Absolutely Continuous Invariant Measures For A Class of Meromorphic Functions

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ABSTRACT

ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR A CLASS OF MEROMORPHIC FUNCTIONS

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We are going to consider a meromorphic function g: $\Re \to \Re$, which has a constant sign in the upper half plan. We will show that it has a special form

$$g(z) = A + \varepsilon\left[Bz - \sum_s p_s\left(\frac{1}{z-c_s} + \frac{1}{c_s}\right)\right]$$

where the poles are real and simples. Subsequently, we will demonstrate that it has an absolutely continuous invariant measure. Finally, we will present an example to emphasise the use of this transformation.

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To my parents
and my family.

Table of Contents

List of Symbols

$\Re$	The set of real numbers,
$\mathbf{C}$	The set of complex numbers,
π_+	The upper half plane,
π_-	The lower half plane,
τ	A transformation of the measure space $(\Re, \lambda)$,
λ	Denotes Lebesgue measure,
L	Line parallel to the real axis,
I	The unit interval = [0, 1].

INTRODUCTION

One of the most active areas in the ergodic theory and dynamical systems is the theory of absolutely continuous invariant measures (**acim**) with respect to Lebesgue measure λ for a transformation of an interval and the entire real line. In 1975, Kemperman considered the transformations $\tau : \Re \to \Re$ which are restrictions to $\Re$ of the same class of meromorphic functions. He announced his existence results in [4] but the proofs were never published. This thesis provides a complete proof for Kemperman's results.

Rényi [6] was the first to introduce a class of transformations of the unit interval I and proved that it has an absolutely continuous invariant measures.

In 1973, Lasota and Yorke [5] proved, using the bounded variation techniques, an important generalization of Rényi's result: if $\tau : \mathrm{I} \to \mathrm{I}$ is piecewise C^2 transformation, then there exists a τ-invariant measure μ which is absolutely continuous with Lebesgue measure.

Later, Lasota and Jablonski [3] proved an analogous theorem for transformations of the real line. It was further generalized by Jablonski, Gora and Boyarsky [2].

In Chapter 1, I will recall preliminary definitions and theorems from measure theory and complex analysis relevant to this thesis.

In Chapter 2, I present a review of the spaces of functions and measures and of ergodic theory.

In Chapter 3, we use the Frobenius-Perron operator to study the absolutely continuous invariant measures. Let f be a probability density function for the random variable X. If $\tau : \mathrm{I} \to \mathrm{I}$ is a transformation, then $P_\tau f$ can be as the probability density function for the random variable $\tau(\mathrm{X})$, where P_τ depends on τ.

In Chapter 4, which is the main part of this thesis, we find an absolutely continuous invariant measure for a class of meromorphic transformations.

1

Preliminaries

In this chapter, we will briefly review some basic definitions and theorems from measure theory and complex analysis.

1.1 Review of Measure Theory

We recall some fundamental notions from measure theory. Let X be a set.

Definition 1.1.1

A family Σ of subsets of X is called a *σ-algebra* if and only if:

1) $X \in \Sigma$;
2) for any $A \in \Sigma$, $X \setminus A \in \Sigma$;
3) if $A_n \in \Sigma$, for n = 1, 2, ..., then $\bigcup_{n=1}^{\infty} A_n \in \Sigma$.

Elements of Σ are usually referred to as *measurable sets*.

Definition 1.1.2

A function $\mu : \Sigma \to \Re^+$ is called a *measure* on Σ if and only if:

1) $\mu(\emptyset) = 0$;
2) for any sequence $\{A_n\}$ of disjoint measurable sets, $A_n \in \Sigma$, n = 1, 2, ...,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Definition 1.1.3

If Σ is a σ-algebra of X and μ is a measure on Σ, then (X, Σ, μ) is called a *measure space*.

Definition 1.1.4

If X is a countable union of sets of finite μ-measure, then we say that μ is a σ-finite measure. In particular, if $\mu(X) = 1$, we say that the measure space is a *normalized measure space* or *probability space*.

Definition 1.1.5

A family $\Im$ of subsets of X is called an *algebra* if and only if:

1) $X \in \Im$;
2) for any $A \in \Im$, $X \setminus A \in \Im$;
3) for any $A_1, A_2 \in \Im$, $A_1 \bigcup A_2 \in \Im$.

For any family $\wp$ of subsets of X there exists a smallest σ-algebra, Σ, containing $\wp$.

Theorem 1.1.1

Given a set X and an algebra $\wp$ of subsets of X, let $\mu_1: \wp \to \Re^+$ be a function satisfying $\mu_1(X) = 1$ whenever $A_n \in \wp$,

$$\mu_1\left(\bigcup_n A_n\right) = \sum_n \mu_1(A_n)$$

for $n = 1, 2, \ldots$, $\bigcup_{n=1}^{\infty} A_n \in \wp$ and $\{A_n\}$ are disjoint. Then, there exists a unique normalized measure μ on $\Sigma = \sigma(\wp)$ such that $\mu(A) = \mu_1(A)$ whenever $A \in \wp$.

Definition 1.1.6

Let X be a topological space. Let θ denote a family of open sets of X. Then the σ-algebra $\Sigma = \sigma(\theta)$ is called the *Borel σ-algebra* of X and its element, *Borel subsets* of X.

Definition 1.1.7

Let (X, Σ, μ) be a measure space. A real-valued function $f : X \to \Re$ is said to be *measurable* if for every interval $I \subset \Re$, $f^{-1}(I) \in \Sigma$.

Definition 1.1.8

Let $I = X$ be the unit interval, Σ be the Borel σ-algebra and λ be the Lebesgue measure. Then, (X, Σ, λ) is a probability space. Hence, every random variable $f: X \to \Re$ on (X, Σ, λ) is a Borel measurable function.

Definition 1.1.9

Let f be a measurable function on (X, Σ, λ) and let $A \in \Sigma$. Then

$$\lambda_f(A) = \lambda\, \{x : f(x) \in A\}$$

is the probability measure induced by f. Hence, the distribution function $F_f : \Re \to [0, 1]$ is given by

$$F_f(t) = \lambda\, \{x : f(x) \leq t\}$$

for $t \in \Re$. Thus,

$$F_f(b) - F_f(a) = \lambda\, \{x : a < f(x) \leq b\} = \lambda_f((a,b]).$$

for $a < b$.

Definition 1.1.10

The function χ_E defined by

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}$$

is called a *characteristic function* of the set E.

1.2 Review of complex analysis

In this section, we present some fundamental definitions and theorems from complex analysis. Let z be a point in the complex plan **C**.

Definition 1.2.1

Let P be a point in the complex plane **C** corresponding to the complex number (x, y), i.e., P = x + iy. Then from Figure 1.1, we have

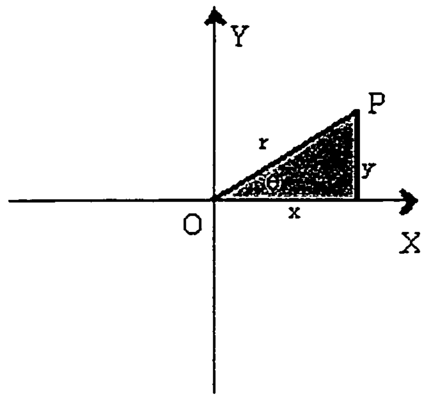

FIGURE 1.1

$$\begin{cases} x = r\cos\theta, \\ \quad \textit{and} \\ y = r\sin\theta \end{cases}$$

where $r = |P| = |x + iy| = \sqrt{x^2 + y^2}$ is called *the modulus* or *absolute value* of x + iy (denotes by *mod z* or |z|); and θ, called the *amplitude* or *argument* of z = x + iy (denoted by *arg z*), is the angle which line *OP* makes with the positive x-axis.

For any complex number $z \neq 0$, there corresponds one and only one value of θ in $0 \leq \theta < 2\pi$. However, any other interval of length 2π, for example $-\pi < \theta \leq \pi$, can be used.

Definition 1.2.2

Let δ be any given positive number. Then, *δ-neighbourhood* of z_0 is the set of all points z such that $|z - z_0| < \delta$. A *deleted δ-neighbourhood* of z_0 is a neighbourhood of z_0 in which z_0 is omitted, i.e., $0 < |z - z_0| < \delta$.

Definition 1.2.3

Let $E \subset X$. z_0 is called a *limit point* or *point of accumulation* of E if every deleted δ-neighbourhood of z_0 contains points of E.

Definition 1.2.4

Let τ be a transformation. Then, z is called a *fixed point* if τ(z) = z.

Definition 1.2.5

Let $f(z)$ be an analytic function inside and on a circle C except at the center z = a. Suppose that $f(z)$ has only a finite number of terms in the principal part of

$$f(z) = a_0 + a_1(z-a) + \dots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \dots + \frac{a_{-n}}{(z-a)^n}$$

where $a_{-n} \neq 0$, then z = a is a *pole of order n*. In particular, if n = 1 then z is called a *simple pole*.

Definition 1.2.6

A function, which is analytic everywhere in the complex plane everywhere except at ∞ is called an *entire or integral function*.

A function, which is analytic everywhere in the complex plane except at a finite number of poles, is called a *meromorphic function*.

Remark 1.2.1

Let C be a curve that has a finite length. Let $f(z)$ be integrable along C, then

1) $\int_C A f(z)\,dz = A\int_C f(z)\,dz$ where A is any constant;
2) $\int_a^b f(z)\,dz = -\int_b^a f(z)\,dz$;
3) $\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$ where C_1 and C_2 represent curves from a to m and m to b respectively.
4) $\left|\int_C f(z)\,dz\right| \leq ML$ where $|f(z)| \leq M$, i.e., M is an *upper bound* of $|f(z)|$ on C, and L is the length of C.

