The Use of Auxiliary Information for Estimation in Finite Population Sampling

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ABSTRACT

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In this thesis we consider the methods of using auxiliary information to increase the efficiency of estimators in the finite population sampling. Three basic methods known as Ratio, Product and Regression estimators are considered with respect to the fixed population approach and superpopulation model approach. We review the properties of these estimators under the two approaches and provide a limited review of further more recent investigations. A numerical study of the three estimators under the Durbin's (1959) superpopulation model is carried out. A class of superpopulation models which may have a greater appeal is proposed and analytical properties of the three estimators under the model are derived.

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DEDICATION

I would like to dedicate this thesis to my parents and Mr. and Mrs. Roger Tang.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

1.1 Use of Auxiliary Information in Sample Survey

If there is one thing that distinguishes sampling theory from general statistical theory, it is the degree of emphasis laid on the use of auxiliary information for improving the precision of estimates. The use of an auxiliary variable x in the estimation of the finite population total or mean of a characteristic variable y based on a sample is usually made through ratio, regression and product estimator. Such estimators take advantage of the correlation between the x variable and the characteristic y.

The ratio estimator works better than the sample mean when the correlation coefficient is highly positive, while the product estimator works better if the correlation coefficient is highly negative (whenever ratio of means is positive). Positive correlations are often encountered in practice, however it is not uncommon to observe highly negative correlations also, e.g. when a transformation: $x \to 1/x$ of x is induced to linearize the regression function. One such example is in estimating Hourly Earnings (y) using the auxiliary variable, years of Experience (x) (see Neter and Wasserman (1974) pp. 130, fig. 4.17). Negative correlations may also arise naturally in industrial situations e.g. in Draper and Smith (1966), pp. 351-352, pounds of steam and Average Atmospheric Pressure are negatively correlated ($r = -0.8452$) and pp. 365-367, amount of tricalcium aluminate and amount of tetracalcium alumino silicate are highly negatively correlated ($r = -0.8241$).

The optimal regression estimator in general takes into account any amount of correlation between the main (y) and the auxiliary (x) characteristics and reduces to the ratio estimator under a specific model (see Chapter II), whereas a generalized product can be shown to be approximately the same as the optimal regression estimator (see Chaubey et al (1984a)). The regression estimator is thus more appealing compared to the product and ratio estimators. However, because of its computations nature, it has gained popularity only recently due to advent of high speed, convenient and cost effective computers. On the other hand, while ratio estimator has been quite popular with practitioners in sample survey methods because of the necessity of estimating ratios (e.g. per capita income), product methods seems to have been ignored, possibly, because it seems a bit artificial.

In this thesis our interest is in making a (limited) survey on the available methods of estimation using the auxiliary information and making a comparison of some of these methods.

Although the methods and procedures discussed in this thesis can be extended to other sampling designs such as stratified sampling, cluster sampling etc., however our interest will be on simple random sampling. Section 1.2 gives the main notations used in this thesis and section 1.3

discusses the two frameworks, the design and the model for comparing the estimators. The last section in this chapter (1.4) presents an outline of the thesis.

1.2 Preliminaries and Notations

Notations:

$y_1 \ldots y_N$	Values of the variable under study of the population.
$x_1 \ldots x_N$	Values of the auxiliary variable of the population.
$\bar{Y} = \frac{1}{N} \sum_{i=1}^{N} y_i$	Population mean of the y-values.
$Y = N\bar{Y}$	Population total of the y-values.
$\bar{X} = \frac{1}{N} \sum_{i=1}^{N} x_i$	Population mean of the x-values.
$X = N\bar{X}$	Population total of the x-values.
(y_i, x_i), $i=1,2,\ldots n$	Sample values.
$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$	Sample mean of the y-values.
$y = n\bar{y}$	Sample total of the y-values.
$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$	Sample mean of the x-values.
$x = n\bar{x}$	Sample total of the x-values.
ρ	Correlation coefficient between x_i and y_i.
$S_y^2 = \frac{1}{N-1} \sum_{i=1}^{N} (y_i - \bar{Y})^2$	Population variance of y_i.

$S_x^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i-\bar{X})^2$	Population variance of x_i.
$S_{xy} = \rho S_x S_y$	Population covariance
$s_y^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i-\bar{y})^2$	Sample variance of y_i.
$s_x^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i-\bar{x})^2$	Sample variance of x_i.
s_{xy}	Sample covariance.
$C_{yy} = S_y^2/\bar{Y}^2$	Square of the coefficient of variation (cv) of y_i.
$C_{xx} = S_x^2/\bar{X}^2$	Square of the coefficient of variation (cv) of x_i.
$C_{xy} = S_{xy}/(\bar{X}\bar{Y})$	Relative covariance.
$R = \bar{Y}/\bar{X}$	Population ratio.
$\hat{R} = \bar{y}/\bar{x}$	Ratio estimate of the population ratio (sample ratio).
$\hat{Y}_R = \bar{y}X/\bar{x}$	Ratio estimate of population total Y.
$\hat{\bar{Y}}_R = \bar{y}\bar{X}/\bar{x}$	Ratio estimate of population mean $\bar{Y}$.
$\hat{\bar{Y}}_p = \bar{y}\bar{x}/\bar{X}$	Product estimate of population mean $\bar{Y}$.
$f = n/N$	Sampling fraction.
E_M	Model Expectation.
E	Design Expectation.
V_M	Variance under the model.

Elementary Theorems (in SRSWOR)

(i) $Var(\bar{y}) = \frac{(1-f)}{n} S_y^2$

(ii) $E(s_y^2) = S_y^2$

(iii) $E(s_{xy}) = S_{xy}$

(iv) $Cov(\bar{x},\bar{y}) = E(\bar{y}-\bar{Y})(\bar{x}-\bar{X})$

$$= (\frac{1-f}{n}) \frac{1}{n-1} \sum_{i=1}^{n} (y_i-\bar{Y})(x_i-\bar{X})$$

$$= (\frac{1-f}{n}) \rho S_x S_y .$$

1.3 Design Based and Model Based Comparisons

The essence of survey sampling consists of selection of a part of a finite collection of units, followed by the making of statements about the entire collection on the basis of the selected part. Two ways of having a finite collection of units are:

i) The Fixed Population Approach: With each population unit is associated a fixed but unknown real number, that is, the value of the variable (y) under study.

ii) The Superpopulation Approach: With each population unit is associated a random variable for which a stochastic structure is specified; the actual value associated with a population unit is treated as the outcome of this random variable.

The Fixed population approach is the "traditional" one in survey sampling. However, there are also early examples of the Superpopulation approach: Cochran (1939, 1946), Deming and Stephan (1941). Many recent important contributions to finite population inference theory take the Super population approach, which is an avenue of great promise in survey sampling. This approach may be meaningful in terms of changing population values with respect to time.

As an example, suppose that the units are farms and that the characteristic under study is the yield of wheat in a given year. One approach to the inference problem, often used in standard texts, is the following: it is assumed that to each farm in the population corresponds a fixed but unknown real number representing the yield of that particular farm in the particular year. When a farm has been selected for the sample, it is furthermore assumed that the fixed real number to this farm is measured without error.

A different approach to the problem is to treat the yields of farms in the population as numbers generated under a stochastic model. Such models often incorporate auxiliary knowledge.

A crude but frequently effective model may simply postulate a linear stochastic relationship, for example, that yield of wheat is, apart from an error term of zero expected value, proportional to size of the farm in acres, x_i, which is assumed known from a previous year. The model is, thus,

$$y_i = \beta x_i + \varepsilon_i \qquad i = 1,2,\ldots,N \qquad E_M(\varepsilon_i | x_i) = 0 \quad (1.1)$$

Moreover, the model considers that the unknown proportionality factor β is common to all farms. It would be determined in the particular year, by the average propensity of farmers to devote acreage to wheat, by average yield per acre that year.

An estimate $\hat{\beta}$ of the unknown proportionality factor can be obtained from a sample of farms. For any one farm not in the sample, the value $\hat{\beta}x_i$ should provide an effective prediction of wheat yield, thereby permitting a prediction of total yield in the population.

In the fixed population approach the variation of the estimator is entirely due to the sampling design chosen to select the units in the sample, however, in the superpopulation approach it will also depend on the stochastic model generating the population. Thus, the basis of comparison of two estimators under the two approaches are $E[(\hat{\theta} - \theta)^2]$, the mean square error (MSE) and $E_M E[(\hat{\theta} - \theta)^2]$, the average MSE under the model, where $\hat{\theta}$ is an estimator of θ. We often strive for unbiased estimators in which case the criterion of comparison becomes the variance.

1.4 Outline of the Thesis

We consider the ratio method of estimation, in Chapter II, where its properties are catalogued and its investigations in the literature are summarized. Chapter III considers the method of product estimator and a similar format as of Chapter II is followed. Chapter IV summarizes the properties and available results about regression estimators. Finally in Chapter V, we investigate the properties of these three estimators under a class of superpopulation models given by Inverse Gaussian distribution. In this chapter we also present a numerical study using the results of Chaubey <u>et al</u> (1984b).

CHAPTER II

RATIO ESTIMATORS

2.1 The Ratio Estimator

In the ratio method an auxiliary variate x_i, correlated with y_i, is obtained for each unit in the sample. The population total must be known. In practice, x_i is often the value of y_i at some previous time when a complete census was taken. The aim in this method is to obtain increased precision by taking advantage of the correlation between y_i and x_i. The ratio estimate of Y is

$$\hat{Y}_R = \frac{yX}{x} = \frac{\bar{y}X}{\bar{x}} \tag{2.1}$$

If x_i is the value of y_i at some previous time the ratio method uses the sample to estimate the relative change Y/X that has occurred since that time. The estimated relative change y/x is multiplied by the known population total X on the previous occasion to provide an estimate of the current population total. If the quantity to be estimated is $\bar{Y}$, the ratio estimate is

$$\hat{\bar{Y}}_R = \frac{y\bar{X}}{x} = \frac{\bar{y}\bar{X}}{\bar{x}} \tag{2.2}$$

For example, suppose it is desired to estimate the total number of inhabitants in N cities during a particular year. If the population total X is known from an earlier year, it would be preferable to estimate the ratio of total number of inhabitants during that year, y, and the total number of inhabitants during the earlier year, x, from a sample of n cities and multiply this figure by the known population total X .

In the next section the results on the bias and mean square error (MSE) of the ratio estimator are given. Section 2.3 gives the condition under which the ratio estimator is better than the sample mean and section 2.4 gives some unbiased alternatives to the ratio estimators. Section 2.5 presents the optimality property of the ratio estimator under a superpopulation model and the last section presents some other investigations.

2.2 Bias and MSE of the Ratio Estimator

Bias

The ratio estimator has, in general, a bias of order 1/n for the corresponding population ratio $R = \bar{Y}/\bar{X}$. Since $\hat{R} = \bar{y}/\bar{x}$ then $\hat{R} - R = \frac{\bar{y}}{\bar{x}} - R = \frac{\bar{y} - R\bar{x}}{\bar{x}}$.

Now

$$\frac{1}{\bar{x}} = \frac{1}{\bar{X} + (\bar{x}-\bar{X})} = \frac{1}{\bar{X}}\left(1 + \frac{\bar{x}-\bar{X}}{\bar{X}}\right)^{-1}$$

$$\simeq \frac{1}{\bar{X}}\left(1 - \frac{\bar{x}-\bar{X}}{\bar{X}}\right), \qquad (2.3)$$

assuming the relative errors $\frac{|\bar{x} - \bar{X}|}{\bar{X}}$ to be small.

Thus

$$\hat{R} - R \simeq \frac{\bar{y} - R\bar{x}}{\bar{X}}\left(1 - \frac{\bar{x} - \bar{X}}{\bar{X}}\right) \tag{2.4}$$

$$\simeq \frac{\bar{y} - R\bar{x}}{\bar{X}} - \left(\frac{\bar{y} - R\bar{x}}{\bar{X}}\right)\left(\frac{\bar{x} - \bar{X}}{\bar{X}}\right) \tag{2.5}$$

$$\simeq \frac{\bar{y} - R\bar{x}}{\bar{X}} + \frac{1}{\bar{X}^2}\left[- \bar{y}(\bar{x}-\bar{X}) + R\bar{x}(\bar{x}-\bar{X})\right] \tag{2.6}$$

Now

$$E\left(\frac{\bar{y} - R\bar{x}}{\bar{X}}\right) = \frac{1}{\bar{X}}(\bar{Y} - R\bar{X}) \tag{2.7}$$

$$= 0 \, .$$

Furthermore,

$$E\left[\frac{\bar{y}(\bar{x}-\bar{X})}{\bar{X}^2}\right] = \frac{1}{\bar{X}^2} E(\bar{y}-\bar{Y})(\bar{x}-\bar{X}) \tag{2.8}$$

$$= \frac{1-f}{n\bar{X}^2} \rho S_y S_x \, . \tag{2.9}$$

Also,

$$E\left[\frac{R\bar{x}(\bar{x}-\bar{X})}{\bar{X}^2}\right] = \frac{R}{\bar{X}^2} E[(\bar{x}-\bar{X})(\bar{x}-\bar{X})] \tag{2.10}$$

$$= \frac{R}{\bar{X}^2} E(\bar{x}-\bar{X})^2 \tag{2.11}$$

$$= \frac{R}{\bar{X}^2}\left(\frac{1-f}{n}\right) S_x^2 \, . \tag{2.12}$$

Therefore,

$$E(\hat{R}-R) = \frac{1-f}{n\bar{X}^2}(RS_x^2 - \rho S_x S_y) \tag{2.13}$$

$$\simeq \frac{1-f}{n}(C_{xx} - C_{xy})R \tag{2.14}$$

Mean Square Error

We now come to the question of the precision of the ratio estimator. Since the estimator is generally biased, its mean square error would be of interest. For large samples

$$MSE(\hat{R}) \simeq Var(\hat{R}) \simeq \frac{1-f}{n\bar{X}^2} \sum_{i=1}^{N} \frac{(y_i - Rx_i)^2}{N-1} \quad (2.15)$$

Thus

$$MSE(\hat{\bar{Y}}_R) \simeq \frac{1-f}{n} \sum_{i=1}^{N} \frac{(y_i - Rx_i)^2}{N-1} \quad (2.16)$$

$$\simeq \frac{1-f}{n(N-1)} \sum_{i=1}^{N} [(y_i - \bar{Y}) - R(x_i - \bar{X})]^2 \quad (2.17)$$

$$\simeq \frac{1-f}{n(N-1)} [\sum_{i=1}^{N} (y_i - \bar{Y})^2 + R^2 \sum_{i=1}^{N} (x_i - \bar{X})^2$$

$$-2R \sum_{i=1}^{N} (y_i - \bar{Y})(x_i - \bar{X})] \quad (2.18)$$

$$\simeq \frac{1-f}{n} (S_y^2 + R^2 S_x^2 - 2R\rho S_x S_y) \quad (2.19)$$

2.3 Comparison with the Sample Mean

The circumstances under which the ratio estimate will be superior to the sample mean will now be pointed out. The variance of the mean $\bar{y}$ from a simple random sample is

$$Var(\bar{y}) = (\frac{1-f}{n}) S_y^2 . \quad (2.20)$$

For the ratio estimate we have from (2.19)

$$\text{Var}(\hat{\bar{Y}}_R) \simeq \frac{1-f}{n}\,(S_y^2 + R^2 S_x^2 - 2R\rho S_x S_y).$$

Hence the ratio estimate is more efficient than the unbiased estimator $\bar{y}$ (in large samples) if

$$S_y^2 + R^2 S_x^2 - 2R\rho S_x S_y < S_y^2 \tag{2.21}$$

that is

$$\rho > \frac{RS_x}{2S_y} \quad \text{or} \quad \rho > \frac{C_x}{2C_y} \tag{2.22}$$

provided x and y are positive.

2.4 Unbiased Ratio-Type Estimators

Another line of research has been to modify the usual ratio estimator itself into one which is unbiased under simple random sampling. Since this estimator is not strictly a ratio estimator in the sense of involving only ratio of estimators, this may be termed a ratio-type estimator.

Hartley and Ross (1954) proposed the following unbiased ratio-type estimator. It is obtained by starting with the mean $\bar{r}$ of the ratios y_i/x_i and correcting it for its bias. Thus, letting

$$\bar{r} = \frac{1}{n}\sum_{i=1}^{n} r_i = \frac{1}{n}\sum_{i=1}^{n} y_i/x_i \tag{2.23}$$

$$\begin{aligned}
\text{Bias}(\bar{r}) &= E(\bar{r}) - R \\
&= E(r_i) - R \\
&= -[R - E(r_i)] \\
&= -\frac{1}{\bar{X}}[\bar{Y} - \bar{X}\,E(r_i)]
\end{aligned}$$

$$= -\frac{1}{\bar{X}}\left[\frac{1}{N}\sum_{i=1}^{N}\frac{y_i}{x_i}x_i - \left(\frac{1}{N}\sum_{i=1}^{N}r_i\right)\bar{X}\right]$$

$$= -\frac{1}{N\bar{X}}\sum_{i=1}^{N}r_i(x_i - \bar{X}). \qquad (2.24)$$

Furthermore, an unbiased sample estimate of $\frac{1}{N-1}\sum_{i=1}^{N}r_i(x_i-\bar{X})$ is

$$\frac{1}{n-1}\sum_{i=1}^{n}r_i(x_i-\bar{x}) = \frac{n}{n-1}(\bar{y}-\bar{r}\bar{x}). \qquad (2.25)$$

Thus, the estimate $\bar{r}$, corrected for its bias, becomes

$$\hat{R}_{HR} = \bar{r} + \frac{n(N-1)}{(n-1)N\bar{X}}(\bar{y}-\bar{r}\bar{x}). \qquad (2.26)$$

The corresponding unbiased estimate of the population mean $\bar{Y}$ is

$$\hat{\bar{Y}}_{HR} = \bar{r}\bar{X} + \frac{n(N-1)}{N(n-1)}(\bar{y}-\bar{r}\bar{x}) \qquad (2.27)$$

By similar agruments, another unbiased estimate (Mickey, 1959) is derived from the n ratios $\hat{R}_i$ obtained by removing each unit in turn from the sample, so that $\hat{R}_i = \Sigma y/\Sigma x$ over the remaining $(n-1)$ members. If $\bar{\hat{R}}_-$ denotes the mean of the $\hat{R}_i$, Mickey's estimate is

$$\hat{R}_M = \bar{\hat{R}}_- + \frac{n(N-n+1)}{N\bar{X}}(\bar{y} - \bar{\hat{R}}_-\bar{x}) \qquad (2.28)$$

Other unbiased ratio-type estimates were introduced by Quenouille (1956)

$$\hat{R}_Q = g\hat{R} - (g-1)\hat{R}_- \qquad (2.29)$$

where $\hat{R}_{-}$ is the average of g quantities $\hat{R}_j$ and $\hat{R}_j$ is the ordinary ratio $\Sigma y/\Sigma x$ from the sample, which has been divided into g groups of size $m = n/g$, after omitting the j^{th} group. Beale's (1962) estimator is

$$\hat{R}_B = \frac{\bar{y} + [(1-f)/n](s_{xy}/\bar{x})}{\bar{x} + [(1-f)/n](s_x^2/\bar{x})} \qquad (2.30)$$

while Tin's (1965) is the closely related quantity.

$$\hat{R}_T = \hat{R}[1 + \frac{(1-f)}{n}(c_{xx} - c_{xy})] . \qquad (2.31)$$

Suppose a random sample of size n is split into s independent subsamples each of size m so that $n = sm$. Let $\bar{x}_i$ and $\bar{y}_i$ be unbiased estimators of $\bar{X}$ and $\bar{Y}$ respectively based on the i^{th} subsample, then Murthy and Nanjamma's (1959) unbiased ratio-type estimator (to the second degree of approximation) is

$$\hat{R}_{MN} = \frac{s\hat{R}_1 - \hat{R}_s}{s - 1} \qquad (2.32)$$

where

$$\hat{R}_1 = \sum_{i=1}^{s} \bar{y}_i / \sum_{i=1}^{s} \bar{x}_i \qquad (2.33)$$

and

$$\hat{R}_s = \frac{1}{s} \sum_{i=1}^{s} \bar{y}_i/\bar{x}_i . \qquad (2.34)$$

T.J. Rao (1966b) considered the two ratio-type estimators $(\bar{y}\bar{X})/\bar{x}$ and $\frac{\bar{X}}{m} \sum_{i=1}^{m} y_i/x_i$ for the estimation of $\bar{Y}$ and showed how the unbiased ratio-type estimators due to Murthy and Nanjamma (1959), Hartley and Ross (1954) and Nieto de Pascual

(1961) can be obtained as linear combinations of these two estimators.

