A Unified Study of Bounds and Asymptotic Estimates for Renewal Equations and Compound Distributions with Applications to Insurance Risk Analysis

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ABSTRACT

A Unified Study of Bounds and Asymptotic Estimates for Renewal Equations and Compound Distributions with Applications to Insurance Risk Analysis

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This thesis consists of a unified study of bounds and asymptotic estimates for renewal equations and compound distributions and gives applications to aggregate claim distributions, stop-loss premiums and ruin probabilities with general claim sizes and especially with heavy-tailed distributions.

Chapter 1 presents the probability models of compound distributions and renewal equations in insurance risk analysis and gives the summary of the results of this thesis.

In Chapter 2, we develop a general method to construct analytical bounds for solutions of renewal equations. Two-sided exponential and linear estimates for the solutions are derived by this method. A generalized Cramér-Lundberg condition is proposed and used to obtain bounds and asymptotic formulae with NWU distributions for the solutions.

Chapter 3 discusses tails of a class of compound distributions introduced by Willmot (1994) and gives uniformly sharper bounds, both with the results obtained in Chapter 2 and renewal theory. The technique of stochastic ordering is employed to get simplified bounds for the tails and to correct the errors of the proofs of some previous results.

In Chapter 4, we derive two-sided estimates for tails of a class of aggregate claim distributions, and especially give upper and lower bounds for compound negative binomial distributions both with adjustment coefficients and with heavy-tailed distributions. For the latter case, Dickson's (1994) condition plays the same role as the Cramér-Lundberg condition.

Chapter 5 is devoted to the aging property of compound geometric distributions and its applications to stop-loss premiums and ruin probabilities. By the aging prop-

erty, general upper and lower bounds for the stop-loss premiums of the class of compound distributions discussed in Chapter 3 are derived, which apply to any claim size distribution. Also, two-sided estimates for the stop-loss premiums of negative binomial sums are obtained both under the Cramér-Lundberg condition and under Dickson's condition. General upper and lower bounds for ruin probabilities are also considered in this chapter.

Chapter 6 gives a detailed discussion of the asymptotic estimates of tails of convolutions of compound geometric distributions. Asymptotic estimates for these tails are given under light, medium and heavy-tailed distributions, respectively. Applications of these results are given to the ruin probability in the diffusion risk model. Also, two-sided bounds for the ruin probability are derived by a generalized Dickson condition, which applies to any positive claim size distribution. Finally, we give some examples and consider numerical comparisons of bounds with asymptotic estimates.

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List of Notation and Symbols

$\Re$	$(-\infty, \infty)$
$(x)_+$	$\max(x, 0)$
$[x]$	the largest integer in x, the integral part of x
$X =^d Y$	X and Y are equal in distribution
$X_n \xrightarrow{D} X$	X_n converges to X in distribution
i.i.d.	independent and identically distributed
$\square$	end of proof
$I_{(A)}$	the indicator function of the set A
$F^{(n)}(x)$	the n-fold convolution of F with itself at x
$F^{(0)}(x)$	0 if $x < 0$ and 1 if $x \geq 0$
$\overline{F}(x)$	$1 - F(x)$, the tail of F
$\overline{F}^{(n)}(x)$	$1 - F^{(n)}(x)$, the tail of $F^{(n)}$
$m_F(s)$	$\int_{-\infty}^{\infty} e^{sy}\, dF(y)$, the moment generating function of F
$g(x) = o(f(x))$	$f(x) \to 0$ and $g(x)/f(x) \to 0$ as $x \to \infty$
$f(x) \sim g(x)$	$f(x)/g(x) \to 1$ as $x \to \infty$
$\Phi(x)$	$\int_{-\infty}^{x} (1/\sqrt{2\pi})\, e^{-y^2/2}\, dy$, the standard normal distribution function

Chapter 1

Introduction

1.1 Compound distributions and renewal equations in insurance risk analysis

Assume that N is a counting random variable with probability mass function

$$\Pr\{N = n\} = p_n, \; n = 0, 1, \cdots$$

and $\{X_i, \, i \geq 1\}$ is a sequence of *i.i.d.* nonnegative random variables independent of N with common distribution F.

The distribution G of the random sum $S = X_1 + \cdots + X_N$ ($S = 0$ if $N = 0$) is called a compound distribution and its distribution function is given by

$$G(x) \;=\; \Pr\{S \leq x\} \;=\; \sum_{n=0}^{\infty} p_n \, F^{(n)}(x) \, , \quad x \geq 0. \tag{1.1.1}$$

The tail probability of the compound distribution G is defined by

$$\overline{G}(x) \;=\; \Pr\{S > x\} \;=\; 1 - G(x) \;=\; \sum_{n=1}^{\infty} p_n \, \overline{F^{(n)}}(x) \, , \quad x \geq 0. \tag{1.1.2}$$

Compound distributions and random sums arise in many applied probability models such as queueing theory and reliability. It is of central importance in insurance risk analysis [e.g. see Bowers *et al.* (1997) and Panjer and Willmot (1992)]. If N represents the number of claims in an insurance portfolio and X_i represents the i-th claim amount, then S is the aggregate claim amount and the tail probability $\overline{G}(x) = \Pr\{S > x\}$ is the probability that the aggregate claim amount exceeds the amount x. There has been great interest in estimating these tail probabilities [e.g.

see Chaubey *et al.* (1998) and Klüppelberg and Mikosch (1997)]. Some well known compound distributions modeling the aggregate claims are compound Poisson and compound negative binomial distributions.

In addition, for the compound Poisson risk model, the risk reserve process is given by

$$R(t) = x + ct - \sum_{i=1}^{N(t)} Y_i \tag{1.1.3}$$

where $\{N(t),\ t \geq 0\}$ is a homogeneous Poisson process with intensity λ; the claim sizes $\{Y_i,\ i \geq 1\}$ are *i.i.d.* nonnegative random variables and have a common distribution function P with $P(0) = 0$ and finite mean $\mu_P > 0$; $\{N(t),\ t \geq 0\}$ and $\{Y_i,\ i \geq 1\}$ are independent; $x \geq 0$ is the initial capital and $c > 0$ is the premium rate.

If we denote the ultimate ruin probability, with initial capital x, by

$$\psi(x) = \Pr\{\, R(t) < 0, \text{ for some } t > 0 \,|\, R(0) = x \,\},$$

then it is well-known that ψ satisfies the following integral equation [see Feller (1971) or Grandell (1991)], that is

$$\psi(x) \;=\; \frac{\overline{F}(x)}{1+\theta} + \frac{1}{1+\theta}\int_0^x \psi(x-y)\, dF(y) \tag{1.1.4}$$

where $F(x) = \int_0^x \overline{P}(y)\, dy/\mu_P$ is called the ladder-height distribution and $\theta > 0$ is the relative safety loading factor, satisfying $(1+\theta)\lambda\mu_P = c$.

(1.1.4) is a defective renewal equation, whose definition is given below.

Definition 1.1 Suppose that G and h are two functions defined on $[0, \infty)$. If h is locally bounded, *i.e.* bounded on any finite interval, and G is a function of bounded variation, *i.e.* the difference of two increasing functions, then the convolution $G * h$ of G and h is a function defined on $[0, \infty)$, given by

$$G * h(x) = \int_0^x h(x-y)\, dG(y).$$

Definition 1.2 A renewal equation is an integral equation of the following form

$$Z(x) = z(x) + q\int_0^x Z(x-y)\, dF(y), \quad x \geq 0,$$

or, equivalently, for $x \geq 0$,

$$Z(x) = z(x) + q\, F * Z(x), \tag{1.1.5}$$

where, $0 < q < \infty$ is a constant, F is a probability distribution on $[0, \infty)$ with $F(0) < 1$, z and Z are defined on $[0, \infty)$ and are locally bounded.

The renewal equation (1.1.5) is called proper if $q = 1$, defective if $0 < q < 1$ and excessive if $q > 1$.

The renewal equation is another important probability model in insurance risk analysis [e.g. see De Vylder (1996), Gerber (1970, 1979) and Schmidli (1997)] and indeed arises in different disciplines [e.g. see Feller (1971), Karlin and Taylor (1975), Resnick (1992) and Ross (1983)]. Many other quantities associated with the compound Poisson risk model also satisfy renewal equations [e.g. see Dickson (1992, 1993), Dickson and Dos Reis (1994, 1996), Dufresne and Gerber (1989), Gerber *et al.* (1987), Gerber and Shiu (1997a, 1997b)].

The ruin probability $\psi(x)$ can also be expressed, equivalently, as the tail of a compound geometric distribution, which is known as Beekman's formula of the ruin probability [see, Goovaerts *et al.* (1990) and Bowers *et al.* (1997)], that is

$$\psi(x) = \frac{\theta}{1+\theta} \sum_{n=1}^{\infty} \left(\frac{1}{1+\theta}\right)^n \overline{F}^{(n)}(x). \tag{1.1.6}$$

Thus, the study of the ruin probability $\psi(x)$ can be reduced to the study of the tail of the compound geometric distribution or the defective renewal equation. In fact, in generalizing ruin in the compound Poisson risk model, the ruin probability in the renewal risk model can also be reduced to the tail of a compound geometric distribution [e.g. see Grandell (1991)]. More generally, the total maximum of a random walk can be reduced to a geometric sum [Feller (1971)]. In addition, a rich variety of applications of compound geometric distributions in reliability, queueing theory and regenerative processes can be found in Asmussen (1987), Brown (1990), Gertsbakh (1984) and Kalashnikov (1994).

However, compound distributions and renewal equations are rarely tractable, neither analytically nor numerically; the analytical closed forms for the compound distribution or the solution of the renewal equations are only available for a few special cases. Hence, some of main probability results for these are probability inequalities or bounds and asymptotic formulae. These provide estimates and insight, both qualitative and quantitative, into the compound distribution and the solution of renewal equations.

Let us first review the results on the ruin probability $\psi(x)$, which is both the tail of a compound geometric distribution and the solution of a defective renewal equation, to see what results can be obtained and which probability methods may be used in deriving these results.

We recall that the Cramér-Lundberg condition is that there exists a constant R, called the adjustment coefficient, satisfying Lundberg's equation

$$\int_0^\infty e^{Rx}\overline{P}(x)\,dx = \frac{c}{\lambda} \tag{1.1.7}$$

or, equivalently,

$$\int_0^\infty e^{Rx}\,dF(x) = 1+\theta. \tag{1.1.8}$$

Under Cramér-Lundberg's condition, the celebrated Lundberg's inequality and Cramér-Lundberg's asymptotic formula hold.

Theorem 1.1 (Lundberg's inequality) In the compound Poisson risk model, for any $x \geq 0$,

$$\psi(x) \;\leq\; e^{-Rx}. \tag{1.1.9}$$

Theorem 1.2 (Cramér-Lundberg asymptotic formula) In the compound Poisson risk model, if

$$\beta = \frac{\lambda\mu_P}{c}\int_0^\infty ye^{Ry}\,dF(y) < \infty, \tag{1.1.10}$$

then

$$\psi(x) \;\sim\; \frac{\theta}{(1+\theta)R\beta}e^{-Rx}, \quad x\to\infty, \tag{1.1.11}$$

in particular, when the claim size distribution P is exponential, then

$$\psi(x) = \frac{1}{1+\theta}\exp\left\{-\frac{\theta}{(1+\theta)\mu_P}x\right\}, \tag{1.1.12}$$

and in this case the Cramér-Lundberg asymptotic formula is exact.

These two results are the pioneering works by Cramér (1930, 1955) and Lundberg (1926). They can be proved in different ways now. For example, the martingale approach [Gerber (1979)], Wald's identity [Ross (1983)] and the induction method [Goovaerts *et al.* (1990)] have been used to prove Lundberg's inequality. Applying the key renewal theorem to the defective renewal equation (1.1.4), a technique introduced by Feller (1971), is used to derive Cramér-Lundberg's asymptotic formula. All these methods are much simpler than the Wiener-Hopf methods used by Cramér (1930, 1955) and Lundberg (1926) and have been used extensively in other disciplines. In particular, the martingale is a powerful tool for deriving exponential inequalities in the form of $c(x)\, e^{-Rx}$ for the ruin probabilities in different risk models [e.g. see Dassios and Embrechts (1989) and Furrer and Schmidli (1994)]. But, the martingale method can not be used to derive asymptotic formulae and the coefficient or function $c(x)$ in the inequality $c(x)\, e^{-Rx}$ derived using the martingale approach is difficult to estimate or to simplify.

The technique of applying the key renewal theorem to defective renewal equations has become a standard method for deriving exponential asymptotic formulae for ruin probabilities. In fact, most exponential asymptotic formulae in risk theory and queueing theory are derived by this technique [e.g. see Asmussen (1987), De Vylder (1996), Ross (1983) and Schmidli (1995)]. Hence, it motivates us to study probability inequalities for the solution of the renewal equation so that we can have a unified method of deriving both asymptotic formulae and bounds for the solution of the renewal equation. Indeed, we develop a general method to construct analytical bounds for the solution of renewal equations, which gives a unified derivation of exponential bounds for the solution of defective and excessive renewal equations and linear bounds for the solution of proper renewal equations.

On the other hand, the Cramér-Lundberg condition plays a critical role in the study of the exponential bounds and asymptotic formulae for the tail of the compound distribution and the solution of defective and excessive renewal equations and for many other applied probability models [e.g. see Bergmann and Stoyan (1976), Glasserman (1997), Kingman (1970), Ross (1974)]. In accordance with the Cramér-Lundberg condition, claim size distributions can be classified into three classes, *i.e.* light, medium and heavy tailed distributions, which are defined as follows.

Definition 1.3 A claim size distribution F (*i.e.* the distribution of a nonnegative random variable) is said to be

1. light tailed (or to have an exponential tail) if (1.1.8) holds;
2. medium tailed if for any $R > 0$, $\int_0^\infty e^{Rx} dF(x) < 1 + \theta$;
3. heavy tailed if the moment generating function of F does not exist.

For example, exponential and gamma distributions are light tailed; certain inverse Gaussian and generalized inverse Gaussian distributions (see, Example 2.7) are medium tailed; Pareto and lognormal distributions are heavy tailed.

Unfortunately, the Cramér-Lundberg condition can not be satisfied by the heavy and medium tailed distributions. Especially, the heavy tailed distributions have attracted much attention in insurance risk analysis due to the presence of the large (catastrophic) claims [e.g. see Asmussen and Klüppelberg (1996), Embrechts and Veraverbeke (1982), Embrechts and Klupelberg (1994), Klupelberg (1993) and Mikosch (1997)]. They have also attracted much interest in queueing theory [e.g. see Asmussen and Klüppelberg (1997) and Asmussen and Teugels (1996)]. Thus, how to generalize Cramér-Lundberg condition and to derive the generalized inequalities and asymptotic formulae for the tail of compound distributions and the solution of the renewal equation is an interesting question. Many works have been devoted to this topic using different methods [e.g. see Cai and Wu (1997b), Dickson (1994), Lin (1996), Willmot (1994, 1996, 1997a, 1997b), Willmot and Lin (1994, 1997a) and Taylor (1976)]. In this thesis, we also give a unified method for the study of this topic using renewal theory and compound geometric distributions. We apply our methods and results to tails of aggregate claim distributions, stop-loss premiums and ruin probabilities, which are all of interest in insurance risk analysis, in particular, our results can apply to large claim size or heavy-tailed distributions.

1.2 Summary

This thesis is organized as follows. In Chapter 2, we develop a general method to construct analytical bounds for the solution of renewal equations. Two-sided exponential and linear bounds for the solution of defective and excessive renewal equations are derived from this method. These results, together with the elementary and key renewal theorems, give both the estimates and the complete description of the behaviour of the solution of renewal equations.

An application of our results gives two-sided exponential bounds and their refinements for the tail of compound geometric distribution; two-sided exponential estimates for the expected number of the age-dependent branching process are also given. Two-sided linear bounds for the renewal function are derived as another application. Many previous results are improved and generalized.

A generalized Cramér-Lundberg condition is presented, under which, bounds are obtained in terms of NWU and NBU distributions for the solution of defective renewal equations. The upper bounds in terms of NWU distributions can be applied to some cases in which Cramér-Lundberg condition can not be satisfied. Asymptotic formulae in terms of NWU distributions for the solution of defective renewal equations are obtained, which include the Cramér-Lundberg asymptotic formula as a special case.

In Chapter 3, we consider the tail probabilities of a class of compound distributions introduced by Willmot (1994). First, the relations between reliability distribution classes and heavy-tailed distributions are discussed. These relations reveal that many previous results on estimating the tail probabilities are not applicable to heavy-tailed distributions. Then, a generalized Wald identity and the identities of compound geometric distributions are presented in terms of renewal processes. The results obtained in Chapter 2 yield uniformly tighter bounds for the tails. Alternatively, using these identities, lower and upper bounds for the tail probabilities are derived for the class of compound distributions, both under the conditions of NBU and NWU tails, which include exponential tails, as well as heavy tails. Many previous results are shown to be special cases of these results. In addition, simplified bounds are derived by the technique of stochastic ordering. It also allows for the correction of errors in the proof of some previous results.

In Chapter 4, we first give the two-sided estimates of the tail probabilities of a class of aggregate claim distributions using the results in Chapter 3. Then, we focus on deriving upper and lower bounds for the tails of compound negative binomial distributions. Sharp upper and lower bounds are obtained for the tails of compound negative binomial distributions. A connection between the compound negative binomial, Poisson and logarithmic distributions is presented, which results in the generalization and improvement of a theorem of Willmot and Lin (1997b). In addition, using Dickson's condition and the technique developed in Chapter 3, we obtain the two-sided bounds for the tails of compound negative binomial distributions with heavy-tailed distributions.

In Chapter 5, we first review the new worse than used (NWU) aging property of the compound geometric distribution, and show that mixed geometric sums also have this aging property. Then, we discuss the relations between the compound geometric distribution and its stop-loss premium. General upper and lower bounds for the stop-loss premium are derived using the aging property.

As the applications of these results, we give bounds for stop-loss premiums of the class of compound distributions discussed in Chapter 3. The stop-loss premiums of compound negative binomial distributions with large claim sizes are also derived. In addition, by the technique of the stochastic ordering, we give general bounds for the stop-loss premiums of the compound distributions with HNBUE and HNWUE claim sizes.

Using the similar method, we also get a general upper bound for the ruin probability. This upper bound is sharper than that of Willmot (1994) and asymptotically sharper than that of Broeckx *et al.* (1986). The asymptotic behaviour of these bounds is discussed. The relationships among sub-exponential, NWU, NBU distributions and upper bounds of the ruin probability and stop-loss premiums are also considered. This chapter is based on Cai and Garrido (1998) and is also an extension of the paper.

In Chapter 6, we consider the tails of convolutions of compound geometric distributions and discuss their asymptotic behaviour in detail. We derive asymptotic estimates for the tail of convolutions of compound geometric distributions with light, medium and heavy tailed distributions, respectively. General lower and upper bounds for the tails are given, which can be used in some cases to determine a closer tail approximation. Applications of these results are given to the ruin problem in the classical risk process perturbed by a diffusion; previous results are easily derived and a theorem of Veraverbeke (1993) is generalized. In addition, using a generalized Dickson's condition, two-sided bounds for the ruin probability in the diffusion risk model with large claim sizes are given, thus Dickson's (1994) bound is extented to the diffusion risk model. Examples are given and numerical comparisons of bounds with asymptotic estimates for the ruin probability in the compound Poisson risk model are also considered.

Chapter 2

Asymptotic Formulae and Inequalities for Renewal Equations

2.1 Preliminaries and limit theorems

In this section, we review some asymptotic formulae for renewal equations. In particular, the elementary and key renewal theorems are presented, which are standard methods for deriving asymptotic formulae in different applied probability models including insurance risk analysis.

Theorem 2.1 Assume $0 \leq qF(0) < 1$.

1. For all $x \geq 0$, $U_q(x) = \sum_{n=0}^{\infty} q^n F^{(n)}(x) < \infty$.
2. The renewal equation (1.1.5) has a unique solution

$$Z(x) = U_q * z(x), \quad x \geq 0 . \tag{2.1.1}$$

In particular, if $q = 1$, the proper renewal equation has a unique solution

$$Z(x) = U * z(x), \quad x \geq 0 , \tag{2.1.2}$$

where

$$U(x) = \sum_{n=0}^{\infty} F^{(n)}(x) = 1 + M(x)$$

and

$$M(x) = \sum_{n=1}^{\infty} F^{(n)}(x) ,$$

is called the renewal function.

If $0 < q < 1$, the defective renewal equation has a unique solution

$$Z(x) = \frac{1}{1-q} G_q * z(x), \quad x \geq 0\,, \tag{2.1.3}$$

where

$$G_q(x) = \sum_{n=0}^{\infty} (1-q)\, q^n\, F^{(n)}(x)$$

is a compound geometric distribution function.

Proof. For the proof of Theorem 2.1, see Resnick (1992). □

Definition 2.1 Let h be a function defined on $[0, \infty)$. For any $a > 0$, let $\overline{m}_n(a)$ be the supremum, and $\underline{m}_n(a)$ the infimum of h over the interval $(n-1)a \leq x \leq na$. We say that h is directly Riemann integrable, if $\sum_{n=1}^{\infty} \overline{m}_n(a)$ and $\sum_{n=1}^{\infty} \underline{m}_n(a)$ are finite for all $a > 0$ and

$$\lim_{a \to 0} a \sum_{n=1}^{\infty} \overline{m}_n(a) = \lim_{a \to 0} a \sum_{n=1}^{\infty} \underline{m}_n(a).$$

The condition of directly Riemann integrability often arises in limit theorems of renewal theory. Any of the following conditions is sufficient for z being directly Riemann integrable [see Feller (1971) and Ross (1983)]:

1. $z(x) \geq 0$, decreasing and Riemann integrable.

2. z is monotonic and absolutely integrable.

3. $z(x) \geq 0$, bounded, continuous and satisfies

$$\sum_{n=1}^{\infty} \overline{m}_n(1) < \infty.$$

4. z is bounded by a directly Riemann integrable function.

5. z is constant on the intervals $[n, n+1]$, $n = 0, 1, \cdots$ and absolutely integrable.

Theorem 2.2 For the proper renewal equation (1.1.5), *i.e.* $q = 1$, suppose that $\mu = \int_0^{\infty} x\, dF(x) \leq \infty$ and Z satisfies the renewal equation (1.1.5).

1. **(The Elementary Renewal Theorem)** If $z_0 = \lim_{x\to\infty} z(x)$ exists, then

$$\lim_{x\to\infty} \frac{Z(x)}{x} = \frac{z_0}{\mu}. \tag{2.1.4}$$

2. **(The Key Renewal Theorem)** If z is directly Riemann integrable and F is not lattice, then

$$\lim_{x\to\infty} Z(x) = \frac{1}{\mu}\int_0^\infty z(x)\,dx, \tag{2.1.5}$$

where, if $\mu = \infty$, the limits in (2.1.4) and (2.1.5) are zero.

Proof. Theorem 2.2 is a classical result in renewal theory, see Feller (1971) for its proof. □

The key renewal theorem gives the limit of the solution of the proper renewal equation and can be used to derive asymptotic formulae for the solution of the defective and excessive renewal equations.

Definition 2.2 Assume that F is a probability distribution on $[0,\infty)$ with $F(0) < 1$. A constant R is called an adjustment coefficient or Lundberg coefficient associated with q and F if it satisfies the following Lundberg equation

$$E\left[e^{RX}\right] = \int_0^\infty e^{Ry}\,dF(y) = \frac{1}{q}, \tag{2.1.6}$$

where X is a random variable with distribution F.

The term "adjustment coefficient" is often used in risk theory. In branching processes, R is called a Malthusian parameter. It plays a critical role in studying the behaviour of the solutions of renewal equations.

We note that if the adjustment coefficient R exists, it is unique since $E\left[e^{sX}\right]$ is strictly monotonic in s. If $q \geq 1$, such an adjustment coefficient always exists, and $R < 0$ if $q > 1$ while $R = 0$ if $q = 1$. If $0 < q < 1$, then R may not exist, but R is positive when it does.

Property 2.1 For the renewal equation (1.1.5), suppose F is not lattice. If there exists an adjustment coefficient R for q and F, and $e^{Rx}z(x)$ is directly Riemann integrable, then

$$Z(x) \sim \frac{\int_0^\infty z(y)\,e^{Ry}\,dy}{q\int_0^\infty ye^{Ry}\,dF(y)}\,e^{-Rx}, \quad x\to\infty. \tag{2.1.7}$$

Proof. Since (1.1.5) implies that for $x \geq 0$,

$$e^{Rx} Z(x) = e^{Rx} z(x) + q \int_0^x e^{R(x-y)} Z(x-y) e^{Ry} dF(y) ,$$

or, equivalently,

$$Z^{\#}(x) = z^{\#}(x) + \int_0^x Z^{\#}(x-y) \, dF_q^{\#}(y) , \tag{2.1.8}$$

where $Z^{\#}(x) = e^{Rx} Z(x)$, $z^{\#}(x) = e^{Rx} z(x)$ and $dF_q^{\#}(y) = q e^{Ry} dF(y)$ is a probability measure, called the Esscher transform of F and

$$\int_0^\infty y \, dF_q^{\#}(y) = q \int_0^\infty y e^{Ry} dF(y).$$

Hence, (2.1.7) follows directly from (2.1.8) and the key renewal theorem. □

Thus, the Cramér-Lundberg asymptotic formula (1.1.11) follows simply from Property 2.1 , a method due to Feller (1971).

Property 2.1 gives a standard method for deriving exponential asymptotic formulae for defective and excessive renewal equations. In particular, defective renewal equations often arise in risk analysis and queueing theory.

From Theorem 2.1, we know that the solutions of proper and defective renewal equations can be expressed in terms of the renewal function and compound geometric distributions, respectively. Interestingly, the renewal function and compound geometric distribution satisfy the following renewal equations, respectively, for $x \geq 0$,

$$M(x) = F(x) + \int_0^x M(x-y) \, dF(y) \tag{2.1.9}$$

and

$$G_q(x) = 1 - q + q \int_0^x G_q(x-y) \, dF(y). \tag{2.1.10}$$

2.2 A general inequality for renewal equations

In this section, we develop a general method for the analytic constructions of bounds for the solution of the renewal equation (1.1.5).

Theorem 2.3 Assume $qF(0) < 1$ and Z satisfies the renewal equation (1.1.5). If a locally bounded function K, defined on $[0, \infty)$, satisfies for any $x \geq 0$,

$$z(x) + q \int_0^x K(x-y) \, dF(y) \ \geq \ K(x), \tag{2.2.1}$$

then, for any $x \geq 0$,

$$Z(x) \;\geq\; K(x). \tag{2.2.2}$$

Conversely, if for any $x \geq 0$,

$$z(x) + q \int_0^x K(x-y)\, dF(y) \;\leq\; K(x), \tag{2.2.3}$$

then, for any $x \geq 0$,

$$Z(x) \;\leq\; K(x). \tag{2.2.4}$$

Proof. By Theorem 2.1, we have

$$Z(x) = \sum_{n=0}^{\infty} q^n\, F^{(n)} * z(x), \quad x \geq 0.$$

Define

$$Z_k(x) = \sum_{n=0}^{k} q^n\, F^{(n)} * z(x), \quad k = 0, 1, 2, \cdots,$$

then, for any $x \geq 0$,

$$\lim_{k\to\infty} Z_k(x) = Z(x) \tag{2.2.5}$$

and the following renewal recursive relations hold

$$Z_{k+1}(x) = z(x) + q\, F * Z_k(x), \quad k = 0, 1, 2, \cdots. \tag{2.2.6}$$

Since (2.2.1) implies that

$$Z_0(x) = z(x) \;\geq\; K(x) - q\, F * K(x),$$

which, together with (2.2.6) and (2.2.1), implies that

$$\begin{aligned} Z_1(x) \;&=\; z(x) + q \int_0^x Z_0(x-y)\, dF(y) \\ &\geq\; z(x) + q \int_0^x K(x-y)\, dF(y) - q^2 \int_0^x F * K(x-y)\, dF(y) \\ &\geq\; K(x) - q^2\, F^{(2)} * K(x). \end{aligned}$$

Thus, by induction, we get for any $k \geq 0$,

$$Z_k(x) \;\geq\; K(x) - q^{k+1}\, F^{(k+1)} * K(x). \tag{2.2.7}$$

We have

$$\left|q^{k+1} F^{(k+1)} * K(x)\right| = q^{k+1} \left|\int_0^x K(x-y)\, dF^{(k+1)}(y)\right| \leq q^{k+1} F^{(k+1)}(x) \sup_{0\leq y\leq x} |K(y)|.$$

However, by Theorem 2.1, we know that $U_q(x) < \infty$ for $x \geq 0$, which implies that for any $x \geq 0$,

$$q^n F^{(n)}(x) \to 0, \quad n \to \infty. \tag{2.2.8}$$

Thus, letting $k \to \infty$ in (2.2.7), we get (2.2.2).

On the other hand, (2.2.3) implies that

$$Z_0(x) = z(x) \leq K(x) - q F * K(x).$$

Thus, similar to the argument above , we get for any $k \geq 0$,

$$Z_k(x) \leq K(x) - q^{k+1} F^{(k+1)} * K(x). \tag{2.2.9}$$

Hence, (2.2.4) follows from (2.2.9) and (2.2.8) as $k \to \infty$. □

The conditions (2.2.1) and (2.2.3) of Theorem 2.3 have similar forms to those in the inequalities of Kingman (1970), which apply to the stationary waiting time distribution in the $GI/G/1$ queue. Theorem 2.3 applies to functions satisfying renewal equations. Although, both Theorem 2.3 and the inequalities of Kingman (1970) are proved by induction, the arguments are different. Kingman (1970) mainly used the property of the stationary waiting time distribution in the $GI/G/1$ queue.

Theorem 2.3 can be used to easily construct bounds for the solution of renewal equations. Its applications are shown in the following sections.

2.3 Exponential bounds for defective and excessive renewal equations

2.3.1 Exponential upper and lower bounds

By Property 2.1, we know that under suitable conditions, the solutions of the defective and excessive renewal equations (1.1.5) have asymptotic exponential forms. Similar to Lundberg's inequality and to the Cramér-Lundberg asymptotic formula, one expects exponential bounds for the solutions, which are also important for one to understand the behavior of the solutions. Precisely, we want exponential bounds in the form of $u(x)\,e^{-Rx}$ for the solutions of defective and excessive renewal equations, where R is the adjustment coefficient. It is easy to construct such a bound by Theorem 2.3 if one chooses u to be an increasing function.

To do so, take $K(x) = u(x)\,e^{-Rx}$ in Theorem 2.3. Since

$$\begin{aligned} z(x) \;&+\; q\int_0^x u(x-y)\,e^{-R(x-y)}\,dF(y) \\ &\le\; z(x) \;+\; q\,u(x)\,e^{-Rx}\int_0^x e^{Ry}\,dF(y), \end{aligned} \tag{2.3.1}$$

by Theorem 2.3, $u(x)\,e^{-Rx}$ is an upper bound if

$$z(x) + q\,u(x)\,e^{-Rx}\int_0^x e^{Ry}\,dF(y) \;\le\; u(x)\,e^{-Rx}. \tag{2.3.2}$$

Since $1 - q\int_0^x e^{Ry}\,dF(y) \;=\; q\int_x^\infty e^{Ry}\,dF(y)$, (2.3.2) is equivalent to

$$u(x) \;\ge\; \frac{z(x)e^{Rx}}{q\int_x^\infty e^{Ry}\,dF(y)}. \tag{2.3.3}$$

Clearly, if we take

$$u(x) \;=\; \sup_{0\le h\le x} \frac{z(h)e^{Rh}}{q\int_h^\infty e^{Ry}\,dF(y)} \;=\; \left\{\inf_{0\le h\le x} \frac{q\int_h^\infty e^{Ry}\,dF(y)}{z(h)e^{Rh}}\right\}^{-1},$$

then u is increasing and satisfies (2.3.3). So, by Theorem 2.3, we get for $x \ge 0$

$$Z(x) \;\le\; u(x)\,e^{-Rx}. \tag{2.3.4}$$

This, together with (2.3.1), implies that

$$\begin{aligned} Z(x) \;&=\; z(x) + q\int_0^x Z(x-y)\,dF(y) \\ &\le\; z(x) + q\int_0^x u(x-y)\,e^{-R(x-y)}\,dF(y) \\ &\le\; z(x) + q\,u(x)\,e^{-Rx}\int_0^x e^{Ry}\,dF(y). \end{aligned} \tag{2.3.5}$$

By (2.3.2), we know that (2.3.5) is a refinement of $u(x)\,e^{-Rx}$.

By exactly similar arguments, if we choose the following decreasing function

$$l(x) = \left\{\sup_{0\le h\le x} \frac{q\int_h^\infty e^{Ry}\,dF(y)}{z(h)e^{Rh}}\right\}^{-1},$$

we get

$$Z(x) \;\ge\; z(x) + q\,l(x)\,e^{-Rx}\int_0^x e^{Ry}\,dF(y) \;\ge\; l(x)\,e^{-Rx}.$$

Summarizing the argument above, we get the following result.

Theorem 2.4 Suppose $qF(0) < 1$ and Z satisfies the renewal equation (1.1.5) and the adjustment coefficient R exists, then for any $x \geq 0$,

$$Z(x) \;\geq\; z(x) + l(x)\, F_R(x)\, e^{-Rx} \;\geq\; l(x)\, e^{-Rx} \tag{2.3.6}$$

and

$$Z(x) \;\leq\; z(x) + u(x)\, F_R(x)\, e^{-Rx} \;\leq\; u(x)\, e^{-Rx}, \tag{2.3.7}$$

where

$$F_R(x) = q \int_0^x e^{Ry}\, dF(y)\ ,$$

$$[l(x)]^{-1} = \sup_{0 \leq u \leq x} \frac{q \int_u^\infty e^{Ry}\, dF(y)}{z(u)\, e^{Ru}} \quad \text{and} \quad [u(x)]^{-1} = \inf_{0 \leq u \leq x} \frac{q \int_u^\infty e^{Ry}\, dF(y)}{z(u)\, e^{Ru}}.$$

Theorem 2.4 gives exponential bounds and their refinements for the solution to defective and excessive renewal equations. Since renewal equations arise in many applied probability models, Theorem 2.4 has many direct applications. We illustrate some in following subsections.

2.3.2 Application to tails of compound geometric distributions

(2.1.10) implies that the tail $\overline{G}_q(x)$ of $G_q(x)$ satisfies the following defective renewal equation,

$$\overline{G}_q(x) = q\,\overline{F}(x) + q \int_0^x \overline{G}_q(x-y)\, dF(y). \tag{2.3.8}$$

Thus, take $z(x) = q\,\overline{F}(x)$ in Theorem 2.4 and notice that $0 < q < 1$ in this case, we get directly the following result.

Corollary 2.1 If the adjustment coefficient R in (2.1.6) exists, then for any $x \geq 0$,

$$\overline{G}_q(x) \;\geq\; q\,\overline{F}(x) + l(x)\, F_R(x)\, e^{-Rx} \;\geq\; l(x)\, e^{-Rx} \tag{2.3.9}$$

and

$$\overline{G}_q(x) \;\leq\; q\,\overline{F}(x) + u(x)\, F_R(x)\, e^{-Rx} \;\leq\; u(x)\, e^{-Rx}\ , \tag{2.3.10}$$

where

$$[l(x)]^{-1} = \sup_{0 \leq u \leq x} \frac{\int_u^\infty e^{Ry}\, dF(y)}{e^{Ru}\,\overline{F}(u)} \quad \text{and} \quad [u(x)]^{-1} = \inf_{0 \leq u \leq x} \frac{\int_u^\infty e^{Ry}\, dF(y)}{e^{Ru}\,\overline{F}(u)}.$$

We note that $R > 0$ if $0 < q < 1$, so for any $0 \leq u \leq x$,

$$\frac{\int_u^\infty e^{Ry}\, dF(y)}{e^{Ru}\,\overline{F}(u)} \;\geq\; \frac{e^{Ru}\,\int_u^\infty dF(y)}{e^{Ru}\,\overline{F}(u)} \;=\; 1,$$

this implies that $[u(x)]^{-1} \geq 1$.