Definition 1.2.7

Let $f(z)$ be an analytic function inside and on a circle C except at the center z = a. Then $f(z)$ has a Laurent series about z = a given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a_n),$$

where

$$\oint_c f(z)\,dz = 2\pi i\, a_{-1}.$$

Then, a_{-1} is called *the residue* of $f(z)$ at z = a.

Theorem 1.2.1 (Residue theorem)

Let $f(z)$ be an analytic function inside and on a simple closed curve C except at a finite number of the singularities a, b, c, ... inside C which have residues given by $a_{-1}, b_{-1}, c_{-1}, \ldots$. Figure 1.2. Then the *residue theorem* states that

$$\oint_c f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \ldots)$$

i.e., the integral of $f(z)$ around C is $2\pi i$ times the sum of the residues of $f(z)$ at the

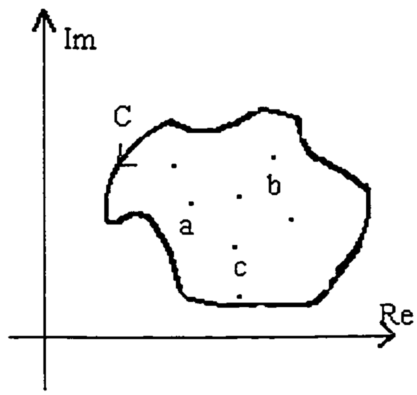

FIGURE 1.2

singularities enclosed by C.

Lemma 1.2.1

Let $g(\bar{z}) = \overline{g(z)}$. Then the residue of a real pole is real.

Proof

Suppose that g(z) has a real pole at $z = c$ such that $g(z) = \frac{f(z)}{z-c}$. If $g(\bar{z}) = \overline{g(z)}$, then $f(\bar{z}) = \overline{f(z)}$ implies $f(c)$ is real whenever c is real.
Now, Residue of $g(z) = \frac{f(z)}{z-c}$ at $z = c$ is $\lim_{z \to c}(z-c)\frac{f(z)}{z-c} = f(c)$.
Therefore, the residue is real.

□

Theorem 1.2.2 (Taylor's theorem)

Let $f(z)$ be analytic inside and on a simple closed curve C and let z and $z + a$ be two points inside C. Thus

$$f(z) = f(a) + (z - a) f'(a) + (z-a)^2 \frac{f''(a)}{2!} + \ldots + (z-a)^n \frac{f^{(n)}(a)}{n!} + \ldots$$

which implies

$$\frac{f(z)-f(a)}{z-a} = f'(a) + (z-a)\frac{f''(a)}{2!} + \ldots + (z-a)^{n-1} \frac{f^{(n)}(a)}{n!} + \ldots \qquad (1.1)$$

Then, the limit of (1.1) when $z \to a$ is $f'(a)$.

Theorem 1.2.3 (Mittag-Leffler's expansion theorem) (See [4], p.175)

1. Suppose that the only singularities of g (z) in the complex plane are the simple poles $c_1, c_2, c_3, \ldots$ arranged in order of increasing absolute value.
2. Let the residues of g(z) at $c_1, c_2, c_3, \ldots$ be $p_1, p_2, p_3, \ldots$
3. Let C_N be circles of radius R_N which do not pass through any poles and on which $|g(z)| < M$, where M is independent of N and $R_N \to \infty$ as $N \to \infty$.

Then , *Mittag-Leffler's expansion theorem* or *Partial-Fraction Theorem* states that

$$g(z) = g(0) + \sum_s p_s \{\frac{1}{z-c_s} + \frac{1}{c_s}\} \qquad (1.2)$$

Proof

Let g(z) have poles at $z = c_s$, s = 1, 2, 3, ..., and suppose that $z = \xi$ is not a pole of g(z). Then the function $\frac{g(z)}{z-\xi}$ has poles at $z = c_s$, s = 1, 2, ... and ξ.

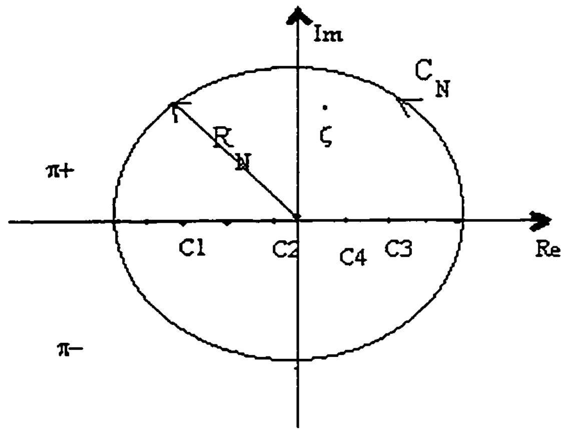

FIGURE 1.3: The real poles in C_N

Residue of $\frac{g(z)}{z-\xi}$ at $z = \xi$ is $\lim_{z\to\xi}(z-\xi)\frac{g(z)}{z-\xi} = g(\xi)$.

Residue of $\frac{g(z)}{z-\xi}$ at $z = c_s$, s = 1, 2, ..., is $\lim_{z\to c_s}(z-c_s)\frac{g(z)}{z-\xi} = \frac{p_s}{c_s-\xi}$.

Then by the residue theorem,

$$\frac{1}{2\pi i}\oint_{C_N}\frac{g(z)}{z-\xi}\,dz = g(\xi)+\sum_s\frac{p_s}{c_s-\xi} \tag{1.3}$$

where the last summation is taken over all real poles inside circle C_N of radius R_N Figure 1.3. Suppose that g(z) is analytic at z = 0. Substituting $\xi = 0$ in (1.3), we get

$$\frac{1}{2\pi i}\oint_{C_N}\frac{g(z)}{z}\,dz = g(0)+\sum_s\frac{p_s}{c_s}. \tag{1.4}$$

Subtraction of (1.4) from (1.3) yields

$$g(\xi)-g(0)+\sum_s p_s\left(\frac{1}{c_s-\xi}-\frac{1}{c_s}\right) = \frac{1}{2\pi i}\oint_{C_N} g(z)\left\{\frac{1}{z-\xi}-\frac{1}{z}\right\}dz$$
$$= \frac{\xi}{2\pi i}\oint_{C_N}\frac{g(z)}{z(z-\xi)}\,dz. \tag{1.5}$$

Now, since $|z-\xi| \geq |z|-|\xi| = R_N - |\xi|$ for z on C_N, we have, if $|g(z)| \leq M$

$$\left| \oint_{C_N} \frac{g(z)}{z(z-\xi)}\, dz \right| \leq \frac{M\, 2\pi R_N}{R_N (R_N - \xi)}.$$

As $N \to \infty$ and $R_N \to \infty$, it follows that the integral on the left approaches zero, i.e.,

$$\lim_{N\to\infty} \oint_{C_N} \frac{g(z)}{z(z-\xi)}\, dz = 0.$$

Hence from (1.5), letting $N \to \infty$, we have

$$g(\xi) = g(0) + \sum_s p_s \left(\frac{1}{\xi - c_s} + \frac{1}{c_s}\right)$$

Therefore, (1.2) will be obtained on replacing ξ by z since it holds for all ξ for which g is analytic.

□

Example 1.2.1

Let $f(z) = \cot z$ and $g(z) = \tan z$. Then, using Theorem 1.2.3, we obtain

$$f(z) = \frac{1}{z} + \sum_n \left(\frac{1}{z - n\pi} + \frac{1}{n\pi} \right),$$

where $n = \pm 1, \pm 2, \ldots$ and

$$g(z) = 2z \sum_n \frac{1}{\left(n + \frac{1}{2}\right)^2 \pi^2 - z^2} = \sum_n \frac{2z}{\left[\left(n + \frac{1}{2}\right)\pi - z\right]\left[\left(n + \frac{1}{2}\right)\pi + z\right]}$$

$$= \sum_s \left\{ \left[\frac{1}{\left(n + \frac{1}{2}\right)\pi + z} - \frac{1}{\left(n + \frac{1}{2}\right)\pi} \right] - \left[\frac{1}{\left(n + \frac{1}{2}\right)\pi - z} - \frac{1}{\left(n + \frac{1}{2}\right)\pi} \right] \right\}$$

since $\tan z = \cot z - 2\cot 2z$.

□

2

REVIEW OF ERGODIC THEORY

In this chapter we will present a brief review from ergodic theory. Let (X, Σ, μ) be a normalized measure space.

2.1 Spaces of Functions and Measures

Definition 2.1.1

Let $1 \leq p < \infty$. Then, the family of real valued measurable functions $f : X \rightarrow \Re$ satisfying

$$\int_X |f(x)|^p \, d\mu < \infty$$

is called the *$L^p(X, \Sigma, \mu)$ space.*

Definition 2.1.2

Let

$$D = D(X, \Sigma, \mu) = \{ f \in L^1(X, \Sigma, \mu) : f \geq 0 \text{ and } \|f\|_1 = 1 \}$$

denote the space of probability density functions. Any function $f \in D$ is called a *density function* or simply a *density.*

Theorem 2.1.1 (Radon-Nikodym theorem)

Let (X, Σ, μ) be a measure space and let ν be another finite measure on Σ with the property that $\nu(A) = 0$ for all $A \in \Sigma$ such that $\mu(A) = 0$. Then, there exist a *unique non-negative integrable function* $f : X \to \Re$ belongs to $L^1(X, \Sigma, \mu)$ such that

$$\nu(A) = \int_A f \, d\mu$$

for all $A \in \Sigma$. f is called the *Randon-Nikodym derivative* of ν with respect to μ and denoted by $\frac{d\nu}{d\mu}$. In the special case when $f \in (X, \Sigma, \mu)$, we also say that f is the *density* of ν and that ν is the normalized measure.

