, Markov posed the following two problems in his paper [3]:

<u>Problem 1</u>. For a given $\xi \in \mathbb{R}$ (real numbers), how large $|P'(\xi)|$ can be for $P(x) \in \Pi_n$ provided $|P(x)| \leq 1$ on $[0,1]$ i.e. for a given $\xi \in \mathbb{R}$ to find a number $N(\xi)$ such that $N(\xi) = \sup|P'(\xi)|$ where supremum is taken over all those $P(x) \in \Pi_n$ which satisfy the condition $|P(x)| \leq 1$ on $[0,1]$.

<u>Problem 2</u>. How large $|P'(x)|$ can be on $[0,1]$ if $P(x) \in \Pi_n$ and $|P(x)| \leq 1$ on $[0,1]$.

Before discussing the work of Markov, we need the following definitions. A polynomial $P_n(x) \in \Pi_n$ is said to be <u>extremal at</u> ξ or <u>an extremal polynomial at the given</u> $\xi \in \mathbb{R}$ for problem 1 if $|P_n(x)| \leq 1$ on $[0,1]$ and at ξ, $P_n'(\xi) = N(\xi)$. It clearly means that if $P_n(x)$ is extremal at ξ, then for every $Q(x) \in \Pi_n$ with $\max_{[0,1]}|Q(x)| = 1$, we have $|Q'(\xi)| \leq P_n'(\xi)$. If $P(x) \in \Pi_n$ and $\max_{[0,1]}|P(x)| = 1$, then a point $\alpha \in [0,1]$ is called a <u>node</u> of $P(x)$ if $|P(\alpha)| = 1$.

In his work, Markov considered the first problem and showed that for $\xi \in \mathbb{R}$, the extremal polynomial $P_n(x)$ of degree n has n or $n+1$ nodes. If the extremal polynomial $P_n(x)$ has $n+1$ nodes then $P_n(x) = \pm T_n(x)$, where $T_n(x) = \cos n \arccos(2x-1)$ is the Čebysev polynomial, and it is extremal at all $\xi \in \mathbb{R} - [0,1]$. Furthermore, $\pm T_n(x)$ is extremal at $\xi \in [0,1]$ if and only if

$$\frac{T_n''(\xi)}{T_n'(\xi)} + \frac{1}{\xi} > 0 \quad \text{and} \quad \frac{T_n''(\xi)}{T_n'(\xi)} + \frac{1}{\xi+1} < 0.$$

If for the point $\xi \in [0,1]$, the extremal polynomial $P_n(x)$ has n nodes, then $P_n(x)$ satisfies the following properties: $P_n(x)$ belongs to the family of polynomials

$$P^*(x) = \pm \cos n \arccos \frac{2x - \alpha_{n+1}}{\alpha_{n+1}}$$

depending on the parameter α_{n+1} or $P_n(x)$ belongs to the family of polynomials

$$P^{**}(x) = \pm \cos n \arccos \frac{2x - (\alpha_0 + 1)}{1 - \alpha_0}$$

depending on the parameter α_0. If $P_n(x) \notin P^*$ and $P_n \notin P^{**}(x)$ then the extremal polynomial $P_n(x)$ must satisfy the following differential equation

$$P_n^2(x) - 1 = \frac{x(x-1)(x-\gamma)(x-\delta)}{n^2(x-\beta)^2} [P'(x)]^2$$

where $|\gamma|, |\delta| > |\beta|$ and $\beta \notin [0,1]$.

From these observations, Markov answered the second problem and proved: <u>For all</u> $P(x) \in \Pi_n$, $|P'(x)| \le 2n^2$ <u>on</u> $[0,1]$ <u>provided</u> $|P(x)| \le 1$ <u>on</u> $[0,1]$.

The proof depends on a number of results; we present them here:

<u>1.2 Theorem</u> <u>Suppose</u> $P_n(x) \in \Pi_n$ <u>is an extremal polynomial at</u> x_0 i.e. <u>for every</u> $Q(x) \in \Pi_n$ <u>with</u> $\max_{[0,1]} |Q(x)| = 1$, <u>at</u> x_0 <u>we have</u>, $|Q'(x_0)| \le P_n'(x_0)$. <u>Suppose also that</u> $0 \le \alpha_1 < \alpha_2 \dots < \alpha_s \le 1$ <u>are the nodes of</u> $P_n(x)$. Then the sequence

$$\frac{P_n(\alpha_2)}{P_n(\alpha_1)}, \frac{P_n(\alpha_3)}{P_n(\alpha_2)}, \dots, \frac{P_n(\alpha_s)}{P_n(\alpha_{s-1})}$$

<u>has at least</u> $n-1$ <u>numbers equal to</u> -1. <u>Hence if</u> $P_n(x)$ <u>is extremal then</u> $s > n-1$ <u>and</u> $\operatorname{sgn} P_n(\alpha_i) = -\operatorname{sgn} P_n(\alpha_{i+1})$, $i = 1, 2, \dots, s-1$.

<u>Proof</u>: Suppose that the extremal polynomial $P_n(x)$ has m nodes

arranged as $0 \leq t_1 < t_2 < \ldots < t_m \leq 1$, (1.1)

and that the sequence $\frac{P_n(t_i)}{P_n(t_{i+1})}$, $i = 1,2,\ldots,m-1$ has $s \leq n-1$ members equal to -1. There is no less of generality in assuming that $P_n(t_1) = 1$. We decompose (1.1) into parts having the following properties:

$$\begin{array}{lll} t_1, \ldots, t_{m_1} & \text{where } P_n(t_i) = +1, & i = 1,2,\ldots,m_1 , \\ t_{m_1+1}, \ldots, t_{m_2} & \text{where } P_n(t_i) = -1, & i = m_1+1,\ldots,m_2, \\ \ldots & \ldots \quad \ldots & \ldots \\ t_{m_s+1}, \ldots, t_m & \text{where } P_n(t_i) = (-1)^{s-1}, & i = m_s+1,\ldots,m . \end{array}$$

Between each segment (t_{m_i}, t_{m_i+1}), $i = 1,2,\ldots,s-1$, we chose a point y_i such that $t_{m_i} < y_i < t_{m_i+1}$.

It is easy to find a polynomial $\phi(x)$ of degree $s-1$ having a simple zero at y_i , $i = 1,2,\ldots,s$ and having $\operatorname{sgn}\phi(t_{m_i}) = -\operatorname{sgn} P_n(t_{m_i})$, $i = 1,2,\ldots,s$. We let

$$0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_s \leq 1$$

be those nodes at which

$$\frac{P_n(\alpha_i)}{P_n(\alpha_{i+1})} = -1 .$$

We define a polynomial

$$\lambda(x) = P_n(x) + \varepsilon(x - x_0)^2 \phi(x) \tag{1.2}$$

where $\varepsilon > 0$ but sufficiently small and $x_0 \in \mathbb{R}$ **at which $P_n(x)$ is extremal**. Recall that $\phi(x)$ is of degree $s-1 \leq n-2$, hence $\lambda(x)$ is a polynomial of degree n we claim that

$$|\lambda(x)| < 1 .$$

Let $I(\alpha_i)$ be the interval around each α_i such that $\phi(x)$ does not change its sign on each $I(\alpha_i)$. On $[0,1] \setminus \bigcup_{i=1}^{s} I(\alpha_i)$, $\max |P_n(x)| = L < 1$. With the choice of a positive ε such that

$\varepsilon(x - x_0)^2|\phi(x)| < 1 - L$ we have on $[0,1] \setminus \bigcup_{i=1}^{s} I(\alpha_i)$, $|\lambda(x)| \le |P_n(x)| + \varepsilon(x - x_0)^2|\phi(x)| < 1$. Moreover on $\bigcup_{i=1}^{s} I(\alpha_i)$, $|\lambda(x)| = |P_n(x) + \varepsilon(x - x_0)^2\phi(x)| < |P_n(x)| \le 1$. Hence $|\lambda(x)| < 1$ on $[0,1]$.

Furthermore, for $x = x_0$ the derivative of (1.2) satisfies

$$\frac{d}{dx}\lambda(x_0) = \frac{d}{dx}P_n(x_0) .$$

We now define a new polynomial $\hat{Q}(x)$ by multiplying $\lambda(x)$ by the number $\dfrac{1}{\max\limits_{[0,1]}|\lambda(x)|}$ which is bigger than 1. That is

$$\hat{Q}(x) = \frac{\lambda(x)}{\max\limits_{[0,1]}|\lambda(x)|} .$$

Clearly $|\hat{Q}(x)| \le 1$ and

$$|\hat{Q}'(x_0)| = \left|\frac{\lambda'(x_0)}{\max\limits_{[0,1]}|\lambda(x)|}\right| > |\lambda'(x_0)| = |P_n'(x_0)| ,$$

which contradicts that $P_n(x)$ is extremal. Therefore $s > n-1$. □

A polynomial $P(x)$ of degree n cannot have more than $n+1$ nodes in $[0,1]$, since otherwise its derivative would vanish at more than $n-1$ interior nodes of $[0,1]$. Consequently, an extremal polynomial $P(x)$ has n or $n+1$ alternating nodes. If the nodes are $n+1$, then 0 and 1 are among the nodes where $P'(x) \ne 0$.

<u>1.3 Theorem</u> <u>If a polynomial $P_n(x)$ of degree n has $n+1$ nodes in $[0,1]$ and $|P_n(x)| \le 1$ on $[0,1]$, then $P_n(x) \equiv \pm T_n(x)$ where $T_n(x)$ is a Čebyšev polynomial of degree n.</u> If $P_n(1) = +1$, $P_n(x) \equiv T_n(x)$; <u>and if</u> $P_n(1) = -1$, $P_n(x) \equiv -T_n(x)$.

<u>Proof</u>: Let us consider the case when $P_n(1) = 1$. Let the $n+1$ nodes of $T_n(x)$ be $(\tau_i)_{i=0}^{n}$;

$$0 = \tau_0 < \tau_1 < \dots < \tau_k < \dots < \tau_n = 1 .$$

Since $(\tau_i)_{i=0}^n$ are alternating nodes of $T_n(x)$, one has

$$T_n(\tau_n) - P_n(\tau_n) = 0$$

$$T_n(\tau_{n-1}) - P_n(\tau_{n-1}) \leq 0$$

$$T_n(\tau_{n-2}) - P_n(\tau_{n-2}) \geq 0$$

$$\dots$$

$$T_n(\tau_0) - P_n(\tau_0) = 0$$

Hence each of the n intervals $[\tau_i,\tau_{i+1}]$, $i = 0,1,\dots,n-1$, contains at least one zero of $R(x) = T_n(x) - P_n(x)$. In fact to every n intervals $[\tau_i,\tau_{i+1}]$, $i = 0,1,\dots,n-1$, there corresponds at least one zero counted once of $R(x)$. In the case τ_i other than $\tau_0 = 0$, $\tau_n = 1$ is a zero of $R(x)$ then τ_i is a node of $P_n(x)$ as well, so τ_i is a double zero of $R(x) = T_n(x) - P_n(x)$ and each of these two zeros can be assigned to each of the intervals $[\tau_{i-1},\tau_i]$ and $[\tau_i,\tau_{i+1}]$. Thus $R(x)$ has at least n zeros.

Let the $n + 1$ nodes of $P_n(x)$ be $(\sigma_i)_{i=0}^n$;

$$0 = \sigma_0 < \sigma_1 < \dots < \sigma_k < \dots < \sigma_n = 1 .$$

If $\tau_{n-1} \leq \sigma_k < \sigma_{k+1} < \dots < \sigma_{n-1} < \sigma_n = \tau_n = 1$ for $k \leq n - 1$ then $R(x)$ has at least two zeros in $[\tau_{n-1},1]$, as observed in Figure 1.

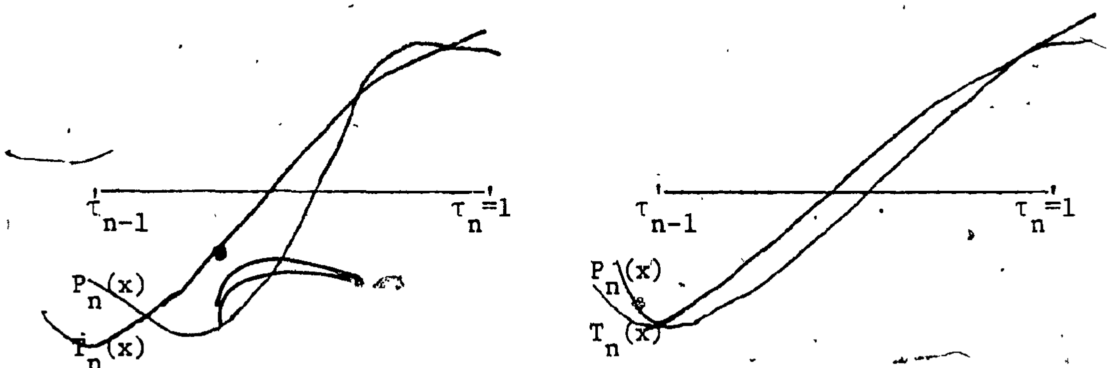

FIGURE 1

So $R(x)$ has at least $n+1$ zeros in $[0,1]$. Consequently, $R(x) \equiv 0$, i.e. $P_n(x) \equiv T_n(x)$.

If $\sigma_{n-1} < \tau_{n-1}$, we interchange the role of $P_n(x)$ and $T_n(x)$ and conclude that $P_n(x) \equiv T_n(x)$[1]. □

If the extremal polynomial $P_n(x)$ has $n+1$ nodes then

$$P_n(x) \equiv \pm T_n(x) = \pm \cos n \arccos (2x - 1) .$$

We will investigate the conditions under which for a given ξ, $\max_{P_n \in \Pi_n} |P_n'(\xi)| = |T_n'(\xi)|$ with $\max_{[0,1]} |P_n(x)| = 1$. We will consider only those polynomials $P_n(x) \in \Pi_n$ such that

$$\operatorname{sgn} P_n'(\xi) = \operatorname{sgn} T_n'(\xi) . \tag{1.3}$$

We put

$$\tau_{n-i} = \frac{1}{2} + \frac{1}{2} \cos \frac{i\Pi}{n} \qquad (i = 0,1,\ldots,n) \tag{1.4}$$

and

$$Q(x) = P_n(x) - T_n(x) . \tag{1.5}$$

Note that the τ_{n-i} in (1.4) are the nodes of $T_n(x)$. We will show that outside $[0,1]$ the extremal polynomials are $\pm T_n(x)$. We need the following Lemma:

<u>1.4 Lemma</u> <u>The zeros of</u> $Q(x)$ <u>are all real, and lying in the interval</u> $[0,1]$.

<u>Proof</u>: Consider the values of $P_n(x)$ and $T_n(x)$ at the points $\tau_0, \tau_1, \ldots, \tau_n$ given in (1.4). For $i = 0$ we find that $\tau_n = \frac{1}{2} + \frac{1}{2} \cos 0 = 1$. Thus $T_n(1) = \cos n \arccos 1 = 1$. Since by definition $|P_n(x)| \leq 1$ we obtain from (1.5) that when $T_n(\tau_n) = +1$, then

1 For another proof of this theorem see page 32 .

$$Q(\tau_n) = Q(1) \le 0 . \qquad (1.6)$$

In the same fashion we get

$$T_n(\tau_{n-1}) = -1 \quad \text{which implies} \quad Q(\tau_{n-1}) \ge 0$$
$$T_n(\tau_{n-2}) = +1 \quad \text{which implies} \quad Q(\tau_{n-2}) \le 0$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$T_n(\tau_1) = (-1)^n \quad \text{which implies} \ (-1)^n Q(\xi_0) \le 0 .$$

From the equation $Q(x) = 0$ it follows that $Q(x)$ must have one root in each interval $[\tau_i, \tau_{i+1}]$ $(i = 0,1,\ldots,n-1)$. In other words the polynomial $Q(x)$ must have all its roots real. Therefore $Q(x)$ can be written as

$$Q(x) = q(x - \eta_1)(x - \eta_2) \ldots (x - \eta_n) \qquad (1.7)$$

where η_i $(i = 1,2,\ldots,n)$ are the roots and

$$0 = \tau_0 \le \eta_1 \le \tau_1 \le \eta_2 \le \ldots \le \tau_{n-1} \le \eta_n \le \tau_n = 1 . \qquad (1.8)$$

Furthermore the coefficient q must be negative, because when $x = 1$, we obtain from (1.6) $Q(1) \le 0$ and from (1.7) all the factors are non-negative. □

From (1.5) $P_n(x) = T_n(x) + Q(x)$. Therefore $P_n'(x) = T_n'(x) + Q'(x)$. By differentiating (1.7) we obtain

$$P_n'(x) = T_n'(x) + \left(\frac{1}{x - \eta_1} + \frac{1}{x - \eta_2} + \ldots \frac{1}{x - \eta_n}\right)Q(x) , \qquad (1.9)$$

where η_i are the roots of $Q(x)$.

The Čebyšev polynomial $T_n(x)$ can be written as

$$T_n(x) = 2^{2n-1}x^n - n2^{2n-3}x^{n-1} + \ldots + \frac{(2n-k-1)\ldots(2n-2k+1)}{k!}2^{2n-2k}x^{n-k} + \ldots + (-1)^{n-1}2nx + (-1)^n .$$

Since $T_n'(x) = 0$ for the points $x = \tau_1, \tau_2, \ldots, \tau_{n-1}$, and the leading coefficient of $T_n(x) = 2^{2n-1}$. Therefore

$$T_n'(x) = 2^{2n-1} n(x - \tau_1)(x - \tau_2) \ldots (x - \tau_{n-1}) \qquad (1.10)$$

<u>1.5 Theorem</u> <u>Outside</u> $[0,1]$ <u>the extremal polynomial is</u> $T_n(x)$, <u>i.e. for</u> $P \in \Pi_n$ <u>with</u> $\max_{[0,1]} |P(x)| = 1$, $|P'(x)| \leq |T_n'(x)|$ <u>for all</u> $x \in \mathbb{R} \setminus [0,1]$.

<u>Proof</u>: We form the following sequence

$$\frac{Q(x)}{x - \eta_1}, \frac{Q(x)}{x - \eta_2}, \ldots, \frac{Q(x)}{x - \eta_n} \qquad (1.11)$$

where $Q(x)$ is defined in (1.7) having $(\eta_i)_{i=1}^n$ as its roots. Each of the expressions in (1.11) has a sign opposite to that of $T_n'(x)$. To show this we consider two cases:

<u>Case 1</u>: Suppose $x > 1$. From the fact that in (1.11) $q \leq 0$, we obtain $Q(x) < 0$. Moreover since $x - \eta_i > 0$, then $\frac{Q(x)}{x - \eta_i} < 0$ for $i = 1,2,\ldots,n$. On the other hand, from (1.10) we obtain that $T_n'(x) > 0$, since the factors $(x - \tau_i)(i = 1,2,\ldots,n-1)$ are all positive.

<u>Case 2</u>: If $x < 0$ then from (1.7) we get $Q(x) < 0$ (recall $q \leq 0$) if n is even, and $Q(x) > 0$ if n is odd. If n is even then $\frac{Q(x)}{x - \eta_i} > 0$ $(i = 1,2,\ldots,n)$. However from (1.10) when n is even then the product $(x - \tau_1)(x - \tau_2) \ldots (x - \tau_{n-1})$ of $n - 1$ negative numbers is negative. Thus to the left of $[0,1]$ when $\frac{Q(x)}{x - \eta_i}$ is positive then $T_n'(x)$ is negative. In the same way it can be shown that when n is odd then $\frac{Q(x)}{x - \eta_i}$ is negative and $T_n'(x)$ is positive.

We show that outside $[0,1]$, $|P'(x)| \leq |T_n'(x)|$. If $P'(x) > 0$ then from (1.3) $T_n'(x) > 0$, and $\sum_{i=1}^n \frac{Q(x)}{x - \eta_i} < 0$, hence from (1.9) $|P'(x)| \leq |T_n'(x)|$. If $P'(x) < 0$, then $T_n'(x) < 0$ and $\sum_{i=1}^n \frac{Q(x)}{x - \eta_1} > 0$. Hence $P'(x) \geq T_n'(x)$ i.e. $|P'(x)| \leq |T_n'(x)|$. □

We will now discuss a sufficient condition for $T_n(x)$ to be extremal at a point $\xi \in [0,1]$. We first prove the following:

<u>1.6 Lemma</u> <u>Suppose</u> $x \in (\tau_{i-1}, \tau_i)$ <u>where</u> τ_i <u>is as in (1.4)</u>. <u>Then</u> $\operatorname{sgn} \frac{Q(x)}{x - \eta_i}$ <u>is opposite to</u> $\operatorname{sgn} T_n'(x)$.

<u>Proof</u>: Suppose that for $x \in (\tau_{i-1}, \tau_i)$, $x \neq \eta_i$ and $x < \eta_i$, then from (1.7) we get

$$\frac{Q(x)}{x - \eta_i} = q(x-\eta_1)(x-\eta_2)\dots(x-\eta_{i-1})(x-\eta_{i+1})\dots(x-\eta_n) .$$

If $Q(x)$ is positive then $\frac{Q(x)}{x - \eta_i}$ is negative, and since q is negative then in (1.7) the product $(x - \eta_1)\dots(x - \eta_n)$ is also negative. Moreover from (1.8) we have $\tau_i \geq \eta_i$ $(i = 1,2,\dots,n)$ and hence the product $(x - \tau_1)\dots(x - \tau_n)$ is also negative. Therefore the product $(x - \tau_1)\dots(x - \tau_{n-1})$ is positive. Consequently by (1.10) $T_n'(x)$ is positive. Suppose now that $Q(x)$ is negative then $\frac{Q(x)}{x - \eta_i}$ is positive and also the product $(x - \eta_1)\dots(x - \eta_n)$ is positive. Hence $(x - \tau_1)\dots(x - \tau_{n-1})$ is negative and by (1.10) $T_n'(x)$ is negative. For the other cases; when $\eta_i > x$, the arguments are the same. Therefore $\operatorname{sgn} \frac{Q(x)}{x - \eta_i}$ is opposite to $\operatorname{sgn} T_n'(x)$ for $x \in (\tau_{i-1}, \tau_i)$. □

<u>1.7 Theorem</u> <u>For a fixed</u> $\xi \in [0,1]$ <u>the extremal polynomial is</u> $T_n(x)$ <u>if and only if</u>

$$\frac{T_n''(\xi)}{T_n'(\xi)} + \frac{1}{\xi} > 0 \tag{1.12}$$

and

$$\frac{T_n''(\xi)}{T_n'(\xi)} + \frac{1}{\xi - 1} < 0 \tag{1.13}$$

<u>Proof</u>: For each $x \in [0,1]$ we can find an interval $[\tau_{i-1}, \tau_i]$ such that $\tau_{i-1} \leq x \leq \tau_i$. Recall from (1.8) that we have $\tau_{i-1} \leq \eta_i \leq \tau_i$.

Let

$$\sum = \frac{x - \eta_i}{x - \eta_1} + \frac{x - \eta_i}{x - \eta_2} + \ldots + \frac{x - \eta_i}{x - \eta_n} \qquad (1.14)$$

using (1.14) we write (1.9) as

$$P_n'(x) = T_n'(x) + \frac{Q(x)}{x - \eta_i}\sum \quad .$$

Therefore

$$\frac{P_n'(x)}{T_n'(x)} = 1 + \frac{Q(x)}{T_n'(x)(x - \eta_i)}\sum \qquad (1.15)$$

as in (1.3) we may take

$$\frac{P_n'(x)}{T_n'(x)} > 0 \quad .$$

We note that the value of $\sum$ is greater than

$$(x - \eta_i)\{\frac{1}{x - \tau_0} + \frac{1}{x - \tau_1} + \ldots + \frac{1}{x - \tau_i} + \ldots + \frac{1}{x - \tau_{n-1}}\} \text{ if } x - \eta_i > 0 \qquad (1.16)$$

or

$$(x - \eta_i)\{\frac{1}{x - \tau_1} + \frac{1}{x - \tau_2} + \ldots + \frac{1}{x - \tau_i} + \ldots + \frac{1}{x - \tau_n}\} \text{ if } x - \eta_i < 0 \qquad (1.17)$$

(1.16) follows from the inequalities $0 < x - \eta_j \leq x - \tau_{j-1}$ for $j = 1,2,\ldots,i-1$ and $0 > x - \tau_{j-1} \geq x - \eta_j$ for $j = i+1,\ldots,n$; and (1.17) follows from the inequalities $0 < x - \tau_j < x - \eta_j$ for $j = 1,2,\ldots,i-1$ and $0 > x - \eta_j \geq x - \tau_j$ for $j = 1+i,\ldots,n$ and $(x - \eta_i) < 0$.