On the other hand,

$$[u(x)]^{-1} \;\leq\; \frac{\int_0^\infty e^{Ry}\, dF(y)}{\overline{F}(0)} \;=\; \frac{1}{q\,\overline{F}(0)},$$

hence,

$$q\,\overline{F}(0) \;\leq\; u(x) \;\leq\; 1. \tag{2.3.11}$$

Similarly,

$$0 \;\leq\; l(x) \;\leq\; q\,\overline{F}(0). \tag{2.3.12}$$

In addition, among reliability distribution classes, we can get exact expressions for $l(x)$ and $u(x)$, see, Willmot (1994) and Property 4.1 of this thesis.

The two-sided bounds in Corollary 2.1 are sharper than both Lundberg's inequality and those obtained by Cai and Wu (1997b), Gerber (1979), Kalashnikov (1996), Lin (1996), Taylor (1976) and Willmot and Lin (1994). Also, the two-sided bounds in Corollary 2.1 are asymptotically exact for small x if $F(0) = 0$, since then

$$\begin{aligned} \lim_{x\to 0}\left\{q\,\overline{F}(x) + u(x)\,F_R(x)\,e^{-Rx}\right\} \;&=\; \lim_{x\to 0}\left\{q\,\overline{F}(x) + l(x)\,F_R(x)\,e^{-Rx}\right\} \\ &=\; q \;=\; \overline{G}_q(0). \end{aligned}$$

2.3.3 Application to age-dependent branching processes

Suppose that an organism at the end of its lifetime produces a random number of offsprings in accordance with the probability distribution $\{p_j,\, j = 0, 1, 2, \cdots\}$. Assume that offsprings act independently of each other and produce their own offsprings in accordance with the same probability distribution $\{p_j\}$. Assume that the lifetimes of the organisms are independent random variables with common distribution F and $F(0) < 1$.

Let $X(t)$ denote the number of organisms alive at t. The stochastic process $\{X(t),\, t \geq 0\}$ is called an age-dependent branching process. Then, an important quantity for such a branching process is the expected number of organisms alive at t, given by $E[X(t)]$. It is known that $E[X(t)]$ satisfies the following renewal equation [see, Ross (1983)]

$$E[X(t)] = \overline{F}(t) + m \int_0^t E[X(t-s)]\, dF(s) \tag{2.3.13}$$

where $m = \sum_{j=0}^{\infty} j p_j$ is the expected offspring of an organism.

Clearly, if $m = 1$, then $E[X(t)] = 1$ is the unique solution of (2.3.13). If $m \neq 1$, by Property 2.1, we get [or, see Theorem 3.4.8 of Ross (1983)] that

$$E[X(t)] \sim \frac{1-m}{m^2 \gamma \int_0^{\infty} y\, e^{\gamma y}\, dF(y)}\, e^{-\gamma t}, \quad t \to \infty, \tag{2.3.14}$$

where γ satisfies

$$\int_0^{\infty} e^{\gamma y}\, dF(y) = \frac{1}{m}.$$

If $m > 1$, then $\gamma < 0$, this implies that the offspring grow at an exponential rate. While, if $m < 1$, then $\gamma > 0$, this implies that the offspring become extinct at an exponential rate. However, by Theorem 2.4, we can give two-sided exponential bounds for $E[X(t)]$, simply by taking $z(t) = \overline{F}(t)$ in Theorem 2.4, we get

Theorem 2.5 For $t \geq 0$,

$$l(t)\, e^{-\gamma t} \;\leq\; E[X(t)] \;\leq\; u(t)\, e^{-\gamma t}, \tag{2.3.15}$$

where

$$[l(t)]^{-1} = \sup_{0 \leq u \leq x} \frac{m \int_u^{\infty} e^{\gamma y}\, dF(y)}{e^{\gamma u}\, \overline{F}(u)} \quad \text{and} \quad [u(t)]^{-1} = \inf_{0 \leq u \leq x} \frac{m \int_u^{\infty} e^{\gamma y}\, dF(y)}{e^{\gamma u}\, \overline{F}(u)}.$$

We note that if $m > 1$, then $\gamma < 0$, this implies that $l(t) \geq 1/m$ since for any $0 \leq u \leq x$,

$$\frac{m \int_u^{\infty} e^{\gamma y}\, dF(y)}{e^{\gamma u}\, \overline{F}(u)} \leq \frac{m\, e^{\gamma u} \int_u^{\infty} dF(y)}{e^{\gamma u}\, \overline{F}(u)} = m,$$

this implies that $[l(t)]^{-1} \leq m$.

Similarly, if $m < 1$, then $\gamma > 0$ and $u(t) \leq 1/m$, thus, by Theorem 2.5, we get the following simple bounds for $E[X(t)]$, namely,

Corollary 2.2 For any $t \geq 0$, if $m > 1$, then

$$E[X(t)] \geq \frac{1}{m} e^{-\gamma t} \tag{2.3.16}$$

and if $m < 1$, then

$$E[X(t)] \leq \frac{1}{m} e^{-\gamma t}. \tag{2.3.17}$$

This result gives interesting estimates for the expected number of the organism alive at t for the age-dependent branching process. In the case of $m < 1$, it is similar to Lundberg's inequality for the ruin probability and for the case of $m > 1$, it is a counterpart of Lundberg's inequality for age-dependent branching processes.

2.4 Linear bounds for proper renewal equations

2.4.1 Linear upper and lower bounds

For the proper renewal equation (1.1.5), *i.e.* $q = 1$, by the elementary renewal theorem, we know that the solution of the proper renewal equation is asymptotically linear under suitable conditions. So, we expect a linear bound for the solution in the form of

$$u(x) + \frac{x}{\mu} \quad \text{or} \quad u(x)\,\frac{x}{\mu}.$$

By methods similar to those used in constructing exponential bounds for defective and excessive renewal equations, we can easily obtain linear bounds for proper renewal equations.

Definition 2.3 The equilibrium distribution function F_e of the life distribution (*i.e.* the distribution of a nonnegative random variable) F is a probability distribution function defined by

$$F_e(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)\,dy,$$

equivalently, the tail of F_e is

$$\overline{F}_e(x) = \frac{1}{\mu}\int_x^\infty \overline{F}(y)\,dy\ ,$$

where $0 < \mu = \int_0^\infty \overline{F}(y)\,dy < \infty$ is the mean of F.

In actuarial mathematics, F_e is called the ladder-height distribution. In renewal theory, it is called a stationary renewal distribution.

Theorem 2.6 Suppose $q = 1$, $0 < \mu < \infty$ and Z satisfies the renewal equation (1.1.5), then for any $x \geq 0$,

$$Z(x) \geq \frac{x}{\mu} + z(x) + l(x)F(x) - F_e(x) \geq \frac{x}{\mu} + l(x) \tag{2.4.1}$$

and

$$Z(x) \leq \frac{x}{\mu} + z(x) + u(x)F(x) - F_e(x) \leq \frac{x}{\mu} + u(x), \tag{2.4.2}$$

where

$$l(x) = \inf_{0 \leq h \leq x} \frac{z(h) - F_e(h)}{\overline{F}(h)} \quad \text{and} \quad u(x) = \sup_{0 \leq h \leq x} \frac{z(h) - F_e(h)}{\overline{F}(h)}.$$

Proof. Using integration by parts, we have

$$\int_0^x y\, dF(y) = \mu F_e(x) - x\overline{F}(x). \tag{2.4.3}$$

Thus, since u is increasing, we get

$$\begin{aligned} z(x) \quad &+ \quad \int_0^x \left\{ u(x-y) + \frac{x-y}{\mu} \right\} dF(y) \\ &\leq z(x) + u(x)F(x) + \frac{x}{\mu}F(x) - \frac{1}{\mu}\int_0^x y\, dF(y) \\ &= z(x) + u(x)F(x) + \frac{x}{\mu} - F_e(x). \end{aligned} \tag{2.4.4}$$

Hence, by Theorem 2.3,

$$u(x) + \frac{x}{\mu}$$

is an upper bound of $Z(x)$ if the following inequality holds,

$$z(x) + u(x)F(x) + \frac{x}{\mu} - F_e(x) \leq u(x) + \frac{x}{\mu}, \tag{2.4.5}$$

which is equivalent to

$$u(x) \geq \frac{z(x) - F_e(x)}{\overline{F}(x)}. \tag{2.4.6}$$

(2.4.6) is evident by the definition of $u(x)$. Thus, we get

$$Z(x) \le \frac{x}{\mu} + u(x),$$

furthermore, this implies that

$$\begin{aligned} Z(x) &= z(x) + \int_0^x Z(x-y)\,dF(y) \\ &\le z(x) + \int_0^x \left\{ u(x-y) + \frac{x-y}{\mu} \right\} dF(y), \end{aligned} \tag{2.4.7}$$

hence, the upper bounds in Theorem 2.6 follow from (2.4.7), (2.4.4) and (2.4.5).

Similarly, we get the lower bounds in Theorem 2.6. □

Theorem 2.7 Suppose $q = 1$, $0 < \mu < \infty$ and Z satisfies the renewal equation (1.1.5), then for any $x \ge 0$,

$$Z(x) \ge l(x)\frac{x}{\mu} + z(x) - l(x)\,F_e(x) \ge l(x)\frac{x}{\mu} \tag{2.4.8}$$

and

$$Z(x) \le u(x)\frac{x}{\mu} + z(x) - u(x)\,F_e(x) \le u(x)\frac{x}{\mu}, \tag{2.4.9}$$

where

$$l(x) = \inf_{0 \le h \le x} \frac{z(h)}{F_e(h)} \quad \text{and} \quad u(x) = \sup_{0 \le h \le x} \frac{z(h)}{F_e(h)}.$$

Proof. Since l is decreasing, by (2.4.3), we have

$$\begin{aligned} z(x) &+ \int_0^x \left[l(x-y)\frac{x-y}{\mu} \right] dF(y) \\ &\ge z(x) + l(x)\int_0^x \frac{x-y}{\mu}\,dF(y) \\ &= z(x) + \frac{l(x)x}{\mu} - l(x)F_e(x). \end{aligned} \tag{2.4.10}$$

Hence, by Theorem 2.3,

$$l(x)\frac{x}{\mu}$$

is a lower bound of Z if the following inequality holds,

$$z(x) + \frac{l(x)x}{\mu} - l(x)F_e(x) \ge l(x)\frac{x}{\mu}, \tag{2.4.11}$$

which is equivalent to

$$l(x) \leq \frac{z(x)}{F_e(x)}. \tag{2.4.12}$$

(2.4.12) is evident by the definition of $l(x)$. Thus, we get

$$Z(x) \geq l(x)\frac{x}{\mu}, \quad x \geq 0,$$

furthermore, this implies that

$$\begin{aligned} Z(x) &= z(x) + \int_0^x Z(x-y)\,dF(y) \\ &\geq z(x) + \int_0^x \left\{ l(x-y)\frac{x-y}{\mu} \right\} dF(y), \end{aligned} \tag{2.4.13}$$

hence, the lower bounds in Theorem 2.7 follow from (2.4.13), (2.4.10) and (2.4.11).

Similarly, we get the upper bounds in Theorem 2.7. □

2.4.2 Application to renewal functions

Since the renewal function M satisfies the proper renewal equation (2.1.9), take $z(x) = F(x)$ in Theorem 2.6 and 2.7, and note that

$$\inf_{0\leq h\leq x} \frac{F(h) - F_e(h)}{\overline{F}(h)} = \inf_{0\leq h\leq x} \frac{\overline{F}_e(h)}{\overline{F}(h)} - 1$$

and

$$\sup_{0\leq h\leq x} \frac{F(h) - F_e(h)}{\overline{F}(h)} = \sup_{0\leq h\leq x} \frac{\overline{F}_e(h)}{\overline{F}(h)} - 1,$$

we get directly the two following results for the renewal function M.

Corollary 2.3 Assume $0 < \mu < \infty$. For any $x \geq 0$,

$$M(x) \geq \frac{x}{\mu} + b_l(x)F(x) - F_e(x) \geq \frac{x}{\mu} + b_l(x) - 1 \tag{2.4.14}$$

and

$$M(x) \leq \frac{x}{\mu} + b_u(x)F(x) - F_e(x) \leq \frac{x}{\mu} + b_u(x) - 1, \tag{2.4.15}$$

where

$$b_l(x) = \inf_{0\leq h\leq x} \frac{\overline{F}_e(h)}{\overline{F}(h)} \quad \text{and} \quad b_u(x) = \sup_{0\leq h\leq x} \frac{\overline{F}_e(h)}{\overline{F}(h)}.$$

Corollary 2.4 Assume $0 < \mu < \infty$. For any $x \geq 0$,

$$M(x) \;\geq\; \frac{x}{\mu} l(x) + F(x) - l(x) F_e(x) \;\geq\; \frac{x}{\mu} l(x) \tag{2.4.16}$$

and

$$M(x) \;\leq\; \frac{x}{\mu} u(x) + F(x) - u(x) F_e(x) \;\leq\; \frac{x}{\mu} u(x), \tag{2.4.17}$$

where

$$l(x) = \inf_{0 \leq h \leq x} \frac{F(h)}{F_e(h)} \quad \text{and} \quad u(x) = \sup_{0 \leq h \leq x} \frac{F(h)}{F_e(h)}.$$

Since $b_l(x) \geq 0$, Corollary 2.3 implies the following simple bound for $M(x)$, namely, for any $x \geq 0$,

$$M(x) \;\geq\; \frac{x}{\mu} - F_e(x). \tag{2.4.18}$$

Since $0 \leq F_e(x) \leq 1$, the bound in (2.4.18) is sharper than the well known lower bound for $M(x)$ [see, Barlow and Proschan (1981) and Stoyan (1983)], namely,

$$M(x) \;\geq\; \frac{x}{\mu} - 1. \tag{2.4.19}$$

The bounds in Corollary 2.3 are also tighter than those of Marshall (1973), in which he showed that

$$\frac{x}{\mu} + b_l(\infty) - 1 \;\leq\; M(x) \;\leq\; \frac{x}{\mu} + b_u(\infty) - 1. \tag{2.4.20}$$

The renewal function M plays a central role in renewal theory and has many applications in reliability, especially in maintenance models. There has been much interest in estimating $M(x)$ for reliability life distribution classes. However, from the general results in Corollary 2.3 and 2.4, we can easily derive refinements of some well known results. Therefore, we first give the definitions of reliability life distribution classes. The most well known classes are the IFR, IFRA, NBU, NBUC, DMRL, NBUE and HNBUE (with their duals: DFR, DFRA, NWU, NWUC, IRML, NWUC and HNWUE), whose definitions are given below. These classes vividly describe the forms of aging of lifetime random variables and have more and more applications in different disciplines such as reliability, queueing theory, inventory theory and risk analysis [e.g. see Barlow and Proschan (1981), Shaked and Shanthikumar (1994), Stoyan (1983) and Szekli (1995)]. We will also discuss their applications in this thesis.

Definition 2.4 A life distribution F is said to have (or to be)

1. increasing failure rate (IFR) if for any $x \geq 0$, the following function
$$\frac{\overline{F}(x+t)}{\overline{F}(t)}$$
is decreasing in $t \geq 0$;

2. increasing failure rate in average (IFRA) if
$$\frac{-\ln \overline{F}(t)}{t}$$
is increasing in $t \geq 0$;

3. new better than used (NBU) if for all $x, y \geq 0$,
$$\overline{F}(x+y) \;\leq\; \overline{F}(x)\,\overline{F}(y);$$

4. new better than used in convex ordering (NBUC) if for all $x, y \geq 0$,
$$\int_{x+y}^{\infty} \overline{F}(t)\,dt \;\leq\; \overline{F}(x)\int_{y}^{\infty} \overline{F}(t)\,dt;$$

5. decreasing mean residual life (DMRL) if
$$\frac{\int_x^\infty \overline{F}(y)\,dy}{\overline{F}(x)}$$
is decreasing in $x \geq 0$;

6. new better than used in expectation (NBUE) if F has the finite mean μ and for any $x \geq 0$,
$$\int_x^\infty \overline{F}(y)\,dy \;\leq\; \mu\,\overline{F}(x);$$

7. harmonic new better than used in expectation (HNBUE) if F has the finite mean μ and for any $x \geq 0$,
$$\int_x^\infty \overline{F}(y)\,dy \;\leq\; \mu\, e^{-x/\mu}.$$

By reversing the inequalities and monotonicity above, the dual classes, decreasing failure rate (DFR), decreasing failure rate in average (DFRA), new worse than used (NWU), new worse than used in convex ordering (NWUC), increasing mean residual life (IMRL), new worse than used in expectation (NWUE) and harmonic new worse than used in expectation (NWUE) are defined.

The relations among the classes above are the following:

$$\begin{array}{cccccccccc}
 & & \text{IFRA} & \longrightarrow & \text{NBU} & & & & & \\
 & \nearrow & & & & \searrow & & & & \\
\text{IFR} & & & & & & \text{NBUC} & \longrightarrow \text{NBUE} & \longrightarrow & \text{HNBUE} \\
 & \searrow & & & & \nearrow & & & & \\
 & & & \text{DMRL} & & & & & &
\end{array}$$

and

$$\begin{array}{cccccccccc}
 & & \text{DFRA} & \longrightarrow & \text{NWU} & & & & & \\
 & \nearrow & & & & \searrow & & & & \\
\text{DFR} & & & & & & \text{NWUC} & \longrightarrow \text{NWUE} & \longrightarrow & \text{HNWUE.} \\
 & \searrow & & & & \nearrow & & & & \\
 & & & \text{IMRL} & & & & & &
\end{array}$$

For the relations and the properties of the classes, see Barlow and Proschan (1981) for IFR (DFR), IFRA (DFRA), NBU (NWU), DMRL (IMRL) and NBUE (NWUE), Cai and Wu (1997c) and Cao and Wang (1991) for NBUC (NWUC) and Cai (1994a, 1994b, 1995), Cai and Wu (1997a) and Klefsjö (1982) for HNBUE (HNWUE).

Some important life distributions with increasing (or decreasing) failure rates are the following:

Example 2.1 Gamma distribution with the density function

$$g(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad x \geq 0, \quad \text{where } \lambda > 0,\ \alpha > 0,$$

is DFR if $0 < \alpha \leq 1$ and IFR if $\alpha \geq 1$.

Example 2.2 Weibull distribution with the distribution function

$$F(x) = 1 - e^{-(\lambda x)^\alpha}, \ x \geq 0, \quad \text{where } \lambda > 0,\ \alpha > 0,$$

is DFR if $0 < \alpha \leq 1$ and IFR if $\alpha \geq 1$.

Example 2.3 Truncated normal distribution with the density function

$$f(x) = \frac{1}{a\sigma\sqrt{2\pi}}\, e^{-(x-u)^2/2\sigma^2}, \ x \geq 0, \quad \text{where } \sigma > 0,\ -\infty < \mu < \infty,\ a = \Phi(u/\sigma),$$

is IFR.

Example 2.4 **Pareto distribution** with the distribution function

$$F(x) = 1 - \left(\frac{\lambda}{\lambda + x}\right)^{\alpha}, \quad x \geq 0, \quad \text{where } \alpha > 0, \lambda > 0,$$

is DFR.

Example 2.5 **Burr distribution** with the distribution function

$$F(x) = 1 - \left(\frac{\lambda}{\lambda + x^{\tau}}\right)^{\alpha}, \; x \geq 0, \quad \text{where } \alpha > 0, \lambda > 0, \tau > 0,$$

is DFR if $0 < \tau \leq 1$.

Now, by Definition 2.4, if F is NBUE, then $\overline{F}_e(u) \leq \overline{F}(u)$, for $u \geq 0$, which is equivalent to $F_e(u) \geq F(u)$, for $u \geq 0$. Hence, Corollary 2.3 and 2.4 imply that if F is NBUE, then

$$M(x) \;\leq\; \frac{x}{\mu} - \overline{F}(x) + \overline{F}_e(x) \;\leq\; \frac{x}{\mu} \tag{2.4.21}$$

and

$$M(x) \;\leq\; \frac{x}{\mu} \sup_{0 \leq u \leq x} \frac{F(u)}{F_e(u)} \;\leq\; \frac{x}{\mu}. \tag{2.4.22}$$

If F is NWUE, then

$$M(x) \;\geq\; \frac{x}{\mu} - \overline{F}(x) + \overline{F}_e(x) \;\geq\; \frac{x}{\mu} \tag{2.4.23}$$

and

$$M(x) \;\geq\; \frac{x}{\mu} \inf_{0 \leq u \leq x} \frac{F(u)}{F_e(u)} \;\geq\; \frac{x}{\mu}. \tag{2.4.24}$$

These bounds are refinements and generalizations of the bounds in Theorem C and E of Section 2.10 of Szekli (1995), in which the following bounds are shown, namely, if F is IFR (DFR), then

$$M(x) \;\leq\; (\geq)\, \frac{x}{\mu} \frac{F(x)}{F_e(x)} \;\leq\; (\geq)\, \frac{x}{\mu}.$$

and if F is NBUE(NWUE), then

$$M(x) \;\leq (\geq)\, \frac{x}{\mu}.$$

In fact, it is straightforward to show that

$$\frac{F(u)}{F_e(u)}$$

is increasing(decreasing) if F is IFR(DFR). Note that

$$\begin{aligned}\left(\frac{F(u)}{F_e(u)}\right)' &= \frac{f(u)\,F_e(u) - F(u)\,\overline{F}(u)/\mu}{[F_e(u)]^2} \\ &= \frac{\overline{F}(u)}{[F_e(u)]^2}\,[r(u)F_e(u) - F(u)/\mu]\ ,\end{aligned}$$

where f is the density function of F and r is its failure rate function, *i.e.* $r(x) = f(x)/\overline{F}(x)$, which is called force of mortality in actuarial mathematics.

Thus, if F is IFR(DFR), then r is increasing (decreasing), this implies that

$$\begin{aligned}F_e(u) &= \frac{1}{\mu}\int_0^u \overline{F}(y)\,dy = \frac{1}{\mu}\int_0^u [r(y)]^{-1} f(y)\,dy \\ &\geq (\leq)\ [r(u)]^{-1}\,F(u)/\mu.\end{aligned}$$

Hence,

$$\left(\frac{F(u)}{F_e(u)}\right)' \geq (\leq)\ 0.$$

In addition, if F is NBUE, then F is HNBUE, *i.e.* $\overline{F}_e(x) \leq e^{-x/\mu}$ for $x \geq 0$, this implies (2.4.21) is also tighter than that of Bhattacharjee (1996), in which he showed that if F is NBUE, then for $x \geq 0$,

$$M(x) \leq \frac{x}{\mu} - (\overline{F}(x) - e^{-\frac{x}{\mu}})_+.$$

The applications above show that the method developed in this chapter is very effective and straightforward. Undoubtedly, we can give more applications since the renewal equations arise extensively in applied probability models.

2.5 Bounds and asymptotic formulae for defective renewal equations in NWU and NBU distributions

2.5.1 Bounds in NWU and NBU distributions

We notice that for the excessive renewal equation, the adjustment coefficient always exists. However, for the defective renewal equations, it may not. For example, this is true for the heavy and medium tailed distributions. Thus, Property 2.1 and Theorem 2.4 do not apply to these cases. However, we can generalize the Lundberg equation (2.1.6) so that it can be satisfied by more general distributions. A natural generalization of the Lundberg equation (2.1.6) is to replace the exponential function by a general life distribution B with $B(0) = 0$. Assume that B satisfies the following equation,

$$\int_0^\infty [\overline{B}(x)]^{-1}\, dF(y) = \frac{1}{q}. \tag{2.5.1}$$

Since the upper bound $u(x)\, e^{-Rx}$ in Theorem 2.4 can be rewritten as

$$u(x)\, e^{-Rx} = \left\{ \inf_{0 \le h \le x} \frac{q\, e^{R(x-h)} \int_h^\infty e^{Ry}\, dF(y)}{z(h)} \right\}^{-1},$$

we guess under condition (2.5.1), the following function is an upper bound for the solution of the defective renewal equations,

$$\left\{ \inf_{0 \le h \le x} \frac{q\, [\overline{B}(x-h)]^{-1} \int_h^\infty [\overline{B}(y)]^{-1}\, dF(y)}{z(h)} \right\}^{-1}.$$

This is valid if we choose B as an NWU distribution. Willmot (1994) first used condition (2.5.1) to study the tails of compound distributions. There have been extensive applications to the study of bounds for tails and ruin probabilities, see, for example, Cai and Wu (1997b), Lin (1996), Willmot (1994, 1996, 1997a, 1997b) and Willmot and Lin (1997a) by different methods.

However, it is easy to prove this general result by the method developed in this Chapter. To do so, we define the following notation

$$K(h, x) = \frac{q\, [\overline{B}(x-h)]^{-1} \int_h^\infty [\overline{B}(y)]^{-1}\, dF(y)}{z(h)} \tag{2.5.2}$$

and

$$[K_1(x)]^{-1} = \sup_{0 \le h \le x} K(h, x) \quad \text{and} \quad [K_2(x)]^{-1} = \inf_{0 \le h \le x} K(h, x).$$

First, we show that the functions K_1 and K_2 have the following properties.

Property 2.2 Suppose that B is a life distribution with $B(0) = 0$ and satisfies (2.5.1).

1. For $x \geq 0$,

$$z(x) \leq q\, K_2(x) \int_x^\infty [\overline{B}(y)]^{-1}\, dF(y). \tag{2.5.3}$$

2. If B is NWU, then for any $0 \leq y \leq x$,

$$K_2(x-y) \leq [\overline{B}(y)]^{-1}\, K_2(x). \tag{2.5.4}$$

Proof. (2.5.3) follows simply from

$$\begin{aligned} [K_2(x)]^{-1} &= \inf_{0 \leq h \leq x} K(h,x) \leq K(x,x) \\ &= q\,[z(x)]^{-1} \int_x^\infty [\overline{B}(y)]^{-1}\, dF(y). \end{aligned}$$

Since B is NWU, for any $0 \leq y \leq x$ and $0 \leq h \leq x-y$,

$$\overline{B}(x-h) \geq \overline{B}(x-y-h)\,\overline{B}(y),$$

which implies that

$$K(h, x-y) \geq \overline{B}(y)\, K(h,x),$$

hence

$$\begin{aligned} [K_2(x-y)]^{-1} &= \inf_{0 \leq h \leq x-y} K(h, x-y) \\ &\geq \overline{B}(y) \inf_{0 \leq h \leq x-y} K(h,x) \\ &\geq \overline{B}(y) \inf_{0 \leq h \leq x} K(h,x) \\ &= \overline{B}(y)\,[K_2(x)]^{-1}, \end{aligned}$$

this implies that (2.5.4) holds. □

Similarly, we have

Property 2.3 Suppose that B is a life distribution with $B(0) = 0$ and satisfies (2.5.1).

1. For $x \geq 0$,

$$z(x) \geq q\, K_1(x) \int_x^\infty [\overline{B}(y)]^{-1}\, dF(y). \tag{2.5.5}$$

2. If B is NBU, then for any $0 \le y \le x$,

$$K_1(x-y) \ \ge \ [\overline{B}(y)]^{-1} K_1(x). \tag{2.5.6}$$

Theorem 2.8 Suppose Z satisfies the defective renewal equation (1.1.5), *i.e.* $0 < q < 1$, and the life distribution B with $B(0) = 0$ satisfies (2.5.1).

(**1**). If B is a NWU distribution, then for any $x \ge 0$,

$$Z(x) \ \le \ z(x) + F_B(x) K_2(x) \ \le \ K_2(x). \tag{2.5.7}$$

(**2**). If B is a NBU distribution, then for any $x \ge 0$,

$$Z(x) \ \ge \ z(x) + F_B(x) K_1(x) \ \ge \ K_1(x)\,, \tag{2.5.8}$$

where

$$F_B(x) = q \int_0^x [\overline{B}(y)]^{-1}\, dF(y).$$

Proof. By (2.5.4), (2.5.3) and (2.5.1), we have for $x \ge 0$,

$$\begin{aligned} z(x) \ &+ \ q \int_0^x K_2(x-y)\, dF(y) \\ &\le \ z(x) + q\, K_2(x) \int_0^x [\overline{B}(y)]^{-1}\, dF(y) && (2.5.9) \\ &\le \ q\, K_2(x) \int_x^\infty [\overline{B}(y)]^{-1}\, dF(y) + q\, K_2(x) \int_0^x [\overline{B}(y)]^{-1}\, dF(y) \\ &= \ q\, K_2(x) \int_0^\infty [\overline{B}(y)]^{-1}\, dF(y) \\ &= \ K_2(x)\,. && (2.5.10) \end{aligned}$$

Thus, by Theorem 2.3, we get

$$Z(x) \ \le \ K_2(x), \quad x \ge 0,$$

this implies that for $x \ge 0$

$$\begin{aligned} Z(x) \ &= \ z(x) + q \int_0^x Z(x-y)\, dF(y) \\ &\le \ z(x) + q \int_0^x K_2(x-y)\, dF(y)\,. \end{aligned}$$

Thus, (2.5.7) follows from (2.5.9) and (2.5.10). Similarly, we get the lower bounds in (2.5.8). □

From Theorem 2.8, we have the following simple upper bound for $Z(x)$.

Corollary 2.5 If B is an NWU distribution, then for $x \geq 0$,

$$Z(x) \leq \left\{ \inf_{0 \leq h \leq x} \frac{q\overline{F}(h)}{z(h)} \right\}^{-1} \overline{B}(x) . \tag{2.5.11}$$

Proof. By the definition (2.5.2) of $K(h,x)$, we have

$$\begin{aligned} K(h,x) &\geq \frac{q\,[\overline{B}(x-h)]^{-1}\,[\overline{B}(h)]^{-1} \int_h^\infty dF(y)}{z(h)} \\ &= \frac{q\,[\overline{B}(x-h)]^{-1}\,[\overline{B}(h)]^{-1}\,\overline{F}(h)}{z(h)} \\ &\geq [\overline{B}(x)]^{-1}\,\frac{q\overline{F}(h)}{z(h)}, \end{aligned}$$

where, the last inequality follows from the definition of NWU.

Thus,

$$[K_2(x)]^{-1} = \inf_{0 \leq h \leq x} K(h,x) \geq [\overline{B}(x)]^{-1} \inf_{0 \leq h \leq x} \frac{q\overline{F}(h)}{z(h)},$$

which implies (2.5.11) holds by Theorem 2.8. □

Applying Theorem 2.8 to the tail $\overline{G}_q(x)$ of the compound distribution in (2.3.8) by taking $z(x) = q\overline{F}(x)$ and Corollary 2.5, we get directly the following result.

Theorem 2.9 (**1**). If a NWU distribution B with $B(0) = 0$ satisfies (2.5.1), then for any $x \geq 0$,

$$\overline{G}_q(x) \leq q\overline{F}(x) + F_B(x)\,K_2(x) \leq K_2(x), \tag{2.5.12}$$

in particular, for any $x \geq 0$,

$$\overline{G}_q(x) \leq \overline{B}(x). \tag{2.5.13}$$

(**2**). If a NBU distribution B with $B(0) = 0$ satisfies (2.5.1), then for any $x \geq 0$,

$$\overline{G}_q(x) \geq q\overline{F}(x) + F_B(x)\,K_1(x) \geq K_1(x), \tag{2.5.14}$$

where

$$F_B(x) = q \int_0^x [\overline{B}(y)]^{-1}\,dF(y).$$

Theorem 2.9 results in new bounds for a class of compound distributions, which will be discussed in Chapter 3. Those new bounds are uniformly sharper than those main results of Lin (1996), Willmot (1994, 1997a, 1997b) and Willmot and Lin (1997a). We will also discuss how to use reliability distribution classes to simplify these bounds.

We note that the condition (2.5.1) can be satisfied by some distributions without adjustment coefficients such as the distributions with only a finite number of moments, Pareto and certain inverse Gaussian distributions. Hence Theorem 2.8 and 2.9 apply to more general cases. Some details are given as follows.

Example 2.6 Assume that F is a life distribution with only a finite number of moments up to $k > 0$, *i.e.* $\int_0^\infty x^i \, dF(x) < \infty$, $1 \le i \le k$ and $\int_0^\infty x^i \, dF(x) = \infty$, $i > k$. For the heavy-tailed distribution F, we can choose B to be a Pareto distribution and $\overline{B}(x) = (1 + \lambda x)^{-\alpha}$ such that α and λ satisfy $\int_0^\infty (1 + \lambda y)^\alpha \, dF(y) = 1/q$.

Example 2.7 Assume that F is the generalized inverse Gaussian distribution with the density function

$$f(x) = \frac{(\mu/\beta)^{\lambda/2}}{2K_\lambda(2\sqrt{\mu\beta})} \, x^{\lambda-1} \, e^{-(\mu x + \beta/x)}, \quad x > 0 \,,$$

where $\mu > 0$, $\beta > 0$, $-\infty < \lambda < \infty$ and K_λ is the modified Bessel function of the third kind with index λ. A special case includes the inverse Gaussian ($\lambda = -\frac{1}{2}$). If X has the distribution F, $\lambda < 0$ and $E(e^{\mu X}) < 1/q$, then the adjustment coefficient does not exist [see Embrechts (1983) for the detail]. However we can choose B to be $\overline{B}(x) = (1 + \kappa x)^{-m} \, e^{-\mu x}$ such that κ and m satisfy $\int_0^\infty (1 + \kappa y)^m \, e^{\mu y} \, dF(y) = 1/q$.

Example 2.8 If F is NWU, we can always choose $\overline{B}(x) = [\overline{F}(x)]^{1-q}$ since

$$\int_0^\infty [\overline{F}(y)]^{q-1} \, dF(y) = \int_0^1 (1-y)^{q-1} \, dy = \frac{1}{q}.$$

2.5.2 Asymptotic formulae in NWU distributions

Corollary 2.5 implies that if (2.5.1) holds for an NWU distribution function B, then the solution Z of the defective renewal equation is bounded by $\overline{B}$. Thus, similar to the relation between Lundberg's inequality and the Cramér-Lundberg asymptotic formula, we guess that under suitable conditions there exists a constant $C^* \ge 0$ such that

$$Z(x) \sim C^* \, \overline{B}(x), \quad x \to \infty, \tag{2.5.15}$$

holds. Indeed, we get the following result.

Theorem 2.10 Suppose that Z satisfies the defective renewal equation (1.1.5), *i.e.* $0 < q < 1$ and there exists a continuous NWU distribution function B with $B(0) = 0$ such that (2.5.1) holds. If the distribution F is non-lattice and the following function

$$\int_0^x \left[1 - \frac{\overline{B}(y)\overline{B}(x-y)}{\overline{B}(x)}\right] [\overline{B}(y)]^{-1} \, dF(y) \tag{2.5.16}$$

is directly Riemann integrable, then there exists a constant $C^* \geq 0$ such that

$$\lim_{x \to \infty} Z(x) \, [\overline{B}(x)]^{-1} = C^*. \tag{2.5.17}$$

The proof of Theorem 2.10 is based on a recent result about a generalized renewal equation discussed by Schmidli (1997), in which he studied the following generalized renewal equation, for $x \geq 0$,

$$Z_1(x) = z(x) + \int_0^x [1 - p(x,y)] \, Z_1(x-y) \, dH(y) \ , \tag{2.5.18}$$

where H is a probability distribution with $H(0) = 0$ and $p(x, y)$ satisfies the conditions that $0 \leq p(x,y) \leq 1$, it is continuous in x and the function $\int_0^x p(x,y) dH(y)$ is directly Riemann integrable.

Let Z_0 be the solution to the ordinary renewal equation

$$Z_0(x) = z(x) + \int_0^x Z_0(x-y) \, dH(y), \tag{2.5.19}$$

Schmidli (1997) proved the two following results.

Lemma 2.1 Assume that $z(x) \geq 0$ is bounded. Let Z_1 satisfy the renewal equation (2.5.18), then

$$0 \ \leq \ Z_1(x) \ \leq \ Z_0(x), \quad x \geq 0. \tag{2.5.20}$$

Theorem 2.11 Assume that z is directly Riemann integrable and that Z_1 satisfies the renewal equation (2.5.18) and is bounded on bounded intervals. If H is non-lattice, then $\lim_{x \to \infty} Z_1(x)$ exists and is finite.