Proposition 2.1.1

If f_1 and f_2 are integrable functions such that

$$\int_B f_1 \, d\mu = \int_B f_2 \, d\mu$$

for all $B \in \Sigma$, then $f_1 = f_2$ a.e.

Definition 2.1.3

Let $f : X \to \Re$. $C^0(X) = C(X)$ with the norm

$$\|f\|_{C^0} = \sup_{x \in X} |f(x)|$$

is the *space of all continuous real functions* f.

Proposition 2.1.2

Two measures μ_1 *and* μ_2 are *identical* if and only if

$$\int_X g \, d\mu_1 = \int_X g \, d\mu_2$$

for all $g \in C(X)$.

Definition 2.1.4

If $f \in D(X, \Sigma, \mu)$, then

$$\mu_f(A) = \int_A f \, d\mu$$

is a measure and f is called the density of μ_f and is written as $f = \frac{d\mu_f}{d\mu}$.

Definition 2.1.5

If $f \in L^1(X, \Sigma, \mu)$ and $f \geq 0$, then the measure

$$\mu_f(A) = \int_A f \, d\mu$$

is said to be *absolutely continuous* with respect to μ.

2.2 Measure Preserving Transformation

Definition 2.2.1

A transformation $\tau : X \to X$ is *measurable* if $\tau^{-1}(\Sigma) \subset \Sigma$, i.e., $A \in \Sigma \Rightarrow \tau^{-1}(A) \in \Sigma$, where $\tau^{-1}(A) \equiv \{ x \in X : \tau(x) \in A\}$ (See [1]).

Definition 2.2.2

A measurable transformation $\tau : X \to X$ on a measure space (X, Σ, μ) is *non-singular* if

$$\mu(\tau^{-1}(A)) = 0$$

for all $A \in \Sigma$ such that $\mu(A) = 0$ (See [1]).

Definition 2.2.3

The measurable transformation $\tau : X \to X$ preserves measure μ or μ is *τ-invariant* if

$$\mu\,(\tau^{-1}(A)) = \mu\,(A)$$

for all $A \in \Sigma$ (See [1]).

Remark 2.2.1

Every measure preserving transformation is necessarily non-singular with respect to its invariant measure.

Example 2.2.1

Let $X = [0, 1]$, Σ = Borel σ-algebra of $[0, 1]$ and λ = Lebesgue measure on $[0, 1]$. Let $\tau : X \to X$ be defined by $\tau(x) = rx \pmod 1$, where r is a positive integer greater than or equal to 2. Then τ is measure preserving.

Proof

For any interval $[a, b] \subset [0, 1]$,

$$\tau^{-1}[a,b] = \bigcup_{i=0}^{r-1}\left[\frac{i+a}{r}, \frac{i+b}{r}\right]$$

Thus, we get

$$\begin{aligned}
\lambda\left(\tau^{-1}[a,b]\right) &= \lambda\left(\bigcup_{i=0}^{r-1}\left[\frac{i+a}{r}, \frac{i+b}{r}\right]\right) = \sum_{i=0}^{r-1}\lambda\left[\frac{i+a}{r}, \frac{i+b}{r}\right] \\
&= \frac{1}{r}\sum_{i=0}^{r-1}\lambda\,[i+a, i+b] = \frac{1}{r}\sum_{i=0}^{r-1}[(i+b)-(i+a)] \\
&= \sum_{i=0}^{r-1}\frac{b-a}{r} = (b - a) = \lambda[a, b].
\end{aligned}$$

□

Definition 2.2.4

Let $\tau : X \to X$ be measure preserving on a normalized space, such that $\tau(A) \in \Sigma$ for every $A \in \Sigma$ and $\mu(A) > 0$. If

$$\lim_{n\to\infty} \mu(\tau^n A) = 1,$$

then τ is called *exact.*

Definition 2.2.5

We say $\tau : (X, \Sigma, \mu) \to (X, \Sigma, \mu)$ is weakly mixing if,

$$\frac{1}{n}\sum_{i=0}^{n-1}\left|\mu\,(\tau^{-i}A\cap B)-\mu(A)\mu(B)\right| \xrightarrow[n\to\infty]{} 0$$

for all $A, B \in \Sigma$.

Definition 2.2.6

Let τ be a measure preserving transformation of a probability space (X, Σ, μ). Then, τ is *strongly mixing* if,

$$\mu(\tau^{-n} A \cap B) \xrightarrow[n\to\infty]{} \mu(A)\mu(B)$$

for all $A, B \in \Sigma$.

Definition 2.2.7

The measure preserving transformation $\tau : (X, \Sigma, \mu) \to (X, \Sigma, \mu)$ is ergodic if for any $A \in \Sigma$, such that $\tau^{-1}A = A$, $\mu(A) = 0$ or $\mu(X\backslash A) = 0$. It can be shown that τ is *ergodic* if and only if,

$$\frac{1}{n}\sum_{i=0}^{n-1}\mu\,(\tau^{-i}A \cap B) \xrightarrow[n\to\infty]{} \mu(A)\mu(B)$$

for all $A, B \in \Sigma$. (See [1])

Example 2.2.2

1) Let us consider the rotation F on the unit circle S^1, where $F(x) = x + \theta$ and $\theta \in [0, 2\pi]$ is constant. Obviously the measure induced by the arc length is invariant under F. Depending on whether θ is rational or irrational, F is not ergodic or ergodic respectively.

2) **Example 2.2.1** is ergodic.

□

Remark 2.2.2

It can be proved that the exactness of τ implies that τ is strongly mixing. Moreover, it is obvious that if τ is strongly mixing $\Rightarrow$ weakly mixing $\Rightarrow \tau$ is ergodic.

□

3

FROBENIUS-PERRON OPERATOR

In this chapter, we begin the study of the main tool we shall use to investigate absolutely continuous invariant measure, their existence and properties, namely the Frobenius-Perron operator. This operator was first introduced by [Kuzmin, 1923] and describes the effect of the transformation τ on a probability density function. This chapter is mostly based on [1].

3.1 Frobenius-Perron operator

Let $f \in L^1(X, \Sigma, \mu)$ and $f \geq 0$. Then, for any measurable set $A \subset [0, 1]$,

$$\text{Prob}\{Y \in A\} = \int_A f \, d\lambda$$

where λ denotes Lebesgue measure on $[0, 1]$. Let $\tau : [0, 1] \to [0, 1]$ be a non-singular transformation. Then $\tau(Y)$ is a random variable and

$$\text{Prob}\{\tau(Y) \in A\} = \text{Prob}\{Y \in \tau^{-1}(A)\} = \int_{\tau^{-1}A} f \, d\lambda .$$

We are going to define $P_\tau : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ as follows: We supposed that $f \in L^1(X, \Sigma, \mu)$ and $f \geq 0$. Then, let us consider

$$\int_{\tau^{-1}(A)} f \, d\mu . \qquad (3.1)$$

Since

$$\tau^{-1}(\bigcup_i A_i) = \bigcup_i \tau^{-1}(A_i),$$

then, it follows from the property of the Lebesgue integral that (3.1) defines a finite measure. Thus, by Theorem 2.1.1, there exists a unique element in $L^1(X, \Sigma, \mu)$, which we denote by $P_\tau f$, such that

$$\int_A P_\tau f \, d\mu = \int_{\tau^{-1}(A)} f \, d\mu$$

for any $A \in \Sigma$.

Now, let $f \in L^1(X, \Sigma, \mu)$ be arbitrary and not necessarily non-negative. Let $f^+ = \max(0, f)$ and $f^- = \max(0, -f)$. Then $f^+, f^- \in L^1(X, \Sigma, \mu)$, $f = f^+ - f^-$ and $|f| = f^+ + f^-$, and define

$$P_\tau f = P_\tau f^+ - P_\tau f^-.$$

Then,

$$\int_A P_\tau f \, d\mu = \int_A P_\tau f^+ \, d\mu - \int_A P_\tau f^- \, d\mu = \int_{\tau^{-1}(A)} f^+ \, d\mu - \int_{\tau^{-1}(A)} f^- \, d\mu$$
$$= \int_{\tau^{-1}(A)} (f^+ - f^-) \, d\mu,$$

that is

$$\int_A P_\tau f \, d\mu = \int_{\tau^{-1}(A)} f \, d\mu \qquad (3.2)$$

for all measurable sets $A \in \Sigma$. Then, by the Radon-Nikodym Theorem and the non-singularity of τ, it follows that equation (3.2) uniquely defines $P_\tau f$. Thus the probability density function f has been transformed on a new probability density function $P_\tau f$. P_τ obviously depends on the transformation τ and is an operator from the space of probability density functions on [0, 1] into itself (See [1]).

□

Definition 3.1.1

Let $J = [a, b]$ and let λ denote the normalized Lebesgue measure on J. Let $\tau : J \to J$ be a non-singular transformation. We define the Frobenius-Perron operator $P_\tau : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ as follows : For any $f \in L^1(X, \Sigma, \mu)$, $P_\tau f$ is the unique function in $L^1(X, \Sigma, \mu)$ such that (3.2) is satisfied for any $A \in \Sigma$.

