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ISBN 0-315-56100-9

The Forcing Relationship for Maps of the Interval

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A Thesis

in

The Department

of

Mathematics

Presented in Partial Fulfillment of the Requirements
for the Degree of Master of Science at
Concordia University
Montréal, Québec, Canada

January 1990

ABSTRACT

The Forcing Relationship

for Maps of the Interval

Mai Chinh

The structures of Stĕfan cycles discovered in 1977 [11] lead Misiurewicz, Block, Hart, Baldwin, etc. [6, 7, 8] to a concept of ordering on patterns of orbits of one-dimentional interval maps, called forcing ($\longrightarrow$).

This thesis introduces $\longrightarrow$ as a partial ordering, studies Stĕfan cycles along with their minimality and discusses an algorithm which verifies the relationship $\longrightarrow$.

Finally, some results of orbits of block structures are exposed under the viewpoint of $\longrightarrow$.

ACKNOWLEDGEMENTS

I am very grateful to my supervisor, Professor W. Byers, for his profound patience, thorough understanding and generous support during the preparation and composition of this thesis Without his encouraging guidance, this work would be impossible

My special thanks go to Professors E. Cohen and A. Malik for their deep concern and continuous encouragement throughout the course of this work.

I am indebted to my family and friends for their moral support during my studies.

In particular I would like to thank my fiancée for her motivation of the work and her careful typing of the manuscript.

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CHAPTERS

Chapter 1

SARKOVSKII'S THEOREM

(STRATIFICATION OF ORBITS IN SIZES)

1.1 Introduction

In this chapter, we shall introduce a remarkable theorem which was proved by Sarkovskii in 1964 This beautiful result classifies families of maps in $C^0(\mathbf{R})$ by arranging them in a certain order We are provided with some strong conclusions about a map (on $\mathbf{R}$) by just assuming continuity

Definition 1.1 (Orbit of a map) *By an orbit of size n of a map f, we mean a set of points*

$$X = \{ x_1 < x_2 < \quad < x_n \}$$

in which f permutes all points of X, cyclically.

Notes:

i) If f has an orbit of size n then we say that f has a point of period n or vice versa. Moreover, a *fixed point* is a point of period 1.

ii) We also denote

$$\mathcal{F}(n) = \{f \in C^0(\mathbf{R}) / f \text{ has a point of period } n\}$$

iii) From now on, we will always assume the continuity of f

1.2 Statement of Sarkovskii's theorem

$$\mathcal{F}(3) \subset \mathcal{F}(5) \subset \mathcal{F}(7) \subset \mathcal{F}(9) \subset \cdots$$
$$\cdots \subset \mathcal{F}(2\cdot 3) \subset \mathcal{F}(2\cdot 5) \subset \cdots$$
$$\cdots \subset \mathcal{F}(2^2\cdot 3) \subset \mathcal{F}(2^2\cdot 5) \subset \cdots$$
$$\cdots \subset \mathcal{F}(2^n) \subset \mathcal{F}(2^{n-1}) \subset \cdots \subset \mathcal{F}(2^2) \subset \mathcal{F}(2) \subset \mathcal{F}(1)$$

thus $\mathcal{F}(1)$ is the largest family and $\mathcal{F}(3)$ is contained in all. In other words, if $f \in \mathcal{F}(3)$ then $f \in \mathcal{F}(n)$ for all n

As a starting point we are going to discuss this statement which is proved by Li and Yorke [1] in 1976, independently of Sarkovskii.

1.3 Propositions

Proposition 1.1 (due to Li and Yorke) *If f has a point of period 3 then f has points of all other periods.*

In order to prove this proposition, let us look at the following two lemmas.

Lemma 1.1 *If $J \subset f(J)$ where J is a closed interval then f has a fixed point in J.*

Proof: Geometrically speaking, in figure 1.1, if $J \subset f(J)$ then the graph of f must cross the line l at some point $x \in J$ which is a fixed point.

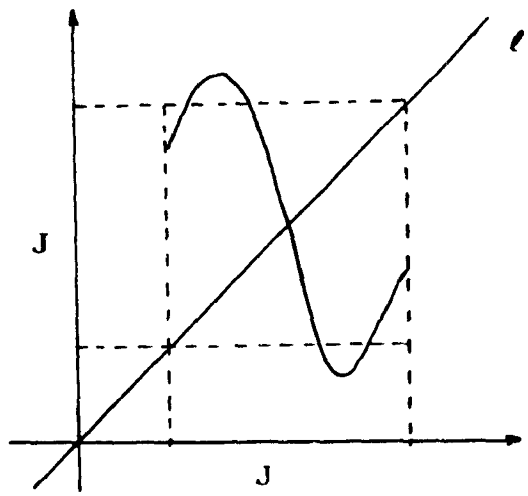


Figure 1.1

Analytically, we can say that

If $J = [a, b]$ and $J \subset f(J)$ then there exists $c, d \in [a, b]$ such that

$$a \leq c \leq d \leq b$$

$$\text{and} \qquad f(c) = a, \qquad f(d) = b$$

therefore by the Intermediate Value Theorem

$$\exists \xi \in [c, d] \ni f(\xi) = \xi$$

Thus f has a fixed point ξ in J. Q.E.D.

Lemma 1.2 *If J_0, J_1,, J_2, ..., J_n is a sequence of closed intervals such that $f(J_i) \supset J_{i+1}$, then there exists a closed interval $I \subset J_0$ such that*

$$f^k(I) \subset J_k, \; k = 1, 2, \ldots, n-1$$

$$\textit{and} \qquad f^n(I) = J_n.$$

Proof: Using induction:

Step 1: The lemma is obviously true when $n = 0$.

Step 2: Assume now that it is true for n. Then in the case $n+1$, we can find a closed interval $I \subset J_0$ such that

$$f^k(I) \subset J_k,\ k = 1, 2, \ldots, n-1$$

$$\text{and} \qquad f^n(I) = J_n$$

Since $\qquad J_{n+1} \subset f(J_n) = f^{n+1}(I)$

thus we can find a closed interval $I' \subset I$ such that

$$J_{n+1} = f^{n+1}(I')$$

$$\text{and} \qquad J_k \supset f^k(I) \supset f^k(I'),\ k = 1, \ldots, n.$$

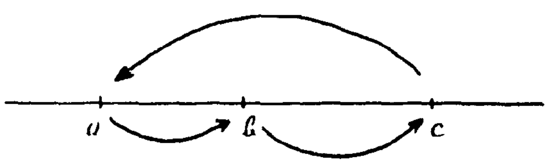
Therefore we get our conclusion. Q.E.D.

With these two lemmas, we can now prove Proposition 1.1.

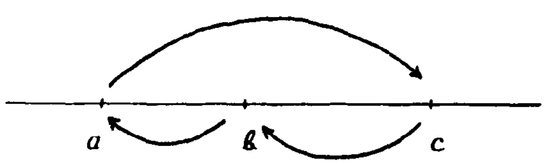
Proof of Proposition 1.1: Let $\{a < b < c\}$ be an orbit of size 3 of f

There are only two possibilities:

Case 1: $f(a) = b,\ f(b) = c,\ f(c) = a$



Case 2: $f(a) = c,\ f(c) = b,\ f(b) = a$



Consider case 1, let $I = [a, b]$ and $J = [b, c]$

then $J \subset f(I)$ and $J \cup f(J)$

Look at the sequence $I, \underbrace{J, J, \ldots, J}_{\text{n-1 times}}, I$.

By lemma 1.2, we can find a closed interval $K \subset I$ such that $f^n(K) = I \supset K$.

By lemma 1.1, there exists a fixed point $x \in K$ of f^n.

Moreover, since $f^i(x) \in J$ and $J \cap I = \{b\}$,

therefore $\{f^i(x) : \ i = 1, 2, \ldots, n\}$ is set of different points.

Hence f has a point of period n.

Similarly in case 2 which is "symmetric" with case 1, we also get the same conclusion. Q.E.D.

We now introduce the concept of a Markov graph which will be used in our subsequent discussion.

Definition 1.2 (Markov graph) *By a Markov graph of a map on points* $\{p_1, \ldots, p_{n+1}\}$ *we mean a directed graph whose vertices are intervals* $I_i = [p_i, p_{i+1}]$ *along with arrows* $\rightarrow$ *defined by:*

$$I_i \rightarrow I_j \qquad \text{iff} \qquad f(I_i) \supset I_j$$

Notes:

i) Using the result of the previous two lemmas, we can say that in a Markov graph of f, if we have a (directed) loop of n arrows then we have a fixed point of f^n: $f^n(\xi) = \xi$.

Moreover if the set of n points $\{f^i(\xi);\ i = 1, 2, \ldots, n\}$ are all different then we say that f has an orbit of size n.

ii) Also note that if $f^m(I_i) \supset I_j$, then there exists a path of length m from I_i to I_j.