Theorem 2.11 is the main result of Schmidli (1997) and has an application to the Björk-Grandell risk model [see, Björk and Grandell (1988)]. Theorem 2.11 is also a generalization of the key renewal theorem to the generalized renewal equation

(2.5.18). Unfortunately, unlike the key renewal theorem for the ordinary renewal equation (2.5.19), no explicit limit of $Z_1(x)$ can be found, we only know by Lemma 2.1 and the key renewal theorem that,

$$0 \le \lim_{x\to\infty} Z_1(x) \le \lim_{x\to\infty} Z_0(x) = \frac{\int_0^\infty z(x)dx}{\int_0^\infty x\,dH(x)}. \tag{2.5.21}$$

However, by Lemma 2.1, we can use the upper bounds of $Z_0(x)$ obtained in this Chapter to estimate $Z_1(x)$.

The proof of Theorem 2.10 is similar to that of Property 2.1. The renewal equation (1.1.5) implies that

$$[\overline{B}(x)]^{-1} Z(x) = [\overline{B}(x)]^{-1} z(x) + q \int_0^x \frac{[\overline{B}(x-y)]^{-1} Z(x-y)\,[\overline{B}(x)]^{-1}\,[\overline{B}(y)]^{-1}}{[\overline{B}(x-y)]^{-1}\,[\overline{B}(y)]^{-1}}\, dF(y)$$

i.e.

$$Z^{\#}(x) = z^{\#}(x) + \int_0^x [1 - b(x,y)]\, Z^{\#}(x-y)\, dF^{\#}(y), \tag{2.5.22}$$

where $Z^{\#}(x) = [\overline{B}(x)]^{-1} Z(x)$, $\quad z^{\#}(x) = [\overline{B}(x)]^{-1} z(x)$, then

$$b(x,y) = 1 - \frac{\overline{B}(y)\,\overline{B}(x-y)}{\overline{B}(x)}$$

and $dF^{\#}(y) = q\,[\overline{B}(y)]^{-1}\, dF(y)$ is a proper probability measure.

However, B is NWU, so $0 \le b(x,y) \le 1$. Thus, Theorem 2.10 follows from (2.5.22) and Theorem 2.11.

Clearly, if $\overline{B}(x) = e^{-Rx}$, then the function in (2.5.16) equals zero and is directly Riemann integrable, so Property 2.1 is a special case of Theorem 2.10. However, a natural question arises. Except for B exponential, are there other NWU distributions such that (2.5.17) holds? This question is interesting but not simple since the function in (2.5.16) is complicated. We can not answer it yet. Once one finds such an NWU distribution B, one gets a new asymptotic formula for the solution of defective renewal equations.

Chapter 3

A Class of Compound Distributions and Renewal Processes

3.1 Introduction

For the compound distribution G in (1.1.1), in a series of works of Lin (1996), Willmot (1994, 1997a, 1997b) and Willmot and Lin (1994, 1997a), have considered a class of compound distributions, for which there exists a constant $0 < \phi < 1$ such that the probability distribution $\{p_n, n \geq 0\}$ of N satisfies

$$a_{n+1} \leq \phi\, a_n, \quad n = 0, 1, 2, \cdots, \tag{3.1.1}$$

or

$$a_{n+1} \geq \phi\, a_n, \quad n = 0, 1, 2, \cdots, \tag{3.1.2}$$

where

$$a_n = \Pr\{N > n\} = \sum_{k=n+1}^{\infty} p_k \,.$$

Such a class of compound distributions includes many interesting models [see, for example, Panjer and Willmot (1992) and Willmot and Lin (1994)]. The main results and methods for estimating the tail probabilities of this class of compound distributions are summarized below.

Denote the tail probability of the class of compound distributions by

$$\psi(x) = \Pr\{S > x\} = \sum_{n=1}^{\infty} p_n\, \overline{F}^{(n)}(x)\,, \; x \geq 0. \tag{3.1.3}$$

Suppose that B is a life distribution with $B(0) = 0$ and satisfies the following equation

$$\int_0^\infty \left[\overline{B}(y)\right]^{-1} dF(y) = \frac{1}{\phi}\,. \tag{3.1.4}$$

By the induction method, Willmot (1994) shows that if (3.1.1) holds and (3.1.4) is satisfied by an NWU distribution function B, then

$$\psi(x) \le \frac{1-p_0}{\phi}\,\alpha_2(x)\,\overline{B}(x), \quad x \ge 0\,, \tag{3.1.5}$$

where $\alpha_2(x)$ is given below.

By the renewal recursive method developed by Cai and Wu (1997b), Willmot (1997a) shows that if (3.1.2) holds and (3.1.4) is satisfied by an NBU distribution function B, then

$$\psi(x) \ge \frac{1-p_0}{\phi}\,\alpha_1(x)\,\overline{B}(x), \quad x \ge 0\,, \tag{3.1.6}$$

where

$$[\alpha_1(x)]^{-1} = \sup_{0\le h\le x,\ \overline{F}(h)>0} \alpha(h)\,, \quad [\alpha_2(x)]^{-1} = \inf_{0\le h\le x,\ \overline{F}(h)>0} \alpha(h)\,,$$

and

$$\alpha(h) = \frac{\int_h^\infty [\overline{B}(y)]^{-1}\,dF(y)}{[\overline{B}(h)]^{-1}\,\overline{F}(h)}\,.$$

Using a generalized Wald's identity, Lin (1996) shows that if (3.1.1) holds and (3.1.4) is satisfied by an NWU distribution function B, then

$$\psi(x) \le \frac{1-p_0}{\phi}\,\Delta_2(x), \quad x \ge 0\,, \tag{3.1.7}$$

while if (3.1.2) holds and (3.1.4) is satisfied by an NBU distribution function B, then

$$\psi(x) \ge \frac{1-p_0}{\phi}\,\Delta_1(x), \quad x \ge 0\,, \tag{3.1.8}$$

where

$$[\Delta_1(x)]^{-1} = \sup_{0\le h\le x,\ \overline{F}(h)>0} \Delta(h)\,, \quad [\Delta_2(x)]^{-1} = \inf_{0\le h\le x,\ \overline{F}(h)>0} \Delta(h)\,,$$

and

$$\Delta(h) = \frac{\int_h^\infty [\overline{B}(x-h+y)]^{-1}\,dF(y)}{\overline{F}(h)}\,.$$

Willmot (1997a) shows that if B has a Decreasing Failure Rate (DFR, a subclass of NWU), then

$$\alpha_2(x)\,\overline{B}(x) \le \Delta_2(x), \quad x \ge 0\,,$$

and if B has an Increasing Failure Rate (IFR, a subclass of NBU), then

$$\alpha_1(x)\,\overline{B}(x) \ge \Delta_1(x), \quad x \ge 0\,.$$

That is to say that the upper bound in (3.1.5) is tighter than that in (3.1.7) if B is DFR and the lower bound in (3.1.6) is tighter than that in (3.1.8) if B is IFR.

As shown by Cai and Wu (1997b), conditions (3.1.1) and (3.1.2) imply respectively that

$$\psi(x) \le \frac{1-p_0}{\phi}\,\psi^*(x), \quad x \ge 0\,, \tag{3.1.9}$$

and

$$\psi(x) \ge \frac{1-p_0}{\phi}\,\psi^*(x), \quad x \ge 0\,, \tag{3.1.10}$$

where

$$\psi^*(x) = \sum_{n=1}^{\infty} (1-\phi)\,\phi^n\,\overline{F}^{(n)}(x)\,, \tag{3.1.11}$$

is the tail of a compound geometric distribution.

Thus, bounds for the tail ψ of the class of compound distributions can be deduced from the bounds for the tail ψ^* of the compound geometric distribution. With this idea, we derive here new lower and upper bounds for $\psi(x)$, which are uniformly sharper than the bounds in (3.1.5), (3.1.6), (3.1.7) and (3.1.8).

In addition, based on the bounds in (3.1.5), (3.1.6), (3.1.7) and (3.1.8), many simplified bounds have been derived in the references mentioned above by imposing additional assumptions on B and F. However, as pointed out by Schmidli (1997b), some proofs of these simplified bounds [for example, those of Willmot and Lin (1997a)] are wrong in general cases, due to an improper use of integrations by parts. But, "by making the same mistake twice, the results turn out to be correct (Schmidli, 1997b)".

In Section 3.5, we use the technique of stochastic ordering to derive simplified bounds. The method is simple and unifying. The error is corrected and these simplified bounds are tighter than those in previous results.

We have pointed out in Section 2.5 that condition (3.1.4) is a generalization of the following Cramér-Lundberg condition, *i.e.* there exists a constant κ such that

$$\int_0^\infty e^{\kappa y}\, dF(y) = \frac{1}{\phi}\,. \tag{3.1.12}$$

Condition (3.1.4) applies to more general distributions including some heavy tailed distributions. Therefore, as we discuss in Section 3.2, many results based on condition (3.1.4) are still not applicable to heavy-tailed distributions. This motivates us to consider other more general conditions that can be satisfied by general distributions, especially by heavy-tailed distributions. For this purpose, two truncated versions of the Cramér-Lundberg condition are proposed.

The first version is obtained by replacing the exponential function $e^{\kappa y}$ in (3.1.12) by a truncated exponential function $\min(e^{\rho_t y},\ e^{\rho_t t})$ that satisfies the following equation

$$\int_0^t e^{\rho_t y}\, dF(y) + e^{\rho_t t}\,\overline{F}(t) = \frac{1}{\phi}\,, \tag{3.1.13}$$

for a given $t > 0$ and ρ_t.

The second type is obtained by replacing the exponential function $e^{\kappa y}$ in (3.1.12) by a truncated exponential function $e^{\kappa_t y}\, I_{[0,\,t)}(y)$ that satisfies the following equation

$$\int_0^t e^{\kappa_t y}\, dF(y) = \frac{1}{\phi}\,, \tag{3.1.14}$$

for a given $t > 0$ and κ_t.

First, we point out the following fact that conditions (3.1.13) and (3.1.14) can be satisfied by any claim size distribution F with positive (possibly infinite) mean, by choosing a sufficiently large value of t. To see this fact, suppose that $0 < \int_0^\infty y\, dF(y) \le \infty$, there must exist a $t > 0$ such that

$$\int_0^t y\, dF(y) > 0\,. \tag{3.1.15}$$

Let

$$h(x) = \int_0^t e^{xy}\, dF(y) + e^{xt}\,\overline{F}(t) - \frac{1}{\phi}\,.$$

Thus

$$\begin{aligned} h(0) &= F(t) + \overline{F}(t) - \frac{1}{\phi} \\ &= 1 - \frac{1}{\phi} < 0\,. \end{aligned} \tag{3.1.16}$$

We have pointed out in Section 2.5 that condition (3.1.4) is a generalization of the following Cramér-Lundberg condition, *i.e.* there exists a constant κ such that

$$\int_0^\infty e^{\kappa y}\, dF(y) = \frac{1}{\phi}\,. \tag{3.1.12}$$

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The first version is obtained by replacing the exponential function $e^{\kappa y}$ in (3.1.12) by a truncated exponential function $\min(e^{\rho_t y},\ e^{\rho_t t})$ that satisfies the following equation

$$\int_0^t e^{\rho_t y}\, dF(y) + e^{\rho_t t}\,\overline{F}(t) = \frac{1}{\phi}\,, \tag{3.1.13}$$

for a given $t > 0$ and ρ_t.

The second type is obtained by replacing the exponential function $e^{\kappa y}$ in (3.1.12) by a truncated exponential function $e^{\kappa_t y}\, I_{[0,t)}(y)$ that satisfies the following equation

$$\int_0^t e^{\kappa_t y}\, dF(y) = \frac{1}{\phi}\,, \tag{3.1.14}$$

for a given $t > 0$ and κ_t.

First, we point out the following fact that conditions (3.1.13) and (3.1.14) can be satisfied by any claim size distribution F with positive (possibly infinite) mean, by choosing a sufficiently large value of t. To see this fact, suppose that $0 < \int_0^\infty y\, dF(y) \le \infty$, there must exist a $t > 0$ such that

$$\int_0^t y\, dF(y) > 0\,. \tag{3.1.15}$$

Let

$$h(x) = \int_0^t e^{xy}\, dF(y) + e^{xt}\,\overline{F}(t) - \frac{1}{\phi}\,.$$

Thus

$$\begin{aligned} h(0) &= F(t) + \overline{F}(t) - \frac{1}{\phi} \\ &= 1 - \frac{1}{\phi} < 0\,. \end{aligned} \tag{3.1.16}$$

given in Section 3.3. As applications, improved lower and upper bounds for $\psi(x)$ are derived.

Section 3.5 gives some simplied bounds, based on the results in Section 3.4, that are derived in a unifying way, by stochastic ordering. This refines the bounds of Willmot and Lin (1997a) and avoids inappropriate uses of integration by parts.

3.2 Some relations between reliability distribution classes and heavy tailed distributions

Property 3.1 If F is a NBUE distribution function, then it can not be a heavy-tailed distribution function.

Proof. Let μ_F be the mean of F. Since NBUE $\rightarrow$ HNBUE , thus by Theorem 3 of Klefsjö (1982), we know that for $x \geq \mu_F$,

$$\overline{F}(x) \;\leq\; \exp\left\{\frac{\mu_F - x}{\mu_F}\right\} . \tag{3.2.1}$$

This implies that F has an exponential tail, *i.e.* there exists some $\alpha > 0$ such that $\int_0^\infty e^{\alpha y}\, dF(y) < \infty$. Hence, F is not heavy-tailed. □

Due to the relations between the reliability distribution classes (see, Section 2.4.2), Property 3.1 implies that all bounds based on the condition that F be a member of IFR, IFRA, DMRL, NBU, NBUC and NBUE distribution classes, are not applicable to heavy-tailed distributions. For example (2.8), (2.13) and (2.14) of Lin (1996) and (2.7) and (3.8) of Willmot and Lin (1997a).

Property 3.2 If B is a NBU distribution function, then condition (3.1.4) can not be satisfied by heavy-tailed distributions F.

Proof. Since condition (3.1.4) implies that $\left[\overline{B}(x)\right]^{-1} \overline{F}(x) \rightarrow 0$ as $x \rightarrow \infty$, there must exist some $M > 0$ such that $\overline{F}(x) \;\leq\; \overline{B}(x)$ for $x \geq M$. Thus, the relation NBU $\rightarrow$ HNBUE and (3.2.1) imply for $x \geq \max(M,\ \mu_B)$ that

$$\overline{F}(x) \;\leq\; \overline{B}(x) \;\leq\; \exp\left\{\frac{\mu_B - x}{\mu_B}\right\} , \tag{3.2.2}$$

where μ_B is the mean of B.

(3.2.2) implies that F has an exponential tail and can not be heavy-tailed. □

Remark 3.1 Property 3.2 implies that all the lower bounds of Lin (1996), Willmot (1994, 1997a, 1997b) and Willmot and Lin (1997a) do not apply to heavy-tailed distributions, since all are based on condition (3.1.4) and B being a NBU distribution function.

3.3 Some useful identities in terms of renewal processes

For $n \geq 0$, define $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ and $N(x) = \sup\{n \geq 0 : S_n \leq x\}$, then $\{N(x), x \geq 0\}$ is a renewal process associated with the underlying random sequence $\{X_i, i \geq 1\}$. For $x \geq 0$

$$\Pr\{N(x) \geq n\} = \Pr\{S_n \leq x\} = F^{(n)}(x) ,$$

and

$$0 \leq S_{N(x)} \leq x < S_{N(x)+1} .$$

The tail probability $\psi^*(x)$ of the compound geometric distribution has a connection to the renewal process $\{N(x), x \geq 0\}$, namely, for $x \geq 0$,

$$\psi^*(x) = \sum_{n=1}^{\infty} (1-\phi)\, \phi^n\, \overline{F}^{(n)}(x) = E\Big[\phi^{N(x)+1}\Big] . \tag{3.3.1}$$

This is a simple and useful representation of the tail of the compound geometric distribution in terms of the renewal process $\{N(x), x \geq 0\}$. Its proof follows simply from summation by parts [for example, see Kalashnikov (1994, 1996)].

Since we can always choose a $t > 0$ such that it is a continuous point of F and $F(t) > 0$, without loss of generality we assume in what follows that these two conditions hold for this given $t > 0$.

Define F_t to be the conditional distribution function of X, given $X \leq t$. That is

$$F_t(x) = \begin{cases} F(x)/F(t), & 0 \leq x < t \\ 1, & x \geq t \end{cases} \tag{3.3.2}$$

There exists a lifetime random variable, say X_t, whose distribution is F_t and survival function

$$\overline{F}_t(x) = 1 - F_t(x) = \begin{cases} \frac{\overline{F}(x)-\overline{F}(t)}{F(t)}, & 0 \leq x < t \\ 0, & x \geq t \end{cases}$$

Thus, condition (3.1.14) is expressed equivalently as

$$\begin{aligned}\frac{1}{\phi F(t)} &= \int_0^t e^{\kappa_t y} \frac{dF(y)}{F(t)} = \int_0^t e^{\kappa_t y} dF_t(y) \\ &= \int_0^\infty e^{\kappa_t y} dF_t(y) = E\left[e^{\kappa_t X_t}\right] .\end{aligned} \tag{3.3.3}$$

Define $X_t^* = \min\{X,\ t\}$, then the distribution function G_t of X_t^* is

$$G_t(x) = \begin{cases} F(x), & 0 \le x < t \\ 1, & x \ge t \end{cases} \tag{3.3.4}$$

and the survival function

$$\overline{G}_t(x) = 1 - G_t(x) = \begin{cases} \overline{F}(x), & 0 \le x < t \\ 0, & x \ge t \end{cases}$$

Thus, condition (3.1.13) is expressed equivalently as

$$\frac{1}{\phi} = \int_0^t e^{\rho_t y} dF(y) + e^{\rho_t t}\overline{F}(t) = \int_0^\infty e^{\rho_t y} dG_t(y) = E\left[e^{\rho_t X_t^*}\right]. \tag{3.3.5}$$

Suppose that $\{X_i^t, i \ge 1\}$ (respectively, $\{X_i^{*t}, i \ge 1\}$) is a sequence of *i.i.d.* non-negative random variables with common distribution function F_t (G_t), $N_t(x) = \sup\{n \ge 0 : X_1^t + \cdots + X_n^t \le x\}$ (resp. $N_t^*(x) = \sup\{n \ge 0 : X_1^{*t} + \cdots + X_n^{*t} \le x\}$) is the renewal process associated with the underlying random sequence $\{X_i^t, i \ge 1\}$ (resp. $\{X_i^{*t}, i \ge 1\}$), thus we have representations of $\psi^*(x)$ in terms of the renewal process $\{N_t(x), x \ge 0\}$ and $\{N_t^*(x), x \ge 0\}$, that will be used in next section.

Property 3.3 For $0 \le x \le t$

$$\psi^*(x) = \frac{(1-\phi)E\left\{[\phi F(t)]^{N_t(x)+1}\right\} + \phi\overline{F}(t)}{1 - \phi + \phi\overline{F}(t)}, \tag{3.3.6}$$

and

$$\psi^*(x) = E\left[\phi^{N_t^*(x)+1}\right] . \tag{3.3.7}$$

Proof. Since, for $0 \le x \le t$, $F_t(x) = F(x)/F(t)$, this implies that for $0 \le x \le t$

$$\begin{aligned} F_t^{(2)}(x) &= \int_0^x F_t(x-y)\, dF_t(y) = \int_0^x F(x-y)\, dF(y) \,/\, [F(t)]^2 , \\ &= F^{(2)}(x) \,/\, [F(t)]^2 . \end{aligned}$$

By induction, we get that for $0 \le x \le t$ and $n \ge 1$

$$F_t^{(n)}(x) = F^{(n)}(x) \,/\, [F(t)]^n \;, \tag{3.3.8}$$

hence, for $0 \le x \le t$

$$\overline{F}_t^{(n)}(x) = 1 - F_t^{(n)}(x) = 1 - \frac{F^{(n)}(x)}{[F(t)]^n} \,. \tag{3.3.9}$$

Replacing ϕ, $F^{(n)}(x)$ and $N(x)$ in (3.3.1) by $\phi\, F(t)$, $F_t^{(n)}(x)$ and $N_t(x)$, respectively, we get for $0 \le x \le t$

$$\begin{aligned}
E\left\{[\phi F(t)]^{N_t(x)+1}\right\} &= \sum_{n=1}^{\infty} [1-\phi F(t)]\,[\phi F(t)]^n\, \overline{F}_t^{(n)}(x)\,, \\
&= \sum_{n=1}^{\infty} [1-\phi F(t)]\,[\phi F(t)]^n \left\{1 - \frac{F^{(n)}(x)}{[F(t)]^n}\right\}\,, \\
&= \phi F(t) - \sum_{n=1}^{\infty} [1-\phi F(t)]\,\phi^n\, F^{(n)}(x)\,, \\
&= \phi F(t) - \sum_{n=1}^{\infty} [1-\phi F(t)]\,\phi^n \left[1 - \overline{F}^{(n)}(x)\right]\,, \\
&= \phi F(t) - \frac{\phi\,[1-\phi F(t)]}{1-\phi} + [1-\phi F(t)] \sum_{n=1}^{\infty} \phi^n\, \overline{F}^{(n)}(x)\,, \\
&= \frac{-\phi\,\overline{F}(t)}{1-\phi} + \frac{[1-\phi F(t)]}{1-\phi}\,\psi^*(x)\,.
\end{aligned}$$

this implies (3.3.6).

Similarly, since for $0 \le x \le t$, $G_t(x) = F(x)$, this implies that for $0 \le x \le t$, $G_t^{(n)}(x) = F^{(n)}(x)$ and $\overline{G}_t^{(n)}(x) = \overline{F}^{(n)}(x)$. Thus, by (3.3.1), we get for $0 \le x \le t$,

$$\begin{aligned}
E\left[\phi^{N_t^*(x)+1}\right] &= \sum_{n=1}^{\infty} (1-\phi)\,\phi^n\, \overline{G}_t^{(n)}(x)\,, \\
&= \sum_{n=1}^{\infty} (1-\phi)\,\phi^n\, \overline{F}^{(n)}(x) = \psi^*(x)\,,
\end{aligned}$$

i.e. (3.3.7) holds. □

Suppose that X has the same distribution function F as $\{X_i, i \ge 1\}$. Define $T(x) = \inf\{n : S_n > x\}$, then $T(x) = N(x)+1$, and (1.14) of Lin (1996), a generalized Wald's identity, can be also expressed in terms of $N(x)$ as the following property.

Property 3.4 If a nonnegative function g on $[0,\infty)$ satisfies

$$E[g(X)] = \int_0^{\infty} g(x)\, dF(x) = \frac{1}{\phi}\,, \tag{3.3.10}$$

then, for any $x \geq 0$

$$E\left[\phi^{N(x)+1} \prod_{i=1}^{N(x)+1} g(X_i)\right] = 1 \,. \tag{3.3.11}$$

3.4 Upper and lower bounds derived from the above identities

The following theorem has been derived in Theorem 2.9 by noting that $E\left[\phi^{N(x)+1}\right] = \overline{G}_\phi(x)$. Here, we give an alternative proof by the generalized Wald's identity, which is of independent interest.

Theorem 3.1 If there exists an NBU distribution function B satisfying (3.1.4), then for any $x \geq 0$

$$E\left[\phi^{N(x)+1}\right] \;\geq\; K_1(x) \,. \tag{3.4.1}$$

If there exists an NWU distribution function B such that (3.1.4) holds, then for any $x \geq 0$,

$$E\left[\phi^{N(x)+1}\right] \;\leq\; K_2(x) \,, \tag{3.4.2}$$

where

$$[K_1(x)]^{-1} = \sup_{0 \leq h \leq x,\, \overline{F}(h)>0} K(h), \qquad [K_2(x)]^{-1} = \inf_{0 \leq h \leq x,\, \overline{F}(h)>0} K(h) \,, \tag{3.4.3}$$

and

$$K(h) = \frac{[\overline{B}(x-h)]^{-1} \int_h^\infty [\overline{B}(y)]^{-1} \, dF(y)}{\overline{F}(h)} \,. \tag{3.4.4}$$

Proof. By (3.1.4), Property 3.4 and the definition of NBU, we get that

$$\begin{aligned}
1 &= E\left\{\phi^{N(x)+1} \prod_{i=1}^{N(x)+1} [\overline{B}(X_i)]^{-1}\right\} \,, \\
&= E\left\{\phi^{N(x)+1} \, [\overline{B}(X_{N(x)+1})]^{-1} \prod_{i=1}^{N(x)} [\overline{B}(X_i)]^{-1}\right\} \,, \\
&\leq E\left\{\phi^{N(x)+1} \, [\overline{B}(X_{N(x)+1})]^{-1} \, [\overline{B}(X_1 + \cdots + X_{N(x)})]^{-1}\right\} \,, && (3.4.5) \\
&= E\left\{\phi^{N(x)+1} \, [\overline{B}(X_{N(x)+1})]^{-1} \, [\overline{B}(S_{N(x)})]^{-1}\right\} \,, \\
&= E\left\{\phi^{N(x)+1} \, E\left\{[\overline{B}(X_{N(x)+1})]^{-1} \mid N(x), S_{N(x)}\right\} [\overline{B}(S_{N(x)})]^{-1}\right\} \,. && (3.4.6)
\end{aligned}$$

By the renewal property, we know that

$$E\left\{[\overline{B}(X_{N(x)+1})]^{-1} \mid N(x),\ S_{N(x)}\right\} = E\left\{[\overline{B}(X_1)]^{-1} \mid X_1 > x - S_{N(x)}\right\}, = \frac{\int_{x-S_{N(x)}}^{\infty} [\overline{B}(y)]^{-1}\, dF(y)}{\overline{F}(x - S_{N(x)})}. \tag{3.4.7}$$

Now, since $0 \le x - S_{N(x)} \le x$, we get that

$$[\overline{B}(S_{N(x)})]^{-1} \times E\left\{[\overline{B}(X_{N(x)+1})]^{-1} \mid N(x),\ S_{N(x)}\right\} = \frac{\{\overline{B}[x-(x-S_{N(x)})]\}^{-1} \int_{x-S_{N(x)}}^{\infty} [\overline{B}(y)]^{-1}\, dF(y)}{\overline{F}(x - S_{N(x)})}, = K(x - S_{N(x)}) \le [K_1(x)]^{-1}. \tag{3.4.8}$$

Substituting in (3.4.6), implies that

$$1 \le E\left[\phi^{N(x)+1}\right][K_1(x)]^{-1},$$

i.e. (3.4.1) holds.

Similarly, we can get (3.4.2) in the case of NWU by reversing inequalities (3.4.5) and (3.4.8) and replacing $K_1(x)$ by $K_2(x)$. □

Combining Theorem 3.1, (3.3.1), (3.1.9) and (3.1.10), we get directly the following corollary.

Corollary 3.1 If (3.1.2) holds and (3.1.4) is satisfied by an NBU distribution function B, then

$$\psi(x) \ge \frac{1-p_0}{\phi} K_1(x), \tag{3.4.9}$$

and if (3.1.1) holds and (3.1.4) is satisfied by an NWU distribution function B, then

$$\psi(x) \le \frac{1-p_0}{\phi} K_2(x). \tag{3.4.10}$$

Remark 3.2 The lower and upper bounds in Corollary 3.1 are uniformly tighter than the lower bounds in (3.1.6) and (3.1.8), and the upper bounds in (3.1.5) and (3.1.7) respectively, which are the main results of Lin (1996) and Willmot (1994, 1997a). These follow easily from the fact that if B is NBU, then for any $x \ge 0$

$$K_1(x) \ge \alpha_1(x)\,\overline{B}(x) \quad \text{and} \quad K_1(x) \ge \Delta_1(x), \tag{3.4.11}$$

and if B is NWU, then for any $x \ge 0$

$$K_2(x) \le \alpha_2(x)\,\overline{B}(x) \quad \text{and} \quad K_2(x) \le \Delta_2(x). \tag{3.4.12}$$

Theorem 3.2 Given $t > 0$, if there exists a constant κ_t such that

$$\int_0^t e^{\kappa_t y}\, dF(y) = \frac{1}{\phi}\,, \tag{3.4.13}$$

then for any $0 \le x \le t$

$$\frac{(1-\phi)\, c_1(x,t) e^{-\kappa_t x} + \phi\, \overline{F}(t)}{1-\phi\ +\ \phi\, \overline{F}(t)} \le \psi^*(x) \le \frac{(1-\phi)\, c_2(x,t) e^{-\kappa_t x} + \phi\, \overline{F}(t)}{1-\phi\ +\ \phi \overline{F}(t)}. \tag{3.4.14}$$

In particular, for any $t > 0$

$$\frac{(1-\phi)\, c_1(t) e^{-\kappa_t t}\ +\ \phi\, \overline{F}(t)}{1-\phi\ +\ \phi \overline{F}(t)} \le \psi^*(t) \le \frac{(1-\phi)\, c_2(t) e^{-\kappa_t t}\ +\ \phi\, \overline{F}(t)}{1-\phi\ +\ \phi \overline{F}(t)}\,, \tag{3.4.15}$$

where

$$[c_1(x,t)]^{-1} = \sup_{0\le h\le x,\ \overline{F}(h)\ne \overline{F}(t)} c(h,t), \quad [c_2(x,t)]^{-1} = \inf_{0\le h\le x,\ \overline{F}(h)\ne \overline{F}(t)} c(h,t),$$

$$c(h,t) = \frac{\int_h^t e^{\kappa_t y}\, dF(y)}{e^{\kappa_t h}\,[\,\overline{F}(h) - \overline{F}(t)\,]},$$

and $c_1(t) = c_1(t,t)$, $c_2(t) = c_2(t,t)$.

Proof. By (3.3.3), we know that (3.4.13) is equivalent to

$$E[e^{\kappa_t X_t}] = \int_0^\infty e^{\kappa_t y}\, dF_t(y) = \frac{1}{\phi F(t)}\,.$$

Thus, taking $\overline{B}(x) = e^{-\kappa_t x}$, by (3.4.1) in Theorem 3.1, we get for $x \ge 0$,

$$E\left\{[\phi F(t)]^{N_t(x)+1}\right\} \ \ge\ \left\{ \sup_{0\le h\le x,\ \overline{F}_t(h)>0} \frac{e^{\kappa_t x}\int_h^\infty e^{\kappa_t y}\, dF_t(y)}{e^{\kappa_t h}\,\overline{F}_t(h)} \right\}^{-1}, \tag{3.4.16}$$

$$\begin{aligned} &= e^{-\kappa_t x} \left\{ \sup_{0\le h\le x,\ \overline{F}(h)\ne\overline{F}(t)} \frac{\int_h^t e^{\kappa_t y}\, dF(y)}{e^{\kappa_t h}\,[\,\overline{F}(h) - \overline{F}(t)\,]} \right\}^{-1}, \\ &= c_1(x,\, t)\, e^{-\kappa_t x}\,. \end{aligned} \tag{3.4.17}$$

Thus, by (3.3.6) and (3.4.17), we get the lower bound in (3.4.14).

The proof of the upper bound in (3.4.14) is similar. Setting $x = t$ in (3.4.14) gives (3.4.15). □

Since

$$c(h,t) \ \ge\ \frac{e^{\kappa_t h}\ \int_h^t dF(y)}{e^{\kappa_t h}\,[\,\overline{F}(h) - \overline{F}(t)\,]} \ =\ 1$$

and

$$c(h,t) \;\leq\; \frac{e^{\kappa_t t} \int_h^t dF(y)}{e^{\kappa_t h}\,[\,\overline{F}(h) - \overline{F}(t)\,]} \;=\; e^{\kappa_t (t-h)},$$

thus, we get for any $0 \leq x \leq t$,

$$e^{\kappa_t t} \;\geq\; \sup_{0 \leq h \leq x} c(h,t) \;\geq\; [c_1(x,t)]^{-1} \;\geq\; [c_2(x,t)]^{-1} \;\geq\; 1\,,$$

i.e.

$$e^{-\kappa_t t} \;\leq\; c_1(x,t) \;\leq\; c_2(x,t) \;\leq\; 1,$$

this implies by taking $x = t$ that

$$e^{-\kappa_t t} \;\leq\; c_1(t) \;\leq\; c_2(t) \;\leq\; 1.$$

Thus we get the following simplified bounds by Theorem 3.2.

Corollary 3.2 Under the conditions and notation of Theorem 3.2, for any $0 \leq x \leq t$

$$\frac{(1-\phi)\, e^{-\kappa_t (x+t)} + \phi\, \overline{F}(t)}{1 - \phi \;+\; \phi\, \overline{F}(t)} \;\leq\; \psi^*(x) \;\leq\; \frac{(1-\phi)\, e^{-\kappa_t x} + \phi\, \overline{F}(t)}{1 - \phi \;+\; \phi\, \overline{F}(t)}. \qquad (3.4.18)$$

In particular, for any $t > 0$

$$\frac{(1-\phi)\, e^{-2\kappa_t t} \;+\; \phi\, \overline{F}(t)}{1 - \phi \;+\; \phi\, \overline{F}(t)} \;\leq\; \psi^*(t) \;\leq\; \frac{(1-\phi)\, e^{-\kappa_t t} \;+\; \phi\, \overline{F}(t)}{1 - \phi \;+\; \phi\, \overline{F}(t)}\,, \qquad (3.4.19)$$

Furthermore, we show that the two-sided bounds in Corollary 3.2 are asymptotically exact for small t.

Let

$$A(t) = \frac{(1-\phi)\, e^{-2\kappa_t t} + \phi\, \overline{F}(t)}{1 - \phi \;+\; \phi\, \overline{F}(t)} \quad \text{and} \quad B(t) = \frac{(1-\phi)\, e^{-\kappa_t t} \;+\; \phi\, \overline{F}(t)}{1 - \phi \;+\; \phi\, \overline{F}(t)}, \qquad (3.4.20)$$

respectively, be the lower and upper bounds in (3.4.19).

Property 3.5 If $F(0) = 0$, then

$$\lim_{t \to 0} A(t) \;=\; \lim_{t \to 0} B(t) \;=\; \psi^*(0) = \phi. \qquad (3.4.21)$$

Proof. By (3.4.13),

$$\frac{1}{\phi} = \int_0^t e^{\kappa_t y}\, dF(y) \le e^{\kappa_t t} F(t),$$

which implies that

$$0 \le e^{-\kappa_t t} \le \phi F(t),$$

thus $e^{-\kappa_t t} \to 0$ as $t \to 0$. Hence Property 3.5 follows from (3.4.19). □

Theorem 3.3 Given $t > 0$, if there exists a constant ρ_t such that

$$\int_0^t e^{\rho_t y}\, dF(y) + e^{\rho_t t}\, \overline{F}(t) = \frac{1}{\phi}, \tag{3.4.22}$$

then for any $0 \le x \le t$

$$\beta_1(x,t)\, e^{-\rho_t x} \le \psi^*(x) \le \beta_2(x,t)\, e^{-\rho_t x}. \tag{3.4.23}$$

In particular, for any $t > 0$

$$\beta_1(t)\, e^{-\rho_t t} \le \psi^*(t) \le \beta_2(t)\, e^{-\rho_t t}, \tag{3.4.24}$$

where

$$[\beta_1(x,t)]^{-1} = \sup_{0 \le h \le x,\ \overline{F}(h) > 0} \beta(h,t), \qquad [\beta_2(x,t)]^{-1} = \inf_{0 \le h \le x,\ \overline{F}(h) > 0} \beta(h,t),$$

$$\beta(h,t) = \frac{\int_h^t e^{\rho_t y}\, dF(y) + e^{\rho_t t}\, \overline{F}(t)}{e^{\rho_t h}\, \overline{F}(h)},$$

and $\beta_1(t) = \beta_1(t,t)$, $\beta_2(t) = \beta_2(t,t)$.