From (1.10) we have $T_n'(x) = 2^{2n-1} n(x - \tau_1) \ldots (x - \tau_{n-1})$ and so,

$$\frac{T_n''(x)}{T_n'(x)} = \frac{1}{x - \tau_1} + \frac{1}{x - \tau_2} + \ldots + \frac{1}{x - \tau_{n-1}} \quad .$$

Since $\tau_0 = 0$ and $\tau_n = 1$ we get

$$\frac{T_n''(x)}{T_n'(x)} + \frac{1}{x} = \frac{1}{x - \tau_0} + \frac{1}{x - \tau_1} + \ldots + \frac{1}{x - \tau_{n-1}} \quad ,$$

and

$$\frac{T_n''(x)}{T_n'(x)} + \frac{1}{x - 1} = \frac{1}{x - \tau_1} + \ldots + \frac{1}{x - \tau_{n-1}} + \frac{1}{x - \tau_n} .$$

Thus we can write (1.16) as

$$(x - \eta_i)\left[\frac{T_n''(x)}{T_n'(x)} + \frac{1}{x}\right] \tag{1.16'}$$

and (1.17) as

$$(x - \eta_i)\left[\frac{T_n''(x)}{T_n'(x)} + \frac{1}{x - 1}\right] . \tag{1.17'}$$

If the smallest value of $\sum > 0$, then clearly all values of $\sum$ are positive. In this case

$$\frac{Q(x)}{T_n'(x)(x - \eta_i)} \sum < 0 .$$

From lemma 1.6 and (1.15) we get

$$0 < \frac{P_n'(x)}{T_n'(x)} < 1$$

Hence $|P'(x)| < |T'(x)|$.

On the other hand, if the smallest value is negative, then from lemma 1.6

$$\frac{Q(x)}{T_n'(x)(x - \eta_i)} \sum > 0 ,$$

and $|P'(x)| > |T_n'(x)|$. Consequently $T_n(x)$ is extremal for $x \in [0,1]$ if and only if $\sum > 0$ i.e. (1.12) and (1.13) hold. To establish only if, we note that when η_i is τ_{i-1} (or τ_i), the value (1.16') or (1.17') is taken by $\sum$. □

We consider the case when $P(x)$ other than $\pm T_n(x)$ is an extremal polynomial for ξ $[0,1]$.

From our preceding discussion we know that $P(x)$ has n alternating nodes. Let these nodes be $0 \le \alpha_1 < \alpha_2 < \ldots < \alpha_n \le 1$. Since not more than $n-1$ nodes are in the interior of $[0,1]$, then $\alpha_1 = 0$ or $\alpha_n = 1$ or both $\alpha_1 = 0$ and $\alpha_n = 1$. Let $\phi(x)$ be another polynomial of degree n $|\phi(x)| \le 1$ on $[0,1]$. Since $|P(x)| \le 1$ for $0 \le x \le 1$, then the polynomial $\psi(x) = \phi(x) - P(x)$ must have one zero η_i in each of the intervals $[\alpha_i, \alpha_{i+1}]$ $i=1,2,\ldots,n-1$. Thus

$$\psi(x) = (qx - r)(x - \eta_1) \ldots (x - \eta_{n-1}) , \qquad (1.18)$$

where $\alpha_1 \le \eta_1 \le \alpha_2 \le \eta_2 \le \ldots \le \eta_{n-1} \le \alpha_n$ and $\frac{r}{q} \geq \alpha_n$ or $\frac{r}{q} \le \alpha_1$. Let $\frac{r}{q} = \eta_n$ thus

$$\phi'(x) = P'(x) + \{\frac{1}{x - \eta_1} + \ldots + \frac{1}{x - \eta_n}\} \psi(x) .$$

<u>1.8 Lemma</u> <u>In (1.18)</u> $\operatorname{sgn}(qx - r)$ <u>is opposite to</u> $\operatorname{sgn} P(\alpha_n)$ <u>for all</u> values $x \in [\alpha_1, \alpha_n]$.

<u>Proof</u>: Suppose that the product in (1.18), $(x - \eta_1) \ldots (x - \eta_{n-1})$ is positive. If $\alpha_n \le \frac{r}{q}$ we obtain $(qx - r) < 0$. Hence $\psi(\alpha_n) = \phi(\alpha_n) - P(\alpha_n) < 0$ and that implies $P(\alpha_n) = +1$. If $\alpha_1 \ge \frac{r}{q}$ then $(qx - r) > 0$ consequently $\psi(\alpha_n) = \phi(\alpha_n) - P(\alpha_n) > 0$ which implies $P(\alpha_n) = -1$. The other case, when the product $(x - \eta_1) \ldots (x - \eta_{n-1})$ is negative, is similarly proven. □

As long as $\operatorname{sgn}(qx - r) = -\operatorname{sgn} P(\alpha_n)$, and $\alpha_1 \le \eta_1 \le \alpha_2 \le \eta_2 \le \ldots \le \alpha_n \le \eta_n$, the numbers $\eta_1, \eta_2, \ldots, \eta_n$ can be given arbitrarily values.

<u>1.9 Theorem</u> <u>Suppose that at a point</u> $x \in [0,1]$ <u>the extremal polynomial</u> $P(x)$ <u>is different from</u> $T_n(x)$ <u>then</u> $x \in [\alpha_1, \alpha_n]$.

<u>Proof</u>: Suppose $\eta_n > x > \alpha_n$. Then the following inequalities obviously hold

$$0 < \frac{1}{x-\eta_1} + \frac{1}{x-\eta_2} + \ldots + \frac{1}{x-\eta_{n-1}} < \frac{1}{x-\alpha_2} + \frac{1}{x-\alpha_3} + \ldots + \frac{1}{x-\alpha_n}$$

and $\frac{1}{x-\eta_n} < 0$.

From (1.8) the numbers $\eta_1, \eta_2, \ldots, \eta_n$ can be taken arbitrarily in such a way that the expression

$$\{\frac{1}{x-\eta_1} + \frac{1}{x-\eta_2} + \ldots + \frac{1}{x-\eta_n}\}\psi(x)$$

can be made positive or negative contradicting that $P(x)$ is extremal. Therefore the case when $x > \alpha_n$ cannot occur. Similarly we can show that $x \nless \alpha_1$. □

1.10 Theorem For each point $x \in [\alpha_1, \alpha_n]$ if the extremal polynomial $P(x) \neq T_n(x)$ then

$$\frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \ldots + \frac{1}{x-\alpha_n} = 0 \qquad (1.19)$$

where α_i, $i = 1, 2, \ldots, n$ are the nodes of $P(x)$ in $[0,1]$.

Proof: Suppose $x \in [\alpha_i, \alpha_{i+1}]$. From the expression

$$\frac{\psi(x)}{x-\eta_i} = (x-\eta_1)\ldots(x-\eta_{i-1})(x-\eta_{i+1})\ldots(x-\eta_{n-1})(qx-r)$$

we get that $\operatorname{sgn}\frac{\psi(x)}{x-\eta_i}$ is opposite to $\operatorname{sgn}(-1)^{n-i-1}P(\alpha_n)$ since $\operatorname{sgn}(qx-r) = -\operatorname{sgn}P(\alpha_n)$ and $x-\eta_j < 0$ $j = i+1, \ldots, n-1$.

In order that $P(x)$ be extremal i.e.

$$|Q'(x)| = |P'(x) + \{\frac{x-\eta_i}{x-\eta_1} + \frac{x-\eta_i}{x-\eta_2} + \ldots + \frac{x-\eta_i}{x-\eta_n}\}\frac{\psi(x)}{x-\eta_i}| \leq |P'(x)| \qquad (1.20)$$

we must have

$$\operatorname{sgn}\{\frac{x-\eta_i}{x-\eta_1} + \ldots + \frac{x-\eta_i}{x-\eta_n}\} = -\operatorname{sgn}\frac{\psi(x)}{x-\eta_i} \cdot P'(x)$$

$$= \operatorname{sgn}(-1)^{n-i-1} P(\alpha_n)P'(x) .$$

But the expression $(-1)^{n-i-1} P(\alpha_n)P'(x) > 0$, because the sign of $(-1)^{n-i-1}P(\alpha_n)$ is identical to the sign of $P(\alpha_{i+1})$ and of $P'(x)$. Recall that $x \in (\alpha_i, \alpha_{i+1})$ and that the nodes α_i are alternating, so

$P'(x) > 0$ if $P(\alpha_{i+1}) > 0$ and $P'(x) < 0$ if $P(\alpha_{i+1}) < 0$. In order for (1.20) to hold we need to find the samllest value of the sum

$$(x - \eta_i)\{\frac{1}{x - \eta_1} + \frac{1}{x - \eta_2} + \ldots + \frac{1}{x - \eta_n}\} . \tag{1.21}$$

It is not hard to see that the smallest value of (1.21) is the smallest of the numbers;

$$(x - \alpha_i)\{\frac{1}{x - \alpha_1} + \frac{1}{x - \alpha_2} + \ldots + \frac{1}{x - \alpha_n}\} ,$$

$$(x - \alpha_{i+1})\{\frac{1}{x - \alpha_1} + \frac{1}{x - \alpha_2} + \ldots + \frac{1}{x - \alpha_n}\} .$$

One of these two must be negative, whereas we have already shown that no matter what η_i is, $\text{sgn}\,(-1)^{n-i-1} P(\alpha_n)P'(x) > 0$. Consequently (1.19) equals to zero. □

In case the extremal polynomial $p(x)$ has n nodes then either $\alpha_1 = 0$ or $\alpha_n = 1$ or both. We now examine the three cases that might occur:

<u>1.11 Theorem</u> <u>If at a fixed point</u> x <u>the extremal polynomial</u> $P(x)$ <u>is different from</u> $T_n(x)$ <u>then the extremal polynomial</u> $P(x)$ <u>satisfies one of the following forms</u>;

(I) <u>If</u> $\alpha_1 = 0$ <u>and</u> $\alpha_n = 1$ <u>then the extremal polynomial</u> $P(x)$ <u>satisfies the following differential equation</u>

$$P^2(x) - 1^2 = \frac{(x)(x - 1)(x - \gamma)(x - \delta)}{n^2(x - \beta)^2} P'^2(x) \tag{1.22}$$

where $|\gamma| > |\beta|$, $|\delta| > |\beta|$.

(II) <u>If</u> $\alpha_1 = 0$ <u>and</u> $\alpha_n < 1$ <u>then the family of extremal polynomial is</u>

$$P^*(x,\alpha_{n+1}) = \pm \cos n \arccos \frac{2x - \alpha_{n+1}}{\alpha_{n+1}}$$

<u>where the parameter</u> α_{n+1} <u>varies from</u> 1 <u>to</u> $\frac{1}{\cos^2 \frac{\Pi}{2n}}$.

III) <u>If</u> $\alpha_1 > 0$ <u>and</u> $\alpha_n = 1$ <u>then the family of extremal polynomial is</u>

$$P^{**}(x,\alpha_0) = \pm \cos n \arccos \frac{2x - \alpha_0 - 1}{1 - \alpha_0}$$

<u>where the parameter</u> α_0 <u>varies from</u>

$$\frac{-\sin^2\frac{\Pi}{2n}}{\cos^2\frac{\Pi}{2n}} \quad \underline{\text{to}} \quad 0 .$$

<u>Proof of I</u>: Suppose $\alpha_1 = 0$ and $\alpha_n = 1$. If $P(x)$ is extremal then its derivative $P'(x)$ is a polynomial of degree $n-1$, with $n-2$ roots: $\alpha_2,\alpha_3,\ldots,\alpha_{n-1}$ between 0 and 1, and one root β outside $[0,1]$. Suppose $\beta > 1$, consider the following polynomial $P^2(x) - 1 = 0$ of degree $2n$. Clearly we have $n-2$ double roots; $\alpha_2,\alpha_3,\ldots,\alpha_{n-1}$ and two simple roots 0 and 1. The other two roots we denote by γ and δ. We will show that $\gamma > \beta$ and $\delta > \beta$. Since $|P(x)| > 1$ for

$x > 1$ then $P^2(x) \to L^2 > 1$ as $x \to \beta$. Since $p'(x)$ has no zeros greater than β and since β is a simple root then for $P(\beta) > 0$, $P(x)$ decreases for $x > \beta$ and whenever $P(\beta) < 0$ $P(x)$ increases for $x > \beta$. Consequently on $[\beta,\alpha)$ $P^2(x)$ first decreases and after vanishing it increases to infinity. Hence $P^2(x)$ attains the value 1 twice at $\gamma > \beta$ and $\delta > \beta$. Hence

$$P^2(x) - 1 = S^2(x-\alpha_2)^2(x-\alpha_3)^2\ldots(x-\alpha_{n-1})^2(x)(x-1)(x-\gamma)(x-\delta) ,$$

where S is a constant. Since

$$P'(x) = n\,s\,(x - \alpha_2)(x - \alpha_3)\ldots(x - \alpha_{n-1})(x - \beta)$$

then

$$P'^2(x) = n^2s^2(x - \alpha_2)^2(x - \alpha_3)^2\ldots(x - \alpha_{n-1})^2(x - \beta)^2$$

consequently we can write

$$P^2(x) - 1 = \frac{x(x-1)(x-\gamma)(x-\delta)}{\eta^2(x-\beta)^2} P'^2(x) .$$

Proof of II: If $\alpha_1 = 0$ and $\alpha_n < 1$ then we can always add to the nodes $(\alpha_i)_{i=1}^n$ another node $\alpha_{n+1} > 1$ such that the extremal polynomial $P(x)$ satisfies

$$P(\alpha_{n+1}) = - P(\alpha_n) .$$

Since the polynomial $P(x) - \cos n \arccos \frac{2x - \alpha_{n+1}}{\alpha_{n+1}}$ vanish $n + 1$ times we have

$$P^*(x)^{2} = \pm \cos n \arccos \frac{2x - \alpha_{n+1}}{\alpha_{n+1}} .$$

The unknown α_{n+1}, according to (1.19) must satisfy the equation

$$\sum_{i=1}^{n} \frac{1}{x - (\frac{\alpha_{n+1}}{2} + \frac{\alpha_{n+1}}{2} \cos i \frac{\Pi}{n})} = 0$$

Since the nodes $\alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of $P^{*\prime}(x)$ then

$$P^{*\prime}(x) = c(x-\alpha_2)(x-\alpha_3) \ldots (x-\alpha_n) ,$$

where c is some constant. Consequently, we get

$$\frac{P^{*\prime\prime}(x)}{P^{*\prime}(x)} = \frac{1}{x-\alpha_2} + \frac{1}{x-\alpha_3} + \ldots + \frac{1}{x-\alpha_n} .$$

Therefore by Theorem 1.10

2 we will write $P^*(x)$ or $P^*(x,\alpha_{n+1})$

$$\frac{P^{*''}(x)}{P^{*'}(x)} + \frac{1}{x} = \frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \dots + \frac{1}{x-\alpha_n} = 0$$

This we can write as

$$xP^{*''}(x) + P^{*'}(x) = 0 . \tag{1.23}$$

Furthermore $\alpha_{n+1} > 1 > \alpha_1 = \frac{\alpha_{n+1}}{2} + \frac{\alpha_{n+1}}{2} \cos \frac{\Pi}{n}$

i.e. $\alpha_{n+1} > 1 > \alpha_{n+1} \cdot \cos^2 \frac{\Pi}{2n}$.

Consequently, in order that the case $\alpha_1 = 0$, $\alpha_n < 1$ occurs, one of the values α_{n+1} satisfying the equation (1.23) must lie between 1 and $\frac{1}{\cos^2 \frac{\Pi}{2n}}$. Further, for each such x where the maximality of $P^*(x)$ is being discussed, there correspond only one α_{n+1}. In fact, consider the sum

$$\sum_{i=1}^{n} \frac{1}{x - (\frac{\alpha_{n+1}}{2} + \frac{\alpha_{n+1}}{2} \cos i \frac{\Pi}{n})} \tag{1.24}$$

as a function of α_{n+1} and note that if α_{n+1} increases this function increases[1]. Hence equation (1.23) can not have more than one root. So we conclude that the case $\alpha_1 = 0$, $\alpha_n < 1$ occurs if and only if for α_{n+1} varying from 1 to $\frac{1}{\cos^2 \frac{\Pi}{2n}}$ the expression $xP^{*''}(x) + P^{*'}(x)$ changes its sign.

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1. Except for those values of α_{n+1} for which (1.24) tends to infinity.

Hence
$$\frac{1}{\cos^2 \frac{\pi}{2n}} > \alpha_{n+1} > 1 .$$

This completes the proof of II. □

The proof of III is similar to the proof of II.

We now give Markov's solution to his first problem.

<u>1.12 Theorem</u> <u>For each</u> $P(x) \in \Pi_n$ <u>with</u> $|P(x)| \leq 1$ on $[0,1]$, $|P'(x)| \leq 2n^2$ <u>for all</u> $x \in [0,1]$.

<u>Proof</u>: Consider $T_n(x) = \cos n \arccos (2x - 1)$ and let $x = \frac{1}{2} + \frac{1}{2}\cos\phi$ then $\frac{d\phi}{dx} = \frac{-2}{\sin\phi}$.

$$T_n(x) = \cos n\phi ,$$
$$T_n'(x) = \frac{2n \sin\phi}{\sin\phi} ,$$
$$T_n''(x) = \frac{2n^2 \cos n\phi \sin\phi - 2n \sin n\phi \cos\phi\,(-2)}{\sin^2\phi}$$
$$= \frac{4n \sin n\phi \cos\phi - 4n^2 \cos\phi \sin\phi}{\sin^3\phi} .$$

Therefore
$$\frac{(x-1)T_n''(x)+T_n'(x)}{x\,T_n''(x)+T_n'(x)} = \frac{(\cos\phi-1)\{\sin n\phi\cos\phi - n\cos n\phi\sin\phi\}+\sin n\phi\sin^2\phi\}}{(1+\cos\phi)\{\sin n\phi\cos\phi - n\cos n\phi\sin\phi\}+\sin n\phi\sin^2\phi\}}$$
$$= \frac{(1-\cos\phi)(\sin n\phi + n\cos n\phi\sin\phi)}{(1+\cos\phi)(\sin n\phi - n\cos n\phi\sin\phi)} .$$

In fact, for $0 < \phi < \frac{\pi}{2n}$,
$$\sin n\phi > n\cos n\phi\sin\phi .$$

This follows by induction. For $n = 1$ it is obvious, let it be true for $n-1$ then
$$\sin n\phi = \sin(n-1)\phi\cos\phi + \cos(n-1)\phi\sin\phi$$
$$\geq (n-1)\cos(n-1)\phi\sin\phi\cos\phi + \cos(n-1)\phi\sin\phi$$

$$\geq n\cos(n-1)\phi\cos\phi\sin\phi$$

$$= n(\cos n\phi + \sin(n-1)\phi\sin\phi)\sin\phi$$

$$\geq n\cos n\phi\sin\phi \quad .$$

Here we have used the fact that $\sin(n-1)\phi, \sin\phi$ and $\cos n\phi$ are positive on $0 < \phi < \frac{\Pi}{2n}$. The case for $\Pi < \phi < \Pi - \frac{\Pi}{2n}$ can be dealt **with similarly.**

Consequently, for $0 < \phi < \frac{\Pi}{2n}$ or $\Pi - \frac{\Pi}{2n} < \phi < \Pi$,

$$\frac{(x-1)T_n''(x) + T_n'(x)}{x\,T_n''(x) + T_n'(x)} > 0$$

from where

$$\frac{\frac{T_n''(x)}{T_n'(x)} + \frac{1}{x-1}}{\frac{T_n''(x)}{T_n'(x)} + \frac{1}{x}} < 0$$

since $\frac{x-1}{x} < 0$. This implies that

$$\frac{T_n''(x)}{T_n'(x)} + \frac{1}{x} > 0$$

and

$$\frac{T_n''(x)}{T_n'(x)} + \frac{1}{x-1} < 0 .$$

Note that $\sum_{i=0}^{n-1} \frac{1}{x-\tau_i} > \sum_{i=1}^{n} \frac{1}{x-\tau_i}$.

Hence by Theorem 1.9 $T_n(x)$ is the extremal polynomial for all $\phi \in (0, \frac{\Pi}{2n}) \cup (\Pi, \Pi - \frac{\Pi}{2n})$, i.e. for all x $(\frac{1}{2} + \frac{1}{2}\cos\frac{\Pi}{2n}, 1) \cup (0, \frac{1}{2} - \frac{1}{2}\cos\frac{\Pi}{2n})$ we have

$$|T_n'(x)| = \left|\frac{2n\sin n\phi}{\sin\phi}\right| \leq 2n^2 .$$

Now suppose that $x \in [\frac{1}{2} - \frac{1}{2}\cos\frac{\Pi}{2n}, \frac{1}{2} + \frac{1}{2}\cos\frac{\Pi}{2n}]$. Then the extremal polynomial $P(x)$ is either $T_n(x)$, $P_1(x)$, $P_2(x)$ or $P(x)$ satisfies the differential equation (1.22). In this case we obtain the following

$$\begin{aligned} x(1-x) &= (\tfrac{1}{2})^2 - (\tfrac{1}{2} - x)^2 \\ &> (\tfrac{1}{2})^2 - (\tfrac{1}{2} - [\tfrac{1}{2} + \tfrac{1}{2}\cos\tfrac{\Pi}{2n}])^2 \\ &= (\tfrac{1}{2})^2 - (\tfrac{1}{2})^2 \cos^2 \tfrac{\Pi}{2n} \\ &= \tfrac{1}{4}\sin^2 \tfrac{\Pi}{2n} > \tfrac{1}{4n^2} \ . \end{aligned}$$

Therefore

$$|T_n'(x)| = \frac{n\,|\sin \operatorname{arc}\cos(2x-1)|}{\sqrt{x(1-x)}} \leq 2n^2 ,$$

$$|P^{*\prime}(x)| = \frac{n\left|\sin \operatorname{arc}\cos \dfrac{(2x-\alpha_{n+1})}{\alpha_{n+1}}\right|}{\sqrt{\dfrac{x}{\alpha_{n+1}}\left(1-\dfrac{x}{\alpha_{n+1}}\right)}} \cdot \frac{1}{\alpha_{n+1}}$$

$$\leq 2n^2$$

$$|P^{**\prime}(x)| = \frac{n\left|\sin \operatorname{arc}\cos \dfrac{(2x-1-\alpha_0)}{1-\alpha_0}\right|}{\sqrt{\dfrac{x}{1-\alpha_0}\left(1-\dfrac{x}{1-\alpha_0}\right)}} \quad \frac{1}{1-\alpha_0}$$

$$\leq 2n^2 ,$$

and from (1.22) we obtain

$$P^{2\prime}(x) = \frac{n^2(x-\beta)^2}{x(1-x)(x-\gamma)(x-\delta)} \quad 1 - P^2(x)$$

$$\leq \frac{n^2}{x(1-x)} \leq (2n^2)^2 \ .$$

Hence $|P'(x)| \leq 2n^2$. Therefore no matter what the extremal polynomial may be for $x \in [0,1]$, we always have $|P'(x)| \leq 2n^2$ for $0 \leq x \leq 1$ provided $|P(x)| \leq 1$ on $0 \leq x \leq 1$. □

CHAPTER II

BERNSTEIN'S THEOREM

2.1 Introduction As we have seen in Theorem 1.11 Markov has found that over the entire closed interval $[0,1]$, $|P'(x)| \le 2n^2$, whenever the degree of $P(x)$ is $\le n$ and $|P(x)| \le 1$ on $[0,1]$. S.N. Bernstein some 23 years later gave a better estimate of $|P'(x)|$ over the open interval $(0,1)$; he has shown that: *if $P(x)$ is a polynomial of degree $\le n$ and $|P(x)| \le 1$ on $[0,1]$ then*

$$|P'(x)| \le \frac{n}{\sqrt{x(1-x)}} .$$

We point out that it is possible to obtain Bernstein's result from Theorem 1.11[3]. This can be seen by examining the original proof given by Bernstein, which we give below.