2.5 Best Linear Unbiasedness

The following theorem brings out the ratio estimator as the best linear unbiased estimator under a superpopulation model (Brewer (1963), Royall (1970)).

Theorem 2.1

The ratio estimator $\hat{Y}_R = \frac{\bar{y}}{\bar{x}} X$ is the best linear unbiased estimator for any sample selected solely according to the values of the x_i, assuming that the N population values (y_i, x_i) are a random sample coming from a superpopulation model in which

$$y_i = \beta x_i + \varepsilon_i , \qquad i = 1,2,\ldots N$$
$$x_i > 0$$

where,

$$\varepsilon_i \text{ are independent of the } x_i \qquad (2.35)$$

and

$$E_M(\varepsilon_i | x_i) = 0, \; E_M(\varepsilon_i^2 | x_i) = \lambda x_i , \; \lambda \text{ being a constant.}$$

Proof:

Since $E_M(\varepsilon_i | x_i) = 0$, it follows from (2.35) that

$$Y = \beta X + \sum_{i=1}^{N} \varepsilon_i \qquad (2.36)$$

Furthermore, with the model (2.35) any linear estimator Y is of the form

$$\hat{Y} = \sum_{i=1}^{n} l_i y_i = \beta \sum_{i=1}^{n} l_i x_i + \sum_{i=1}^{n} l_i \epsilon_i . \qquad (2.37)$$

If we keep the n sample values x_i, fixed in repeated sampling under the model (2.35)

$$E_M(\hat{Y}) = \beta \sum_{i=1}^{n} l_i x_i , \quad V_M(\hat{Y}) = \lambda \sum_{i=1}^{n} l_i^2 x_i . \qquad (2.38)$$

Under the model (2.35), $Y = \beta X + \sum_{i=1}^{N} \epsilon_i$ is a random variable and an estimator $\hat{Y}$ is unbiased if $E(\hat{Y}) = E(Y)$ in repeated selections of the finite population and sample under the model. Such an estimator might be called model-unbiased. Thus from (2.36) and (2.38), $\hat{Y}$ is clearly model-unbiased if $\sum_{i=1}^{n} l_i x_i \equiv X$. Minimizing $V_M(\hat{Y})$ under this condition by a Lagrange multiplier c gives $2l_i x_i = cx_i$, i.e. $l_i = \frac{X}{n\bar{x}}$. Hence the Best Linear Unbiased Estimator $\hat{Y}$ is $\frac{n\bar{y}X}{n\bar{x}} = \frac{\bar{y}}{\bar{x}} X = \hat{Y}_R$ which is the ratio estimator.

2.6 Other Investigations

Durbin (1959) considered the following superpopulation model

$$Y_i = \alpha + \beta x_i + \epsilon_i \qquad i = 1, 2, \ldots , N$$

where

$$E_M(\epsilon_i | x_i) = 0 , \qquad E_M(\epsilon_i^2 | x_i) = \delta x_i^t . \qquad (2.39)$$

and

x_i's are independent gamma variates with parameter h.

This model has been used to compare various ratio estimators and under this same model Rao and Webster (1966) showed that the bias of the ratio estimate, $\hat{\bar{Y}}_R$, is given by

$$E_M E(\hat{\bar{Y}}_R - \bar{Y}) = \frac{\alpha}{nh(nh-1)} \tag{2.40}$$

and that the average MSE of $\hat{\bar{Y}}_R$, obtained by P.S.R.S. Rao (1968), is given by

$$AMSE(\hat{\bar{Y}}_R) = E_M E(\hat{\bar{Y}}_R - \bar{Y})^2$$

$$= \alpha^2 A_r + \delta D_r . \tag{2.41}$$

where

$$A_r = \frac{N-n}{N^2} \cdot \frac{Nnh + 2N - 2n}{(nh-1)(nh-2)},$$

$$D_r = \frac{N-n}{N^2} \cdot \frac{\{(nh+t-1)(nh+t-2) + nh(Nh-nh+1)\}}{(nh+t-1)(nh+t-2)} \cdot \frac{\Gamma(t+h)}{\Gamma(h)}$$

Rao and Rao (1971) have used the model (2.39) to find the relative efficiencies of five estimators for a ratio, namely, the classical ratio estimator, Quenouille's (1956) estimator, Tin's (1965) estimator, the Hartley-Ross (1954) estimator and Mickey's (1959) estimator through a numerical study.

Srivastava et al (1983) have derived the exact bias and mean square error of Beale's (1962) ratio estimator under a bivariate normal model in the form of an infinite series. It is found that some conventional large sample approximations are extremely poor if the relative variances of the auxiliary variable is large. It is also brought out that Beale's estimator of the population mean seems to be more efficient than the usual sample mean under the condition resulting from the large sample comparison of the customary ratio estimator and the usual sample mean.

Hutchison (1971) compared six ratio estimators, the classical ratio estimator, the Hartley-Ross (1954) estimator, Mickey's (1954), Quenouille's (1956), Beale's (1962) and Tin's (1965) estimator by a Monte Carlo technique under two models in each of which the conditional variance of y given x, depends upon t .

T.J. Rao (1966a) derived an expression for the variance of the ratio estimator for the sampling scheme of Midzuno (1952) and Sen (1952). T.J. Rao (1972) then studied some of the interesting properties of the coefficients involved in the expression obtained before which depend on the auxiliary information. The use of these coefficients is made of in finding out an exact expression for the bias and mean square error of the ratio estimator under simple random sampling without replacement. Later T.J. Rao (1977) estimated the variance of the ratio estimator for the sampling scheme of Midzuno (1952) and Sen (1952).

S.K. Ray, Ashok Sahai and Ajit Sahai (1979) proposed a transformed ratio estimator obtained through parametric linear combination of the ratio and the usual unbiased estimator of the mean for any sample design. To the first degree of approximation, the proposed estimator has a smaller mean square error than that of the ratio and usual unbiased estimator for suitable choice of the parameter. The transformed ratio estimator is given by

$$\hat{\bar{Y}}_{RA} = (1 - A)\bar{Y} - A\,\hat{\bar{Y}}_R \qquad (2.42)$$

where "A" (> 0) is a scalar constant.

S.K. Ray and Ashok Sahai (1980) obtained two parameter families of ratio-type and product-type estimators for a finite population mean based on simple random samples of observations on the variable of interest and a concomitant variable. Using some prior information it is shown that the families contain estimators which have in practical situations lower mean squared error than the usual ratio, product and sample mean estimators.

S.K. Ray and Ajit Sahai (1980) suggested a transformed estimator which is even more efficient than the ratio-cum-product type estimators proposed by Srivastava (1967) and Reddy (1973) for a wide range of the value of the correlation coefficient between the main and auxiliary variables.

Sahoo and Swain (1980) proposed a Hartley-Ross (1954) type unbiased ratio-cum-product estimator of a finite population mean $\bar{Y}$. It is observed to be a particular case of generalized unbiased estimators due to Williams (1961, 1963) and Mickey (1959). The conditions for the proposed estimator to be asymptotically more efficient than M.P. Singh's (1967) biased estimator are also derived.

Royall and Eberhardt (1975) studied four estimators for the variance of the ratio estimator under various linear prediction (superpopulation) models. These are(i) the conventional statistic, (ii) the Jack-Knife estimator, (iii) a weighted least squares estimator from linear prediction theory and (iv) a new estimator obtained by adjusting the conventional estimator in ways suggested by linear prediction theory. Previous studies have indicated that the weighted least squares estimator is better than the conventional one for setting confidence intervals when the model most often used in ratio estimation studies applies. Here two more estimators are studied under this model and some effects of the failure of the model are examined. The conventional estimator appears to be the worst of the four. The new estimator and the jack-knife estimator share some important advantages when the actual variance structure is not that represented in the usual model.

M.P. Singh (1967) suggested some methods of estimation which may be considered as combination of ratio and product methods. The mean square errors of these estimators utilizing two supplementary variables are compared with (i) simple unbiased estimator (p = 0), (ii) usual ratio and product methods of estimators (p = 1) and (iii) multivariate ratio and multivariate product estimators (p = 2), where p is the number of supplementary variables utilized. Conditions for their efficient use have been obtained for each case and the extension to general case of p-variables are briefly discussed.

Sengupta (1981) compared the mean square error of the almost unbiased ratio-type estimator, obtained with the help of the Jack-Knifing technique for simple random sampling in two phases, with the usual (biased) estimator and it has been found that they are approximately the same.

Shah and Shah (1978) suggested a new ratio-cum-product estimator for estimating the ratio of two population means using auxiliary information on two other variables. Its bias and mean square error to the order of $(1/n)$ are obtained and the efficiency of the estimator is compared using optimum and simple weights, with the classical ratio estimator and the estimators suggested by M.P. Singh (1965, 1967, 1969).

Srivenkataramana (1980) proposed a dual to ratio estimator

$$\bar{Y}_a = (\bar{y}\hat{\bar{X}}^*)/\bar{X} \qquad (2.43)$$

which is essentially a product type estimator of Murthy (1964) where $\hat{\bar{X}}^* = (N\bar{X} - n\bar{x})/(N-n)$ and is based on the complement of the sample.

Chaubey et al (1985 c) investigated the dual to ratio estimator discussed by Srivenkataramana (1980) in double sampling. They obtained the expressions for the bias and mean square error up to order of $(1/n)$ and also the relative efficiency under a linear cost structure.

Srivenkataramana and Tracy (1979) considered four estimators suited for cases where the positive correlation is only moderate and gave a rule of thumb for choosing among these and the classical ratio estimator. Such a choice needs a good guess of the interval containing a certain parameter k, which may not be hard in survey practice. A numerical example has also been included for illustration.

Reddy (1974) proposed an alternative estimator to the ratio estimator

$$\hat{Y}_\theta = \frac{\bar{y}\bar{X}}{\bar{X} + \theta(\bar{x}-\bar{X})} \tag{2.44}$$

which is based on the supplementary information where θ is a scalar such that $\left|\frac{\theta(\bar{x}-\bar{X})}{\bar{X}}\right| < 1$. This estimator is almost unbiased and has a smaller bias and mean square error than the classical ratio estimator for a fairly wide range of θ.

Nanjamma, Murthy and Sethi (1959) have modified many of the selection procedures commonly adopted in practice, namely, equal probability sampling, varying probability sampling, stratified sampling and multi-stage sampling, which, while retaining the form of the usual biased ratio estimators, make then unbiased. This modification of a given sampling scheme consists essentially in first selecting one unit with probability proportional to its value of the characteristic, x, occurring in the denominator of the ratio and then the remaining units in the sample according to the original sampling scheme.

Nieto de Pascual (1961) obtained several unbiased ratio estimators in stratified random sampling. The "separate" unbiased ratio estimator is a direct application of the Hartley-Ross unbiased ratio estimator within strata. The "combined" unbiased ratio estimator, for equal sample size k per stratum, is obtained by means of a complete replacement, repeated k times, of a sample of size one per stratum. The relative efficiencies of these estimators with respect to the usual biased ratio estimators are discussed. Finally an approximately unbiased ratio estimator is developed and it is shown that, for large samples, it is as efficient as the "combined" biased ratio estimator.

Lahiri (1951) showed that the ratio estimator would become an unbiased estimator of the population ratio if the probability of selecting the sample is made proportional to its mean or total size. This can be achieved by selecting one unit with probability proportional to x and the rest with simple random sampling without replacement from the remaining units from the population.

CHAPTER III

PRODUCT ESTIMATORS

3.1 The Product Estimator

Let $\bar{y}$ and $\bar{x}$ be the sample means corresponding to the variate under consideration and the auxiliary variate respectively and let the value $\bar{X}$ be known, then the product estimator of the population mean is given by

$$\hat{\bar{Y}}_p = \frac{\bar{y}\bar{x}}{\bar{X}} . \tag{3.1}$$

This estimator is suggested, as it is complementary to the ratio estimator $\hat{\bar{Y}}_R = \frac{\bar{y}\bar{X}}{\bar{x}}$ and hence is likely to be useful in situations, where the ratio estimator is not efficient, that is, where the estimators $\bar{y}$ and $\bar{x}$ are negatively correlated. However the product estimator has not acquired the same popularity as the ratio estimator due to the misconceptions about occurrence of negative correlation in practice and apparent superiority of ratio estimator (under positive correlation) under design based comparisons. It is true that positive correlations are often encountered in practice but negative correlations are often induced through inverse transformations of the regressor variable (see Neter and Wasserman (1974) fig. 4.17).

Robson (1957) studied a modification of (3.1) and Goodman (1960) considered the question of obtaining variances and variance estimators of products only for infinite populations. Even after the work of Murthy (1964) the product estimator of the form (3.1) did not catch enough attention of practitioners and it is getting attention of researchers only recently. In this chapter we study some of the properties of the product estimator and review some of the investigations carried out on this estimator. Section 3.2 gives the bias and MSE and section 3.3 presents its comparison with the sample mean. Section 3.4 presents some modifications of (3.1) and the last section gives other investigations.

3.2 Bias and MSE of the Product Estimator

Bias

As we have seen in Chapter II in the case of ratio estimator, the product estimator has also a bias of order 1/n.

Now

$$E\left(\frac{\bar{y}\bar{x}}{\bar{X}} - \bar{Y}\right) = \frac{1}{\bar{X}} E(\bar{y}\bar{x} - \bar{Y}\bar{X})$$

$$= \frac{1}{\bar{X}} \mathrm{Cov}(\bar{x},\bar{y})$$

$$= \frac{1}{\bar{X}} \left(\frac{1-f}{n}\right) S'_{xy} \quad . \qquad (3.2)$$

Mean Square Error

Theorem 3.1

For moderately large sample size, the mean square error of the product estimator is given by

$$MSE(\hat{\bar{Y}}_p) \doteq (\frac{1-f}{n})(S_y^2 + R^2S_x^2 + 2R\rho S_y S_x) \quad (3.3)$$

__Proof__

$$MSE(\hat{\bar{Y}}_p) = E(\frac{\bar{x}\bar{y}}{\bar{X}} - \bar{Y})^2$$

$$= E[\frac{\bar{x}\bar{y} - \bar{X}\bar{Y}}{\bar{X}}]^2$$

$$= \frac{1}{\bar{X}^2} E\{\bar{X}\bar{Y}[\{(\frac{\bar{x}-\bar{X}}{\bar{X}})+1\}\{(\frac{\bar{y}-\bar{Y}}{\bar{Y}})+1\}-1]\}^2$$

$$= \bar{Y}^2 E[(\frac{\bar{x}-\bar{X}}{\bar{X}}) + (\frac{\bar{y}-\bar{Y}}{\bar{Y}}) + (\frac{\bar{x}-\bar{X}}{\bar{X}})(\frac{\bar{y}-\bar{Y}}{\bar{Y}})]^2$$

$$= \frac{\bar{Y}^2}{\bar{X}^2} E(\bar{x}-\bar{X})^2 + E(\bar{y}-\bar{Y})^2 + 2\frac{\bar{Y}}{\bar{X}} E(\bar{x}-\bar{X})(\bar{y}-\bar{Y})$$

$$+ \; E(\frac{\bar{x}-\bar{X}}{\bar{X}})^2(\bar{y}-\bar{Y})^2 + 2\bar{Y}\; E(\frac{\bar{x}-\bar{X}}{\bar{X}})^2(\bar{y}-\bar{Y})$$

$$+ \; 2E(\frac{\bar{x}-\bar{X}}{\bar{X}})(\bar{y}-\bar{Y})^2$$

$$= (\frac{1-f}{n})(R^2S_x^2+S_y^2+2R\rho S_y S_x) + 0(n^{-2}) \quad (3.4)$$

which can be approximated by

$$MSE(\hat{\bar{Y}}_p) \doteq (\frac{1-f}{n})(S_y^2 + R^2S_x^2 + 2R\rho S_x S_y)$$

when n is large.

3.3 Comparison with the Sample Mean

In large samples, with simple random sampling, the product estimator $\hat{\bar{Y}}_p$ is more efficient than the unbiased estimator $\bar{y}$. This can be easily seen as follows. Since variance of the mean $\bar{y}$ is given by

$$\text{Var}(\bar{y}) \doteq \left(\frac{1-f}{n}\right) S_y^2 \tag{3.5}$$

and from (3.4) we have

$$\text{MSE}(\hat{\bar{Y}}_p) \simeq \left(\frac{1-f}{n}\right)(S_y^2 + R^2 S_x^2 + 2R\rho S_x S_y) ,$$

therefore the product estimator is more efficient than $\bar{y}$ if

$$S_y^2 + R^2 S_x^2 + 2R\rho S_x S_y < S_y . \tag{3.6}$$

The condition in (3.6) is equivalent to

$$\rho < -\frac{RS_x}{2S_y} \tag{3.7}$$

or

$$\rho < -\frac{C_x}{2C_y} \tag{3.8}$$

in case both the main and auxiliary variables are negative or positive, and it becomes

$$\rho < \frac{C_x}{2C_y} \tag{3.9}$$

in case either of the two variables is negative and the other is positive.

3.4 Unbiased Product-Type Estimators

Robson (1957) proposed the following product estimator obtained by subtracting an unbiased estimate of bias. Since an unbiased estimator of S_{xy} is $s_{xy} = \frac{1}{n-1} \sum_{i=1}^{n} (x_i-\bar{x})(y_i-\bar{y})$ we have the following unbiased product type estimator

$$\hat{\hat{Y}}_R = \frac{\bar{y}\bar{x}}{\bar{X}} - \left(\frac{1-f}{n\bar{X}}\right) s_{xy} \qquad (3.10)$$

Murthy (1964) has considered simple random sampling from infinite population and has provided an unbiased product type estimator, combining the two definitions of the product estimator, which are the duals of the definitions of ratio estimators. This is detailed below.

Suppose a random sample of size n is split into s independent subsamples each of size m so that $n = sm$. Let $\bar{x}_i$ and $\bar{y}_i$ be unbiased estimators of $\bar{X}$ and $\bar{Y}$ respectively based on the i^{th} subsample, then Murthy's unbiased estimator is

$$\hat{\hat{Y}}_M = \frac{1}{s-1}(s\hat{\hat{Y}}_1 - \hat{\hat{Y}}_2) \qquad (3.11)$$

where

$$\hat{\hat{Y}}_1 = \frac{\bar{y}.\bar{x}.}{\bar{X}} = \frac{1}{\bar{X}}\left(\frac{1}{s}\sum_{i=1}^{s}\bar{y}_i\right)\left(\frac{1}{s}\sum_{i=1}^{s}\bar{x}_i\right) \qquad (3.12)$$

and

$$\hat{Y}_2 = \frac{\sum_{i=1}^{s}\bar{y}_i\bar{x}_i}{s\bar{X}}\,. \qquad (3.13)$$

It is shown below that $\hat{\hat{Y}}_M$ is unbiased.