Proof. By (3.3.5), we know that (3.4.22) is equivalent to

$$\int_0^\infty e^{\rho_t y}\, dG_t(y) = E[e^{\rho_t X_t^*}] = \frac{1}{\phi}.$$

Thus, taking $\overline{B}(x) = e^{-\rho_t x}$, by (3.4.1) in Theorem 3.1, we get that for $x \ge 0$,

$$E\left[\phi^{N_t^*(x)+1}\right] \ge \left\{ \sup_{0 \le h \le x,\ \overline{G}_t(h) > 0} \frac{e^{\rho_t x} \int_h^\infty e^{\rho_t y}\, dG_t(y)}{e^{\rho_t h}\, \overline{G}_t(h)} \right\}^{-1}, \tag{3.4.25}$$

$$= e^{-\rho_t x} \left\{ \sup_{0 \le h \le x,\ \overline{F}(h) > 0} \frac{\int_h^t e^{\rho_t y}\, dF(y) + e^{\rho_t t}\, \overline{F}(t)}{e^{\rho_t h}\, \overline{F}(h)} \right\}^{-1},$$

$$= \beta_1(x,t)\, e^{-\rho_t x}. \tag{3.4.26}$$

Thus, the lower bound in (3.4.23) follows from (3.3.7) and (3.4.26).

The proof of the upper bound in (3.4.23) is similar. Setting $x = t$ in (3.4.23) gives (3.4.24). □

Since

$$\begin{aligned}\beta(h,t) &\geq \frac{e^{\rho_t h} \int_h^t dF(y) + e^{\rho_t t}\overline{F}(t)}{e^{\rho_t h}\overline{F}(h)} \\ &= \frac{e^{\rho_t h}[\overline{F}(h) - \overline{F}(t)] + e^{\rho_t t}\overline{F}(t)}{e^{\rho_t h}\overline{F}(h)} \geq 1\end{aligned}$$

and

$$\begin{aligned}\beta(h,t) &\leq \frac{e^{\rho_t t} \int_h^t dF(y) + e^{\rho_t t}\overline{F}(t)}{e^{\rho_t h}\overline{F}(h)} \\ &= \frac{e^{\rho_t t}[\overline{F}(h) - \overline{F}(t)] + e^{\rho_t t}\overline{F}(t)}{e^{\rho_t h}\overline{F}(h)} \\ &= e^{\rho_t (t-h)},\end{aligned}$$

thus,

$$e^{\rho_t t} \geq \sup_{0 \leq h \leq x} \beta(h,t) \geq [\beta_1(x,t)]^{-1} \geq [\beta_2(x,t)]^{-1} \geq 1,$$

i.e.

$$e^{-\rho_t t} \leq \beta_1(x,t) \leq \beta_2(x,t) \leq 1,$$

which implies by taking $x = t$ that

$$e^{-\rho_t t} \leq \beta_1(t) \leq \beta_2(t) \leq 1.$$

Thus we get the following simplified bounds by Theorem 3.3.

Corollary 3.3 Under the conditions and notation of Theorem 3.3, for any $0 \leq x \leq t$

$$e^{-\rho_t (x+t)} \leq \psi^*(x) \leq e^{-\rho_t x}. \tag{3.4.27}$$

In particular, for any $t > 0$

$$e^{-2\rho_t t} \leq \psi^*(t) \leq e^{-\rho_t t}. \tag{3.4.28}$$

Combining Corollary 3.2 and 3.3 with expressions (3.1.9) and (3.1.10) directly gives the following two corollaries.

Corollary 3.4 If (3.1.2) holds, then for any $0 \le x \le t$

$$\psi(x) \;\ge\; \frac{(1-p_0)\left[(1-\phi)\,e^{-\kappa_t\,(x+t)} \;+\; \phi\,\overline{F}(t)\right]}{\phi\,\left[1-\phi \;+\; \phi\,\overline{F}(t)\right]} \tag{3.4.29}$$

and, in particular, for any $t > 0$

$$\psi(t) \;\ge\; \frac{(1-p_0)\left[(1-\phi)\,e^{-2\kappa_t\,t} \;+\; \phi\,\overline{F}(t)\right]}{\phi\,\left[1-\phi \;+\; \phi\,\overline{F}(t)\right]}\,. \tag{3.4.30}$$

Alternatively, if (3.1.1) holds, then for any $0 \le x \le t$

$$\psi(x) \;\le\; \frac{(1-p_0)\left[(1-\phi)\,e^{-\kappa_t\,x} \;+\; \phi\,\overline{F}(t)\right]}{\phi\,\left[1-\phi \;+\; \phi\,\overline{F}(t)\right]} \tag{3.4.31}$$

and, in particular, for any $t > 0$

$$\psi(t) \;\le\; \frac{(1-p_0)\left[(1-\phi)\,e^{-\kappa_t\,t} \;+\; \phi\,\overline{F}(t)\right]}{\phi\,\left[1-\phi \;+\; \phi\,\overline{F}(t)\right]}\,. \tag{3.4.32}$$

Corollary 3.5 If (3.1.2) holds, then for any $0 \le x \le t$

$$\psi(x) \;\ge\; \frac{1-p_0}{\phi}\;e^{-\rho_t\,(x+t)}\,, \tag{3.4.33}$$

and, in particular, for any $t > 0$

$$\psi(t) \;\ge\; \frac{1-p_0}{\phi}\;e^{-2\rho_t\,t}\,. \tag{3.4.34}$$

Alternatively, if (3.1.1) holds, then for any $0 \le x \le t$

$$\psi(x) \;\le\; \frac{1-p_0}{\phi}\;e^{-\rho_t\,x}\,, \tag{3.4.35}$$

and, in particular, for any $t > 0$

$$\psi(t) \;\le\; \frac{1-p_0}{\phi}\;e^{-\rho_t\,t}\,. \tag{3.4.36}$$

Remark 3.3 1. It is easily shown that if Cramér-Lundberg's condition (3.1.12) holds, then $\kappa_t \searrow \kappa$ and $\rho_t \searrow \kappa$ as $t \to \infty$. Thus, Lundberg's inequality (1.1.9) can be obtained and improved by letting $t \to \infty$ in the upper bounds in (3.4.14) and (3.4.23).

2. The upper bounds in (3.4.23) and (3.4.27) are uniformly sharper on $[0, t]$ than that in Lundberg's inequality (1.1.9), since $0 \le \beta_2(x,t) \le 1$, and (3.4.22) with (3.1.12) means that

$$\int_0^\infty e^{\kappa y}\, dF(y) = \int_0^\infty e^{\rho_t \min\{y,t\}}\, dF(y) = 1/\phi.$$

This implies that $\rho_t \ge \kappa$.

3. The upper bounds of (1.9) of Broeckx *et al.* (1986), (2.1) of Dickson (1994) and (16) of Taylor (1976) are all special cases of the upper bounds in (3.4.14) and (3.4.23), and improved upon by the new upper bounds. In addition, corresponding lower bounds are given in Theorems 3.2 and 3.3. Cai and Garrido (1997) gives a special case of Theorem 3.2, for the ruin probability in the compound Poisson risk model, derived by the renewal recursive method.

4. Putting (3.4.22) and (3.4.13) together, we have

$$\int_0^t e^{\kappa_t y}\, dF(y) = \int_0^t e^{\rho_t y}\, dF(y) + e^{\rho_t t}\overline{F}(t) = 1/\phi\,,$$

which implies that $\kappa_t \ge \rho_t$. This suggests that the upper bounds in (3.4.14) and (3.4.15) may be tighter than those in (3.4.23) and (3.4.24). A numerical example of Dickson (1994) shows that in that special case, the upper bounds in (3.4.14) and (3.4.15) are superior to those in (3.4.23) and (3.4.24) for large values of x, but inferior for small values of x. From the same example, we can see the lower bounds in (3.4.14) and (3.4.15) are also superior to those in (3.4.23) and (3.4.24) for large values of x, but inferior for small values of x.

5. The bounds in Theorems 3.2 and 3.3, Corollaries 3.4 and 3.5 can apply to any life distribution F with positive (possibly infinite) mean. Especially, as shown by numerical examples in Section 6.5, the bounds in Theorem 3.2 and Corollary 3.4 are very effective for heavy-tailed distributions. Even when the Cramér-Lundberg condition holds, in some cases, they are also superior to Lundberg's inequality.

3.5 Simplified bounds derived from stochastic orderings

In this section, we derive simplified bounds for $\psi(x)$ by studying the function K in (3.4.4) and the use of stochastic ordering. Indeed, stochastic ordering is a very useful tool for deriving probability inequalities in applied probability models [see, e.g. Khalil and Falin (1994), Stoyan (1983) and Szekli (1995)]. This approach helps unify the theory, is simpler than integrations by parts, used by Lin (1996) and Willmot and Lin (1997a) as it does not require continuity conditions, and yields new, sharper bounds for $\psi(x)$.

In this section, we denote by F_h the residual life distribution function of F, *i.e.*

$$F_h(x) = \Pr\{X \leq x + h | X > h\} = \frac{F(x+h) - F(h)}{\overline{F}(h)}, \tag{3.5.1}$$

and δ_d the degenerate distribution function of the probability measure concentrated at d, *i.e.*

$$\delta_d(x) = \begin{cases} 0, & x < d \\ 1, & x \geq d \end{cases}$$

Thus, given $h \geq 0$, note that $F(0) = 0$ and we know for any $y \in [0, \infty)$,

$$\begin{aligned} F(y+h) &= \overline{F}(h) \frac{F(y+h) - F(h)}{\overline{F}(h)} + F(h)\, \delta_{-h}(y)\,, \\ &= \overline{F}(h)\, F_h(y) + F(h)\, \delta_{-h}(y)\,. \end{aligned} \tag{3.5.2}$$

Hence, by (3.4.4), (3.5.2) and $\int_0^\infty [\overline{B}(y+h)]^{-1} F(h)\, d\delta_{-h}(y) = 0$ for $h \geq 0$, we get for any $0 \leq h \leq x$,

$$\begin{aligned} K(h) &= \frac{[\overline{B}(x-h)]^{-1} \int_h^\infty [\overline{B}(y)]^{-1}\, dF(y)}{\overline{F}(h)}\,, \\ &= \frac{[\overline{B}(x-h)]^{-1} \int_0^\infty [\overline{B}(y+h)]^{-1}\, dF(y+h)}{\overline{F}(h)}\,, \qquad (3.5.3) \\ &= [\overline{B}(x-h)]^{-1} \int_0^\infty [\overline{B}(y+h)]^{-1}\, dF_h(y)\,. \qquad (3.5.4) \end{aligned}$$

Theorem 3.4 Suppose that condition (3.1.1) holds and (3.1.4) is satisfied by an NWU distribution function B.

1. If Q_x is an increasing function satisfying, for any $y \geq 0$,

$$Q_x(y) \leq \inf_{0 \leq h \leq x} \left[\overline{B}(x-h)\, \overline{B}(y+h)\right]^{-1}, \tag{3.5.5}$$

and H_x is a distribution function satisfying $H_x(0) = 0$ and, for any $y \geq 0$,

$$\overline{H}_x(y) \leq \inf_{0 \leq h \leq x} \overline{F}_h(y)\,, \tag{3.5.6}$$

then for any $x \geq 0$

$$\psi(x) \leq \frac{1 - p_0}{\phi} \left[\int_0^\infty Q_x(y)\, dH_x(y)\right]^{-1}. \tag{3.5.7}$$

2. If Q_x is an increasing convex function satisfying (3.5.5) and H_x is a distribution function satisfying $H_x(0) = 0$ and, for any $y \geq 0$,

$$\int_y^\infty \overline{H}_x(u)\,du \;\leq\; \inf_{0\leq h\leq x} \int_y^\infty \overline{F}_h(u)\,du\,, \tag{3.5.8}$$

then, for any $x \geq 0$, (3.5.7) holds.

Proof. **1.** (3.5.6) implies that for any $y \geq 0$ and $0 \leq h \leq x$, $\overline{H}_x(y) \leq \overline{F}_h(y)$, *i.e.* $H_x <_{st} F_h$. Since Q_x is increasing, by the equivalent condition of "$<_{st}$" [see, for example, Theorem B of Szekli (1995), page 6] and (3.5.4), we get for any $0 \leq h \leq x$,

$$K(h) \;\geq\; \int_0^\infty Q_x(y)\,dF_h(y) \;\geq\; \int_0^\infty Q_x(y)\,dH_x(y)\,. \tag{3.5.9}$$

This, together with (3.4.3), implies that in (3.4.10), $[K_2(x)]^{-1} \geq \int_0^\infty Q_x(y)\,dH_x(y)$. Thus (3.5.7) follows from (3.4.10).

2. (3.5.8) implies that for any $y \geq 0$ and $0 \leq h \leq x$, $\int_y^\infty \overline{H}_x(u)\,du \leq \int_y^\infty \overline{F}_h(u)\,du$, *i.e.* $H_x <_{icx} F_h$. Thus, since Q_x is an increasing convex function, by the equivalent condition of "$<_{icx}$" [see, for example, Theorem 3.A.1. of Shaked and Shanthikumar (1994)] and (3.5.4), we get for any $0 \leq h \leq x$,

$$K(h) \;\geq\; \int_0^\infty Q_x(y)\,dF_h(y) \geq \int_0^\infty Q_x(y)\,dH_x(y)\,. \tag{3.5.10}$$

This implies now that $[K_2(x)]^{-1} \geq \int_0^\infty Q_x(y)\,dH_x(y)$, and hence (3.5.7) still follows from (3.4.10) when (3.5.8) holds. □

Similarly, it is easy to prove the following result.

Theorem 3.5 Suppose that condition (3.1.2) holds and (3.1.4) is satisfied by an NBU distribution function B.

1. If Q_x is an increasing function satisfying, for any $y \geq 0$,

$$Q_x(y) \geq \sup_{0\leq h\leq x} \left[\overline{B}(x-h)\,\overline{B}(y+h)\right]^{-1}\,, \tag{3.5.11}$$

and H_x is a distribution function satisfying $H_x(0) = 0$ and, for any $y \geq 0$,

$$\overline{H}_x(y) \;\geq\; \sup_{0\leq h\leq x} \overline{F}_h(y)\,, \tag{3.5.12}$$

then, for any $x \geq 0$,

$$\psi(x) \;\geq\; \frac{1-p_0}{\phi}\left[\int_0^\infty Q_x(y)\,dH_x(y)\right]^{-1}\,. \tag{3.5.13}$$

2. If Q_x is an increasing convex function satisfying (3.5.11) and H_x is a distribution function satisfying $H_x(0) = 0$ and, for any $y \geq 0$,

$$\int_y^\infty \overline{H}_x(u)\, du \;\geq\; \sup_{0\leq h\leq x} \int_y^\infty \overline{F}_h(u)\, du\,, \tag{3.5.14}$$

then, for any $x \geq 0$, (3.5.13) holds.

Corollary 3.6 Under the conditions of Theorem 3.4-1,

1. If F has an Increasing Failure Rate (IFR), then for $x \geq 0$,

$$\psi(x) \;\leq\; \frac{1-p_0}{\phi}\left[\int_x^\infty Q_x(y-x)\, dF(y)\right]^{-1} \overline{F}(x)\,. \tag{3.5.15}$$

2. If F is NWU, then for $x \geq 0$,

$$\psi(x) \;\leq\; \frac{1-p_0}{\phi}\left[\int_0^\infty Q_x(y)\, dF(y)\right]^{-1}. \tag{3.5.16}$$

3. If F is Used Better than Aged (UBA) with $L_F > 0$ [*i.e.* for any $h \geq 0$ and $y \geq 0$, $\overline{F}_h(y) \;\geq\; e^{-L_F y}$, see, Alzaid (1994)], then for $x \geq 0$,

$$\psi(x) \;\leq\; \frac{1-p_0}{\phi L_F}\left[\int_0^\infty e^{-L_F y}\, Q_x(y)\, dy\right]^{-1}, \tag{3.5.17}$$

where, $L_F = \lim_{x\to\infty} r_F(x)$ and $r_F(x)$ is the failure rate function of F.

Proof. 1. Since F is IFR if and only if $\overline{F}_h$ is decreasing in $h \geq 0$, then $\inf_{0\leq h\leq x} \overline{F}_h(y) = \overline{F}_x(y)$. Thus, setting $\overline{H}_x(y) = \overline{F}_x(y)$ in Theorem 3.4-1. gives, for any $x \geq 0$,

$$\begin{aligned}
\psi(x) \;&\leq\; \frac{1-p_0}{\phi}\left[\int_0^\infty Q_x(y)\, dF_x(y)\right]^{-1},\\
&=\; \frac{1-p_0}{\phi}\left[\int_0^\infty Q_x(y)\, dF(y+x)\right]^{-1} \overline{F}(x)\,,\\
&=\; \frac{1-p_0}{\phi}\left[\int_x^\infty Q_x(u-x)\, dF(u)\right]^{-1} \overline{F}(x)\,,
\end{aligned}$$

where the first equality follows from (3.5.2), similarly to the argument for (3.5.4).

2. Since F is NWU if and only if for any $h \geq 0$ and $y \geq 0$,

$$\overline{F}(y) \;\leq\; \overline{F}_h(y)\,,$$

this implies that $\overline{F}(y) \le \inf_{0\le h\le x} \overline{F}_h(y)$. Thus, setting $\overline{H}_x(y) = \overline{F}(y)$ in Theorem 3.4-1. gives (3.5.16).

3. Since F is UBA if and only if for any $h \ge 0$ and $y \ge 0$,

$$\overline{F}_h(y) \ \ge \ e^{-L_F y} ,$$

which implies that $e^{-L_F y} \le \inf_{0\le h\le x} \overline{F}_h(y)$. Thus, setting $\overline{H}_x(y) = e^{-L_F y}$ in Theorem 3.4-1. gives (3.5.17). □

Corollary 3.7 Under the conditions of Theorem 3.4-2,

1. If F has a Decreasing Mean Residual Life (DMRL), then for $x \ge 0$,

$$\psi(x) \ \le \ \frac{1-p_0}{\phi} \left[\int_x^\infty Q_x(y-x)\, dF(y)\right]^{-1} \overline{F}(x) \, . \qquad (3.5.18)$$

2. If F is New Worse than Used in Convex ordering (NWUC), then for $x \ge 0$,

$$\psi(x) \ \le \ \frac{1-p_0}{\phi} \left[\int_0^\infty Q_x(y)\, dF(y)\right]^{-1} . \qquad (3.5.19)$$

Proof. 1. By the equivalence of these conditions [see, for example, Theorem 3.A.18 of Shaked and Shanthikumar (1994)], we know that F is DMRL if and only if, for any $0 \le h_1 \le h_2$

$$F_{h_2} \ <_{icx} \ F_{h_1} \, ,$$

which implies that, for any $0 \le h \le x$ and $y \ge 0$,

$$\int_y^\infty \overline{F}_x(u)\, du \le \int_y^\infty \overline{F}_h(u)\, du \, ,$$

and hence, $\int_y^\infty \overline{F}_x(u)\, du \ \le \ \inf_{0\le h\le x} \int_y^\infty \overline{F}_h(u)\, du$. Thus, setting $\overline{H}_x(u) = \overline{F}_x(u)$ in Theorem 3.4-2. gives (3.5.18).

2. By the definition of NWUC, F is NWUC if and only if, for any $h \ge 0$ and $y \ge 0$,

$$\int_y^\infty \overline{F}(u)\, du \le \int_y^\infty \overline{F}_h(u)\, du \, ,$$

which implies that $\int_y^\infty \overline{F}(u)\, du \ \le \ \inf_{0\le h\le x} \int_y^\infty \overline{F}_h(u)\, du$. So here, setting $\overline{H}_x(u) = \overline{F}(u)$ in Theorem 3.4-2. gives (3.5.19). □

Similarly, using the dual classes DFR, NBU, UWA, IMRL and NBUC to those in Corollaries 3.6 and 3.7 and by use of Theorem 3.5, we get the following two corollaries.

Corollary 3.8 Under the conditions of Theorem 3.5-1,

1. If F has a Decreasing Failure Rate (DFR), then for $x \geq 0$,

$$\psi(x) \;\geq\; \frac{1-p_0}{\phi}\left[\int_x^\infty Q_x(y-x)\,dF(y)\right]^{-1}\overline{F}(x)\,. \tag{3.5.20}$$

2. If F is NBU, then for $x \geq 0$,

$$\psi(x) \;\geq\; \frac{1-p_0}{\phi}\left[\int_0^\infty Q_x(y)\,dF(y)\right]^{-1}. \tag{3.5.21}$$

3. If F is Used Worse than Aged (UWA) with $L_F > 0$ [*i.e.* for any $h \geq 0$ and $y \geq 0$, $\overline{F}_h(y) \;\leq\; e^{-L_F\,y}$, see, Alzaid (1994)], then for $x \geq 0$,

$$\psi(x) \;\geq\; \frac{1-p_0}{\phi\,L_F}\left[\int_0^\infty e^{-L_F\,y}\,Q_x(y)\,dy\right]^{-1}, \tag{3.5.22}$$

where $L_F = \lim_{x\to\infty} r_F(x)$ and $r_F(x)$ is the failure rate function of F.

Corollary 3.9 Under the conditions of Theorem 3.5-2,

1. If F has an Increasing Mean Residual Life (IMRL), then for $x \geq 0$,

$$\psi(x) \;\geq\; \frac{1-p_0}{\phi}\left[\int_x^\infty Q_x(y-x)\,dF(y)\right]^{-1}\overline{F}(x)\,. \tag{3.5.23}$$

2. If F is New Better than Used in Convex ordering (NBUC), then for $x \geq 0$,

$$\psi(x) \;\geq\; \frac{1-p_0}{\phi}\left[\int_0^\infty Q_x(y)\,dF(y)\right]^{-1}. \tag{3.5.24}$$

Remark 3.4 1. It is clear that if B is NWU, then

$$[\overline{B}(x+y)]^{-1} \leq \inf_{0\leq h\leq x}\left[\overline{B}(x-h)\,\overline{B}(y+h)\right]^{-1},$$

while, if B is NBU, then

$$[\overline{B}(x+y)]^{-1} \geq \sup_{0\leq h\leq x}\left[\overline{B}(x-h)\,\overline{B}(y+h)\right]^{-1}.$$

Thus, Theorems 1 and 2 of Willmot and Lin (1997a) are obtained as special cases of Theorems 3.4-1. and 3.5-1., respectively, by setting $Q_x(y) = [\overline{B}(x+y)]^{-1}$. With stochastic orderings, the proofs do not require the condition that B and F, in $\int_x^\infty [\overline{B}(y)]^{-1}\,dF(y)$, have no common discontinuities [the proof of Corollaries 2.2 and 3.5 of Lin (1996) would also require it].

2. Theorems 3.1 and 5.1 of Willmot (1997b) are also special cases of Theorems 3.4-2. and 3.5, respectively, by setting $Q_x(y) = [\overline{B}(x+y)]^{-1}$. Note that the condition that B be twice differentiable is not necessary here which yields a simpler proof.

   Similarly, Corollaries 2.2, 2.4 and 3.5 of Lin (1996), Corollaries 1 and 3 of Willmot and Lin (1997a) and Corollaries 3.1, 3.2 , 5.1 and 5.2 of Willmot (1997b) can all be obtained as special cases of Corollaries 3.6, 3.7, 3.8 and 3.9 by setting $Q_x(y) = [\overline{B}(x+y)]^{-1}$.

3. The bounds in Theorems 3.4 and 3.5 are sharp, in the sense that if F and B are exponential distributions and N is a geometric random variable with $\Pr\{N = n\} = (1-\phi)\phi^n, n = 0, 1, 2, \cdots$, then the inequalities become equalities.

Chapter 4

Bounds for Tails of Aggregate Claim Distributions

4.1 Introduction

Let

$$\overline{G}(x) = \sum_{n=1}^{\infty} p_n \overline{F}^{(n)}(x) \,, \quad x \geq 0 \,,$$

be the tail of the compound distribution G in (1.1.2) and denote the tail of an aggregate claim distribution with $F(0) = 0$. If the probability function $\{p_n, \; n \geq 0\}$ satisfies (3.1.1) or (3.1.2), then the upper bounds for $\overline{G}(x)$ under condition (3.1.1) and the lower bounds for $\overline{G}(x)$ under condition (3.1.2) have been derived in Chapter 3.

Thus, if there exist two constants $0 < \phi_1 \leq \phi_2 < 1$ such that at the same time

$$\phi_1 \, a_n \;\leq\; a_{n+1} \;\leq\; \phi_2 \, a_n, \quad n = 0, 1, 2, \cdots, \tag{4.1.1}$$

where

$$a_n \;=\; \sum_{k=n+1}^{\infty} p_k,$$

then, we can obtain two-sided bounds for $\overline{G}(x)$. Fortunately, condition (4.1.1) can be satisfied by many interesting probability functions such as geometric, logarithmic, certain negative binomial distributions, etc.

First, we give general results about two-sided bounds for $\overline{G}(x)$ under condition (4.1.1), then focus on their applications to the tail probabilities of compound negative binomial distributions, which is an important stochastic model in insurance risk analysis and applied probability.

In this chapter, we denote the adjustment coefficient as a function of ϕ, *i.e.* adopt the following notation.

Definition 4.1 For any constant $0 < \phi < 1$, constant κ_ϕ satisfying

$$\int_0^\infty e^{\kappa_\phi y} \, dF(y) = \frac{1}{\phi} \tag{4.1.2}$$

is called an adjustment coefficient of the distribution F.

Definition 4.2 Denote

$$\delta_\phi^{-1} = \sup_{x \geq 0} \frac{\int_x^\infty e^{\kappa_\phi y} \, dF(y)}{e^{\kappa_\phi x} \overline{F}(x)} \quad \text{and} \quad \theta_\phi^{-1} = \inf_{x \geq 0} \frac{\int_x^\infty e^{\kappa_\phi y} \, dF(y)}{e^{\kappa_\phi x} \overline{F}(x)}. \tag{4.1.3}$$

By the definitions of δ_ϕ and θ_ϕ, we know that [cf. (2.3.11) and (2.3.12)],

$$0 \leq \delta_\phi \leq \phi \quad \text{and} \quad \phi \leq \theta_\phi \leq 1. \tag{4.1.4}$$

Property 4.1 If F is NBUC, then

$$\delta_\phi = \phi \tag{4.1.5}$$

and if F is NWUC, then

$$\theta_\phi = \phi. \tag{4.1.6}$$

Proof: By taking $\overline{B}(y) = e^{-\kappa_\phi y}$ in (3.5.4), we get for $h \geq 0$,

$$\frac{\int_h^\infty e^{\kappa_\phi y} \, dF(y)}{e^{\kappa_\phi h} \overline{F}(h)} = \int_0^\infty e^{\kappa_\phi y} \, dF_h(y),$$

where F_h is defined in (3.5.1).

Since $e^{\kappa_\phi y}$ is increasing and convex, by the equivalent condition of NBUC [see, Cao and Wang (1991)], we get

$$\int_0^\infty e^{\kappa_\phi y} \, dF_h(y) \;\leq\; \int_0^\infty e^{\kappa_\phi y} \, dF(y) = \frac{1}{\phi},$$

which implies that $\delta_\phi^{-1} \leq 1/\phi$, hence (4.1.5) holds by (4.1.4) . Similarly, we get (4.1.5). □

Lemma 4.1 Suppose that (4.1.1) holds. If there exist adjustment coefficients κ_{ϕ_1} and κ_{ϕ_2}, then for any $x \geq 0$,

$$\frac{(1-p_0)\,\delta_{\phi_1}}{\phi_1}\, e^{-\kappa_{\phi_1} x} \;\leq\; \overline{G}(x) \;\leq\; \frac{(1-p_0)\,\theta_{\phi_2}}{\phi_2}\, e^{-\kappa_{\phi_2} x}. \tag{4.1.7}$$

Proof. Lemma 4.1 follows from (3.1.9), (3.1.10) and Corollary 2.1 by noticing that $l(x) \leq l(\infty)$ and $u(x) \geq u(\infty)$ in (2.3.9) and (2.3.10), respectively. □

For the case where F is heavy-tailed, adjustment coefficients in (4.1.2) do not exist and Lemma 4.1 does not apply. Therefore, we define a generalized adjustment coefficient of F as follows, which was first used by Dickson (1994).

Definition 4.3 Given $t > 0$, for any constant $0 < \phi < 1$, constant $\kappa_\phi(t)$ satisfying

$$\int_0^t e^{\kappa_\phi(t)\, y}\, dF(y) = \frac{1}{\phi} \tag{4.1.8}$$

is called a generalized adjustment coefficient of the distribution F.

Definition 4.4 Denote

$$[\delta_\phi(t)]^{-1} = \sup_{0 \leq x \leq t} \frac{\int_x^t e^{\kappa_\phi(t)\, y}\, dF(y)}{e^{\kappa_\phi(t)\, x}\,[\overline{F}(x) - \overline{F}(t)]} \tag{4.1.9}$$

and

$$[\theta_\phi(t)]^{-1} = \inf_{0 \leq x \leq t} \frac{\int_x^t e^{\kappa_\phi(t)\, y}\, dF(y)}{e^{\kappa_\phi(t)\, x}\,[\overline{F}(x) - \overline{F}(t)]}. \tag{4.1.10}$$

Similar to (4.1.4), it is not hard to show by the definitions of $\delta_\phi(t)$ and $\theta_\phi(t)$ and (4.1.8) [cf. (2.3.11) and (2.3.12)] that

$$e^{-\kappa_\phi(t)\, t} \;\leq\; \delta_\phi(t) \;\leq\; \phi\, F(t) \quad \text{and} \quad \phi\, F(t) \;\leq\; \theta_\phi(t) \;\leq\; 1. \tag{4.1.11}$$

Lemma 4.2 Given $t > 0$, suppose that (4.1.1) holds, then for any $0 \leq x \leq t$,

$$\overline{G}(x) \;\geq\; \frac{(1-p_0)\,\left[(1-\phi_1)\, e^{-\kappa_{\phi_1}(t)\,(x+t)} + \phi_1\, \overline{F}(t)\right]}{\phi_1[1 - \phi_1 + \phi_1\, \overline{F}(t)]} \tag{4.1.12}$$

and

$$\overline{G}(x) \;\leq\; \frac{(1-p_0)\,\left[(1-\phi_2)\, e^{-\kappa_{\phi_2}(t)\, x} + \phi_2\, \overline{F}(t)\right]}{\phi_2[1 - \phi_2 + \phi_2\, \overline{F}(t)]}. \tag{4.1.13}$$

Proof. Lemma 4.2 follows from (3.1.9), (3.1.10) and Corollary 3.4. □

Since for $n \geq 0$, $a_n = \sum_{k=n+1}^{\infty} p_k$ and $a_{n+1} = \sum_{k=n+2}^{\infty} p_k = \sum_{k=n+1}^{\infty} p_{k+1}$, hence, if for any $k \geq 1$, $a \leq p_{k+1}/p_k \leq b$, then for any $n \geq 0$, $a\, a_n \leq a_{n+1} \leq b\, a_n$. Using this property, it is easy to check the following facts.

1. For the geometric distribution with

$$p_n = (1-q)\, q^n, \quad n = 0, 1, 2, \cdots, \; 0 < q < 1,$$

$$a_{n+1} \;=\; q\, a_n, \quad n \geq 0. \tag{4.1.14}$$

2. For the logarithmic distribution with

$$p_n = \frac{\theta^n}{-n \ln(1-\theta)}, \quad n = 1, 2, \cdots, \; 0 < \theta < 1,$$

$$\frac{\theta}{2}\, a_n \;\leq\; a_{n+1} \;\leq\; \theta\, a_n, \quad n \geq 0. \tag{4.1.15}$$

3. For the negative binomial distribution with

$$p_n = \binom{n+\alpha-1}{n} (1-q)^{\alpha}\, q^n, \quad n = 0, 1, 2, \cdots,$$

   where, $\alpha > 0$ and $0 < q < 1$.

   (a) If $0 < \alpha \leq 1$, then

$$\alpha q\, a_n \leq a_{n+1} \leq \; q\, a_n, \quad n \geq 0. \tag{4.1.16}$$

   (b) If $\alpha \geq 1$, then

$$q\, a_n \leq \; a_{n+1} \;\leq\; \alpha q\, a_n, \quad n \geq 0. \tag{4.1.17}$$

Thus, Lemmas 4.1 and 4.2 can apply directly to compound geometric distributions, compound logarithmic distributions, compound negative binomial distributions with $0 < \alpha < 1/q$.

The purpose of this chapter is to use the convolution technique, as in Willmot and Lin (1997b), and the changing distributions technique for Dickson's condition, developed in Chapter 3, to derive two-sided bounds for the tails of general compound negative binomial distributions with $\alpha > 0$.

We derive upper and lower bounds for the tails of compound negative binomial distributions with the generalized adjustment coefficients of F. Sharp bounds are obtained for the tail of the compound negative binomial distribution. A connection between the compound negative binomial, Poisson and logarithmic distributions is presented, this results in the generalization and improvement of Theorem 3 of Willmot and Lin (1997b).

4.2 Notation and preliminaries

Denote the gamma distribution function with scale parameter $\alpha > 0$ and shape parameter $\beta > 0$ by $H_{\alpha,\beta}(x)$ and define $\overline{H}_{\alpha,\beta}(x) = 1 - H_{\alpha,\beta}(x)$ as the tail of $H_{\alpha,\beta}(x)$, *i.e.*

$$\overline{H}_{\alpha,\beta}(x) = \int_x^\infty \frac{\alpha(\alpha y)^{\beta-1} e^{-\alpha y}}{\Gamma(\beta)} dy, \quad x \geq 0. \tag{4.2.1}$$

If $\beta = n$ is a postive integer, then $H_{\alpha,n}(x)$ is an Erlang distribution function and

$$\overline{H}_{\alpha,n}(x) = e^{-\alpha x} \sum_{k=0}^{n-1} \frac{(\alpha x)^k}{k!}, \quad x \geq 0, \quad n = 1, 2, \cdots. \tag{4.2.2}$$

For notational convention, we define $H_{\alpha,0}(x)$ as a discrete distribution function with unit mass at 0, *i.e.* $H_{\alpha,0}(x) = 1$ if $x \geq 0$, 0 otherwise. Or equivalently, $\overline{H}_{\alpha,0}(x) = 0$ if $x \geq 0$, 1 otherwise.

Definition 4.5 A distribution function $B_{\alpha,\lambda}$ is said to be a pseudo exponential distribution function if

$$B_{\alpha,\lambda}(x) = \begin{cases} 0, & x < 0 \\ 1 - \alpha e^{-\lambda x}, & x \geq 0 \end{cases} \tag{4.2.3}$$

where $0 \leq \alpha \leq 1$ and $\lambda \geq 0$. Or, equivalently,

$$\overline{B}_{\alpha,\lambda}(x) = \begin{cases} 1, & x < 0 \\ \alpha e^{-\lambda x}, & x \geq 0 \end{cases} \tag{4.2.4}$$

If $\alpha = 1$ and $\lambda > 0$, then $B_{\alpha,\lambda}$ reduces to an exponential distribution function. Similar to the property that the convolution of exponential distributions is a gamma distribution, we can derive the expression for the convolution of pseudo exponential distribution functions, which is a mixture of Erlang distributions. It is the basis of the bounds for the tails of compound negative binomial distributions.

The following property can be found in the proof of Theorem 1 of Willmot and Lin (1997b). It is a generalization of the convolution of exponential distributions, so we state it with its proof.

Property 4.2 Let $B_{\alpha_i,\lambda}$, $i = 1, \cdots, n$ be n pseudo exponential distribution functions. For any $x \geq 0$,

$$B_{\alpha_1,\lambda} * \cdots * B_{\alpha_n,\lambda}(x) = \sum_{i=0}^{n} b_i \, H_{\lambda,i}(x) \tag{4.2.5}$$

or, equivalently,

$$\overline{B_{\alpha_1,\lambda} * \cdots * B}_{\alpha_n,\lambda}(x) = \sum_{i=1}^{n} b_i \, \overline{H}_{\lambda,i}(x) \tag{4.2.6}$$

where $b_i \geq 0$, $i = 0, 1, \cdots, n$ satisfy that for any $x \geq 0$,

$$\sum_{i=0}^{n} b_i \, x^i = \prod_{i=1}^{n} (1 - \alpha_i + \alpha_i x), \tag{4.2.7}$$

in particular, if $\alpha_i = \alpha$, $i = 1, \cdots, n$, then

$$\overline{B}^{(n)}_{\alpha,\lambda}(x) = \sum_{i=1}^{n} \binom{n}{i} (1-\alpha)^{n-i} \alpha^i \, \overline{H}_{\lambda,i}(x). \tag{4.2.8}$$

Proof. Since the Laplace transform of $B_{\alpha,\lambda}$ is

$$\begin{aligned} \int_0^\infty e^{-sx} \, dB_{\alpha,\lambda}(x) &= 1 - \alpha + \int_0^\infty e^{-sx} \, \alpha\lambda \, e^{-\lambda x} \, dx \\ &= 1 - \alpha + \alpha \frac{\lambda}{\lambda + s}. \end{aligned}$$

Hence, the Laplace transform of $B_{\alpha_1,\lambda} * \cdots * B_{\alpha_n,\lambda}$ is

$$\prod_{i=1}^{n} \left(1 - \alpha_i + \alpha_i \frac{\lambda}{\lambda + s}\right) = \sum_{i=0}^{n} b_i \left(\frac{\lambda}{\lambda + s}\right)^i, \tag{4.2.9}$$

where, the equality follows from (4.2.7).