2.2 Theorem *Let $P_n(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial of degree n such that $\max_{0\le x\le 1} |P_n'(x)\sqrt{x(1-x)}| = M$. Then it does not follow that $|P_n(x)| < \frac{M}{n}$ for all $x \in [0,1]$, i.e. we can find a point $x_0 \in [0,1]$ such that*

$$|P_n(x_0)| \ge \frac{M}{n} .$$

Proof: Let $\mathbf{P}$ be the collection of all polynomials of degree n such that for each $P_n(x) \in \mathbf{P}$, $\max_{0\le x\le 1} |P_n'(x)\sqrt{x(1-x)}| = M$. Let $P(x)$ be the polynomial of least deviation from zero in $\mathbf{P}$. Suppose $\max_{0\le x\le 1} |P(x)| = L$ and let $(\alpha_i)_{i=1}^{k}$ be all the nodes of $P(x)$ in $[0,1]$, that is $|P(\alpha_i)| = L$ $i=1,2,\ldots,k$ with $0 \le \alpha_1 < \alpha_2 < \ldots < \alpha_k \le 1$. Let $\xi \in [0,1]$ have the property that $|P'(\xi)\sqrt{\xi(1-\xi)}| = M$. We claim that there is no polynomial $F_n(x) \in \mathbf{P}$ of degree n which satis-

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3 In the 20th session of Nato's advanced study institute, in the summer of 1981 at Université de Montréal, Professor A. Goncar of the Steklov institute of Moscow brought to our attention that Markov was aware of Bernstein's results at the time of publication of his paper in 1889. This can be seen from the fact that the key to Bernstein's proof is line (2.8) which is identical to (1.22) in Markov's paper.

fies simultaneously the following $k+1$ equations

$$P(\alpha_1)=F_n(\alpha_1)\ ,\ P(\alpha_2)=F_n(\alpha_2),\dots,P(\alpha_k)=F_n(\alpha_k) \quad \text{and} \quad F_n'(\xi)=0\ . \quad (2.1)$$

Suppose that we have a polynomial $F_n(x) \in \mathbf{P}$ satisfying (2.1) we associate with each node α_i an interval A_i with the property that $\alpha_i \in A_i$ and for all $x \in \mathring{A}_i$ $\operatorname{sgn} P(x) = \operatorname{sgn} F_n(x)$. If we delete all A_i from $[0,1]$ then $\max |P(x)| < L' < L$ for $x \in [0,1] \setminus \bigcup_{i=1}^{k} A_i$. Let $\delta = L - L'$ and for a sufficiently small λ we get $|\lambda F_n(x)| < \delta$. We form the polynomial

$$P(x) - \lambda F_n(x)\ .$$

Since $P(x)$ and $F_n(x)$ have the same sign on $\bigcup_{i=1}^{k} A_i$ we get $|P(x) - \lambda F_n(x)| \le |P(x)| < L$ on $\bigcup_{i=1}^{k} A_i$ and on $[0,1] \setminus \bigcup_{i=1}^{k} A_i$, $|P(x) - \lambda F_n(x)| < L' + \delta = L$. Therefore

$$|P(x) - \lambda F_n(x)| < L \quad \text{for all} \quad x \in [0,1]\ .$$

On the other hand, since $(P'(\xi) - \lambda F_n'(\xi))\sqrt{\xi(1-\xi)} = M$ then, whenever $\max |[P'(x) - \lambda F_n'(x)]\sqrt{x(1-x)}| = M_1$ we get $M_1 \ge M$. Hence the polynomial

$$\hat{P}(x) = \frac{M}{M_1}\,(P(x) - \lambda F_n(x))$$

is such that

$$\max_{0 \le x \le 1} |\hat{P}'(x)\sqrt{x(1-x)}| = M \quad \text{and} \quad |\hat{P}(x)| < L\ .$$

This implies that the deviation of $\hat{P}(x)$ is less than that of $P_n(x)$, but this is a contradiction.

We now show that the number of nodes is greater than $n-1$, that is $s > n-1$. We assume that $s \le n-1$. By the Lagrange interpolation formula we can construct a polynomial

$$Q(x) = \frac{(x-x_2)(x-x_3)\dots(x-x_s)}{(x_1-x_3)(x_1-x_3)\dots(x_1-x_s)} P(x_1) + \dots + \frac{(x-x_1)(x-x_2)\dots(x-x_{s-1})}{(x_s-x_1)(x_s-x_2)\dots(x_s-x_{s-1})} P(x_s) \ .$$

$Q(x)$ is of degree $s-1$ satisfying the first s equations in (2.1). Putting $R(x) = (x-x_1)(x-x_2)\dots(x-x_s)$ we see that the polynomial

$$F_{s+1}(x) = Q(x) + (Ax+B)R(x)$$

of degree $s+1$, also satisfies the first s equations in (2.1). We can always choose the numbers A and B such that

$$F'_{n+1}(\xi) = Q'(\xi) + AR(\xi) + (A\xi+B)R'(\xi) = 0 \ .$$

This is possible because $R(x)$ has no double roots, $R'(\xi) = R(\xi) = 0$ cannot occur. Hence we have otbained a polynomial $F_{s+1}(x)$ of degree at most n satisfying all $s+1$ equations of (2.1). As we have already seen this is not possible. Therefore s must be greater than $n-1$. If $s=n+2$ then $P'(x)$ being a polynomial of degree $n-1$ must vanish at the n interior nodes $(\alpha_i)_{i=2}^{s-1}$ which is impossible. Hence $s \neq n+2$. Therefore we can only have $s=n$ or $s=n+1$.

<u>Case I</u> Suppose $s=n$. Let

$$F_n(x) = Q(x) + BR(x)$$

where B is such that

$$Q'(\xi) + BR'(\xi) = 0 \ .$$

Clearly $F_n(x)$ is a polynomial of degree n satisfying the $s+1$ equations of (2.1). In order that the last equation in (2.1) fails, i.e. $F'_n(\xi) \neq 0$, it is necessary to have

$$R'(\xi) = 0 \ . \qquad (2.2)$$

Since at the nodes $(\alpha_i)_{i=2}^{n-1}$ of $P(x), P'(\alpha_i) = 0$, and $P'(x)$ is a polynomial of degree $n-1$. Hence we must have a node at 0 or at 1 or at both 0 and 1. In all three cases

$$R(x) = \frac{C(x(1-x))P'(x)}{x-\beta}, \qquad (2.3)$$

where C and β are constants and when the nodes $\alpha_1 = 0$ or $\alpha_n = +1$, we take $\beta = 0$ or $+1$ respectively. If however both nodes $\alpha_1 = 0$ and $\alpha_n = +1$ then β is the root of $P'(x)$ outside $[0,1]$. From (2.2) and (2.3) we get

$$R'(\xi) = \frac{d}{dx}\left[\frac{x(1-x)}{x-\beta} P'(x)\right]_{x=\xi}$$

$$= \frac{d}{dx}\left(\frac{\sqrt{x(1-x)}}{x-\beta}\left[\sqrt{x(1-x)}\, P'(x)\right]\right)\Big|_{x=\xi}$$

$$= \sqrt{x(1-x)}\, P'(x) \frac{d}{dx}\left(\frac{\sqrt{x(1-x)}}{x-\beta}\right)\Big|_{x=\xi}$$

$$+ \frac{\sqrt{x(1-x)}}{x-\beta} \frac{d}{dx}\left(\sqrt{x(1-x)}\, P'(x)\right)\Big|_{x=\xi} = 0 .$$

Consequently

$$\frac{d}{dx}\frac{\sqrt{x(1-x)}}{x-\beta}\Big|_{x=\xi} = \frac{\frac{1}{2}\frac{(1-2x)(x-\beta)}{\sqrt{x(1-x)}} - \sqrt{x(1-x)}}{(x-\beta)^2}\Bigg|_{x=\xi}$$

$$= \frac{(1-2\xi)(\xi-\beta) - 2\xi(1-\xi)}{2(\xi-\beta)^2\sqrt{\xi(1-\xi)}}$$

$$= \frac{-\beta(1-2\xi) - \xi}{(\xi-\beta)^2\sqrt{\xi(1-\xi)}} = 0 ,$$

so, $\beta = \frac{\xi}{2\xi-1}$. Since $\xi \in (0,1)$ we have $\beta \notin [0,1]$.

Moreover two of the nodes of $P(x)$ are at 0 and 1, and the remaining $n-2$ nodes are zeros of $P'(x)$ in $(0,1)$.

We need the following observations. The polynomial $L^2 - P^2(x)$ is of degree $2n$ and has zeros at the nodes of $P(x)$. The two zeros at 0 and 1 are simple and $n-2$ zeros inside $(0,1)$ are double zeros. Hence

$$\frac{[P'(x)]^2 x(1-x)}{(x-\beta)^2} \text{ divides } L^2 - P^2(x)$$

having the quotient $ax^2 + bx + c$ where a, b and c are constants. Thus, we get

$$L^2 - P^2(x) = \frac{[P'(x)]^2 x(1-x)(ax^2+bx+c)}{(x-\beta)^2}. \tag{2.4}$$

We show that

$$L^2 > \frac{x(1-x)[P'(x)]^2}{n^2}. \tag{2.5}$$

To prove (2.5), suppose $\beta > 1$. We observe that since $|P(x)| > L$ for $x > 1$ then as $x > 1$ tends to β, $[P(x)]^2$ tends to a number $L_1^2 > L^2$. Since $P'(x)$ has no zeros greater than β and since β is a simple zero of $P'(x)$, then $P(x)$ decreases for $x > \beta$ if $P(\beta) > 0$. And $P(x)$ increases for $x > \beta$ if $P(\beta) < 0$. Consequently on $[\beta, \infty)$, $[P(x)]^2$ first decreases and after vanishing it increases to infinity. Therefore $[P(x)]^2$ attains the value L^2 twice at $x = \gamma$ and $x = \alpha$ where $\gamma > \beta$ and $\alpha > \beta$. From (2.4) we obtain that γ and α must be the zeros of $ax^2 + bx + c = 0$. Furthermore the coefficient of the highest degree term of $P'(x)$ is n times the coefficient of the highest degree of $P(x)$. Therefore (2.4) can be written as

$$L^2 - P^2(x) = \frac{[P'(x)]^2 x(1-x)(x-\gamma)(x-\alpha)}{n^2(x-\beta)^2}. \tag{2.6}$$

Since $\gamma > \beta > 1$ and $\alpha > \beta > 1$ then for every $x \in [0,1]$ we have $L^2 > \left(\frac{[P'(x)]^2 x(1-x)}{n^2}\right)$. Thus $L > \frac{M}{n}$. This proves the first case.

Case II Suppose $s = n + 1$. Since $P'(x)$ is a polynomial of degree $n-1$ then $(\alpha_i)_{i=2}^{s-1}$, the $n-1$ interior nodes are the zeros of $P'(x)$ and the two other nodes of $P(x)$ are $\alpha_1 = 0$ and $\alpha_{n+1} = 1$. Therefore from (2.6) $P(x)$ satisfies the differential equation

$$L^2 - P^2(x) = \frac{x(1-x)[P'(x)]^2}{n^2}$$

which implies

$$\frac{n}{\sqrt{x(1-x)}} = \frac{P'(x)}{\sqrt{L^2 - P^2(x)}}$$

Hence

$$n \arccos(2x-1) = \arccos P(x)/L \quad .$$

Consequently,

$$P(x) = L \cos n \arccos(2x-1)$$

which is the Čebyšev polynomial. Since

$$P'(x) = nL \sin n \arccos \frac{(2x-1)}{\sqrt{x(1-x)}}$$

we obtain

$$L = \frac{M}{n} \; .$$

From Theorem 2.2 Bernstein deduced the Theorem: If $P(x)$ is a polynomial of degree n, and $|P(x)| \leq 1$ for $0 \leq x \leq 1$, then

$$|P'(x)| \leq \frac{n}{\sqrt{x(1-x)}} \quad \text{for} \quad 0 < x < 1 \; .$$

CHAPTER III

MARKOV - VORONOVSKAJA THEOREM

PART I: CEBYSEV POLYNOMIALS ARE EXTREMALS

<u>3.1 Introduction</u> In Chapter I we mentioned that Markov attempted to investigate how large $|P'(\xi)|$ can be for the polynomial $P(x)$ bounded by 1 on $[0,1]$. He obtained conditions under which for a given $\xi \in \mathbf{R}$, the Cebysev polynomials $\pm T_n(x)$ is extremal. However, he did not complete this study. E.V. Voronovskaja studied the provlem posed by Markov and unified it with the methods of Functional Analysis. In this chapter and in the subsequent chapter we present the solution of Markov's problem as presented by E.V. Voronovskaja. It is already established in Chapter I that the extremal polynomial $P(x)$ to our problem has nodes $s = n + 1$ or n. We discuss the case when $s = n + 1$ in the present chapter and the case $s = n$ in the next chapter.

Let $C[0,1]$ be the vector space of all continuous functions defined on $[0,1]$ with the distance between two elements $f_1, f_2 \in C[0,1]$ given as

$$d(f_1,f_2) = \max_{0 \le x \le 1} |f_1(x) - f_2(x)| .$$

It is well known that the convergence in $C[0,1]$ is uniform convergence of the sequence $\{f_n(x)\}$ and $C[0,1]$ is a complete metric space.

Let $F: C[0,1] \to R$ (real numbers) be a continuous linear functional i.e. F satisfies the following conditions:

(1) $F(f_1 + f_2) = F(f_1) + F(f_2)$ for all $f_1, f_2 \in C[0,1]$ **(additivity)**

(2) $F(\alpha f) = \alpha F(f)$ for all $f \in C[0,1]$ and all $\alpha \in R$ **(homogeneity)**

(3) $\lim_{n\to\infty} F(f_n) = F(f)$, if f_n converges to f uniformly **(continuity)**

It is easily shown that an additive homogenous functional is continuous if and only if it is bounded, i.e. there exists a constant K

such that

$$|F(f)| \leq K \max_{0\leq x\leq 1} |f(x)| \quad \text{for all} \quad f \in C[0,1] .$$

The smallest possible number K is called the norm of F and is denoted by $\|F\|$ or N. Thus

$$N = \|F\| = \sup |F(f)|$$

where sup is taken over all $f \in C[0,1]$ with $\max_{0\leq x\leq 1} |f(x)| = 1$.

The following representation theorem of continuous linear functional defined on $C[0,1]$ is fundamental to our work.

Risez Representation Theorem: Every continuous linear functional F on $C[0,1]$ can be represented in the form

$$F(f) = \int_0^1 f(t)\, dH(t) ,$$

where $H(t)$ is a function of bounded variation on $[0,1]$. Moreover

$$N = \|F\| = \int_0^1 |dH(t)| .$$

Any such functional evaluated on the sequence of powers $1, x, x^2 \ldots x^n, \ldots$ generates a moment sequence

$$F(x^n) = \int_0^1 t^n\, dH(t) = U_n .$$

It follows from Weierstrass approximation theorem that a continuous linear functional is completely determined by a moment sequence.

We now consider a linear functional

$$F_n : \Pi_n \to \mathbb{R}$$

defined by a sequence of $n+1$ real numbers, i.e.

$$F_n = U_0, U_1, \ldots, U_n$$

such that

$$F_n[Q_n(x)] = q_0 U_0 + q_1 U_1 + \ldots + q_n U_n ,$$

where $Q_n(x) = q_0 + q_1 x + \ldots + q_n x^n \in \Pi_n$. Recall that Π_n is the space of all polynomials of degree $\leq n$ with $d(P_1, P_2) = \max_{0\leq x\leq 1} |P_1(x) - P_2(x)|$.

for $P_1, P_2 \in \Pi_n$, defined on $[0,1]$. Since Π_n is a finite-dimensional vector space the norm N_n of the functional F_n is attained by some reduced polynomial $Q_n(x)$ which is then an extremal polynomial. That is $Q_n(x)$ is extremal for the functional F_n if and only if $Q_n(x) \not\equiv 1$ is a reduced polynomial i.e. $\max_{[0,1]} |Q_n(x)| = 1$ and $F_n[Q_n(x)] = +N_n$[4]. Of course the extremal polynomial $Q_n(x)$ does not have to be unique. In fact even the integrator function corresponding to the functional $F_n = U_0, U_1, \ldots, U_n$ need not be unique. In what follows we will give necessary and sufficient conditions for a given Functional F_n to have a unique integrator function and a unique extremal polynomial.

For a moment sequence $(U_i)_{i=0}^{\infty}$, let $U_n = U_{n,0}$ and $U_{m,n+1} = U_{m,n} - U_{m+1,n}$. If none of the differences $(U_{m,n})(m = 0,1,2,\ldots,; n = 0,1,2,\ldots)$ is negative then we call $(U_i)_{i=0}^{\infty}$ an <u>absolutely-monotonic moment sequence</u>, otherwise we call $(U_i)_{i=0}^{\infty}$ <u>non-absolutely-monotonic moment sequence</u>. Every moment sequence $(U_i)_{i=0}^{\infty}$ can be represented uniquely as the difference of two non-zero absolutely-monotonic moment sequence $(\alpha_i)_{i=0}^{\infty}$ and $(\beta_i)_{i=0}^{\infty}$, called the <u>minimal components</u> of $(U_i)_{i=0}^{\infty}$. When the functional F is defined by an absolutely-monotonic moment sequence $(\alpha_i)_{i=0}^{\infty}$ we have the norm $N = \alpha_0$. For the functional F defined by a non-absolutely-monotonic moment sequence $(U_i)_{i=0}^{\infty} = (\alpha_i)_{i=0}^{\infty} - (\beta_i)_{i=0}^{\infty}$, the norm $N = \alpha_0 + \beta_0$[5].

If F is a bounded linear functional given by a moment sequence $(U_i)_{i=0}^{\infty}$, then we denote $(U_i)_{i=0}^{\infty}$ by $\bar{U}$, and the value $F[Q_n(x)]$ as $Q_n(\bar{U})$, where $Q_n(x) \in \Pi_n$. We now give the following:

<u>5</u> see [7 p. 8-12]

<u>4</u> we return to our main problem on page 48.

3.2 Theorem [7, Theorem 1, p.14]. An absolutely monotonic moment sequence $\bar{\alpha} = (\alpha_i)_{i=0}^{\infty}$ has an extremal polynomial $P(x) \not\equiv 1$ if and only if the integrator function $g(t)$ of $\bar{\alpha}$ is a step function with a finite number of jumps $(\delta_i)_{i=1}^{k}$ and if, $(\sigma_i)_{i=0}^{s}$ are the points of discontinuity of $g(t)$ then $P(\sigma_i) = +1$ for $i=1,2,\ldots,s$.

Proof: Since $\bar{\alpha}$ is absolutely monotonic moment sequence then $N = \operatorname{Var} g(t) = \int_0^1 dg(t) = \alpha_0$. Since $P(x) \not\equiv 1$, there exists a point x_0 and a closed interval $[\alpha,\beta]$ such that $0 \leq \alpha \leq x_0 \leq \beta \leq 1$ and $P(x) < 1$ for all x $[\alpha,\beta]$. We apply the mean value theorem and obtain

$$\begin{aligned}\int_\alpha^\beta P(t)\, dg(t) &< P(\xi) \int_\alpha^\beta dg(t) \\ &= P(\xi) \operatorname*{var}_{[\alpha,\beta]} g(t) \\ &< \operatorname*{var}_{[\alpha,\beta]} g(t) .\end{aligned}$$

Since $N = \alpha_0$ we have

$$\begin{aligned}\alpha_0 &= \int_0^1 P(t)\, dg(t) \\ &= \int_0^\alpha P(t)\, dg(t) + \int_\alpha^\beta P(t)\, dg(t) + \int_\beta^1 P(t)\, dg(t) \\ &= P(\xi_1) \operatorname*{var}_{[0,\alpha]} g(t) + P(\xi_2) \operatorname*{var}_{[\alpha,\beta]} g(t) + P(\xi_3) \operatorname*{var}_{[\beta,1]} g(t) .\end{aligned}$$

If $\int_\alpha^\beta P(t)\, dg(t) > 0$ then from the fact that $P(\xi_2) < 1$ we get

$$\alpha_0 < P(\xi_1) \operatorname*{var}_{[0,\alpha]} g(t) + \operatorname*{var}_{[\alpha,\beta]} g(t) + P(\xi_3) \operatorname*{var}_{[\beta,1]} g(t) ,$$

which means that $\alpha_0 < N$. Since $\alpha_0 = \operatorname*{var}_{[0,1]} g(t)$, we must have that $\operatorname*{var}_{[\alpha,\beta]} g(t) = 0$ or more explicitly $g(t)$ on $[\alpha,\beta]$ is a constant function. Consequently $g(t)$ is a step function and can have discontinuities only at finitely many points where $p(x) = 1$.

Assume now that the step function $g(t)$ is the integrator function for the functional $\bar{\alpha}$, and suppose $(\sigma_i)_{i=1}^{s}$ are the points on the abscissa where the step function is discontinuous. We make the following observations. If $0 < \sigma_i < 1$ then $|g(\sigma_i^+) - g(\sigma_i^-)| = \delta_i > 0$. If $\sigma_1 = 0$ then $g(0) = 0$ and $g(0^+) = \delta_1 > 0$. If $\sigma_s = 1$ then $g(1) - g(1^-) = \delta_s$. We then have

$$\alpha_0 = \int_0^1 d\,g(t) = \sum_{i=1}^{s} \delta_i \ ,$$

and

$$\alpha_k = \int_0^1 t^k \, d\,g(t) = \sum_{i=1}^{s} \sigma_i^k \delta_i \ .$$

Therefore if $P_n(x) = \sum_{k=0}^{n} a_k x^k$ is a reduced polynomial of degree n i.e. $\max_{[0,1]} |P(x)| = 1$. Then

$$\begin{aligned}
P_n(\bar{\alpha}) &= \int_0^1 P_n(t)\, d\,g(t) \\
&= \int_0^1 \sum_{k=0}^{n} a_k t^k \, d\,g(t) \\
&= \int_0^1 (a_0 + a_1 t + \ldots + a_n t^n) d\,g(t) \\
&= \int_0^1 a_0 \, d\,g(t) + \int_0^1 a_1 t \, d\,g(t) + \ldots + \int_0^1 a_n t^n \, d\,g(t) \\
&= \sum_{i=1}^{s} a_0 \delta_i + \sum_{i=1}^{s} a_1 \sigma_i \delta_i + \ldots + \sum_{i=1}^{s} a_n \sigma_i^n \delta_i \\
&= \sum_{i=1}^{s} a_0 \delta_i + a_1 \sigma_i \delta_i + \ldots + a_n \sigma_i^n \delta_i \\
&= \sum_{i=1}^{s} P_n(\sigma_i) \delta_i \ .
\end{aligned}$$

Hence for $P_n(\bar{\alpha}) = \alpha_0 = \sum_{i=1}^{s} \delta_i$ we must have $P_n(\sigma_i) = +1$. This completes the proof of our theorem. □

The extremal polynomial of smallest degree, we call the <u>principal polynomial</u>.

<u>Corollary</u> [7, Corollary, p. 15] <u>If</u> $P_n(x)$ <u>is a principal polynomial for the functional</u> $\bar{\alpha} = (\alpha_i)_{i=0}^{\infty}$ <u>where</u> $\bar{\alpha}$ <u>is an absolutely-monotonic moment sequence. Then the general form for</u> $P_n(x)$ <u>is easily obtained depending on the distribution of the discontinuities</u> $(\sigma_i)_{i=1}^{s}$ <u>on</u> $[0,1]$. <u>For</u>

(1) $0 < \sigma_1 < \ldots < \sigma_s < 1$, $P_n(x) = 1 - c \prod_{i=1}^{s} (x - \sigma_i)^2$

(2) $0 = \sigma_1 < \ldots < \sigma_s < 1$, $P_n(x) = 1 - cx \prod_{i=2}^{s} (x - \sigma_i)^2$

(3) $0 < \sigma_1 < \ldots < \sigma_s = 1$, $P_n(x) = 1 - c(1-x) \prod_{i=1}^{s-1} (x - \sigma_i)^2$

(4) $0 = \sigma_1 < \ldots < \sigma_s = 1$, $P_n(x) = 1 - cx(1-x) \prod_{i=2}^{s-1} (x - \sigma_i)^2$.

In the four cases c is a positive constant. The degree of the polynomial $P_n(x)$ defined above cannot be decreased since every point where $P_n(x) = +1$ is a discontinuity of the integrator function $g(t)$.

3.3 Definitions and Remarks The polynomial $\prod_{i=1}^{s} (x - \sigma_i)$ which we denote as $R_s(x)$ is called the resolvent of the extremal polynomial $P_n(x)$. The polynomial $\prod_{i=1}^{s} (x - \sigma_i)^2$, is called squared resolvent and is denoted by $R_s^2(x)$.

Remark 1 Let $\bar{U}$ be a moment sequence with a principal polynomial $P_n(x)$. If $f(x)$ is also a reduced extremal polynomial of degree higher than $P_n(x)$, we can then express $f(x)$ in the form $f(x) = 1 - \hat{\phi}(x) R_s^2(x)$ where $R_s^2(x)$ is the squared resolvent of $P_n(x)$, and $\hat{\phi}(x) > 0$. Since $f(x)$ is a reduced polynomial that is $|f(x)| \leq 1$ on $[0,1]$ then $1 - f(x) \leq 2$ and $1 - (1 - \hat{\phi}(x) R_s^2(x)) \leq 2$ which implies $0 \leq \hat{\phi}(x) \cdot R_s^2(x) \leq 2$.