1.4 Proof of Sarkovskii's Theorem

We present the proof in 8 steps. Steps 1 and 2 are valid for any n. Steps 3, 4, and 5 deal with odd values of n. Finally steps 6, 7, and 8 complete the proof with the case where n is even.

Step 1: Suppose f has an orbit of size n

$$X = \{x_1 < x_2 < \ldots < x_n\}$$

Since $f(x_n) < x_n$, we can pick the largest i such that

$$f(x_i) > x_i \text{ and } f(x_{i+1}) < x_{i+1}$$

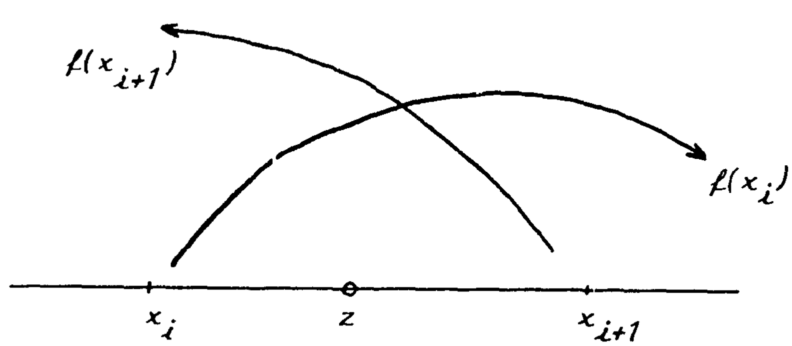


Figure 1 2

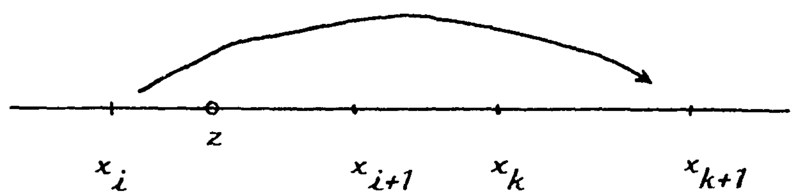
Let $I_1 = [x_i, x_{i+1}]$, then figure 1.2 gives us $I_1 \subset f(I_1)$

By lemma 1.1 there exists a fixed point $z \in I_1$.

Step 2:

Claim: In the Markov graph of f, there is a path from I_1 to every subinterval (vertex) I_k.

Proof: Let $I_k = [x_k, x_{k+1}]$.



If $I_k \geq I_1$ then $\exists m \ni f^m(x_i) = x_{k+1}$.

Since $f^m(z) = z$ therefore $I_k \subset f^m(I_1)$.

Similarly if $I_k \leq I_1$ then

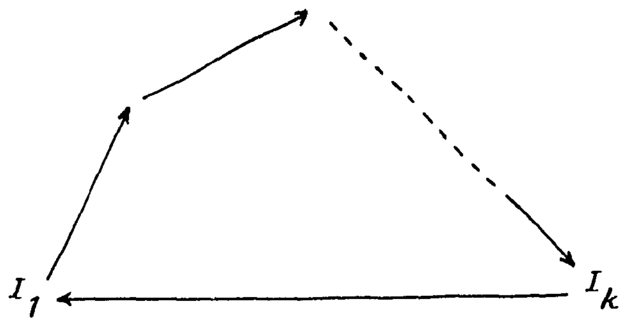
$$\exists m' \ni f^{m'}(I_1) \supset I_k \; (f^m(x_{i+1}) = x_k)$$

By note (ii) following the definition of Markov graph, the proof is complete.

Case n is odd:

Step 3: If $n > 1$ is odd, then there exists a number k such that $I_1 \subset f(I_k)$

(Thus in the Markov graph we can find a loop



)

Proof (Step 3): Consider the fixed point z.

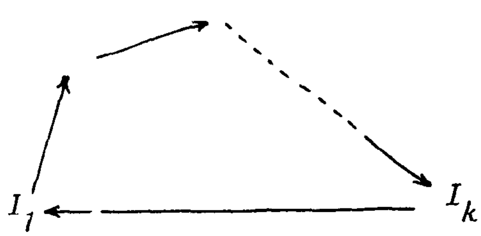
After one iteration of f, if every point x_j switches sides about z, then z must have an equal number of points x_j's at both sides, hence X has an *even* number of points which contradicts our assumption.

Therefore there must be some point whose image under f is on the *same* side of z.

Moreover we know that x_i, x_{i+1} switch sides about z. Therefore we can find x_k, x_{k+1} whose images are on *both* sides of z.

$$\Longrightarrow \; I_1 \subset f(I_k)$$

Consider the loop in the Markov graph of f



then we can pick the smallest loop (i.e. no inner loop) such that all vertices are different.

Step 4: If n is the smallest odd integer such that $f \in \mathcal{F}(n)$ then $k = n - 1$.

Proof: Suppose that $k < n - 1$ then one of the loops

$$I_1 \to I_2 \to \ldots \to I_k \to I_1 \to I_1$$

$$\text{or} \qquad I_1 \to I_2 \to \ldots \to I_k \to I_1$$

will provide a fixed point of f^m where m is odd, $m < n$.

Also that point must have odd period m $(< n)$ under f (since I_i's consist of only x_j's in common) which contradicts the assumption.

Therefore $$k = n - 1$$

Moreover the above argument also shows us that in the Markov graph there must be *no* short-cuts (see figure 1.3).

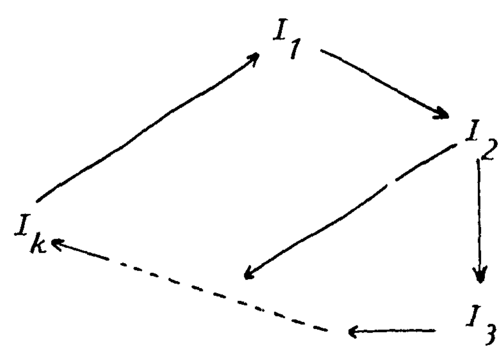
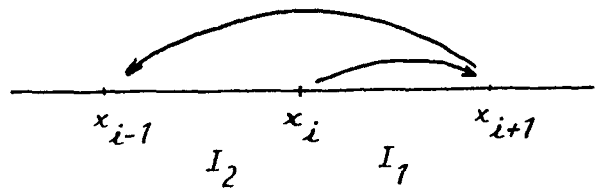


Figure 1.3

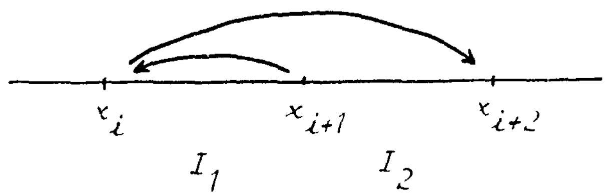
Q.E.D.

Step 5: Analyzing the type of orbit X (in step 4),

we have $I_1 \cup I_2 \subset f(I_1)$ since there is no short-cut Thus there are only two situations



or



For the first situation we can see that $f(I_2)$ contains I_3 and nothing else. This implies that

$$f(x_{i-1}) = x_{i+2}$$

where $$I_3 = [x_{i+1}, x_{i+2}]$$

$$\vdots$$

$$I_{n-1} = [x_1, x_2]$$

with $f(x_1) = x_i$ and $f(x_2) = x_n$

We then can construct the type of orbit of X as follows:

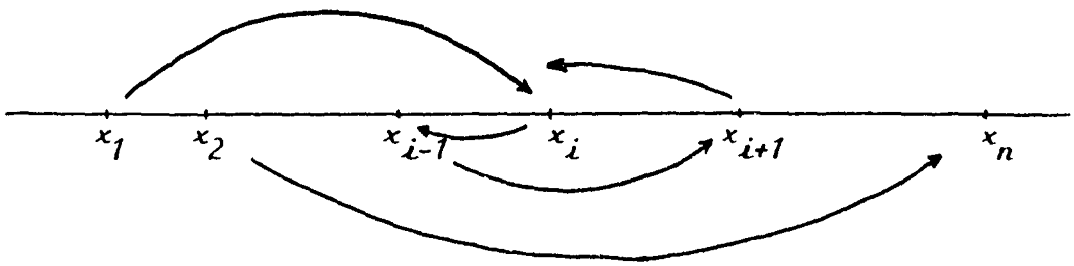
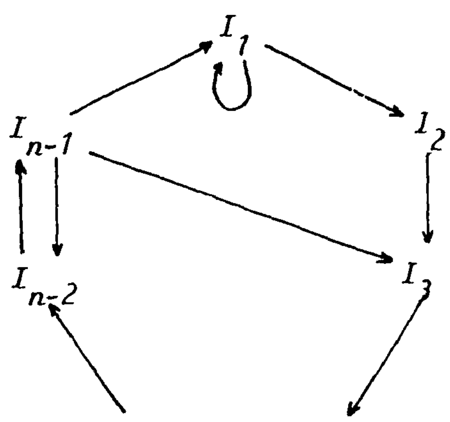


Figure 1.4

along with its Markov graph.