Applying result of equation (3.3) to the estimator in (3.12) we get

$$B(\hat{\hat{Y}}_1) = \frac{1}{\bar{X}} E(\bar{y}. - \bar{Y})(\bar{x}. - \bar{X})$$

$$= \frac{1}{\bar{X}}\left\{\frac{1}{s^2}\sum_{i=1}^{s} E(\bar{y}_i-\bar{Y})(\bar{x}_i-\bar{X})\right\} \qquad (3.14)$$

(B(.) represents the bias)

since $\bar{y}_i$ and $\bar{x}_j$ are uncorrelated for $i \neq j$.

Hence we have

$$B(\hat{\bar{Y}}_1) = \frac{1}{s^2} \sum_{i=1}^{s} B\left(\frac{\bar{y}_i \bar{x}_i}{\bar{X}}\right) . \tag{3.15}$$

Now the bias of $\hat{\bar{Y}}_2$ is given by

$$B(\hat{\bar{Y}}_2) = \frac{1}{s} \sum_{i=1}^{s} B\left(\frac{\bar{y}_i \bar{x}_i}{\bar{X}}\right). \tag{3.16}$$

Comparing (3.15) and (3.16) we see that the bias of $\hat{\bar{Y}}_2$ is s times the bias of $\hat{\bar{Y}}_1$, i.e.

$$B(\hat{\bar{Y}}_2) = sB(\hat{\bar{Y}}_1) . \tag{3.17}$$

Hence,

$$E(\hat{\bar{Y}}_2 - \hat{\bar{Y}}_1) = B(\hat{\bar{Y}}_2) - B(\hat{\bar{Y}}_1)$$

$$= (s-1)B(\hat{\bar{Y}}_1) \tag{3.18}$$

and an unbiased estimator of the bias of $\hat{\bar{Y}}_1$ is given by

$$\hat{B}(\hat{\bar{Y}}_1) = \frac{\hat{\bar{Y}}_2 - \hat{\bar{Y}}_1}{s-1} . \tag{3.19}$$

This unbiased estimator of the bias of $\hat{\bar{Y}}_1$ may be used to correct it for its bias, thereby obtaining an unbiased product estimator $\hat{\bar{Y}}_M$ given by

$$\hat{\bar{Y}}_M = \frac{s\hat{\bar{Y}}_1 - \hat{\bar{Y}}_2}{s-1} . \tag{3.20}$$

Shukla (1976) obtained an unbiased product type estimator (to the first degree of approximation) with the help of the technique developed by Quenouille (1956) and has established that this new estimator is better than the product estmator suggested by Murthy (1964) in the mean square error sense.

Chaubey et al (1981) presented a product-type estimator based on harmonic mean of the auxiliary character and demonstrated its unbiasedness under the sampling scheme of Midzuno (1952) and Sen (1952). An unbiased estimator of its variance is also found.

3.5 Other Investigations

Chaubey et al (1984b) have used Durbin's (1959) superpopulation model to compare the efficiency of product and ratio estimator and showed that the bias of the product estimator is given by

$$E_M E(\hat{\bar{Y}}_p - \bar{Y}) = \beta(\frac{1}{n} - \frac{1}{N}) \frac{Nh}{Nh+1} \qquad (3.21)$$

and that the average MSE is

$$AMSE(\hat{\bar{Y}}_p) = E_M E(\hat{\bar{Y}}_p - \bar{Y})^2$$

$$= \alpha^2 A_p + \beta^2 B_p + 2\alpha\beta C_p + \delta D_p \qquad (3.22)$$

where

$$A_p = \left(\frac{1}{n} - \frac{1}{N}\right)\frac{N}{Nh+1}$$

$$B_p = \frac{N^2 h}{n^3 (Nh+2)(Nh+3)} \{(h+1)(h+2)(h+3)+(n-1)h(h+1)(7h+11)$$

$$+ 6(n-1)(n-2)h^2(h+1)$$

$$+ (n-1)(n-2)(n-3)h^3\}$$

$$- \frac{2h(nh+1)}{n} + \frac{h(Nh+1)}{N}$$

$$C_p = \frac{N^2 h\{(h+1)(h+2) + 3(n-1)h(h+1) + (n-1)(n-2)h^2\}}{n^2(Nh+1)(Nh+2)}$$

$$- \frac{Nh(nh+1)}{n(Nh+1)}$$

$$D_p = \frac{N^2}{n^3(Nh+t)(Nh+t+1)} \cdot \frac{\Gamma(t+h)}{\Gamma(h)} \{(t+h)(t+h+1)$$

$$+ (n-1)h(3h+2t+1) + (n-1)(n-2)h^2\}$$

$$+ \frac{\Gamma(t+h)}{N\Gamma(h)} - \frac{2(nh+t)}{n(Nh+t)} \cdot \frac{\Gamma(t+h)}{\Gamma(h)} .$$

They have also obtained the expected variance of the unbiased product estimator of Robson (1957) and a theoretical comparison is made under an infinite population (see also correction to this paper, Chaubey et al (1985b)).

S.K. Ray, Ashok Sahai and Ajit Sahai (1979) proposed a transformed product estimator obtained through parametric linear combination of the product and the usual unbiased estimator of the mean for any sample design. To the first degree of approximation, the proposed estimator has a smaller mean square error than that of the product and usual unbiased estimator for suitable choice of the parameter. The transformed product estimator is given by

$$\hat{\bar{Y}}_{pA} = (1+A)\bar{Y} - A\hat{\bar{Y}}_p .$$

Sengupta (1981) compared the mean square error of the almost unbiased product-type estimator, obtained with the help of the Jack-Knifing technique for simple random sampling in two phases, with the usual (biased) estimator and it has been found that they are approximately the same.

Shukla et al (1981) compared the product estimators proposed by Robson (1957) and Murthy (1964) and it is found that the Robson's estimator gives a better performance.

Srivastava (1966) considered the product estimator defined by Murthy (1964) and obtained certain results analogous to that of the ratio estimator, namely, asymptotic distribution of the estimator and in different sampling schemes.

Chaubey et al (1985a) considered the usual product estimator adjusted for its bias, considered by Robson (1957), and obtained the exact variance of this estimator through a direct method. Its superiority over the sample mean and the product estimator is demonstrated.

Shah and Shah (1979) obtained an unbiased product-type estimator by using the technique employed by T.J. Rao (1966) for unbiased ratio-type estimator. The variance of the estimator is derived using the expectations of symmetric functions of sample observations. The results are also extended in the case of stratified sampling.

CHAPTER IV

REGRESSION ESTIMATORS

4.1 The Linear Regression Estimator

In the previous two chapters, we considered the question of improving the conventional unbiased estimator $\bar{y}$ by the use of the auxiliary information, by multiplying it with the factor $\bar{X}/\bar{x}$ and $\bar{x}/\bar{X}$ respectively. Here we examine the possibility of improving $\bar{y}$ by considering the estimator $\bar{X} - \bar{x}$ which is a zero function in the sense that its expected value is zero, from which the linear regression estimate is obtained.

Suppose that y_i and x_i are each obtained for each unit in the sample and that the population mean $\bar{X}$ for the x_i is known. The linear regression estimate of $\bar{Y}$ is given by

$$\hat{\bar{Y}}_{lr} = \bar{y} + b(\bar{X} - \bar{x}) \qquad (4.1)$$

where the subscript "lr" denotes linear regression and b is an estimate of the change in y when x is increased by unity.

Watson (1937) used a regression of leaf area on leaf weight to estimate the average area of the leaves on a plant. The procedure was to weigh all the leaves on a plant. For a small sample of leaves, the area and the weight of each leaf were determined. The sample mean leaf area was then adjusted by means of the regression on leaf weight. The point of the application is, of course, that the weight of a leaf can be found quickly but determination of its area is more time consuming.

The regression estimator is not commonly used in practice due to the fact that the calculation of the estimate of the regression coefficient, b, in large-scale surveys become cumbersome and time-consuming. Further, since the regression line passes through the origin or close to the origin in most of the cases usually met with, the ratio estimator is generally used instead of the more complicated regression estimator. But, in general, the estimator $\hat{\hat{Y}}_{1r}$ is optimum with respect to MSE criterion in a class considered by Chaubey et al (1984 a). Moreover, with new efficient computing techniques and software, this estimator is gaining more popularity (see Särndal (1984)).

We discuss the properties of $\hat{\hat{Y}}_{1r}$ with fixed b and estimated b from the sample in sections 4.2 and 4.3 respectively. Section 4.4 makes its MSE comparison with the sample mean and section 4.5 considers a superpopulation regression model. Finally, section 4.6 gives other investigations concerning this estimator.

4.2 Properties of the Regression Estimator with Preassigned b

Although b is estimated from the results of the sample, it is sometimes reasonable to choose the value of b in advance. In repeated surveys, previous calculations may have been shown that the sample values of b remain fairly constant. We now present the bias and MSE properties in the following theorems.

Theorem 4.1

In simple random sampling, in which $b = b_0$ is a preassigned constant, the linear regression estimate is unbiased.

Proof:

Since b_0 is constant in repeated sampling

$$E(\hat{\bar{Y}}_{1r}) = E\,[\bar{y} + b_0(\bar{X} - \bar{x})]$$

$$= E(\bar{y}) + b_0\; E(\bar{X} - \bar{x})$$

$$= \bar{Y}\;. \qquad (4.2)$$

Theorem 4.2

For a given value b_0 of b

$$V(\hat{\bar{Y}}_{1r}) = \left(\frac{1-f}{n}\right)(S_y^2 + b_0^2 S_x^2 - 2b_0 S_{xy}) \qquad (4.3)$$

Proof:

$$V(\hat{\bar{Y}}_{1r}) = E(\bar{Y}_{1r} - \bar{Y})^2 \qquad (4.4)$$

$$= E\{(\bar{y} - \bar{Y}) - b_0(\bar{x} - \bar{X})\}^2$$

$$= E\{(\bar{y}-\bar{Y})^2 + b_0^2(\bar{x}-\bar{X})^2 - 2b_0(\bar{x}-\bar{X})(\bar{y}-\bar{Y})\}$$

$$= E(\bar{y}-\bar{Y})^2 + b_0^2 E(\bar{x}-\bar{X})^2 - 2b_0 E(\bar{x}-\bar{X})(\bar{y}-\bar{Y})$$

$$= \left(\frac{1-f}{n}\right)S_y^2 + b_0\left(\frac{1-f}{n}\right)S_x^2 - 2b_0\left(\frac{1-f}{n}\right)S_{xy}$$

$$= \left(\frac{1-f}{n}\right)\{S_y^2 + b_0 S_x^2 - 2b_0 S_{xy}\} \qquad (4.5)$$

Theorem 4.3

The value of b_0 that minimizes $V(\hat{\bar{Y}}_{1r})$ is

$$b_0 = B = \frac{S_{xy}}{S_x^2} = \frac{\sum_{i=1}^{N} (y_i-\bar{Y})(x_i-\bar{X})}{\sum_{i=1}^{N} (x_i-\bar{X})^2} \tag{4.6}$$

which may be called the linear regression coefficient of y on x in the finite population. Note that B depends on the population values, hence it is not computable but could theoretically be preassigned. The resulting minimum variance is

$$V_{Min}(\hat{\bar{Y}}_{1r}) = (\frac{1-f}{n})S_y^2(1 - \rho^2) \tag{4.7}$$

Proof:

Let $g(b) = V(\hat{\bar{Y}}_{1r})$

$$= (\frac{1-f}{n})(S_y^2 - 2bS_{xy} + b^2S_x^2) . \tag{4.8}$$

Now,

$$\frac{dg(b)}{db} = (\frac{1-f}{n})[-2S_{xy} + 2bS_x^2] \tag{4.9}$$

and

$$\frac{d^2g(b)}{db^2} = (\frac{1-f}{n})2S_x^2 > 0 . \tag{4.10}$$

Thus $g(b)$ is a minimum for b which is a solution of $\frac{dg(b)}{db} = 0$; solving which we have $b = \frac{S_{xy}}{S_x^2} = B$, the population regression coefficient.

Substituting B in place of b in (4.8) we get, since $S_{xy} = \rho S_x S_y$,

$$V_{Min}(\hat{\bar{Y}}_{1r}) = (\frac{1-f}{n})[S_y^2 - 2\rho\frac{S_y}{S_x}\rho S_x S_y + \rho^2 \frac{S_y^2}{S_x^2} S_x^2]$$

$$= (\frac{1-f}{n})(S_y^2 - \rho^2 S_y^2),$$

$$= (\frac{1-f}{n}) S_y^2(1 - \rho^2) \qquad (4.11)$$

4.3 Properties of the Regression Estimator when b is Computed from the Sample

Theorem 4.3 suggests that if b must be computed from the sample an effective estimate is likely to be the familiar least squares estimate of B , that is,

$$b = \frac{\sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} . \qquad (4.12)$$

Therefore, we have

$$E(\hat{\bar{Y}}_{1r}) = \bar{Y} - E[b(\bar{x} - \bar{X})]$$

and

$$B(\hat{\bar{Y}}_{1r}) = - E[b(\bar{x} - \bar{X})]$$

$$= - Cov(b, \bar{x}). \qquad (4.13)$$

Now we demonstrate that the bias is of order $1/n$. Letting $e_i = y_i - \bar{Y} - B(x_i - \bar{X})$, note that,

$$\sum_{i=1}^{N} e_i(x_i-\bar{X}) = \sum_{i=1}^{N} (y_i-\bar{Y})(x_i-\bar{X}) - B\sum_{i=1}^{N} (x_i-\bar{X})^2 = 0 \qquad (4.14)$$

and

$$\sum_{i=1}^{N} e_i = 0 . \qquad (4.15)$$

We can also write b as

$$b = \frac{\sum_{i=1}^{n} y_i(x_i-\bar{x})}{\sum_{i=1}^{n} (x_i-\bar{x})^2} \qquad (\because \sum_{i=1}^{n} \bar{y}(x_i-\bar{x}) = 0)$$

$$= \frac{\sum_{i=1}^{n} [\bar{Y}+B(x_i-\bar{X}) + e_i](x_i-\bar{x})}{\sum_{i=1}^{n} (x_i-\bar{x})^2}$$

$$= B + \frac{\sum_{i=1}^{n} e_i(x_i-\bar{x})}{\sum_{i=1}^{n} (x_i-\bar{x})^2} . \qquad (4.16)$$

Now replace $\sum_{i=1}^{n} (x_i-\bar{x})^2$ by its leading term nS_x^2 and write

$$\sum_{i=1}^{n} e_i(x_i-\bar{x}) = \sum_{i=1}^{n} e_i(x_i-\bar{X}) + n\bar{e}(\bar{X}-\bar{x}) . \qquad (4.17)$$

Hence the leading term in the bias $-E[b(\bar{x} - \bar{X})]$ of $\hat{\bar{Y}}_{1r}$ is the average of

$$\left[\frac{-\sum_{i=1}^{n} e_i(x_i-\bar{X})(\bar{x}-\bar{X})}{nS_x^2} + \frac{\bar{e}(\bar{x}-\bar{X})^2}{S_x^2}\right] . \qquad (4.18)$$

Let $u_i = e_i(x_i - \bar{X})$. By (4.14) its population mean $\bar{U} = 0$, thus the average value of the first term in (4.18) may therefore be written

$$-\frac{E(\bar{u}-\bar{U})(\bar{x}-\bar{X})}{S_x^2} = -\left(\frac{1-f}{n}\right)\frac{E(u_i-\bar{U})(x_i-\bar{X})}{S_x^2} . \quad (4.19)$$

This in turn equals

$$-\left(\frac{1-f}{n}\right)\frac{E[e_i(x_i-\bar{X})^2]}{S_x^2} \quad (4.20)$$

which is of order $1/n$. In the second term in (4.18), $\bar{e}$ is of order $1/\sqrt{n}$ and $(\bar{x}-\bar{X})^2$ is $O(1/n)$, so that this term is of smaller order than (4.20). Thus (4.20) is the leading term in the bias of $\hat{\bar{Y}}_{1r}$.

<u>Theorem 4.4</u>

If b is the least squares estimate of B and

$$\hat{\bar{Y}}_{1r} = \bar{y} + b(\bar{X}-\bar{x}) \quad (4.21)$$

then with n large

$$V(\hat{\bar{Y}}_{1r}) \simeq \left(\frac{1-f}{n}\right) S_y^2(1-\rho^2) \quad (4.22)$$

<u>Proof:</u>

The sampling error of $\hat{\bar{Y}}_{1r}$ arises from the quantity

$$\hat{\bar{Y}}_{1r} - \bar{Y} = \bar{y} - \bar{Y} + b(\bar{X}-\bar{x}) . \quad (4.23)$$

As an approximation, replace $\hat{\bar{Y}}_{1r}$ by

$$\tilde{\bar{Y}}_{1r} = \bar{y} + B(\bar{X}-\bar{x}), \quad (4.24)$$

where B is the population linear regression coefficient of y on x. The error committed in this approximation is $(B - b)(\bar{X} - \bar{x})$. This quantity is of order $1/n$ in a simple random sample of size n, since $(b - B)$ and $(\bar{x} - \bar{X})$ are both of order of $1/\sqrt{n}$. But the sampling error in $\hat{\bar{Y}}_{1r}$ is of order $1/\sqrt{n}$, since it is the error in the sample mean of the variable $(y_i - Bx_i)$. Hence the leading term of $E(\hat{\bar{Y}}_{1r} - \bar{Y})^2$ is $V(\hat{\bar{Y}}_{1r})$. By (4.7), in large samples, thus,

$$E(\hat{\bar{Y}}_{1r} - \bar{Y})^2 \simeq V(\hat{\bar{Y}}_{1r}) = (\frac{1-f}{n})S_y^2(1 - \rho^2) \qquad (4.25)$$

4.4 Comparison with the Sample Mean

For this comparison the sample size n must be large enough so that the approximate formula for the variance of the regression estimate is valid. The sampling variance of the regression estimate of the population mean $\bar{Y}$ to terms of order $1/n$ has been shown to be

$$V(\hat{\bar{Y}}_{1r}) = (\frac{1-f}{n})S_y^2(1 - \rho^2) \qquad (4.26)$$

while that of the unbiased estimate is given by

$$V(\bar{y}) = (\frac{1-f}{n})S_y^2 . \qquad (4.27)$$

Comparing the regression estimate with the simple unbiased estimate, we notice that the regression estimate is always better or more efficient than the simple unbiased estimate because $S_y(1-\rho^2) < S_y^2$ since $(1-\rho^2) < 1$.