The validity of this definition, i.e., the existence and the uniqueness of $P_\tau f$, follows by the Radon-Nikodym Theorem (See [1]).

Now, let us consider the transformation τ on $[0, 1]$, which is differentiable and invertible. Since $f \in L^1(X, \Sigma, \mu)$, then $P_\tau f \in L^1(X, \Sigma, \mu)$. Hence $P_\tau : L^1(X, \Sigma, \mu) \rightarrow L^1(X, \Sigma, \mu)$ is a well-defined operator. If we let $A = [0, x] \subset [0, 1]$, we get

$$\int_0^x P_\tau f \, d\mu = \int_{\tau^{-1}[0,x]} f \, d\mu$$

On differentiating both sides with respect to x, we obtain

$$P_\tau f(x) = \frac{d}{dx} \int_{\tau^{-1}[0,x]} f \, d\mu$$

□

Example 3.1.1

Let

$$\tau(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ -x + \frac{3}{2}, & \frac{1}{2} \le x \le 1, \end{cases}$$

as shown in Figure 3.1.

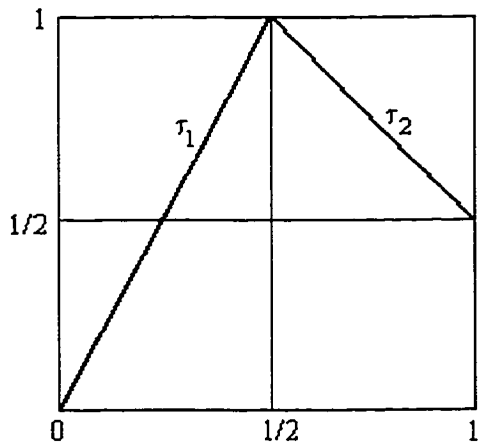

FIGURE 3.1

It is easy to see that

$$\tau^{-1}([0,x]) = [0,\frac{1}{2}x], \qquad \text{if } x < \frac{1}{2}$$

and

$$\tau^{-1}([0,x]) = [0,\frac{1}{2}x] \cup [\frac{3}{2}-x,1], \qquad \text{if } x \ge \frac{1}{2}$$

Hence, we have

$$\tau^{-1}([0,\ x]) = [0,\ \frac{1}{2}x] \cup \{[\frac{3}{2}-x,1] \cap B\}, \qquad 0 \le x \le 1$$

where $B = [\frac{1}{2}, 1]$. Hence, for any $f \in L^1(X, \Sigma, \mu)$,

$$\int_{\tau^{-1}[0,x]} P_\tau f \, d\lambda = \int_0^{\frac{x}{2}} f \, d\lambda + \int_{\frac{3}{2}-x}^{1} f \, \chi_B \, d\lambda.$$

We derive

$$(P_\tau f)(x) = \frac{1}{2} f(\frac{x}{2}) + f(\frac{3}{2}-x)\chi_J(x)$$

where $J = \tau(B) = [\frac{1}{2}, 1]$.

□

3.2 Properties of the Frobenius-Perron Operator

In this section, we define and present the basic properties of the Frobenius-Perron operator.

Proposition 3.2.1 (Linearity)

$P_\tau : L^1(X, \Sigma, \mu) \rightarrow L^1(X, \Sigma, \mu)$ is a linear operator, i.e., $P_\tau(\alpha f + \beta g) = \alpha P_\tau f + \beta P_\tau g$ a.e. for all $f, g \in L^1(X, \Sigma, \mu)$ and $\alpha, \beta \in \Re$.

Proposition 3.2.2 (Continuity)

Let $P_\tau : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$. Then P_τ is continuous.

Proposition 3.2.3 (Positivity)

Let $f \in L^1(X, \Sigma, \mu)$ and assume that $f \geq 0$. Then $P_\tau f \geq 0$.

Proposition 3.2.4 (Preservation of Integrals)

For all $f \in L^1(X, \Sigma, \mu)$,

$$\int_0^1 P_\tau f \, d\lambda = \int_0^1 f \, d\lambda$$

Proposition 3.2.5 (Contraction Property)

$P_\tau : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$ is a contraction, i.e., $\|P_\tau f\| \leq \|f\|$ for any $f \in L^1(X, \Sigma, \mu)$.

Proposition 3.2.6 (Composition Property)

If $\tau^n = \underbrace{\tau \circ \ldots \circ \tau}_{n}$, then $P_{\tau^n} = P_\tau{}^n$, where P_τ is the Frobenius-Perron operator induced by τ.

Proposition 3.2.7 (Adjoint Property)

If $f \in L^1(X, \Sigma, \mu)$ and $g \in L^\infty(X, \Sigma, \mu)$, then $\langle P_\tau f, g \rangle = \langle f, U_\tau g \rangle$, i.e.,

$$\int_I (P_\tau f) . g \, d\lambda = \int_I f . U_\tau \, g \, d\lambda$$

where $U_\tau g = g \circ \tau$

□

The next proposition says that every density f is a fixed point of P_τ, i.e., f is P_τ-invariant, if and only if $\mu = f \cdot \lambda$ is τ-invariant.

Proposition 3.2.8

Let $\tau : I \to I$ be non-singular. Then, $P_\tau f = f$ a.e., if and only if the measure $\mu = f \cdot \lambda$, defined by $\mu(A) = \int_A f \, d\lambda$, is τ-invariant, i.e., if and only if $\mu(\tau^{-1}A) = \mu(A)$ for all measurable sets A, where $f \geq 0$, $f \in L^1(X, \Sigma, \mu)$ and $\|f\|_1 = 1$.

Proof

$\Rightarrow$

Assume that μ is τ-invariant. Then by the definition of the invariant measure,

$$\mu(\tau^{-1}A) = \mu(A)$$

for all measurable set A. Then ,

$$\int_{\tau^{-1}A} f \, d\lambda = \int_A f \, d\lambda$$

However, by the definition of the Frobenius-Perron operator, we have

$$\int_{\tau^{-1}A} f \, d\lambda = \int_A P_\tau f \, d\lambda$$

and therefore,

$$\int_A f \, d\lambda = \int_A P_\tau f \, d\lambda$$

Since $A \in \Sigma$ is arbitrary, then $P_\tau f = f$ a.e.

$\Leftarrow$

Assume that $P_\tau f = f$ a.e. Then,

$$\int_A P_\tau f \, d\lambda = \int_A f \, d\lambda = \mu(A)$$

By definition,

$$\int_A P_\tau f \, d\lambda = \int_{\tau^{-1}A} f \, d\lambda = \mu(\tau^{-1}A)$$

and so,

$$\mu(\tau^{-1}A) = \mu(A).$$

□

3.3 Representation of the Frobenius-Perron Operator

In this section, we derive an extremely useful representation for the Frobenius-Perron operator for a large class of one-dimensional piecewise monotonic transformations.

Definition 3.3.1

The transformation $\tau : [0, 1] \rightarrow [0, 1]$ is called *piecewise monotonic* if there exists a finite partition P of [0, 1], $0 = a_1 < \ldots < a_q < 1$, and a number $r \geq 1$ such that :

1) $\tau|_{(a_{i-1}, a_i)}$ is a C^r function, i = 1, …, q which can be extended to a C^r function on $[a_{i-1}, a_i]$, i = 1, …, q, and
2) $|\tau'(x)| > 0$ on (a_{i-1}, a_i), i = 1, …, q. (See [1])

Example 3.3.1

A good example of such a transformation is Figure 3.2.

Theorem 3.3.1

Let $\tau|_{I_j} \in C^1[a_{j-1}, a_j]$ (first derivative of τ exists and continuous) be monotone, j = 1, 2, …, n, where $0 = a_1 < \ldots < a_{r-1} < 1$. Then, we have

$$P_\tau f = \sum_j f(\sigma_j x)\psi_j(x)\chi_{J_j}(x)$$

where σ_j is the inverse of τ, i.e., $\sigma_j = \tau^{-1}$, over $J_j = \tau(I_j)$; $\psi_j(x) = |\sigma_j'(x)|$; χ_j is the indicator function of J_j.