(There are arrows from I_{n-1} to every I_j, j odd)

Similarly for the second situation, the type of X looks almost alike with the order reversed. On the other hand, X has the same Markov graph.

Now by using the Markov graph, we can see that there exist loops of length $k \geq n-1$, for any given k (by repeating I_1). Also there are loops of even length $\leq n-1$ by considering arrows from I_{n-1}.

In other words, we have proved that

$$\mathcal{F}(3) \subset \mathcal{F}(5) \subset \mathcal{F}(7) \subset \cdots \subset \{\mathcal{F}(even)\}$$

Remarks:

i) We have shown that if the orbit X is of smallest odd size n then its Markov graph has a certain structure and has only two possible forms.

These types of orbit X are called *Stĕfan cycles* and will be studied more carefully in chapter 3.

ii) As a corollary, if X with odd size n is not of Stĕfan type then we must have an orbit of size $n-2$.

Case n is even:

For the case of even n in steps 6, 7, and 8, we will borrow the concept of *separated orbit of order* r which will be introduced in chapter 3.

Step 6:

Claim: Suppose f has *only* points of *even* periods then every orbit X is separated by a fixed point of f.

Proof: Recall in step 1, there exists an interval I_1 such that $I_1 \subset f(I_1)$ and also there exists a fixed point $z \in I_1$.

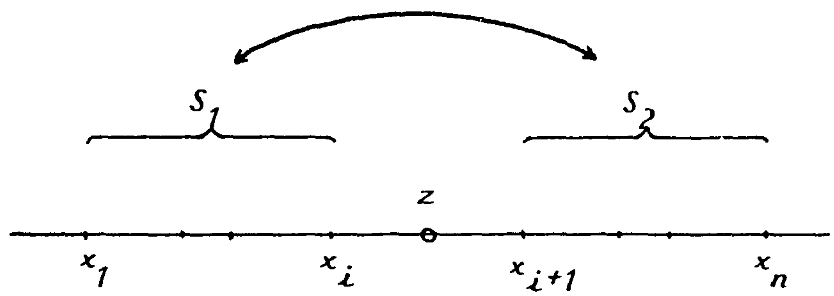
Regarding that point z, if *not all* points switch sides around z (after one iteration) then there must be an interval I_k such that

$$I_1 \subset f(I_k)$$

However in step 2, we have concluded that there exists a loop

$$I_1 \to I_2 \to \ldots \to I_k \to I_1 \qquad \underbrace{\ldots \to I_1 \ldots}_{\text{as many times as we want}}$$

In other words this loop contradicts our assumption that f has only even periods. Equivalently, all points x_i's switch around z.



And thus X is separated of order 1 by z. Q.E.D.

As a consequence, we have the following result:

Corollary 1.1 *If f has any point of even period then f has a point of period 2.*

Proof: In fact, if all periodic points of f are of even size then we have the conclusion as above. Otherwise we have the proof of case n odd. Q.E.D.

Step 7 (Order on $\mathcal{F}(2^m q)$, with q odd):

Proposition 1.2 *If all orbits are of even size $|X| = 2^r q$ (q odd ≥ 1) then each of them is separated of order r by a periodic point of period 2^{r-1}. Also, f has a point of period 2^r.*

Proof: We will use induction on r.

i) $r = 1$ is verified by step 6 and its corollary.

ii) Assume that it is true for r.

Now for $r + 1$, each orbit X is separated of order 1,

$$X = S_1 \cup S_2$$

and f^2 reduced on S_1 and S_2 satisfies the assumption, i.e. f^2 separates S_1 and S_2 of order $r - 1$. Also f^2 has a point of period 2^r.

Thus f separates X of order r and f has a point of period 2^{r+1}.

(*Note*: This type of orbit is of another Stēfan form.) Q.E.D.

Step 8:

Proposition 1.3 *If f has a point of period 2^m then f has also a point of period 2^l, with $l < m$*

Proof: The statement would be true if we show that

If f has period 2^m then f has period 2^{m-1}

We will use induction on m.

Suppose $m > 2$, then consider $g = f^{2^{m-2}}$

$$\Longrightarrow g \text{ has a point of even period}$$

By corollary 1.1 in step 6,

$$\Longrightarrow g \text{ has a point of period } 2$$

which we can see is a point of period 2^{m-1} of f.

Thus in this step, we have proved

$$\cdots \mathcal{F}(2^n) \subset \mathcal{F}(2^{n-1}) \subset \cdots \subset \mathcal{F}(2) \subset \mathcal{F}(1)$$

which completes the proof of Sarkovskii's theorem. Q.E.D.

Note: For convenience, we also write the Sarkovskii's ordering as following

$$3 \triangleright 5 \triangleright \ldots 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2^n \triangleright \ldots \triangleright 1$$

Chapter 2

FORCING ON CYCLES

(STRATIFICATION OF ORBITS IN PATTERNS)

2.1 Introduction

In this and following chapters, we will continue to study orbits of maps in $C(\mathbf{R})$ where their structures are taken into account. This new viewpoint creates certain ordering on the orbits and is more complicated in the sense that we do not have generally a relationship among orbits.

Some concepts such as *cycles* and *forcing* are basic ones and will be regarded carefully.

2.2 Forcing on cycles

Definition 2.1 (Cycles) *Let $X = \{x_1 < x_2 < \ldots < x_n\}$ be an orbit of a map f. Let $p = \{1, 2, \ldots, n\}$ be a permutation on the set $\{1, 2, \ldots, n\}$. Now if we have*

$$f(x_i) = x_j \text{ iff } p(i) = j$$

Then we call the set $\{1, p(1), p^2(1), \ldots, p^{n-1}(1)\}$ in this order a cycle of type p.

Notes:

i) Obviously, our notation now is on the behaviour of the dynamics of X, i.e. the way it is arranged instead of its size, e.g.

$$\{x_1 \longrightarrow x_2 \longrightarrow x_3 \longrightarrow x_1\} \text{ and } \{x_1 \longrightarrow x_3 \longrightarrow x_2 \longrightarrow x_1\}$$

are different ones.

ii) Without loss of generality, from now on we mean *cycle of type p* every time we mention orbit X.

iii) Denote by C_n as the set of all cycles *of size* n then C_n has $(n-1)!$ elements

Definition 2.2 (Forcing) *Let p and q be cycles (which may be of different sizes). We then say that p* forces *q, denoted by* $p \longrightarrow q$ *if and only if for every continuous map* $f \in C(\mathbf{R})$, *if f has cycle p then f has cycle q.*

If we denote C as the set of all cycles, $C = \bigcup C_n$, then this concept of forcing provides a binary relation which will be proved to be a partial ordering on C.

Moreover, by Sarkovskii's theorem, we know that each cycle p of size n must force some cycle q of suitable size m. Therefore $\longrightarrow$ creates a *network on C*.

Our main target is to study the general picture of this network. What are its starting points as well as its end-points where $\longrightarrow$ is regarded as a *greater than* relation.

Definition 2.3 (Primitive map) *Let p be a cycle of size n,* $p \in C_n$. *The* primitive map *of p, denoted by* T_p, *is a function in* $C(\mathbf{R})$ *defined by*

$$T_p(x) = \begin{cases} p(1) & x \leq 1 \\ (p(i+1) - p(i))x + (i+1)p(i) - ip(i+1) & i \leq x \leq i+1 \\ p(n) & x \geq n \end{cases}$$

In other words, T_p is the piecewise linear map which join all the dots $(i, p(i))$ in the simplest way.

Clearly T_p has p as a cycle and indeed is minimal (in the sense of the following lemma)

Lemma 2.1 *Let $f \in C(\mathbf{R})$ with cycle p, thus induces T_p.*

If T_p has q as a cycle, then f also has cycle q.

Proof: Consider the simple case in figure 2.1.

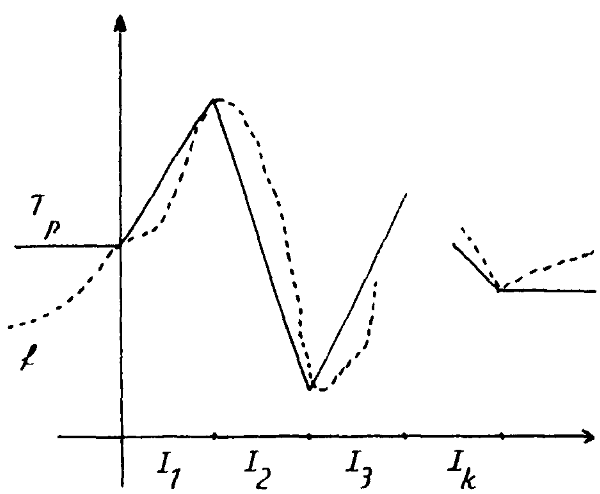


Figure 2.1

Since f is continuous therefore

$$\text{if } I_i \subset \mathcal{T}(T_p) \text{ then } I_i \subset f(I_j) \tag{1}$$

Now if T_p has cycle q, then the Markov graph of T_p must have a loop

$$I_{i_1} \longrightarrow I_{i_2} \longrightarrow \dots \longrightarrow I_{i_m} \longrightarrow I_{i_1}$$

corresponds to q

By (1), the Markov graph of f must also have this loop, i.e f also has cycle q

Q.E.D.