The Laplace transform of the gamma distribution functions $H_{\lambda,i}$ is $\left(\frac{\lambda}{\lambda+s}\right)^i$, $i = 0, 1, \cdots, n$. Hence, (4.2.9) is also the Laplace transform of $\sum_{i=0}^{n} b_i \, H_{\lambda,i}$. Thus, the uniqueness of the Laplace transform implies that (4.2.5) holds.

Take $x = 1$ in (4.2.7), we get

$$\sum_{i=0}^{n} b_i = 1.$$

Thus, by $\overline{H}_{\lambda,0}(x) = 0$ for $x \geq 0$, we get for $x \geq 0$,

$$\overline{B_{\alpha_1,\lambda} * \cdots * B_{\alpha_n,\lambda}}(x) = \sum_{i=0}^{n} b_i - \sum_{i=0}^{n} b_i H_{\lambda,i}(x) = \sum_{i=1}^{n} b_i \overline{H}_{\lambda,i}(x).$$

□

Definition 4.6 Suppose random variables X and Y follow distributions F and G, respectively.

1. We say that X is stochastically smaller than Y, written $X <_{st} Y$ or $F <_{st} G$ if for all x,

$$\overline{F}(x) \leq \overline{G}(x).$$

2. We say that X is smaller than Y in stop-loss ordering, written $X <_{sl} Y$ or $F <_{sl} G$ if for all x,

$$\int_x^\infty \overline{F}(y)\,dy \leq \int_x^\infty \overline{G}(y)\,dy.$$

Property 4.3 Let M and N be two nonnegative integer valued random variables, $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be sequences of independent nonnegative random variables. Assume that M and N are independent of $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$, respectively.

1. If $M <_{st} N$, $X_i <_{st} Y_i$, $i \geq 1$, then

$$\sum_{i=1}^{M} X_i <_{st} \sum_{i=1}^{N} Y_i, \tag{4.2.10}$$

in particular, for any $n \geq 1$,

$$\sum_{i=1}^{n} X_i <_{st} \sum_{i=1}^{n} Y_i. \tag{4.2.11}$$

2. If $M <_{sl} N$, $X_i <_{sl} Y_i$, $i \geq 1$, then

$$\sum_{i=1}^{M} X_i <_{sl} \sum_{i=1}^{N} Y_i, \tag{4.2.12}$$

in particular, for any $n \geq 1$,

$$\sum_{i=1}^{n} X_i <_{sl} \sum_{i=1}^{n} Y_i. \tag{4.2.13}$$

Proof. This property is well known as the preservation of "$<_{st}$" and "$<_{sl}$" under random sums, see Shaked and Shanthikumar (1994) or Jean-Marie and Liu (1992). □

4.3 Bounds for tails of compound negative binomial distributions with adjustment coefficients

Denote the tail of the compound negative binomial distribution by

$$\overline{G}_\alpha(x) = \sum_{n=1}^{\infty} \binom{n+\alpha-1}{n} (1-q)^\alpha q^n \overline{F}^{(n)}(x), \quad x \geq 0.$$

In this section, we derive lower and upper bounds for the tail of compound negative binomial distributions when the adjustment coefficients exist.

Definition 4.7 Define

$$l_\alpha = \frac{\delta_q [1-(1-q)^\alpha]}{q} \quad \text{and} \quad u_\alpha = \frac{\theta_q [1-(1-q)^\alpha]}{q}. \tag{4.3.1}$$

We note that $0 \leq l_\alpha \leq 1$ since $0 \leq \delta_q \leq q$ [c.f. (4.1.4)]. In addition, $0 \leq u_\alpha \leq 1$ if $0 \leq \alpha \leq 1$ since $q \leq \theta_\alpha \leq 1$ [c.f. (4.1.4)] and $1-q \leq (1-q)^\alpha$ if $0 \leq \alpha \leq 1$.

Theorem 4.1 (1) If $0 < \alpha \leq 1$, then for any $x \geq 0$,

$$\frac{\delta_{\alpha q} [1-(1-q)^\alpha]}{\alpha q} e^{-\kappa_{\alpha q} x} \leq \overline{G}_\alpha(x) \leq \frac{\theta_q [1-(1-q)^\alpha]}{q} e^{-\kappa_q x}. \tag{4.3.2}$$

(2) For $\alpha \geq 1$, suppose that m and n are two positive integers such that

$$1 \leq m \leq \alpha, \quad \alpha_j \geq 1, \quad \text{for} \quad j = 1, \cdots, m \quad \text{and} \quad \sum_{j=1}^{m} \alpha_j = \alpha$$

while

$$n \geq \alpha, \quad 0 < \alpha_j^* \leq 1, \quad \text{for} \quad j = 1, \cdots, n \quad \text{and} \quad \sum_{j=1}^{n} \alpha_j^* = \alpha,$$

then for any $x \geq 0$,

$$\sum_{i=1}^{m} \gamma_i \overline{H}_{\kappa_q, i}(x) \leq \overline{G}_\alpha(x) \leq \sum_{i=1}^{n} d_i \overline{H}_{\kappa_q, i}(x), \tag{4.3.3}$$

where $\gamma_i \geq 0, \ i = 1, \cdots, m$ and $d_j \geq 0, \ j = 1, \cdots, n$ satisfy

$$\sum_{i=0}^{m} \gamma_i x^i = \prod_{i=1}^{m} (1 - l_{\alpha_i} + l_{\alpha_i} x), \quad x \geq 0,$$

$$\sum_{j=0}^{n} d_j y^j = \prod_{j=1}^{n} \left(1 - u_{\alpha_j^*} + u_{\alpha_j^*} y\right), \quad y \geq 0.$$

Proof. (1) (4.3.2) follows simply from Lemma 4.1 and (4.1.16) by taking $\phi_1 = \alpha q$, $\phi_2 = q$ and $p_0 = (1-q)^\alpha$.

(2) For any $\alpha \geq 1$, by Lemma 4.1, (4.1.17) and taking $\phi_1 = q$, $p_0 = (1-q)^\alpha$, we get by (4.2.4) that for $x \geq 0$,

$$\overline{G}_\alpha(x) \;\geq\; \frac{\delta_q\left[1-(1-q)^\alpha\right]}{q} e^{-\kappa_q x} \;=\; \overline{B}_{l_\alpha, \kappa_q}(x). \tag{4.3.4}$$

Suppose that f^* is the Laplace transform of the distribution F, then the Laplace transform of G_α is

$$\left(\frac{1-q}{1-qf^*(s)}\right)^\alpha = \prod_{i=1}^{m}\left(\frac{1-q}{1-qf^*(s)}\right)^{\alpha_i},$$

this implies that for $x \geq 0$,

$$G_\alpha(x) = G_{\alpha_1} * \cdots * G_{\alpha_m}(x),$$

or, equivalently, for $x \geq 0$,

$$\overline{G}_\alpha(x) = \overline{G_{\alpha_1} * \cdots * G}_{\alpha_m}(x).$$

Since $\alpha_j \geq 1$, $j = 1, \cdots, m$, by (4.3.4) and Property 4.3, we get for $x \geq 0$,

$$\overline{G}_\alpha(x) \;\geq\; \overline{B_{l_{\alpha_1}, \kappa_q} * \cdots * B}_{l_{\alpha_m}, \kappa_q}(x),$$

which, together with Property 4.2, implies that the lower bound in (4.3.3) holds.

For the proof of the upper bound in (4.3.3), see Theorem 1 of Willmot and Lin (1997b). □

We note that if $\alpha_1 = \cdots = \alpha_m = 1$ and $\alpha_1^* = \cdots = \alpha_n^* = 1$, then,

$$l_{\alpha_i} = \delta_q, \; i = 1, \cdots, m \quad \text{and} \quad u_{\alpha_j^*} = \theta_q, \; j = 1, \cdots, n,$$

thus, by Theorem 4.1 and (4.2.8), we get immediately the following result.

Corollary 4.1 If α is a positive integer, then for any $x \geq 0$,

$$\overline{G}_\alpha(x) \;\geq\; \sum_{j=1}^{\alpha} \binom{\alpha}{j} (1-\delta_q)^{\alpha-j}\, \delta_q^j\, \overline{H}_{\kappa_q, j}(x) \tag{4.3.5}$$

and

$$\overline{G}_\alpha(x) \;\leq\; \sum_{j=1}^{\alpha} \binom{\alpha}{j} (1-\theta_q)^{\alpha-j}\, \theta_q^j\, \overline{H}_{\kappa_q, j}(x). \tag{4.3.6}$$

The lower and upper bounds in Corollary 4.1 are sharp in that if F is an exponential distribution, then $\delta_q = \theta_q = q$ and $\overline{G}_\alpha(x)$ equals the lower and upper bounds.

We notice that (4.3.4) gives a simple lower bound for $\overline{G}_\alpha(x)$ for any $\alpha \geq 1$. However, we know that if $i < j$, then for any $x \geq 0$,

$$\overline{H}_{\kappa_q, i}(x) \;\leq\; \overline{H}_{\kappa_q, j}(x), \quad x \geq 0$$

and

$$\overline{H}_{\kappa_q, i}(x) = o\left(\overline{H}_{\kappa_q, j}(x)\right), \quad x \to \infty.$$

This implies that the lower bound in (4.3.3) is better than that in (4.3.4) at least for large x.

Hence, to get the best lower bound in (4.3.3), one should choose the largest m satisfying $1 \leq m \leq \alpha$, *i.e.* take $m = [\alpha]$. The bounds derived in this chapter are based on Lemma 4.1. However, the lower bound in Lemma 4.1 can reduce to sharp lower bounds for the tail of compound geometric distributions, *i.e.* for $\alpha = 1$. Thus, the lower bound in Lemma 4.1 is better if α approximates 1. Hence, we shall decompose $\alpha = \alpha_1 + \cdots + \alpha_{[\alpha]}$, $\alpha_j \geq 1$, for $j = 1, \cdots, [\alpha]$ such that α_j approaches 1 as much as possible and in this way, we can get better lower bounds for $\overline{G}_\alpha(x)$.

Thus, natural choices of $(\alpha_1, \cdots, \alpha_{[\alpha]})$ are

$$\alpha_j = 1, \quad \text{for } j = 1, \cdots, [\alpha] - 1 \quad \text{and} \quad \alpha_{[\alpha]} = 1 + \alpha - [\alpha], \tag{4.3.7}$$

with this choice of $(\alpha_1, \cdots, \alpha_{[\alpha]})$, we get

$$\gamma_j = (1 - l_{1+\alpha-[\alpha]})\, a_j + l_{1+\alpha-[\alpha]}\, a_{j-1}, \quad \text{for } j = 0, 1, \cdots, [\alpha],$$

where

$$a_0 = a_{-1} = 0 \;\; \text{and} \;\; a_j = \binom{[\alpha] - 1}{j} [1 - \delta_q]^{[\alpha]-1-j}\, \delta_q^j, \quad \text{for } j = 1, \cdots, [\alpha] - 1.$$

Another simple choice of $(\alpha_1, \cdots, \alpha_{[\alpha]})$ is to set all α_j equal, but not less than 1, *i.e.*

$$\alpha_j = \frac{\alpha}{[\alpha]}, \quad \text{for } j = 1, \cdots, [\alpha], \tag{4.3.8}$$

with this choice of $(\alpha_1, \cdots, \alpha_{[\alpha]})$, we get

$$\gamma_j = \binom{[\alpha]}{j} \left(1 - l_{\alpha/[\alpha]}\right)^{[\alpha]-j} l^j_{\alpha/[\alpha]}, \quad \text{for } j = 0, 1, \cdots, [\alpha].$$

Similarly, for the best upper bound in (4.3.3), one should choose the smallest n satisfying $n \geq \alpha$, *i.e.* take $n = \alpha$ if it is an integer, $[\alpha]+1$ if it is not. Similarly to the proof of Theorem 2 of Willmot and Lin (1997b), we can show that the choice (4.3.7) of $(\alpha_1, \cdots, \alpha_{[\alpha]})$ is optimal under the condition of $\delta_q = q$, which can be satisfied when F is NBUC.

4.4 A connection between compound negative binomial, logarithmic and Poisson distributions and its application

It is well known that a negative binomial distribution is a mixture of Poisson distributions. This implies that a compound negative binomial distribution is a mixture of compound Poisson distributions, as well. However, another important fact is that a negative binomial distribution is a Poisson sum of logarithmic random variables [see, e.g. Johnson *et al.* (1992)]. Thus, we claim that a compound negative binomial distribution is a compound Poisson distribution with a compound logarithmic distribution as the underlying distribution. We present this fact in the following property, which results in simple two-sided bounds for $\overline{G}_\alpha(x)$. This upper bound generalizes and improves Theorem 3 of Willmot and Lin (1997b).

Property 4.4 Let F_q be a compound logarithmic distribution with

$$F_q(x) = \sum_{n=1}^{\infty} \frac{q^n}{-n\ln(1-q)} F^{(n)}(x), \quad x \geq 0,$$

then

$$\overline{G}_\alpha(x) = \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \overline{F}_q^{(n)}(x), \quad x \geq 0 \tag{4.4.1}$$

where $\lambda = -\alpha\ln(1-q)$.

Proof. Suppose that $\{Y_i,\ i \geq 1\}$ is a sequence of *i.i.d.* logarithmic random variables with

$$\Pr\{Y_1 = n\} = \frac{q^n}{-n\ln(1-q)}, \quad n = 1, 2, \cdots,$$

N_λ is a Poisson random variable and independent of $\{Y_i,\ i \geq 1\}$ with

$$\Pr\{N_\lambda = n\} = \frac{\lambda^n e^{-\lambda}}{n!}, \quad n = 0, 1, 2, \cdots.$$

Hence,

$$F_q(x) = \Pr\left\{\sum_{i=1}^{Y_1} X_i \leq x\right\}.$$

It is known by the Laplace transform that $N_\alpha = Y_1 + \cdots + Y_{N_\lambda}$ is a negative binomial random variable with [see, Johnson *et al.* (1992)],

$$\Pr\{N_\alpha = n\} = \binom{n+\alpha-1}{n}(1-q)^\alpha q^n, \quad n = 0, 1, 2, \cdots.$$

Denote the Laplace transforms of F and F_q by $\tilde{f}$ and $\tilde{f}_q$, respectively. Thus, the Laplace transform of $F_q^{(n)}$ is $\tilde{f}_q^n$ and by the definition of $\tilde{f}_q$, we know

$$\begin{aligned}
\tilde{f}_q(s) &= E\left[e^{-s\sum_{i=1}^{Y_1} X_i}\right] = E\left\{E\left[e^{-s\sum_{i=1}^{Y_1} X_i} \,|\, Y_1\right]\right\} \\
&= E\left\{E\left[(\tilde{f}(s))^{Y_1} \,|\, Y_1\right]\right\} \\
&= E\left[(\tilde{f}(s))^{Y_1}\right].
\end{aligned}$$

However, the Laplace transform of $\sum_{i=1}^{Y_1+\cdots+Y_n} X_i$ is

$$\begin{aligned}
E\left[e^{-s\sum_{i=1}^{Y_1+\cdots+Y_n} X_i}\right] &= E\left\{E\left[e^{-s\sum_{i=1}^{Y_1+\cdots+Y_n} X_i} \,|\, Y_1+\cdots+Y_n\right]\right\} \\
&= E\left\{E\left[(\tilde{f}(s))^{Y_1+\cdots+Y_n} \,|\, Y_1+\cdots+Y_n\right]\right\} \\
&= E\left[(\tilde{f}(s))^{Y_1+\cdots+Y_n}\right] = \left(E\left[(\tilde{f}(s))^{Y_1}\right]\right)^n \\
&= (\tilde{f}_q(s))^n.
\end{aligned}$$

This implies that

$$F_q^{(n)}(x) = \Pr\left\{\sum_{i=1}^{Y_1+\cdots+Y_n} X_i \leq x\right\}.$$

Thus,

$$\begin{aligned} G_\alpha(x) &= \Pr\left\{\sum_{i=1}^{N_\alpha} X_i \le x\right\} = \Pr\left\{\sum_{i=1}^{Y_1+\cdots+Y_{N_\lambda}} X_i \le x\right\} \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \Pr\left\{\sum_{i=1}^{Y_1+\cdots+Y_n} X_i \le x\right\} \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} F_q^{(n)}(x). \end{aligned}$$

This implies that (4.4.1) holds. □

Theorem 4.2 Let $\lambda = -\alpha \ln(1-q)$. Then

$$\overline{G}_\alpha(x) \ge \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \left\{\sum_{j=1}^{n} \binom{n}{j} (1-a_1)^{n-j} a_1^j \, \overline{H}_{\kappa_{q/2}, j}(x)\right\} \tag{4.4.2}$$

and

$$\overline{G}_\alpha(x) \le \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \left\{\sum_{j=1}^{n} \binom{n}{j} (1-a_2)^{n-j} a_2^j \, \overline{H}_{\kappa_q, j}(x)\right\}. \tag{4.4.3}$$

where

$$a_1 = \frac{2\delta_{q/2}}{q} \quad \text{and} \quad a_2 = \frac{\theta_q}{q}.$$

Proof. For the probability function of the logarithmic random variable Y_1, by Lemma 4.1, (4.1.15) and $p_0 = 0$, we get for $x \ge 0$,

$$\frac{\delta_{q/2}}{q/2} e^{-\kappa_{q/2} x} \le \overline{F}_q(x) \le \frac{\theta_q}{q} e^{-\kappa_q x},$$

i.e. for any $x \ge 0$

$$a_1 e^{-\kappa_{q/2} x} \le \overline{F}_q(x) \le a_2 e^{-\kappa_q x},$$

or

$$\overline{B}_{a_1, \kappa_{q/2}}(x) \le \overline{F}_q(x) \le \overline{B}_{a_2, \kappa_q}(x). \tag{4.4.4}$$

By Property 4.3, we know that (4.4.4) implies that

$$\overline{B}^{(n)}_{a_1, \kappa_{q/2}}(x) \le \overline{F}^{(n)}_q(x) \le \overline{B}^{(n)}_{a_2, \kappa_q}(x),$$

this implies by Property 4.2 that

$$\sum_{j=1}^{n} \binom{n}{j} (1-a_1)^{n-j} a_1^j \overline{H}_{\kappa_{q/2},j}(x) \le \overline{F}_q^{(n)}(x) \le \sum_{j=1}^{n} \binom{n}{j} (1-a_2)^{n-j} a_2^j \overline{H}_{\kappa_q,j}(x)$$

this, together with (4.4.1) implies (4.4.2) and (4.4.3). □

Remark 4.1 Under the condition of $\theta_q = q$, Theorem 3 of Willmot and Lin (1997b) states that

$$\overline{G}_\alpha(x) \le \sum_{n=1}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} \overline{H}_{\kappa_q,n}(x), \quad x \ge 0,$$

however,

$$\sum_{j=1}^{n} \binom{n}{j} (1-a_2)^{n-j} a_2^j \overline{H}_{\kappa_q,j}(x) \le (1-a_2)^n \overline{H}_{\kappa_q,n}(x) \le \overline{H}_{\kappa_q,n}(x).$$

So, the upper bound in (4.4.3) improves Theorem 3 of Willmot and Lin (1997b). Also, the lower bound is given without the unnecessary condition that $\theta_2 = q$.

4.5 Bounds for tails of compound negative binomial distributions with large claim sizes

In this section, we use the technique developed in Chapter 3 and Dickson's condition to derive two-sided bounds for tails of compound negative binomial distributions with large claim sizes.

Given $t > 0$, let $G_\alpha(x, qF(t))$ be the compound negative binomial distribution with parameter α, $qF(t)$ and the underlying distribution F_t. Equivalently,

$$\overline{G}_\alpha(x, qF(t)) = \sum_{n=1}^{\infty} \binom{n+\alpha-1}{n} [1-qF(t)]^\alpha [qF(t)]^n \overline{F}_t^{(n)}(x), \quad x \ge 0\,,$$

where F_t is defined in (3.3.2).

We first give the following identities.

Property 4.5 For any $0 \le x \le t$,

$$\overline{G}_\alpha(x) = \overline{H}(t) + H(t)\, \overline{G}_\alpha(x, qF(t))\,, \tag{4.5.1}$$

where

$$H(x) = \left(\frac{1-q}{1-qF(x)}\right)^\alpha, \quad x \ge 0\,,$$

is a probability distribution function.

Proof. By (3.3.9), we get for any $0 \leq x \leq t$,

$$\overline{F}_t^{(n)}(x) = 1 - \frac{1}{[F(t)]^n} + \frac{\overline{F}^{(n)}(x)}{[F(t)]^n}, \tag{4.5.2}$$

which implies that for any $0 \leq x \leq t$,

$$\begin{aligned} \overline{G}_\alpha(x, qF(t)) &= \sum_{n=1}^{\infty} \binom{n+\alpha-1}{n} [1-qF(t)]^\alpha [qF(t)]^n \left[1 - \frac{1}{[F(t)]^n} + \frac{\overline{F}^{(n)}(x)}{[F(t)]^n}\right] \\ &= 1 - [1-qF(t)]^\alpha - \left(\frac{1-qF(t)}{1-q}\right)^\alpha [1-(1-q)^\alpha] \\ &\quad + \left(\frac{1-qF(t)}{1-q}\right)^\alpha \overline{G}_\alpha(x) \\ &= 1 - [H(t)]^{-1} + [H(t)]^{-1}\overline{G}_\alpha(x), \end{aligned}$$

furthermore, this implies that (4.5.1) holds. □

Definition 4.8 Define

$$l_\alpha(t) = \frac{\delta_q(t)[1-(1-qF(t))^\alpha]}{qF(t)} \quad \text{and} \quad u_\alpha(t) = \frac{\theta_q(t)[1-(1-qF(t))^\alpha]}{qF(t)}. \tag{4.5.3}$$

By (3.3.3), we know that the condition (4.1.8) is equivalent to

$$\int_0^\infty e^{\kappa_\phi(t)\,y}\, dF_t(y) = \frac{1}{\phi F(t)},$$

furthermore, by (4.1.9), we get

$$\sup_{x\geq 0} \frac{\int_x^\infty e^{\kappa_\phi(t)\,y}\, dF_t(y)}{e^{\kappa_\phi(t)\,x}\,\overline{F}_t(x)} = \sup_{0\leq x\leq t} \frac{\int_x^t e^{\kappa_\phi(t)\,y}\, dF(y)}{e^{\kappa_\phi(t)\,x}\,[\overline{F}(x) - \overline{F}(t)]} = [\delta_\phi(t)]^{-1}.$$

Similarly, we have

$$\inf_{x\geq 0} \frac{\int_x^\infty e^{\kappa_\phi(t)\,y}\, dF_t(y)}{e^{\kappa_\phi(t)\,x}\,\overline{F}_t(x)} = \inf_{0\leq x\leq t} \frac{\int_x^t e^{\kappa_\phi(t)\,y}\, dF(y)}{e^{\kappa_\phi(t)\,x}\,[\overline{F}(x) - \overline{F}(t)]} = [\theta_\phi(t)]^{-1},$$

thus, apply Property 4.5 and Theorem 4.1 to the tail $\overline{G}_\alpha(x, qF(t))$ of the compound negative binomial, respectively, we get immediately the two following results.

Theorem 4.3 For given $t > 0$, if $0 < \alpha \leq 1$, then for any $0 \leq x \leq t$,

$$\overline{G}_\alpha(x) \;\geq\; \overline{H}(t) + H(t)\, l_\alpha^*(t)\, e^{-\kappa_{\alpha q}(t)\,x} \tag{4.5.4}$$

and

$$\overline{G}_\alpha(x) \;\le\; \overline{H}(t) + H(t)\, u_\alpha(t)\, e^{-\kappa_q(t)\, x}, \tag{4.5.5}$$

where,

$$l^*_\alpha(t) = \frac{\delta_{\alpha q}(t)[1-(1-qF(t))^\alpha]}{\alpha q F(t)},$$

in particular, for any $x > 0$,

$$\overline{G}_\alpha(x) \;\ge\; \overline{H}(x) + H(x)\, l^*_\alpha(x)\, e^{-\kappa_{\alpha q}(x)\, x} \tag{4.5.6}$$

and

$$\overline{G}_\alpha(x) \;\le\; \overline{H}(x) + H(x)\, u_\alpha(x)\, e^{-\kappa_q(x)\, x}. \tag{4.5.7}$$

Theorem 4.4 For $\alpha \ge 1$, suppose that m and n are two positive integers such that

$$1 \le m \le \alpha, \quad \alpha_j \ge 1, \quad j = 1, \cdots, m \quad \text{and} \quad \sum_{j=1}^{m} \alpha_j = \alpha$$

and

$$n \ge \alpha, \quad 0 < \alpha^*_j \le 1, \quad j = 1, \cdots, n \quad \text{and} \quad \sum_{j=1}^{n} \alpha^*_j = \alpha.$$

Given $t > 0$, then for any $0 \le x \le t$,

$$\overline{G}_\alpha(x) \;\ge\; \overline{H}(t) + H(t) \sum_{i=1}^{m} \gamma_i(t)\, \overline{H}_{\kappa_q(t),\, i}(x) \tag{4.5.8}$$

and

$$\overline{G}_\alpha(x) \;\le\; \overline{H}(t) + H(t) \sum_{i=1}^{n} d_i(t)\, \overline{H}_{\kappa_q(t),\, i}(x), \tag{4.5.9}$$

in particular, for any $x > 0$,

$$\overline{G}_\alpha(x) \;\ge\; \overline{H}(x) + H(x) \sum_{i=1}^{m} \gamma_i(x)\, \overline{H}_{\kappa_q(x),\, i}(x) \tag{4.5.10}$$

and

$$\overline{G}_\alpha(x) \;\le\; \overline{H}(x) + H(x) \sum_{i=1}^{n} d_i(x)\, \overline{H}_{\kappa_q(x),\, i}(x), \tag{4.5.11}$$

where $\gamma_i(t) \geq 0,\ i = 1, \cdots, m$ and $d_j(t) \geq 0,\ j = 1, \cdots, n$ satisfy that

$$\sum_{i=0}^{m} \gamma_i(t)\, z^i = \prod_{i=1}^{m} \left[1 - l_{\alpha_i}(t) + l_{\alpha_i}(t)\, z\right], \quad z \geq 0,$$

and

$$\sum_{j=0}^{n} d_j(t)\, z^j = \prod_{j=1}^{n} \left[1 - u_{\alpha_j^*}(t) + u_{\alpha_j^*}(t)\, z\right], \quad z \geq 0.$$

Since if $\alpha_1 = \cdots = \alpha_m = 1$ and $\alpha_1^* = \cdots = \alpha_n^* = 1$, then,

$$l_{\alpha_i}(t) = \delta_q(t),\ i = 1, \cdots, m \quad \text{and} \quad u_{\alpha_j^*}(t) = \theta_q(t),\ j = 1, \cdots, n,$$

thus, by Theorem 4.4, we get immediately the following result.

Corollary 4.2 Given $t > 0$, if α is an integer, *i.e.* $\alpha = 1, 2, \cdots$, then for any $0 \leq x \leq t$,

$$\overline{G}_\alpha(x) \ \geq\ \overline{H}(t) + H(t) \sum_{j=1}^{\alpha} \binom{\alpha}{j} [1 - \delta_q(t)]^{\alpha-j} \, [\delta_q(t)]^j \, \overline{H}_{\kappa_q(t), j}(x) \tag{4.5.12}$$

and

$$\overline{G}_\alpha(x) \ \leq\ \overline{H}(t) + H(t) \sum_{j=1}^{\alpha} \binom{\alpha}{j} [1 - \theta_q(t)]^{\alpha-j} \, [\theta_q(t)]^j \, \overline{H}_{\kappa_q(t), j}(x), \tag{4.5.13}$$

in particular, for any $x > 0$,

$$\overline{G}_\alpha(x) \ \geq\ \overline{H}(x) + H(x) \sum_{j=1}^{\alpha} \binom{\alpha}{j} [1 - \delta_q(x)]^{\alpha-j} \, [\delta_q(x)]^j \, \overline{H}_{\kappa_q(x), j}(x) \tag{4.5.14}$$

and

$$\overline{G}_\alpha(x) \ \leq\ \overline{H}(x) + H(x) \sum_{j=1}^{\alpha} \binom{\alpha}{j} [1 - \theta_q(x)]^{\alpha-j} \, [\theta_q(x)]^j \, \overline{H}_{\kappa_q(x), j}(x). \tag{4.5.15}$$

It is well known that if F is subexponential, then $\overline{G}_\alpha(x)$ has the following asymptotic formula [see, for example, Embrechts *et al.* (1979)],

$$\overline{G}_\alpha(x) \ \sim\ \frac{\alpha q}{1-q}\, \overline{F}(x), \quad x \to \infty \tag{4.5.16}$$

It is easy to show by L'Hôspital's rule that

$$\frac{\alpha q}{1-q}\, \overline{F}(x) \ \sim\ \overline{H}(x), \quad x \to \infty \tag{4.5.17}$$

Cai and Garrido (1998) (see also Chapter 5 of this thesis) show that $\overline{H}(x)$ is the lower bound of $\overline{G}_\alpha(x)$. Evidently, the lower bounds in Theorem 4.3 and Theorem 4.4 are tighter and the upper bounds are also given here.

Chapter 5

Aging Properties of Geometric Sums and Bounds for Stop-Loss Premiums and Ruin Probabilities

5.1 Introduction

Definition 5.1 For a random variable X with distribution F,

$$E[(X-d)_+] = \int_d^\infty \overline{F}(y)\,dy,$$

is called stop-loss premium (or stop-loss transform) of X (or F) with retention d, denoted by $\pi_X(d)$ [or $\pi_F(d)$].

In insurance risk analysis, one is interested in the stop-loss premium for aggregate claims $S = X_1 + \cdots + X_N$, which has distribution function G and tail probability $\overline{G}(x)$ given in (1.1.1) and (1.1.2), respectively.

The stop-loss premium of S is

$$\pi_G(d) = E(S-d)_+ = \int_d^\infty \overline{G}(y)\,dy. \tag{5.1.1}$$

The premium $\pi_G(d)$ is paid in exchange for stop-loss reinsurance coverage. An insurance company can reduce its risk by the use of reinsurance, *i.e.* by a stop-loss contract with deductible $d \geq 0$. The amount paid by the reinsurer to the ceding insurance company is

$$(S-d)_+ = \begin{cases} 0, & S \leq d, \\ S-d, & S > d, \end{cases}$$

Thus, the amount retained by the insurance company is bounded by d and $\pi_S(d)$ represents the expected amount of the reinsurer's losses, a quantity of interest in insurance risk analysis.

In this chapter, we consider stop-loss premiums for the class of compound distributions discussed in Chapter 3, *i.e.* compound distributions satisfying (3.1.1) or (3.1.2). The stop-loss premiums for compound negative binomial distributions are also considered. Similarly to the study of tail probabilities for the class of compound distributions in Chapter 3, the bounds for stop-loss premiums can be deduced from the bounds for stop-loss premiums of compound geometric distributions, by (3.1.9) and (3.1.10).

Assume that N_0 is a geometric random variable independent of the sequence $\{X_i,\, i \geq 1\}$ with

$$\Pr\{N_0 = n\} = \frac{\theta}{1+\theta}\left(\frac{1}{1+\theta}\right)^n,\; n = 0, 1, \cdots,$$

where $\theta > 0$.

Let $S_0 = \sum_{i=1}^{N_0} X_i$ be the geometric sum. In this chapter, we denote the tail probability of the geometric sum S_0 by ψ, *i.e.*

$$\psi(x) = \Pr\{S_0 > x\} = \frac{\theta}{1+\theta}\sum_{n=1}^{\infty}\left(\frac{1}{1+\theta}\right)^n \overline{F}^{(n)}(x), \tag{5.1.2}$$

and the stop-loss premium of the geometric sum S_0 by

$$\pi_{S_0}(x) = \int_x^{\infty} \psi(y)\, dy.$$

With the notation above, we can view $\psi(x)$ and θ as the ruin probability and the relative safety loading factor in the compound Poisson risk model, respectively (see Chapter 1). Alternately, we call $\psi(x)$ ruin probability or the tail of the compound geometric distribution in this chapter.

In principle, the bounds for $\overline{G}(y)$ enable one to deduce bounds for the stop-loss premium $\pi_S(d)$. Precisely, if for $x \geq 0$,

$$L(x) \;\leq\; \overline{G}(x) \;\leq\; U(x), \tag{5.1.3}$$

then, for $x \geq 0$,

$$\int_x^{\infty} L(y)\, dy \;\leq\; \pi_G(x) \;\leq\; \int_x^{\infty} U(y)\, dy.$$

That is also one of the reasons why one is interested in bounds for the tail $\overline{G}(y)$ of the compound distribution. For example, for the class of compound distributions discussed in Chapter 4, *i.e.* the compound distribution G satisfing (4.1.1), if there exist adjustment coefficients κ_{ϕ_1} and κ_{ϕ_2}, then by Lemma 4.1, we get two-sided exponential bounds for the stop-loss premium $\pi_G(x)$, namely, for any $x \geq 0$,

$$\frac{(1-p_0)\,\delta_{\phi_1}}{\phi_1\,\kappa_{\phi_1}}\,e^{-\kappa_{\phi_1}x} \;\leq\; \pi_G(x) \;\leq\; \frac{(1-p_0)\,\theta_{\phi_2}}{\phi_2\,\kappa_{\phi_2}}\,e^{-\kappa_{\phi_2}x}. \tag{5.1.4}$$

Furthermore, we can get the bounds for the stop-loss premiums of compound negative binomial distributions with this idea. We denote the stop-loss premium of the compound negative binomial distribution by $\pi_{G_\alpha}(x)$. To get the bounds, we note that the stop-loss premium for the Erlang distribution $H_{\alpha,n}$ [see (4.2.2)] is, for $x \geq 0$, given by

$$\begin{aligned}\int_x^\infty \overline{H}_{\alpha,n}(y)\,dy &= \int_x^\infty e^{-\alpha y}\sum_{k=0}^{n-1}\frac{(\alpha y)^k}{k!}\,dy\\ &= \frac{1}{\alpha}\sum_{k=0}^{n-1}\int_x^\infty \frac{\alpha(\alpha y)^{(k+1)-1}\,e^{-\alpha y}}{\Gamma(k+1)}\,dy\\ &= \frac{1}{\alpha}\sum_{k=0}^{n-1}\overline{H}_{\alpha,k+1}(x) \;=\; \frac{1}{\alpha}\sum_{k=1}^{n}\overline{H}_{\alpha,k}(x),\end{aligned} \tag{5.1.5}$$

thus we get the following result.

Corollary 5.1 Under the conditions and notations of Theorem 4.1, for any $x \geq 0$,

(1) If $0 < \alpha \leq 1$, then

$$\frac{\delta_{\alpha q}\,[1-(1-q)^\alpha]}{\alpha q\kappa_{\alpha q}}\,e^{-\kappa_{\alpha q}x} \;\leq\; \pi_{G_\alpha}(x) \;\leq\; \frac{\theta_q\,[1-(1-q)^\alpha]}{q\kappa_q}\,e^{-\kappa_q x}. \tag{5.1.6}$$

(2) If $\alpha \geq 1$, then

$$\frac{1}{\kappa_q}\sum_{i=1}^{m} f_i\,\overline{H}_{\kappa_q,i}(x) \;\leq\; \pi_{G_\alpha}(x) \;\leq\; \frac{1}{\kappa_q}\sum_{i=1}^{n} g_i\,\overline{H}_{\kappa_q,i}(x), \tag{5.1.7}$$

where $f_i = \sum_{j=i}^m \gamma_j$ and $g_i = \sum_{j=i}^n d_j$.