Remark 2 If we have an absolutely monotonic moment sequence $\bar{U} = (U_i)_{i=0}^{\infty}$, and $\bar{U}$ has an extremal $P_n(x) \not\equiv 1$ which we know, and moreover $P_n(x)$ is principal, then we can construct the integrator function $g(t)$ of the functional determined by $\bar{U}$. We first find all the nodes $\sigma_1, \sigma_2, \ldots, \sigma_s$, of $P_n(x)$ on $[0,1]$ i.e. these are the points where $P_n(x) = +1$. We then solve the system

$$\sum_{i=1}^{s} \sigma_i^k \delta_i = U_k \qquad k = 0,1,2,\ldots,s-1$$

for the jumps δ_i of $g(t)$. If we write V for the Vandermonde determinant of $(\sigma_i)_{i=1}^{s}$ and $V^{(i)}$ for the determinant of the matrix formed by replacing the ith column of V by $(u_0,u_1,\ldots,u_{s-1})$. Then by Cramer's Rule $\delta_i = \frac{V^{(i)}}{V_i}$, $i = 0,1,\ldots,s-1$. Hence $g(t)$ is determined with jumps δ_i at σ_i.

We now extend Theorem 3.2 to the case when we have a moment sequence $(U_i)_{i=0}^{\infty}$ not necessarily absolutely monotonic. We first give the following lemma:

3.4 Lemma [7, p. 16] Let $\bar{U} = (U_i)_{i=1}^{\infty}$ be a moment sequence with $\bar{\alpha} = (\alpha_i)_{i=0}^{\infty}$ and $\bar{\beta} = (\beta_i)_{i=0}^{\infty}$ its minimal components. Let the monotonic step function $g_1(t)$ be the integrator function for $\bar{\alpha}$ and let the monotonic step function $g_2(t)$ be the integrator function for $\bar{\beta}$. Then $g_1(t)$ and $g_2(t)$ have no common points of discontinuity.

Proof: Let the points of discontinuity of $g_1(t)$ be $(a_i)_{i=1}^{s_1}$ and let $(b_i)_{i=1}^{s_2}$ be the points of discontinuity of $g_2(t)$. Suppose that $a_k = b_k = c$ with corresponding jumps δ_a and δ_b. We can always construct an absolutely monotonic sequence $(\gamma_i)_{i=0}^{\infty}$ for which the integrator $g_3(t)$ is a step function with a single positive jump δ_c at the point c. We let δ_c be the smaller of δ_a and δ_b. It is easy to see that both $g_1(t) - g_3(t)$ and $g_2(t) - g_3(t)$ are non-decreasing. We construct two new sequences

$$(\alpha_i')_{i=0}^{\infty} = (\alpha_i - \gamma_i)_{i=0}^{\infty}$$

and

$$(\beta_i')_{i=0}^{\infty} = (\beta_i - \gamma_i)_{i=0}^{\infty} .$$

Clearly

$$u_i = \alpha_i' - \beta_i' = \alpha_i - \gamma_i - \beta_i + \gamma_i = \alpha_i - \beta_i .$$

Thus $(\alpha_i')_{i=0}^{\infty}$ and $(\beta_i')_{i=0}^{\infty}$ are two other components for $(U_i)_{i=0}^{\infty}$. We now compute $\alpha_0' + \beta_0'$. We have

$$\alpha_0' + \beta_0' = \int_0^1 d\,g_1(t) - \int_0^1 d\,g_3(t) + \int_0^1 d\,g_2(t) - \int_0^1 d\,g_3(t)$$
$$= N - 2\gamma_0 < N -$$

which is impossible. □

3.5 Theorem [7, Theorem 2, p. 16] <u>The moment sequence</u> $\bar{U} = (U_i)_{i=0}^{\infty}$ <u>has an extremal polynomial</u> $Q_n(x) \not\equiv \pm 1$ <u>if and only if its minimal components</u> $\bar{\alpha} = (\alpha_i)_{i=0}^{\infty}$ <u>and</u> $\bar{\beta} = (\beta_i)_{i=0}^{\infty}$ <u>both have extremal polynomials</u>.

<u>Proof necessity</u>: Assume that the functional defined by $\bar{U}$ has an extremal polynomial $Q_n(x)$ of degree n. Then

$$Q_n(\bar{U}) = + N = \alpha_0 + \beta_0 .$$

Since we are given that the absolutely monotonic moment sequences $\bar{\alpha}$ and $\bar{\beta}$ are the minimal components of $\bar{U}$. Then $Q_n(\bar{\alpha}) = \alpha_0$ and $-Q_n(\bar{\beta}) = \beta_0$. Hence

$$\alpha_0 + \beta_0 = Q_n(\bar{\alpha}) - Q_n(\bar{\beta}) .$$

consequently $Q_n(x)$ is the extremal polynomial for $\bar{\alpha}$ and $-Q_n(x)$ is the extremal polynomial for $\bar{\beta}$.

<u>Sufficiency</u> Let $Q_{P_1}(x)$ and $Q_{P_2}(x)$ be reduced polynomials of lowest degree such that $Q_{P_1}(\bar{\alpha}) = \alpha_0$ and $Q_{P_2}(\bar{\beta}) = \beta_0$. That is $Q_{P_1}(x)$ is the principal polynomial for $\bar{\alpha}$ and $Q_{P_2}(x)$ is the principal polynomial for $\bar{\beta}$. We will construct an extremal polynomial $Q(x)$ for the functional $\bar{U}$. By the corollary 3.2 every principal polynomial has one of the forms (1),(2),(3),(4). Let $(a_i)_{i=1}^{s_1}$ and $(b_i)_{i=1}^{s_2}$ be the points of discontinuity of $g_1(x)$ and $g_2(x)$ where $g_1(x)$ and $g_2(x)$ are the integrator functions of the functionals given by the

corresponding minimal components $\bar{\alpha}$ and $\bar{\beta}$ of $\bar{U}$. By Lemma 3.4 $(a_i)_{i=1}^{s_1}$ and $(b_i)_{i=1}^{s_2}$ have no points in common. Thus if $Q(x)$ is the extremal polynomial of $\bar{U}$ then by the first remark of 3.3, the resolvent of $Q(x)$ must vanish at least $s_1 + s_2 = s$ points. Moreover $Q(x)$ is an extremal polynomial for both $\bar{\alpha}$ and $\bar{\beta}$. Let the resolvent of $Q_{P_1}(x)$ be $R_{S_1}(x)$, and $R_{S_2}(x)$ be the resolvent of $Q_{P_2}(x)$. Since $R_{S_1}^2(x)$ and $R_{S_2}^2(x)$ are two relatively primed polynomials, then by the Euclidean algorithm for polynomials we can obtain unique polynomials $\phi(x)$ and $\psi(x)$ such that

$$\phi(x)R_{S_1}^2(x) + \psi(x)R_{S_2}^2(x) \equiv 2 \text{ .} \tag{3.1}$$

We have two cases to consider.

<u>Case 1</u>. Suppose $\phi(x) \geq 0$ and $\psi(x) \geq 0$ for all $x \in [0,1]$. From (3.1) we obtain

$$\phi(x)R_{S_1}(x) \equiv 2 - \psi(x)R_{S_2}^2(x) \text{ .}$$

Thus

$$Q(x) = 1 - \phi(x)R_{S_1}^2(x) \equiv -1 + \psi(x)R_{S_2}^2(x) \text{ .}$$

$Q(x)$ is a reduced polynomial because $\phi(x)$ and $\psi(x)$ are non-negative.

<u>Case 2</u>. Suppose that $\phi(x)$ and $\psi(x)$ are not both non-negative. Then from (3.1) we obtain

$$\frac{\phi(x)}{R_{S_2}^2(x)} + \frac{\psi(x)}{R_{S_1}^2(x)} \equiv \frac{2}{R_{S_1}^2(x)R_{S_2}^2(x)} \text{ .}$$

We must construct a polynomial $\lambda(x)$ such that

$$\frac{-\phi(x)}{R_{S_2}^2(x)} \leq \lambda(x) \leq \frac{\psi(x)}{R_{S_1}^2(x)} \tag{3.2}$$

That is $\lambda(x)$ satisfies the following

$$[\phi(x) + \lambda(x)R_{S_2}^2(x)]R_{S_1}^2(x) + [\psi(x) - \lambda(x)R_{S_1}^2(x)]R_{S_2}^2(x) \equiv 2 \text{ ,} \tag{3.3}$$

where $[\phi(x)+\lambda(x)R^2_{S_2}(x)]$ and $[\psi(x)-\lambda(x)R^2_S(x)]$ are non-negative. Expanding (3.3) we obtain

$$\phi(x)R^2_{S_1}(x)+\lambda(x)R^2_{S_2}(x)R^2_{S_1}(x)+\psi(x)R^2_{S_1}(x)-\lambda(x)R^2_{S_1}(x)R^2_{S_2}(x)\equiv 2$$

which clearly equals (3.1). Thus (3.3) holds. We call (3.2) the <u>zone of reduction</u> for $\lambda(x)$, it is of width $\frac{2}{R^2_{S_1}(x)R^2_{S_2}(x)} > 0$. To obtain $\lambda(x)$ explicitly we take an arbitrary continuous curve inside the zone of reduction, then by Weierstrass's Theorem we approximate it by a polymonial $\lambda(x)$ which lies also inside the zone. After choosing $\lambda(x)$ we can write the extremal polynomial $Q(x)$ as

$$Q(x) = 1 - [\phi(x) + \lambda(x)R^2_{S_2}(x)]R^2_{S_1}(x)$$

$$= -1+[\psi(x) - \lambda(x)R^2_{S_1}(x)]R^2_{S_2}(x) .$$

This completes the proof. □

<u>Corollary</u> [7, Corollary, p. 18] <u>If</u> $(U_i)_{i=0}^{\infty}$ <u>is a moment sequence with</u> $Q_n(x)$ <u>as its extremal polynomial then</u>

$$U_k = \sum_{i=1}^{s} \delta_i \sigma_i^k \qquad k = 0,1,2,...$$

<u>and</u> $0 \le \sigma_i \le 1$ <u>with</u> $Q_n(\sigma_i) \simeq \text{sgn}\,\delta_i$, <u>and</u> $(\sigma_i)_{i=1}^{s}$ <u>are the points of discontinuity of the integrator function</u> $H(t)$.

Let $\bar{U}_n = (U_i)_{i=0}^{n}$ be a finite sequence of real numbers we call this sequence a <u>segment-functional</u>. As we had in the introduction (3.1) for a continuous linear functional $F_n = \bar{U}_n : \Pi_n \to \mathbb{R}$ we have

$$F_n[Q_n(x)] = Q_n(\bar{U}_n) = \sum_{i=0}^{n} q_i U_i ,$$

where $$Q_n(x) = \sum_{i=1}^{n} q_i x^i \in \Pi_n .$$

By the Hahn-Banach extension theorem the functional $F_n = (U_i)_{i=0}^{n}$ can be extended to Π_{n+1} by one number U_{n+1} such that the functional

$$F_{n+1} \equiv (U_i)_{i=0}^{n+1} : \Pi_{n+1} \to \mathbb{R}$$

has the same norm N_n as F_n.

A segment-functional $\bar{U}_n = (U_i)_{i=0}^{n}$ is called <u>absolutely-monotonic segment functional</u> if there is at least one absolutely monotonic moment sequence $\bar{U} = (U_i)_{i=0}^{\infty}$ of which $\bar{U}_n = (U_i)_{i=0}^{n}$ forms the first $n+1$ terms of $\bar{U}$. If among all the extensions none is absolutely monotonic moment sequence, we say that $\bar{U}_n = (U_i)_{i=0}^{n}$ is <u>non-absolutely-monotonic segment functional</u>. If $\bar{U}_n = (U_i)_{i=0}^{n}$ is a non-absolutely-monotonic segment functional with $P(x)$ as its extremal polynomial, then $P(x)$ cannot be equal to the constant 1. Had it been $\equiv 1$ the integrator function $H(t)$ is monotonic and $1(\bar{U}_n) = U_0 = N$ and then $(U_i)_{i=0}^{n}$ becomes an absolutely monotonic segment functional. Furthermore a segment-functional $(U_n)_{i=0}^{n}$ has a corresponding integrating function however the integrating function need not be unique[5]. In the next theorem we will show that if $\bar{U}_n = (U_i)_{i=0}^{n}$ is non-absolutely-monotonic segment functional then the integrating function is in fact unique.

<u>3.6 Theorem</u> [7, Theorem 12, p. 33] <u>If $\bar{U}_n = (U_i)_{i=0}^{n}$ is non-absolutely-monotonic segment functional then $\bar{U}_n$ has a unique best extension, i.e. there is just one number U^*_{n+1} such that the segment $U_0, U_1, \ldots, U_n, U^*_{n+1}$ defines a functional F_{n+1} with the same norm N_n.</u>

<u>Proof</u>: We must show that there is a unique number U^*_{n+1} such that the segment $U_0, U_1, \ldots, U_n, U^*_{n+1}$ defines a functional F_{n+1} with the same norm as F_n. Let $Q_m(x)$ be an arbitrary extremal polynomial of $\bar{U}_n$. By Hahn-Banach we form an arbitrary best extension

$$U_0, U_1, \ldots, U_n, U_{n+1}, \ldots, U_p, \ldots .$$

5 see [6, TM 6, p. 22] for more details.

We then obtain a moment sequence where

$$U_p = \int_0^1 t^p \, dH(t) \quad p = 0,1,2,\ldots \tag{3.6}$$

By the corollary to Theorem 3.5 $H(t)$ is a step function with discontinuities at the points $(\sigma_i)_{i=1}^{s}$ and with jumps $(\delta_i)_{i=1}^{s}$.

Let $(\hat{\sigma}_i)_{i=1}^{s_1}$ be all the points of deviation of $Q_m(x)$ on the closed interval $[0,1]$. That is $|Q_m(\hat{\sigma}_i)| = 1$ for $i = 1,2,\ldots,s_1$. Clearly $2 \leq S \leq S_1 \leq m+1$. Then the $(\delta_i)_{i=1}^{s_1}$ at the points $\hat{\sigma}_i$ are determined uniquely by the system of s_1 equations

$$\sum_{i=1}^{s_1} \hat{\sigma}_i^k \delta_i = U_k \qquad k = 0,1,2,\ldots,s_1 - 1$$

Therefore by (3.6) all U_p with $p \geq S_1$ can be uniquely defined by[6]

$$U_p = \sum_{i=1}^{s_1} \hat{\sigma}_i^p \delta_i \quad .$$

Therefore the segment $(U_i)_{i=0}^{n}$ has a unique sequence of best extensions which we denote by

$$U_0, U_1, \ldots, U_n, U_{n+1}^*, U_{n+2}^*, \ldots$$

where each U_{n+1}^* is unique, and every extremal polynomial $P(x)$ is such that the points $(\sigma_i)_{i=1}^{s}$ of discontinuity of the unique $H(t)$ are some of the points of deviation not necessarily all of them. ■□

<u>Corollary 1</u> [7, Corollary 1, p. 34] <u>To each non-absolutely-monotonic segment functional</u> $\bar{U}_n$ <u>there corresponds a definite set</u> $(\sigma_i)_{i=1}^{s}$, <u>the points of discontinuity of the step-function</u> $H(t)$, <u>where</u> $0 \leq \sigma_i \leq 1$.

<u>Corollary 2</u> [7, Corollary 2, p. 34] <u>If</u> $\bar{U}_n$ <u>is a non-absolutely-monotonic segment functional and</u> $(\sigma_i)_{i=1}^{s}$ <u>are the points of discontinuity</u>

6 If a point σ_1' is not a point of discontinuity of $H(t)$, then its jump δ is zero.

then there is a reduced extremal polynomial $Q_m(x)(m < n)$ with $(\sigma_i)_{i=1}^{s}$ as its nodes and $Q_m(\sigma_i) = \text{sig}\,\delta_i$.

From now on we agree to call a non-absolutely-monotonic segment functional simply a segment functional. Suppose that $\bar{U}_n = (U_i)_{i=0}^{n}$ is a segment functional and $Q_m(x)$ is its principal polynomial with $m < n$. By the Hahn-Banach Theorem from U_m onward the segment functional has a best extension $U^*_{m+1}, U^*_{m+2}, \ldots$. Consequently the segment functional $\bar{U}_n = (U_i)_{i=0}^{n}$ can be replaced by a truncated segment $\bar{U}_m = (U_i)_{i=0}^{m}$. Therefore we make the assumption that the principal polynomials of the segment functional $\bar{U}_n$ are precisely of degree n, and in that case the segment functional is said to be irreducible.

3.7 Theorem [7, Theorem 14, p. 36] If $Q(x)$ is an extremal polynomial of the segment functional $\bar{U}_n = (U_i)_{i=0}^{n}$, then every other extremal polynomial is of the form

$$L(x) = Q(x) + \phi(x)R_S^2(x)$$

where $\phi(x)$ is a polynomial that ensures that $L(x)$ is reduced. The function $R_2^2(x)$ is the squared resolvent of the segment $\bar{U}_n$ see Definition 3.3.

Proof: Let $L(x) = Q(x) + \hat{\phi}(x)$. If $(\sigma_i)_{i=1}^{s}$ are the nodes of the segment $\bar{U}_n$ then $\text{sgn}\,L(\sigma_i) = \text{sgn}\,Q(\sigma_i)$ which implies $\hat{\phi}(\sigma_i) = 0$. Furthermore, by the fact that the polynomials $Q(x)$ and $L(x)$ have extrmas at $\sigma_i \in (0,1)$, the derivatives evaluated at σ_i of $Q(x)$ and $L(x)$ equals to zero. That is $Q'(\sigma_i) = 0$ and $L'(\sigma_i) = 0$. This implies that $\hat{\phi}'(\sigma_i) = 0$. It therefore follows that $\hat{\phi}(x)$ is a multiple of $R_s^2(x)$. This completes the proof. □

<u>3.8 Remark</u> Let $\bar{U}_n = (U_i)_{i=0}^n$ be a segment functional. Then a reduced polynomial $Q_n(x) \not\equiv 1$ is extremal for the given segment functional $\bar{U}_n$ if and only if the system

$$\sum_{i=1}^{s} \delta_i \sigma_i^k = U_k \qquad k = 0,1,\ldots,n$$

of $n+1$ equations in s unknown δ_i satisfies the following two conditions:

(1) the above system is consistent

(2) and $(\sigma_i)_{i=1}^s$ are the nodes of $Q_n(x)$ with $\operatorname{sgn} Q_n(\sigma_i) = \operatorname{sgn} \delta_i$ $\delta_i \neq 0$, generally not all $\delta_i = 0$.

The above we call <u>criterion for extremality</u>. In this case, the integrating function $H(t)$ corresponding to the segment functional $\bar{U}_n = (U_i)_{i=0}^n$ is a step function having points of discontinuities at $(\sigma_i)_{i=1}^s$ and

$$U_k = \int_0^1 t^k \, dH(t) = \sum_{i=1}^{s} \delta_i \sigma_i^k, \qquad k = 0,1,\ldots,n .$$

We now give the following definitions. To each node σ_i we assign the sign of the corresponding jump δ_i. This we denote by $(\sigma_i^{\pm})_{i=1}^s$ and call it the <u>distribution</u> of the segment $\bar{U}_n$. For a fix natural number n we divide the family of all segments into two classes depending on the number of nodes. If $s \leq \frac{n}{2} + 1$ we have a segment of <u>class I</u>. If $s > \frac{n}{2} + 1$ we have a segment of <u>class II</u>. We extend the concepts of classes determined by segments to polynomials. We say that a polynomial $P(x) \in \Pi_n$ is of class II if $s > \frac{n}{2} + 1$ otherwise $P(x)$ is of class I. It is clear that if $Q_m(x)$ is a principal polynomial for the segment $\bar{U}_n$ and if this segment is of class II with s_1 nodes then $Q_m(x)$ is a polynomial of class II with s_2 nodes where $s_2 \geq s_1$.

<u>3.9 Theorem</u> [7, Corollary 1, p. 37] <u>For every segment functional</u> $\bar{U}_n$

<u>of class II the extremal polynomial of degree</u> $\leq n$ <u>is unique</u>.

<u>Proof</u>: Assume that $Q_m(x)$ with $m < n$ is an extremal polynomial of the segment $\bar{U}_n$. Then by Theorem 3.7 we have that all other extremal polynomials have the form

$$L(x) = Q_m(x) + \phi(x)R_s^2(x) .$$

Since $\bar{U}_n$ belongs to class II, i.e. $s > \frac{n}{2} + 1$, the degree of $L(x)$ must be greater that n. □

<u>3.10</u> Let us return to our problem and consider the derivative $P'(x)$ at $x = \xi$ as a continuous linear functional F_ξ on Π_n, i.e. for every $P(x) \in \Pi_n$

$$F_\xi[P(x)] = P'(\xi) .$$

Since $F_\xi(x^k) = k\xi^{k-1}$, we identify F_ξ by the segment functional

$$F_\xi \equiv \bar{U}_n = 0,1,2\xi,\dots,k\xi^{k-1},\dots,n\xi^{n-1} .$$

We call F_ξ the <u>derivative functional</u>. The problem of finding the $\max|P'(\xi)|$ over all polynomials $P(x)$ of degree $\leq n$ with $\max\limits_{0\leq x\leq 1}|P(x)| = 1$ is in fact finding the norm $\|F_\xi\|$. We denote the norm by $N_n(\xi)$. If no confusion arises we write $N(\xi)$ for $N_n(\xi)$. Therefore for every reduced polynomial $P(x)$

$$N(\xi) = \|F_\xi\| = F_\xi(P) = |P'(\xi)| .$$

Because of finite dimensionality of Π_n, the space of all polynomials of degree $\leq n$, an extremal polynomial for each F_ξ exists. In Chapter I we have already established that the number of nodes s of an extremal polynomial must be n or $n+1$. Here for a given $\xi \in \mathbb{R}$ the extremal polynomial is unique (note: $s > \frac{n}{2} + 1$).

First we consider the case when $s = n+1$. The other case where the extremal polynomial has n nodes will be discussed in Chapter IV.

Since the Čebyšev polynomials $\pm T_n(x) = \pm\cos n \arccos(2x-1)$ are the only polynomials with $n+1$ nodes, our problem is reduced to the study of those ξ for which $\pm T_n(x)$ are extremal.

Let the nodes of $\pm T_n(x)$ be $(\tau_i)_{i=0}^n$. From the criterion of extremality (Remark 3.8), the integrator function $H(t)$ corresponding to the functional F_ξ, has discontinuity at $(\tau_i)_{i=0}^n$, and the jumps δ_i take the sign of $T_n(\tau_i)$. Thus we are led to determine ξ for which the system

$$\sum_{i=0}^{n} \delta_i \tau_i^k = k\xi^{k-1} \quad (k = 0,1,\ldots,n)$$

when solved for δ_i, gives δ_i with alternating sign since $\operatorname{sgn} T_n(\tau_i) = -\operatorname{sgn} T_n(\tau_{i+1})$ $i = 0,1,\ldots n-1$.

The interval I such that for $\xi \in I$, the extremal polynomial is a Čebyšev polynomial will be referred to as the <u>Čebyšev interval</u>. We now have;

<u>3.11 Theorem</u> [7, p. 158] <u>The domain where the Čebyšev polynomials $\pm T_n(x)$ are extremal consists of n separate closed subintervals of $\mathbb{R}$, called the Čebyšev intervals.</u>

<u>Proof</u>: Let $(\tau_i)_{i=0}^n$ be the nodes of $T_n(x)$ and let the Čebyšev resolvent be

$$R_{n+1}(x) = \prod_{i=0}^{n} (x - \tau_i) .$$

We will solve the following $n+1$ linear equations in $n+1$ unknown δ_i,

$$\sum_{i=0}^{n} \delta_i \tau_i^k = k\xi^{k-1} \quad (k = 0,1,\ldots,n) . \tag{3.5}$$

By Cramer's Rule, from the system (3.5), we get

$$\delta_k = \frac{\dfrac{d}{d\xi}\begin{vmatrix} 1_0, 1_1, \ldots, 1_k, \ldots, 1_n \\ \tau_0\ \tau_1, \ldots, \xi\ , \ldots, \tau_n \\ \tau_0^2\ \tau_1, \ldots, \xi^2, \ldots, \tau_n^2 \\ \vdots \qquad \vdots \qquad \vdots \\ \tau_0^n, \tau_1^n, \ldots, \xi^n, \ldots, \tau_n^n \end{vmatrix}}{\prod\limits_{0 \le j \le i \le n} (\tau_i - \tau_j)} . \tag{3.6}$$

The numerator of (3.6) is written as the derivative of a Vandermonde determinant, that is

$$\delta_k = \frac{\dfrac{d}{d\xi}\Big(\prod\limits_{0\le j\le i\le n} (\tau_i - \tau_j) \quad \text{where} \quad \tau_k = \xi\Big)}{\prod\limits_{0\le j\le i\le n}(\tau_i - \tau_j)}$$

$$= \frac{[\prod_{i\neq k}(\xi - \tau_i)]'}{\prod_{i\neq k}(\tau_k - \tau_i)}$$

$$= \frac{(-1)^{n-k}}{\prod\limits_{i\neq k} |\tau_k - \tau_i|} \; \frac{[\prod_{i\neq k} (\xi - \tau_i)(\xi - \tau_k)]'}{\xi - \tau_k}$$

$$= \frac{(-1)^{n-k}}{\prod\limits_{i\neq k} |\tau_k - \tau_i|} \; \left(\frac{R_{n+1}(\xi)}{(\xi - \tau_k)}\right)'$$

$$= \frac{(-1)^{n-k}}{\prod\limits_{i\neq k} |\tau_k - \tau_i|} \cdot \frac{R'_{n+1}(\xi)(\xi - \tau_k) - R_{n+1}(\xi)}{(\xi - \tau_k)^2} . \tag{3.7}$$

If $\xi < 0$, we note that all the nodes are on the right hand side of ξ, so $R'_{n+1}(\xi)(\xi - \tau_k) - R_{n+1}(\xi)$ does not change its sign with k, hence δ_k alternately changes its signs with k. The same conclusion holds if $\xi > 1$. Consequently for $\xi \notin [0,1]$, the extremal polynomial for F_ξ is the Čebyšev polynomials.