Lemma 2.1 is a simple but critical result. It restricts our attention to primitive maps in the study of $\longrightarrow$. These T_p's which have the same dynamics with respect to forcing as any other f_p's, are nicely behaved maps whose monotonicity on each interval I_k gives us some initial control.

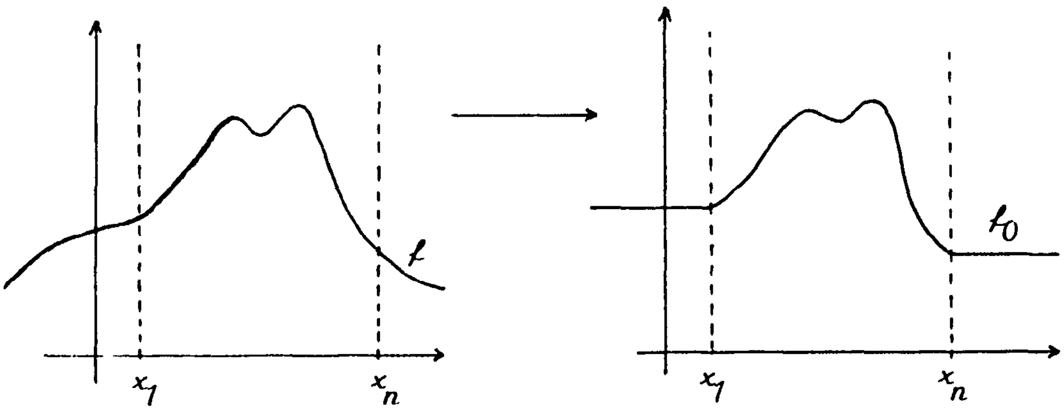
2.3 Properties of $\longrightarrow$

Theorem 2.1 $\longrightarrow$ *is a partial ordering.*

Proof:

i) Reflexivity and transitivity are obvious from the definition of $\longrightarrow$.

ii) Antisymmetry: Suppose that $p \longrightarrow q$ and $q \longrightarrow p$. Now let f be a polynomial which has an orbit of type p. Since each orbit must satisfy the equation $f^n(x) = x$ which has only finitely many roots, we can see that f has only a finite number of those orbits of type p. Pick orbit $X = \{x_1, x_2, \ldots, x_n\}$ of smallest diameter Consider a modified function f_0 derived from f as follows:



$$\text{i.e., } f_0 \simeq f \text{ in } [x_1, x_n]$$

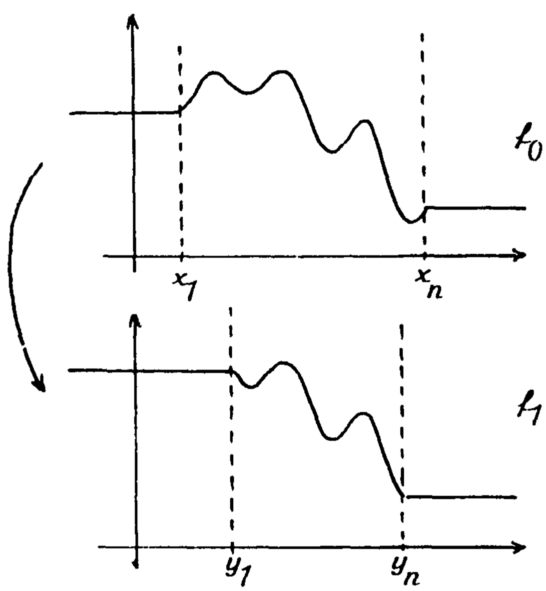
$$\simeq \text{ illustrated constants outside } [x_1, x_n]$$

Now f_0 has an orbit of type p then it also has an orbit of type q:

$$Y = \{y_1, y_2, \ldots, y_m\}$$

and assume $y_1 < y_2 < \ldots < y_m$ and $[y_1, y_m] \subseteq [x_1, x_n]$

Consider another function f_1 derived from f_0 as follows: f_0 is cut off by considering *smaller intervals* as when we squeeze f_0 from f.



Now f_1 has a cycle of type q, thus it also has a cycle of type p whose orbit is given by

$$X'_p = \{x'_1 < x'_2, \ldots, x'_n\}$$

Obviously,

$$[x'_1, x'_n] \subseteq [y_1, y_m] \subseteq [x_1, x_n]$$

Moreover f_1 by construction, is identical with f in the interval $[x'_1, x'_n]$ thus we imply that f also has X'_p as an orbit of type p.

However by minimality of X_p we imply $X'_p = X_p$. Therefore $[x_1, x_n] = [x'_1, x'_n]$. Thus the endpoints $y_1 \simeq x_1$ and $y_m \simeq x_n$ and since $f_0 \simeq f$ in the intervals $[x_1, x_n] = [y_1, y_m]$, we imply that

$$X_p = Y_q$$

$$\text{thus} \qquad p \simeq q$$

and therefore our proof is complete. Q.E.D.

Notation: Let $p \in C_n$ be a cycle with primitive map $\mathcal{T}_p$. Denote

$$z(\max p) = \# \text{ of locally maximum points of } \mathcal{T}_p$$

$$z(\min p) = \# \text{ of locally minimum points of } \mathcal{T}_p$$

$$z(p) = \# \text{ of extreme points of } \mathcal{T}_p$$

If $z(\max p) = z(p) = 1$ then we say that p is *unimodal*.

Theorem 2.2 *Let p and q be two cycles such that $p \longrightarrow q$. Then*

a) $z(\max p) \geq z(\max q)$

b) $z(min\ p) \geq z(min\ q)$

and as a conseqtence $z(p) \geq z(q)$.

Proof: The proof is simple by merely using the graph in figure 2.2.

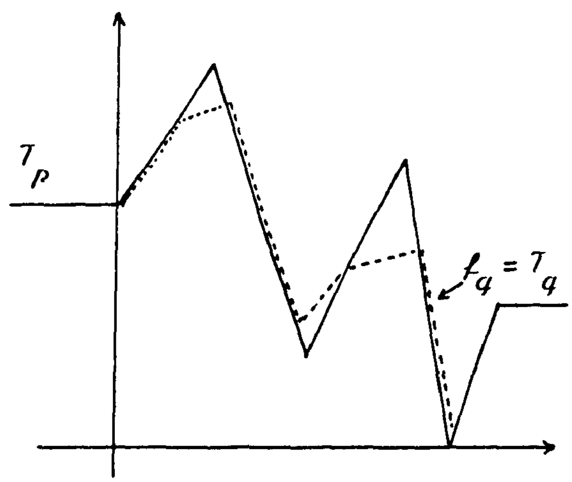


Figure 2.2

If $p \longrightarrow q$ then $\mathcal{T}_p$ can be embedded as a piecewise linear map joining dots on $\mathcal{T}_p$. From figure 2.2 we can see that between any two points which create a local maximum point of $\mathcal{T}_p$, there must be a local maximum point for $\mathcal{T}_p$, i.e.

$$z(\max p) \geq z(\max q)$$

Smilarly for local minimum points we have

$$z(\min p) \geq z(\min q)$$

Q.E.D.

Geometrically this theorem tells us that if $p \longrightarrow q$ then $\mathcal{T}_p$ must have a piece whose shape "looks like" $\mathcal{T}_q$.

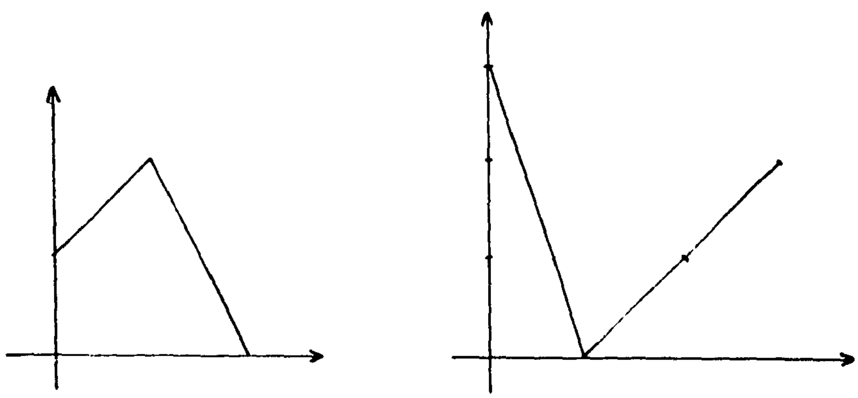
Thus the theorem gives us a way to rule out $p \longrightarrow q$ by looking at their variations.