4.5 The Linear Regression Estimator Under a Superpopulation Model

Suppose that the finite population values $y_i (i = 1, 2, \ldots, N)$ are randomly drawn from an infinite superpopulation in which

$$y_i = \alpha + \beta x_i + \varepsilon_i \quad , i = 1, 2, \ldots, N \qquad (4.28)$$

where the ε_i's are independent, with means 0 and variance σ_ε^2 for fixed x_i. By the method of least squares we find that

$$b = \frac{\sum_{i=1}^{n} (y_i-\bar{y})(x_i-\bar{x})}{\sum_{i=1}^{n} (x_i-\bar{x})^2} \qquad (4.29)$$

$$= \frac{\sum_{i=1}^{n} y_i(x_i-\bar{x})}{\sum_{i=1}^{n} (x_i-\bar{x})^2} \qquad (4.30)$$

$$= \beta + \frac{\sum_{i=1}^{n} \varepsilon_i(x_i-\bar{x})}{\sum_{i=1}^{n} (x_i-\bar{x})^2} \qquad (4.31)$$

$$\hat{\bar{Y}}_{lr} - \bar{Y} = (\bar{\varepsilon}_n-\bar{\varepsilon}_N) + (\bar{X}-\bar{x})\frac{\sum_{i=1}^{n} \varepsilon_i(x_i-\bar{x})}{\sum_{i=1}^{n} (x_i-\bar{x})^2} \qquad (4.32)$$

where $\bar{\varepsilon}_n$ and $\bar{\varepsilon}_N$ are means over the sample and the finite population respectively. It follows from (4.32) that under this model $E(\hat{\bar{Y}}_{lr} - \bar{Y}) = 0$, so that $\hat{\bar{Y}}_{lr}$ is model unbiased for any size of sample.

As regards variance, it follows from (4.32) that for a given set of x_i's

$$V(\hat{\bar{Y}}_{1r}) = E(\hat{\bar{Y}}_{1r} - \bar{Y})^2$$

$$= \sigma_\varepsilon^2 [(\frac{1}{n} - \frac{1}{N}) + \frac{(\bar{X}-\bar{x})^2}{\sum_{i=1}^{n} (x_i-\bar{x})^2}] . \qquad (4.33)$$

This result holds for any $n > 1$ and any sample selected solely by the values of x.

If we make further assumption that x_i's are independent gamma variates with parameter h as in sections 2.6 and 3.5, then we get the following results.

By theorem 4.1, we have seen that the linear regression estimator is unbiased for a preassigned b. Similarly the bias of the linear regression estimator under Durbin's (1959) model is also zero. The variance of $\hat{\bar{Y}}_{1r}$ is given in the following theorem.

Theorem 4.5

Under the model in section 2.6, the average variance of $\hat{\bar{Y}}_{1r}$ is given by

$$AVAR(\hat{\bar{Y}}_{1r}) = (\frac{1-f}{n}) \{ (b-\beta)^2 h + \delta \frac{\Gamma(t+h)}{\Gamma(h)} \} . \qquad (4.34)$$

Proof:

We can write

$$\text{AVAR}(\hat{\bar{Y}}_{1r}) = E_M E(\hat{\bar{Y}}_{1r} - \bar{Y})^2$$

$$= E_M E[(b-\beta)^2(\bar{X}-\bar{x})^2 + (\bar{\varepsilon}_n - \bar{\varepsilon}_N)^2$$

$$+ 2(b-\beta)(\bar{X}-\bar{x})(\bar{\varepsilon}_n - \bar{\varepsilon}_N)]$$

$$= (b-\beta)^2 E_M(\text{Var } \bar{x}) + E_M E(\bar{\varepsilon}_n - \bar{\varepsilon}_N)^2$$

$$+ 2(b-\beta) E_M[\text{Cov}(\bar{x}, \bar{\varepsilon}_n)]$$

$$= (b-\beta)^2 E_M(\text{Var } \bar{x}) + E_M E(\bar{\varepsilon}_n - \bar{\varepsilon}_N)^2 \quad . \quad (4.35)$$

Now,

$$E_M(\text{Var } \bar{x}) = \left(\frac{1-f}{n}\right) E_M(S_x^2)$$

$$= \left(\frac{1-f}{n}\right) E_M\left[\frac{1}{N-1}\left(\sum_{i=1}^{N} x_i^2 - \frac{1}{N}\left(\sum_{i=1}^{N} x_i\right)^2\right)\right]$$

Since, x_i i.i.d. Gamma (h), $\sum_{i=1}^{N} x_i$ i.i.d. Gamma (Nh), and we have

$$E_M(\text{Var } \bar{x}) = \left(\frac{1-f}{n}\right)\left[\frac{1}{N-1}\left\{Nh(1+h) - \frac{Nh(1+Nh)}{N}\right\}\right]$$

$$= \left(\frac{1-f}{n}\right) h. \quad (4.36)$$

Expectation in the second term of (4.35) can be obtained by noting that

$$(\bar{\varepsilon}_n - \bar{\varepsilon}_N)^2 = \bar{\varepsilon}_n^2 + \bar{\varepsilon}_N^2 - 2\bar{\varepsilon}_n \bar{\varepsilon}_N \, . \quad (4.37)$$

Furthermore,

$$E_M E(\bar{\varepsilon}_n^2) = E_M E\left(\frac{\sum_{i=1}^{n} \varepsilon_i}{n}\right)^2$$

$$= \frac{1}{n^2} E_M E\left[\sum_{i=1}^{n} \varepsilon_i^2 + \sum_{i \neq j=1}^{n} \varepsilon_i \varepsilon_j\right]$$

$$= \frac{1}{n^2} E_M\left[\frac{n}{N} \sum_{i=1}^{N} \varepsilon_i^2 + \frac{n(n-1)}{N(N-1)} \sum_{i \neq j=1}^{N} \varepsilon_i \varepsilon_j\right]$$

$$= \frac{1}{nN} \sum_{i=1}^{N} E_M E(\varepsilon_i^2 | x_i) + \frac{(n-1)}{nN(N-1)} \sum_{i \neq j=1}^{N} E_M E(\varepsilon_i \varepsilon_j | x_i, x_j)$$

$$= \frac{1}{nN} \sum_{i=1}^{N} E_M E(\varepsilon_i^2 | x_i) \quad (\because E(\varepsilon_i | x_i) = 0)$$

$$= \frac{1}{nN} \sum_{i=1}^{N} E_M(\delta x_i^t)$$

$$= \frac{\delta}{n} \cdot \frac{\Gamma(t+h)}{\Gamma(h)} \qquad (4.38)$$

Also,

$$E_M E(\bar{\varepsilon}_N^2) = E_M E\left(\frac{\sum_{i=1}^{N} \varepsilon_i}{N}\right)^2$$

$$= \frac{1}{N^2} E_M\left(\sum_{i=1}^{N} \varepsilon_i^2 + \sum_{i \neq j=1}^{N} \varepsilon_i \varepsilon_j\right)$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} E_M E(\varepsilon_i^2 | x_i)$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} E_M(\delta x_i^t)$$

$$= \frac{\delta}{N} \cdot \frac{\Gamma(t+h)}{\Gamma(h)} \qquad (4.39)$$

and finally,

$$E_M E(\bar{\varepsilon}_n \bar{\varepsilon}_N) = E_M E\left[\bar{\varepsilon}_n \left(\frac{(N-n)\bar{\varepsilon}_{N-n} + n\bar{\varepsilon}_n}{N}\right)\right]$$

$$= E_M E\left(\frac{n}{N}\bar{\varepsilon}_n^2\right) + E_M E\left(\frac{N-n}{N}\bar{\varepsilon}_n \bar{\varepsilon}_{N-n}\right) .$$

Because the second term is zero and the first is simplified in (4.38) we get

$$E_M E(\bar{\varepsilon}_n \bar{\varepsilon}_N) = \frac{n}{N} E_M E(\bar{\varepsilon}_n^2)$$

$$= \frac{\delta}{N} \cdot \frac{\Gamma(t+h)}{\Gamma(h)} . \qquad (4.40)$$

Hence,

$$E_M E(\bar{\varepsilon}_n - \bar{\varepsilon}_N)^2 = \delta \frac{\Gamma(t+h)}{\Gamma(h)}\left(\frac{1}{n} + \frac{1}{N} - \frac{2}{N}\right)$$

$$= \delta \frac{\Gamma(t+h)}{\Gamma(h)} \left(\frac{1}{n} - \frac{1}{N}\right)$$

$$= \left(\frac{1-f}{n}\right) \delta \frac{\Gamma(t+h)}{\Gamma(h)} . \qquad (4.41)$$

Therefore combining (4.36) and (4.41) we get the result in (4.34).

4.6 Other Investigations

Tikkiwal (1960) examined the various classical results in the theory of regression and double sampling estimation and extended them to the study of a finite population.

R. Singh and Sukhatme (1973) considered the problem of optimum stratification on an auxiliary variable x when the information on the auxiliary variable x is also used to estimate the population mean $\bar{Y}$ using ratio or regression methods of estimation. Assuming the form of the regression of the main variable y on the auxiliary variable x and also the form of the conditional variance function Var $(y|x)$, the problem of determining optimum strata boundaries is shown to be a particular case of optimum stratification on the auxiliary variable for stratified simple random sampling. Later R. Singh (1977) considered the problem of determining approximately optimum strata boundaries on x when the sample size in each stratum is equal. A numerical investigation into the relative efficiency of equal allocation with respect to Neyman and proportional allocations has also been made and it has been found that it is equal to one.

Fuller (1975) investigated the estimation of regression equations for a sample selected from a finite population where the finite population is treated as a sample from an infinite population. The regression coefficients are shown to be asymptotically normal, given mild assumptions and relatively simple expressions for the covariance matrix of the regression coefficients are also presented.

Konijn (1979) obtained expressions for the bias and mean square error of the regression estimator with definite bounds of error and without assuming a particular superpopulation model.

Scott and Wu (1981) gave a formal derivation of the asymptotic normality of regression estimator of a finite population total for simple random sampling and for sampling with probability proportional to size. The results are all well known and widely used, of course, but a vigorous development, spelling out relatively simple sufficient conditions under which the standard results are valid, were not available.

Royall and Cumberland (1981) examined the usual variance estimator for the linear regression estimator of a finite population total under the same superpopulation models. Its bias is compared with that of the least squares variance estimator and three bias-robust alternatives, one of which is the Jack-Knife variance estimator, are also described. The theoretical results are supported by an empirical study in which simple and restricted random samples as well as some nonrandom samples are drawn from six real populations.

Chaubey et al (1984 a) derived the regression estimator through an optimality consideration over a class of estimators generating Generalised Product (see Ray and Sahai (1980)) and dual to ratio estimators (see Srivenkataramana (1980)).

Tamhane (1978) considered the problem of hypothesis using the regression estimator in double sampling. Test procedures are provided when the covariance matrix between the primary and the auxiliary variables is either partially known or completely unknown. For the latter case a new "studentized" version of the regression estimator is proposed.

as a test statistic. An approximation to the null distribution is derived in the general case studied by means of the Monte-Carlo method and the problem of choosing between the double sample regression estimator and the single sample mean is also discussed.

Singh and Srivastava (1980) proposed a sampling scheme for which the usual regression estimator is unbiased. Another sampling scheme with an unbiased regression-type estimator is also considered. On comparing the efficiencies of these sampling strategies with some existing strategies, the performance of the first of the new schemes is found to be highly satisfactory.

CHAPTER V

COMPARISON OF THE RATIO, PRODUCT AND REGRESSION ESTIMATORS UNDER A CLASS OF SUPERPOPULATION MODELS

5.1 Introduction

Comparison of the ratio, product and regression methods of estimation under the fixed population model can be done through large sample studies. The superpopulation models are models imposed on these populations which may seem to be real and allow small sample investigations. The model of Durbin (1959) (see section 2.6) has been of a particular interest to the researchers in this respect, for example J.N.K. Rao (1967), J.N.K. Rao and Webster (1966), P.S.R.S. Rao (1968), P.S.R.S. Rao and J.N.K. Rao (1971), Hutchison (1971) and Chaubey <u>et al</u> (1984 b). However J.N.K. Rao (1967), P.S.R.S. Rao and J.N.K. Rao (1971) and Hutchison (1971) paid particular interest to ratio estimators and Chaubey <u>et al</u> (1984 b) have given analytical properties of the product and ratio estimators.

In this chapter, firstly, we are interested in using this model for comparing the three estimators in finite samples; secondly, we propose another superpopulation model which has more general appeal than Durbin's model. The next section presents the inverse Gaussian distribution which is used in this superpopulation model and some of its properties. The analytical properties of the ratio, product and regression estimators are given in sections 5.3, 5.4 and 5.5 respectively.

Finally section 5.6 presents a numerical comparison of the three estimators.

5.2 Inverse Gaussian Distribution and Some of its Properties

In this section we introduce the inverse Gaussian distribution which we'll use for the analytical properties of the three estimators under the following superpopulation model;

$$y_i = \alpha + \beta x_i + \varepsilon_i \, , \quad i = 1, 2, \ldots, N$$

where

$$E_M(\varepsilon_i | x_i) = 0 \, , \quad E_M(\varepsilon_i^2 | x_i) = \delta x_i^t \qquad (5.1)$$

and x_i's follow a general inverse Gaussian distribution with parameters μ and λ, defined below. (This model will be referred as the inverse Gaussian regression superpopulation model.)

The probability density function of the inverse Gaussian distribution with parameters μ and λ is given by

$$f(x;\mu,\lambda) = [\lambda/2\pi x^3]^{\frac{1}{2}} \exp\{-\lambda(x-\mu)^2/2\mu^2 x\}, \quad x > 0 \, . \qquad (5.2)$$

The population or the random variable with the above distribution will be denoted by $IG(\mu,\lambda)$.

We need the first four moments of (5.2) for later use, which are given below,

$$\mu'_1 = \mu ,$$

$$\mu'_2 = \mu^2 + \frac{\mu^3}{\lambda} ,$$

$$\mu'_3 = \mu^3 + \frac{3\mu^4}{\lambda} + \frac{3\mu^5}{\lambda^2} , \qquad (5.3)$$

$$\mu'_4 = \mu^4 + \frac{6\mu^5}{\lambda} + \frac{15\mu^6}{\lambda^2} + \frac{15\mu^7}{\lambda^3} .$$

For simplifying the AMSE formulae under the model (5.1), it is useful to find the form of $E(\tilde{T}|g)$ where $g = \lambda n/\bar{x}$ and $\tilde{T}$ is an unbiased estimator of $T(\theta) = \theta^{-s}$ (s being an integer ≥ 1). Such a result is available from Tweedie (1957 b) which we quote in lemmas 1 and 2.

__Lemma 1__

Let $\tilde{T}$ be an unbiased estimator of $T(\theta) = \theta^{-s}$ based on a random sample $(x_1, x_2, \ldots, x_n)$ from $IG(\mu,\lambda)$ distribution, where $\theta = \lambda n/\mu$; then for $s \geq 2$

$$E_g^{-1}(\theta^{-s}) = E(\tilde{T}|g) = g^{-1} e^{\frac{1}{2}g} \int_1^\infty (u-1)^{s-2} e^{-\frac{1}{2}gu^2}\, du/(s-2)! \qquad (5.4)$$

__Lemma 2__

$$\int_a^\infty f(u) e^{-\frac{u^2}{2b}}\, du = (1-bDP)^{-1} f(u)\Big|_{u=0} \int_a^\infty e^{-\frac{u^2}{2b}}\, du$$

$$+ bP(1-bDP)^{-1} f(u)\Big|_{u=a}\, e^{-\frac{a^2}{2b}} , \qquad (5.5)$$

where a and b are positive constants, $Pf(u) = [f(u)-f(0)]/u$ and D is the differential operator, with respect to u and $(1-bDP)^{-1}$ is an abbreviation for the series $1 + bDP + b^2(DP)^2 + \ldots$

Lemma 2 is used in simplifying results of (5.4) for a particular s, as reported in the following corollaries.

Corollaries

$$
\begin{aligned}
&\text{(i)} && E_g^{-1}(\theta^{-1}) = g^{-1} , \\
&\text{(ii)} && E_g^{-1}(\theta^{-2}) = g^{-1} I_g , \\
&\text{(iii)} && E_g^{-1}(\theta^{-3}) = g^{-1}(g^{-1} - I_g) , \\
&\text{(iv)} && E_g^{-1}(\theta^{-4}) = \tfrac{1}{2} g^{-1}\{(g^{-1}+1) I_g - g^{-1}\} , \\
&\text{(v)} && E_g^{-1}(\theta^{-5}) = \tfrac{1}{6} g^{-1}\{g^{-1} + 2g^{-2} - (1+3g^{-1}) I_g\} , \\
&\text{(vi)} && E_g^{-1}(\theta^{-6}) = \tfrac{1}{24} g^{-1}\{(1+6g^{-1}+3g^{-2}) I_g - g^{-1} - 5g^{-2}\} , \\
&\text{(vii)} && E_g^{-1}(\theta^{-7}) = \tfrac{1}{120} g^{-1}\{g^{-1} + 9g^{-2} + 8g^{-3} - (1+10g^{-1}+15g^{-2}) I_g\}
\end{aligned}
\qquad (5.6)
$$

where $I_g = e^{\frac{1}{2}g} \int_1^\infty e^{-\frac{1}{2}gu^2} du$.

Proofs of Corollaries

(i) can be verified directly by putting $\tilde{T} = \bar{x}/\lambda n = g^{-1}$, with $a = 1$ and $b = g^{-1}$, and (ii) can be obtained directly from lemma 1 by putting $s = 2$. Now consider $s = 3,4,\ldots,7$ successively.

For $s = 3$, $f(u) = u - 1$, $f(0) = -1$, $Pf(u) = 1$, $DPf(u) = 0$, hence,

$$E_g^{-1}(\theta^{-3}) = [f(0)\int_1^\infty e^{-\frac{1}{2}gu^2}du + g^{-1}\ e^{-\frac{1}{2}g}]g^{-1}\ e^{-\frac{1}{2}g}$$

$$= -g^{-1}\ I_g + g^{-2}\ ,$$

$$= g^{-1}(g^{-1} - I_g)\ . \qquad (5.7)$$

The results in (iv) - (vii) are obtained similarly.

We will also need other expectations which we quote directly from Tweedie (1957 b).

(i) $\theta E(g^{-1}) = 1$, (5.8)

(ii) $\theta^3 E(g^{-2}) = E(g) = \theta + 1$, (5.9)

(iii) $\theta^5 E(g^{-3}) = E(g^2) = \theta^2 + 3\theta + 3$, (5.10)

(iv) $E(gI_g) = \frac{1}{2}(1 + \theta - \theta^2 I^*_g)$, (5.11)

(v) $E(g^{-1}I_g) = \theta^{-2}$, (5.12)

where $I^*_g = E(I_g) = e^\theta \int_\theta^\infty x^{-1}e^{-x}\ dx$ and can be obtained numerically or by the use of incomplete gamma tables.

5.3 Bias and MSE of the Ratio Estimator, $\hat{\hat{Y}}_R$, under the Model (5.1)

The bias of $\hat{\hat{Y}}_R$ under the inverse-Gaussian regression superpopulation model, is given in the following theorem.

Theorem 5.1

$$\text{Bias}(\hat{\hat{Y}}_R) = \alpha(\frac{1-f}{n})\frac{\mu}{\lambda}\ . \qquad (5.13)$$

Proof:

First note that

$$\text{Bias}(\hat{\bar{Y}}_R) = \alpha E(\frac{\bar{X}}{\bar{x}} - 1) + E(\frac{\bar{\varepsilon}_n \bar{X}}{\bar{x}} - \bar{\varepsilon}_N). \tag{5.14}$$

Now, writing $N\bar{X} = n\bar{x} + (N-n)\bar{x}_{N-n}$, where $\bar{x}_{N-n}$ denotes the mean of x on the complement of the sample, we have

$$E(\frac{\bar{X}}{\bar{x}}) = (1-f)\ E(\frac{\bar{x}_{N-n}}{\bar{x}_n}) + \frac{n}{N}. \tag{5.15}$$

The expectation in the above equation can be obtained by conditioning on the sample $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$ and we get

$$\begin{aligned} E(\frac{\bar{X}}{\bar{x}}) &= (1-f)\ E_g[g^{-1}E(\bar{x}_{N-n}|g)] + f \\ &= (1-f)\ E_g(g^{-1})\mu + f \\ &= (1-f)(\frac{1}{\mu} + \frac{1}{\lambda n})\mu + f \\ &= (1-f)(1 + \frac{\mu}{\lambda n}) + f\ . \end{aligned} \tag{5.16}$$

Similarly, we can also simplify $E(\bar{\varepsilon}_n \bar{X}/\bar{x})$ as being equal to zero. Using these facts we get the result.