Let $(\sigma_j)_{j=1}^n$ be the extrema of $R_{n+1}(\xi) = \prod_{i=0}^{n}(\xi - \tau_i)$, where

$(\tau_i)_{i=0}^n$ are the nodes of Čebyšev polynomial. In the numerator of (3.7) at $\xi = \sigma_j$,

$$R'_{n+1}(\sigma_j)(\sigma_j - \tau_k) - R_{n+1}(\sigma_j) = - R_{n+1}(\sigma_j)$$

does not change its sign with k, hence $(\delta_k)_{k=0}^n$ changes its sign alternately. Consequently for each σ_j; the derivative functional F_{σ_j} has for its extremal polynomial one of the polynomials $\pm T_n(x)$. Furthermore since δ_k is a continuous function of ξ, and $\delta_k \neq 0$, when we fix $\xi = \sigma_k$; we note that all δ_k will remain non-zero and keeps its sign changing alternately with k over an interval containing σ_j. This means that if one of the polynomials $\pm T_n(x)$ is extremal at $\xi = \sigma_j$, then it is extremal over an interval containing σ_j. These intervals we have called the Čebyšev intervals. One should note that none of the intervals would contain any of τ_k because $T'_n(\tau_k) = 0$, $k = 1,\dots,n-1$. This implies that there are n seperate intervals $[\alpha,\beta]$ such that for ξ in these intervals the Čebyšev polynomial is extremal. We also observe that when $\xi = 0$ or $\xi = 1$ we have

$$F_{\xi=0} = (0,1,0,\dots,0) \quad \text{and} \quad F_{\xi=1} = (0,1,2,\dots,n)$$

respectively, and so $\delta_k \neq 0$ for all k and changes its sign alternately. Hence we have two numbers $\alpha < 1$ and $\beta > 0$ such that for each ξ in the two intervals $(\alpha,+\infty)$ and $(-\infty,\beta)$ called the <u>boundary Čebyšev intervals</u>, $\pm T_n(x)$ is extremal for F_ξ. □

We are now going to describe the end points of the Čebyšev intervals $[\alpha,\beta]$. Of course, for the end points at least one of the δ_k must vanish and then the Čebyšev polynomial ceases to be extremal. We will show that at the end points of $[\alpha,\beta]$, the first δ_k to vanish is either δ_0 or δ_n . For this we need:

3.12 Theorem [, TM 63, p. 158] For each k $(k = 0,1,\ldots,n)$ put

$$R'_{n+1}(\xi)(\xi - \tau_k) - R_{n+1}(\xi) = \Delta_k(\xi) .$$

I) Suppose $R'_{n+1}(\xi) > 0$ then if $\Delta_0(\xi) \leq 0$ we have $\Delta_k(\xi) < 0$ $(k \neq 0)$; if $\Delta_n(\xi) \geq 0$ then $\Delta_k(\xi) > 0$ $(k \neq n)$. Furthermore if $\Delta_k(\xi) = 0$ then $\Delta_{k-i}(\xi) > 0$ and $\Delta_{k+i}(\xi) < 0$ $(i > 0,\ k \neq 0,n)$

II) Suppose $R'_{n+1}(\xi) < 0$ then if $\Delta_0(\xi) \geq 0$ we have $\Delta_k(\xi) > 0$ $(k \neq 0)$, if $\Delta_n(\xi) \leq 0$ then $\Delta_k(\xi) < 0$ $(k \neq n)$. If $\Delta_k(\xi) = 0$ for $k \neq 0,n$ then $\Delta_{k-i}(\xi) < 0$ and $\Delta_{k+i}(\xi) > 0$.

Proof: Let $R'_{n+1}(\xi) > 0$. Since $(\xi - \tau_0) > (\xi - \tau_k)$ for $k \neq 0$ and for any $\xi \in (0,1]$.

$$R'_{n+1}(\xi)(\xi - \tau_0) - R_{n+1}(\xi) \geq R'_{n+1}(\xi)(\xi - \tau_k) - R_{n+1}(\xi)$$

from where $\Delta_k(\xi) < 0$ if $\Delta_0(\xi) \leq 0$. Further, we note that $(\xi - \tau_n) < (\xi - \tau_k)$ for $\xi \neq n$ and for any $\xi \in [0,1)$,

$$R'_{n+1}(\xi)(\xi - \tau_k) - R_{n+1}(\xi) < R'_{n+1}(\xi)(\xi - \tau_n) - R_{n-1}(\xi) .$$

Hence $\Delta_k(\xi) > 0$ if $\Delta_n(\xi) \geq 0$. Since $\tau_{k-i} < \tau_k < \tau_{k+1}$, we have $\xi - \tau_{k-1} > \xi - \tau_k > \xi - \tau_{k+1}$ and so

$$\Delta_{k-i}(\xi) > \Delta_k(\xi) > \Delta_{k+i}(\xi) .$$

If $\Delta_k(\xi) = 0$, $\Delta_{k-i}(\xi) > 0$ and $\Delta_{k+i}(\xi) < 0$. The proof of (II) follows in a similar fashion. □

3.13 Corollary 1 [7, Corollary 1, p. 159] Let α and β be the left-hand and right-hand ends of some Čebyšev interval. Then at the ends of $[\alpha,\beta]$ one of the boundary nodes loses its weight. Further for $\xi = \alpha$ $\delta_0 = 0$ and for $\xi = \beta$ $\delta_n = 0$.

Proof: Suppose that for $\xi = \alpha$, $\delta_k = 0$ for $k \neq 0,n$ then $\Delta_k(\xi) = 0$. In case $R'_{n+1}(\xi) > 0$ (when $R'_{n+1}(\xi) < 0$ the argument is the same), from Theorem 3.12, we have $\Delta_{k-1}(\xi) > 0$ and $\Delta_{k+1}(\xi) < 0$. Therefore

$$\operatorname{sgn} \delta_{k-1} = -\operatorname{sgn} \delta_{k+1}$$

because

$$\delta_{k-1} = \frac{(-1)^{n-k+1}\Delta_{k-1}(\xi)}{\prod_{i \neq k} |\tau_i - \tau_k|} \quad ;$$

$$\delta_{k+1} = \frac{(-1)^{n-k-1}\Delta_{k-1}(\xi)}{\prod_{i \neq k} |\tau_i - \tau_k|} \quad .$$

We note that for $\xi \in [\alpha,\beta]$ but sufficiently close to α, the Čebyšev polynomial is extremal and δ_i alternates. By the use of continuity of δ_i, we must have $\operatorname{sgn} \delta_{k-1} = -\operatorname{sgn} \delta_k = \operatorname{sgn} \delta_{k+1}$. Consequently for $\xi = \alpha$ $\delta_0 = 0$ or $\delta_n = 0$.

Finally we show that for $\xi = \alpha$, $\delta_n \neq 0$ thus $\delta_0 = 0$. Since $R_{n+1}(\xi) = \prod_{i=0}^{n}(\xi - \tau_i)$ has simple zeros at τ_i and $R'_{n+1}(\sigma_i) = 0$ where $(\sigma_i)_{i=1}^{n}$ are the extrema of $R_{n+1}(\xi)$, each Čebyšev interval $[\alpha,\beta]$ contains only one σ_i, i.e. $\tau_i < \alpha < \sigma_i < \beta < \tau_{i+1}$. Thus in the left of σ_i, at α we must have $R'_{n+1}(\alpha) > 0$ if $R_{n+1}(\alpha) > 0$ or $R'_{n+1}(\alpha) < 0$ if $R_{n+1}(\alpha) < 0$; see Figure 2.

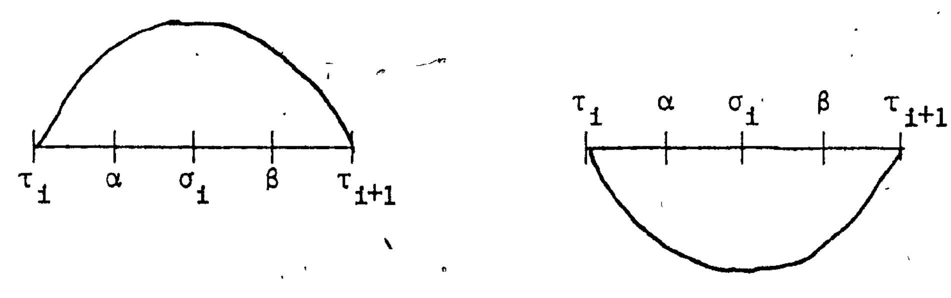

FIGURE 2

As $(\alpha - \tau_n) < 0$, we conclude that $\Delta_n(\alpha) = R'_{n+1}(\alpha)(\alpha - \tau_n) - R_{n+1}(\alpha)$

is either positive or negative but never zero. Hence corresponding to α, $\delta_n \neq 0$. The other case corresponding to β is dealt similarly. □

It follows from corollary 1 that by setting $\delta_0 = 0$ in (3.8) the roots of $R'_{n\ 1}(\xi)\xi - R_{n+1}(\xi) = 0$ (except for $\xi = 0$) are in fact the left end point α of the Čebyšev intervals. Thus we have:

Corollary 2. [7, Cor 2 p. 159] The end points of every Čebyšev interval $[\alpha,\beta]$ are respectively the roots of

$$R'_{n+1}(\alpha) \cdot \alpha - R_{n+1}(\alpha) = 0$$

$$R'_{n\ 1}(\beta)(\beta - 1) - R_{n+1}(\beta) = 0$$

with the double root $\alpha = 0$ excluded from the first equation, and the double root $\beta = 1$ excluded from the second. □

3.14 Remark Recall that Markov has shown that for $\xi \in [0,1]$ the extremal polynomial is $T_n(x)$ iff (1.12) and (1.13) hold i.e.

$$\frac{T''_n(\xi)}{T'_n(\xi)} + \frac{1}{\xi} > 0 \qquad (1.12)$$

and

$$\frac{T''_n(\xi)}{T'_n(\xi)} + \frac{1}{\xi - 1} < 0 \qquad (1.13)$$

If for a given $\xi \in [0,1]$,(1.12) and (1.13) hold, then by continuity it also holds over an interval containing ξ , and at the end points of the interval the right hand side of (1.12) or (1.13) must become zero. So, Markov's observation should conform with corollary 2 of 3.13: Since

$$R(\xi) = n\,2^{2n-1}\, T'_n(\xi)\,\xi(\xi - 1)$$

we have

$$R'(\xi)\xi - R(\xi) = n\, 2^{2n-1}\{T''(\xi)\xi^2(\xi-1) + T_n'(\xi)(2\xi-1)\xi - T_n'(\xi)\xi(\xi-1)\}$$
$$= n\, 2^{2n-1}\xi^2\{T_n''(\xi)(\xi-1) + T_n'(\xi)\}$$
$$= n\, 2^{2n-1}\frac{\xi}{T'(\xi)(\xi-1)}\{\frac{T_n''(\xi)}{T_n'(\xi)} + \frac{1}{\xi-1}\}$$

and

$$R'(\xi)(\xi-1) - R(\xi) = n\, 2^{n-1}\{T_n''(\xi)\xi(\xi-1)^2 + T_n'(\xi)(2\xi-1)(\xi-1) - T_n'(\xi)\xi(\xi-1)\}$$
$$= n\, 2^{2n-1}(\xi-1)^2\{T_n''(\xi)\xi + T_n'(\xi)\}$$
$$= n\, 2^{2n-1}\frac{(\xi-1)^2}{T_n'(\xi)\xi}\{\frac{T_n''(\xi)}{T_n'(\xi)} + \frac{1}{\xi}\} .$$

Of course $T_n'(\xi) \neq 0$ hence the assertion is verified.

The Čebyšev intervals we will denote by

$$[0,\beta_1], [\alpha_2,\beta_2],\ldots,[\alpha_{n-1},\beta_{n-1}], \qquad [\alpha_n,1] \equiv (I_i)_{i=1}^n \equiv E_T$$

where $\alpha_1 = 0$, $\beta_n = 1$. Now we are going to show that for $\xi \in [\alpha_n,1]$, F_ξ has its extremal $+T_n(x)$ and for $\xi \in [\alpha_{n-1},\beta_{n-1}]$ F_ξ has its extremal $-T_n(x)$ and so on alternately. We note that at $\sigma_n \in [\alpha_n,1]$, the extrema of $R_{n+1}(x)$, $R_{n+1}(\sigma_\eta) < 0$ because $R_{n+1}(x) > 0$ for $x > 1$ and $\tau_n = 1$ is a simple zero of $R_{n+1}(x)$. Thus corresponding to σ_n, from (3.7)

$$\delta_n = \frac{-R_{n+1}(\sigma_n)}{\prod_{i\neq k}|\tau_i - \tau_k|(\sigma_n - \tau_n)^2} > 0 .$$

This gives that the value of the Čebyšev polynomial (which is extremal) at $\tau_n = 1$ must be $+1$. Thus $+T_n(x)$ is extremal for the entire interval $[\alpha_n,1]$. Since at $\sigma_{n-1} \in [\alpha_{n-1},\beta_{n-1}]$, $R_{n+1}(\sigma_{n-1}) > 0$ we have $-T_n(x)$ as extremal for $\xi \in [\alpha_{n-1},\beta_{n-1}]$, and so on.

We now give a complete description of the norm of the derivative functional F_ξ over the Čebyšev interval. At this stage we need:

3.15 Theorem [7, p. 157] If $Q(x)$ is an extremal polynomial for F_ξ, then the polynomial $Q(1-x)$ is extremal for the segment functional

$$(U_{0,i})_{i=1}^{n} = 0, -1, -2(1-\xi), \dots, -n(1-\xi)^{n-1}$$

and consequently $N(\xi) = N(1-\xi)$. We need the following lemmas:

Lemma 1 [7, p. 3] If $(U_i)_{i=0}^{\infty}$ is a moment sequence then

$$U_{m,n} = U_{m,0} - \binom{n}{1} U_{m+1,0} + \binom{n}{2} U_{m+2,0} - \dots - (1)^n U_{m+n} \tag{3.8}$$

Proof: The proof is by induction on n. The number m is arbitrary. For $n = 0$ we have $U_{m,0} = U_{m,0}$ for all m. We next assume that for n no matter what m is (3.8) holds. We will show that

$$U_{m,n+1} = U_{m,0} - \binom{n+1}{1} U_{m+1,0} + \binom{n+2}{2} U_{m+2,0} + \dots (-1)^{n+1} U_{m+n+1,0}.$$

Since $U_{m,n+1} = U_{m,n} - U_{m+1,n}$ we obtain

$$\begin{aligned} U_{m,n} - U_{m+1,n} &= U_{m,0} - \binom{n}{0} U_{m+1,0} + \binom{n}{2} U_{m+2,0} - \dots (-1)^n U_{m+n,0} \\ &\quad - [U_{m+1,0} - \binom{n}{1} U_{m+2,0} + \binom{n}{2} U_{m+3,0} - \dots - (-1)^n U_{m+n+1,0}] \\ &= U_{m,0} - [\binom{n}{1} + \binom{n}{0}] U_{m+1,0} + [\binom{n}{2} + \binom{n}{1}] U_{m+2,0} \\ &\quad - \dots - (-1)^{n+1} U_{m+n+1,0}. \end{aligned}$$

Since $\binom{p+1}{q} = \binom{p}{q} + \binom{p}{q-1}$ we obtain

$$\begin{aligned} U_{m,n} - U_{m+1,n} &= U_{m,0} - \binom{n+1}{1} U_{m+1,0} + \binom{n+1}{2} U_{m+1,0} \\ &\quad - \dots (-1)^{n+1} U_{m+n+1,0} \\ &= U_{m,n+1}. \end{aligned}$$

□

Lemma 2 If $(U_i)_{i=0}^{n} = 0, 1, 2\xi, 3\xi^2, \dots, n\xi^{n-1}$ then

$$(U_{0,1})_{i=0}^{n} = 0, -1, -2(1-\xi), -3(1-\xi)^3, \dots -n(1-\xi)^{n-1}$$

Proof: From Lemma 1 we have

$$U_{0,n} = 0 - \binom{n}{1} + \binom{n}{2}2\xi - \binom{n}{3}3\xi^2 + \ldots (-1)^n n\xi^{n-1}$$

$$= -n + n\binom{n}{1}\xi - n\binom{n}{2}\xi^2 \ldots (-1)^n n\xi^{n-1}$$

$$= -n\{1 - \binom{n}{1}\xi + \binom{n}{2}\xi^2 - \ldots (-1)^n (-\xi)^{n-1}\}$$

$$= -n(1-\xi)^{n-1} \quad .$$

Proof of Theorem 3,15 By the definition of a moment sequence there exists a function $H(t)$ of bounded variation such that

$$U_m = \int_0^1 t^m \, dH(t) \quad .$$

We shall show that

$$U_{m,n} = \int_0^1 t^m (1-t)^n \, dH(t) \quad .$$

This we show by induction on n . Since $U_{m,0} - U_{m+1,0} = U_{m,1}$ we obtain that

$$U_{m,0} - U_{m+1,0} = \int_0^1 t^m \, dH(t) - \int_0^1 t^{m+1} \, dH(t)$$

$$= \int_0^1 t^m (1-t) \, dH(t) \quad .$$

We assume that

$$U_{m,n} = \int_0^1 t^m (1-t)^n \, dH(t) \; .$$

We will show that

$$U_{m,n+1} = \int_0^1 t^m (1-t)^{n+1} \, dH(t) \quad .$$

Since $U_{m,n+1} = U_{m,n} - U_{m+1,n}$ then

$$U_{m,n+1} = \int_0^1 t^m (1-t)^n \, dH(t) - \int_0^1 t^{m+1} (1-t)^n \, dH(t)$$

$$= \int_0^1 (t^m (1-t)^n - t^{m+1} (1-t)^n) \, dH(t)$$

$$= \int_0^1 t^m (1-t)^{n+1} \, dH(t) \quad .$$

Since m was arbitrary we have for $m = 0$

$$U_{0,n} = \int_0^1 (1-t)^n \, dH(t) .$$

We replace $(1-t)$ by γ , hence

$$U_{0,n} = \int_0^1 \gamma^n \, d\,[H(1) - H(1-\gamma)]$$

$$= \int_0^1 \gamma^n \, dh(\gamma) ,$$

where

$$h(\gamma) = H(1) - H(1-\gamma) .$$

Consequently the integrator function $h(\gamma)$ has discontinuities at the points $(1-\sigma_i)$ with jumps δ_i, where the σ_i's are the discontinuities of $H(t)$. Thus for the extremal polynomial $Q(x)$, for the functional F_ξ we have

$$N(\xi) = F_\xi(Q(x)) = \int_0^1 Q(x)\, dH(x) = \Sigma\, |\delta_i| = \Sigma\, \delta_i\, Q(\sigma_i)$$

$$= \Sigma\, \delta_i\, Q(1-(1-\sigma_i)) = \int_0^1 Q(1-x)\, dh(x) = F_{1-\xi}(Q(1-x))$$

$$= N(1-\xi) . \qquad \square$$

From Theorem 3.15 it is enough to consider the norm on half the interval $[0,1]$ i.e. $[\frac{1}{2},1]$.

<u>3.16 Theorem</u> [7, Theorem 65, p. 162] <u>Let $(\gamma_i)_{i=1}^{n-2}$ be the zeros of $T''_n(x)$ and $(I_i)_{i=2}^{n-1}$ be the Čebyšev intervals. Then</u> $\gamma_i \in I_i$ $(i = 2,\ldots,n-2)$. <u>That is each interior Čebyšev interval contains only one zero of $T''_n(x)$</u> .

<u>Proof</u>: Let $[\alpha,\beta]$ be one of the interior Čebyšev intervals with $\alpha > \frac{1}{2}$. It follows from (1.10) that

$$n\, 2^{2n-1} R_{n+1}(x) = T'_n(x)\, x(x-1) , \tag{3.9}$$

also $R_{n+1}(x)$ and $-T'_n(x)$ have the same sign on $[0,1]$. Moreover, $\operatorname{sgn} R_{n+1}(\alpha) = \operatorname{sgn} R_{n+1}(\beta)$ because $\tau_i < \alpha < \beta < \tau_{i+1}$ and $T'_n(x)$

does not change its sign within consecutive nodes. Therefore from (3.9) we get

$$n\,2^{2n-1} R'_{n+1}(x) = T''_n(x)\,x\,(x-1) + T'_n(x)(2x-1) . \tag{3.10}$$

From (3.7) and (3.8) we obtain

$$R'_{n+1}(\alpha)\alpha = R_{n+1}(\alpha) , \tag{3.11}$$

and

$$R_{n+1}(\beta)(\beta-1) = R_{n+1}(\beta) . \tag{3.12}$$

By substituting (3.10) in (3.11) and (3.12) we get

$$n\,2^{2n-1} R_{n+1}(\alpha) = T''_n(\alpha)\alpha^2(\alpha - 1) + T'_n(\alpha)\alpha(2\alpha - 1) ,$$

and

$$n\,2^{2n-1} R_{n+1}(\beta) = T''_n(\beta)\beta(\beta-1)^2 + T'_n(\beta)(2\beta-1)(\beta-1) .$$

If $R_{n+1}(\alpha) > 0$, then from above we have $\operatorname{sgn} R_{n+1}(\alpha) = \operatorname{sgn} - T'_n(\alpha)$ so $T'_n(\alpha)\alpha(2\alpha-1) < 0$, hence we obtain that $T''_n(\alpha)\alpha^2(\alpha-1) > 0$ and $T''_n(\alpha) < 0$. If $R_{n+1}(\beta) > 0$, then from above we get $T'_n(\beta)(2\beta-1)(\beta-1) > 0$. From (3.9) we have $n\,2^{2n-1}R_{n+1}(\beta) > T'_n(\beta)(2\beta-1)(\beta-1)$ because $\beta > 2\beta-1$ that is $\beta < 1$. Thus $T''_n(\beta)\beta(\beta-1)^2 > 0$, and consequently $T''_n(\beta) > 0$. Since $\operatorname{sgn} R_{n+1}(\alpha) = \operatorname{sgn} R_{n+1}(\beta)$, there is a zero of $T''_n(x)$ between α and β. If $R_{n+1}(\alpha) < 0$ and $R_{n+1}(\beta) < 0$ the proof is similar. This completes the proof. □

We further note that when $\operatorname{sgn} R_{n+1}(\alpha) = \operatorname{sgn} R_{n+1}(\beta) = +\text{ve}$, the extremal polynomial is $-T_n(x)$ and $N(\xi) = -T'_n(\xi)$. In this case $-T''_n(\alpha) > 0$ and $-T''_n(\beta) < 0$, because of τ_k, the norm $N(\xi)$ takes its maximum at $\xi = \tau_k$. Thus we have:

<u>Corollary</u> [7, corollary, p. 164] <u>In each of the interior Čebyšev interval the norm takes its maximum</u>

$$N(\gamma_k) = |T_n'(\gamma_k)|$$

just once. In the boundary Čebyšev interval the norm decreases monotonically from outside in, that is

$$\max N(\xi) = + T_n'(1) = |T_n'(0)| \ .$$

3.17 Theorem [7, Theorem 66, p. 163] Let $(\sigma_i)_{i=1}^n$ be the roots of $T_n(x)$ then $\sigma_i \in I_i$ $(i = 1,2,\ldots,n)$ i.e. each Čebyšev interval contains exactly one root of $T_n(x)$.