Corollary 2.1 *Unimodal maps can force only unimodal ones.*

Proof: By definitions and the above remark.

Corollary 2.2 $\longrightarrow$ *is not totally ordering.*

Proof: Consider the cycles (123) and (1432) in the figure below.



Then theorem 2.2 proves that neither of them forces the other.

However if we look at unimodal cycles only then $\longrightarrow$ forms a linear ordering. This is proved using the kneading theory [15]. Q.E.D.

Theorem 2.3 *Let U be the set of all unimodal cycles. The $(U, \longrightarrow)$ is a total ordering.*

This theorem also gives us a fact that $(U, \longrightarrow)$ has a largest element which can be proved to be the cycle (1342). Also note that $(1342) \notin U$.

2.4 An algorithm for $\longrightarrow$

This section presents a theorem due to S Baldwin [7] which is used to verify whether $p \longrightarrow q$ or not.

Definition 2.4 (Balwin graph) *A* Balwin graph $G_p = (G, sgn)$ *of a cycle p of size n is defined as follows:*

i) G is the Markov graph of p whose nodes are the set $\{1, 2, \ldots, n-1\}$ *and arrow from i to j if* $T_p[i, i+1] \supseteq [j, j+1]$.

ii) $sgn : G \longrightarrow \{-1, 1\}$ *is a function with*

$$sgn(i) = \begin{cases} 1 & \text{if } T_p \uparrow \text{ in } [i, i+1] \\ -1 & \text{if } T_p \downarrow \text{ in } [i, i+1] \end{cases}$$

Therefore Baldwin graph is merely Markov graph where monotonicity of the function (i.e. the primitive map) is considered.

Definition 2.5 (Closed walk) *Let* G_p *be a Balwin graph. A* closed walk of length *K is a loop* $(a_1 \frown a_2 \frown \ldots \frown a_k \frown a_1) = \bar{a}$ *on the (directed) Markov graph of* G_p.

Definition 2.6 *Let* $\bar{a}$ *and* $\bar{b}$ *be two closed walk of same length K. Then we say that* $\bar{a} < \bar{b}$ *if*

$$\left(\prod_{i=1}^{j-1}(sgn(a_i)\right) a_j < \left(\prod_{i=1}^{j-1}(sgn(a_i)\right) b_j$$

where j is the least *integer such that* $a_j \neq b_j$.

Note: $<$ is then a total ordering on closed walks of the same length.

Definition 2.7 *A* shift-operation sh *on closed walk* $(a_1 \frown a_2 \frown .. \frown a_k) = \bar{a}$ *is defined by*

$$sh(\bar{a}) = (a_2 \frown a_3 \frown .. \frown a_1)$$

i.e. $sh(\bar{a})$ *is the same loop whose first node is the second one of* $\bar{a}$.

Definition 2.8 *Let* W *be any (finite) collection of* m *closed walks* $\bar{a}$ *which is closed under* sh *(thus these walks are of same length).*

The type of W, *denoted by* $t(W)$, *is a cycle* q *defined as follows:*

a) Let ord: $W \longrightarrow \{1, 2, \ldots, m\}$ *be the unique 1-1 onto order-preserving function, i.e.,* $\bar{a} < \bar{b}$ *if and only if* $ord(\bar{a}) < ord(\bar{b})$.

b) Define $q(i) = ord \circ \text{sh} \circ ord^{-1}(i)$.

In other words, $t(W)$ *is a cycle* q *whose structure is defined by the shift-operation on orders of the closed walks.*

Definition 2.9 *Let* $\bar{a}$ *be a closed walk. Let* W *be the smallest set which contains* $\bar{a}$ *and which is closed under* sh. *Then the type of* $\bar{a}$, *defined by* $t(\bar{a}) = t(W)$.

Note: We choose W to be smallest to prevent the situation where $\bar{a}$ is a repetitive loop.

Theorem 2.4 (Balwin's Theorem) *Let* $p \in C_n$ *and* $q \in C_m$. *Then* $p \longrightarrow q$ *if and only if either* $p = q$ *or for some closed walk* $\bar{a}$ *of length* m *in* G_p, $t(\bar{a}) = q$.

The theorem then gives us a way to check if $p \longrightarrow q$.

ALGORITHM

Let $p \in C_n$ and $q \in C_m$ with $p \neq q$.

Step 1: Construct the Balwin graph G_p of p (by definition 2.4).

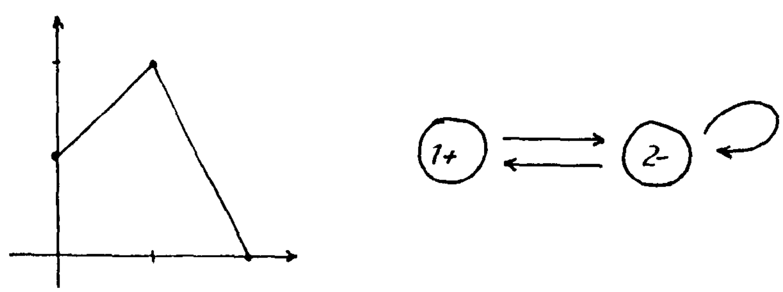
Step 2: Find all closed walks $\bar{a}$ of length m from G_p.

Step 3: For each $\bar{a}$, construct $t(\bar{a})$ (by definitions 2.7, 2.8, 2.9).

Thus the algorithm gives all cycles $q' \in C_m$ such that $p \longrightarrow q'$.

Example: Find all cycles of size 5 which are forced by (123).

Solution: First of all by Sarkovskii's theorem, we know that there exists such a cycle q.



Balwin graph of (123)

Now there are only two relevant closed walks of size 5 (upto equivalent modulo sh), namely (22121) and (21221). (Note that (22222) is repetitive). We have

$$(12212) < (12122) < (22121) < (21212) < (21221)$$

therefore $t(\bar{a}) = (13425)$ with $\bar{a} = (22121)$.

Similarly we also have $t(\bar{b}) = (13425)$ with $\bar{b} = (12222)$.

And since $\bar{a}, \bar{b}$ are not equivalent (*mod sh*), this implies that there exist only two distinct orbits of the same type (13425) which are forced by (123).

However in practice even when p and q are of small sizes we still have to handle a massive computation in order to determine whether $p \longrightarrow q$.

We now consider the end points of the chain $\longrightarrow$ of the subnetwork on C given by cycles of a given length m.

Definition 2.10 (Primary cycles) *A cycle p of length n is said to be* primary *if and only if $p \longrightarrow q$ implies that $p \equiv q$ for $|p| = |q|$, that is the only cycle of length n which can be forced by a primary cycle of same size is itself.*

We will see later that primary cycles in fact are the simplest ones in the sense of Markov graph and have a certain structure.

It is also true that the primary cycles minimize a certain topological invariant called the *topological entropy* [12, 13] which is usually considered to measure the complexity of a given dynamical system.

Chapter 3

SIMPLE AND STĚFAN CYCLES

In this chapter, we will study cycles whose structures are arranged in a "nice" way. These cycles have been referred to in the proof of Sarkovskii's theorem in chapter 1. Now we shall examine them in relation to the $\longrightarrow$ ordering.

3.1 Definitions

Definition 3.1 (Block structure) *Let* $X = \{x_1 < x_2 < \ldots < x_n\}$ *be an orbit of type* p *and* $Y = \{y_1 < y_2 < \ldots < y_m\}$ *be an orbit of type* q. *We say that* X *has a* block structure *over* Y *if and only if*

i) $n = sm$

ii) We can write

$$X = X_1 \cup X_2 \cup \ldots \cup X_m$$

$$\textit{where} \qquad X_i = \{x_{is+1} < x_{is+2} < \ldots < x_{is+s}\}$$

$$\textit{and} \qquad p(X_i) = X_j \quad \text{iff} \quad q(Y_i) = Y_j.$$

In other words, the dynamics of X_i*'s under* p *are the same as of* Y_i*'s under* q.

Definition 3.2 (z-extension) *Let* z *be a cycle.* X *is said to have a* z-extension *over* Y *if*

i) X *has a block structure over* Y, *ii) p is monotonic on all blocks X_i's except at exactly one k where*

$$p_m | X_k \simeq z$$

Notes:

i) For convenience, we shall also say that p is a z-extension of q.