Proof

Let $A_j(x) = \sigma_j([0,x]) \bigcap I_j$. Then

$$\int_{A_j(x)} f(s)\,ds = \pm \int_{\sigma_j(0)}^{\sigma_j(x)} f(s)\,\chi_{I_j}(s)\,ds. \tag{3.3}$$

We want that the left side of (3.3) be ≥ 0 when $f \geq 0$. $\tau\big|_{I_j}$ is monotone, σ_j is monotone where $\tau\big|_{I_j}$ and σ_j are either increasing or decreasing. Therefore,

$$\frac{\sigma_j'(x)}{\left|\sigma_j'(x)\right|} = \frac{\sigma_j'(y)}{\left|\sigma_j'(y)\right|}$$

for all x, y $\in$ [0, 1]. We use this to set the sign in (3.3), thus

$$\int_{A_j(x)} f(s)\,ds = \frac{\sigma_j'(x)}{\left|\sigma_j'(x)\right|} \int_{\sigma_j(0)}^{\sigma_j(x)} f(s)\,\chi_{I_j}(s)\,ds.$$

This implies

$$\begin{aligned}
\frac{d}{dx}\int_{A_j(x)} f(s)\,ds &= \frac{\sigma_j'(x)}{\left|\sigma_j'(x)\right|}\frac{d}{dx}\int_{\sigma_j(0)}^{\sigma_j(x)} f(s)\,\chi_{I_j}(s)\,ds \\
&= \frac{\sigma_j'(x)}{\left|\sigma_j'(x)\right|} f(\sigma_j(x))\,\chi_{I_j}(\sigma_j(x))\,\sigma_j'(x) \\
&= \frac{\left|\sigma_j'(x)\right|^2}{\left|\sigma_j'(x)\right|} f(\sigma_j(x))\,\chi_{I_j}(\sigma_j(x)) \\
&= f(\sigma_j(x))\,\psi_j(x)\,\chi_{I_j}(\sigma_j(x)).
\end{aligned}$$

Note that

$$\begin{aligned}
\chi_{I_j}(\sigma_j(x)) = 1 &\Leftrightarrow \sigma_j(x) \in I_j \\
&\Leftrightarrow x \in \tau(I_j) = J_j \Leftrightarrow \chi_{J_j} = 1.
\end{aligned}$$

Therefore,

$$\chi_{I_j}(\sigma_j(x)) = \chi_{J_j},$$

and we have

$$\frac{d}{dx}\int_{A_j(x)} f(s)\,ds = \sum_j f(\sigma_j x)\psi_j(x)\chi_{J_j}(x).$$

For

$$\tau = \sum_{i=1}^{n} \tau|_{I_j} \chi_{I_j},$$

and

$$\sigma([0,x]) = \bigcup_{j=0}^{n-1} A_j(x),$$

where A_j are disjoint since I_j are disjoint (Figure 3.2).

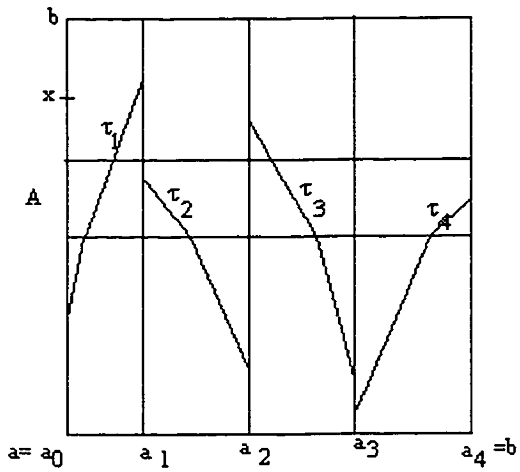

FIGURE 3.2. Piecewise monotonic transformation

Thus

$$\begin{aligned} P_\tau f &= \frac{d}{dx}\int_{\sigma([0,x])} f(s)\,ds = \frac{d}{dx}\sum_{j=0}^{n}\int_{A_j(x)} f(s)\,ds \\ &= \sum_j f(\sigma_j x)\psi_j(x)\chi_{J_j}(x). \end{aligned}$$

Therefore,

$$P_\tau f = \sum_j f(\sigma_j x)\psi_j(x)\chi_{J_j}(x).$$

□

3.4 Existence of Absolutely Continuous Invariant Measures

In this section, we will state the conditions that guarantee that absolutely continuous invariant measures exist. Rényi [1956] was the first who defined a class of transformations that have an absolutely continuous invariant measure (**acim**). In 1973, Lasota and Yorke proved an important generalization of Rényi's result. Their essential observation was that, for piecewise expanding transformations, the Frobenius-Perron operator is a contraction on a space of functions. Below, we state the most recent result on the existence of absolutely continuous invariant measures for real line transformations (see [2] p.185).

Theorem 3.4.1

Let $\tau : \bigcup_{i=1}^{\infty} I_i \to A$ be a transformation which satisfies the following conditions :

1) I_i, i = 1, 2, ..., are open intervals, $I_i \subset A$,
2) $I_i \cap I_j = \varnothing$, for $i \neq j$,
3) $\sup_{i\geq 1} \lambda(I_i) < +\infty$, where λ is the Lebesgue measure on $\Re$,
4) A is an interval, not necessarily bounded, and $\lambda(A \setminus \bigcup_{i=1}^{\infty} I_i) = 0$,
5) $\tau_i = \tau \big|_{I_i}$ is of class C^1, i = 1, 2, ...,
6) $|\tau_i'| \geq \lambda > 2$, i = 1, 2, ...,
7) $\sum_{n=1}^{\infty} d_n < +\infty$, where

$$d_n = \sup_{J \in P^{(n)}} \underset{J}{osc} |\tau'|,$$

$$P^{(n)} = \bigcup_{i=0}^{n} \tau^{-i}\left(P^{(0)}\right) \ \ and \ \ P^{(0)} = \{I_i\}_{i=1}^{\infty}.$$

8) $\sup_{i\geq 1} |\psi_i'(x)|$ is integrable on A, where $\psi_i = \tau_i^{-1}$, i = 1, 2, ..., i.e.,

$$\psi_i = \begin{cases} \tau_i^{-1}, & if\ x \in \tau(I_i), \\ 0, & if\ x \notin \tau(I_i), \end{cases}$$

9) $\sup_{x\in A} \sup_{i\geq 1} \frac{|\psi_i'(x)|}{\lambda(I_i)} = K < +\infty$ and $\sup_{i\geq 1} \frac{\int_{|x|>u} |\psi_i'(x)|}{\lambda(I_i)} \leq K(u)$ and $K(u) \to 0$, as $u \to \infty$.

Then, there exists a finite absolutely continuous measure μ on A invariant with respect to τ.

□

Example 3.4.1

Let

$$\tau(x) = 3\tan(x), \qquad x \in \Re.$$

Then, Theorem 3.4.1 states that τ has an absolutely continuous invariant measure.

□

4

A class of meromorphic transformations

We are now ready to start the main part of our discussion. In this chapter, we are going to consider a class of meromorphic transformation and find its absolutely continuous invariant measure.

4.1. Meromorphic functions

In this section, we are going to prove three propositions characterizing a class of meromorphic functions.

Proposition 4.1.1

Let g(z) be a meromorphic function defined for all complex z and such that Im{g(z)} has a constant sign $\varepsilon = \pm 1$ in the upper half plane $\mathrm{Im}\{z\} > 0$. This is equivalent to the following : the meromorphic function g has only real and simple poles c_s and is of the form

$$g(z) = A + \varepsilon \left[Bz - \sum_s p_s \left(\frac{1}{z - c_s} + \frac{1}{c_s} \right) \right] \qquad (4.1)$$

where A, B and p_s are real constants such that $B \geq 0$, $p_s > 0$ and $\sum_s \frac{p_s}{c_s^2} < \infty$. Moreover, $\{c_s\}$ has no finite accumulation points (See [4]).

Proof

$\Rightarrow$

We know that g is a meromorphic function defined for all complex z such that the imaginary part of g has a constant sign in π_+.

First, we are going to prove that g has only *simple poles* c_s. Let us suppose that g has poles of order p where $p > 1$. Then, in the neighbourhood of c_s,

$$g(z) = \frac{1}{(z-c_s)^p} f(z), \qquad (4.2)$$

where $f(z)$ is holomorphic at c_s and $f(c_s) \neq 0$.
When we substitute $c_s + re^{i\theta}$ for z, (4.2) becomes

$$g(c_s + re^{i\theta}) = \frac{1}{\left(c_s + re^{i\theta} - c_s\right)^p} f(c_s + re^{i\theta}) = \frac{1}{r^p} e^{-ip\theta} f(c_s + re^{i\theta})$$

where $\theta \in [0, \pi]$ and $f(c_s + re^{i\theta}) \xrightarrow[r \to 0^+]{} f(c_s)$.

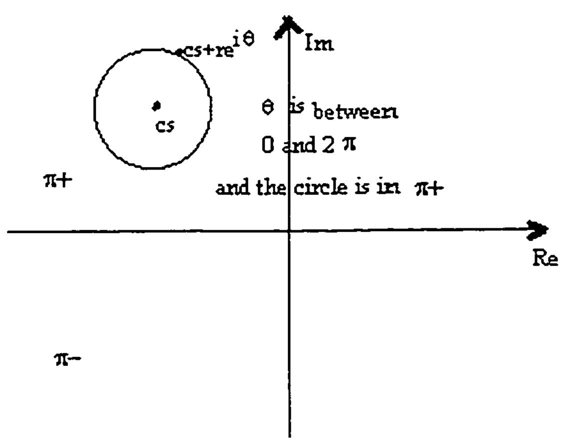

FIGURE 4.1. A complex pole.

This is a contradiction because if $p \geq 2$, then $-p\theta \in [-p\pi, 0]$ and Arg{g(z)} changes over an interval of length $p\pi$ which means that Im{g(z)} will not have a constant sign in the neighbourhood of c_s, s = 1, 2, 3, ... (Figure 4.1). Thus, p = 1 and therefore g(z) has only simple poles.

□

Now, let us prove that g has only *real poles*. Assume that g has a complex pole c_s. Then, in the neighbourhood of c_s,

$$g(z) = \frac{1}{z - c_s} f(z), \qquad (4.3)$$

where $f(z)$ is holomorphic at c_s and $f(c_s) \neq 0$.