The MSE of $\hat{\bar{Y}}_R$ is given in the theorem below.

Theorem 5.2

$$\text{AMSE}(\hat{\bar{Y}}_R) = \alpha^2 A_{ri} + \delta D_{rit}, \tag{5.17}$$

where

$$A_{ri} = (f^2-1)+[1+\frac{3}{n\phi}+3(\frac{1}{n\phi})^2][(1-f)^2+(\frac{1-f}{N\phi})]+2(1-f)[f-\frac{(1-f)}{n\phi}] \tag{5.18}$$

$$\begin{aligned} D_{rit} = {} & \frac{\mu_t'}{N}+\frac{n}{N^2}\mu_t'+k_1E_g[g^2E_g^{-1}(\mu_t')]+k_2E_g[gE_g^{-1}(\mu_t')] \\ & -2(\frac{n}{N^2}\mu_t'+k_3E_g[gE_g^{-1}(\mu_t')]), \end{aligned} \tag{5.19}$$

with ϕ, k_1, k_2 and k_3 defined as,

$$\phi = \lambda/\mu \ ,$$

$$k_1 = \frac{1}{\lambda^2 n}\left(\frac{1-f}{n}\right)^2\left(\frac{\mu^3}{\lambda(N-n)} + \mu^2\right) \ ,$$

$$k_2 = \frac{2}{\lambda n}\left(\frac{1-f}{N}\right)\mu \ , \tag{5.20}$$

$$k_3 = \frac{2n}{\lambda}\left(\frac{1-f}{N}\right)\mu \ .$$

<u>Proof</u>:

We'll use the following results to prove the above theorem.

$$E\left(\frac{\bar{X}}{\bar{x}}\right)^2 = f^2 + \left[1+\frac{3}{n\phi}+3\left(\frac{1}{n\phi}\right)^2\right]\left[(1-f)^2+\frac{(1-f)}{N\phi}\right] + 2f(1-f)\left(1+\frac{1}{n\phi}\right) \ , \tag{5.21}$$

$$E(\bar{\varepsilon}_N^2) = \frac{\mu_t'}{N} \ , \tag{5.22}$$

$$E\left(\frac{\bar{\varepsilon}_n\bar{X}}{\bar{x}}\right)^2 = \frac{n\mu_t'}{N^2} + k_1 E_g[g^2 E_g^{-1}(\mu_t')] + k_2 E_g[g E_g^{-1}(\mu_t')] \ , \tag{5.23}$$

$$E\left(\frac{\bar{\varepsilon}_n\bar{\varepsilon}_N\bar{X}}{\bar{x}}\right) = \frac{n}{N^2}\mu_t' + k_3 E_g[g E_g^{-1}(\mu_t')] . \tag{5.24}$$

Now, note that

$$A_{ri} = E\left(\frac{\bar{X}}{\bar{x}} - 1\right)^2$$
$$= E\left[\left(\frac{\bar{X}}{\bar{x}}\right)^2 - 2\left(\frac{\bar{X}}{\bar{x}}\right) + 1\right] . \tag{5.25}$$

Hence, using (5.16) and (5.21) we get,

$$A_{ri} = f^2 + [1+\frac{3}{n\phi}+3(\frac{1}{n\phi})^2][(1-f)^2+\frac{(1-f)}{N\phi}]+2f(1-f)(1+\frac{1}{n\phi})$$

$$- 2[1+\frac{(1-f)}{n\phi}] + 1$$

$$= (f^2-1)+[1+\frac{1}{n\phi}+3(\frac{1}{n\phi})^2][(1-f)^2+\frac{(1-f)}{N\phi}]+2(1-f)[f-\frac{(1-f)}{n\phi}] \tag{5.26}$$

Next,

$$D_{rit} = E(\frac{\bar{\varepsilon}_n\bar{X}}{\bar{x}} - \bar{\varepsilon}_N)^2$$

$$= E[(\frac{\bar{\varepsilon}_n\bar{X}}{\bar{x}})^2 - 2\frac{\bar{\varepsilon}_n\bar{\varepsilon}_N\bar{X}}{\bar{x}} + \bar{\varepsilon}_N^2], \tag{5.27}$$

and from equations (5.22), (5.23) and (5.24) we get,

$$D_{rit} = \frac{\mu_t'}{N} + \frac{n}{N^2}\mu_t' + k_1E_g[g^2E_g^{-1}(\mu_t')] + k_2E_g[gE_g^{-1}(\mu_t')]$$

$$- 2(\frac{n}{N^2}\mu_t' + k_3E_g[gE_g^{-1}(\mu_t')]). \tag{5.28}$$

For $t = 0, 1, 2$ we can simplify D_{rit} in a more explicit form as follows.

<u>t = 0</u>,

$$\mu_0' = 1, \quad E_g[g^2E_g^{-1}(1)] = \theta^2+3\theta+3, \quad E_g[gE_g^{-1}(1)] = \theta+1,$$

and

$$D_{ri0} = \frac{1}{N} + \frac{n}{N^2} + k_1(\theta^2+3\theta+3)+k_2(\theta+1) - 2[\frac{n}{N^2}+k_3(\theta+1)]$$

$$= \frac{(1-f)}{N} + k_1(\theta^2+3\theta+3) + (k_2-2k_3)(\theta+1). \tag{5.29}$$

t = 1 ,

$$\mu_1' = \mu,\quad E_g[g^2 E_g^{-1}(\lambda n\theta^{-1})] = \lambda n(\theta+1),\quad E_g[gE_g^{-1}(\lambda n\theta^{-1})] = \lambda n.$$

$$D_{ri1} = \frac{(1-f)}{N}\mu + \lambda n k_1(\theta+1) + (k_2-2k_3)\lambda n \tag{5.30}$$

t = 2 ,

$$\mu_2' = \mu^2 + \frac{\mu^3}{\lambda}$$

$$\begin{aligned} E_g[g^2 E_g^{-1}(\lambda^2 n^2\theta^{-2}+\lambda^2 n^3\theta^{-3})] &= \frac{\lambda^2 n^2}{2}(1+\theta-\theta^2 I_g^*) \\ &\quad + \lambda^2 n^3[1-\tfrac{1}{2}(1+\theta-\theta^2 I_g^*)] \\ &= \lambda^2 n^3 - \frac{\lambda^2 n^2(n-1)}{2}(1+\theta-\theta^2 I_g^*) , \end{aligned}$$

$$E_g[gE_g^{-1}(\lambda^2 n^2\theta^{-2}+\lambda^2 n^3\theta^{-3})] = \lambda^2 n^2(\theta+1)+\lambda^2 n^3(\frac{1}{\theta}-I_g^*) ,$$

$$\begin{aligned} D_{ri2} &= \frac{(1-f)}{N}(\mu^2+\frac{\mu^3}{\lambda})+k_1[\lambda^2 n^2 - \frac{\lambda^2 n^2(n-1)}{2}(1+\theta-\theta^2 I_g^*)] \\ &\quad + (k_2-2k_3)[\lambda^2 n^2(\theta+1) + \lambda^2 n^3(\frac{1}{\theta} - I_g^*)] . \end{aligned} \tag{5.31}$$

5.4 Bias and MSE of the Product Estimator, $\hat{\hat{Y}}_P$, under the Model (5.1)

The bias of $\hat{\hat{Y}}_P$ under the inverse Gaussian regression superpopulation model, is given by in the following theorem.

Theorem 5.3

$$\text{Bias}(\hat{\hat{Y}}_P) = \beta N(\frac{1-f}{n})(\mu - \lambda N I_G^*) . \tag{5.32}$$

where $G = \lambda N/\bar{X}$ and I_G^* is defined in the same way as I_g^*.

Proof:

Since, $\bar{y} = \alpha + \beta\bar{x} + \bar{\varepsilon}_n$, where $\bar{\varepsilon}_n$ is the mean of errors ε_i in the sample, we have

$$E(\hat{\bar{Y}}_p) = E_M E(\alpha\frac{\bar{x}}{\bar{X}} + \beta\frac{\bar{x}^2}{\bar{X}} + \frac{\bar{\varepsilon}_n\bar{x}}{\bar{X}})$$

$$= \alpha + \beta E_M[\frac{(\frac{1-f}{n})S_x^2 + \bar{X}^2}{\bar{X}}]$$

$$= \alpha + \beta E_G[\frac{1-f}{n\bar{X}} E_G^{-1}(\mu_2) + \bar{X}]$$

where μ_2 is the second cumulant of $IG(\mu,\lambda)$ and $G = \lambda N/\bar{X}$. Using the corollary (iii), since $\mu_2 = \lambda^2 N^3 \theta_N^{-3}$, we get

$$E(\hat{\bar{Y}}_p) = \alpha + \beta E_G[(\frac{1-f}{n})G\frac{\lambda^2 N^3}{\lambda N} G^{-1}(G^{-1} - I_G) + \lambda N G^{-1}]$$

$$= \alpha + \beta[(\frac{1-f}{n})\lambda N^2(\theta_N^{-1} - I_G^*) + \lambda N\theta_N^{-1}] , \quad (5.33)$$

where $\theta_N = \lambda N/\mu$.

Hence,

$$\text{Bias}(\hat{\bar{Y}}_p) = \beta N(\frac{1-f}{n})(\mu - \lambda N I_G^*) . \quad (5.34)$$

The MSE of $\hat{\bar{Y}}_p$ is given in the theorem below.

Theorem 5.4

$$\text{AMSE}(\hat{\bar{Y}}_p) = \alpha^2 A_{pi} + \beta^2 B_{pi} + 2\alpha\beta C_{pi} + \delta D_{pit} \quad (5.35)$$

where

$$A_{pi} = (\frac{1-f}{n})N[I^* - \tfrac{1}{2}(1+\theta_N - \theta_N^2 I_G^*)] \quad (5.36)$$

$$B_{pi} = \mu^2\left(1+\frac{\mu}{\lambda n}\right)+\lambda^2 N^2 \sum_{j=1}^{4} b_j B_j - 2\mu^2\left[1+\left(\frac{1-f}{n}\right)\left(1+\frac{1}{N}\right)\frac{\mu}{\lambda}\right], \quad (5.37)$$

$$C_{pi} = \lambda N \sum_{j=1}^{3} c_j C_j - \left[\left(\frac{n-N}{n}\right)\lambda N I_G^* + \frac{N}{n}\mu\right], \quad (5.38)$$

$$D_{pit} = \frac{\mu_t'}{N} - \frac{2}{n\lambda N^2} E_G[G\, E_G^{-1}(\mu_{t+1}' + (n-1)\mu_t'\, \mu]$$

$$+ \frac{1}{n^3(\lambda N)^2} E_G[G^2 E_G^{-1}(\mu_{t+2}' + 2(n-1)(\mu_t'\mu_2' + \mu_{t+1}'\mu)$$

$$+ (n-1)(n-2)\mu_t'\mu^2)] \quad (5.39)$$

and

$$b_1 = \frac{1+(n-1)(n^2-n+1)}{n^3},$$

$$b_2 = \frac{6N[1+(n-1)n]}{n^3},$$

$$b_3 = \frac{15N^2}{n^2}, \quad (5.40)$$

$$b_4 = \frac{15N^3}{n^3};$$

$$B_1 = \tfrac{1}{2}\left[I_G^*\left(1 - \frac{\theta_N^2}{2}\right) + \tfrac{1}{2}(\theta_N - 1)\right],$$

$$B_2 = \frac{1}{6}(1+2\theta_N^{-1}) - \frac{1}{12}(1+\theta_N) + \frac{1}{12}(\theta_N^2-6)I_G^*, \quad (5.41)$$

$$B_3 = \frac{1}{48}(1+\theta_N) - \frac{1}{24}(1+5\theta_N^{-1}-3\theta_N^{-2}) - \frac{1}{48}(\theta_N^2-12)I_G^*,$$

$$B_4 = \frac{1}{120}\left[\tfrac{1}{2}-\tfrac{1}{2}\theta_N+9\theta_N^{-1} - 7\theta_N^{-2}+8\theta_N^{-3} + \tfrac{1}{2}(\theta_N^{-2}-20)I_G^*\right],$$

$$c_1 = \frac{1+(n-1)^2}{n^2} ,$$

$$c_2 = \frac{N(3+(n-1)^2)}{n^2} , \qquad (5.42)$$

$$c_3 = \frac{3N^2}{n^2} ,$$

$$C_1 = \tfrac{1}{2}(1-\theta_N) + \tfrac{1}{2}\theta_N^2 I_G^* , \qquad (5.43)$$

$$C_2 = B_1 ,$$

$$C_3 = B_2 .$$

<u>Proof</u>:

Since $(\bar{\hat{Y}}_p-\bar{Y})^2 = \alpha^2(\frac{\bar{x}}{\bar{X}}-1)^2+\beta^2(\frac{\bar{x}^2}{\bar{X}}-\bar{X})^2+2\alpha\beta(\frac{\bar{x}}{\bar{X}}-1)(\frac{\bar{x}^2}{\bar{X}}-\bar{X})$

$$+ \delta(\frac{\bar{\varepsilon}_n\bar{x}}{\bar{X}} - \bar{\varepsilon}_N)^2 ,$$

hence, $A_{pi} = E(\frac{\bar{x}}{\bar{X}}-1)^2$, $B_{pi} = E(\frac{\bar{x}^2}{\bar{X}}-\bar{X})^2$, $C_{pi} = E(\frac{\bar{x}}{\bar{X}}-1)(\frac{\bar{x}^2}{\bar{X}}-\bar{X})$

and

$$D_{pit} = E(\frac{\bar{\varepsilon}_n\bar{x}}{\bar{X}} - \bar{\varepsilon}_N)^2 .$$

Now,

$$A_{pi} = E(\frac{\bar{x}^2}{\bar{X}^2} - 2\frac{\bar{x}}{\bar{X}} + 1)$$

$$= E_M[\frac{(\frac{1-f}{n})S_x^2 + \bar{X}^2}{\bar{X}^2}] - 1$$

$$= (\frac{1-f}{n})E_M(\frac{S_x^2}{\bar{X}^2})$$

$$= (\frac{1-f}{n})N[1 - \tfrac{1}{2}(1+\theta_N-\theta_N^2 I_G^*)] . \qquad (5.44)$$

$$B_{pi} = E(\frac{\bar{x}^4}{\bar{X}^2} - 2\bar{x}^2 + \bar{X}^2).$$

We'll evaluate each term of the above expression individually;

$$E(\bar{X}^2) = \mu^2(1 + \frac{\mu}{\lambda N}) \qquad (5.45)$$

$$E(\bar{x}^2) = (\frac{1-f}{n})E_M(S_x^2) + E_M(\bar{X}^2)$$

$$= (\frac{1-f}{n})\frac{\mu^3}{\lambda} + \mu^2(1+\frac{\mu}{\lambda N})$$

$$= (\frac{1-f}{n})(1+\frac{1}{N})\frac{\mu^3}{\lambda} + \mu^2 , \qquad (5.46)$$

and

$$E(\frac{\bar{x}^4}{\bar{X}^2}) = \frac{1}{n^4}E_G\{\frac{1}{\bar{X}^2}[nE(X_i^4)+n(n-1)\{3E(X_i^2X_j) + 4E(X_i^3X_j)\}$$
$$+ 6(n-1)(n-2)E(X_i^2X_jX_k)$$
$$+ n(n-1)(n-2)(n-3)E(X_iX_jX_kX_l)] \,|\, G\} . \qquad (5.47)$$

Now since,

$$E(X_i^4) = \mu_4' = \lambda^4N^4(\theta_N^{-4}+6N\theta_N^{-5}+15N^2\theta_N^{-6}+15N^3\theta_N^{-7}) , \qquad (5.48)$$

$$E(X_i^2X_j^2|\bar{X}) = E_G^{-1}(\mu_2'^2) = E_G^{-1}[\lambda^4N^4(\theta_N^{-4}+2N\theta_N^{-5}+N^2\theta_N^{-7})] , \qquad (5.49)$$

$$E(X_i^3X_j|\bar{X}) = E_G^{-1}(\mu_3'\mu_1') = E_G^{-1}[\lambda^4N^4(\theta_N^{-4}+3N\theta_N^{-5}+3N^2\theta_N^{-6})] , \qquad (5.50)$$

$$E(X_i^2X_jX_k|\bar{X}) = E_G^{-1}(\mu_2'\mu_1'^2) = E_G^{-1}[\lambda^4N^4(\theta_N^{-4}+N\theta_N^{-5})] , \qquad (5.51)$$

and

$$E(X_iX_jX_kX_l|\bar{X}) = E_G^{-1}(\mu^4) = E_G^{-1}(\lambda^4N^4\theta_N^{-4}) . \qquad (5.52)$$

Therefore, (5.47) can be simplified as

$$E(\frac{\bar{x}^4}{\bar{X}^2}) = \lambda^2N^2\sum_{j=1}^{4} b_jE_G[G^2E_G^{-1}(\theta_N^{-j-3})]$$

where b_j are the coefficients of $E_G[G^2 E_G^{-1}(\theta_N^{-j-3})]$ $j = 1, 2, 3, 4$. Since,

$$B_1 = E_G[G^2 E_G^{-1}(\theta_N^{-4})]$$

$$= \tfrac{1}{2}[I_G^*(1-\tfrac{1}{2}\theta_N^2) + \tfrac{1}{2}(\theta_N - 1)] , \qquad (5.53)$$

$$B_2 = E_G[G^2 E_G^{-1}(\theta_N^{-5})]$$

$$= \frac{1}{6}(1+2\theta_N^{-1}) - \frac{1}{12}(1+\theta_N) + \frac{1}{12}(\theta_N^2 - 6) I_G^* , \qquad (5.54)$$

$$B_3 = E_G[G^2 E_G(\theta_N^{-6})]$$

$$= \frac{1}{48}(1+\theta_N) - \frac{1}{24}(1+5\theta_N^{-1} - 3\theta_N^{-2}) - \frac{1}{48}(\theta_N^2 - 12) I_G^* , \qquad (5.55)$$

$$B_4 = E_G[G^2 E_G(\theta_N^{-7})]$$

$$= \frac{1}{120}[\tfrac{1}{2} - \tfrac{1}{2}\theta_N + 9\theta_N^{-1} - 7\theta_N^{-2} + 8\theta_N^{-3} + \tfrac{1}{2}(\theta_N^2 - 20) I_G^*] , \qquad (5.56)$$

where the use of the corollaries (iv) - (vii) and equations (5.8)-(5.12) has been made. Combining (5.45), (5.46) and (5.53) to (5.56) we have B_{pi} .