Proof. The proof of Corollary 5.1 follows directly from Theorem 4.1, (5.1.5) and the following formula of summation by parts, namely,

$$\sum_{j=1}^{m} a_j \sum_{i=1}^{j} b_i \;=\; \sum_{i=1}^{m} b_i \sum_{j=i}^{m} a_j \tag{5.1.8}$$

where m is a positive integer or ∞. □

However, the method above fails if L and U in (5.1.3) are not integrable. This occurs with heavy-tailed distributions and general bounds, which apply to any claim size distribution. For example, Broeckx *et al.* (1986) derived a general upper bound for $\psi(x)$, which states that for any $x > 0$

$$\psi(x) \leq \frac{\overline{F}(x) + \int_0^x y\,dF(y)/x}{\theta + \overline{F}(x) + \int_0^x y\,dF(y)/x} \stackrel{def.}{=} A_1(x). \tag{5.1.9}$$

Moreover, Willmot's (1994) Theorem 8 provides another general upper bound for $\psi(x)$, which states that for $x \geq 0$,

$$\psi(x) \leq \frac{E(S_0)}{E(S_0) + x} \stackrel{def.}{=} A_2(x). \tag{5.1.10}$$

Clearly, $A_2(x)$ is not integrable on $[0, \infty)$. It can also be seen that $A_1(x)$ is not integrable on $[0, \infty)$ since for $x \geq 1$

$$A_1(x) \geq \frac{\int_0^x y\,dF(y)}{(1+\theta)x + \int_0^x y\,dF(y)} \geq \frac{\int_0^1 y\,dF(y)}{(1+\theta)x + \int_0^1 y\,dF(y)}.$$

Hence, an upper bound for $\pi_{S_0}(x)$ does not follow from (5.1.9) and (5.1.10). As for the case of $\psi(x)$, it is useful to obtain general bounds for the stop-loss premium $\pi_{S_0}(x)$. In particular, the bounds for $\pi_{S_0}(x)$ can produce bounds for the stop-loss premiums of the class of compound distributions. The related results about the bounds for stop-loss premiums can be found in other works. For example, Runnenburg and Goovaerts (1985) discuss general upper bounds on the tails and stop-loss premiums of compound Poisson and negative binomial distributions by using Chebyshev's inequality. The bounds on stop-loss premiums of compound Poisson distributions for bounded compound distributions are derived in Steenackers and Goovaerts (1991). Sundt and Dhaene (1996) also give bounds for differences in stop-loss premiums of two compound distributions.

In this chapter, we first review the new worse than used (NWU) aging property of the compound geometric distribution, and show that the mixed geometric sums also have this aging property in Section 5.2. Then, in Section 5.3, we discuss the relations between the compound geometric distribution and its stop-loss premium. General upper and lower bounds for the stop-loss premium are derived by the aging property in Section 5.4.

As applications of these results we give bounds for the stop-loss premiums for the class of the compound distributions that satisfy (3.1.1) and (3.1.2). The stop-loss premiums for compound negative binomial distributions with large claim sizes are also derived in this section. In addition, by the technique of stochastic ordering, we give general bounds for the stop-loss premiums of the compound distributions with HNBUE and HNWUE claim sizes. In Section 5.5, we get a general upper bound for $\psi(x)$, this upper bound is sharper than $A_2(x)$ and asymptotically sharper than $A_1(x)$. The relationships among sub-exponential, NWU, new better than used (NBU) distributions and upper bounds on $\psi(x)$ and $\pi_{S_0}(x)$ are also considered in Section 5.6.

5.2 Aging properties of geometric and mixed geometric sums

The following Lemma is well-known, we state it with its proof by Brown (1990) to see how the renewal theory and the property of the geometric distribution are used in the proof.

Lemma 5.1 If N_0 is a geometric random variable with $\Pr\{N_0 = n\} = (1-q)q^n$, for $n = 0, 1, \cdots$ and $\{X_i,\ i \geq 1\}$ is a sequence of *i.i.d.* nonnegative random variables independent of N_0, then $Y = \sum_{i=1}^{N_0} X_i$ is NWU.

Proof. For $t > 0$, define $M_t = \min\{k : S_k > t\}$, where $S_k = \sum_{i=1}^{k} X_i$. Since M_t is independent of N_0, it follows from the lack of memory property of the geometric distribution that the conditional distribution of $N_0 - M_t$ given $N_0 \geq M_t$ is the same as that of N_0. Furthermore, since M_t is a stopping time, $\{X_{M_t+i},\ i \geq 1\} =^d \{X_i,\ i \geq 1\}$.

Thus, the conditional distribution of $\sum_{M_t+1}^{N_0} X_i$ given $N_0 \geq M_t$ is the same as that of $Y = \sum_{i=1}^{N_0} X_i$. Note that the events $\{Y > t\}$ and $\{N_0 \geq M_t\}$ are equivalent. Thus their indicator functions are equal, *i.e.* $I_{(Y>t)} = I_{(N_0 \geq M_t)}$. Now,

$$\begin{aligned}(Y-t)I_{(Y>t)} &= \left[\left\{\left(\sum_{i=1}^{M_t} X_i\right) - t\right\} + \sum_{i=M_t+1}^{N_0} X_i\right] I_{(Y>t)} \\ &\geq \left(\sum_{i=M_t+1}^{N_0} X_i\right) I_{(N_0 \geq M_t)}.\end{aligned}$$

Thus, for $x \geq 0$,

$$\Pr\{Y > t + x\} = \Pr\{(Y-t)\, I_{(Y>t)} > x\}$$

$$
\begin{aligned}
&\geq \Pr\left\{\left(\sum_{i=M_t+1}^{N_0} X_i\right) I_{(N_0 \geq M_t)} > x\right\} \\
&= \Pr\left\{\sum_{i=M_t+1}^{N_0} X_i > x,\, N_0 \geq M_t\right\} \\
&= \Pr\left\{\sum_{i=M_t+1}^{N_0} X_i > x | N_0 \geq M_t\right\} \Pr\{N_0 \geq M_t\} \\
&= \Pr\{Y > x\} \Pr\{N_0 > M_t\} \;=\; \Pr\{Y > x\} \Pr\{Y > t\},
\end{aligned}
$$

this implies that Y is NWU. □

The proof of Lemma 5.1 uses the lack of memory property of the geometric distribution. We know that the geometric distribution is the only discrete distribution having this property, hence one might think that Lemma 5.1 holds only for geometric sums. But, this is not true. Indeed, by the preservation of NWU under mixtures of distributions that do not cross, we show that the mixed geometric sums are also NWU.

Definition 5.2 If $\mathcal{F} = \{F_\alpha : \alpha \in \mathcal{A}\}$ is a family of probability distributions and G is a probability distribution defined on $\mathcal{A}$, then $F(t) = \int_{\mathcal{A}} F_\alpha(t)\, dG(\alpha)$ is called a mixture of probability distributions from $\mathcal{F}$.

Lemma 5.2 Suppose F is the mixture of F_α, $\alpha \in \mathcal{A}$, each F_α is NWU, with no two distinct F_{α_1}, F_{α_2} crossing on $(0, \infty)$, then F is NWU.

Proof. See Theorem 5.7 of Barlow and Proschan (1981). □

Property 5.1 Let N be a mixture of geometric random variables, *i.e.*

$$
\Pr\{N = n\} = \int_0^1 (1-q)q^n\, dB(q),\ n = 0, 1, 2, \cdots,
$$

where $0 < q < 1$ and B is a probability distribution defined on $[0,\ 1]$. If $\{X_i,\ i \geq 1\}$ is a sequence of *i.i.d.* nonnegative random variables and independent of N, then $\sum_{i=1}^{N} X_i$ is NWU.

Proof. Let

$$
\overline{H}(x) = \Pr\{\sum_{i=1}^{N} X_i > x\}
$$

and

$$
\overline{G}_q(x) = \sum_{n=1}^{\infty} (1-q)q^n\, \overline{F}^{(n)}(x)
$$

be the tail of the compound geometric distribution, where F is the distribution of X_i.

Thus,

$$\overline{H}(x) = \int_0^1 \overline{G}_q(x)\, dB(q), \tag{5.2.1}$$

hence, by Lemma 5.1 and Lemma 5.2, we need only to show that no two distinct G_{q_1} and G_{q_2} cross on $(0, \infty)$. In fact, we show that if $0 \leq q_1 < q_2 \leq 1$, then $\overline{G}_{q_1}(x) < \overline{G}_{q_2}(x)$ for $x > 0$.

Suppose that $N(x)$ is the renewal process associated with the sequence $\{X_i, i \geq 1\}$, then [c.f. (3.3.1)] we have,

$$\overline{G}_{q_i}(x) = E\left[q_i^{N(x)+1}\right], \quad \text{for } i = 1, 2. \tag{5.2.2}$$

Thus,

$$\begin{aligned}
\overline{G}_{q_2}(x) - \overline{G}_{q_1}(x) &= E\left[q_2^{N(x)+1} - q_1^{N(x)+1}\right] \\
&= (q_2 - q_1)\, E\left[\sum_{j=0}^{N(x)} q_2^{N(x)-j}\, q_1^j\right] \\
&\geq (q_2 - q_1)\, E\left[q_2^{N(x)}\right] \\
&\geq (q_2 - q_1)\, q_2^{E[N(x)]} && (5.2.3) \\
&> 0 && (5.2.4)
\end{aligned}$$

where (5.2.3) and (5.2.4) follow from Jensen's inequality and the renewal function $E[N(x)]$ being finite for any $x \geq 0$, respectively. □

Remark 5.1 Since the ruin probability $\psi(x)$ in the Poisson model is the tail of the compound geometric distribution (see Chapter 1), where $\theta > 0$ is the relative safety loading factor, thus it follows from (5.2.4) that the ruin probability is strictly decreasing in the relative safety loading factor. This fact is reasonable.

In addition, another aging property of the geometric sum is that if the distribution of X_i is DFR, then the geometric sum $\sum_{i=1}^{N_0} X_i$ is also DFR [see Shanthikumar (1988)]. For other properties of geometric sums, see Gertsbakh (1984), Johnson *et al.* (1992), Khalil *et al.* (1991) and Szekli (1995).

5.3 Relations between ruin probabilities and stop-loss premiums

In this section, we show that the stop-loss premium $\pi_{S_0}(x)$ satisfies a defective renewal equation. These results, together with Lemma 5.5 below, are used to derive general upper and lower bounds for $\psi(x)$ and $\pi_{S_0}(x)$.

Now, consider the following facts which are direct implications of the assumptions and notation of Section 5.1.

(i) $E(N_0) = 1/\theta$ and $E(N_0^2) = (2+\theta)/\theta^2$,

(ii) $E(S_0) = E(N_0)E(X) = E(X)/\theta < \infty$ and

$$E(S_0) = \int_0^\infty \Pr\{S_0 > x\}\, dx = \pi_{S_0}(0)\;. \tag{5.3.1}$$

(iii) If $E(X^2) < \infty$, then,

$$E(S_0^2) = \frac{\theta\,\mathrm{Var}(X) + (2+\theta)[E(X)]^2}{\theta^2} < \infty \tag{5.3.2}$$

and

$$\begin{aligned}\int_0^\infty \pi_{S_0}(x)\, dx &= \int_0^\infty \int_x^\infty \psi(u)\, du dx \\ &= \int_0^\infty u \Pr\{S_0 > u\}\, du = E(S_0^2)/2\;.\end{aligned} \tag{5.3.3}$$

Lemma 5.3 For any $x \geq 0$,

(a) $\pi_{S_0}(x) \;\geq\; E(S_0)\,\psi(x)$,

(b) $\int_0^x \psi(y)\, dy \;\leq\; E(S_0)\,[1-\psi(x)]$,

(c) $\int_0^x \pi_{S_0}(y)\, dy \;\leq\; E(S_0)\,[\frac{E(S_0^2)}{2E(S_0)} - \pi_{S_0}(x)]$.

Proof: (a) Since $S_0 = \sum_{i=1}^{N_0} X_i$ is a geometric sum, by Lemma 5.1 we know that S_0 is NWU $\longrightarrow$ NWUE, *i.e.* for any $x, y \geq 0$,

$$\int_x^\infty \psi(t)\, dt \;\geq\; E(S_0)\,\psi(x),$$

this implies that (a) of Lemma 5.3 holds.

(b) Since

$$\pi_{S_0}(x) \;=\; \int_x^\infty \psi(y)\, dy \;=\; E(S_0) - \int_0^x \psi(y)\, dy, \tag{5.3.4}$$

thus, by (5.3.4) and (a) above, we get

$$E(S_0) - \int_0^x \psi(y)\, dy \;\geq\; E(S_0)\,\psi(x),$$

which implies (b).

(c) It follows directly from $\int_0^x \pi_{S_0}(y)\, dy = \int_0^\infty \pi_{S_0}(y)\, dy - \int_x^\infty \pi_{S_0}(y)\, dy$, (5.3.3) and (a) above. □

Lemma 5.4 The stop-loss premium $\pi_{S_0}(x)$ satisfies the following defective renewal equation, namely, for $x \geq 0$,

$$\pi_{S_0}(x) = \frac{\pi_F(x) + E(S_0)\,\overline{F}(x)}{1+\theta} + \frac{1}{1+\theta}\int_0^x \pi_{S_0}(x-y)\,dF(y) \tag{5.3.5}$$

where $\pi_F(x) = \int_x^\infty \overline{F}(y)\,dy$ is the stop-loss transform of F.

Proof: It is known that [see (1.1.4)] ψ satisfies the following defective renewal equation

$$\psi(x) = \frac{\overline{F}(x)}{1+\theta} + \frac{1}{1+\theta}\int_0^x \psi(x-y)\,dF(y). \tag{5.3.6}$$

From this we get

$$\pi_{S_0}(t) = \int_t^\infty \psi(x)\,dx = \frac{\pi_F(t)}{1+\theta} + \frac{1}{1+\theta}\int_t^\infty \int_0^x \psi(x-y)\,dF(y)\,dx. \tag{5.3.7}$$

By Fubini's Theorem and (5.3.7), we get

$$\begin{aligned}
\pi_{S_0}(t) &= \frac{\pi_F(t)}{1+\theta} + \frac{1}{1+\theta}\int_0^t \int_t^\infty \psi(x-y)\,dx\,dF(y) \\
&\quad + \frac{1}{1+\theta}\int_t^\infty \int_y^\infty \psi(x-y)\,dx\,dF(y) \\
&= \frac{\pi_F(t)}{1+\theta} + \frac{1}{1+\theta}\int_0^t \int_{t-y}^\infty \psi(u)\,du\,dF(y) \\
&\quad + \frac{1}{1+\theta}\int_t^\infty \int_0^\infty \psi(u)\,du\,dF(y) \\
&= \frac{\pi_F(t)}{1+\theta} + \frac{1}{1+\theta}\int_0^t \pi_{S_0}(t-y)\,dF(y) + \frac{E(S_0)}{1+\theta}\overline{F}(t) \\
&= \frac{\pi_F(t)}{1+\theta} + \frac{E(S_0)\,\overline{F}(t)}{1+\theta} + \frac{1}{1+\theta}\int_0^t \pi_{S_0}(t-y)\,dF(y)
\end{aligned}$$

So (5.3.5) holds. □

Lemma 5.5 If f and g are integrable functions with different monotonicity on $[a, b]$, then

$$\int_a^b f(x)g(x)\,dx \leq \frac{1}{b-a}\int_a^b f(x)\,dx \int_a^b g(x)\,dx \tag{5.3.8}$$

Proof: It follows from the fact that $\int_a^b \int_a^b [f(x) - f(y)][g(x) - g(y)]\,dxdy \leq 0$. □

Now we are ready to derive general upper and lower bounds for the stop-loss premium $\pi_{S_0}(x)$ and the ruin probability $\psi(x)$.

5.4 General upper and lower bounds for stop-loss premiums

In this section, we suppose that $E(X^2) < \infty$, which implies that $E(S_0^2) < \infty$ by (5.3.2).

5.4.1 Bounds for the stop-loss premiums of the class of the compound distributions

Theorem 5.1 Assume that F has a decreasing density. For any $x > 0$,

$$\frac{\pi_F(x) + E(S_0)\,\overline{F}(x)}{\theta + \overline{F}(x)} \;\le\; \pi_{S_0}(x) \;\le\; \frac{\pi_F(x) + E(S_0)\,\overline{F}(x) + E(S_0^2)F(x)/(2x)}{1+\theta+E(S_0)F(x)/x} \tag{5.4.1}$$

Proof: Since $\pi_{S_0}(x-y) = \int_{x-y}^{\infty}\psi(u)\,du$ is non-decreasing in y for $0 \le y \le x$, by (5.3.5), we get

$$\pi_{S_0}(x) \;\ge\; \frac{\pi_F(x) + E(S_0)\,\overline{F}(x)}{1+\theta} + \frac{F(x)}{1+\theta}\pi_{S_0}(x), \tag{5.4.2}$$

thus, (5.4.2) implies that

$$[1 - \frac{F(x)}{1+\theta}]\,\pi_{S_0}(x) \;\ge\; \frac{\pi_F(x) + E(S_0)\,\overline{F}(x)}{1+\theta} \tag{5.4.3}$$

and (5.4.3) implies that the lower bound of (5.4.1) holds.

On the other hand, by (5.3.5), we get

$$\pi_{S_0}(x) = \frac{\pi_F(x) + E(S_0)\,\overline{F}(x)}{1+\theta} + \frac{1}{1+\theta}\int_0^x \pi_{S_0}(x-y)f(y)\,dy \tag{5.4.4}$$

where f is the density of F. Since f is non-increasing and $\pi_{S_0}(x-y)$ is non-decreasing in y, for $0 \le y \le x$, by (5.4.4) and (5.3.8), we get for $x > 0$,

$$\begin{aligned}
\pi_{S_0}(x) &\le \frac{\pi_F(x) + E(S_0)\,\overline{F}(x)}{1+\theta} + \frac{1/x}{1+\theta}\int_0^x f(y)\,dy\int_0^x \pi_{S_0}(x-y)\,dy \\
&= \frac{\pi_F(x) + E(S_0)\,\overline{F}(x)}{1+\theta} + \frac{F(x)/x}{1+\theta}\int_0^x \pi_{S_0}(u)\,du \\
&\le \frac{\pi_F(x) + E(S_0)\,\overline{F}(x)}{1+\theta} + \frac{F(x)/x}{1+\theta}\left[\frac{E(S_0^2)}{2} - E(S_0)\,\pi_{S_0}(x)\right]
\end{aligned}$$

The last inequality is obtained by (c) of Lemma 5.3. This implies that

$$\left[1 + \frac{E(S_0)F(x)/x}{1+\theta}\right]\pi_{S_0}(x) \;\le\; \frac{\pi_F(x) + E(S_0)\,\overline{F}(x) + E(S_0^2)F(x)/(2x)}{1+\theta} \tag{5.4.5}$$

which gives the upper bound in (5.4.1). □

Remark 5.2 The condition that the distribution has a decreasing density can be satisfied by many distributions. An example is the class of the equilibrium distributions F_e (see Definition 2.3) with decreasing density function $f(x) = \overline{F}(x)/\mu$, which often arises in risk analysis, reliability, queueing theory, and renewal theory. In addition, the class of DFR distributions have decreasing density functions since $f(x) = r(x)\,\overline{F}(x)$, where $r(x) \geq 0$ is the failure rate function of F, which is decreasing in this case.

Now denote by $A_{S_0}(x)$ and $B_{S_0}(x)$ the upper and lower bound in (5.4.1), respectively, *i.e.* for $x > 0$

$$A_{S_0}(x) = \frac{\pi_F(x) + E(S_0)\,\overline{F}(x) + E(S_0^2)F(x)/(2x)}{1 + \theta + E(S_0)F(x)/x}$$

and

$$B_{S_0}(x) = \frac{\pi_F(x) + E(S_0)\,\overline{F}(x)}{\theta + \overline{F}(x)}.$$

The following property implies that the lower bound $B_{S_0}(x)$ for the stop-loss premium $\pi_{S_0}(x)$ is asymptotically exact for small x and is asymptotically exact for large x if F is a sub-exponential distribution.

Property 5.2 (i) $\lim_{x \to 0} B_{S_0}(x) = \pi_{S_0}(0)$,

(ii) If F is a sub-exponential distribution, then

$$\pi_{S_0}(x) \;\sim\; B_{S_0}(x) \tag{5.4.6}$$

Proof: (i) By (5.3.1), we get

$$\begin{aligned}\lim_{x \to 0} B_{S_0}(x) &= \frac{\pi_F(0) + E(S_0)}{1 + \theta} \\ &= \frac{E(X) + E(S_0)}{1 + \theta} = \frac{\theta\, E(S_0) + E(S_0)}{1 + \theta} \\ &= E(S_0) = \pi_{S_0}(0).\end{aligned}$$

(ii) It is known [see, for example, Embrechts and Klüppelberg (1994)] that if F is a sub-exponential distribution, then

$$\psi(x) \;\sim\; E(N_0)\,\overline{F}(x) = \frac{\overline{F}(x)}{\theta}. \tag{5.4.7}$$

Then by L'Hôspital's rule, we get

$$\int_x^\infty \psi(u)\,du \;\sim\; \frac{1}{\theta}\int_x^\infty \overline{F}(u)\,du \tag{5.4.8}$$

i.e.

$$\pi_{S_0}(x) \sim \frac{\pi_F(x)}{\theta} \tag{5.4.9}$$

but, (5.4.1) implies that

$$\pi_{S_0}(x) \geq B_{S_0}(x) \geq \frac{\pi_F(x)}{\theta + \overline{F}(x)} \sim \frac{\pi_F(x)}{\theta} \tag{5.4.10}$$

thus, (5.4.9) and (5.4.10) together imply (5.4.6).

On the other hand, since $E(X^2) = 2 \int_0^\infty x\,\overline{F}(x)\,dx < \infty$ implies that

$$\lim_{x\to\infty} \int_x^\infty y\,\overline{F}(y)\,dy = 0 \tag{5.4.11}$$

which, in turn, implies that

$$\lim_{x\to\infty} x\,\pi_F(x) = 0 \tag{5.4.12}$$

since for any $x \geq 0$,

$$0 \leq x\,\pi_F(x) = x \int_x^\infty \overline{F}(y)\,dy \leq \int_x^\infty y\,\overline{F}(y)\,dy \tag{5.4.13}$$

and it follows that

$$A_{S_0}(x) \sim \frac{\pi_F(x)}{\theta} + \frac{E(S_0^2)F(x)}{2(1+\theta)x}. \tag{5.4.14}$$

Together, (5.4.14) and (5.4.9) imply that if F is sub-exponential, the upper bound $A_{S_0}(x)$ of $\pi_{S_0}(x)$ is asymptotically equal to $\pi_{S_0}(x)$ plus an error term tending to 0 for large x. □

Applying Theorem 5.1, (3.1.9) and (3.1.10) to the class of the compound distributions satisfying (3.1.1) and (3.1.2), we get immediately the following result.

Corollary 5.2 **(1)** Suppose the compound distribution G in (1.1.1) satisfies (3.1.1), then its stop-loss premium $\pi_G(x)$ satisfies for any $x > 0$,

$$\pi_G(x) \leq \frac{(1-p_0)\left[\pi_F(x) + \mu_1\,\overline{F}(x) + \mu_2\,F(x)/(2x)\right]}{1 + \phi\,\mu_1\,F(x)/x}. \tag{5.4.15}$$

(2) Suppose the compound distribution G in (1.1.1) satisfies (3.1.2), then its stop-loss premium $\pi_G(x)$ satisfies for any $x \geq 0$,

$$\pi_G(x) \geq \frac{(1-p_0)\left[\pi_F(x) + \mu_1\,\overline{F}(x)\right]}{1 - \phi + \phi\,\overline{F}(x)}. \tag{5.4.16}$$

where $\mu_1 = \phi\,\mu/(1-\phi)$, $\mu_2 = \phi\,[\,(1-\phi)\,\sigma^2 + (1+\phi)\,\mu^2\,]/(1-\phi)^2$, μ and σ^2 are the mean and variance of F, respectively.

Proof. Take $1/(1+\theta) = \phi$ in (3.1.11) or $\theta = (1-\phi)/\phi$, by (5.3.1) and (5.3.2), we get

$$E(S_0) = \frac{\mu}{\theta} = \frac{\phi\,\mu}{1-\phi} = \mu_1$$

and

$$\begin{aligned} E(S_0^2) &= \frac{\theta\,\sigma^2 + (2+\theta)\,\mu^2}{\theta^2} \\ &= \frac{(1-\phi)\,\phi^{-1}\,\sigma^2 + [\,2 + (1-\phi)\,\phi^{-1}\,]\,\mu^2}{(1-\phi)^2\,\phi^{-2}} \\ &= \mu_2, \end{aligned}$$

hence, by (3.1.9) and Theorem 5.1, we have

$$\begin{aligned} \pi_G(x) &\le \frac{1-p_0}{\phi}\int_x^\infty \psi^*(y)\,dy \\ &\le \left\{\frac{1-p_0}{\phi}\right\}\frac{\pi_F(x) + E(S_0)\,\overline{F}(x) + E(S_0^2)F(x)/(2x)}{1+\theta+E(S_0)F(x)/x} \\ &= \frac{(1-p_0)\,[\,\pi_F(x) + \mu_1\,\overline{F}(x) + \mu_2\,F(x)/(2x)\,]}{1+\phi\,\mu_1\,F(x)/x}. \end{aligned}$$

Similarly, we get (5.4.16). □

5.4.2 Bounds for the stop-loss premiums of compound negative binomial distributions with large claim sizes

For the stop-loss premiums of compound negative binomial distributions without the adjustment coefficients κ_q, $\kappa_{\lambda/n}$ and $\kappa_{\lambda/2n}$, we can get two-sided bounds using the results of Chapter 4 and Dickson's condition.

Corollary 5.3 Under the conditions and notations of Theorem 4.3, for given $t > 0$, if $0 < \alpha \le 1$, then for any $0 \le x \le t$,

$$\pi_{G_\alpha}(x) \le \frac{\alpha q \mu}{1-q} - x\,\overline{H}(t) - \frac{l_\alpha^*(t)\,H(t)}{\kappa_{\alpha q}(t)}\left[1 - e^{-\kappa_{\alpha q}(t)\,x}\right] \tag{5.4.17}$$

and

$$\pi_{G_\alpha}(x) \ge \frac{\alpha q \mu}{1-q} - x\,\overline{H}(t) - \frac{u_\alpha(t)\,H(t)}{\kappa_q(t)}\left[1 - e^{-\kappa_q(t)\,x}\right], \tag{5.4.18}$$

in particular, for any $x > 0$,

$$\pi_{G_\alpha}(x) \le \frac{\alpha q \mu}{1-q} - x\,\overline{H}(x) - \frac{l_\alpha^*(x)\,H(x)}{\kappa_{\alpha q}(x)}\left[1 - e^{-\kappa_{\alpha q}(x)\,x}\right] \tag{5.4.19}$$

and

$$\pi_{G_\alpha}(x) \;\geq\; \frac{\alpha q\mu}{1-q} - x\,\overline{H}(x) - \frac{u_\alpha(x)\,H(x)}{\kappa_q(x)}\left[1-e^{-\kappa_q(x)\,x}\right]. \tag{5.4.20}$$

Proof. Let S_α be the negative binomial sum, *i.e.* $S_\alpha = \sum_{i=1}^{N_\alpha} X_i$, where N_α is the negative random variable independent of the sequence $\{X_i,\ i \geq 1\}$ with

$$\Pr\{N_\alpha = n\} = \binom{n+\alpha-1}{n}(1-q)^\alpha\,q^n,\quad n = 0, 1, 2, \cdots.$$

Since

$$E(S_\alpha) = E(N_\alpha)\,E(X_i) = \frac{\alpha q\,\mu}{1-q}$$

and

$$\pi_{G_\alpha}(x) = \int_x^\infty \overline{G}_\alpha(y)\,dy = E(S_\alpha) - \int_0^x \overline{G}_\alpha(y)\,dy, \tag{5.4.21}$$

(5.4.17) and (5.4.18) follow from (4.5.4) and (4.5.5). Take $x = t$ in (5.4.17) and (5.4.18), we get (5.4.19) and (5.4.20), respectively. □

Corollary 5.4 Under the conditions and notations of Theorem 4.4, for given $t > 0$, if $\alpha \geq 1$, then for any $0 \leq x \leq t$,

$$\begin{aligned}\pi_{G_\alpha}(x) \;&\leq\; \frac{\alpha q\mu}{1-q} - x\,\overline{H}(t) - \frac{H(t)\,\sum_{j=1}^m j\gamma_j(t)}{\kappa_q(t)} \\ &\quad + \frac{H(t)}{\kappa_q(t)}\sum_{j=1}^m f_j(t)\,\overline{H}_{\kappa_q(t),j}(x)\end{aligned} \tag{5.4.22}$$

and

$$\begin{aligned}\pi_{G_\alpha}(x) \;&\geq\; \frac{\alpha q\mu}{1-q} - x\,\overline{H}(t) - \frac{H(t)\,\sum_{j=1}^n j d_j(t)}{\kappa_q(t)} \\ &\quad + \frac{H(t)}{\kappa_q(t)}\sum_{j=1}^n g_j(t)\,\overline{H}_{\kappa_q(t),j}(x),\end{aligned} \tag{5.4.23}$$

in particular, for any $x > 0$,

$$\begin{aligned}\pi_{G_\alpha}(x) \;&\leq\; \frac{\alpha q\mu}{1-q} - x\,\overline{H}(x) - \frac{H(x)\,\sum_{j=1}^m j\gamma_j(x)}{\kappa_q(x)} \\ &\quad + \frac{H(x)}{\kappa_q(x)}\sum_{j=1}^m f_j(x)\,\overline{H}_{\kappa_q(x),j}(x)\end{aligned} \tag{5.4.24}$$

and

$$\pi_{G_\alpha}(x) \geq \frac{\alpha q\mu}{1-q} - x\,\overline{H}(x) - \frac{H(x)\sum_{j=1}^{n} j d_j(x)}{\kappa_q(x)} + \frac{H(x)}{\kappa_q(x)}\sum_{j=1}^{n} g_j(x)\,\overline{H}_{\kappa_q(x),j}(x), \tag{5.4.25}$$

where $f_j(t) = \sum_{i=j}^{m} \gamma_i(t)$ and $g_j(t) = \sum_{i=j}^{n} d_i(t)$.

Proof. Since the mean of the Erlang distribution $H_{\alpha,n}$ is n/α, by (5.1.5), we get for $x \geq 0$,

$$\begin{aligned}\int_0^x \overline{H}_{\alpha,n}(y)\,dy &= \frac{n}{\alpha} - \int_x^\infty \overline{H}_{\alpha,n}(y)\,dy \\ &= \frac{n}{\alpha} - \frac{1}{\alpha}\sum_{k=1}^{n} \overline{H}_{\alpha,k}(x),\end{aligned}$$

thus, by (5.4.21) and Theorem 4.4, we get for $0 \leq x \leq t$,

$$\begin{aligned}\pi_{G_\alpha}(x) &\leq \frac{\alpha q\mu}{1-q} - x\,\overline{H}(t) - H(t)\sum_{j=1}^{m}\gamma_j(t)\int_0^x \overline{H}_{\kappa_q(t),j}(y)\,dy \\ &= \frac{\alpha q\mu}{1-q} - x\,\overline{H}(t) - H(t)\sum_{j=1}^{m}\gamma_j(t)\left\{\frac{j}{\kappa_q(t)} - \frac{1}{\kappa_q(t)}\sum_{i=1}^{j}\overline{H}_{\kappa_q(t),i}(x)\right\},\end{aligned}$$

this implies (5.4.22) by (5.1.8).

The proof of (5.4.23) is similar. Take $x = t$ in (5.4.22) and (5.4.23), we get (5.4.24) and (5.4.25), respectively. □

In addition, by the stochastic ordering technique, we can give general bounds for the stop-loss premium of the compound distribution G in (1.1.1) with HNBUE and HNWUE claim sizes.

Theorem 5.2 Suppose F is HNBUE (HNWUE) with the finite mean $1/\alpha$. Then the stop-loss premium $\pi_G(x)$ in (5.1.1) satisfies that for any $x \geq 0$,

$$\pi_G(x) \leq (\geq)\ \frac{1}{\alpha}\sum_{n=1}^{\infty} a_{n-1}\,\overline{H}_{\alpha,n}(x) \tag{5.4.26}$$

where $a_n = \sum_{k=n+1}^{\infty} p_k$ and $\overline{H}_{\alpha,n}(x)$ is the tail of the Erlang distribution defined in (4.2.2).

Proof. By the definition of HNBUE, we get for $x \geq 0$,

$$\int_x^\infty \overline{F}(y)\,dy \leq \int_x^\infty e^{-\alpha y}\,dy, \tag{5.4.27}$$

this means

$$X <_{sl} Y,$$

where X and Y have distribution F and exponential distribution with the mean $1/\alpha$, respectively.

Suppose $\{X_i,\, i \geq 1\}$ and $\{Y_i,\, i \geq 1\}$ are two sequences of *i.i.d.* nonnegative random variables independent of N with the same distribution as X and Y, respectively, then,

$$X_i <_{sl} Y_i,\ i = 1, 2, \cdots,$$

thus, by Property 4.3, we get

$$\sum_{i=1}^{N} X_i <_{sl} \sum_{i=1}^{N} Y_i,$$

this implies that for $x \geq 0$,

$$\pi_G(x) = \int_x^{\infty} \Pr\{\sum_{i=1}^{N} X_i > y\}\, dy \leq \int_x^{\infty} \Pr\{\sum_{i=1}^{N} Y_i > y\}\, dy \tag{5.4.28}$$

however,

$$\Pr\{\sum_{i=1}^{N} Y_i > y\} = \sum_{n=1}^{\infty} p_n \overline{H}_{\alpha,n}(y)$$

thus, by (5.1.5), we get

$$\pi_G(x) \leq \sum_{n=1}^{\infty} p_n \left\{\frac{1}{\alpha} \sum_{k=1}^{n} \overline{H}_{\alpha,k}(x)\right\} = \frac{1}{\alpha} \sum_{n=1}^{\infty} \overline{H}_{\alpha,k}(x) \left\{\sum_{k=n}^{\infty} p_k\right\}$$

the last equality follows from the summation by parts [see (5.1.8)].

Reversing the inequalities in the proof above, we get the bound for the HNWUE case. □

With a method similar to that used in the proof of Theorem 5.2, we can derive bounds for the ruin probability in the Poisson model, see, for example, Cai and Wu (1997b).

5.5 General upper and lower bounds for ruin probabilities

A method similar to that used in Section 5.4 for the stop-loss premium $\pi_{S_0}(x)$, allows us to derive here a general upper bound for the ruin probability $\psi(x)$. This upper bound is sharper than $A_2(x)$ and asymptotically sharper than $A_1(x)$. De Vylder and Goovaerts's (1984) general lower bound for $\psi(x)$ is also re-derived using two simple, different methods.

Theorem 5.3 Assume that F has a decreasing density . For any $x > 0$,

$$\frac{\overline{F}(x)}{\theta+\overline{F}(x)} \;\le\; \psi(x) \;\le\; \frac{\overline{F}(x)+E(S_0)F(x)/x}{1+\theta+E(S_0)F(x)/x}. \tag{5.5.1}$$

Proof: Since $\psi(x-y) = \Pr\{S_0 > x-y\}$ is non-decreasing in y for $0 \le y \le x$, by (5.3.6), we get

$$\psi(x) \;\ge\; \frac{\overline{F}(x)}{1+\theta} + \frac{\psi(x)}{1+\theta}F(x) \tag{5.5.2}$$

which is equivalent to the following inequality

$$[1-\frac{1}{1+\theta}F(x)]\,\psi(x) \;\ge\; \frac{\overline{F}(x)}{1+\theta} \tag{5.5.3}$$

and the lower bound in (5.5.1) holds.