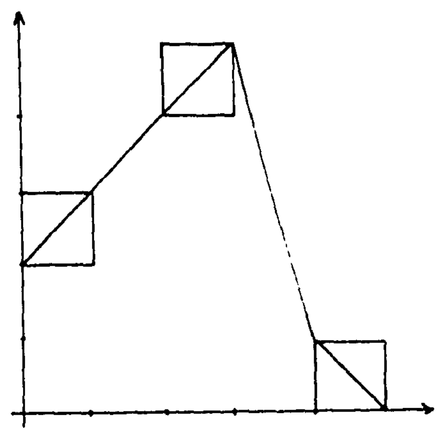
ii) Extensions are not commutative, i.e., in general

$$z\text{-extension of } q \neq q\text{-extension of } z.$$

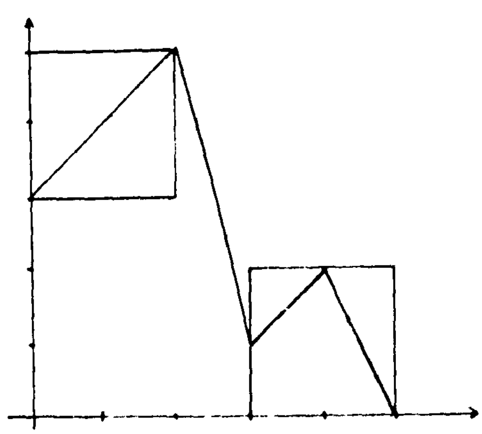
as we can see in the following example.

Example: Let $z = (12)$ and $q = (123)$ then $p_1 = z$-extension of q and $p_2 = q$-extension of z are defined and graphed as follows:

$$p_1 = \begin{pmatrix} 1\,2 & 3\,4 & 5\,6 \\ 3\,4 & 5\,6 & 2\,1 \end{pmatrix}$$



$$p_2 = \begin{pmatrix} 1\,2\,3 & 4\,5\,6 \\ 4\,5\,6 & 2\,3\,1 \end{pmatrix}$$



Note: Those p_1 and p_2 are not the only extensions from z, q as we will see in the following theorem.

Lemma 3.1 *Let z and q be two cycles of orders k and m respectively. Then*

i) there are $[(s-1)!]^m s^{m-1}$ cycles of order $n = sm$ which has block structure over q.

ii) there are $m \cdot 2^{m-1}$ cycles which are z-extension of q.

Proof:

i) Let p be of block structure over q. Then each block (with s points) of p has $s!$ ways to map to another block. However to make p into a cycle, the last block has only $(s-1)!$ ways to go.

ii) Given z and q, there are m choices of the "special block" (which is not monotonic), each of the others has 2 choices for monotonicity (either $\uparrow$ or $\downarrow$). Also the special block is determined by z and $(m-1)$ of others. Thus we have our result. Q.E.D.

Note: The idea of block structure and z-extension is shared by Misiurewicz [8], Baldwin [7], and Block [4].

Definition 3.3 (Separated orbit of order r) *Let $X = \{x_1 < x_2 < \ldots < x_n\}$ be an orbit.*

i) We say that X is separated of order 1 if $n = 2m$ and X has a block structure over cycle (12).

ii) Moreover we say that X is separated of order r if X is separated of order 1 and each of the blocks $X_1 = \{x_1 < x_2 < \ldots < x_m\}$ and $X_2 = \{x_{m+1} < x_{m+2} < \ldots < x_n\}$ is separated of order $r-1$ under f^2.

Notes:

i) Obviously if X is separated of order r then $2^r | n$.

ii) By the proof of Sarkovskii's theorem, we can see that there exists a point of period 2^{r-1} which separates 2^r blocks of X.

Definition 3.4 (Simple cycles) *Let X be an orbit of f of size $n = 2^r m$, with m odd. We say that X is of* simple type *if*

i) X is separated of order r, and

ii) We can write $X = X_1 \cup X_2 \cup \ldots \cup X_{2^r}$ with each X_i contains m points in that order i. Moreover

$$X_i = \{x_{(i-1)m+1} < x_{(i-1)m+2} < \cdots < x_{(i-1)m+m}\}$$

is periodic under $f^{(2^r)}$, with an arrangement which looks like the one discussed in step 5 of Sarkovskii's theorem.

In 1983, Block and Hart [5] proved the following theorem.

Theorem A *If f has a point of period n, then f must have a point of simple type of period n.*

Therefore from the view point of forcing we can see that every cycle of size n must force a simple cycle of the same size. In other words, simple cycles are "smaller" in the subnetwork $(C_n, \longrightarrow)$. We shall now examine those which are "smallest".

Definition 3.5 (Stĕfan cycles) *An orbit X with size $n = 2^r m$, with m odd is called a Stĕfan cycle if*

i) X is separated of order r and

ii) f is monotonic on all blocks X_i's, except at exactly one block X_k, $f^{(2^r)}$ is simple of order 0.

Notes:

i) Stĕfan cycles are obviously simple.

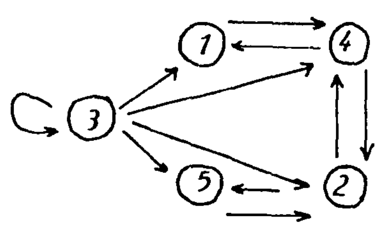
ii) If $n = 2^r$ or n is odd, simple cycles are also Stĕfan cycles.

iii) Stĕfan cycles are r numbers of extensions of two-point cycles from a one-point cycle and then an extension of at most one odd simple cycle.

In 1986, Block and Coppel proved a stronger theorem [6].

Theorem B *If continous f has an orbit of size n then f has also a Stĕfan orbit of the same size.*

Examples: Cycles of order 6 (Stĕfan and simple)



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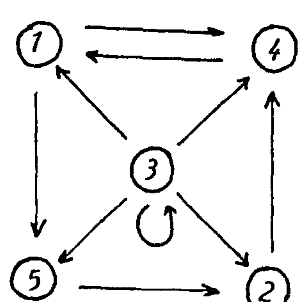

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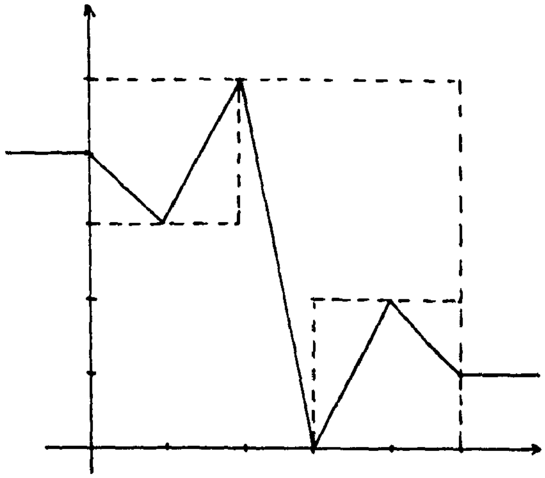
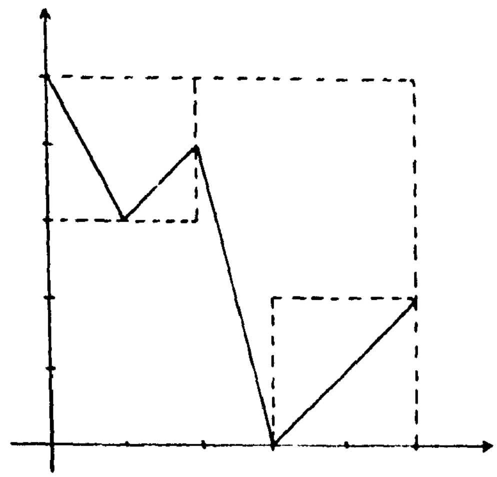




3.2 Minimality of Stẽfan cycles

Theorem 3.1 *A cycle p is primary if and only if it is Stẽfan.*

Proof:

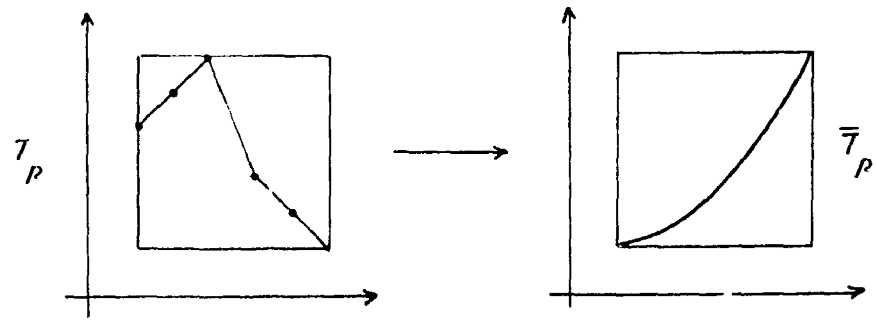
a) ($\Rightarrow$) Given p is primary then by theorem B, p must force a Stẽfan cycle q of the same size. By definition of primality, this implies that p is Stẽfan.

b) ($\Leftarrow$) Now suppose that p is a Stẽfan cycle.

Case 1: If the size $p = n$ with n odd, then recall in the proof of Sarkovskii's theorem, the Markov graph of p has a certain structure, and the maximal loop (closed walk) of length n is p itself. That is p cannot force any cycle of the same length. (Note that in this case, we can also use the Balwin Algorithm to show that $p \longrightarrow q$ then $q \simeq p$, with q of the same size)

Case 2: If $|p| = n = 2^m l$, l odd ≥ 3, then consider the primitive map of $p:\ \overline{\mathcal{T}}_p = \mathcal{T}_p$.