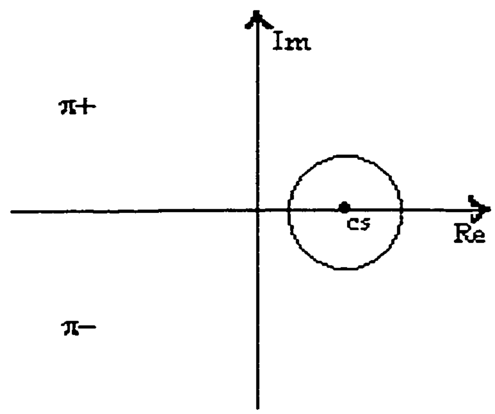

FIGURE. 4.2: Im{g(z)} change sign and fall into π_-

When we substitute $c_s + re^{i\theta}$ for z, (4.3) becomes

$$g(c_s + re^{i\theta}) = \frac{1}{c_s + re^{i\theta} - c_s} f(c_s + re^{i\theta}) = \frac{1}{r} e^{-i\theta} f(c_s + re^{i\theta})$$

where $\theta \in [0, 2\pi]$ and $f(c_s + re^{i\theta}) \xrightarrow[r \to 0^+]{} f(c_s)$.
This is a contradiction, because if $\theta \in [0, 2\pi]$, then $-\theta \in [-2\pi, 0]$ and Arg{g(z)} changes over an interval of length bigger than π which means that Im{g(z)} will not have a constant sign in the neighbourhood of c_s, s = 1, 2, 3, ... (Figure 4.2). Therefore, all the poles are real.

□

Then, we are going to show that g *cannot have terms of the form* z^p where p > 1. Let us suppose that g has a term of form z^p, p >1. Then, g will be

$$g(z) = \sum_{p=0}^{\infty} a_p z^p \qquad (4.4)$$

which have poles of order p at ∞.

When we substitute $re^{i\theta}$ for z where r is large enough, then (4.4) becomes

$$g(re^{i\theta}) = \sum_{p=0}^{\infty} a_p (re^{i\theta})^p$$

where $\theta \in [0, \pi]$.
Again, this is a contradiction because if $p \geq 2$, then $p\theta \in [0, p\pi]$ and Arg{g(z)} changes over an interval of length grater than π which means that Im{g(z)} will not have a constant sign in the neighbourhood of c_s, s = 1, 2, 3, ... Therefore, p = 1 and (4.4.) becomes

$$g(z) = a_0 + a_1 z \qquad (4.5)$$

Assume that a_0 is a *real number*, then a_1 should be a *real number* too. If a_1 is not a real number, then a_1 is a complex number and it is equal to u + iv where u and v are real. When we substitute u + iv for a_1 and x + iy for z, (4.5) becomes

$$g(x + iy) = a_0 + (u + iv)(x + iy) = a_0 + ux - vy + i(uy + vx)$$

Then Im{g(x + iy)}, which is equal to uy + vx, does not have a constant sign in the upper half plane π_+, Im(z) = y > 0 because x, u and v are arbitrary real numbers. Therefore, $g(z) = a_0 + a_1 z$ where a_0 and a_1 are real.

We know now that a_0 and a_1 are real. Then, when we substitute x + iy for z, (4.5) becomes

$$g(x + iy) = a_0 + a_1(x + iy) = a_0 + a_1 x + i a_1 y$$

Hence, Im{g(x + iy)} = a_1y where Im(z) = y >0. Since Im{g(x + iy)} has to have a constant sign in π_+, then a_1 should be ≥ 0. Therefore, $a_1 \geq 0$.

□

Next, we know that g have real and simple poles at c_s, s = 1, 2, 3, ... Then from Theorem 1.3.2 (Mittag-Leffler's expansion Theorem), g becomes

$$g(z) = g(0) - \sum_s p_s \left\{ \frac{1}{z-c_s} + \frac{1}{c_s} \right\}, \qquad (4.6)$$

where p_s are the *residues* at c_s, s = 1, 2, 3, ...

Suppose now that p_s are complex residues, i.e., $p_s = u_s + iv_s$. Then, when we substitute $u_s + iv_s$ for p_s, g(0) for A and x + iy for z, (4.6) becomes

$$\begin{aligned} g(x+iy) &= A - \sum_s (u_s + iv_s) \left\{ \frac{1}{(x+iy)-c_s} + \frac{1}{c_s} \right\} \\ &= A - \sum_s (u_s + iv_s) \left\{ \frac{(x-c_s)-iy}{(x-c_s)^2+y^2} + \frac{1}{c_s} \right\} \\ &= A - \sum_s \frac{u_s(x-c_s)}{(x-c_s)^2+y^2} - \sum_s \frac{u_s(-iy)}{(x-c_s)^2+y^2} - \sum_s \frac{iv_s(x-c_s)}{(x-c_s)^2+y^2} \\ &\quad - \sum_s \frac{(iv_s)(-iy)}{(x-c_s)^2+y^2} - \sum_s \frac{u_s}{c_s} - \sum_s \frac{iv_s}{c_s} \\ &= A - \sum_s \frac{u_s(x-c_s)}{(x-c_s)^2+y^2} - \sum_s \frac{u_s}{c_s} - \sum_s \frac{v_s y}{(x-c_s)^2+y^2} \\ &\quad + i\sum_s \left[\frac{u_s y}{(x-c_s)^2+y^2} - \frac{v_s}{c_s} - \frac{v_s(x-c_s)}{(x-c_s)^2+y^2} \right] \end{aligned}$$

Then Im{g(x + iy)}, which is equal to $\sum_s \left[\frac{u_s y}{(x-c_s)^2+y^2} - \frac{v_s}{c_s} - \frac{v_s(x-c_s)}{(x-c_s)^2+y^2} \right]$, does not have a constant sign in the upper half plane π_+ Im(z) = y > 0 because x, c_s, u_s and v_s are arbitrary real numbers. Therefore, p_s are *real*.

We know now that p_s are *real*. Then, when we substitute x + iy for z, (4.6) becomes

$$\begin{aligned} g(x+iy) &= A - \sum_s p_s \left\{ \frac{1}{(x+iy)-c_s} + \frac{1}{c_s} \right\} = A - \sum_s p_s \left\{ \frac{x-c_s-iy}{(x-c_s)^2+y^2} + \frac{1}{c_s} \right\} \\ &= A - \sum_s p_s \frac{x-c_s-iy}{(x-c_s)^2+y^2} - \sum_s \frac{p_s}{c_s} \\ &= A - \sum_s p_s \left[\frac{x-c_s}{(x-c_s)^2+y^2} + \frac{1}{c_s} \right] + iy\sum_s \frac{p_s}{(x-c_s)^2+y^2} \end{aligned}$$

Then the imaginary part of g(x + iy), which is equal to $y\sum_s \frac{p_s}{(x-c_s)^2+y^2}$, does not have a constant sign in the upper half plane π_+ Im(z) = y > 0 unless $p_s > 0$.

We just proved that g can be written as (4.5), i.e., g(z) = a_0 + a_1z. Then, by adding (4,5) to (4.6), (4.1) will appear where ε is equal to ± 1 and a_1 = B.

To finish our discussion, we need to check if there exist any conditions on g itself. We know that g is a meromorphic function, i.e., g is analytic everywhere except at the poles, i.e., $g'(z)$ should exist. Then, by taking a common denominator in (4.1), g becomes

$$g(z) = A + \varepsilon\left[Bz - \sum_s p_s \frac{c_s + z - c_s}{c_s(z-c_s)}\right] = A + \varepsilon\left[Bz - \sum_s p_s \frac{z}{c_s(z-c_s)}\right]$$
$$= A + \varepsilon\left[Bz - \sum_s \frac{p_s}{c_s}\left(1 + \frac{c_s}{z-c_s}\right)\right]. \quad (4.7)$$

Differentiating both sides of (4.7), g' becomes

$$g'(z) = \varepsilon\left[B + \sum_s \frac{p_s}{(z-c_s)^2}\right]$$

where B and p_s are positive real constant. g(z) is analytic at z = 0, then g' exist if and only if B + $\sum_s \frac{p_s}{c_s^2}$ is finite i.e., $\sum_s \frac{p_s}{c_s^2} < \infty$.

Finally, $\{c_s\}$ can only have an infinite accumulation points, since otherwise g would have a finite real singularity, which would contradict the definition of meromorphic function. Then, $\{c_s\}$ has no finite accumulation points.

Therefore, if g is a meromorphic function defined for all complex z such that Im{g(z)} has a constant sign in the upper half plane π_+ Im(z) = y > 0, then g has only real and simple poles and is of the form

$$g(z) = A + \varepsilon\left[Bz - \sum_s p_s\left(\frac{1}{z-c_s} + \frac{1}{c_s}\right)\right],$$

where A, B and p_s are real constants such that B ≥ 0, $p_s > 0$ and $\sum_s \frac{p_s}{c_s^2} < \infty$.

Moreover, $\{c_s\}$ has no finite accumulation points.

□

$\Leftarrow$

If we substitute x + iy for z, then (4.1) becomes

$$g(x+iy) = A + \varepsilon\left[B(x+iy) - \sum_s p_s\left(\frac{1}{(x+iy)-c_s} + \frac{1}{c_s}\right)\right]$$
$$= A + \varepsilon\left[Bx + iBy - \sum_s p_s\left(\frac{1}{(x-c_s)-iy} + \frac{1}{c_s}\right)\right]$$
$$= A + \varepsilon Bx + i\varepsilon By - \varepsilon\sum_s p_s\left(\frac{x-c_s-iy}{(x-c_s)^2+y^2}\right) - \varepsilon\sum_s \frac{p_s}{c_s}$$
$$= A + \varepsilon Bx - \varepsilon\sum_s \frac{p_s}{c_s} + i\varepsilon By - \varepsilon\sum_s\left[\frac{p_s(x-c_s)}{(x-c_s)^2+y^2}\right] + i\varepsilon\sum_s\left(\frac{p_s y}{(x-c_s)^2+y^2}\right)$$
$$= A + \varepsilon Bx - \varepsilon\sum_s \frac{p_s}{c_s} - \varepsilon\sum_s\left[\frac{p_s(x-c_s)}{(x-c_s)^2+y^2}\right] + i\varepsilon y\left[B + \sum_s\left(\frac{p_s}{(x-c_s)^2+y^2}\right)\right]$$

where $B \geq 0$, $p_s > 0$, $y > 0$ and $\varepsilon = 1$.