On the other hand, by (5.3.6) and (5.3.8), for any $x > 0$, we have

$$\begin{aligned}\psi(x) \;&\le\; \frac{\overline{F}(x)}{1+\theta} + \frac{1/x}{1+\theta}\int_0^x \psi(x-y)\,dy\int_0^x F'(y)\,dy \\ &=\; \frac{\overline{F}(x)}{1+\theta} + \frac{F(x)/x}{1+\theta}\int_0^x \psi(u)\,du \\ &\le\; \frac{\overline{F}(x)}{1+\theta} + \frac{F(x)/x}{1+\theta}E(S_0)[1-\psi(x)]\end{aligned}$$

where the last inequality follows from (b) of Lemma 5.3. Thus, we get

$$\psi(x)\left[1+\frac{E(S_0)F(x)/x}{1+\theta}\right] \;\le\; \frac{\overline{F}(x)+E(S_0)F(x)/x}{1+\theta}$$

and the upper bound in (5.5.1) also holds. □

Remark 5.3 The lower bound in (5.5.1) was derived by De Vylder and Goovaerts (1984) by considering the error function of an approximation for $\psi(x)$. As pointed out by them, this lower bound is asymptotically exact for $\psi(x)$ as $x \to \infty$ if F is sub-exponential. It is also asymptotically exact for $\psi(x)$ as $x \to 0$ since

$$\lim_{x\to 0} \frac{\overline{F}(x)}{\theta + \overline{F}(x)} = \frac{1}{1+\theta} = \psi(0).$$

It should be noted that this lower bound can also be derived by the following simple probability inequality, namely,

$$\overline{F^{(n)}}(x) \;\geq\; 1 - [F(x)]^n. \tag{5.5.4}$$

By (5.1.2) and (5.5.4), we hence get

$$\psi(x) \;\geq\; \frac{\theta}{1+\theta}\sum_{n=1}^{\infty}\left(\frac{1}{1+\theta}\right)^n \{1-[F(x)]^n\} \;=\; \frac{\overline{F}(x)}{\theta+\overline{F}(x)}. \tag{5.5.5}$$

This idea indicates clearly why the lower bound of (5.5.1) is asymptotically exact for $\psi(x)$ as $x \to \infty$ if F is sub-exponential, because then $\overline{F^{(n)}}(x) \sim 1-[F(x)]^n$. This method can also be used to derive general lower bounds for any compound distribution and get asymptotically exact lower bounds for the tail of the compound distribution, as $x \to \infty$, if the compound distribution is sub-exponential. For example, assume that $\{W_i,\, i \geq 1\}$ are *i.i.d.* random variables with common distribution H and are independent of a counting random variable M. Denote by $Z = \sum_{i=1}^{M} W_i$, then if M has a Poisson distribution with $\alpha > 0$, we have

$$\Pr\{Z > x\} \geq 1 - e^{-\alpha \overline{H}(x)} \tag{5.5.6}$$

while if M has a negative binomial distribution with

$$\Pr\{M = k\} = \binom{k+\alpha-1}{k}\left(\frac{\beta}{1+\beta}\right)^{\alpha}\left(\frac{1}{1+\beta}\right)^{k},$$

for $k = 0, 1, \cdots$ and $\alpha > 0$, $\beta > 0$, then

$$\Pr\{Z > x\} \geq 1 - \left[\frac{\beta}{\beta+\overline{H}(x)}\right]^{\alpha} \tag{5.5.7}$$

However, the improved bound (5.5.7) has been obtained in Theorem 4.3 and 4.4.

Now we discuss the properties of the upper bound in (5.5.1), namely

$$A(x) \;=\; \frac{\overline{F}(x) + E(S_0)F(x)/x}{1+\theta+E(S_0)F(x)/x}$$

Then the following property shows that the upper bound $A(x)$ of $\psi(x)$ is sharper than $A_2(x)$ in (5.1.10) and is asymptotically sharper than $A_1(x)$ in (5.1.9).

Property 5.3 (i) For any $x > 0$,

$$A(x) \leq \frac{E(S_0)}{E(S_0)+x} = A_2(x) \tag{5.5.8}$$

(ii) If $E(X^2) < \infty$, then

$$A(x) \sim \frac{1}{1+\theta} A_1(x). \tag{5.5.9}$$

Proof: (i) By Markov's inequality and (5.3.1), we get that for any $x > 0$,

$$\overline{F}(x) = \Pr\{X > x\} \leq E(X)/x = \theta E(S_0)/x \tag{5.5.10}$$

which implies that

$$x\overline{F}(x) \leq \theta E(S_0). \tag{5.5.11}$$

Using $F(x) + \overline{F}(x) = 1$ and (5.5.11), we have

$$E(S_0)\overline{F}(x) + x\overline{F}(x) + E(S_0)F(x) \leq (1+\theta)E(S_0). \tag{5.5.12}$$

This implies that $A(x) \leq A_2(x)$.

(ii) Since $E(X^2) < \infty$ implies that $x\overline{F}(x) \to 0$ as $x \to \infty$, then

$$\begin{aligned} \lim_{x\to\infty} \frac{A(x)}{A_1(x)} &= \lim_{x\to\infty} \frac{\overline{F}(x) + E(S_0)F(x)/x}{\overline{F}(x) + x^{-1}\int_0^x y\,dF(y)} \left(\frac{\theta}{1+\theta}\right) \\ &= \left(\frac{\theta}{1+\theta}\right) \frac{E(S_0)}{E(X)} = \frac{1}{1+\theta}. \end{aligned}$$

Also, it follows that

$$A(x) \sim \frac{\overline{F}(x)}{\theta} + \frac{E(S_0)F(x)}{(1+\theta)x} \tag{5.5.13}$$

which implies that the upper bound $A(x)$ is asymptotically equal to $\psi(x)$ plus an error term tending to 0 for large x when F is sub-exponential. □

5.6 Relations between upper bounds, NWU, NBU and sub-exponential distributions

In Sections 5.4 and 5.5, we derived general upper and lower bounds for the stop-loss premium $\pi_{S_0}(x)$ and the ruin probability $\psi(x)$, and also discussed the asymptotic behaviour of the bounds in the case where F is sub-exponential.

As shown in Sections 5.3, 5.4 and 5.5, the NWU aging property of $\psi(x)$ plays an important role in deriving those bounds. Interestingly, $\psi(x)$ can not hold a new better than used (NBU) upper bound if F is sub-exponential, which generalizes De Vylder and Goovaerts's (1984) assertion that no exponential upper bound exists for $\psi(x)$ if F is sub-exponential, since exponential distributions are NBU distributions. At the same time, we can also claim that no exponential upper bound exists for $\pi_{S_0}(x)$ if F is sub-exponential. These assertions are summarized in the following results.

Theorem 5.4 If F is sub-exponential, then the stop-loss premium $\pi_{S_0}(x)$ does not admit an exponential upper bound, *i.e.* no constant $r > 0$ and bounded function c are such that for all $x \geq 0$

$$\pi_{S_0}(x) \leq c(x)\, e^{-rx}. \tag{5.6.1}$$

Proof: If (5.6.1) were true, then by (a) of Lemma 5.3, we would get

$$E(S_0)\, \psi(x) \leq c(x)\, e^{-rx} \tag{5.6.2}$$

which would imply that $\psi(x) \leq c\, e^{-rx}$, where $c = \sup_{x \geq 0} c(x)/E(S_0)$, a contradiction to De Vylder and Goovaerts's (1984) result. □

Corollary 5.5 If F is sub-exponential, then $\psi(x)$ can not admit a NBU upper bound, *i.e.* no NBU distribution function B with finite mean and bounded function c exist, such that for all $x \geq 0$

$$\psi(x) \leq c(x)\, \overline{B}(x) \tag{5.6.3}$$

Proof: If (5.6.3) holds, then we have

$$\pi_{S_0}(x) = \int_x^\infty \psi(u)\, du \leq \int_x^\infty c(u)\, \overline{B}(u)\, du \leq c \int_x^\infty \overline{B}(u)\, du \tag{5.6.4}$$

where $c = \sup_{x \geq 0} c(x)$.

Since $B(x)$ is NBU $\longrightarrow$ HNBUE, for all $x \geq 0$

$$\int_x^\infty \overline{B}(y)\,dy \;\leq\; m\,e^{-\frac{x}{m}} \tag{5.6.5}$$

where $m = \int_0^\infty \overline{B}(x)\,dx$. By (5.6.5) and (5.6.4) we get

$$\pi_{S_0}(x) \;\leq\; c\,m\,e^{-\frac{x}{m}}$$

a contradiction to Theorem 5.4. □

Remark 5.4 Corollary 5.5 suggests that a reasonable choice of upper bound for the ruin probability $\psi(x)$ with the form in (5.6.3) may be a NWU distribution function B. Indeed, we have derived upper bounds for $\psi(x)$ in terms of NWU distribution functions which generalize and improve the classical inequality of Lundberg (see Chapter 2 and 3 of this thesis). Many recent works tend in this direction, see, for example, Cai and Wu (1997b), Lin (1996), Willmot (1994, 1996, 1997a, 1997b) and Willmot and Lin (1997a).

The other method for deriving bounds of ruin probabilities is to use a truncating condition about $F(x)$. Bounds based on this condition are applicable to any claim distributions and are especially useful for heavy-tailed distributions, where the adjustment coefficient does not exist. For example, Dickson (1994) and Chapter 3 of this thesis.

Chapter 6

Asymptotic Estimates for Tails of Convolutions of Compound Geometric Distributions and Diffusion Risk Models

6.1 Introduction

Let $\{X_i, i \geq 1\}$ be a sequence of *i.i.d.* non-negative random variables with common distribution function F and $F(0) = 0$. Further, let N be a geometric random variable with $\Pr\{N = n\} = qp^n$, for $n = 0, 1, 2, \dots$ and $p = 1 - q$, for $0 < q < 1$, which is independent of $\{X_i, i \geq 1\}$. $S_N = \sum_{i=1}^{N} X_i$ is said to be compound geometric, where $S_N = 0$ if $N = 0$. Its distribution function is denoted by $H(x) = \Pr\{S_N \leq x\}$.

Suppose that Y is another non-negative random variable with distribution G and $G(0) = 0$, where Y, N and $\{X_i, i \geq 1\}$ are independent. Then, the convolution $H * G$ of the compound geometric distribution H and distribution G, *i.e.* the distribution of $S_N + Y$, arises in many applied probability models, such as regenerative processes [Cohen (1976), Kalashnikov (1994) and Keilson (1966)], insurance risk analysis [Dufresne and Gerber (1991), Sundt and Teugels (1995) and Veraverbeke (1993)] and queueing theory [Asmussen (1987), van Hoorn (1984) and Szekli (1986, 1995)]. Many distributions of interest in these works can be expressed in the form $G * H$ of the distribution of $S_N + Y$.

For $W(x) = G * H(x)$, consider $\overline{W}(x) = 1 - W(x)$, the tail or survival function of $W(x)$. It is well-known (Rényi's theorem) that if F has a finite mean, then for any

$x \geq 0$, $\overline{H}(x)/\int_0^\infty \overline{H}(y)\,dy \to e^{-x}$ as $q \to 0$ or equivalently $E(N) \to \infty$. Furthermore, Keilson (1966) [also see Kalashnikov (1994)] showed that if F has a finite second moment and G has a finite mean, then a similar limit theorem holds for any $\overline{W}(x)$, namely for any $x \geq 0$, $\overline{W}(x)/\int_0^\infty \overline{W}(y)\,dy \to e^{-x}$ as $E(N) \to \infty$. However, in many applications, we are interested in the large deviation probability of $S_N + Y$, or the asymptotic form for $\overline{W}(x)$ as $x \to \infty$.

The asymptotic behavior of the tail $\overline{H}(x)$ of the compound geometric distribution function is well-known. The purpose of this Chapter is to give a more complete description of the asymptotic behavior for $\overline{W}(x)$, obtain the asymptotic estimates for $\overline{W}(x)$ under various situations, and consider the applications of these results in diffusion risk models.

This chapter is organized as follows: in Section 6.2, we derive an exponential asymptotic form for $\overline{W}(x)$ in terms of Lundberg's coefficient using Property 2.1.

In Section 6.3, we consider subexponential asymptotic forms for $\overline{W}(x)$. Here, general lower and upper bounds for $\overline{W}(x)$ are given first, the bounds indicate the possible asymptotic form for $\overline{W}(x)$ and are also used to determine a closer approximation for $\overline{W}(x)$ in some cases.

In Section 6.4, we discuss asymptotic forms for $\overline{W}(x)$ in the intermediate case, *i.e.* the distribution has exponential moments but Lundberg's coefficient does not exist.

In Section 6.5, as an application of the results for $\overline{W}(x)$, we consider the ruin probability in the diffusion risk model. The asymptotic estimates of the ruin probability derived by Dufresne and Gerber (1991), Gerber (1970) and Veraverbeke (1993) are easily obtained. A theorem of Veraverbeke (1993) is also generalized. In addition, two-sided bounds for the ruin probability with large claim sizes are given, thus Dickson's (1994) bound is extented to the diffusion risk model.

6.2 Asymptotic estimates with light-tailed distributions

In general, $\overline{W}(x)$ does not admit an exponential asymptotic form, for example, see Remark 6.5 of this Chapter. But if conditions similar to those in Cramér-Lundberg's asymptotic formula hold, then an asymptotic exponential form exists as given in the following theorem.

Theorem 6.1 Suppose that F is non-lattice and there exists a constant κ such that

$$\int_0^\infty e^{\kappa x}\, dF(x) = 1/p \tag{6.2.1}$$

and $m_G(\kappa) = \int_0^\infty e^{\kappa x}\, dG(x) < \infty$. If $\beta = \int_0^\infty x e^{\kappa x}\, dF(x) < \infty$, then

$$\overline{W}(x) \sim \frac{q\, m_G(\kappa)}{p\kappa\beta} e^{-\kappa x} \tag{6.2.2}$$

and if $\beta = \infty$, then

$$\overline{W}(x) = o(e^{-\kappa x}). \tag{6.2.3}$$

Proof. Since H is the compound geometric distribution, for any $x \geq 0$,

$$W(x) = G * H(x) = \sum_{n=0}^{\infty} qp^n\, G * F^{(n)}(x). \tag{6.2.4}$$

Thus by (6.2.4), we get that for any $x \geq 0$,

$$\begin{aligned}\int_0^x W(x-y)\, dF(y) &= \sum_{n=0}^{\infty} qp^n \int_0^x G * F^{(n)}(x-y)\, dF(y) \\ &= \sum_{n=0}^{\infty} qp^n\, G * F^{(n+1)}(x) = \frac{1}{p}\,[W(x) - qG(x)].\end{aligned}$$

This implies that W satisfies the following defective renewal equation, *i.e.* for any $x \geq 0$,

$$W(x) = q\, G(x) + p \int_0^x W(x-y)\, dF(y), \tag{6.2.5}$$

or equivalently,

$$\overline{W}(x) = q\, \overline{G}(x) + p\, \overline{F}(x) + p \int_0^x \overline{W}(x-y)\, dF(y). \tag{6.2.6}$$

Hence, (6.2.2) and (6.2.3) follow directly from Property 2.1, (6.2.6) and $\int_0^\infty e^{\kappa x}\, dG(x) = 1 + \kappa \int_0^\infty e^{\kappa x}\, \overline{G}(x)\, dx$. □

6.3 Asymptotic estimates with heavy-tailed distributions

In this Section, we consider the case when F or G are heavy-tailed, in particular, subexponential distributions. First, the following two theorems give general lower and upper bounds for $\overline{W}(x)$, which indicate the possible asymptotic forms used to determine, in some cases, a closer approximation for $\overline{W}(x)$.

Theorem 6.2 For any $x \geq 0$,

$$\overline{W}(x) \;\geq\; \frac{p\,\overline{F}(x) + q\,\overline{G}(x)}{p\,\overline{F}(x) + q}. \qquad (6.3.1)$$

Proof. Since $\overline{W}(x) = \Pr\{S_N + Y > x\}$ is the survival function of the random variable $S_N + Y$, $\overline{W}(x)$ is decreasing. By (6.2.6), we get that for any $x \geq 0$,

$$\begin{aligned}\overline{W}(x) \;&\geq\; q\,\overline{G}(x) + p\,\overline{F}(x) + p\,\overline{W}(x) \int_0^x dF(y) \\ &=\; q\,\overline{G}(x) + p\,\overline{F}(x) + p\,\overline{W}(x) F(x),\end{aligned}$$

this implies that (6.3.1) holds. □

Remark 6.1 Take $Y = 0$ in Theorem 6.2, we get a lower bound for the tail $\overline{H}(x)$ of the compound geometric distribution function H, namely, for any $x \geq 0$,

$$\overline{H}(x) \;\geq\; \frac{p\,\overline{F}(x)}{p\,\overline{F}(x) + q}. \qquad (6.3.2)$$

This gives a derivation of the lower bound in Theorem 5.3, which is a result known in the form of the ruin probability in the classical risk process [see for example Theorem 3.1 of De Vylder and Goovaerts (1984)].

Theorem 6.3 If F has a finite mean $E(X_1)$ and G has a decreasing density function, then for any $x > 0$,

$$\overline{W}(x) \;\leq\; \frac{p\,\overline{F}(x) + q\,\overline{G}(x) + \delta(x)}{p\,\overline{F}(x) + q}, \qquad (6.3.3)$$

where $\delta(x) = pG(x)\{x^{-1}E(X_1) - \overline{F}(x)\} \to 0$ as $x \to \infty$.

Proof. Since $W(x) = G * H(x) = \Pr\{Y + S_N \leq x\}$, by conditioning on the value of Y, we get that for any $x \geq 0$,

$$\overline{W}(x) \;=\; \overline{G}(x) + \int_0^x \overline{H}(x-y)\, dG(y) \qquad (6.3.4)$$

$$\;=\; \overline{G}(x) + \int_0^x \overline{H}(x-y) G'(y)\, dy. \qquad (6.3.5)$$

Thus, by (6.3.5), Lemma 5.5 and the fact that $\overline{H}(x-y)$ is increasing in y over $[0, x]$, we get that for any $x > 0$,

$$\overline{W}(x) \;\leq\; \overline{G}(x) + \frac{1}{x}\int_0^x \overline{H}(x-y)\, dy \int_0^x G'(y)\, dy \qquad (6.3.6)$$

$$\begin{aligned}&=\; \overline{G}(x) + \frac{G(x)}{x} \int_0^x \overline{H}(t)\, dt \\ &=\; \overline{G}(x) + \frac{G(x)}{x} \left[E(S_N) - \int_x^\infty \overline{H}(t)\, dt\right].\end{aligned} \qquad (6.3.7)$$

Since S_N is a geometric sum, the distribution H of S_N is NWU $\longrightarrow$ NWUE, *i.e.* for any $x \geq 0$,

$$\int_x^\infty \overline{H}(t)\,dt \;\geq\; E(S_N)\,\overline{H}(x). \tag{6.3.8}$$

Thus, by (6.3.7), (6.3.8) and (6.3.2), we get that for any $x > 0$,

$$\begin{aligned} \overline{W}(x) &\leq \overline{G}(x) + x^{-1}G(x)E(S_N)[1 - \overline{H}(x)] \\ &\leq \overline{G}(x) + x^{-1}G(x)E(S_N)\left[1 - \frac{p\,\overline{F}(x)}{p\,\overline{F}(x) + q}\right] \\ &= \overline{G}(x) + \frac{x^{-1}G(x)E(S_N)q}{p\,\overline{F}(x) + q}. \end{aligned} \tag{6.3.9}$$

But $E(S_N) = E(\sum_{i=1}^N X_i) = E(N)E(X_i) = pE(X_1)/q$, so $qE(S_N) = pE(X_1)$, this, together with (6.3.9), implies that (6.3.3) holds. □

Remark 6.2 Combining Theorems 6.2 and 6.3, we get that under the conditions and notation of Theorem 6.3, for any $x > 0$,

$$\frac{p\,\overline{F}(x) + q\,\overline{G}(x)}{p\,\overline{F}(x) + q} \;\leq\; \overline{W}(x) \;\leq\; \frac{p\,\overline{F}(x) + q\,\overline{G}(x) + \delta(x)}{p\,\overline{F}(x) + q}. \tag{6.3.10}$$

Since $\delta(x) \to 0$ as $x \to \infty$, (6.3.10) indicates that

$$\frac{p\,\overline{F}(x) + q\,\overline{G}(x)}{p\,\overline{F}(x) + q}$$

may be an asymptotic form for $\overline{W}(x)$ as $x \to \infty$. Indeed, below we will show that under various situations, this is precisely the large deviation probability of $S_N + Y$, *i.e.* the asymptotic form of $\overline{W}(x)$ as $x \to \infty$. In addition, for the distributions with decreasing densities, see Remark 5.2.

Corollary 6.1 Under the conditions of Theorem 6.3, if $x\,\overline{G}(x) \to \infty$ as $x \to \infty$, then

$$\overline{W}(x) \;\sim\; \frac{p\,\overline{F}(x) + q\,\overline{G}(x)}{p\,\overline{F}(x) + q} \;\sim\; \overline{G}(x). \tag{6.3.11}$$

Proof. By (6.3.10), we get for any $x > 0$,

$$1 \;\leq\; \overline{W}(x)\frac{p\,\overline{F}(x) + q}{p\,\overline{F}(x) + q\,\overline{G}(x)} \;\leq\; 1 + \frac{pG(x)\{E(X_1) - x\,\overline{F}(x)\}}{px\,\overline{F}(x) + qx\,\overline{G}(x)}. \tag{6.3.12}$$

Since F has a finite mean, $x\overline{F}(x) \to 0$ as $x \to \infty$. Thus, (6.3.12) and $x\overline{G}(x) \to \infty$ as $x \to \infty$ imply that

$$\overline{W}(x) \sim \frac{p\overline{F}(x) + q\overline{G}(x)}{p\overline{F}(x) + q}.$$

Also,

$$\frac{\overline{F}(x)}{\overline{G}(x)} = \frac{x\overline{F}(x)}{x\overline{G}(x)} \to 0 \text{ as } x \to \infty,$$

implies that

$$\frac{p\overline{F}(x) + q\overline{G}(x)}{p\overline{F}(x) + q} \sim \overline{G}(x),$$

hence, (6.3.11) holds. □

Remark 6.3 There exist classes of distributions that satisfy the conditions of Corollary 6.1, for example the Pareto distribution with density function

$$g(x) = \frac{\alpha c^{\alpha}}{(c+x)^{1+\alpha}}, \quad x \geq 0, \quad \text{where } 0 < \alpha < 1,\ c > 0,$$

and Burr distribution with distribution function

$$F(x) = 1 - \left(\frac{\lambda}{\lambda + x^{\tau}}\right)^{\alpha}, \quad x \geq 0, \quad \text{where } \lambda > 0,\ 0 < \alpha \leq 1,\ 0 < \tau \leq 1.$$

But, it should be pointed out that the condition $x\overline{G}(x) \to \infty$ as $x \to \infty$ is restrictive, since it implies that G has no finite mean. On the other hand, under the conditions of Corollary 6.1, $\{p\overline{F}(x) + q\overline{G}(x)\}/\{p\overline{F}(x) + q\}$ gives a closer approximation to $\overline{W}(x)$ than $\overline{G}(x)$ does, as seen by Theorem 6.2 and the fact that for any $x \geq 0$,

$$\frac{p\overline{F}(x) + q\overline{G}(x)}{p\overline{F}(x) + q} \geq \overline{G}(x).$$

Furthermore, we notice that no relation between F and G is required in Corollary 6.1. If some relation between them is assumed and subexponentiality (as defined below) is further imposed on F or G, we can derive additional results for $\overline{W}(x)$.

Definition 6.1 A distribution B on $[0, \infty)$ is said to be subexponential, denoted by $B \in \mathcal{S}$, if $\overline{B^{(2)}}(x) \sim 2\overline{B}(x)$.

Subexponential distributions are heavy tailed; typical examples are the Pareto and Lognormal distributions. The following Lemma is a combination of Proposition 1 of Embrechts *et al.* (1979) and Theorem 2 of Chistyakov (1964).

Lemma 6.1 Suppose that F_1 and F_2 are two distributions on $[0, \infty)$.

(i) If $F_2 \in \mathcal{S}$ and $\overline{F_1}(x) = o(\overline{F_2}(x))$, then $F_1 * F_2 \in \mathcal{S}$ and $\overline{F_1 * F_2}(x) \sim \overline{F_2}(x)$.

(ii) If $F_1 * F_2 \in \mathcal{S}$ and $\overline{F_1}(x) = o(\overline{F_1 * F_2}(x))$, then $F_2 \in \mathcal{S}$ and $\overline{F_2}(x) \sim \overline{F_1 * F_2}(x)$.

(iii) If $F_2 \in \mathcal{S}$, then for any $\varepsilon > 0$, $e^{\varepsilon x}\overline{F_2}(x) \to \infty$ as $x \to \infty$, *i.e.* $e^{-\varepsilon x} = o(\overline{F_2}(x))$.

Theorem 6.4 Suppose that $\overline{G}(x) = o(\overline{F}(x))$. The following three assertions are equivalent:

(i) $F \in \mathcal{S}$,

(ii) $W \in \mathcal{S}$,

(iii) $\overline{W}(x) \sim \{p\overline{F}(x) + q\overline{G}(x)\}/\{p\overline{F}(x) + q\} \sim p\overline{F}(x)/q$.

Proof. First, it is clear that $\overline{G}(x) = o(\overline{F}(x))$ implies

$$\frac{p\overline{F}(x) + q\overline{G}(x)}{p\overline{F}(x) + q} \sim \frac{p\overline{F}(x)}{q}. \tag{6.3.13}$$

By Corollary 6.1 of Embrechts *et al.* (1979), we know that the following three conditions are equivalent:

(a) $F \in \mathcal{S}$, (b) $H \in \mathcal{S}$, (c) $\overline{H}(x) \sim p\overline{F}(x)/q$

Hence, if $F \in \mathcal{S}$, by (b), (c) and $\overline{G}(x) = o(\overline{F}(x))$, we get that $H \in \mathcal{S}$ and

$$\frac{\overline{G}(x)}{\overline{H}(x)} = \frac{\overline{G}(x)}{\overline{F}(x)} \times \frac{\overline{F}(x)}{\overline{H}(x)} \to 0 \text{ as } x \to \infty$$

i.e. $\overline{G}(x) = o(\overline{H}(x))$, hence, (c) and (i) of Lemma 6.1 imply that $W = G * H \in \mathcal{S}$ and $\overline{W}(x) \sim \overline{H}(x) \sim p\overline{F}(x)/q$.

Conversely, if $W = G * H \in \mathcal{S}$, by Theorem 6.2 and $\overline{G}(x) = o(\overline{F}(x))$, we get that

$$0 \leq \frac{\overline{G}(x)}{\overline{W}(x)} \leq \frac{\{p\overline{F}(x) + q\}\overline{G}(x)}{p\overline{F}(x) + q\overline{G}(x)} = \frac{\{p\overline{F}(x) + q\}\overline{G}(x)/\overline{F}(x)}{p + q\overline{G}(x)/\overline{F}(x)} \longrightarrow 0 \text{ as } x \to \infty$$

this is to say that $\overline{G}(x) = o(\overline{W}(x))$, thus, (ii) of Lemma 6.1 and (c) imply that $H \in \mathcal{S}$ and $\overline{W}(x) \sim \overline{H}(x) \sim p\overline{F}(x)/q$.

So, we have shown that $F \in \mathcal{S} \Leftrightarrow W \in \mathcal{S}$. In addition, the above proof also showed that $F \in \mathcal{S} \Rightarrow \overline{W}(x) \sim p\overline{F}(x)/q$. Thus, in order to complete the proof of Theorem 6.4, we still need to prove that

$$\overline{W}(x) \sim \frac{p}{q}\overline{F}(x) \Rightarrow F \in \mathcal{S}. \tag{6.3.14}$$

To do it, by (6.2.4), we get

$$\overline{W}(x) = \sum_{n=0}^{\infty} qp^n \, \overline{G * F^{(n)}}(x)\,,$$

which implies that

$$\overline{G * F^{(2)}}(x) = \frac{1}{qp^2}\left[\overline{W}(x) - \sum_{n \neq 2} qp^n \, \overline{G * F^{(n)}}(x)\right]. \tag{6.3.15}$$

Since Y is a non-negative random variable, for any integer $k \geq 1$,

$$\begin{aligned}
\overline{G * F^{(k)}}(x) &= \Pr\{Y + X_1 + \cdots + X_k > x\} \\
&\geq \Pr\{X_1 + \cdots + X_k > x\} = \overline{F^{(k)}}(x) \\
&\geq \Pr\{\max(X_1, \ldots, X_k) > x\} = 1 - [F(x)]^k\,,
\end{aligned}$$

i.e.

$$\overline{G * F^{(k)}}(x) \geq \overline{F^{(k)}}(x) \geq 1 - [F(x)]^k = \overline{F}(x)\sum_{n=0}^{k-1}[F(x)]^n\,. \tag{6.3.16}$$

(6.3.16) implies that

$$\liminf_{x \to \infty} \frac{\overline{G * F^{(k)}}(x)}{\overline{F}(x)} \geq \liminf_{x \to \infty} \frac{\overline{F^{(k)}}(x)}{\overline{F}(x)} \geq k. \tag{6.3.17}$$

Clearly, (6.3.16) and (6.3.17) are also true for $k = 0$, thus by (6.3.15) and (6.3.16), we get that

$$\frac{\overline{F^{(2)}}(x)}{\overline{F}(x)} \leq \frac{\overline{G * F^{(2)}}(x)}{\overline{F}(x)} \leq \frac{1}{qp^2}\left[\frac{\overline{W}(x)}{\overline{F}(x)} - \sum_{n \neq 2} qp^n \frac{\overline{F^{(n)}}(x)}{\overline{F}(x)}\right]. \tag{6.3.18}$$

Thus by (6.3.18), (6.3.17) and $\overline{W}(x) \sim p\overline{F}(x)/q$, we get that

$$\begin{aligned}
\limsup_{x \to \infty} \frac{\overline{F^{(2)}}(x)}{\overline{F}(x)} &\leq \frac{1}{qp^2}\left[\frac{p}{q} - \sum_{n \neq 2} qp^n \liminf_{x \to \infty} \frac{\overline{F^{(n)}}(x)}{\overline{F}(x)}\right] \\
&\leq \frac{1}{qp^2}\left[\frac{p}{q} - \sum_{n \neq 2} nqp^n\right] \\
&= \frac{1}{qp^2}\left[\frac{p}{q} + 2qp^2 - \sum_{n=0}^{\infty} nqp^n\right] = 2\,,
\end{aligned} \tag{6.3.19}$$

hence, (6.3.17) and (6.3.19) imply that $\lim_{x\to\infty} \overline{F^{(2)}}(x)/\overline{F}(x) = 2$, *i.e.* $F \in \mathcal{S}$. □

It is interesting to note that (6.3.14) holds and is independent of the condition that $\overline{G}(x) = o(\overline{F}(x))$. In addition, if G is an exponential distribution, using Lemma 6.1(iii) and following the proof of Theorem 6.4, we get directly the following corollary.

Corollary 6.2 If G is an exponential distribution, then Theorem 6.4 holds.

Theorem 6.5 Suppose that $\overline{F}(x) = o(\overline{G}(x))$ and $F \in \mathcal{S}$. The two following assertions are equivalent:

(i) $G \in \mathcal{S}$,

(ii) $W \in \mathcal{S}$,

and either one of them implies that

(iii) $\overline{W}(x) \sim \{p\overline{F}(x) + q\overline{G}(x)\}/\{p\overline{F}(x)+q\} \sim \overline{G}(x)$.

Proof. By (c) in the proof of Theorem 6.4, we know that

$$\frac{\overline{H}(x)}{\overline{G}(x)} = \frac{\overline{H}(x)}{\overline{F}(x)} \times \frac{\overline{F}(x)}{\overline{G}(x)} \to 0 \text{ as } x\to\infty ,$$

i.e. $\overline{H}(x) = o(\overline{G}(x))$, hence, if $G \in \mathcal{S}$, by Lemma 6.1(i), we get that $W = H * G \in \mathcal{S}$ and $\overline{W}(x) \sim \overline{G}(x)$. But, $\overline{F}(x) = o(\overline{G}(x))$ implies that

$$\frac{p\overline{F}(x) + q\overline{G}(x)}{p\overline{F}(x)+q} \sim \overline{G}(x).$$

Conversely, if $W = H * G \in \mathcal{S}$, by Theorem 6.2 and the proof of Theorem 6.4(c), we get that

$$0 \le \frac{\overline{H}(x)}{\overline{W}(x)} \le \frac{\{p\overline{F}(x)+q\}\overline{H}(x)}{p\overline{F}(x)+q\overline{G}(x)} = \frac{\{p\overline{F}(x)+q\}\overline{H}(x)/\overline{G}(x)}{q\overline{F}(x)/\overline{G}(x)+q} \to 0 \text{ as } x\to\infty$$

i.e. $\overline{H}(x) = o(\overline{W}(x))$, thus , Lemma 6.1(ii) implies that $G \in \mathcal{S}$ and $\overline{W}(x) \sim \overline{G}(x)$. □

Remark 6.4 We know that

$$\frac{p\overline{F}(x) + q\overline{G}(x)}{p\overline{F}(x)+q} \ge \frac{p\overline{F}(x)}{q}$$

if and only if $q^2\overline{G}(x) \ge [p\overline{F}(x)]^2$. Thus, in view of Theorem 6.4(iii), if $F \in \mathcal{S}$, $\overline{G}(x) = o(\overline{F}(x))$ and $q^2\overline{G}(x) \ge [p\overline{F}(x)]^2$ as $x\to\infty$, (for example, $\overline{G}(x) = [\overline{F}(x)]^{3/2}$), then by Theorem 6.2, we know that $\{p\overline{F}(x) + q\overline{G}(x)\}/\{p\overline{F}(x)+q\}$ is a closer approximation for $\overline{W}(x)$ than $p\overline{F}(x)/q$ is. By an argument similar to that in Remark 6.3, we also know that under the conditions of Theorem 6.5 and if $G \in \mathcal{S}$, then $\{p\overline{F}(x) + q\overline{G}(x)\}/\{p\overline{F}(x)+q\}$ is a closer approximation for $\overline{W}(x)$ than $\overline{G}(x)$ is.

Theorem 6.6 If $\overline{G}(x) \sim \overline{F}(x)$ and $F \in \mathcal{S}$, then

$$\overline{W}(x) \sim \frac{p\overline{F}(x) + q\overline{G}(x)}{p\overline{F}(x) + q} \sim \frac{\overline{F}(x)}{q} \sim \frac{\overline{G}(x)}{q}.$$

Proof. $F \in \mathcal{S}$ implies that $H \in \mathcal{S}$ and $\overline{H}(x) \sim p\overline{F}(x)/q$. Hence,

$$\frac{\overline{G}(x)}{\overline{H}(x)} = \frac{\overline{G}(x)}{\overline{F}(x)} \times \frac{\overline{F}(x)}{\overline{H}(x)} \to \frac{q}{p} \text{ as } x \to \infty.$$

Thus, by Theorem 1 of Cline (1986), we know that

$$\overline{W}(x) = \overline{G * H}(x) \sim (\frac{q}{p} + 1)\overline{H}(x) = \frac{\overline{H}(x)}{p} \sim \frac{\overline{F}(x)}{q}.$$

On the other hand, $\overline{F}(x) \sim \overline{G}(x)$ implies that $\{p\overline{F}(x) + q\overline{G}(x)\}/\{p\overline{F}(x) + q\} \sim \overline{F}(x)/q \sim \overline{G}(x)/q$, so Theorem 6.6 holds. □

Remark 6.5 The results of this Section also show that the lower bound in Theorem 6.2 is asymptotically exact for $\overline{W}(x)$ and the upper bound in Theorem 6.3 is asymptotically equal to $\overline{W}(x)$ plus an error term tending to zero as $x \to \infty$ for subexponential distributions. The bounds for $\overline{W}(x)$ were also considered by Kalashnikov (1994) and Willmot and Lin (1996). The bounds for $\overline{W}(x)$ in (5.1) and (5.2) of Kalashnikov (1994) are asymptotically exact as $E(N) \to \infty$. The bounds of Willmot and Lin (1996) are applicable to the tail of convolutions of more general compound distributions; their bounds are based on the generalized Lundberg coefficient and in terms of NWU distributions.