The expression for C_{pi} can be obtained by writing C_{pi} as

$$C_{pi} = E(\frac{\bar{X}^3}{\bar{X}^2} - \frac{\bar{X}^2}{\bar{X}}) . \qquad (5.57)$$

Writing the first term of (5.57) as

$$E(\frac{\bar{X}^3}{\bar{X}^2}) = \frac{1}{n^3} E_G\{\frac{1}{\bar{X}^2}[nE(X_i^3) + n(n-1)E(X_i^2 X_j) + n(n-1)(n-2)E(X_i X_j X_k)] | G\}, \qquad (5.58)$$

After simplification, similarly as in the case of B_{pi} , we get,

$$E\left(\frac{\bar{x}^3}{\bar{X}^2}\right) = \lambda N \sum_{j=1}^{3} c_j C_j \tag{5.59}$$

where $C_j = E_G[G^2 E_G^{-1}(\theta_N^{-j-2})]$ $j = 1, 2, 3$ and these can be simplified using the results in corollaries (iii) to (v) and equations (5.8) - (5.12) as

$$C_1 = \tfrac{1}{2}(1-\theta_N) + \tfrac{1}{2}\theta_N^2 I_G^* , \tag{5.60}$$

$$C_2 = B_1 , \tag{5.61}$$

$$C_3 = B_2 . \tag{5.62}$$

Also, from (5.33) we can obtain

$$E\left(\frac{\bar{x}^2}{\bar{X}}\right) = \left(\frac{n-N}{n}\right)\lambda N I_G^* + \frac{N}{n}\mu . \tag{5.63}$$

Now combining (5.59) to (5.63) we get C_{pi} .

Next, writing D_{pit} as

$$D_{pit} = \frac{1}{\delta} E\left(\frac{\bar{\varepsilon}_n^2 \bar{x}^2}{\bar{X}^2} - 2\,\frac{\bar{\varepsilon}_n \bar{\varepsilon}_N \bar{x}}{\bar{X}} + \bar{\varepsilon}_N^2\right) . \tag{5.64}$$

Since,

$$E\left(\frac{\bar{\varepsilon}_n \bar{\varepsilon}_N \bar{x}}{\bar{X}}\right) = E\left(\frac{n\bar{\varepsilon}_n^2 \bar{x}}{N\bar{X}}\right) + \frac{N-n}{N} E\left(\frac{\bar{\varepsilon}_n \bar{\varepsilon}_{N-n} \bar{x}}{\bar{X}}\right)$$

$$(\because E(\bar{\varepsilon}_n \bar{\varepsilon}_{N-n} | \underline{x}) = 0)$$

$$= \frac{n}{N} E\left(\frac{\bar{\varepsilon}_n^2 \bar{x}}{\bar{X}}\right) . \tag{5.65}$$

Now,

$$E(\bar{\varepsilon}_n^2\bar{x}) = \frac{1}{n^3}\, E\left(\sum_{i=1}^{n} \varepsilon_i^2 + \sum_{i\neq j}^{n} \varepsilon_i\varepsilon_j\right)\left(\sum_{i=1}^{n} x_i\right)$$

$$= \frac{\delta}{n^3}\, E\left(\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i^t\right)$$

$$= \frac{\delta}{n^3}\, E\left(\sum_{i=1}^{n} x_i^{t+1} + \sum_{i\neq j}^{n} x_i^t x_j\right)$$

$$= \frac{\delta}{n^2}\left(\frac{1}{N} \sum_{i=1}^{N} X_i^{t+1} + \frac{(n-1)}{N(N-1)} \sum_{i\neq j}^{N} X_i^t X_j\right), \qquad (5.66)$$

hence (5.65) becomes

$$E\left(\frac{\bar{\varepsilon}_n\bar{\varepsilon}_N\bar{x}}{\bar{X}}\right) = \frac{\delta}{nN}\left[E\left(\frac{X_i^{t+1}}{\bar{X}}\right) + (n-1)E\left(\frac{X_i^t X_j}{\bar{X}}\right)\right]$$

$$= \frac{\delta}{n\lambda N^2}\, E_G[GE_G^{-1}(\mu'_{t+1} + (n-1)\mu'_t\mu)]. \qquad (5.67)$$

Similarly, since,

$$\bar{\varepsilon}_n^2\bar{x}^2 = \frac{1}{n^4}\left(\sum_{i=1}^{n} \varepsilon_i^2 + \sum_{i\neq j}^{n} \varepsilon_i\varepsilon_j\right)\left(\sum_{k=1}^{n} x_k^2 + \sum_{k\neq l}^{n} x_k x_l\right)$$

$$= \frac{1}{n^4}\Big[\sum_{i=1}^{n} \varepsilon_i^2 x_i^2 + 2\sum_{i\neq j}^{n} (\varepsilon_i^2 x_j^2 + \varepsilon_i\varepsilon_j x_i x_j + \varepsilon_i\varepsilon_j x_i^2 + \varepsilon_i^2 x_i x_j)$$

$$+ \sum_{i\neq j\neq k}^{n} (\varepsilon_i^2 x_j x_k + 2\varepsilon_i\varepsilon_j x_i x_k + \varepsilon_i\varepsilon_j x_k^2 + 2\varepsilon_i\varepsilon_j x_j x_k)$$

$$+ \sum_{i\neq j\neq k\neq l}^{n} \varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_l\Big],$$

then,

$$E(\bar{\varepsilon}_n^2\bar{x}^2) = \frac{\delta}{n^4}E\Big[\sum_{i=1}^{n} x_i^{t+2} + 2\sum_{i\neq j}^{n} (x_i^t x_j^2 + x_i^{t+1} x_j) + \sum_{i\neq j\neq k}^{n} x_i^t x_j x_k\Big]$$

$$= \frac{\delta}{n^3}\left[\frac{1}{N}\sum_{i=1}^{N} X_i^{t+2} + \frac{2(n-1)}{N(N-1)}\sum_{i\neq j}^{N}(X_i^t X_j^2 + X_i^{t+1}X_j)\right.$$

$$\left. + \frac{(n-1)(n-2)}{N(N-1)(N-2)}\sum_{i\neq j\neq k}^{N}(X_i^t X_j X_k)\right] ,$$

therefore

$$E_M E\left(\frac{\bar{\varepsilon}_n^2 \bar{x}^2}{\bar{X}^2}\right) = \frac{\delta}{n^3(\lambda N)^2} E_G[G^2 E_G^{-1}\{\mu'_{t+2} + 2(n-1)(\mu'_{t+1}\mu'_2 + \mu'_{t+1}\mu)$$

$$+ (n-1)(n-2)\mu'_t \mu^2\}] . \quad (5.68)$$

Thus, combining (5.22), (5.67) and (5.68) we get D_{pit}. For $t = 0, 1, 2$ the explicit formulae for D_{pit} are given below.

<u>t = 0</u> ,

$$D_{pi0} = \frac{1}{N} - \frac{2}{n\lambda N^2} E_G[G\, E_G^{-1}(\lambda N\theta_N^{-1} + (n-1)\lambda N\theta_N^{-1})]$$

$$+ \frac{1}{n^3(\lambda N)^2} E_G[G^2 E_G^{-1}\{\lambda^2 N^2\theta_N^{-2} + \lambda^2 N^3\theta_N^{-3}$$

$$+ 2(n-1)(\lambda^2 N^2\theta_N^{-2} + \lambda^2 N^3\theta_N^{-3} + \lambda^2 N^2\theta_N^{-2})$$

$$+ (n-1)(n-2)\lambda^2 N^2\theta_N^{-2}\}]$$

$$= \frac{1}{N} - \frac{2}{N} E_G[G\, E_G^{-1}(\theta_N^{-1})] + \frac{1}{n^3} E_G[G^2 E_G^{-1}(d_{01}\theta_N^{-2} + d_{02}\theta_N^{-3})] \quad (5.69)$$

where $d_{01} = 1 + (n-1)(n+2)$,

$d_{02} = N(2n-1)$.

Since, $E_G[G\, E_G^{-1}(\theta_N^{-1})] = 1$,

$$D_{01} = E_G[G^2 E_G^{-1}(\theta_N^{-2})]$$

$$= \tfrac{1}{2}(1 + \theta_N - \theta_N^2 I_G^*) ,$$

$$D_{02} = E_G[G^2 E_G^{-1}(\theta_N^{-3})]$$
$$= \tfrac{1}{2}(1 - \theta_N + \theta_N^{-2} I_G^*) ,$$

hence,

$$D_{pi0} = -\frac{1}{N} + \frac{1}{n^3} \sum_{k=1}^{2} d_{0k} D_{0k} . \tag{5.70}$$

<u>t = 1</u> ,

$$E_G[G\ E_G^{-1}(\mu'_{t+1}+(n-1)\mu'_t\mu)] = \lambda^2 N^2 E_G[G\ E_G^{-1}(n\theta_N^{-2}+N\theta_N^{-3})]$$
$$= \lambda^2 N^2 [N\theta_N^{-1}-(N-n)I_G^*] \tag{5.71}$$

$$E_G[G^2 E_G^{-1}(\mu'_{t+2}+2(n-1)(\mu'_t\mu'_2+\mu'_{t+1}\mu)+(n-1)(n-2)\mu'_t\mu^2)]$$
$$= \lambda^3 N^3 E_G[G^2 E_G^{-1}\{(\theta_N^{-3}+3N\theta_N^{-4}+3N^2\theta_N^{-5}) + 4(n-1)(\theta_N^{-3}+\theta_N^{-4})$$
$$+ (n-1)(n-2)\theta_N^{-3}\}]$$
$$= \lambda^3 N^3 E_G[G^2 E_G^{-1}(d_{11}\theta_N^{-3}+d_{12}\theta_N^{-4}+d_{13}\theta_N^{-5})] , \tag{5.72}$$

where

$$d_{11} = 1 + (n-1)(n+2) ,$$
$$d_{12} = N(4n-1) ,$$
$$d_{13} = (n-1)(n-2) .$$

Now noting that $D_{11} = E_G[G^2 E_G^{-1}(\theta_N^{-3})] = C_1$, $D_{12} = E_G[G^2 E_G^{-1}(\theta_N^{-4})] = B_1$ and $D_{13} = E_G[G^2 E_G^{-1}(\theta_N^{-5})] = B_2$, we get

$$D_{pi1} = \lambda\theta_N^{-1} - \frac{2\lambda}{n}[N\theta_N^{-1}-(N-n)I_G^*] + \frac{\lambda N}{n^3} \sum_{k=1}^{3} d_{1k} D_{1k} . \tag{5.73}$$

<u>t = 2</u> ,

$$E_G[G\,E_G^{-1}(\mu'_{t+1}+(n-1)\mu'_t\mu)] = \lambda^3N^3E_G[G\,E_G^{-1}(n\theta_N^{-3} + N(n+2)\theta_N^{-4}+3N^2\theta_N^{-5})]$$

$$= \frac{\lambda^3N^3}{2}[\,(2n-N(n+2)+N^2)\theta_N^{-1} + (N(n+2)+N^2)\theta_N^{-2} + 2N^2\theta_N^{-3}+(N(n+2)-2n-N^2)I_G^*] . \quad (5.74)$$

$$E_G[G^2E_G^{-1}(\mu'_{t+2}+2(n-1)(\mu'_t\mu'_2+\mu'_{t+1}\mu) + (n-1)(n-2)\mu'_t\mu^2)]$$

$$= \lambda^4N^4E_G[G^2E_G^{-1}\{\theta_N^{-4}+6N\theta_N^{-5}+15N^2\theta_N^{-6}+15N^3\theta_N^{-7}+2(n-1)(\theta_N^{-4} + 2N\theta_N^{-5}+N^2\theta_N^{-6}+\theta_N^{-4}+3N\theta_N^{-5}+3N^2\theta_N^{-6})+(n-1)(n-2)(\theta_N^{-4}+N\theta_N^{-5})\}]$$

$$= \lambda^4N^4\sum_{k=1}^{4} d_{2k}\,D_{2k} , \quad (5.75)$$

where $d_{21} = 1 + (n-1)(n+2)$,

$d_{22} = N[\,6 + (n-1)(n+8)]$,

$d_{23} = N^2(8n+7)$,

$d_{24} = 15N^3$,

and

$$D_{2k} = E_G[G^2E_G^{-1}(\theta_N^{-k-3})] = B_k .$$

Therefore,

$$D_{pi2} = \lambda^2N^2(\theta_N^{-2}+N\theta_N^{-3}) - \frac{\lambda^2N}{n}\,(2n-N(n+2)+N^2)\theta_N^{-1} + (N(n+2)+N^2)\theta_N^{-2}+2N^2\theta_N^{-3}+(N(n+2)-2n-N^2)I_G^*] + \frac{\lambda^2N^2}{n^3}\sum_{k=1}^{4} d_{2k}\,D_{2k} . \quad (5.76)$$

5.5 Bias and MSE of the Regression Estimator, $\hat{\bar{Y}}_{lr}$, under the Model (5.1)

Similarly as in section 4.5, the bias of the regression estimator, for a fixed b, under model (5.1) is zero. The expected variance (~~EVAR) of~~ $\hat{\bar{Y}}_{lr}$ is given in the theorem below.

Theorem 5.5

$$EVAR(\hat{\bar{Y}}_{lr}) = (\frac{1-f}{n})[(b-\beta)^2 \frac{\mu^3}{\lambda} + \delta\mu_t'] \quad (5.77)$$

which can be simplified for $t = 0, 1, 2$ as follows:

$t = 0$,

$$EVAR(\hat{\bar{Y}}_{lr}) = (\frac{1-f}{n})[(b-\beta)^2 \frac{\mu^3}{\lambda} + \delta]. \quad (5.78)$$

$t = 1$,

$$EVAR(\hat{\bar{Y}}_{lr}) = (\frac{1-f}{n})[(b-\beta)^2 \frac{\mu^3}{\lambda} + \delta\mu]. \quad (5.79)$$

$t = 2$,

$$EVAR(\hat{\bar{Y}}_{lr}) = (\frac{1-f}{n})[(b-\beta)^2 \frac{\mu^3}{\lambda} + \delta(\mu^2+\frac{\mu^3}{\lambda})]. \quad (5.80)$$

Proof:

From section 4.5 we have

$$\begin{aligned} EVAR(\hat{\bar{Y}}_{lr}) &= (b-\beta)^2 E(Var\ \bar{x}) + E(\bar{\varepsilon}_n-\bar{\varepsilon}_N)^2 \\ &= (\frac{1-f}{n})(b-\beta)^2 E(S_x^2) + E(\bar{\varepsilon}_n-\bar{\varepsilon}_N)^2 \\ &= (\frac{1-f}{n})(b-\beta)^2 \frac{\mu^3}{\lambda} + E(\bar{\varepsilon}_n-\bar{\varepsilon}_N)^2 \end{aligned}$$

Now,

$$E(\bar{\varepsilon}_n-\bar{\varepsilon}_N)^2 = E(\bar{\varepsilon}_n^2-2\bar{\varepsilon}_n\bar{\varepsilon}_N + \bar{\varepsilon}_N^2) \quad (5.81)$$

and,

$$E(\bar{\varepsilon}_n\bar{\varepsilon}_N) = E[\frac{\bar{\varepsilon}_n(n\bar{\varepsilon}_n+(N-n)\bar{\varepsilon}_{N-n})}{N}]$$

$$= \frac{n}{N} E(\bar{\varepsilon}_n^2) + (\frac{N-n}{N}) E(\bar{\varepsilon}_n\bar{\varepsilon}_{N-n})$$

$$(\because E(\bar{\varepsilon}_n\bar{\varepsilon}_{N-n}|x) = 0)$$

$$= \frac{n}{N} E(\bar{\varepsilon}_n^2) \tag{5.82}$$

$$= \frac{1}{nN} E(\sum_{i=1}^{n} \varepsilon_i^2 + \sum_{i\neq j}^{n} \varepsilon_i\varepsilon_j)$$

$$= \frac{\delta}{N} \mu_t' . \tag{5.83}$$

Hence from (5.22), (5.82) and (5.83) we get

$$E(\bar{\varepsilon}_n - \bar{\varepsilon}_N)^2 = \delta\mu_t'(\frac{1}{n} - \frac{2}{N} + \frac{1}{N})$$

$$= \delta(\frac{1-f}{n})\mu_t' . \tag{5.84}$$

and therefore,

$$EVAR(\bar{\bar{Y}}_{1r}) = (\frac{1-f}{n})[(b-\beta)^2 \frac{\mu^3}{\lambda} + \delta\mu_t'] . \tag{5.85}$$

5.6 A Numerical Comparison of the Ratio, Product and Regression Estimators

In this section we present a numerical study for selected values of n, N, α, β, δ, t and h of the three estimators under the Gamma superpopulation model (see sections 2.6, 3.5 and 4.5).

These numerical studies are done with object of ascertaining the range of parameters where one estimator may be better than another. Some of the analytical properties are available in Chaubey et al (1984b) but it does not exhaust all cases.

A similar study can be performed on model (5.1), however this study is not undertaken here in this thesis for reasons of space. We plan to do this study separately by using the formulae developed in this chapter. However, for large samples, similar performance can be expected.

The bias and MSE for the three estimators are given in Tables 1 and 2 respectively for $\delta = 1$ which corresponds to the constant variance model for which the analytical properties are reported in Chaubey et al (1984 b) paper for an infinite population. We find from this numerical study that similar properties also hold for a finite population of moderate size. However the following points are highlighted below.

(i) The most significant conclusion is that the regression estimator comes out to be the best in terms of MSE if the estimate of the slope used is not far away from the true slope. The gain in efficiency being the most when the estimator is the same as the true value.

(ii) The bias of the ratio estimator depends only on α whereas for the product estimator it depends on β. The bias of the ratio is generally smaller than that of the product estimator except when β is exactly zero and then the bias is exactly zero.

(iii) When β is in the neighbourhood of zero, the product estimator is better than the ratio estimator and the large gains can be expected for higher values of α .