If $|p| = n = 2^m$, then we can modify T_p as $\overline{T}_p$ by making the map expand in one block



We will show that $\overline{T}_p$ has no other orbit of size n, i.e. p forces no other cycle.

Consider the following observations: Let $X = \{1\ 2 \ldots l \vdots l+1\ l+2 \ldots\}$ be the cycle p. If $Y = \{y_1 < y_2 < \ldots < y_n\}$ be any orbit of $\overline{T}_p$, then

i) y_i's are not integer.

ii) y_i cannot belong to those repelling intervals $[kl, kl+1]$. In other words y_i's occur in blocks $[1,l]$, $[l+1, 2l], \ldots$ only.

iii) In exactly one interval in the crucial block, $\overline{T}_p$ is expanding. In all others, $\overline{T}_p$ maps end points to end points, with length preserving, i.e. in all other intervals, $|x - y| = |\overline{T}_p(x) - \overline{T}_p(y)|$.

Therefore consider any interval $[[y_1], y_1]$ with its inverse image $\overline{T}_p^{-n}$, we can see that

$$\begin{aligned} \overline{T}_p^{-n}[[y_1], y_1] &= [\overline{T}_p^{-n}[y_1], \overline{T}_p^{-n}[y_1]] \\ &\not\subseteq [[y_1], y_1] \end{aligned}$$

In other words, $\overline{T}_p^{-n}$ squeezes the length of all intervals $[[t], t]$. Therefore

$\overline{T}_p^{-n}$ cannot have any periodic point (fixed point) except at integer end points. This implies that $\overline{T}_p$ has no interior (non integer) periodic point of period n. Q.E.D.

3.3 Some counting arguments

Lemma 3.2 (Number of simple cycles) *Let $S(n)$ be the number of simple cycles in C_n.*

a) If n is odd then $S(n) = 2$

b) If $n = 2^m$, $S(n) = 2^{2^m - m - 1}$

c) If $n = 2^m(l)$, l odd, $S(n) = (2^{2^m - m})(2l)^{2^m - 1}$

Proof:

a) The result is obvious from the definition of simple cycles.

b) The proof is done by induction on m.

i) When $m = 1$ then $S(n) = 1$.

ii) Assume that the proposition is true when $n = 2^{m-1}$. Consider simple cycles of size $n = 2^m$. They are separated of order $m - 1$, thus we can write

$$X = X_1|X_2|\ldots|X_{2^{m-1}}$$

where each X_i's is a 2-point cycle. Moreover the dynamics on blocks X_i are the same as a simple cycle of size 2^{m-1}. Also on every block, except *exactly* *one*, there are two possibilities to map to the other, ↑ or ↓. Therefore

$$S(2^m) = S(2^{m-1}) \cdot 2^{2^{m-1}-1} = 2^{2^m-m-1}$$

c) $n = 2^m l$, with l odd.

First of all, simple cycles are separated of order m, thus by part b) there are 2^{2^m-m-1} ways to arrange the blocks.

Next, there are $2j$ ways to map the points in each of the $2^m - 1$ blocks in order to get f^{2^m} be Stĕfan on them. Moreover there are only 2 ways to map points in the last block (since there are only 2 kinds of Stĕfan cycles for f^{2^m} on this block). And therefore we have our result. Q.E.D.

Lemma 3.3 (Number of Stĕfan cycles) *Let $S(n)$ be the number of Stĕfan cycles in C_n, then*

a) If n is odd or n is a power of 2, then $S(n) = s(n)$

b) If $n = 2^m l$, l odd then $S(n) = 2^{2^{m+1}-1}$.

Proof:

a) The result is obvious from definition 3.4 of simple cycles.

b) Again there are 2^{2^m-m-1} ways to arrange the blocks. Next there are 2^m ways to choose the "crucial block". On the crucial block there are two kinds of odd Stĕfan type. On the other blocks, each can be either ↑ or ↓. Thus

$$S(n) = 2^{2^m-m-1} \cdot 2^m \cdot 2 \cdot 2^{2^m-1} = 2^{2^{m+1}-1}$$

Q.E.D.

Chapter 4

FORCING ON BLOCK STRUCTURES

In this chapter, we will study cycles which have block structures. Some relationship will come up in the network $(C, \longrightarrow)$ and we will examine the minimality of Stĕfan cycles from a different viewpoint.

For convenience, we will use capital letters to denote cycles, and small letters to denote their sizes.

Recall (Block structures) *Given a cycle R of order m. We say that a cycle P has a* block structure *over R if*

i) $n = sm$ where n is the order of P;

ii) We can write

$$P = B_1 \cup B_2 \cup \ldots \cup B_m$$

$$\textit{where} \qquad B_i = \{is + k, k = 1 \ldots s\}$$

$$\textit{and} \qquad P(B_i) = B_j \quad \text{iff} \quad R(i) = j.$$

Notes:

i) This definition is exactly the same as definition 3.1. We just want to simplify the notation by taking the pattern of the indices instead of the orbit itself - which is topologically equivalent.

ii) If P has a block structure over R then we write $R \mid P$ or $P \xrightarrow{\circlearrowleft} R$.

Remark: The relationship $\xrightarrow{O}$ is also a partial ordering in C (the set of all cycles) and is stronger than $\longrightarrow$ in the following sense.

Lemma 4.1 *If $P \xrightarrow{O} R$ then $P \longrightarrow R$.*

Proof: Define a function $h: [1,n] \longrightarrow [1,m]$ as following

$$h(x) = \begin{cases} i+1 & is < x \leq is+s \\ x+i-is & is \leq x \leq is+1 \end{cases}$$

then h is a continuously non-decreasing function on $[1,n]$. Thus h is a semi-conjugacy which squeezes blocks of T_p into points of T_r. And we can see that there exists a point $x \in B$ whose orbit is of type R. Q.E.D.

Proposition 4.1 *Let $P \xrightarrow{O} R$. If $P \longrightarrow Q$ then*

i) either $Q \xrightarrow{O} R$,

ii) or $R \longrightarrow Q$.

Proof: Since $P \longrightarrow Q$ then T_p must have a point x whose orbit is of type Q.

Now T_p permutes B_i into B_j, therefore if x belongs to some block then its orbit must stay in the blocks forever (since x is periodic). Otherwise, $T_p^n(x)$ lies totally outside of B_i's.

Case 1: If $x \in B_i$ then clearly the orbit of x is a block structure over R, i.e. $Q \xrightarrow{O} R$.

Case 2: If $x \notin B_i$, $\forall i$, then consider the semi-conjugacy h defined in lemma 1, we can see that T_p has a point $y = h(x)$ whose orbit is of type Q, i.e. $R \longrightarrow Q$.

Q.E.D.

Proposition 4.1 gives several interesting results as we shall explore in the following corollaries.

Corollary 4.1 *Let $R \mid P$. Consider $Q \in C_l$ where l is not a multiple of m. Then*

$$P \longrightarrow Q \qquad \text{if and only if} \qquad R \longrightarrow Q$$

Proof: One way is clear since if $P \xrightarrow{O} R$ and $R \longrightarrow Q$ then $P \longrightarrow Q$.

On the other hand, if $P \longrightarrow q \in C_l$ then Q cannot be a block structure of R. Thus by proposition 4.1, $R \longrightarrow Q$. Q.E.D.

Example 1: Let $R = (123)$ and $P = (147268359)$ then although P is complicated (in fact, it is a 6-modal map) but P can force *only* unimodal maps in C_5, since R is unimodal.

Example 2: If $R = (12)$ then we can say that any cycle which is forced by a cycle P where $R \mid P$ must have two blocks.

Corollary 4.2 *Let $R \mid Q$ and $P \longrightarrow Q$, where R is a simple cycle of size 2^m. Then either Q has block structure over R (i.e., Q is separated of order m) or Q itself is simple of size 2^s, $s \leq m$.*

Proof: Suppose Q does not have a block structure over R, then $R \longrightarrow Q$.

i) The size of Q must be less than the size of R, in Sarkovski's order, i.e., $|Q| = 2^s$, $s \leq m$. Otherwise, Q must force some cycle of order 2^m, and thus R must also force that cycle of order 2^m, which contradicts to the minimality of R.

ii) Q must be simple, otherwise by Block and Coppel [6], Q must force a cycle of size $3 \cdot 2^{s-2}$.

Therefore, by Sarkovski's theorem, Q must force some cycle of order 2^m which again leads to a contradiction.

Thus $|Q| \mid |R|$ and Q is simple. Q.E.D.

Corollary 4.3 *Let P, Q, R be as in proposition 4.1, and suppose that $n = 2m$. Then*

$$\textit{either} \qquad R \longrightarrow Q \qquad \textit{or} \qquad Q = P.$$

Proof: Suppose $R \not\longrightarrow Q$. Consider the point x whose orbit is of type Q in the proof of proposition 4.1, then $x \in B_i = [i, i+1]$ for some i.