Proof: Let $[\alpha,\beta]$ be an interior Čebyšev interval with $\alpha > \frac{1}{2}$. It can be verified that $T_n(x) = \cos n \arccos (2x-1)$ satisfy

$$x(1-x)T_n''(x) - (x-\tfrac{1}{2})T_n'(x) + n^2 T_n(x) = 0 \ . \qquad (3.13)$$

Let γ be a root of $T_n''(x)$ in this interval. Suppose that the extremal polynomial is $+T_n(x)$ then $T_n'(x) > 0$, (otherwise $-T_n(x)$ would be the extremal polynomial). It also follows from (3.13) that $T_n(\gamma) > 0$, hence for $\beta > \gamma$, $T_n(\beta) > 0$ because $T_n'(x) > 0$. We will show that $T_n(\alpha) < 0$. From (3.9) and (3.11) we have that $R_{n+1}'(\alpha) < 0$ and so from (3.10):

$$n\, 2^{2n-1} R_{n+1}'(\alpha) = T_n''(\alpha)\alpha(\alpha-1) + T_n'(\alpha)(2\alpha-1) \ .$$

We get $T_n''(\alpha) \neq 0$ (in fact > 0). Using (3.13), we have

$$\begin{aligned} n^2 T_n(\alpha) &= (\alpha-\tfrac{1}{2})T_n'(\alpha) - \alpha(1-\alpha)T_n''(\alpha) \\ &= (\alpha-\tfrac{1}{2})T_n'(\alpha) + \alpha(\alpha-1)T_n''(\alpha) \\ &= -\tfrac{3}{2}(\alpha-\tfrac{1}{2})T_n'(\alpha) + (2\alpha-1)T_n'(\alpha) + \alpha(\alpha-1)T_n''(\alpha) \\ &< 0 \ . \end{aligned}$$

Thus $T_n(\alpha) < 0$. Hence in each of the interior Čebyšev intervals $T_n(x)$ has one root.

For the last Čebyšev interval $[\alpha,1]$, $T_n'(x) > 0$, so $T_n'(\alpha) > 0$ and $T_n''(x) \neq 0$, thus again $T_n(\alpha) < 0$ where as $T_n(1) = 1 > 0$. □

<u>3.18 Remark</u> Since we are taking the derivative functional over [0,1], the Čebyšev polynomial is given by $T_n(x) = \cos n \arccos (2x-1)$ and its roots are $\sigma_i = \frac{1}{2}\cos(\frac{2i-1}{2n})\Pi + \frac{1}{2}$ $(i = 1,\ldots,n)$.

When n is odd there is a Čebyšev interval of the form $[\alpha, 1-\alpha]$ containing the point $\frac{1}{2}$. From (3.13) we obtain that $\frac{1}{2}$ is also the root of $T_n''(x)$. Hence Theorem 3.16 and Theorem 3.17 remain valid. In each of the other Čebyšev intervals to the right of $\frac{1}{2}$ the roots $(\gamma_i)_{i=1}^{n-2}$ of $T_n''(x)$ are such that $\gamma_i > \sigma_i$ where $(\sigma_i)_{i=1}^{n}$ are the roots of $T_n(x)$. This follows from (3.13) because $\operatorname{sgn} T_n(\gamma_i) = \operatorname{sgn} T_n'(\gamma_i)$ and $T_n'(x)$ does not change its sign. So $\operatorname{sgn} T_n(x) = \operatorname{sgn} T_n'(\gamma_i)$ for $\gamma_i < x$, thus $\sigma_i < \gamma_i$. Hence over a Čebyšev interval $[\alpha,\beta]$

$$N(\xi) = |T_n'(\xi)| = \frac{n|\sin n \arccos (2\xi-1)|}{\sqrt{\xi(1-\xi)}}$$

We can easily see the following inequality

$$N(\xi) \leq \frac{n}{\sqrt{\xi(1-\xi)}} \quad \text{for } \xi \ [\alpha,\beta] \ ,$$

with equality taking place only at the points where $|\sin n \arccos (2\xi-1)| = 1$, which are the zeros of $T_n(x) =$ $T_n(x) = \cos n \arccos (2x-1)$, (see Figure 3).

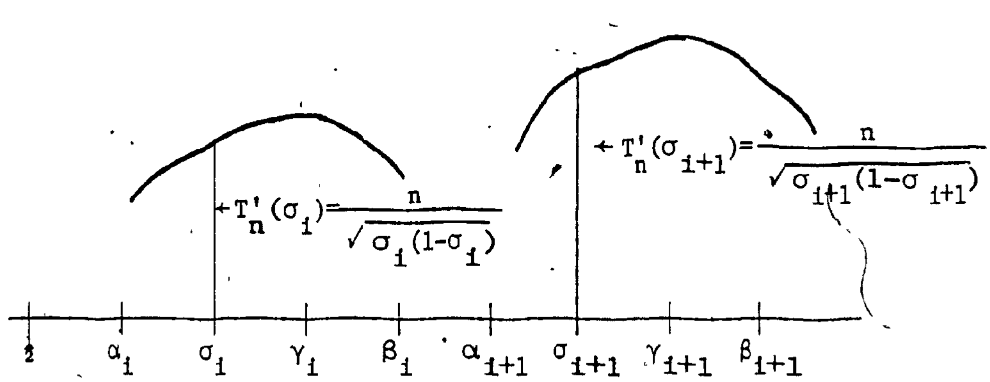

FIGURE 3

CHAPTER IV

MARKOV – VORONOVSKAJA THEOREM

PART II: ZOLOTOREV POLYNOMIALS ARE EXTREMALS

4.1 Introduction We have already seen in Chapter I that for $\xi \in [0,1]$ the extremal polynomial for the derivative functional F_ξ has n or $n+1$ alternating nodes. The case for $n+1$ nodes has been discussed in the preceding chapter. We are now going to discuss the case when the extremal polynomial has n alternating nodes.

Let $P(x)$ be a reduced polynomial of degree n having s nodes. If the signs associated with two consecutive nodes are the same then this interval is called the interval of repetition. The total number of the intervals of repetition we denote by P. Then the numbers n,s,P written as $[n,s,P]$ is called the passport of the polynomial $P(x)$.

The Čebyšev polynomials $\pm T_n(x)$ are the only polynomials with passport $[n,n+1,0]$. Hence outside the Čebyšev intervals the derivative functional F_ξ has as extremals those polynomials which are of passport $[n,n,0]$. We will investigate those polynomials in some detail by studying those properties that are needed for our work. We will show that the polynomials of passport $[n,n,0]$ form a family of polynomials depending on a single parameter, which can be taken to be the leading coefficients.

These polynomials take the form

$$\sigma x^n + y_{n-1}(\sigma)x^{n-1} + \ldots + y_1(\sigma)x + y_0(\sigma) \quad ,$$

where $-2^{2n-1} < \sigma < 2^{2n-1}$. For $\sigma = \pm 2^{2n-1}$ these are the Čebyšev polynomials. If $0 < \sigma < 2^{2n-1}$, we denote the family by $Q_n(x,\sigma)$. For $-2^{2n-1} < \sigma < 0$ we have the polynomials $(-1)^{n-1}Q_n(1-x,\sigma)$ and for $\sigma = 0$

$$Q(x,0) = -T_{n-1}(x) \quad .$$

The polynomials of passport $[n,n,0]$ are of the form $\pm Q_n(x,\sigma)$ and $\pm Q_n(1-x,\sigma)$. We call these the General Zolotorev Polynomials. They fall into two classes. One class consists of transformations of $T_n(x)$, that is they are of the form $\pm T_n(\nu x)$ and $\pm T_n(\nu(1-x))$, where $\cos^2 \frac{\pi}{2n} \leq \nu < 1$. The relation between the parameters ν and σ is given by

$$\sigma = 2^{2n-1} \nu^n .$$

The second class which we denote by $Z_n(x,\sigma)$, are called the Zolotorev polynomials. The collection of all such polynomials are $\pm Z_n(x,\sigma)$ and $\pm Z_n(1-x,\sigma)$. The polynomials $Z_n(x,\sigma)$ are connected with their resolvent

$$R_n(x,\sigma) = \prod_{i=1}^{n} (x-\sigma_i)$$

by the relation

$$\frac{\delta Z_n(x,\sigma)}{\delta\sigma} = R_n(x,\sigma) .$$

By the theorem on continuous deformation (Theorem 4.9) as σ decreases continuously from 2^{2n-1} to -2^{2n-1} the polynomial $Q_n(x,\sigma)$ is deformed continuously from $+T_n(x)$ to $-T_n(x)$ following the sequence $T_n(\nu x)$, $Z_n(x,\sigma)$, through $-T_{n-1}(x)$ then $(-1)^{n-1} Z_n(1-x,\sigma)$, $(-1)^{n-1} T_n(\nu(1-x))$, and finally $-T_n(x)$ as $\sigma = -2^{2n-1}$.

We start our investigations into the class of polynomials of passport $[n,n,0]$ by noting an important property of the polynomial of passport $[n,n+1,0]$. This will enable us to turn to segment functionals with a variable parameter, i.e. segment functionals of the form $U_0,U_1,\ldots,U_{n-1},\theta$. Furthermore we will show how we can construct the family of all polynomials of passport $[n,n,0]$ by means of simple segment functionals.

4.2 Theorem [7, Theorem 17, p. 42] Let $\bar{U}_n = (U_i)_{i=0}^n$ be a segment functional and suppose that $|T_n(\bar{U}_n)| < N_n$. Then we can find a number $h_1 > 0$ so large that $+T_n(x)$ is extremal for the segment functional

$$U_0, U_1, \ldots, U_{n-1}, U_n + h_1 .$$

We can also find a number $h_2 > 0$ such that $-T_n(x)$ is extremal for the segment functional

$$U_0, U_1, \ldots, U_{n-1}, U_n - h_2 .$$

Proof: Let

$$0 = \tau_0 < \tau_1 < \ldots < \tau_n = 1$$

be the nodes of $T_n(x)$ so that $T_n(\tau_i) = (-1)^{n-i}$, where $\tau_i = \sin^2(\frac{i\pi}{2n})$. We solve $\sum_{i=1}^{n} \delta_i \tau_i^k = U_k$ for $k = 0,1,2,\ldots,n$.

That is we solve for δ_i in

$$\begin{bmatrix} 1_0, 1_1, 1_2, \ldots, 1_n \\ \tau_0, \tau_1, \tau_2, \ldots, \tau_n \\ \tau_0^2, \tau_1^2, \tau_2^2, \ldots, \tau_n^2 \\ \vdots \qquad \vdots \qquad \vdots \\ \tau_0^n, \tau_1^n, \tau_2^n, \ldots, \tau^n \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \vdots \\ \delta_{n+1} \end{bmatrix} = \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix} .$$

By Cramer's rule we obtain

$$\delta_i = \frac{\begin{vmatrix} 1_0, 1_1, \ldots, 1_{i-1}, U_0, 1_{i+1}, \ldots, 1_n \\ \tau_0, \tau_1, \ldots, \tau_{i-1}, U_1, \tau_{i+1}, \ldots, \tau_n \\ \tau_0^2, \tau_1^2, \ldots, \tau_{i-1}^2, U_2, \tau_{i+1}^2, \ldots, \tau_n^2 \\ \vdots \quad \vdots \quad \vdots \\ \tau_0^n \; \tau_1^n, \ldots, \tau_{i-1}^n, U_n, \tau_{i+1}^n, \ldots, \tau_n^n \end{vmatrix}}{\begin{vmatrix} 1_0, 1_1, \ldots, 1_n \\ \tau_0, \tau_1, \ldots, \tau_n \\ \tau_0^2, \tau_1^2, \ldots, \tau_1^2 \\ \vdots \quad \vdots \quad \vdots \\ \tau_0^n, \tau_1^n, \ldots, \tau_n^n \end{vmatrix}} \qquad (4.1)$$

To simplify our work we introduce the following notations: $V_{n+1}(\tau_0,\ldots,\tau_n)$ is the Vandermonde determinant of $(\tau_i)_{i=0}^n$. $V_n^{(i)}(\bar{U}_n)$ is the determinant obtained by replacing the ith column of V_{n+1} by a column of the segment functional $\bar{U}_n$. $V_{n,i}$ is the minor of τ_i^n in the determinant $V_{n+1}(\tau_0,\ldots,\tau_n)$. Then (4.1) has the form

$$\delta_i = \frac{V_n^{(i)}(\bar{U}_n)}{V_{n+1}(\tau_0,\ldots,\tau_n)} \qquad (i = 0,1,\ldots,n) .$$

We now replace U_n by U_{n+h} . We thus obtain

$$\delta_i' = \frac{V_n^{(i)}(\bar{U}) + (-1)^{n-i} h \cdot V_{n,i}}{V_{n+1}(\tau_0,\ldots,\tau_n)} . \qquad (4.2)$$

The minor $V_{n,i}$ of τ_i^n in the determinant $V_{n+1}(\tau_o,\ldots,\tau_n)$, is itself a Vandermonde determinant. Hence

$$(-1)^{n-i} h \cdot V_{n,i} = (-1)^{n-i} h \cdot \prod_{1\le j\le k\le n} (\tau_k - \tau_j) \qquad k,j \neq i .$$

The product $\prod_{1\le j\le k\le n} (\tau_k - \tau_j)$ $k,j \neq i$ is always positive, since

$\tau_j > \tau_k$ for $j > k$. Thus we can see that in the numerator of (4.2) we can always take h so large that the numerator alternates in sign. That is $\operatorname{sgn} \delta_i' = \operatorname{sgn}(-1)^{n-i} = \operatorname{sgn} T_n(\tau_i)$. Therefore $+T_n(x)$ is an extremal polynomial for the segment functional $U_0, U_1, \ldots, U_{n-1}, U_n + h$. Similarly we can verify that the second assertion also holds. □

Corollary 1 [7, Corollary, p. 43] *If $+T_n(x)$ is an extremal polynomial for the segment functional $(U_i)_{i=0}^n$, it remains extremal if U_n is increased. However if $-T_n(x)$ is extremal it remains extremal if U_n is decreased.*

In the above (Theorem 4.2) let the number h_1'' be the **smallest** among h_1, and h_2'' be the **largest** among h_2. We denote $U_n'' = U_n + h_1''$ and $U_n' = U_n - h_2''$, clearly $U_n' \le U_n''$. We call the open interval (U_n', U_n'') the critical interval for U_n. Therefore the polynomials $\pm T_n(x)$ cannot be extremal for a segment $(U_i)_{i=0}^n$ if $U_n' < U_n < U_n''$.

Corollary 2 [7, Corollary 3, p. 44) *If the segment $(U_i)_{i=0}^n$ is such that the element U_n lies in its critical interval, that is $U_n' < U_n < U_n''$. Then a given polynomial $Q_n(x)$ other than $\pm T_n(x)$ is extremal at most for one value of U_n. In other words if $Q_n(x)$ is extremal it is so only for one value U_n* ***lying*** *in the critical interval (U_n', U_n'').*

Proof: Suppose $Q_n(x) \neq T_n(x)$ and let the number of nodes of $Q_n(x)$ be s. Assume $(\overset{\pm}{\sigma}_i)_{i=1}^{s_1}$ is the distribution of the segment $(U_i)_{i=0}^n$ for which $Q_n(x)$ is extremal. Therefore the number of nodes s is less than $n+1$ and s_1 is less than or equals s. Consequently the system

$$\sum_{i=1}^{s_1} \delta_i \sigma_i^k = U_k \qquad k = 0,1,\ldots,n \tag{4.3}$$

is overdetermined that is we have more equations than unknowns. Therefore the jumps δ_i can be determined by any s_1 equations from the system (4.3). Choose $U_{n_1} \neq U_{n_2}$ where $U'_n < U_{n_1} < U''_n$ and $U'_n < U_{n_2} < U''_n$. Assume $Q_n(x)$ is an extremal for the two segments:

$$U_0, U_1, \ldots, U_n, U_{n_1} \quad ,$$

$$U_0, U_1, \ldots, U_n, U_{n_2} \quad .$$

We solve for the jumps δ_i by choosing s_1 equations not one of which equals to U_{n_1} or U_{n_2}. Then we must have that

$$\sum_{i=1}^{s_1} \delta_i \sigma_i^n = U_{n_1} = U_{n_2} \quad ,$$

which is impossible. □

Theorem 4.2 can be generalized to any term U_k $(k > 0)$.

Furthermore for any segment functional

$$\bar{\alpha} \equiv 0_1, 0_2, \ldots, 0_{k-1}, \pm 1_k, 0_{k+1}, \ldots, 0_n \quad (k > 0)$$

the extremal polynomials are $\pm T_n(x)$. This can be directly verified by solving for δ_i in the system $\sum_{i=0}^{n} \delta_i \tau_i^j = \begin{cases} 0 & k = 0,1,\ldots k-1,k+1,\ldots n \\ 1 & j = k \end{cases}$ where $(\tau_i)_{i=0}^n$ are the nodes of $\pm T_n(x)$. Since $\bar{\alpha}(\sum_{k=1}^{n} a_k t^k) = a_k$, we have a simple proof of the following well known result;

<u>4.3 Theorem</u> <u>Let</u> $P(x) = \sum_{k=0}^{n} a_k x^k$ <u>be a polynomial of degree</u> n <u>such that</u> $|P(x)| \leq 1$ <u>on</u> $[0,1]$ <u>then</u>

$$|a_k| \leq |t_k|$$

<u>where</u> t_k <u>is the coefficient of the Čebyšev polynomial</u> $T_n(x) = \cos n \arccos (2x - 1) = \sum_{k=0}^{n} t_k x^k$. (see page 12).

Since the leading coefficient of $T_n(x) = 2^{2n-1}$ we immediately conclude that for all reduced polynomials $p(x) = \sum_{k=0}^{n} a_k x^k$,

$-2^{2n-1} \leq a_n \leq 2^{2n-1}$.

4.4 Theorem [7, Theorem 5, p. 21] For any segment $(U_i)_{i=0}^{n}$ if one of the moments U_k taken as a parameter is varied, then the norm $N_n(U_k)$ of the segment-functional is a continuous function of U_k .

Proof: Let h be a real number. We will examine $|N_n(u_k+h) - N_n(u_k)|$. Let $\max|P_k|$ be the largest k'th coefficient of all reduced polynomials belonging to Π_n . Fix U_k in the segment $(U_i)_{i=0}^{n}$ and then let $Q_n(x)$ be the extremal polynomial. The norm $N_n(U_k) = q_0U_0 + q_1U_1 +\ldots+ q_kU_k +\ldots+ q_nU_n$. Also let $Q_n^*(x) = q_0^* + q_1^*x +\ldots+ q_k^*x^i +\ldots+ q_n^*x^n$ be the extremal polynomial for the segment $U_0, U_1, \ldots, U_k + h, \ldots, U_n$. Hence the norm $N_n(U_k + h) = q_0^*U_0 + q_1^*U_1 +\ldots+ q_k^*(U_k + h) +\ldots+ q_n^*U_n$.

We now observe that

$$\begin{aligned} N_n(U_k) - \max|P_k|\cdot|h| &= q_0U_0 + q_1U_1 + \ldots + q_kU_k + \ldots + q_nU_n - \max|P_k|\cdot|h| \\ &\leq q_0U_0 + q_1U_1 + \ldots + q_k(U_k + h) + \ldots + q_nU_n \\ &\leq N_n(U_k+h) = q_0^*U_0 + q_1^*U_1 + \ldots + q_k^*(U_k+h) + \ldots + q_n^*U_n \\ &\leq q_0^*U_0 + q_1^*U_1 + \ldots + q_1^*U_k + \ldots + q_n^*U_n + \max|P_k|\cdot|h| \\ &\leq N_n(U_k) + \max|P_k|\cdot|h| . \end{aligned}$$

Thus we get

$$N_n(U_k) - \max|P_k|\cdot|h| \leq N_n(U_k + h) \leq N_n(U_k) + \max|P_k|\cdot|h| .$$

Thus $|N_n(U_k + h) - N_n(U_k)| < \max|P_k|\cdot|h|$.

Since the set $\{P_k\}$ of all the k'th coefficients from all the reduced polynomials belonging to Π_n is bounded by the norm N_n of the segment

$$\bar{\alpha} = 0_0, 0_1, \ldots, 0_{k-1}, 1_k, 0_{k+1}, \ldots, 0_n .$$

That is $Q_n(\bar{\alpha}) = P_k < \max|P_k| = N_n$, for all $Q_n(x) \in \Pi_n$ (see **Theorem 4.3**)
Therefore we have that for any $\varepsilon = N_n \cdot |h| > 0$ there exist a number $\delta \geq |h|$ such that $|N_n(U_k + h) - N_n(U_k)| \leq \varepsilon$ whenever $|U_k - (U_k + h)| < \delta$. This proves that the norm $N_n(U_k)$ is a continuous function of U_k. □

4.5 Definitions Suppose the segment $(U_i)_{i=0}^{n}$ is given. If we consider a moment U_k $(0 \leq k \leq n)$ as a variable, that is $U_k = \theta$. Then we denote the segment

$$U_0, U_1, \ldots, U_{k-1}, \theta, U_{k+1}, \ldots, U_n \equiv (U_{i,\theta=U_k})_{i=0}^{n} \equiv \bar{U}_{n,\theta=U_k} .$$

We call $\bar{U}_{n,\theta=U_k}$ a variable segment. The norm of a variable segment we denote by $N_n(U_k)$ as was done in Theorem 4.4.

4.6 Theorem [7, Theorem 19, p. 49] If the variable segment $\bar{U}_{n,\theta=U_k}$ does not become absolutely monotonic for any value of $U_k = \theta$ then there is a unique number U_k^* called the focus of the variable element U_k, such that at the k'th place the smallest norm is obtained when $U_k = U_k^*$.

Proof: Since $N_n(U_k)$ is continuous there is a number U_k^* that minimizes the norm. We now show its uniqueness. Suppose that there are two numbers $U_k^{(1)}$ and $U_k^{(2)}$ that minimize the norm $N_n(U_k)$. Then the segments

$$\overline{\nu_n} = U_0, \ldots, U_{k-1}, U_k^{(1)}, U_{k+1}, \ldots, U_n \quad ;$$

$$\overline{\lambda_n} = U_0, \ldots, U_{k-1}, U_k^{(2)}, U_{k+1}, \ldots, U_n \quad ,$$

have the same minimum norn $N_n(U_k^*)$. Take any $\alpha \geq 0$ and $\beta \geq 0$, not both zero, and construct a third segment

$$\overline{\xi_n} = U_0, \ldots, U_{k-1}, \frac{\alpha U_n^{(1)} + \beta U_n^{(2)}}{\alpha + \beta}, U_{k+1}, \ldots, U_n .$$

That is, we obtain $\overline{\xi_n}$ by multiplying $\overline{\nu_n}$ and $\overline{\lambda_n}$ respectively by $\frac{\alpha}{\alpha+\beta}$ and $\frac{\beta}{\alpha+\beta}$ and adding them term by term. Since the norm of the two segments does not exceed the sum of their norms, then the norm of $\overline{\xi_n}$ which we denote by $N_n^{(\overline{\xi_n})}$ satisfies

$$N_n^{(\overline{\xi_n})} \leq \frac{\alpha\, N_n(U_k^*)}{\alpha+\beta} + \frac{\beta\, N_n(U_k^*)}{\alpha+\beta} = N_n(U_k^*) .$$

Consequently $N_n^{(\overline{\xi_n})} = N_n(U_k^*)$. By a suitable choice of α and β the number $\frac{\alpha U_k^{(1)} + \beta U_k^{(2)}}{\alpha+\beta}$ can be any number in the closed interval $[U_k^{(1)}, U_k^{(2)}]$. It follows that every number in this interval minimizes the norm.

Let $Q_n(x) = \sum_{i=0}^{n} q_i x^i$ be an extremal polynomial of $\overline{\xi_n}$ with a fixed U_k where $U_k^{(1)} < U_k < U_k^{(2)}$. That is

$$Q_n(\overline{\xi_n}) = N_n^{(\overline{\xi_n})} .$$

We have three cases to consider;

<u>Case 1</u> Suppose that the coefficient $q_k < 0$. If we add a positive number h to U_k such that $U_k + h \leq U_k^{(2)}$ then the norm is not increased, but

$$Q_n(\overline{\xi_n}) = N_n(U_k^*) + q_k \cdot h > N_n(U_k^*)$$

which is impossible.

<u>Case 2</u> Suppose $q_k < 0$. If we substract a positive number h from U_k such that $U_k - h \geq U_k^{(1)}$, we obtain

$$Q_n(\xi_n) = N_n(U_k^*) + |q_k| \cdot h > N_n(U_k^*)$$

which is impossible.