Table 1.a

				h					
				1		2		4	
N	n	α	β	Ratio	Product	Ratio	Product	Ratio	Product
100	10	1	-2	0.011111	-0.178217	0.002631	-0.179104	0.000641	-0.179551
			-1	0.011111	-0.089109	0.002631	-0.039552	0.000641	-0.089775
			0	0.011111	0.000000	0.002631	0.000000	0.000641	0.000000
			1	0.011111	0.089109	0.002631	0.089552	0.000641	0.089775
			2	0.011111	0.178217	0.002631	0.179104	0.000641	0.179551
		5	-2	0.055556	-0.178217	0.013157	-0.179104	0.003205	-0.179551
			-1	0.055556	-0.089109	0.013157	-0.039552	0.003205	-0.089775
			0	0.055556	0.000000	0.013157	0.000000	0.003205	0.000000
			1	0.055556	0.089109	0.013157	0.089552	0.003205	0.089775
			2	0.055556	0.178217	0.013157	0.179104	0.003205	0.179551
	20	1	-2	0.002631	-0.079207	0.000641	-0.079602	0.000158	-0.079800
			-1	0.002631	-0.039604	0.000641	-0.039801	0.000158	-0.039900
			0	0.002631	0.000000	0.000641	0.000000	0.000158	0.000000
			1	0.002631	0.039604	0.000641	0.039801	0.000158	0.039980
			2	0.002631	0.079207	0.000641	0.079602	0.000158	0.079800
		5	-2	0.013158	-0.079207	0.003205	-0.079602	0.000791	-0.079800
			-1	0.013158	-0.039604	0.003205	-0.039801	0.000791	-0.039900
			0	0.013158	0.000000	0.003205	0.000000	0.000791	0.000000
			1	0.013158	0.039604	0.003205	0.039801	0.000791	0.039900
			2	0.013158	0.079207	0.003205	0.079602	0.000791	0.079300

Table 1.b

				h					
				1		2		4	
N	n	α	β	Ratio	Product	Ratio	Product	Ratio	Product
500	50	1	-2	0.004082	-0.035928	0.000101	-0.035964	0.000025	-0.035982
			-1	0.004082	-0.017964	0.000101	-0.017982	0.000025	-0.017991
			0	0.004082	0.000000	0.000101	0.000000	0.000025	0.000000
			1	0.004082	0.017964	0.000101	0.017982	0.000025	0.017991
			2	0.004082	0.035928	0.000101	0.035964	0.000025	0.035982
		5	-2	0.002041	-0.035928	0.000505	-0.035964	0.000125	-0.035982
			-1	0.002041	-0.017964	0.000505	-0.017982	0.000125	-0.017991
			0	0.002041	0.000000	0.000505	0.000000	0.000125	0.000000
			1	0.002041	0.017964	0.000505	0.017982	0.000125	0.017991
			2	0.002041	0.035928	0.000505	0.035964	0.000125	0.035982
	100	1	-2	0.000101	-0.015968	0.000025	-0.015984	0.000006	-0.015992
			-1	0.000101	-0.007984	0.000025	-0.007992	0.000006	-0.007996
			0	0.000101	0.000000	0.000025	0.000000	0.000006	0.000000
			1	0.000101	0.007984	0.000025	0.007992	0.000006	0.007996
			2	0.000101	0.015968	0.000025	0.015984	0.000006	0.015992
		5	-2	0.000505	-0.015968	0.000125	-0.015984	0.000031	-0.015992
			-1	0.000505	-0.007984	0.000125	-0.007992	0.000031	-0.007996
			0	0.000505	0.000000	0.000125	0.000000	0.000031	0.000000
			1	0.000505	0.007984	0.000125	0.007992	0.000031	0.007996
			2	0.000505	0.015968	0.000125	0.015984	0.000031	0.015992

Table 1.c

				h					
				1		2		4	
N	n	α	β	Ratio	Product	Ratio	Product	Ratio	Product
1000	100	1	-2	0.000101	-0.017982	0.000025	-0.017991	0.000006	-0.017995
			-1	0.000101	-0.008991	0.000025	-0.008995	0.000006	-0.008997
			0	0.000101	0.000000	0.000025	0.000000	0.000006	0.000000
			1	0.000101	0.008991	0.000025	0.008995	0.000006	0.008997
			2	0.000101	0.017982	0.000025	0.017991	0.000006	0.017995
		5	-2	0.000505	-0.017982	0.000125	-0.017991	0.000031	-0.017995
			-1	0.000505	-0.008991	0.000125	-0.008995	0.000031	-0.008997
			0	0.000505	0.000000	0.000125	0.000000	0.000031	0.000000
			1	0.000505	0.008991	0.000125	0.008995	0.000031	0.008997
			2	0.000505	0.017982	0.000125	0.017991	0.000031	0.017995
	200	1	-2	0.000025	-0.007992	0.000006	-0.007996	0.000001	-0.007998
			-1	0.000025	-0.003996	0.000006	-0.003998	0.000001	-0.003999
			0	0.000025	0.000000	0.000006	0.000000	0.000001	0.000000
			1	0.000025	0.003996	0.000006	0.003998	0.000001	0.003999
			2	0.000025	0.007992	0.000006	0.007996	0.000001	0.007998
		5	-2	0.000125	-0.007992	0.000031	-0.007996	0.000007	-0.007998
			-1	0.000125	-0.003996	0.000031	-0.003998	0.000007	-0.003999
			0	0.000125	0.000000	0.000031	0.000000	0.000007	0.000000
			1	0.000125	0.003996	0.000031	0.003998	0.000007	0.003999
			2	0.000125	0.007992	0.000031	0.007996	0.000007	0.007998

Table 2a

t = 0

					e			h					
					1			2					
N	n	α	β	b	Ratio	Product	Reg.	Ratio	Product	Reg.	Ratio	Product	Reg.
100	10	1	-1	-1	0.270250	0.246980	0.09	0.161631	0.569514	0.09	0.122077	1.270725	0.09
				0	0.270250	0.246980	0.18	0.161631	0.569514	0.27	0.122077	1.270725	0.45
				1	0.270250	0.246980	0.45	0.161631	0.569514	0.81	0.122077	1.270725	1.53
			0	-1	0.270250	0.188019	0.18	0.161631	0.139253	0.27	0.122077	0.114688	0.45
				0	0.270250	0.188019	0.09	0.161631	0.139253	0.09	0.122077	0.114688	0.09
				1	0.270250	0.188019	0.18	0.161631	0.139253	0.27	0.122077	0.114688	0.45
			1	-1	0.270250	1.015763	0.45	0.161631	1.314305	0.81	0.122077	2.003222	1.53
				0	0.270250	1.015763	0.18	0.161631	1.314305	0.2[illegible]	0.122077	2.003222	0.45
				1	0.270250	1.015763	0.09	0.161631	1.314305	0.09	0.122077	2.003222	0.09
		5	-1	-1	3.810250	0.848029	0.09	1.538473	0.154559	0.09	0.731307	0.344384	0.09
				0	3.810250	0.848029	0.18	1.538473	0.1545[illegible]	0.27	0.731307	0.344384	0.45
				1	3.810250	0.848029	0.45	1.538473	0.154559	0.81	0.731307	0.344384	1.53
			0	-1	3.810250	2.326633	0.18	1.538473	1.213880	0.27	0.731307	0.653341	0.45
				0	3.810250	2.326633	0.09	1.538473	1.213880	0.09	0.731307	0.653341	0.09
				1	3.810250	2.326633	0.18	1.538473	1.213880	0.27	0.731307	0.653341	0.45
			1	-1	3.810250	4.691943	0.45	1.538473	3.878514	0.81	0.731307	4.006870	1.53
				0	3.810250	4.691943	0.18	1.538473	3.878514	0.27	0.731307	4.006870	0.45
				1	3.810250	4.691943	0.09	1.538473	3.878514	0.09	0.731307	4.006870	0.09
	20	1	-1	-1	0.096421	0.092786	0.04	0.065220	0.233666	0.04	0.051933	0.544051	0.04
				0	0.096421	0.092786	0.08	0.065220	0.233666	0.12	0.051933	0.544051	0.20
				1	0.096421	0.092786	0.20	0.065220	0.233666	0.36	0.051933	0.544051	0.68
			0	-1	0.096421	0.081589	0.08	0.065220	0.060895	0.12	0.051933	0.050473	0.20
				0	0.096421	0.081589	0.04	0.065220	0.060895	0.04	0.051933	0.050473	0.04
				1	0.096421	0.081589	0.08	0.065220	0.060895	0.12	0.051933	0.050473	0.20
			1	-1	0.096421	0.418936	0.20	0.065220	0.556803	0.36	0.0[illegible]	0.865635	0.68
				0	0.096421	0.418936	0.08	0.065220	0.556803	0.12	[illegible]	0.865635	0.20
				1	0.096421	0.418936	0.04	0.065220	0.556803	0.04	0.0519[illegible]	0.865635	0.04
		5	-1	-1	1.309052	0.390981	0.04	0.604167	0.065004	0.04	0.306188	0.140284	0.04
				0	1.309052	0.390981	0.08	0.604167	0.065004	0.12	0.306188	0.140284	0.20
				1	1.309052	0.390981	0.20	0.604167	0.065004	0.36	0.306188	0.140284	0.68
			0	-1	1.309052	1.032079	0.08	0.604167	0.538507	0.12	0.306188	0.289875	0.20
				0	1.309052	1.032079	0.04	0.604167	0.538507	0.04	0.306188	0.289875	0.04
				1	1.309052	1.032079	0.08	0.604167	0.538507	0.12	0.306188	0.289875	0.20
			1	-1	1.309052	2.021732	0.20	0.604167	1.680688	0.36	0.306188	1.748205	0.68
				0	1.309052	2.021732	0.08	0.604167	1.680688	0.12	0.306188	1.748205	0.20
				1	1.309052	2.021732	0.04	0.604167	1.680688	0.04	0.306188	1.748205	0.04

t = 0

Table 2b

					h								
					1			2					
h	n	a	B	b	Ratio	Product	Reg.	Ratio	Product	Reg.	Ratio	Product	Reg.
500	50	1	-1	-1	0.038879	0.038610	0.018	0.027959	0.101950	0.018	0.022864	0.241615	0.018
				0	0.038879	0.038610	0.036	0.027959	0.101950	0.054	0.022864	0.241615	0.090
				1	0.038879	0.038610	0.090	0.027959	0.101950	0.162	0.022864	0.241615	1.530
			0	-1	0.038879	0.036323	0.036	0.027959	0.027170	0.054	0.022864	0.022587	0.090
				0	0.038879	0.036323	0.018	0.027959	0.027170	0.018	0.022864	0.022587	0.018
				1	0.038879	0.036323	0.036	0.027959	0.027170	0.054	0.022864	0.022587	0.090
			1	-1	0.038879	0.184613	0.090	0.027959	0.246955	0.162	0.022864	0.386118	1.530
				0	0.038879	0.184613	0.036	0.027959	0.246955	0.054	0.022864	0.386118	0.090
				1	0.038879	0.184613	0.018	0.027959	0.246955	0.018	0.022864	0.386118	0.018
		5	-1	-1	0.514594	0.177742	0.018	0.254601	0.027725	0.018	0.133440	0.060550	0.018
				0	0.514594	0.177742	0.036	0.254601	0.027725	0.054	0.133440	0.060550	0.090
				1	0.514594	0.177742	0.090	0.254601	0.027725	0.162	0.133440	0.060550	1.530
			0	-1	0.514594	0.467461	0.036	0.254601	0.242955	0.054	0.133440	0.130533	0.090
				0	0.514594	0.467461	0.018	0.254601	0.242955	0.018	0.133440	0.130533	0.018
				1	0.514594	0.467461	0.036	0.254601	0.242955	0.054	0.133440	0.130533	0.090
			1	-1	0.514594	0.907756	0.090	0.254601	0.752749	0.162	0.133440	0.783070	1.530
				0	0.514594	0.907756	0.036	0.254601	0.752749	0.054	0.133440	0.783070	0.090
				1	0.514594	0.907756	0.018	0.245601	0.752749	0.018	0.133440	0.783070	0.018
	100	1	-1	-1	0.016590	0.016511	0.008	0.012198	0.049547	0.008	0.010075	0.106562	0.008
				0	0.016590	0.016511	0.016	0.012198	0.049547	0.024	0.010075	0.106562	0.040
				1	0.016590	0.016511	0.040	0.012198	0.049547	0.072	0.010075	0.106562	0.136
			0	-1	0.016590	0.016063	0.016	0.012198	0.012036	0.024	0.010075	0.010019	0.040
				0	0.016590	0.016063	0.008	0.012198	0.012036	0.008	0.010075	0.010019	0.008
				1	0.016590	0.016063	0.016	0.012198	0.012036	0.024	0.010075	0.010019	0.040
			1	-1	0.016590	0.080765	0.040	0.012198	0.108675	0.072	0.010075	0.170626	0.136
				0	0.016590	0.080765	0.016	0.012198	0.108675	0.024	0.010075	0.170626	0.040
				1	0.016590	0.080765	0.008	0.012198	0.108675	0.008	0.010075	0.170626	0.008
		5	-1	-1	0.217654	0.079620	0.008	0.110435	0.012196	0.008	0.058631	0.026411	0.008
				0	0.217654	0.079620	0.016	0.110435	0.012196	0.024	0.058631	0.026411	0.040
				1	0.217654	0.079620	0.040	0.110435	0.012196	0.072	0.058631	0.026411	0.136
			0	-1	0.217654	0.207680	0.016	0.110435	0.107940	0.024	0.058631	0.057995	0.040
				0	0.217654	0.207680	0.008	0.110435	0.107940	0.008	0.058631	0.057995	0.008
				1	0.217654	0.207680	0.016	0.110435	0.107940	0.024	0.058631	0.057995	0.040
			1	-1	0.217654	0.400890	0.040	0.110435	0.332834	0.072	0.058631	0.346730	0.136
				0	0.217654	0.400890	0.016	0.110435	0.332834	0.024	0.058631	0.346730	0.040
				1	0.217654	0.400890	0.008	0.110435	0.332834	0.008	0.058631	0.346730	0.008

Table 2c

t = 0

					h = 1			h = 2			h = 4		
N	n	α	β	b	Ratio	Product	Reg.	Ratio	Product	Reg.	Ratio	Product	Reg.
1000	100	1	-1	-1	0.018701	0.018650	0.009	0.013736	0.050236	0.009	0.011340	0.120028	[illegible]
				0	0.018701	0.018650	0.018	0.013736	0.050236	0.027	0.011340	0.120028	[illegible]
				1	0.018701	0.018650	0.045	0.013736	0.050236	0.081	0.011340	0.120028	[illegible]
			0	-1	0.018701	0.018080	0.018	0.013736	0.013542	0.027	0.011340	0.011271	[illegible]
				0	0.018701	0.018080	0.009	0.013736	0.013542	0.009	0.011340	0.011271	0.009
				1	0.018701	0.018080	0.018	0.013736	0.013542	0.027	0.011340	0.011271	0.045
			1	-1	0.018701	0.091152	0.045	0.013736	0.122488	0.081	0.011340	0.192154	0.153
				0	0.018701	0.091152	0.018	0.013736	0.122488	0.027	0.011340	0.192154	0.045
				1	0.018701	0.091152	0.009	0.013736	0.122488	0.009	0.011340	0.192154	0.009
		5	-1	-1	0.245343	0.089429	0.009	0.124362	0.013679	0.009	0.065992	0.029763	0.009
				0	0.245343	0.089429	0.018	0.124362	0.013679	0.027	0.065992	0.029763	0.045
				1	0.245343	0.089429	0.045	0.124362	0.013679	0.081	0.065992	0.029763	0.153
			0	-1	0.245343	0.233865	0.018	0.124362	0.121488	0.027	0.065992	0.065258	0.045
				0	0.245343	0.233865	0.009	0.124362	0.121488	0.009	0.065992	0.065258	0.009
				1	0.245343	0.233865	0.018	0.124362	0.121488	0.027	0.065992	0.065258	0.045
			1	-1	0.245343	0.451941	0.045	0.124362	0.374937	0.081	0.065992	0.390392	0.153
				0	0.245343	0.451941	0.018	0.124362	0.374937	0.027	0.065992	0.390392	0.045
				1	0.245343	0.451941	0.009	0.124362	0.374937	0.009	0.065992	0.390392	0.009
	200	1	-1	-1	0.008145	0.008128	0.004	0.006049	0.022137	0.004	0.005018	0.053140	0.004
				0	0.008145	0.008128	0.008	0.006049	0.022137	0.012	0.005018	0.053140	0.020
				1	0.008145	0.008128	0.020	0.006049	0.022137	0.036	0.005018	0.053140	0.068
			0	-1	0.008145	0.008016	0.008	0.005049	0.006009	0.012	0.005018	0.005004	0.020
				0	0.008145	0.008016	0.004	0.005049	0.006009	0.004	0.005018	0.005004	0.004
				1	0.008145	0.008016	0.008	0.005049	0.006009	0.012	0.005018	0.005004	0.020
			1	-1	0.008145	0.040191	0.020	0.005049	0.054168	0.036	0.005018	0.085156	0.068
				0	0.008145	0.040191	0.008	0.005049	0.054168	0.012	0.005018	0.085156	0.020
				1	0.008145	0.040191	0.004	0.005049	0.054168	0.004	0.005018	0.085156	0.004
		5	-1	-1	0.106382	0.039904	0.004	0.054604	0.006049	0.004	0.029157	0.013102	[illegible]
				0	0.106382	0.039904	0.008	0.054604	0.006049	0.012	0.029157	0.013102	0.020
				1	0.106382	0.039904	0.020	0.054604	0.006049	0.036	0.029157	0.013102	0.068
			0	-1	0.106382	0.103920	0.008	0.054604	0.053985	0.012	0.029157	0.028998	[illegible]
				0	0.106382	0.103920	0.004	0.054604	0.053985	0.004	0.029157	0.028998	0.004
				1	0.106382	0.103920	0.008	0.054604	0.053985	0.012	0.029157	0.028998	0.020
			1	-1	0.106382	0.200223	0.020	0.054604	0.166208	0.036	0.029157	0.173182	0.068
				0	0.106382	0.200223	0.008	0.054604	0.166208	0.012	0.029157	0.173182	0.020
				1	0.106382	0.200223	0.004	0.054604	0.166208	0.004	0.029157	0.173182	0.004

Table 2d

t = 1

					h = 1			h = 2			h = 4		
N	n	α	B	b	Ratio	Product	Reg.	Ratio	Product	Reg	Ratio	Product	Reg
100	10	1	-1	-1	0.247500	0.264418	0.09	0.245842	0.680820	0.18	0.394615	1.561975	0.36
				0	0.247500	0.264418	0.18	0.245842	0.680820	0.36	0.394615	1.561975	0.72
				1	0.247500	0.264418	0.45	0.245842	0.680820	0.90	0.394615	1.561975	1.80
			0	-1	0.247500	0.205457	0.18	0.245842	0.250560	0.36	0.394615	0.407938	0.72
				0	0.247500	0.205457	0.09	0.245842	0.250560	0.18	0.394615	0.407938	0.36
				1	0.247500	0.205457	0.18	0.245842	0.250560	0.36	0.394615	0.407938	0.72
			1	-1	0.247500	1.033200	0.45	0.245842	1.425611	0.90	0.394615	2.296472	1.80
				0	0.247500	1.033200	0.18	0.245842	1.425611	0.36	0.394615	2.296472	0.72
				1	0.247500	1.033200	0.09	0.245842	1.425611	0.18	0.394615	2.296472	0.36
		5	-1	-1	3.787500	0.865466	0.09	1.623684	0.265866	0.18	1.003846	0.637634	0.36
				0	3.787500	0.865466	0.18	1.623684	0.265866	0.36	1.003846	0.637634	0.72
				1	3.787500	0.865466	0.45	1.623684	0.265866	0.90	1.003846	0.637634	1.80
			0	-1	3.787500	2.344071	0.18	1.623684	1.325186	0.36	1.003846	0.946591	0.72
				0	3.787500	2.344071	0.09	1.623684	1.325186	0.18	1.003846	0.946591	0.36
				1	3.787500	2.344071	0.18	1.623684	1.325186	0.36	1.003846	0.946591	0.72
			1	-1	3.787500	1.709380	0.45	1.623684	3.989820	0.90	1.003846	4.300120	1.80
				0	3.787500	1.709380	0.18	1.623684	3.989820	0.36	1.003846	4.300120	0.72
				1	3.787500	1.709380	0.09	1.623684	3.989820	0.18	1.003846	4.300120	0.36
	20	1	-1	-1	0.092631	0.096071	0.04	0.104507	0.277904	0.08	0.172619	0.668769	0.16
				0	0.092631	0.096071	0.08	0.104507	0.277904	0.16	0.172619	0.668769	0.32
				1	0.092631	0.096071	0.20	0.104507	0.277904	0.40	0.172619	0.668769	0.80
			0	-1	0.092631	0.084869	0.08	0.104507	0.105133	0.16	0.172619	0.175191	0.32
				0	0.092631	0.084869	0.04	0.104507	0.105133	0.08	0.172619	0.175191	0.16
				1	0.092631	0.084869	0.08	0.104507	0.105133	0.16	0.172619	0.175191	0.32
			1	-1	0.092631	0.422221	0.20	0.104507	0.001041	0.40	0.172619	0.990353	0.80
				0	0.092631	0.422221	0.08	0.104507	0.001041	0.16	0.172619	0.990353	0.32
				1	0.092631	0.422221	0.04	0.104507	0.001041	0.08	0.172619	0.990353	0.16
		5	-1	-1	1.305263	0.394205	0.04	0.643454	0.109243	0.08	0.426874	0.265003	0.16
				0	1.305263	0.394205	0.08	0.643454	0.109243	0.16	0.426874	0.265003	0.32
				1	1.305263	0.394205	0.20	0.643454	0.109243	0.40	0.426874	0.265003	0.80
			0	-1	1.305263	1.035364	0.08	0.643454	0.582745	0.16	0.426874	0.414593	0.32
				0	1.305263	1.034364	0.04	0.643454	0.582745	0.08	0.425874	0.414593	0.16
				1	1.305263	1.034364	0.08	0.643454	0.582745	0.16	0.425874	0.414593	0.32
			1	-1	1.305263	2.025017	0.20	0.643454	1.724927	0.40	0.425874	1.872923	0.80
				0	1.305263	2.025017	0.08	0.643454	1.724927	0.16	0.425874	1.872923	0.32
				1	1.305263	2.025017	0.04	0.643454	1.724927	0.08	0.425874	1.872923	0.16