Since T_p is linear and the length of the blocks is equal to 1, this implies that the orbit of x is either at the midpoints of B_i's (i.e. $Q = R$) or at the endpoints of B_i's (i.e $Q = P$). Q.E.D.

In corollary 4.3, P is 2-extension of R. It is called the *immediate successor* of R in the meaning that there is no cycle between them in the network $(C, \longrightarrow)$.

Proposition 4.2 *Given that P is a block structure over R. The P is an immediate successor of R if and only if P is a 2-extension of R (denoted by $2 * R$).*

Proof: The first part was proven previously in corollary 4.3. We now prove the opposite direction of the proposition.

Suppose that the blocks B_i's of P are of s symbols. Consider $P^m \in C_s$, $s \geq 3$.

$$P^m \text{ must then have a point } x \text{ of order } 2$$

$$\Longrightarrow x \text{ has order } 2m, \text{ since } P \text{ preserves the blocks}$$

Thus P has a point of period $2m$ whose orbit is of type $Q \neq R$, i.e if P is not a 2-extension of R, then there exist a cycle such that $P \longrightarrow Q \xrightarrow{\bigcirc} R$. Q.E.D.

The idea of *immediate successor* was studied by Bernhardt [9] in 1986 who gave a different proof on proposition 4.2.

Now using that proposition, we can also show the minimality of Stĕfan cycles of order 2^n by induction.

Corollary 4.4 *The Stĕfan cycle $\tilde{S}(2^n)$ is minimal.*

Proof: When $n = 1$, it is true that there is only one cycle of order 2.

Suppose that it is true for n Then consider any Stĕfan cycle of order 2^{n+1} which is a 2-extension of some Stĕfan cycle of order 2^n by definition.

Thus if $\tilde{S}(2^{n+1}) \longrightarrow Q$, with $|Q| \rhd 2^{n+1}$ where $\rhd$ is the Sarkovski's order

Then $\tilde{S}(2^n) \longrightarrow Q$ which leads to a contradiction of Sarkovski's theorem. Q E D

Proposition 4.3 *If P has block structures over cycles R and T then either T must have block structure over R or vice versa. We then can express the statement by the following diagram*

$$\begin{cases} P \xrightarrow{\bigcirc} R \quad \textit{and} \\ P \xrightarrow{\bigcirc} T \end{cases} \Longrightarrow \begin{cases} T \xrightarrow{\bigcirc} R \\ R \xrightarrow{\bigcirc} T \end{cases}$$

Proof: We have $P \xrightarrow{\bigcirc} T$ and $P \xrightarrow{\bigcirc} R$, therefore

i) either $R \xrightarrow{O} T$, or

ii) $T \xrightarrow{O} R$, but in this case we also have

$$P \xrightarrow{O} R \qquad \text{and} \qquad P \longrightarrow T$$

therefore either $R \longrightarrow T$ which implies $R \equiv T$ or $T \xrightarrow{O} R$. Q.E.D.

Corollary 4.5 *Given $P \xrightarrow{O} R$ and $Q \xrightarrow{O} T$. Then if $P \longrightarrow Q$ then either $R \longrightarrow T$ or $R \mid T$.*

Proof: By proposition 4.1, if $P \longrightarrow Q$ then

i) either $R \longrightarrow Q$ and thus $R \longrightarrow T$, or

ii) $Q \xrightarrow{O} R$, but then by proposition 4.3

- either $T \mid R$ hence $R \longrightarrow T$, or
- $R \mid T$.

Thus we have this diagram

$$\begin{cases} P \xrightarrow{O} R \text{ and} \\ Q \xrightarrow{O} T \end{cases} \Longrightarrow \begin{cases} R \longrightarrow Q \xrightarrow{O} T \\ R \xrightarrow{O} T \quad \text{or} \quad T \xrightarrow{O} R \end{cases}$$

Q.E.D.

We now introduce a new concept of minimality.

Definition 4.1 *Let $\mathcal{F}$ be a family of cycles and $P \in \mathcal{F}$. We say that P is* primary with respect to $\mathcal{F}$ *if the only cycle (in $\mathcal{F}$) which is forced by P is P itself.*

Note: *If P is primary with respect to $\mathcal{F}$ then P is also primary with respect to $\mathcal{F}'$, with $\mathcal{F}'$ be any non-empty subset of $\mathcal{F}$.*

Proposition 4.4 *Denote by $\tilde{S}(n)$ a Stēfan cycle of order n.*

Then $\tilde{S}(n)$ is primary with respect to $\mathcal{F} = \bigcup_k C_k$, $k \rhd n$ where $\rhd$ is the Sarkovski's order.

Proof: Suppose $\tilde{S}(n) \longrightarrow Q$, where $Q \in \mathcal{F}$, with $\mathrm{ord}(Q) = k$. Thus by Sarkovski, Q must force a cycle of order n, i.e.

$$\exists R \ni Q \longrightarrow R, \qquad \mathrm{ord}(R) = n$$

Moreover, Block and Coppel [6] (theorem B, chapter 3) proved that R must force a Stēfan cycle of the same order.

Thus $\tilde{S}(n) \longrightarrow$ some other Stēfan cycle, which contradicts the minimality of $\tilde{S}(n)$.

Q.E.D

Proposition 4.5 *Let P be a z-extension over R and that $P^m \mid B_i, \forall i$ is primary with respect to $\mathcal{F}_i$.*

Then P is primary with respect to $\mathcal{F} = \mathcal{F}' \bigcup \mathcal{F}''$ where

$$\mathcal{F}' = \{Q : R \not\longrightarrow Q\}$$

$$\textit{and} \qquad \mathcal{F}'' = \{Q : R \mid Q \quad \textit{and} \quad Q^m \mid B_i \in \mathcal{F}_i\}$$

Proof: We use the same notation as in proposition 4.1.

Claim 1: $\mathcal{T}_{p^m} = (\mathcal{T}_p)^m$

Proof: Consider all the integral points $i = 1, 2, \ldots, n$. Then clearly $\mathcal{T}_{p^m} = [\mathcal{T}_p(i)]^m$

Now at every subinterval of length 1 $I_i = [i, i+1]$, $\mathcal{T}_p$ is linear, therefore $(\mathcal{T}_p)^m$ is also linear on every I_i.

On the other hand, by definition, $\mathcal{T}_{p^m}$ is linear on I_i.

Therefore we have $\mathcal{T}_{p^m} = (\mathcal{T}_p)^m$.

Claim 2: P is primary with respect to $\mathcal{F}$.

Proof: Assume that P is not primary with respect to $\mathcal{F}$, i.e. there exists a cycle $Q \in \mathcal{F}$ such that $P \longrightarrow Q$. Then we have two cases:

Case 1: $Q \in \mathcal{F}'$, i.e., $R \not\longrightarrow Q$. But then this implies that $R \mid Q$ by proposition 4.1.

Now in the proof of proposition 4.1, $\mathcal{T}_p$ has a point x whose orbit is of type Q. Also, x must belong to some block $\mathcal{B}_i$, otherwise we must have $R \longrightarrow Q$.

Consider the primitive map $\mathcal{T} = \mathcal{T}_{p^m}$, then by claim 1, we have $\mathcal{T} = (\mathcal{T}_p)^m$, i.e., $\mathcal{T}$ has a point of period of type Q^m on every block $\mathcal{B}_i$.

By the given condition that P^m is primary on each of the block, this implies P^m and Q^m are isomorphic on each $\mathcal{B}_i$, i.e., P and Q are isomorphic.

Hence P can force only itself in this case.

Case 2: If $Q \in \mathcal{F}''$, i.e., if $R \mid Q$, then again P has a point x whose orbit of type Q lies totally either inside or outside of the blocks $\mathcal{B}_i$'s.

In the first situation, we then have case 1, i.e. $Q \equiv P$.

In the latter case, we have $R \longrightarrow Q$, hence $R \equiv Q$.

However, R^m on any block is a fixed point, i.e. $R^m \ni \mathcal{F}''$ which completes our proof. Q.E.D.

Note: The inverse of proposition 4.5 is not true, i.e.,

If $P \longrightarrow R$ and P is primary with respect to $\mathcal{F}$ then it is not sufficient to imply that P is a z-extension of R.

Example: Take $P = (162435)$ and $R = (12)$

Then P which is a simple cycle, is not a z-extension of R and we still have P is primary with respect to $\mathcal{F}$.

Corollary 4.6 $P = \tilde{S}(n)$ *is primary on* C_n.

Proof: Suppose that $n = s \cdot 2^m$, where s odd ≥ 3. Then P has a block structure over R, where R is a simple cycle of order 2^m.

If $P \longrightarrow q \in C_n$ then we must have $Q \xrightarrow{O} R$, since R is simple.

However P^{2^m} is an odd Stĕfan cycle, which is primary with respect to C_s by Sarkovski's proof, i.e.

P is primary with respect to C_n. Q.E.D.

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