<u>Case 3</u> Suppose $q_k = 0$ then for every $U_k \in (U_k^{(1)}, U_k^{(2)})$ we have

$$Q_n(\overline{\xi_n}) = N_n(U_k^*) \quad .$$

Thus we have that $Q_n(x)$ remains extremal for all U_k in the interval $(U_k^{(1)}, U_k^{(2)})$. Consequently by corollary 2 of 4.6 we conclude that since $Q_n(x) \not\equiv \pm T_n(x)$ then $U_k^{(1)} = U_k^* = U_k^{(2)}$. □

For the index $k > 0$, we call the open interval (U_k', U_k^*) left-hand part of the critical interval and the interval (U_k^*, U_k'') we call the right-hand part of the critical interval.

4.7 Theorem [7, Theorem 20, p. 50] If $\bar{U}_{n,\theta=U_k}$ is a variable segment with $U_k = \theta$ and $U_k' \leq \theta \leq U_k''$. Then the following holds;

1) The k'th coefficient $q_k(\theta)$ of any principal polynomial $Q_n(x,\theta) = \sum_{i=0} q_i(\theta)x^i$ increases on the whole critical interval (U_k', U_k'') with the property that in the left-hand part of the critical interval $q_k(\theta) < 0$ and in the right-hand part $q_k(\theta) > 0$.

2) The norm $N_n(\theta)$ is a continuous function which decreases monotonically in the left-hand part of the critical interval and increases monotonically in the right-hand part.

Proof: Let U_k^* be the focus of the variable segment $\bar{U}_{n,\theta=U_k}$. Choose $\theta = U_k^* + h$, h positive or negative number. Then by the definition of U_k^*

$$N_n(U_k^*) < N_n(U_k^* + h) \quad . \tag{4.4}$$

Let $Q_n(x, U_k^* + h) = \sum_{i=0}^{n} q_i(U_k^* + h)x^i$ be the extremal polynomial for the segment $\bar{U}_{n,\theta=U_k^*+h}$. Then

$$\begin{aligned} N_n(U_k^* + h) &= q_0(U_k^* + h)U_0 + q_1(U_k^* + h)U_1 + \ldots + q_k(U_k^* + h)[U_k^* + h] \\ &\quad + \ldots + q_n(U_k^* + h)U_n \; . \\ &= q_0(U_k^* + h)U_0 + q_1(U_k^* + h)U_1 + \ldots + q_k(U_k^* + h)U_k^* \\ &\quad + \ldots + q_n(U_k^* + h)U_n + q_k(U_k^* + h) \cdot h \end{aligned}$$

$$= Q_n(\bar{U}_{n,\theta=U_k^*}, U_k^* + h) + q_k(U_k^* + h) \cdot h \quad .$$

Hence

$$N(U_k^* + h) < N(U_k^*) + q_k(U_k^* + h) \cdot h \quad .$$

Since $Q_n(x, U_k^* + h)$ is not an extremal polynomial for the segment $\bar{U}_{n,\theta=U_k^*} = U_0, U_1, \ldots, U_k^*, \ldots, U_n$. Therefore

$$Q_n(\bar{U}_{n,\theta=U_k^*}) < N_n(U_k^*) \quad . \tag{4.5}$$

From (4.4) and (4.5) we obtain that if $q_k(\theta) > 0$ then $h > 0$ and if $q_k(\theta) < 0$ then $h < 0$. We must consider two cases the first when $U_k > U_k^*$ and the second when $U_k \leq U_k^*$.

<u>Case 1</u> Let $U_k > U_k^*$, we will show that the norm $N_n(U_k)$ increases to the right of the focus and that the coefficient $q_k(\theta)$ increases.

Take $\theta = U_k + h$ with $h > 0$ and denote the segment $U_0, U_1, \ldots, U_k, \ldots, U_n$ by $\bar{U}_n$. Let $Q_n(x, U_k) = \sum_{i=0}^{n} q_i(U_k) x^i$ be the extremal polynomial of $\bar{U}_n$, and let $L_n(x) = \sum_{i=0}^{n} \ell_i x^i$ be the extremal polynomial for the segment

$$\bar{\nu} = U_0, U_1, \ldots, U_k + h, \ldots, U_n \quad .$$

Therefore

$$\begin{aligned} L_n(\bar{\nu}) &= \ell_0 U_0 + \ell_1 U_1 + \ldots + \ell_k (U_k + h) + \ldots + \ell_n U_n \\ &= \ell_0 U_0 + \ell_1 U_1 + \ldots + \ell_k U_k + \ldots + \ell_n U_n + \ell_k \cdot h \\ &= L_n(\bar{U}_n) + \ell_k \cdot h \\ &= N_n(U_k + h) \quad . \end{aligned}$$

And

$$\begin{aligned} Q_n(\bar{\nu}, U_k) &= q_0(U_k)U_0 + q_1(U_k)U_1 + \ldots + q_k(U_k)(U_k + h) + \ldots + q_n(U_k) \cdot h \\ &= q_0(U_k)U_0 + q_1(U_k)U_1 + \ldots + q_k(U_k)U_k + \ldots + q_n(U_k)U_k + q_k(U_k) \cdot h \\ &= Q_n(\bar{U}_n, U_k) + q_k(U_k) \cdot h \\ &< L_n(\bar{\nu}) \quad . \end{aligned}$$

Therefore

$$N_n(U_k) + q_k(U_k) \cdot h < N_n(U_k + h) \quad .$$

Since $h > 0$ then we must have the coefficient $q_k > 0$. Hence

$$N_n(U_k + h) = L_n(\bar{U}_n) + \ell_k \cdot h$$
$$> N_n(U_k) + q_k(U_k) \cdot h \quad .$$

The function $L_n(x)$ is not an extremal polynomial for the segment $\bar{U}_n$. Therefore $L_n(\bar{U}_n) < N_n(U_k)$. Hence

$$N_n(U_k) + \ell_k \cdot h > L_n(\bar{U}_n) + \ell_k \cdot h > N(U_k) + q_k(U_k) \cdot h \quad .$$

Consequently the norm increases and $\ell_k > q_k$ to the right of the focus.

Case 2 We show that to the left of the focus U_k^* , the norm $N_n(U_k)$ decreases and that the coefficient $q_k(\theta)$ increases.

Let $U_k \leq U_k^*$ and put $\theta = U_k + h$ where $h < 0$. Let $\hat{L}_n(x) = \sum_{i=0}^{n} \hat{\ell}_i x^i$ be the extremal polynomial for the following segment; $\overline{P_n} = U_0, U_1, \ldots, U_k + h, \ldots, U_n,$ where $h < 0$. We then have

$$\hat{L}_n(\bar{P}_n) = \hat{L}_n(\bar{U}_n) + \hat{\ell}_k \cdot h$$

and

$$Q_n(\bar{P}_n, U_k) = Q_n(\bar{U}_n, U_k) + q_k \cdot h \; .$$

Since $Q_n(x, U_k)$ is not an extremal polynomial of the segment $\bar{P}_n$ we have

$$N_n(U_k) + q_k(U_k) \cdot h < N_n(U_k + h) \; .$$

Consequently

$$N_n(U_k + h) = \hat{L}_n(\bar{U}_n) + \hat{\ell}_k \cdot h > N_n(U_k) + q_k(U_k) \cdot h$$

and

$$N_n(U_k) + \hat{\ell}_k \cdot h > N_n(U_k) + q_k(U_k) h \quad .$$

Since $h < 0$ we have $q_k(\theta) < 0$. Observe that $N(U_k)$ decreases when q_k increases.

We have also shown that

$$q_k(\theta) \cdot h \leq N_n(\theta + h) - N_n(\theta) \leq q_k(\theta + h) \cdot h \quad . \qquad (h \gtrless 0)$$

Therefore if $h < 0$, $q_k(\theta + h) \cdot h > 0$ and $N_n(U_k + h) - N_n(U_k) > 0$. This means that $N_n(U_k)$ decreases monotonically in the left hand side of the focus U_k^* . If $h > 0$ then $q_k(\theta + h) \cdot h > 0$ and $N_n(U_k + h) - N_n(U_k) > 0$. This means that the norm $N_n(U_k)$ increases monotonically on the right hand side of the focus U_k^* . This completes the proof. □

4.8 Theorem [7, Theorem 27, p. 64] (Theorem on continuous deformation) Suppose $\bar{U}_{n,\theta=U_k} = (U_{i,\theta=U_k})_{i=0}^n$ is a variable segment functional with a variable element $\theta = U_k$ whose domain is the closed interval $[\alpha,\beta]$ such that $\bar{U}_{n,\theta=U_k}$ belongs to Class II. Then the principal polynomial

$$Q_n(x,\theta) = \sum_{i=0}^{n} q_i(\theta)x^i$$

is unique at each point $U_k = \theta$ such that $q_i(\theta)(i=0,1,\ldots,n)$ is continuous in the closed interval $[\alpha,\beta]$.

Proof: We fix a point θ_0 belonging to the closed interval $[\alpha,\beta]$. We will show that

$$\lim_{\theta\to\theta_0} q_i(\theta) = q_i(\theta_0) \ ,$$

this will prove continuity of $q_i(\theta)$. Suppose $Q_n(x,\theta)$ is a reduced polynomial. From Th 4.3, we have for each $i(0\leq i\leq n)$

$$|q_i| \leq |t_i|$$

where t_i are coefficients of the Čebyšev polynomial $T_n(x) = \sum_{i=0}^{n} t_i x^i$. Then for any variable moment θ_0 the function $q_i(\theta)$ attains its upper and lower bound on $[\alpha,\beta]$. (We consider two cases, the first is that each $q_i(\theta)$ has a limit as $\theta\to\theta_0$ and the second case is that for some i, $q_i(\theta)$ has no limit as $\theta\to\theta_0$.

<u>Case 1</u> Assume that each $q_i(\theta)$ has a limit as $\theta\to\theta_0$. Let

$$\lim_{\theta\to\theta_0} q_i(\theta) = P_i \quad (i = 0,1,\ldots,n). \tag{4.6}$$

Then

$$\lim_{\theta\to\theta_0} Q_n(x,\theta) = \sum_{i=0}^{n} P_i x^i = P_n(x) \quad .$$

For any closed interval $A \le x \le B$ we note the difference

$$|P_n(x) - Q_n(x)| = \left|\sum_{i=0}^{n} P_i x^i - \sum_{i=0}^{n} q_i(\theta)x^i\right|$$

$$= \left|\sum_{i=0}^{n}[P_i - q_i(\theta)]x^i\right|$$

$$\le \sum_{i=0}^{n} |P_i - q_i(\theta)| M \quad ,$$

where $M = \max_{[A,B]}\{|x^i|;\ i = 0,1,\ldots,n\}$. By (4.6) we may suppose that for each i, $|P_i - q_i(\theta)| < \frac{\varepsilon}{n+1} M$ for sufficiently small $|\theta - \theta_0|$. Therefore on $[A,B]$

$$|P_n(x) - Q_n(x,\theta)| < \varepsilon \quad . \tag{4.7}$$

$Q_n(x,\theta)$ is a reduced polynomial, that is $\max_{[0,1]} |Q_n(x,\theta)| = 1$. If there exists a point $x_0 \in [0,1]$ such that $|P_n(x)| > 1$ then by (4.7) for a sufficiently small $|\theta - \theta_0|$, we obtain that $|Q_n(x_0,\theta)| > 1$ which is impossible. Therefore $|P_n(x_0)|$ cannot be greater than 1. In fact we will show that there are points on the closed interval $[0,1]$ such that at those points $P_n(x) = 1$. Since the norm is a continuous function of the variable element $\theta = U_k$ we have

$$P_n(\bar{U}_{n,\theta=U_k}) = \lim_{\theta\to\theta_0} Q_n(\bar{U}_{n,\theta=U_k},\theta)$$

$$= \lim_{\theta\to\theta_0} N_n(\theta)$$

$$= N_n(\theta_0) \quad .$$

Furthermore every polynomial $P(x)$ of degree n satisfies the following inequality

$$P(\bar{U}_{n,\theta=U_k}) \leq N_n(\theta) \max_{[0,1]} |P(x)| \quad . \tag{4.8}$$

Since $|\underline{P}_n(x)| \leq 1$ for $0 \leq x \leq 1$ we must have $\max_{[0,1]} |\underline{P}_n(x)| = 1$ if (4.8) is to hold. Thus $\underline{P}_n(x)$ is an extremal polynomial when $\theta = \theta_0$. Since $\bar{U}_{n,\theta=U_k}$ is of class II then by Theorem 4.2 it must be that $\underline{P}_n(x) \equiv Q_n(x,\theta_0)$.

<u>Case 2</u> Assume that for some i, $\lim_{\theta\to\theta_0} q_i(\theta)$ does not exist. Choose a subsequence $\theta_{k_1}, \theta_{k_2}, \ldots, \theta_{k_n}, \ldots$ converging to θ_0 such that

$$\lim_{n\to\infty} q_i(\theta_{k_n}) = P_i \; .$$

Then the polynomial $Q_n(x,\theta_{k_n})$ tends to $\underline{P}_n(x)$ and by the same argument as in case 1 we have

$$P_n(x) \equiv Q_n(x,\theta_0) \; .$$

We now take a different subsequence $\theta_{P_1}, \theta_{P_2}, \theta_{P_3}, \ldots, \theta_{P_n}, \ldots$ converging to θ_0 such that

$$\lim_{n\to\infty} q_i(\theta_{P_n}) = \hat{P}_i \; .$$

We thus obtain $Q_n(x,\theta_{P_n})$ tending to $\hat{\underline{P}}_n(x) \not\equiv P_n(x)$. That is we have obtained a different extremal polynomial which is not possible. Thus we have $\lim_{\theta\to\theta_0} q_i(\theta) = q_i(\theta_0)$ for each coefficient. This means that $q_i(\theta)$ is continuous in the closed interval $[0,1]$. □

<u>4.9 Remark</u> We now consider the segment

$$\bar{\nu}_{n,\theta=\nu_n} = (\nu_{i,\theta=\nu_n})_{i=0}^{n} = 0_0, 0_1, \ldots, -1_{n-1}, \theta$$

and let (θ',θ'') be the critical interval $\bar{\nu}_{n,\theta=U_n}$. For $\theta \in (\theta',\theta'')$

the extremal polynomial is not $\pm T_n(x)$. We show that $\tilde{\nu}_{n,\theta}=U_n$ determines the family of polynomials of passport $[n,n,0]$ for $\theta \in (\theta',\theta'')$. The system

$$\sum_{i=0}^{s} \delta_i \sigma_i^k = U_k$$

where $U_k = 0$ for $k = 0,1,\dots,n-2$; $U_{n-1} = -1$, $U_n = \theta$ can have a unique solution if $s \geq n$. Since the possibility of $\pm T_n(x)$ is ruled out; $s = n$. Furthermore the jumps δ_i alternates. Hence the extremal polynomial is of passport $[n,n,0]$.

For any given reduced polynomial of passport $[n,n,0]$, with nodes $\sigma_0,\sigma_1,\dots,\sigma_{n-1}$ we can solve the system

$$\begin{aligned}
\delta_0 \quad &+ \delta_1 \quad + \dots + \delta_{n-1} = 0 \\
\delta_0\sigma_0 \quad &+ \delta_1\sigma_0 \quad + \dots + \delta_{n-1}\sigma_{n-1} = 0 \\
&\vdots \\
\delta_0\sigma_1^{n-1} + \dots &+ \delta_1\sigma_1^{n-1} + \dots + \delta_{n-1}\sigma_{n-1}^{n-1} = -1 ,
\end{aligned}$$

with alternating δ_i, thus determining $\theta = \sum_{i=0}^{n-1} \delta_i \sigma_i^n$ uniquely in the critical interval.

Hence, there is a one-one correspondence between $\theta \in (\theta',\theta'')$ and the family of all polynomials of passport $[n,n,0]$.

Any polynomial with n nodes in $[0,1]$ must have either end or both ends as its nodes, so the polynomials of passport $[n,n,0]$ are of two types. When 1 is not the node, they are of type $\pm T_n(\nu x)$ because if $P(\alpha) = 1$ for $\alpha > 1$ then $P(x) = \pm T_n(\ x\ )$. When 0 is not a node $P(x) = (-1)^{n-1}\ T_n(\nu(1-x))$. The other types are those when both the ends are the nodes, and cannot be of type one.

In order to describe how the polynomials changes its form with θ, we must determine the critical interval for θ in the segment

$\bar{\nu}_{n,\theta=\nu_n}$ given above.

4.10 Lemma [5, Theorem 10, p. 315] The critical interval for $(\nu_{i,\theta=\nu_n})_{i=0}^{n-1} = 0_0, 0_1, \ldots, 0_{n-2}, -1_{n-1}, \theta$ is (θ', θ'') where $\theta' = -\frac{1}{2}(n+1)$ and $\theta'' = -\frac{1}{2}(n-1)$.

Proof: We use the same technique as in Theorem 4.2. We use the formula $U_k = \sum_{i=1}^{n+1} \tau_i^k \delta_i$ $(i = 0,1,\ldots,n)$, where $\tau_i = \sin^2(\frac{i\Pi}{2n})$ are the nodes of $T_n(x)$. That is we solve for δ_i in

$$\begin{bmatrix} 1 & ,1 & ,\ldots,1 \\ \tau_0 & ,\tau_1 & ,\ldots,\tau_n \\ \tau_0^2 & ,\tau_1^2 & ,\ldots,\tau_n^2 \\ \cdot & \cdot & \cdots \\ \tau_0^{n-1} & ,\tau_1^{n-1} & ,\ldots,\tau_n^{n-1} \\ \tau_0^n & ,\tau_1^n & ,\ldots,\tau_n^n \end{bmatrix} \begin{bmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \\ \vdots \\ \delta_{n-1} \\ \delta_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ -1 \\ \theta \end{bmatrix}$$

We denote by $S_{n-m}^{(k)}$ the sum of the factors of $\tau_0, \tau_1, \ldots, \tau_{k-1}, \tau_{k+1}, \ldots, \tau_n$ taken $n-m$ at a time. Consequently

$$\delta_k = \frac{(-1)^{k-1} V_{n,k}[(-1)^{n-1} S_{n-(n-1)}^{(k)}(-1) + (-1)^n S_{n-n}^{(k)} \theta]}{V_{n+1}(\tau_0, \ldots, \tau_n)}$$

$$= \frac{(-1)^{k-1} V_{n,k}[(-1)^{n-2}(\tau_0 + \tau_1 + \ldots + \tau_{k-1} + \tau_{k+1} + \ldots + \tau_n) + (-1)^n \theta]}{V_{n+1}(\tau_0, \ldots, \tau_n)} \tag{4.9}$$

For $\delta_k = 0$ we obtain

$$(-1)^{n+k-2} V_{n,k} \cdot \theta = (-1)^{n+k-3} V_{n,k}(\tau_0 + \tau_1 + \ldots + \tau_{k-1} + \tau_{k+1} + \ldots \tau_n).$$

Hence

$$\theta = -1(\tau_0 + \tau_1 + \ldots + \tau_{k-1} + \tau_{k+1} + \ldots + \tau_n).$$

To obtain θ minimum we take $S_1^{(0)}$. Hence

$$\theta' = -(\tau_1+\tau_2+\ldots+\tau_n)$$

$$= -(\sin^2(\frac{\Pi}{2n}) + \sin^2(\frac{2\Pi}{2n}) + \sin^2(\frac{3\Pi}{2n}) + \ldots + \sin^2(\frac{n-3}{2n})\Pi$$

$$+ \sin^2(\frac{n-2}{2n})\Pi + \sin^2(\frac{n-1}{2n})\Pi + \sin^2\frac{\Pi}{n} \quad . \qquad (4.10)$$

Since $\sin x = \cos(\frac{\Pi}{2} - x)$ we have that $\sin(\frac{i\Pi}{2n}) = \cos(\frac{n-i}{2n})\Pi$.

Hence (4.10) equals to

$$-[\cos^2\frac{(n-1)\Pi}{2n} + \cos^2\frac{(n-2)\Pi}{2n} + \cos^2\frac{(n-3)\Pi}{2n} + \ldots + \sin^2\frac{(n-3)\Pi}{2n}$$

$$+ \sin^2\frac{(n-2)\Pi}{2n} + \sin^2\frac{(n-1)\Pi}{2n} + 1] \ .$$

If n is odd then the first $n-1$ terms add up to $\frac{n-1}{2}$. Therefore $\theta' = -(\tau_1+\tau_2+\ldots+\tau_n) = -[\frac{n+1}{2}]$. If n is even then the middle term of the first $n-1$ terms is $\frac{n}{2}$, and hence $\sin^2\frac{\Pi}{4} = \frac{1}{2}$. Therefore $\theta' = \tau_1+\tau_2+\ldots+\tau_n = -[\frac{n-2}{2}+\frac{1}{2}+1] = -[\frac{n+1}{2}]$.

Consequently $\theta' = -[\frac{n+1}{2}]$. To obtain θ maximum we take $S_1^{(n)}$, and in the same way we show that

$$\theta'' = -(\tau_0+\tau_1+\ldots+\tau_{n-1}) = -[\frac{n-1}{2}] \ .$$

Therefore the critical interval

$$(\theta',\theta'') = (-(\frac{n+1}{2}), -(\frac{n-1}{2})) \ . \qquad \square$$

<u>**4.11 Theorem** [5, Theorem 10, p. 315] Let $(\nu_{i,\theta=\nu_n})_{i=0}^{n}$ be the same variable segment functional that was introduced in Lemma 4.10. If we denote by $Q_n(x,\theta=\nu_n)$ the extremal polynomial corresponding to the interval $-\frac{n}{2} < \theta < -\frac{(n-1)}{2}$ then the extremal polynomials corresponding to $-\frac{(n+1)}{2} < \theta < -\frac{n}{2}$ are $(-1)^{n-1}Q_n(1-x,\theta=\nu_n)$ moreover the focus of $(\nu_{i,\theta=\nu_n})_{i=0}^{n}$ is $\theta^* = -\frac{n}{2}$.</u>

<u>Proof</u>: Suppose $\theta \in [-\frac{n}{2}, -\frac{(n-1)}{2}]$. We write $\theta = -\frac{n}{2} + \gamma$ where

$0 < \gamma < \frac{1}{2}$. Suppose that the polynomial

$$Q_n(x,\theta=\nu_n) = a_0(\theta) + a_1(\theta)x + \ldots + a_{n-1}(\theta)x^{n-1} + a_n(\theta)x^n$$

is the extremal for the segment $\bar{\alpha} = 0_0, 0_1, \ldots, -1_{n-1}, -\frac{n}{2} + \gamma$. Then for any reduced polynomial $P_n(x) = b_0 + b_1x + \ldots + b_nx^n$ we get

$$-a_{n-1} + (-\frac{n}{2}+\gamma)a_n > -b_{n-1} + (-\frac{n}{2}+\gamma)b_n \quad . \qquad (4.11)$$

Since the norm is a positive number and $-\frac{n}{2}+\gamma$ is negative, we must have $-a_{n-1}$ and a_n both negative.

Suppose now that the polynomial

$$(-1)^{n-1}Q_n(1-x,\theta=\nu_n) = (-1)^{n-1}\{a_0 + a_1(1-x) + \ldots + a_n(1-x)^n\} \quad (4.12)$$

is extremal for the segment functional $\bar{\beta} = 0_0, 0_1, \ldots, -1_{n-1}, -\frac{n}{2} - \gamma$. By expanding (4.12) and collecting terms we can write $(-1)^{n-1}Q_n(1-x,\theta=\nu_n)$ as

$$a_0 + \ldots + (a_{n-1} + n\,a_n)x^{n-1} - a_nx^n \ .$$

Hence $(-1)^{n-1}Q_n(\bar{\beta},\theta=\nu_n) > P_n(\bar{\beta})$ that is

$$-(a_{n-1} + n\,a_n) - a_n(-\frac{n}{2} - \gamma) > -b_{n-1} + b_n(-\frac{n}{2} - \gamma) \quad .$$

Since $b_n < 0$ then

$$-b_{n-1} + b_n(-\frac{n}{2} - \gamma) > -b_{n-1} + b_n(-\frac{n}{2} + \gamma) \ . \qquad (4.13)$$

We also have

$$-(a_{n-1} + n\,a_n) - a_n(-\frac{n}{2} - \gamma) = -a_{n-1} + a_n(-\frac{n}{2} + \gamma) \quad .$$

If we denote the norm of $\bar{\alpha}$ by $N_n(-\frac{n}{2}+\gamma)$ and the norm of $\bar{\beta}$ by $N_n(-\frac{n}{2}-\gamma)$ then we get by (4.11),(4.12) and (4.13) that

$$N_n(-\frac{n}{2}+\gamma) = N_n(-\frac{n}{2}-\gamma) \quad \text{for} \quad 0 < \gamma < \tfrac{1}{2} \quad .$$

By the fact that the norm is a continuous function of θ it follows that $N_n(-\frac{1}{2})$ is the minimum. That is the focus $\theta^* = -\frac{n}{2}$. □

<u>4.12 Theorem</u> [7, Theorem 39, p. 85 and 5, Theorem 11, p. 317]

<u>The family of polynomials</u> $Q_n(x,\theta=\nu_n)$ <u>disintegrate into two</u>

families of different form:

I) for $\theta \in \left(\frac{-(n-1)}{2\cos^2(\frac{\Pi}{2n})}, \frac{-(n-1)}{2}\right)$ the extremal polynomials are $T_n(\alpha x)$ where $\alpha = \frac{-(n-1)}{2\theta}$.