Then Im{g(x + iy)}, which is equal to $\varepsilon y[B + \sum_s(\frac{p_s}{(x-c_s)^2+y^2})]$, has a constant sign in the upper half plane.

□

Proposition 4.1.2

$g'(x)$ is a positive function on the real line, i.e., $g'(x) = |g'(x)|$ for all $x \in \Re$.

Proof

We know from (4.1) that

$$g(z) = A + \varepsilon\left[Bz - \sum_s p_s\left(\frac{1}{z-c_s} + \frac{1}{c_s}\right)\right].$$

Then

$$g'(z) = \varepsilon\left[B + \sum_s p_s \frac{1}{(z-c_s)^2}\right].$$

Substituting x for z where $x \in \Re$, g' becomes

$$g'(x) = \varepsilon\left[B + \sum_s p_s \frac{1}{(x-c_s)^2}\right]$$

implies $g'(x) = |g'(x)|$ because $B \geq 0$, $p_s > 0$ and $\varepsilon = 1$.

□

Proposition 4.1.3

Let $B > 1$, then g does not have any real fixed point x_0 with $|g'(x_0)| \leq 1$.

Proof

We know that $B > 1$. Suppose now that g has a real fixed point x_0, i.e., $g(x_0) = x_0$. Then from (4.1), g becomes

$$g(z) = A + \varepsilon\left[Bz - \sum_s p_s\left(\frac{1}{z-c_s} + \frac{1}{c_s}\right)\right],$$

and

$$g'(z) = \varepsilon\left[B + \sum_s \frac{p_s}{(z-c_s)^2}\right].$$

Substituting x_0 for z, we obtain

$$g'(x_0) = \varepsilon\left[B + \sum_s \frac{p_s}{(x_0-c_s)^2}\right].$$

Hence,

$$|g'(x_0)| = \left|B + \sum_s \frac{p_s}{(x_0-c_s)^2}\right| = B + \sum_s \frac{p_s}{(x_0-c_s)^2} > 1$$

since $B > 1$ and $p_s > 0$. Therefore, g does not have any real fixed point z_0.

□

4.2 Cauchy density and meromorphic transformations

Let the transformation τ be given by $\tau x = g(x)$ ($x \in \Re$), where g(x) is defined by

$$g(x) = A + \varepsilon \left[Bx - \sum_s p_s \left(\frac{1}{x - c_s} + \frac{1}{c_s} \right) \right].$$

We assume that $B > 1$ or $B < 1$ and g does not have any real fixed point x_0 with $|g'(x_0)| \leq 1$. Then, g(z) has a unique fixed point z_0 with $\mathrm{Im}(z_0) > 0$.

It turns out that then the transformation $\tau: \Re \to \Re$ has at most one invariant probability measure ν that is equivalent to λ. Moreover, the transformation τ of the measure space $(\Re, \nu)$ is exact and thus mixing of all order and ergodic (See [4]).

Next, the τ-invariant $\psi = \dfrac{d\nu}{d\lambda}$ is a Cauchy density of the form

$$\psi(x) = \left(\frac{q}{\pi}\right)\{(x-p)^2 + q^2\}^{-1}, \qquad (4.8)$$

with p and q real constants, $q > 0$, where $z_0 = p + iq$ is precisely the unique complex fixed point in the upper half plane of either the transformation $z' = g(z)$ or $z' = \overline{g(z)}$, depending on whether $\varepsilon = +1$ or $\varepsilon = -1$, respectively (See [4]).

□

Theorem 4.21

The density $\psi(x) = \left(\frac{q}{\pi}\right)\{(x-p)^2 + q^2\}^{-1}$ is τ-invariant.

Proof

It is enough to show that the function $\psi(y)$ does satisfy the functional equation

$$\sum_{\tau x = y} \psi(x) \left| \left(\frac{d}{dx}\right) \tau x \right|^{-1} = \psi(y) \qquad (4.9)$$

for almost all $y \in R$. The proof makes use of the integral

$$(2\pi i)^{-1} \int_L \psi(z)(g(z) - y)^{-1} dz \qquad (4.10)$$

where L is a line parallel to the real axis. Let

$$C_1 = L_\delta + \Gamma_{R,\delta} \tag{4.11}$$

where C_1 is the contour indicated in Figure 4.3, $L_\delta = L: z = x - i\delta \ (x \in [-R, R])$, $\Gamma_{R,\delta} : z = Re^{i\theta} - i\delta \ (\theta \in [0, \pi])$ and $R \to \infty$.
We consider only a sequence of contours $C_{1,N} = L_{\delta,N} + \Gamma_{R_N,\delta}$ such that $\Gamma_{R_N,\delta}$ omit zeros of $g(z) - y$. Then,

$$\oint_{C_1} \frac{\psi(z)}{g(z) - y} dz = \int_{L_\delta} \frac{\psi(z)}{g(z) - y} dz + \int_{\Gamma_{R,\delta}} \frac{\psi(z)}{g(z) - y} dz$$

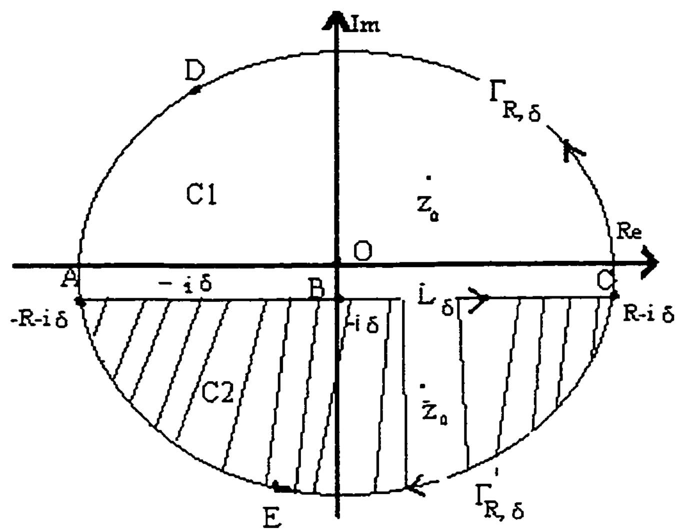

FIGURE 4.3 : The figure of C_1 and C_2.

For $\Gamma_{R,\delta}$,

$$\psi(z) = O(\frac{1}{z^2}) \text{ and } g(z) = O(\frac{1}{z}) \text{ for } |z| \text{ big enough.}$$

Therefore,

$$(2\pi i)^{-1} \int_{\Gamma_{R,\delta}} \frac{\psi(z)}{g(z) - y} dz \approx (2\pi i)^{-1} \int_{\Gamma_{R,\delta}} \frac{1}{z^2} \cdot \frac{1}{z} dz = (2\pi i)^{-1} \int_{\Gamma_{R,\delta}} \frac{1}{z^3} dz$$

Substituting $Re^{i\theta} - i\delta$ for z, then

$$(2\pi i)^{-1} \int_{\Gamma_{R,\delta}} \frac{\psi(z)}{g(z) - y} dz \approx (2\pi i)^{-1} \int_0^\pi \frac{1}{(Re^{i\theta} - i\delta)^3} Re^{i\theta} d\theta$$

$$\approx (2\pi i)^{-1} \frac{1}{R^2} \int_0^\pi \frac{1}{e^{2i\theta}} d\theta \xrightarrow[R \to \infty]{} 0$$

That means that

$$\lim_{R\to\infty}\oint_{C_1}\frac{\psi(z)}{g(z)-y}\,dz = \int_{L_\delta}\frac{\psi(z)}{g(z)-y}\,dz = \int_{L}\frac{\psi(z)}{g(z)-y}\,dz$$

Now, let

$$C_2 = L_\delta + \Gamma'_{R,\delta}$$

where C_2 is the contour indicated in Figure 4.3, L_δ = L : z = x - iδ (x ∈ [-R, R]), $\Gamma'_{R,\delta}$: z = $\mathrm{Re}^{i\theta}$ - iδ (θ ∈ [0, -π]) and R → ∞.