$\overline{W}(x)$ cannot admit exponential asymptotic forms and exponential upper bounds if F or G is subexponential, *i.e.* no constant $c > 0$ and $\varepsilon > 0$ such that $\overline{W}(x) \sim ce^{-\varepsilon x}$ or $\overline{W}(x) \le ce^{-\varepsilon x}$, for all $x \ge 0$. For example, if F is subexponential, then we know that $H \in \mathcal{S}$ and $e^{\varepsilon x}\overline{H}(x) \to \infty$ as $x \to \infty$, hence $e^{\varepsilon x}\overline{W}(x) \to \infty$ as $x \to \infty$ because of $\overline{W}(x) = \overline{H * G}(x) \ge \overline{H}(x)$.

6.4 Asymptotic estimates with medium-tailed distributions

We consider here the intermediate case, *i.e.* when $m_F(s) < \infty$ for some $s > 0$, but Lundberg's coefficient does not exist, as $m_F(s) = 1/p$ can not be satisfied but $m_F(s) < 1/p$ holds. For example, the inverse Gaussian and the generalized inverse Gaussian with certain parameters are in this case (see, Example 2.7). First, recall the definition of the $\mathcal{S}(\alpha)$ class and its properties.

Definition 6.2 A distribution B on $[0, \infty)$ is said to belong the $\mathcal{S}(\alpha)$ class for $\alpha \geq 0$, denoted by $B \in \mathcal{S}(\alpha)$, if

(i) $\lim_{x \to \infty} \overline{B}^{(2)}(x)/\overline{B}(x) = 2\,m_B(\alpha) < \infty$,

(ii) $\lim_{x \to \infty} \overline{B}(x-y)/\overline{B}(x) = e^{\alpha y}$, for all $y \in \Re$.

Clearly, $B \in \mathcal{S}(0) \Leftrightarrow B \in \mathcal{S}$. A class of distributions in $\mathcal{S}(\alpha)$ is the class of generalized inverse Gaussian distribution $N^{-1}(a,b,c)$ with $a < 0$, $b > 0$ and $c \geq 0$, since $N^{-1}(a,b,c) \in \mathcal{S}(c/2)$ [for details, see Embrechts (1983)]. The following proposition recalls some properties of $\mathcal{S}(\alpha)$, which will be used here. Its proof is in Lemma 2.4 and Theorem 2.7 of Embrechts and Goldie (1982) and page 268 of Klüppelberg (1989).

Property 6.1 Suppose that $B \in \mathcal{S}(\gamma)$.

(i) For any $\varepsilon > 0$, $e^{\gamma+\varepsilon}\,\overline{B}(x) \to \infty$ as $x \to \infty$,

(ii) If L is a distribution on $[0, \infty)$ and $\lim_{x \to \infty} \overline{L}(x)/\overline{B}(x) = c$, where $0 < c < \infty$, then $L \in \mathcal{S}(\gamma)$,

(iii) If $\gamma > 0$, B has a finite mean m, and B_1 is the stationary renewal distribution of B, *i.e.* $B_1(x) = \int_0^x \overline{B}(y)\,dy/m$, then $B_1 \in \mathcal{S}(\gamma)$ and $\overline{B_1}(x) \sim \overline{B}(x)/(\gamma m)$.

Theorem 6.7 Suppose that $F \in \mathcal{S}(\gamma)$ for some $\gamma > 0$ and that $m_F(\gamma) = \int_0^\infty e^{\gamma x} dF(x) < 1/p$. If $\overline{G}(x)/\overline{F}(x) \to \alpha$ as $x \to \infty$, then $W \in \mathcal{S}(\gamma)$ and

$$\overline{W}(x) \sim \frac{q\{p\,m_G(\gamma) + \alpha[1 - p\,m_F(\gamma)]\}}{[1 - p\,m_F(\gamma)]^2}\,\overline{F}(x). \tag{6.4.1}$$

Proof. Since $0 < p\,m_F(\gamma) < 1$, there exists some $\varepsilon > 0$ such that $0 < p[m_F(\gamma) + \varepsilon] < 1$, thus $\sum_{n=0}^\infty qp^n[m_F(\gamma) + \varepsilon]^n < \infty$. Hence, by Theorem 2.13 of Cline (1987), we get that $H \in \mathcal{S}(\gamma)$ and

$$\overline{H}(x) \sim c\,\overline{F}(x) \tag{6.4.2}$$

where $c = \sum_{n=1}^\infty nqp^n[m_F(\gamma)]^{n-1} = pq[1 - pm_F(\gamma)]^{-2}$. Thus,

$$\frac{\overline{G}(x)}{\overline{H}(x)} = \frac{\overline{G}(x)}{\overline{F}(x)} \times \frac{\overline{F}(x)}{\overline{H}(x)} \to \frac{\alpha}{c} \quad \text{as } x \to \infty.$$

By Theorem 1 of Cline (1986), we get that

$$\lim_{x\to\infty}\frac{\overline{W}(x)}{\overline{H}(x)}=\lim_{x\to\infty}\frac{\overline{H*G}(x)}{\overline{H}(x)}=m_G(\gamma)+\frac{\alpha}{c}m_H(\gamma). \tag{6.4.3}$$

But,

$$\begin{aligned} m_H(\gamma) &= E(e^{\gamma S_N})=\sum_{n=0}^{\infty}qp^nE\left[e^{\gamma\sum_{i=1}^n X_i}\right] \\ &= \sum_{n=0}^{\infty}qp^n[m_F(\gamma)]^n=q[1-pm_F(\gamma)]^{-1}. \end{aligned}$$

Hence, by (6.4.2) and (6.4.3), we get that

$$\begin{aligned} \overline{W}(x) &\sim [m_G(\gamma)+\frac{\alpha}{c}m_H(\gamma)]\,\overline{H}(x) \\ &\sim [cm_G(\gamma)+\alpha m_H(\gamma)]\,\overline{F}(x) \\ &= \left\{pq[1-pm_F(\gamma)]^{-2}m_G(\gamma)+\alpha q[1-pm_F(\gamma)]^{-1}\right\}\overline{F}(x)\,, \end{aligned}$$

i.e. (6.4.1) holds. $W\in\mathcal{S}(\gamma)$ follows directly from (6.4.1) and (ii) of Proposition 6.1. □

6.5 Ruin probabilities in diffusion risk models

Consider the compound Poisson risk process perturbed by a Wiener process, *i.e.* the surplus $R(t)$ at time t is

$$R(t)=x+ct-S(t)+W(t)\,,$$

where $x\geq 0$ is the initial risk reserve, $c>0$ is the premium rate, $S(t)$ is the compound Poisson process representing the total claims at t [with rate $1/d>0$, and the independent individual claim sizes with common distribution function B and $B(0)=0$], and $\{W(t)\}$ is a Wiener process, independent of $\{S(t)\}$, with infinitesimal drift 0 and infinitesimal variance $2D>0$. Assume $c>\lambda/d$, where $\lambda=\int_0^\infty\overline{B}(x)\,dx$ is the expected claim size, then the relative security loading $q=1-\lambda/(cd)$ is such that $0<q<1$.

Let $\psi_d(x)$ denote the probability of ultimate ruin, starting with initial reserve x, *i.e.*

$$\psi_d(x)=\Pr\{\inf_{t\geq 0}R(t)<0\}\,.$$

Define $\varphi_d(x) = 1-\psi_d(x)$. Then, Dufresne and Gerber (1991) [also see Veraverbeke (1993)] have shown that for any $x \geq 0$,

$$\varphi_d(x) = \sum_{n=0}^{\infty} qp^n F^{(n)} * G(x) = H * G(x) ,$$

or, equivalently,

$$\psi_d(x) = \sum_{n=0}^{\infty} qp^n \, \overline{F^{(n)} * G}(x) = \overline{H * G}(x) , \tag{6.5.1}$$

where $H(x) = \sum_{n=0}^{\infty} qp^n F^{(n)}(x)$, $G(x) = 1 - e^{-cx/D}$ for $x \geq 0$, $F(x) = G * B_1(x)$, $B_1(x) = \int_0^x \overline{B}(y)\,dy/\lambda$, for $x \geq 0$, $p = \lambda/(cd)$ and $q = 1 - p = 1 - \lambda/(cd)$.

Suppose that ξ and η are independent random variables with distributions G and B_1, respectively, then $\xi + \eta$ has distribution $F = G * B_1$ and for any $s \in \Re$,

$$\begin{aligned} m_F(s) &= \int_0^{\infty} e^{sx}\,dF(x) = E[e^{s(\xi+\eta)}] \\ &= E(e^{s\xi})E(e^{s\eta}) = m_G(s)\, m_{B_1}(s) . \end{aligned} \tag{6.5.2}$$

Thus, if there exists a constant R such that

$$m_F(R) = m_G(R)\, m_{B_1}(R) = \frac{1}{p}, \tag{6.5.3}$$

then (6.5.3) implies that $R < \frac{c}{D}$,

$$\begin{aligned} m_G(R) = E(e^{R\xi}) &= \frac{c}{D}\int_0^{\infty} e^{Rx}e^{-\frac{cx}{D}}\,dx \\ &= \frac{c}{c-RD} < \infty \end{aligned} \tag{6.5.4}$$

and

$$\begin{aligned} E(\xi e^{R\xi}) &= \frac{c}{D}\int_0^{\infty} xe^{Rx}e^{-\frac{cx}{D}}\,dx \\ &= \frac{cD}{(c-RD)^2} = \frac{D}{c-RD}\, m_G(R) . \end{aligned} \tag{6.5.5}$$

By (6.5.3) and (6.5.4), we get that

$$E(e^{R\eta}) = m_{B_1}(R) = \frac{1}{p\,m_G(R)} = \frac{c-RD}{pc} .$$

Hence,

$$\begin{aligned} \beta &= \int_0^{\infty} xe^{Rx}\,dF(x) = E[(\xi+\eta)e^{R(\xi+\eta)}] \\ &= E(\xi e^{R\xi})E(e^{R\eta}) + E(e^{R\xi})E(\eta e^{R\eta}) \\ &= m_G(R)\left[\frac{D}{pc} + \frac{1}{\lambda}\int_0^{\infty} xe^{Rx}\,\overline{B}(x)\,dx\right] . \end{aligned} \tag{6.5.6}$$

6.5.1 Asymptotic estimates of the ruin probability in the diffusion risk model

We know that if G and B_1 have density functions, so does $F = G * B_1$. This implies that F is non-lattice. Thus by Theorem 6.1, we get the following exponential formula for the ruin probability $\psi_d(x)$.

Corollary 6.3 Suppose that $m_F(R) = 1/p$. If $\int_0^\infty xe^{Rx}\overline{B}(x)\,dx < \infty$, then

$$\begin{aligned}\psi_d(x) &\sim \frac{qm_G(R)}{pR\beta}e^{-Rx} \\ &= (1-\frac{\alpha}{cd})\left[\frac{RD}{c}+\frac{R}{cd}\int_0^\infty xe^{Rx}\overline{B}(x)\,dx\right]^{-1}e^{-Rx}\end{aligned} \tag{6.5.7}$$

and if $\int_0^\infty xe^{Rx}\overline{B}(x)\,dx = \infty$, then

$$\psi_d(x) = o(e^{-Rx}). \tag{6.5.8}$$

This Corollary includes Theorem 4.1 of Gerber (1970), the results of Section 7 of Dufresne and Gerber (1991) and the results of Section 3 of Veraverbeke (1993).

Since $G(x) = 1 - e^{-cx/D}$ is an exponential distribution function, by (i) and (ii) of Lemma 6.1, we know that $B_1 \in \mathcal{S} \Leftrightarrow F = G * B_1 \in \mathcal{S}$. Furthermore, by Corollary 6.2 and Lemma 6.1, we know that

$$\begin{aligned}B_1 \in \mathcal{S} \Rightarrow \psi_d(x) &\sim \frac{p}{q}\,\overline{F}(x) = \frac{\lambda}{cd-\lambda}\,\overline{F}(x) \\ &\sim \frac{\lambda}{cd-\lambda}\,\overline{B}_1(x) = \frac{1}{cd-\lambda}\int_x^\infty \overline{B}(y)\,dy.\end{aligned}$$

In addition, using the fact $\overline{F}(x) = \overline{G * B_1}(x) \geq \overline{B}_1(x)$ and following the proof of (6.3.14), it is clear that

$$\psi_d(x) \sim \frac{p}{q}\,\overline{B}_1(x) = \frac{\lambda}{cd-\lambda}\,\overline{B}_1(x) \Rightarrow B_1 \in \mathcal{S},$$

thus, Corollary 6.2 implies the following theorem.

Theorem 6.8 For the diffusion risk model, the following conditions are equivalent:

(i) $B_1 \in \mathcal{S}$,

(ii) $\varphi_d \in \mathcal{S}$,

(iii) $\psi_d(x) \sim \frac{1}{cd-\lambda}\int_x^\infty \overline{B}(y)\,dy = p\overline{B}_1(x)/q.$

Remark 6.6 Theorem 6.8 generalizes Theorem 1 of Veraverbeke (1993), where it is shown that (i) and (ii) are equivalent and that either one of them implies (iii). Conditions enabling $B_1 \in \mathcal{S}$ can be found, expressed in terms of B, in Emberechts and Omey (1984) and Klüppelberg (1988).

Finally, for the intermediate case, suppose that for some $\gamma > 0$, $B \in \mathcal{S}(\gamma)$ and $m_F(\gamma) = m_G(\gamma)m_{B_1}(\gamma) < 1/p$, then by Proposition 6.1, we get that $B_1 \in \mathcal{S}(\gamma)$ and $\overline{B_1}(x) \sim \overline{B}(x)/(\gamma\lambda)$, in addition,

(1) $m_G(\gamma) = (c/D)\int_0^\infty e^{\gamma x}e^{-cx/D}\,dx = c/(c-D\gamma) < \infty,$

(2) $m_{B_1}(\gamma) = \int_0^\infty e^{\gamma x}\,dB_1(x) = (1/\lambda)\int_0^\infty e^{\gamma x}\,\overline{B_1}(x)\,dx < \infty$ and $\gamma < c/D,$

hence, there exists some $\varepsilon > 0$ such that $\gamma + \varepsilon < c/D$, thus $m_G(\gamma+\varepsilon) = c/[c-(\gamma+\varepsilon)D] < \infty$, and by Proposition 6.1, we get that

$$\frac{\overline{G}(x)}{\overline{B}_1(x)} = \frac{e^{\gamma+\varepsilon}\,\overline{G}(x)}{e^{\gamma+\varepsilon}\,\overline{B}_1(x)} \longrightarrow 0 \text{ as } x \to \infty.$$

Thus, by Theorem 1 of Cline (1986) and (ii) of Proposition 6.1, we get that

$$\frac{\overline{F}(x)}{\overline{B}_1(x)} = \frac{\overline{G * B_1}(x)}{\overline{B}_1(x)} \longrightarrow m_G(\gamma) \text{ as } x \to \infty$$

and

$$F \in \mathcal{S}(\gamma) \text{ and } \overline{F}(x) \sim m_G(\gamma)\,\overline{B}_1(x) \sim \frac{m_G(\gamma)}{\gamma\mu}\,\overline{B}(x).$$

Thus, $\overline{G}(x) = o(\overline{F}(x))$ and Theorem 6.7 imply that $\varphi_d \in \mathcal{S}(\gamma)$ and

$$\psi_d(x) \sim \frac{pq m_G(\gamma)}{[1-pm_F(\gamma)]^2}\,\overline{F}(x) \tag{6.5.9}$$

$$\sim \frac{pq[m_G(\gamma)]^2}{\lambda\gamma[1-pm_F(\gamma)]^2}\,\overline{B}(x) \tag{6.5.10}$$

$$= \frac{1}{cd\gamma}\left(1-\frac{\lambda}{cd}\right)\left[1-\frac{\gamma D}{c}-\frac{1}{cd}\int_0^\infty e^{\gamma y}\,\overline{B}(y)\,dy\right]^{-2}\overline{B}(x)\,, \tag{6.5.11}$$

thus, (6.5.11) yields Theorem 2 of Veraverbeke (1993).

6.5.2 Bounds for the ruin probability in the diffusion risk model with large claim sizes

Under condition (6.5.3), the exponential bounds for the ruin probability $\psi_d(x)$ have been derived by Dufresne and Gerber (1991). Here, we use Dickson's condition to derive the bounds for $\psi_d(x)$ with large claim sizes. General upper and lower bounds for $\psi_d(x)$ can be obtained directly by Theorem 6.2 and Theorem 6.3 since G is exponential and has a decreasing density.

Given $t > 0$, suppose that R_t satisfies

$$m_G(R_t) \int_0^t e^{R_t y} \, dB_1(y) = \frac{1}{p} \,, \tag{6.5.12}$$

or equivalently,

$$\int_0^t e^{R_t y} \, dB_1(y) = \frac{c - R_t D}{cp}. \tag{6.5.13}$$

We first state the following property about R_t.

Property 6.2 For any claim size distribution B with $B(0) = 0$, there exists a unique solution $R_t \in (0, c/D)$ for equation (6.5.12).

Proof. Let

$$h(x) = m_G(x) \int_0^t e^{xy} \, dB_1(y) - \frac{1}{p}.$$

Since $H(0) = B_1(t) - 1/p < 0$ and

$$\lim_{x \uparrow c/D} m_G(x) = \lim_{x \uparrow c/D} \frac{c}{c - xD} = \infty,$$

this implies that

$$\lim_{x \uparrow c/D} h(x) = \infty.$$

Thus, the existence of the unique root of h follows since h is continuous and strictly increasing. □

Since B_1 is continuous in this case, it is clear that the condition (6.5.12) is equivalent to

$$m_G(R_t) \int_0^\infty e^{R_t y} \, dB_t(y) = \frac{1}{pB_1(t)} \,, \tag{6.5.14}$$

where,

$$B_t(x) = \begin{cases} B_1(x)/B_1(t), & 0 \le x < t \\ 1, & x \ge t \end{cases} \tag{6.5.15}$$

Let

$$\varphi_t(x) = \sum_{n=0}^{\infty} q_t\, p_t^n\, F_t^{(n)} * G(x)\ , \tag{6.5.16}$$

or, equivalently,

$$\psi_t(x) = 1 - \varphi_t(x) = \sum_{n=1}^{\infty} q_t\, p_t^n\, \overline{F^{(n)} * G}(x)\ , \tag{6.5.17}$$

where $p_t = pB_1(t)$, $q_t = 1 - p_t$ and $F_t = B_t * G$. That is to say $\psi_t(x)$ is the ruin probability in the diffusion risk model with corresponding parameters p_t, q_t, F_t and G.

Since for any $0 \le x \le t$, $B_t^{(n)}(x) = B_1^{(n)}(x)/[B_1(t)]^n$ [c.f. (3.3.8)], we get for any $0 \le x \le t$,

$$\begin{aligned} \varphi_t(x) &= \sum_{n=0}^{\infty} q_t\, p_t^n B_t^{(n)} * G^{(n)} * G(x) \\ &= \sum_{n=0}^{\infty} \frac{q_t\, p_t^n}{[B_1(t)]^n}\, B_1^{(n)} * G^{(n)} * G(x) \\ &= \frac{q_t}{q} \sum_{n=0}^{\infty} q\, p^n F^{(n)} * G(x) \\ &= \frac{q_t}{q}\, \varphi_d(x)\ . \end{aligned}$$

Thus, for any $0 \le x \le t$,

$$\begin{aligned} \psi_t(x) &= 1 - \varphi_t(x) = 1 - \frac{q_t}{q}\, \varphi_d(x) \\ &= 1 - \frac{q_t}{q}\, [1 - \psi_d(x)]\ , \end{aligned}$$

which implies the following property.

Property 6.3 For any $0 \le x \le t$,

$$\psi_d(x) = \frac{p\, \overline{B}_1(t)}{q + p\, \overline{B}_1(t)} + \frac{q\, \psi_t(x)}{q + p\, \overline{B}_1(t)}\ . \tag{6.5.18}$$

Using Property 6.3, we can easily get the following result.

Theorem 6.9 Suppose R_t satisfies (6.5.12), then for any $0 \le x \le t$,

$$\frac{p\,\overline{B}_1(t)}{q + p\,\overline{B}_1(t)} \;\le\; \psi_d(x) \;\le\; \frac{p\,\overline{B}_1(t)}{q + p\,\overline{B}_1(t)} + \frac{q\,e^{-R_t\,x}}{q + p\,\overline{B}_1(t)}\,. \tag{6.5.19}$$

In particular, for any $x > 0$,

$$\frac{p\,\overline{B}_1(x)}{q + p\,\overline{B}_1(x)} \;\le\; \psi_d(x) \;\le\; \frac{p\,\overline{B}_1(x)}{q + p\,\overline{B}_1(x)} + \frac{q\,e^{-R_x\,x}}{q + p\,\overline{B}_1(x)}. \tag{6.5.20}$$

Proof. The lower bound in (6.5.19) follows from $\psi_t(x) \ge 0$ and (6.5.18). On the other hand, if a constant R satisfies

$$m_G(R)\int_0^\infty e^{Ry}\,dB_1(y) = \frac{1}{p} \tag{6.5.21}$$

then, Dufresne and Gerber (1991) show that for any $x \ge 0$,

$$\psi_d(x) \;\le\; e^{-Rx}\,. \tag{6.5.22}$$

Thus, apply (6.5.22) to $\psi_t(x)$ with condition (6.5.14), to get for any $x \ge 0$,

$$\psi_t(x) \;\le\; e^{-R_t\,x}\,.$$

This, together with (6.5.18), implies that the upper bound in (6.5.19) holds. Taking $x = t$ in (6.5.19) gives (6.5.20). □

Since

$$\frac{p\,\overline{B}_1(x)}{q + p\,\overline{B}_1(x)} \sim \frac{p\,\overline{B}_1(x)}{q}\,,$$

by Theorem 6.8, we know that the lower bound in (6.5.20) is asymptotically exact for large x if B_1 is a subexponential distribution.

In addition, if $D = 0$, the diffusion risk model is reduced to the compound Poisson risk model, thus Dickson's bound is derived as a special case of Theorem 6.9 and is improved upon.

6.6 Examples and numerical results

Let

$$L(x) = \frac{\theta\, e^{-2x\kappa(x)} + \overline{B}_1(x)}{\theta + \overline{B}_1(x)} \quad \text{and} \quad U(x) = \frac{\theta\, e^{-x\kappa(x)} + \overline{B}_1(x)}{\theta + \overline{B}_1(x)}\,, \tag{6.6.23}$$

respectively, be the lower and upper bounds for the ruin probability $\psi(x)$ in Corollary 3.2, where $\theta = q/p$, *i.e.* $p = 1/(1+\theta)$, is the relative safety loading factor. Also let

$$L_1(x) = \frac{\overline{B}_1(x)}{\theta + \overline{B}_1(x)} \quad \text{and} \quad U_1(x) = e^{-x\kappa(x)} + \frac{\overline{B}_1(x)}{\theta + \overline{B}_1(x)}\,,$$

be the lower and upper bounds of De Vylder and Goovaerts (1984) and Dickson (1994), respectively. Now denote by

$$A_h(x) = \frac{\overline{B}_1(x)}{\theta}\,,$$

$$A_m(x) = \frac{\theta\gamma\mu}{[1 + (1+\theta)\gamma\mu - m_B(\gamma)]^2}\,\overline{B}(x)\,,$$

$$A_l(x) = \frac{\theta\mu}{m'_B(\kappa) - \mu(1+\theta)}\, e^{-\kappa x}\,,$$

the asymptotic formulas for the ruin probability $\psi(x)$ when the claim size distribution $B(y)$ is heavy-tailed, medium-tailed and light-tailed respectively, where $\gamma > 0$ satisfies $m_B(\gamma) < 1 + \mu(1+\theta)$ while $m_B(t) = \infty$ for any $t > \gamma$.

For example, if B is a Pareto or lognormal distribution, we have [see Embrechts and Veraverbeke (1982) or Panjer and Willmot (1992)]

$$\psi(x) \sim A_h(x)\,.$$

On the other hand, if B is an inverse Gaussian distribution with a medium-tail, *i.e.* $m_B(\gamma) < 1 + \mu(1+\theta)$ and $m_B(t) = \infty$ for any $t > \gamma$, then [again, see Embrechts and Veraverbeke (1982) or Panjer and Willmot (1992)]

$$\psi(x) \sim A_m(x)\,.$$

Finally, if B is a distribution that admits an adjustment coefficient κ and $m'_B(\kappa) < \infty$, then [see Feller (1971), Grandell (1991) or Panjer and Willmot (1992)]

$$\psi(x) \sim A_l(x) .$$

Although these formulas approximate the ruin probability $\psi(x)$ when x is large, the accuracy of such asymptotic approximations is not known, as pointed by Kalashnikov (1996). This is one reason why bounds for $\psi(x)$ are useful.

In this section, we use $L(x)$ and $U(x)$ in (6.6.23) to calculate lower and upper bounds for Pareto and lognormal claim size distributions, which do not admit finite moment generating functions, and for inverse Gaussian claim size distributions with medium-tails, which admit finite moment generating functions, but for which the adjustment coefficient κ does not exist.

Note that in all these cases, Lundberg's bound is not available. We will also consider bounds for the inverse Gaussian claim size distributions with a light-tail. Here the adjustment coefficient κ exists and Lundberg's bound applies. However, we show that the upper bound $U(x)$ is sharper in this case. The numerical values of the bounds are also compared with those from the above asymptotic formulas.

Example 6.1 Let $B(y) = 1-(1+y)^{-2}$, $y \geq 0$ be a Pareto distribution function with mean 1 and the relative safety loading factor $\theta = 0.1$, then Table 6.1 gives numerical values of the lower and upper bounds as well as the asymptotic formulas for $\psi(x)$.

$L(x)$ is uniformly larger than the lower bound $L_1(x)$ of De Vylder & Goovaerts (1984). Similarly, $U(x)$ is uniformly sharper than the upper bound $U_1(x)$ of Dickson (1994). In both cases, the difference is more significant for small initial reserves x, while they give very comparable bounds for the tail of $\psi(x)$.

The asymptotic values $A_h(x)$, meaningless for small x, seem to indicate a greater accuracy of lower bounds $L(x)$ than the corresponding upper bounds $U(x)$.

Example 6.2 Let B be the lognormal distribution with mean 1 and variance 3, and let $\theta = 0.1$. Table 6.2 gives the corresponding bound and asymptotic values for $\psi(x)$.

The same remarks about $L(x)$ and $U(x)$ as for Example 6.1 apply. However, here the asymptotic values $A_h(x)$, are closer to $L_1(x)$ than to $L(x)$. In fact $L(x)$ produces lower bounds consistently larger than the asymptotic value of $\psi(x)$.

Example 6.3 Let $B'(y) = \mu\,[2\pi\beta y^3]^{-\frac{1}{2}} \exp\left\{-\frac{(y-\mu)^2}{2\beta y}\right\}$, $y > 0$ be a medium-tailed inverse Gaussian density, with mean $\mu = 1$ and variance $\mu\beta = 12$ and $\theta = 1.1$. Table 6.3 gives the desired values.

Again $L(x)$ and $U(x)$ provide sharper bounds, while the medium-tailed asymptotic values $A_m(x)$, are completely meaningless for the chosen parameter and x values.

Example 6.4 Let $B(x)$ be another medium-tailed inverse Gaussian distribution, with mean 1, variance 5 and let $\theta = 2.5$. Table 6.4 reports the appropriate values. Again $L(x)$ and $U(x)$ provide sharper bounds than $L_1(x)$ and $U_1(x)$. Here, most asymptotic values $A_m(x)$, lay outside the interval between (either) lower and upper bounds.

In Examples 6.3 and 6.4, the loading factors are choosed so that the inverse Gaussian distribution is medium tailed.

Example 6.5 Let $B(x)$ be an inverse Gaussian with a light-tail, mean 1, variance 4 and $\theta = 0.1$. Here the adjustment coefficient, κ exists and Lundberg's bound in (1.1.9) is available; it appears in Table 6.5 under the column title Lundberg.

Even when Lundberg's bound exists, $L(x)$ and $U(x)$ provide the sharpest bounds. Here the asymptotic values $A_l(x)$, provide meaningful values for large x.

6.7 Conclusions

In this thesis, we develop general methods for studying bounds and asymptotic estimates for the solution of renewal equations and the tail probabilities of compound distributions using distribution theory, renewal theory, stochastic ordering and the aging property of compound geometric distributions. Generalized Cramér-Lundberg conditions in terms of NWU distributions and Dickson's condition are used to derive the bounds and asymptotic estimates in the presence of heavy tailed distributions. This study yields new results, which improve and generalize many previous results and are applied to analyze aggregate claim distributions, stop-loss premiums and ruin probabilities.

Dickson's condition can be applied to any positive claim size distribution. The numerical examples in Section 6.7 show those bounds derived by Dickson's condition are most appropriate for heavy and medium-tailed distributions, and their numerical evaluation is very simple. It plays the same role in the presence of heavy and medium tailed distributions as the Cramér-Lundberg condition for the exponential tail cases. The examples also show that bounds are more useful than asymptotic estimates in the presence of heavy and medium tailed distributions. In fact, the bounds can be used in decision making and risk management [see, e.g. De Vylder (1996), Gerber (1979) and Panjer and Willmot (1992)]. They are safe in practice since if upper bounds for ruin probability, stop-loss premiums and tail probabilities of the aggregate claims are less than some α, then these quantities are also less than α, respectively. But the asymptotic formulae with heavy and medium tailed distributions are not safe. A numerical example [see, page 177 of De Vylder (1996)] shows also that for the ruin probability in the compound Poisson risk model, even the value of asmyptotic formula is less than 0.00014, the ruin probability is still greater than 0.1 for Pareto claim size distributions.

As shown in this thesis, generalized Cramér-Lundberg conditions are effective for the insurance risk analysis in the compound Poisson risk model and the diffusion risk model, so we hope to further this study to general risk models. Willmot (1996) applies condition (3.1.4) to study the ruin probability in the renewal risk model [see, also Dickson (1998), Dickson and Hipp (1998) and Grandell (1991) for this model] and obtained upper bounds in terms of NWU distributions for the ruin probability. It is evident that condition (6.5.12) can be used in the renewal risk model, but the method deriving the two-sided bounds in Theorem 6.9 for the ruin probability in the diffusion risk model does not apply to the renewal risk model (at least for the author of this thesis). Hence, other methods are needed to solve this problem.

In addition, applying the generalized Cramér-Lundberg conditions to risk models in an economic and random environment [e.g. see Chukova *et al.*(1993), Garrido(1988, 1989), Garrido and Nana (1997), Paulsen (1993), Sundt and Teugels (1995) and Willmot (1989)] and discounted random sums [see, e.g. Aebi *et al.* (1994), Artikis and Malliaris (1990) and Dufresne (1990)] is an interesting topic, which will be studied by the author of this thesis.

Table 6.1: Bounds for the ruin probability with Pareto claim sizes of mean 1 and $\theta = 0.1$

x	$\kappa(x)$	$L_1(x)$	$L(x)$	$U(x)$	$U_1(x)$	$A_h(x)$
5	0.211511	0.6249995	0.6702318	0.7552383	0.9723026	1.6666630
10	0.101777	0.4761888	0.5446036	0.6654939	0.8375883	0.9090847
50	0.030937	0.1639256	0.2018299	0.3419449	0.3768484	0.1960658
100	0.021316	0.0900751	0.1028847	0.1980369	0.2087243	0.0989918
500	0.008734	0.0195442	0.0197021	0.0319871	0.0322351	0.0199338
1000	0.005468	0.0098573	0.0098749	0.0140359	0.0140775	0.0099554
1500	0.004059	0.0065716	0.0065767	0.0088271	0.0088420	0.0066151
2000	0.003256	0.0049343	0.0049365	0.0064121	0.0064194	0.0049588
5000	0.001556	0.0018909	0.0018912	0.0023073	0.0023081	0.0018946
10000	0.000868	0.0008848	0.0008849	0.0010554	0.0010556	0.0008856

Table 6.2: Bounds for the ruin probability with lognormal claim sizes of mean 1, variance 3 and $\theta = 0.1$

x	$\kappa(x)$	$L_1(x)$	$L(x)$	$U(x)$	$U_1(x)$	$A_h(x)$
5	0.157564	0.4790302	0.5868062	0.7159860	0.9338661	0.9194969
10	0.078677	0.2371247	0.3952768	0.5844723	0.6924384	0.3108302
25	0.048911	0.0460106	0.1287013	0.3268773	0.3404234	0.0482297
50	0.042198	0.0082844	0.0228649	0.1285326	0.1295372	0.0083536
100	0.038703	0.0010644	0.0014987	0.0218935	0.0219157	0.0010655
150	0.036739	0.0002606	0.0002770	0.0043022	0.0043032	0.0002607
200	0.034927	0.0000457	0.0000466	0.0009710	0.0009711	0.0000457

Table 6.3: Bounds for the ruin probability with medium-tailed inverse Gaussian claim sizes of mean 1, variance 12 and $\theta = 1.1$

x	$\kappa(x)$	$L_1(x)$	$L(x)$	$U(x)$	$U_1(x)$	$A_m(x)$
1	3.545990	0.3771012	0.3776193	0.3950656	0.4059413	21555.62
5	0.560508	0.2378071	0.2406113	0.2840386	0.2984630	5643.261
10	0.266822	0.1523864	0.1564659	0.2111901	0.2217621	2505.392
50	0.074008	0.0099846	0.0105893	0.0344520	0.0346988	86.95006
60	0.067690	0.0055307	0.0058258	0.0226602	0.0227555	46.05902
70	0.063339	0.0031263	0.0032667	0.0149594	0.0149965	25.13600
80	0.060174	0.0017935	0.0018593	0.0098948	0.0099094	14.02842
100	0.055904	0.0006116	0.0006255	0.0043429	0.0043452	4.589443
150	0.050553	0.0000460	0.0000463	0.0005551	0.0005551	0.335443

Table 6.4: Bounds for the ruin probability with medium-tailed inverse Gaussian of mean 1, variance 5 and $\theta = 2.5$

x	$\kappa(x)$	$L_1(x)$	$L(x)$	$U(x)$	$U_1(x)$	$A_m(x)$
1	3.838749	0.1812962	0.1816754	0.1989151	0.2028167	3.374153
5	0.704724	0.0680559	0.0688665	0.0955412	0.0975484	0.607592
10	0.383118	0.0268631	0.0273207	0.0479647	0.0485472	0.184272
20	0.236196	0.0055888	0.0056672	0.0144194	0.0144691	0.031232
30	0.190161	0.0013809	0.0013919	0.0047061	0.0047107	0.007040
40	0.167798	0.0003742	0.0003757	0.0015901	0.0015905	0.001802
50	0.154587	0.0001085	0.0001087	0.0005482	0.0005482	0.000496
60	0.145857	0.0000315	0.0000315	0.0001897	0.0001897	0.000144

Table 6.5: Bounds for the ruin probability with light-tailed inverse Gaussian of mean 1, variance 4 and $\theta = 0.1$

x	$\kappa(x)$	κ	$L_1(x)$	$L(x)$	Lundberg
1	1.84358	0.03422	0.840108	0.844112	0.966358
5	0.17809	0.03422	0.594651	0.662941	0.842733
10	0.07033	0.03422	0.320677	0.487092	0.710198
50	0.03435	0.03422	0.000651	0.032847	0.180675
100	0.03422	0.03422	0.000004	0.001070	0.032644
150	0.03422	0.03422	0.000000	0.000004	0.005898
x	$\kappa(x)$	κ	$U(x)$	$U_1(x)$	$A_l(x)$
1	1.84358	0.03422	0.865411	0.998358	0.821617
5	0.17809	0.03422	0.761028	1.005104	0.716508
10	0.07033	0.03422	0.656905	0.815623	0.603825
50	0.03435	0.03422	0.180024	0.180141	0.153614
100	0.03422	0.03422	0.032643	0.032644	0.027754
150	0.03422	0.03422	0.005871	0.005871	0.005014

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