II) for $\theta \in \left(-\frac{n}{2}, -\frac{(n-1)}{2\cos^2(\frac{\Pi}{2n})}\right)$ we obtain a new family of polynomials which we denote as $Z_n(x,\theta)$. The points 0 and 1 always enter as their nodes and

$$\lim_{\theta \to -\frac{n}{2}+} Z_n(x,\theta) = -T_{n-1}(x) .$$

Proof of I: By (4. 9) the polynomial $T_n(x)$ loses its weight at $\tau_n = 1$, at the end point $\theta'' = -\frac{1}{2}(n-1)$ of the critical interval $(-\frac{1}{2}(n+1), -\frac{1}{2}(n-1))$. We also have that at $\tau_n = 1$, $T_n(x)$ preserves the signs of the weights δ_k at the other nodes. Consequently by Theorem 4.8 (on Continuous Deformation), the extremal polynomials are $T_n(\alpha x)$ with $\alpha > 0$. The nodes of $T_n(\alpha x)$ are

$$0 = \frac{\tau_0}{\alpha} < \frac{\tau_1}{\alpha} < \ldots < \frac{\tau_{n-2}}{\alpha} < \frac{\tau_{n-1}}{\alpha} ,$$

where the nodes of $T_n(x)$ are given by the formula

$\tau_i = \sin^2(\frac{i\Pi}{2n}) = \cos^2\frac{(n-i)\Pi}{2n}$, $(i = 0,1,\ldots,n)$. For $i = n-1$ we obtain $\tau_{n-1} = \sin^2\frac{(n-1)\pi}{2n} = \cos^2\frac{\Pi}{2n}$.

If the number of nodes is $s = n$ then

$$\frac{\cos^2(\frac{\Pi}{2n})}{\alpha} < 1 .$$

Therefore $\cos^2(\frac{\Pi}{2n}) \leq \alpha < 1$, and if $\cos^2(\frac{\Pi}{2n}) \leq \alpha < 1$ then clearly the number of nodes $s = n$. Hence

$$\cos^2(\frac{\Pi}{2n}) \leq \alpha < 1$$

is the exact interval in which $T_n(\alpha x)$ is extremal.

We first establish the relation between α and θ. The basis of $(\nu_{i,\theta=U_n})_{i=0}^{n}$ is $(\nu_i)_{i=0}^{n-1}$ $0_0, 0_1, \ldots, 0_{n-2}, -1_{n-1}$. We denote by Δ_i the numbers satisfying the equations

$$\nu_k = \frac{1}{\alpha^k} \sum_{i=0}^{n} \Delta_i \tau_i^k .$$

That is

$$\alpha^k \nu_k = \sum_{i=0}^{n-1} \Delta_i \tau_i^k .$$

But $\sum_{i=0}^{n-1} \Delta_i \tau_i^k$ is the decomposition of $0_0, 0_1, \ldots, 0_{n-2}, -\alpha^{n-1}, \alpha^n \theta$ in terms of the nodes of $T_n(x)$, with $\tau_n = 1$ omitted. Consequently, since $\theta' = \frac{-1(n-1)}{2}$ we obtain $-\frac{1}{2}(n-1) = \alpha\theta$. That is $\theta = -\frac{1}{2\alpha}(n-1)$. Since $\alpha = -\frac{n-1}{2\theta}$ and $\cos^2(\frac{\Pi}{2n}) \le \alpha < 1$ we obtain

$$\cos^2(\frac{\Pi}{2n}) \le -\frac{n-1}{2\theta} < 1 .$$

Hence

$$-\frac{n-1}{2\cos^2(\frac{\Pi}{2n})} \le \theta < -\frac{n-1}{2} . \qquad (4.14)$$

Consequently $T_n(\alpha x)$ are extremal only in the interval (4.14).

<u>Proof of II</u>: Since $Q_n(x,\theta=U_n)$ is extremal in the right hand part of the critical interval and $(-1)^{n-1} Q_n(1-x,\theta=U_n)$ in the left hand part $-(\frac{n+1}{2}) < \theta < -\frac{(n+1)}{2\cos^2(\frac{\Pi}{2n})}$. It is enough to consider the right hand part

$$-\frac{n}{2} < \theta < \frac{-(n-1)}{2\cos^2(\frac{\Pi}{2n})} .$$

In this interval the extremal polynomial cannot be $T_n(\alpha x)$ by Part I. These new polynomials of passport $[n,n,0]$ we denote by $Z_n(x,\theta)$. We now show that $Z_n(x,\theta) \to -T_{n-1}(x)$ as $\theta \to -\frac{n}{2}+$. From the formula $U_k = \sum_{i=1}^{n} \delta_i \sigma_i^k$, where $U_k = 0$, $k = 0, 1, 2, \ldots, n-2$, $U_{n-1} = -1$ and $U_n = \theta$. We obtain as in (4.9) that

$$\theta + \sigma_1 + \sigma_2 + \dots + \sigma_n = 0 \quad .$$

Since $(\cos^2 \frac{n-1-i}{2(n-1)} \Pi)_{i=0}^{n-1}$ are the nodes of $T_{n-1}(x)$. Then as in (4.10)

$$\sum_{i=1}^{n} \sigma_i = \sum_{i=1}^{n} \cos^2 \frac{n-1-i}{2(n-1)} \Pi = \frac{n}{2} \quad .$$

Therefore whenever $\theta = -\frac{n}{2}$ we obtain

$$\theta + \sum_{i=1}^{n} \sigma_i = 0 \quad .$$

Hence

$$\lim_{\theta \to -n/2+} Z_n(x,\theta) = - T_{n-1}(x) \quad . \qquad \square$$

4.13 Remark The family of all polynomials of passport $[n,n,0]$ we call the General Zolotorev polynomials, and they are of two types. The first type are $Q_n(x,\theta=U_n)$ and $Q_n(1-x,\theta=U_n)$. The second type we denote by $Z_n(x,\theta)$ and $Z_n(1-x,\theta)$. We also have that a Zolotorev interval is subdivided into the following four subintervals.

$$\left(-\frac{n+1}{2}, -n+\frac{n-1}{2\cos^2(\frac{\Pi}{2n})}\right), \left(-n+\frac{n-1}{2\cos^2(\frac{\Pi}{2n})}, -\frac{n}{2}\right),$$

$$\left(-\frac{n}{2}, -\frac{n-1}{2\cos^2(\frac{\Pi}{2n})}\right), \left(-\frac{n-1}{2\cos^2(\frac{\Pi}{2n})}, -\frac{n-1}{2}\right),$$

having respectively as extremal polynomials

$(-1)^{n-1}T_n(\alpha(1-x)) \equiv Q_n(1-x,\theta=U_n)$, $(-1)^{n-1}Z_n(1-x,\theta)$, $Z_n(x,\theta)$ and $T_n(\alpha x) \equiv Q_n(x,\theta=U_n)$. We remind the reader that we did not yet examine the Zolotorev polynomials of type $Z_n(x,\theta)$ and $Z_n(1-x,\theta)$ over the middle two Zolotorev subintervals given above. It is only necessary to discuss the polynomials $Z_n(x,\theta)$ defined by the segment $0_0, 0_1, \dots, 0_{n-1}, -1_{n-1}, \theta$ in the interval $-\frac{n}{2} < \theta < -\frac{(n-1)}{2\cos^2(\frac{\Pi}{2n})}$.

We will show that $Z_n(x,\theta)$ form a family of polynomials depending on a single parameter, which can be taken to be the leading coefficient.

That is the polynomial $Z_n(x,\theta)$ takes the form

$$\sigma x^n + y_{n-1}(\sigma)x^{n-1}+\ldots+y_n(\sigma)x + y_0(\sigma) \quad .$$

We know that for each θ in the critical interval (θ',θ''), we have one and only one Zolotorev polynomial. So this family depends on the single parameter θ. Let $Z_n(x,\theta) = \sum_{i=0}^{n} q_i(\theta)x^i$. In Theorems 4.7 and 4.8 we have that $Z_n(x,\theta)$, as a single valued function of θ has the property that its leading coefficient $q_n(\theta)$ is a single valued continuous function increasing with respect to θ in the critical interval (θ',θ''). Hence $\sigma = q_n(\theta)$ is in one-one correspondence with the family $Z_n(x,\theta)$ and so σ can be taken as a parameter for the family $\xi_n(x,\sigma) = Z_n(x,\theta)$ where σ and θ are inverse continuous monotonic functions. Put $\theta = \psi(\sigma)$ hence

$$Z_n(x,\theta) = \xi_n(x,\sigma) = \sigma x^n + y_{n-1}(\sigma)x^{n-1} + \ldots + y_1(\sigma)x + y_0(\sigma) \quad ,$$

where $y_i(\theta) = q_i(\psi(\sigma))$.

Concerning the properties of the coefficients $y_k(\sigma)$ we mention without giving a proof the following:

<u>4.14 Theorem</u> [7, Theorem 41, p. 95] <u>Denote the leading coefficient of</u> $Z_n(x,\theta)$ <u>by</u> σ, <u>and then take</u> σ <u>as a parameter, so that</u>

$$Z_n(x,\theta) \equiv \xi_n(x,\sigma) \; .$$

<u>Then</u> $\xi_n(x,\sigma)$ <u>has the following properties</u>:

I) <u>Its coefficients are differentiable functions of</u> σ, <u>and</u>

II) <u>the resolvent</u> $R_n(x,\sigma)$ <u>of</u> $\xi_n(x,\sigma)$ <u>equals to the derivative of</u> $\xi_n(x,\sigma)$ <u>with respect to</u> σ. <u>That is</u>

$$R_n(x,\sigma) = \frac{\partial}{\partial\sigma}\xi_n(x,\sigma) \; .$$

<u>4.15 Remark</u> We have seen that the polynomials of passport $[n,n,0]$ form a family of polynomials depending on a single parameter which can

be taken to be the leading coefficient. That is the polynomials of passport [n,n,0] take the form

$$\sigma x^n + y_{n-1}(\sigma)x^{n-1} + \ldots + y_1(\sigma)x + y_0(\sigma) \quad .$$

Since the leading coefficient of $T_n(x)$ is 2^{2n-1}, then the leading coefficient σ cannot be greater than 2^{2n-1} (see Theorem 4.3). Hence $-2^{2n-1} \le \sigma \le 2^{2n-1}$. In Theorem 4.12 we saw that $\pm T_n(\nu x)$ and $\pm T_n(\nu(1-x))$ are in the family of polynomials of passport [n,n,0] whenever $\cos^2(\frac{\Pi}{2n}) \le \nu < 1$. The relation between ν and σ is $\sigma = 2^{2n-1}\nu^n$. Hence for $0 < \sigma < 2^{2n-1}\cos^{2n}(\frac{\Pi}{2n})$ we have the second class of polynomials $Z_n(x,\theta)$.

We now investigate the derivative functional F_ξ at the points of [0,1] outside the Čebyšev intervals. We have already observed that the extremal polynomial for F_ξ must be of passport [n,n,0] . It is interesting to further observe that every polynomial of passport [n,n,0] must be extremal for the derivative functional corresponding to some ξ outside the Čebyšev interval. This follows from:

<u>4.16 Theorem</u> [7, Theorem 64, p. 61] <u>Each polynomial</u> $L_n(x)$ <u>of passport</u> [n,n,0] <u>is an extremal of</u> F_ξ <u>at precisely the</u> $n-1$ <u>points</u> ξ <u>at which the derivative of its resolvent is zero that is</u>

$$R_n'(\xi) = \sum_{k=0}^{n} \gamma_k \xi^k = 0$$

<u>Proof</u>: Suppose $L_n(x)$ is an arbitrary polynomial of passport [n,n,0] having $R_n(x) = \sum_{k=0}^{n} \gamma_k x^k$ as its resolvent. Let ξ_0 be the point where $R_n'(\xi_0) = 0$. We shall show that $L_n(x)$ is extremal for F_{ξ_0} . Let $(\sigma_i^{\pm})_{i=1}^{n}$ be the distribution of $L_n(x)$; we solve the system of n equations

$$\sum_{i=1}^{n} \delta_i \sigma_i^k = k\xi_0^{k-1} \quad k = 0,1,\ldots,n-1$$

to get

$$\delta_k = \frac{(-1)^{n-k} R_n(\xi_0)}{\Pi\,|\,\sigma_k - \sigma_i\,|\,(\xi_0 - \sigma_k)^2} \qquad (4.15)$$

recall (3.7) ; note $R_n'(\xi_0) = 0$. Hence δ_k alternates and so the second criteria for extremality (page 47) is satisfied. In order to establish that $L_n(x)$ is extremal, we are to show that δ_k obtained from (4.15) also satisfy

$$\sum_{i=1}^{n} \delta_i \sigma_i^n = n\,\xi_0^{n-1} \qquad (4.16)$$

i.e. the system is consistent. We see that

$$\begin{aligned}
F_{\xi_0}(L_n) &= R_n'(\xi_0) \\
&= \sum_{k=0}^{n} \gamma_k k \xi_0^{k-1} \\
&= \sum_{k=0}^{n-1} \gamma_k k \xi_0^{k-1} + \gamma_n n \xi_0^{n-1} \\
&= \sum_{k=0}^{n-1} \gamma_k \sum_{i=1}^{h} \delta_i \sigma_i^k + \gamma_n n \xi_0^{n-1} \\
&= \sum_{k=0}^{n} \gamma_k \sum_{i=1}^{n} \delta_i \sigma_i^k + \gamma_n n \xi_0^{n-1} - \gamma_n \sum_{i=1}^{n} \delta_i \sigma_i^n \\
&= \sum_{i=0}^{n} \delta_i \sum_{k=0}^{n} \gamma_k \sigma_i^k + \gamma_n \left(n \xi_0^{n-1} - \sum_{i=1}^{n} \delta_i \sigma_i^n\right) \\
&= \sum_{i=1}^{n} \delta_i R_n(\sigma_i) + \gamma_n \left(n \xi^{n-1} - \sum_{i=1}^{n} \delta_i \sigma_i^n\right) = 0 \ .
\end{aligned}$$

Since $R_n(\sigma_i) = 0$, because the zeros of the resolvent are the nodes of $L_n(x)$ and $R_n'(\xi_0)=0$. Hence (4.16) holds.

Thus as ξ varies over the Zolotorev interval, the entire family of polynomials of passport $[n,n,0]$ is described. As a consequence of the above theorem we have:

<u>Corollary</u> [7, Corollary, p. 162] <u>Since the derivative functional</u> F_ξ <u>loses its weight at</u> $\tau_n = 1$ <u>or</u> $\tau_0 = 0$ <u>at the end points</u> β <u>and</u> α ,

then as ξ varies from β to α within (β,α) the types of extremal polynomial passes from one general Zolotorev polynomial to another in the order indicated above (Remark 4.13). By the theorem on continuous deformation we see that as ξ varies the extremal polynomials passes through all Zolotorev polynomials ending with $\mp T_n(x)$ at α if $\pm T_n(x)$ is extremal at β.

We now give a description of the norm $N(\xi)$ over the Zolotorev intervals. From Theorem 4.16 and Theorem 4.12 each Zolotorev interval (β,α) contains a unique point, which we label ξ^* for which $-T_{n-1}(x)$ is extremal. We first see the following:

4.17 Theorem [7, Theorem 67, p. 165] Suppose (β,α) is a Zolotorev interval. Then the norm $N(\xi)$ varies monotonically at each point $\xi \in (\beta,\alpha)$ at which the second derivative of the extremal polynomial is not zero.

Proof: Without loss of generality we take $N(\beta) = +T_n'(\beta)$. By Theorem 4.16 there is a Zolotorev interval (β,A), where A satisfies the inequality $\beta < \xi < A < \xi^*$, and the extremal polynomials of F_ξ are $T_n(\nu\xi)$ for $\cos^2(\frac{\pi}{2n}) < \nu < 1$. That is we have (see Theorem 4.12)

$$N(\xi) = T_n'(\nu\xi)\cdot\nu \quad \text{for all} \quad \nu \quad (\cos^2(\tfrac{\pi}{2n}), 1). \tag{4.16}$$

Since the point ξ is a relative maximum of the norm N we differentiate (4.16) with respect to ξ, and we obtain

$$N'(\xi) = + T_n''(\nu\xi)\nu^2 . \tag{4.17}$$

But $T_n'(\nu\xi)\nu$ is also a relative maximum with respect to ν for fixed ξ, thus

$$T_n''(\nu\xi)\nu\xi + T_n'(\nu\xi) = 0 . \tag{4.18}$$

This means that

$$\frac{d}{d(\nu\xi)}\,(T_n'(\nu\xi)\nu\xi) = 0 \quad .$$

Consequently $\nu\xi$ is a constant. Since $\nu = 1$ for $\xi = \beta$ we get $\nu = \beta/\xi$, and from (4.18) we have

$$N'(\xi) = T_n''(\nu\xi)\nu^2 = -\,T'(\nu\xi)\,\frac{\beta}{\xi^2} \quad .$$

Since $T_n'(\nu\xi) > 0$, $N'(\xi) < 0$, i.e. $N(\xi)$ decreases. For the other Čebyšev transformations there is a similar proof that the norm is monotonic. Thus the extrema of $N(\xi)$ lies on the part of the Zolotorev interval (β,α) where F_ξ is served by the polynomial $Z_n(x,\theta)$. It remains to establish Theorem 4.17 for these polynomials.

Let $A < \xi_1 < \xi^*$ and let an extremal polynomial for F_{ξ_1} be $Z_n(x,\theta_{\xi_1})$ with resolvent $R_n(x,\theta_{\xi_1})$. Then

$$N(\xi_1) = \left(\frac{\partial Z_n(\xi,\theta_{\xi_1})}{\partial\xi}\right)_{\xi=\xi_1} \tag{4.19}$$

and by Theorem 4.16

$$\left(\frac{\partial R_n(\xi,\theta_{\xi_1})}{\partial\xi}\right)_{\xi=\xi_1} = 0 \tag{4.20}$$

Since we have

$$N'(\xi) = \frac{\partial^2 Z_n(\xi,\theta)}{\partial\xi^2} + \frac{\partial^2 Z_n(\xi,\theta)}{\partial\xi\,\partial\theta}\,\frac{d\theta}{d\xi} ,$$

by Theorem 4.14 and lines (4.19) and (4.20) we get

$$N'(\xi_1) = \left(\frac{\partial^2 Z_n(\xi,\theta)}{\partial\xi^2}\right)_{\substack{\xi=\xi_1\\ \theta=\theta_{\xi_1}}} + \left(\frac{\partial R_n(\xi,\theta)}{\partial\xi}\,\frac{d\theta}{d\xi}\right)_{\substack{\xi=\xi_1\\ \theta=\theta_{\xi_1}}}$$

$$= Z_n''(\xi_1,\theta_{\xi_1}) + 0 \, . \tag{4.21}$$

This completes the proof. □

Since, from one Čebyšev interval to the next Cebysev interval the norm $N(\xi)$ becomes larger, and, on the Zolotorev interval in between it first decreases, we must have a point $\xi_0 \in (\beta,\alpha)$ where $N(\xi)$ is a minimum. This is observed in the following:

4.18 Theorem [7, Theorem 68, p. 165] Suppose (β,α) is a Zolotorev interval. Then in each interval (β,α) there is a unique point $\xi_0 \in (\beta,\alpha)$ satisfying the property that

$$N'(\xi_0) = 0$$

and

$$N(\xi_0) = \min_{(\beta,\alpha)} N(\xi) \leq |T'_{n-1}(\xi^*)| .$$

Moreover if $\beta > \frac{1}{2}$ then $\beta < \xi_0 < \xi^*$, and if $\alpha < \frac{1}{2}$ then $\xi^* < \xi_0 < \alpha$.

Proof: From the definition of the resolvent $R_n(x,\theta)$ we have

$$Z'_n(x,\sigma) = \frac{n\sigma R_n(x,\sigma)(x-\lambda)}{x(x-1)} \tag{4.22}$$

where λ is the zero of $Z'_n(x,\sigma)$ outside $[0,1]$, and σ is the leading coefficient of $Z_n(x,\sigma)$, taken as a parameter

. Clearly λ is a function of σ that is $\lambda = \lambda(\sigma)$. From Remark 4.15 if σ decreases from $2^{2n-1}\cos^{2n}(\frac{\Pi}{2n})$ to zero, then by rewriting (4.22) as

$$\sigma = \frac{x(x-1)Z'_n(n,\sigma)}{n R_n(x,\sigma)(x-\lambda)}$$

we see that λ increases from 1 to ∞ . Let $Z''_n(\xi_0,\theta_{\xi_0}) = 0$. Since $Z'_n(\xi_0,\sigma_{\xi_0}) > 0$ it follows that $R_n(\xi_0,\sigma_{\xi_0}) > 0$ and from (4.21) and (4.22)

$$N'(\xi_0) = Z''_n(\xi_0,\sigma_{\xi_0}) = n\sigma R_n(\xi_0,\sigma_{\xi_0})\left[\frac{\xi-\lambda}{\xi(\epsilon-1)}\right]'_{\xi=\xi_0}$$

$$= n\sigma R_n(\xi_0,\sigma_{\xi_0})[\lambda(2\xi_0-1)-\xi_0^2] .$$

Consequently ξ_0 can be found from the equation

$$\lambda(2\xi_0 - 1) - \xi_0^2 = 0 \quad . \tag{4.23}$$

When the Zolotorev interval (β,α) is left of $\frac{1}{2}$ i.e. $\alpha < \frac{1}{2}$, there is no ξ_0 satisfying (4.23) as σ varies from $2^{2n-1}\cos^{2n}\frac{\Pi}{2n}$ to zero or ξ varies from A to ξ^*. Hence the necessary condition $\min N(\xi) = N(\xi_0)$ is possible only when $\beta > \frac{1}{2}$. This proves that $\beta < A < \xi_0 < \xi^*$. By the symmetry of $N(\xi)$ this proves that if $\alpha < \frac{1}{2}$ then $\xi^* < \xi_0 < \alpha$. We now find ξ_0 for $Z_n(x,\sigma)$ when $\beta > \frac{1}{2}$. For $\xi \in (A,\xi^*)$ λ increases monotonically hence we put $\lambda = \phi(\xi)$, that is we consider λ as a function of ξ. Therefore at the point ξ_0 we have simultaneously $\lambda = \phi(\xi_0)$ and $\lambda = \dfrac{\xi_0^2}{(2\xi_0-1)}$. Since the curve of $\lambda = \phi(\xi)$ increases monotonically from 1 to ∞ on (A,ξ^*) and the curve of $\lambda = \dfrac{\xi^2}{2\xi-1}$ decreases monotonically from $+\infty$ to 1 on $(\frac{1}{2},1)$, it follows that the point ξ_0 is the point of intersection and that is unique. This proves the theorem. □

We note that for n even the Zolotorev interval is of the form $(\beta,1-\beta)$. Hence we have $\xi_0 = \xi^* = \frac{1}{2}$ and

$$N(\xi_0) = \min N(\xi) = T'_{n-1}(\tfrac{1}{2}) = 2(n-1) .$$

We also note that two successive curves of $N(\xi)$ say $N_n(\xi)$ and $N_{n-1}(\xi)$ cannot have more than one intersection (see figure 4) in each Zolotorev interval for the derivative functional corresponding to $N_n(\xi)$, since otherwise there would be a contradiction of the theorem on the uniqueness of the extremal polynomial. The intersection takes place at ξ^* where

$$N_n(\xi^*) = |T'_{n-1}(\xi^*)| = N_{n-1}(\xi^*) .$$

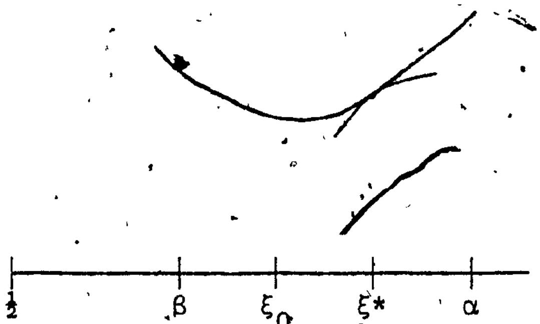

FIGURE 4

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