The same consideration gives

$$\lim_{R\to\infty}\oint_{C_2}\frac{\psi(z)}{g(z)-y}\,dz = -\int_{L_\delta}\frac{\psi(z)}{g(z)-y}\,dz = -\int_{L}\frac{\psi(z)}{g(z)-y}\,dz,$$

since C_2 is a circle in the negative (clockwise) direction. That means that

$$\lim_{R\to\infty}\oint_{C_1}\frac{\psi(z)}{g(z)-y}\,dz = -\lim_{R\to\infty}\oint_{C_2}\frac{\psi(z)}{g(z)-y}\,dz. \qquad (4.12)$$

We know that z_0 is precisely the unique complex fixed point in the upper half plane, i.e., $g(z_0) = z_0$. Now, $\frac{1}{g(z)-y}$ has only real and simple poles and $\psi(z)$ has 2 simple poles : z_0 = p + iq and $\bar{z}_0$ = p - iq. Therefore,

$$\lim_{R\to\infty}(2\pi i)^{-1}\oint_{C_1}\frac{\psi(z)}{g(z)-y}\,dz = \sum_{x:g(x)=y}\mathrm{Re}\,s(\frac{\psi(z)}{g(z)-y},x) + \mathrm{Re}\,s(\frac{\psi(z)}{g(z)-y},z_0)$$

and (4.13)

$$\lim_{R\to\infty}(2\pi i)^{-1}\oint_{C_2}\frac{\psi(z)}{g(z)-y}\,dz = \mathrm{Re}\,s(\frac{\psi(z)}{g(z)-y},\bar{z}_0).$$

Then, for any x such that g(x) = y,

$$\mathrm{Re}\,s(\frac{\psi(z)}{g(z)-y},x) = \lim_{z\to x}(z-x)\frac{\psi(z)}{g(z)-y} = \lim_{z\to x}\psi(z)\frac{1}{\frac{g(z)-y}{(z-x)}}.$$

Thus, from Theorem 1.2.1 and Proposition 1.2.1,

$$\operatorname{Res}\left(\frac{\psi(z)}{g(z)-y}, x\right) = \psi(x) \left|\left(\frac{d}{dx}\right) g(x)\right|^{-1}.$$

Then,

$$\begin{aligned}
\operatorname{Res}\left(\frac{\psi(z)}{g(z)-y}, z_0\right) &= \lim_{z\to z_0}(z - z_0)\frac{\psi(z)}{g(z)-y} \\
&= \lim_{z\to z_0}(z - z_0)\left[\frac{q}{\pi}\frac{1}{(z-z_0)(z-\bar{z}_0)}\right]\frac{1}{g(z)-y} \\
&= \frac{q}{\pi}\lim_{z\to z_0}\frac{1}{(z-\bar{z}_0)}\frac{1}{g(z)-y} = \frac{q}{\pi}\frac{1}{(z_0-\bar{z}_0)}\frac{1}{g(z_0)-y} \\
&= \frac{q}{\pi}\frac{1}{p+iq-(p-iq)}\frac{1}{z_0-y} = \frac{q}{\pi}\frac{1}{p+iq-p+iq}\frac{1}{p+iq-y} \\
&= \frac{1}{2\pi i}\frac{1}{(p-y)+iq},
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Res}\left(\frac{\psi(z)}{g(z)-y}, \bar{z}_0\right) &= \lim_{z\to \bar{z}_0}(z - \bar{z}_0)\frac{\psi(z)}{g(z)-y} \\
&= \lim_{z\to \bar{z}_0}(z - \bar{z}_0)\left[\frac{q}{\pi}\frac{1}{(z-\bar{z}_0)(z-z_0)}\right]\frac{1}{g(z)-y} \\
&= \frac{q}{\pi}\lim_{z\to \bar{z}_0}\frac{1}{(z-z_0)}\frac{1}{g(z)-y} = \frac{q}{\pi}\frac{1}{(\bar{z}_0-z_0)}\frac{1}{g(\bar{z}_0)-y} \\
&= \frac{q}{\pi}\frac{1}{p-iq-(p+iq)}\frac{1}{\overline{g(z_0)}-y} = \frac{q}{\pi}\frac{1}{p-iq-p-iq}\frac{1}{\bar{z}_0-y} \\
&= \frac{-1}{2\pi i}\frac{1}{(p-y)-iq},
\end{aligned}$$

since the coefficient of g are real, i.e., $g(\bar{z}_0) = \overline{g(z_0)}$. Now, by substituting τx for g(x), then by (4.12) and (4.13)

$$\sum_{\tau x=y}\psi(x)\left|\left(\frac{d}{dx}\right)\tau x\right|^{-1} + \frac{1}{2\pi i}\frac{1}{(p-y)+iq} = -\left[\frac{-1}{2\pi i}\frac{1}{(p-y)-iq}\right],$$

which implies

$$\sum_{\tau x=y}\psi(x)\left|(\frac{d}{dx})\tau x\right|^{-1} = \frac{-1}{2\pi i}\frac{1}{(p-y)+iq}+\frac{1}{2\pi i}\frac{1}{(p-y)-iq}$$
$$= \frac{1}{2\pi i}[\frac{-1}{(p-y)+iq}+\frac{1}{(p-y)-iq}]$$
$$= \frac{1}{2\pi i}(\frac{-p+y+iq+p-y+iq}{(p-y)^2+q^2})$$
$$= \frac{q}{\pi}(\frac{1}{(p-y)^2+q^2}) = \psi(y).$$

Therefore,

$$\sum_{\tau x=y}\psi(x)\left|(\frac{d}{dx})\tau x\right|^{-1} = \psi(y).$$

□

Example 4.2.1

a) Let $\tau : \Re \rightarrow \Re$ be given by

$$\tau(x) = a\tan(x), \quad x \neq k\frac{\pi}{2},$$

where $k = \pm(2n - 1)$, n is an integer and $a > 1$. We show that the density function $f(x) = \frac{q/\pi}{q^2+x^2}$ is τ-invariant for $q > 0$ satisfying the equation $a \cdot \tanh(q) = q$.

b) More generally, let

$$\tau(x) = a[\tan(bx+c)], \quad x \neq \frac{k}{b}\cdot\frac{\pi}{2},$$

where $k = \pm 1, \pm 3, \ldots$ We will show that if $a \cdot b > 1$, then the density $f(x) = \frac{q/\pi}{(x-p)^2+q^2}$ is τ-invariant.

Proof

a) It is enough to show that τ has a fixed point, i.e., to prove that τ(iq) = iq. We know that

$$\tan(z) = \frac{e^{iz} - e^{-iz}}{i\left(e^{iz} + e^{-iz}\right)},$$

then

$$\tan(iz) = -i\frac{e^{-z} - e^{z}}{\left(e^{z} + e^{z}\right)} = i \cdot \tanh(z).$$

Now, τ (iq) = $a \cdot \tan(iq)$ = $a \cdot i \cdot \tanh(q)$ = $i \cdot [a \cdot \tanh(q)]$ = iq. Then, the equation $a \cdot \tanh(q)$ = q has a unique solution q for any a > 1. By Theorem 4.2.1, the density function f is τ-invariant.

b) As in the first part, it is enough to show that τ has a fixed point, i.e., to prove that τ(p + iq) = p + iq. We know that if tan(z) = p + iq, then

$$p = \frac{\sin(2x)}{\cos(2x) + \cosh(2y)}$$

and

$$q = \frac{\sinh(2x)}{\cos(2x) + \cosh(2y)},$$

which implies

$$p = a \cdot \frac{\sin[2(bp+c)]}{\cos[2(bp+c)] + \cosh(2bq)} \qquad (4.14)$$

and

$$q = a \cdot \frac{\sinh[2(bp+c)]}{\cos[2(bp+c)] + \cosh(2bq)}. \qquad (4.15).$$

Now, for x > 0, we know that $\cosh(x) = \dfrac{e^{x} + e^{-x}}{2} > 1$ and $\sinh(x) = \dfrac{e^{x} - e^{-x}}{2} > 0$.
Therefore, $\cos[2(bp+c)] + \cosh(2bq) \neq 0$ for $q > 0$ since $-1 \leq \cos[2(bp+c)] \leq 1$.
Thus, for $q > 0$ and for any value of p, $a \cdot \dfrac{\sinh[2(bp+c)]}{\cos[2(bp+c)] + \cosh(2bq)} > 0$.
On the other hand, $\lim\limits_{q \to +\infty} a \cdot \dfrac{\sinh[2(bp+c)]}{\cos[2(bp+c)] + \cosh(2bq)}$ is 0. Therefore, (4.15) has a positive solution $q_0 > 0$.

Now, let us consider the function

$$G(p) = a \cdot \frac{\sin[2(bp+c)]}{\cos[2(bp+c)]+\cosh(2bq)}.$$

Then, G(p) has zeros when $2(bp + c) = k\pi \Rightarrow p = \frac{1}{b}\left(\frac{k\pi}{2} - c\right)$, where k is an integer. Therefore, $G(p) = p$ has a solution, say p_0. Thus, the system (4.14) and (4.15) has a solution (p_0,q_0), which defines a fixed point $p_0 + iq_0$ of τ. By Theorem 4.2.1, the density function f is τ-invariant.

□

References

[1] Boyarsky, A., Gora, P., *Laws of Chaos: Invariant Measures and Dynamical Systems in One Dimension*, Birkhauser, New York, 1997.

[2] Jablonski, M., Gora, P., Boyarsky, A., *A general existence theorem for absolutely continuous invariant measures on bounded and unbounded intervals*, Nonlinear World **3** (1996), 183-200.

[3] Jablonski, M., Lasota, A., *Absolutely continuous invariant measures for transformations on the real line*, Zeszyty Nauk Univ. Jagiellon Prace Mat. Zeszyt **22** (1981), 7-13.

[4] Kemperman, J. H . B., *The ergodic behavior of a class of real transformations*, Stochastic processes and related topics. Proc. Summer Res. Inst. Statist. Inference for Stochastic Processes, Indiana Univ., Bloomington, Ind., 1974, Vol. 1, dedicated to Jerzy Neyman, Academic Press, New York, 1975, 249-258.

[5] Lasota, A. and Yorke, J.A., *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. **186** (1973), 481-488.

[6] Rényi, A., *Representation for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. **8** (1957), 477-493.

[7] Spiegel, M. R., *Schaum's Outline of Theory and Problems of Complex Variables*, Schaum Publishing Company, 1964.