Table 2e

t = 1

					h								
					1			2			4		
N	n	α	β	b	Ratio	Product	Reg.	Ratio	Product	Reg.	Ratio	Product	Reg.
500	50	1	-1	-1	0.038188	0.039268	0.018	0.045807	0.120783	0.036	0.076971	0.296535	0.072
				0	0.038188	0.039268	0.036	0.045807	0.120783	0.072	0.076971	0.296535	0.144
				1	0.038188	0.039268	0.090	0.045807	0.120783	0.180	0.076971	0.296535	0.360
			0	-1	0.038188	0.036981	0.036	0.045807	0.046003	0.072	0.076971	0.077508	0.144
				0	0.038188	0.036981	0.018	0.045807	0.046003	0.036	0.076971	0.077508	0.072
				1	0.038188	0.036981	0.036	0.045807	0.046003	0.072	0.076971	0.077508	0.144
			1	-1	0.038188	0.185271	0.090	0.045807	0.265788	0.180	0.076971	0.441038	0.360
				0	0.038188	0.185271	0.036	0.045807	0.265788	0.072	0.076971	0.441038	0.144
				1	0.038188	0.185271	0.018	0.045807	0.265788	0.036	0.076971	0.441038	0.072
		5	-1	-1	0.513903	0.178400	0.018	0.272449	0.046558	0.036	0.187597	0.115475	0.072
				0	0.513903	0.178400	0.036	0.272449	0.046558	0.072	0.187597	0.115475	0.144
				1	0.513903	0.178400	0.090	0.272449	0.046558	0.180	0.187597	0.115475	0.360
			0	-1	0.513903	0.468119	0.036	0.272449	0.261788	0.072	0.187597	0.185454	0.144
				0	0.513903	0.468119	0.018	0.272449	0.261788	0.036	0.187597	0.185454	0.072
				1	0.513903	0.468119	0.036	0.272449	0.261788	0.072	0.187597	0.185454	0.144
			1	-1	0.513903	0.908414	0.090	0.272449	0.771581	0.180	0.187597	0.837991	0.360
				0	0.513903	0.908414	0.036	0.272449	0.771581	0.072	0.187597	0.837991	0.144
				1	0.513903	0.908414	0.018	0.272449	0.771581	0.036	0.187597	0.837991	0.072
	100	1	-1	-1	0.016458	0.016640	0.008	0.020173	0.052716	0.016	0.034103	0.130751	0.032
				0	0.016458	0.016640	0.016	0.020173	0.052716	0.032	0.034103	0.130751	0.064
				1	0.016458	0.016640	0.040	0.020173	0.052716	0.080	0.034103	0.130751	0.160
			0	-1	0.016458	0.016192	0.016	0.020173	0.020204	0.032	0.034103	0.034207	0.064
				0	0.016458	0.016192	0.008	0.020173	0.020204	0.016	0.034103	0.034207	0.032
				1	0.016458	0.016192	0.016	0.020173	0.020204	0.032	0.034103	0.034207	0.064
			1	-1	0.016458	0.080894	0.040	0.020173	0.116843	0.080	0.034103	0.194814	0.160
				0	0.016458	0.080894	0.016	0.020173	0.116843	0.032	0.034103	0.194814	0.064
				1	0.016458	0.080894	0.008	0.020173	0.116843	0.016	0.034103	0.194814	0.032
		5	-1	-1	0.217522	0.079749	0.008	0.118410	0.020365	0.016	0.082658	0.050599	0.032
				0	0.217522	0.079749	0.016	0.118410	0.020365	0.032	0.082658	0.050599	0.064
				1	0.217522	0.079749	0.040	0.118410	0.020365	0.080	0.082658	0.050599	0.160
			0	-1	0.217522	0.207809	0.016	0.118410	0.116108	0.032	0.082658	0.082183	0.064
				0	0.217522	0.207809	0.008	0.118410	0.116108	0.016	0.082658	0.082183	0.032
				1	0.217522	0.207809	0.016	0.118410	0.116108	0.032	0.082658	0.082183	0.064
			1	-1	0.217522	0.401019	0.040	0.118410	0.341002	0.080	0.082658	0.370918	0.160
				0	0.217522	0.401019	0.016	0.118410	0.341002	0.032	0.082658	0.370918	0.064
				1	0.217522	0.401019	0.008	0.118410	0.341002	0.016	0.082658	0.370918	0.032

Table 2f

t = 1

N	n	α	β	b	h = 1 Ratio	h = 1 Product	h = 1 Reg.	h = 2 Ratio	h = 2 Product	h = 2 Reg.	h = 4 Ratio	h = 4 Product	h = 4 Reg.
1000	100	1	-1	-1	0.018534	0.018813	0.009	0.022699	0.059444	0.018	0.038367	0.147258	0.036
				0	0.018534	0.018813	0.018	0.022699	0.059444	0.036	0.038367	0.147258	0.072
				1	0.018534	0.018813	0.045	0.022699	0.059444	0.090	0.038367	0.147258	0.180
			0	-1	0.018534	0.018244	0.018	0.022699	0.022750	0.036	0.038367	0.038501	0.072
				0	0.018534	0.018244	0.009	0.022699	0.022750	0.018	0.038367	0.038501	0.036
				1	0.018534	0.018244	0.018	0.022699	0.022750	0.036	0.038367	0.038501	0.072
			1	-1	0.018534	0.091315	0.045	0.022699	0.131695	0.090	0.038367	0.219384	0.180
				0	0.018534	0.091315	0.018	0.022699	0.131695	0.036	0.038367	0.219384	0.072
				1	0.018534	0.091315	0.009	0.022699	0.131695	0.018	0.038367	0.219384	0.036
		5	-1	-1	0.245176	0.089593	0.009	0.133325	0.022887	0.018	0.093019	0.056992	0.036
				0	0.245176	0.089593	0.018	0.133325	0.022887	0.036	0.093019	0.056992	0.072
				1	0.245176	0.089593	0.045	0.133325	0.022887	0.090	0.093019	0.056992	0.180
			0	-1	0.245176	0.234028	0.018	0.133325	0.130696	0.036	0.093019	0.092488	0.072
				0	0.245176	0.234028	0.009	0.133325	0.130696	0.018	0.093019	0.092488	0.036
				1	0.245176	0.234028	0.018	0.133325	0.130696	0.036	0.093019	0.092488	0.072
			1	-1	0.245176	0.452104	0.045	0.133325	0.384145	0.090	0.093019	0.417622	0.180
				0	0.245176	0.452104	0.018	0.133325	0.384145	0.036	0.093019	0.417622	0.072
				1	0.245176	0.452104	0.009	0.133325	0.384145	0.018	0.093019	0.417622	0.036
	200	1	-1	-1	0.008113	0.008160	0.004	0.010043	0.026179	0.008	0.017025	0.065187	0.016
				0	0.008113	0.008160	0.008	0.010043	0.026179	0.016	0.017025	0.065187	0.032
				1	0.008113	0.008160	0.020	0.010043	0.026179	0.040	0.017025	0.065187	0.080
			0	-1	0.008113	0.008048	0.008	0.010043	0.010051	0.016	0.017025	0.017051	0.032
				0	0.008113	0.008048	0.004	0.010043	0.010051	0.008	0.017025	0.017051	0.016
				1	0.008113	0.008048	0.008	0.010043	0.010051	0.016	0.017026	0.017051	0.032
			1	-1	0.008113	0.040223	0.020	0.010043	0.058210	0.040	0.017025	0.097203	0.080
				0	0.008113	0.040223	0.008	0.010043	0.058210	0.016	0.017025	0.097203	0.032
				1	0.008113	0.040223	0.004	0.010043	0.058210	0.008	0.017025	0.097203	0.016
		5	-1	-1	0.105349	0.039936	0.004	0.058598	0.010091	0.008	0.041164	0.025149	0.016
				0	0.105349	0.039936	0.008	0.058598	0.010091	0.016	0.041164	0.025149	0.032
				1	0.105349	0.039936	0.020	0.048498	0.010091	0.040	0.041164	0.025149	0.080
			0	-1	0.105349	0.103952	0.008	0.048498	0.058027	0.016	0.041164	0.041045	0.032
				0	0.105349	0.103952	0.004	0.048498	0.058027	0.008	0.041164	0.041045	0.016
				1	0.105349	0.103952	0.008	0.048498	0.058027	0.016	0.041164	0.041045	0.032
			1	-1	0.105349	0.200255	0.020	0.048498	0.170250	0.040	0.041164	0.185229	0.080
				0	0.105349	0.200255	0.008	0.048498	0.170250	0.016	0.041164	0.185229	0.032
				1	0.105349	0.200255	0.004	0.048498	0.170250	0.008	0.041164	0.185229	0.016

Table 2g

t = 2

N	n	α	β	b	h = 1 Ratio	h = 1 Product	h = 1 Reg.	h = 2 Ratio	h = 2 Product	h = 2 Reg.	h = 4 Ratio	h = 4 Product	h = 4 Reg.
100	10	1	-1	-1	0.314409	0.417984	0.18	0.576797	1.144723	0.54	1.790262	3.190117	1.80
				0	0.314409	0.417984	0.27	0.576797	1.144723	0.72	1.790262	3.190117	2.16
				1	0.314409	0.417984	0.54	0.576797	1.144723	1.26	1.790262	3.190117	3.25
			0	-1	0.314409	0.359023	0.27	0.576797	0.714462	0.72	1.790262	2.034080	2.16
				0	0.314409	0.349023	0.18	0.576797	0.714462	0.54	1.790262	2.034080	1.80
				1	0.314409	0.349023	0.27	0.576797	0.714462	0.72	1.790262	2.034080	2.16
			1	-1	0.314409	1.186767	0.54	0.576797	1.889514	1.26	1.790262	3.922614	3.24
				0	0.314409	1.186767	0.27	0.576797	1.889514	0.72	1.790262	3.922614	2.16
				1	0.314409	1.186767	0.18	0.576797	1.889514	0.54	1.790262	3.922614	1.80
		5	-1	-1	3.854409	1.019032	0.18	1.953639	0.729768	0.54	2.399493	2.263776	1.80
				0	3.854409	1.019032	0.27	1.953639	0.729768	0.72	2.399493	2.263776	2.16
				1	3.854409	1.019032	0.54	1.953639	0.729768	1.26	2.399493	2.263776	3.25
			0	-1	3.854409	2.497637	0.27	1.953639	1.789089	0.72	2.399493	2.572733	2.16
				0	3.854409	2.497637	0.18	1.953639	1.789089	0.54	2.399493	2.572733	1.80
				1	3.854409	2.497637	0.27	1.953639	1.789089	0.72	2.399493	2.572733	2.16
			1	-1	3.854409	4.862946	0.54	1.953639	4.453722	1.26	2.399493	5.926262	3.24
				0	3.854409	4.862946	0.27	1.953639	4.453722	0.72	2.399493	5.926262	2.16
				1	3.854409	4.862946	0.18	1.953639	4.453722	0.54	2.399493	5.926262	1.80
	20	1	-1	-1	0.128240	0.148076	0.08	0.238943	0.458236	0.24	0.804668	1.345862	0.80
				0	0.128240	0.148076	0.12	0.238943	0.458236	0.32	0.804668	1.345862	0.96
				1	0.128240	0.148076	0.24	0.238943	0.458236	0.56	0.804668	1.345862	1.44
			0	-1	0.128240	0.136874	0.12	0.238943	0.285465	0.32	0.804668	0.852284	0.96
				0	0.128240	0.136874	0.08	0.238943	0.285465	0.24	0.804668	0.852284	0.80
				1	0.128240	0.136874	0.12	0.238943	0.285465	0.32	0.804668	0.852284	0.96
			1	-1	0.128240	0.474226	0.24	0.238943	0.781373	0.56	0.804668	1.667446	1.44
				0	0.128240	0.474226	0.12	0.238943	0.781373	0.32	0.804668	1.667446	0.96
				1	0.128240	0.474226	0.08	0.238943	0.781373	0.24	0.804668	1.667446	0.80
		5	-1	-1	1.340872	0.446271	0.08	0.797891	0.289574	0.24	1.058923	0.942095	0.80
				0	1.340872	0.446271	0.12	0.797891	0.289574	0.32	1.058923	0.942095	0.96
				1	1.340872	0.446271	0.24	0.797891	0.289574	0.56	1.058923	0.942095	1.44
			0	-1	1.340872	1.087369	0.12	0.797891	0.763077	0.32	1.058923	1.091686	0.96
				0	1.340872	1.087369	0.08	0.797891	0.763077	0.24	1.058923	1.091686	0.80
				1	1.340872	1.087369	0.12	0.797891	0.763077	0.32	1.058923	1.091686	0.96
			1	-1	1.340872	2.077022	0.24	0.797891	1.905258	0.56	1.058923	2.550016	1.44
				0	1.340872	2.077022	0.12	0.797891	1.905258	0.32	1.058923	2.550016	0.96
				1	1.340872	2.077022	0.08	0.797891	1.905258	0.24	1.058923	2.550016	0.80

Table 2h

t = 2

					h = 1			h = 2			h = 4		
N	n	α	β	b	Ratio	Product	Reg.	Ratio	Product	Reg.	Ratio	Product	Reg.
500	50	1	-1	-1	0.055256	0.039622	0.036	0.116588	0.196783	0.108	0.363176	0.591842	0.360
				0	0.055256	0.039622	0.054	0.115588	0.196783	0.144	0.363176	0.591842	0.432
				1	0.055256	0.039622	0.108	0.115588	0.196783	0.252	0.363176	0.591842	0.648
			0	-1	0.055256	0.057335	0.054	0.115588	0.122003	0.144	0.363176	0.372815	0.432
				0	0.055256	0.057335	0.036	0.115588	0.122003	0.108	0.363176	0.372815	0.360
				1	0.055256	0.057335	0.054	0.115588	0.122003	0.144	0.343176	0.372815	0.432
			1	-1	0.055256	0.205625	0.108	0.115588	0.341788	0.252	0.343176	0.736346	0.648
				0	0.055256	0.205625	0.054	0.115588	0.341788	0.144	0.343176	0.735346	0.432
				1	0.055256	0.205625	0.036	0.115588	0.341788	0.108	0.343176	0.735346	0.360
		5	-1	-1	0.530971	0.198754	0.036	0.343229	0.122558	0.108	0.473802	0.410782	0.360
				0	0.530971	0.198754	0.054	0.343229	0.122558	0.144	0.473802	0.410782	0.432
				1	0.530971	0.198754	0.108	0.343229	0.122558	0.252	0.473802	0.410782	0.648
			0	-1	0.530971	0.488473	0.054	0.343229	0.337788	0.144	0.473802	0.480761	0.432
				0	0.530971	0.488473	0.036	0.343229	0.337788	0.108	0.473802	0.480761	0.360
				1	0.530971	0.488473	0.054	0.343229	0.337788	0.144	0.473802	0.480761	0.432
			1	-1	0.530971	0.928768	0.108	0.343229	0.847582	0.252	0.473802	1.133298	0.648
				0	0.530971	0.928768	0.054	0.343229	0.847582	0.144	0.473802	1.133298	0.432
				1	0.530971	0.928768	0.036	0.343229	0.847582	0.108	0.473802	1.133298	0.360
	100	1	-1	-1	0.024282	0.025108	0.016	0.051949	0.085518	0.048	0.161783	0.260225	0.160
				0	0.024282	0.025108	0.024	0.051949	0.085518	0.064	0.161783	0.260225	0.192
				1	0.024282	0.025108	0.048	0.051949	0.085518	0.112	0.161783	0.260225	0.288
			0	-1	0.024282	0.024660	0.024	0.051949	0.053007	0.064	0.161783	0.163681	0.192
				0	0.024282	0.024660	0.016	0.051949	0.053007	0.048	0.161783	0.163681	0.160
				1	0.024282	0.024660	0.024	0.051949	0.053007	0.064	0.161783	0.163681	0.192
			1	-1	0.024282	0.089362	0.048	0.051949	0.149646	0.112	0.161783	0.324289	0.288
				0	0.024282	0.089362	0.024	0.051949	0.149646	0.064	0.161783	0.324289	0.192
				1	0.024282	0.089362	0.016	0.051949	0.149646	0.048	0.161783	0.324289	0.160
		5	-1	-1	0.225346	0.088216	0.016	0.150186	0.053167	0.048	0.210339	0.180073	0.160
				0	0.225346	0.088216	0.024	0.150186	0.053167	0.064	0.210339	0.180073	0.192
				1	0.225346	0.088216	0.048	0.150186	0.053167	0.112	0.210339	0.180073	0.288
			0	-1	0.225346	0.216276	0.024	0.150186	0.148911	0.064	0.210339	0.211657	0.192
				0	0.225346	0.216276	0.016	0.150186	0.148911	0.048	0.210339	0.211657	0.160
				1	0.225346	0.216276	0.024	0.150186	0.148911	0.064	0.210339	0.211657	0.192
			1	-1	0.225346	0.409486	0.048	0.150186	0.148911	0.112	0.210339	0.211657	0.288
				0	0.224346	0.409486	0.024	0.150186	0.148911	0.064	0.210339	0.211657	0.192
				1	0.224346	0.409486	0.016	0.150186	0.148911	0.048	0.210339	0.211657	0.160

Table 21

$t = 2$

					h								
					1			2			4		
N.	n	α	β	b	Ratio	Product	Reg.	Ratio	Product	Reg.	Ratio	Product	Reg.
1000	100	1	-1	-1	0.027300	0.028395	0.018	0.058394	0.096439	0.054	0.181918	0.293080	0.180
				0	0.027300	0.028395	0.027	0.058394	0.096439	0.072	0.181918	0.293080	0.216
				1	0.027300	0.028395	0.054	0.058394	0.096439	0.126	0.181918	0.293080	0.324
			0	-1	0.027300	0.027826	0.027	0.058394	0.059745	0.072	0.181918	0.184324	0.216
				0	0.027300	0.027826	0.018	0.058394	0.059745	0.054	0.181918	0.184324	0.180
				1	0.027300	0.027826	0.027	0.058394	0.059745	0.072	0.181918	0.184324	0.216
			1	-1	0.027300	0.100898	0.054	0.058394	0.168691	0.126	0.181918	0.365200	0.324
				0	0.027300	0.100898	0.027	0.058394	0.168691	0.072	0.181918	0.365200	0.216
				1	0.027300	0.100898	0.018	0.058394	0.168691	0.054	0.181918	0.365200	0.180
		5	-1	-1	0.253942	0.099175	0.018	0.169020	0.059882	0.054	0.236570	0.202815	0.180
				0	0.253942	0.099175	0.027	0.169020	0.059882	0.072	0.236570	0.202815	0.216
				1	0.253942	0.099175	0.054	0.169020	0.059882	0.126	0.236570	0.202815	0.324
			0	-1	0.253942	0.243610	0.027	0.169020	0.167691	0.072	0.236570	0.238310	0.216
				0	0.253942	0.243610	0.018	0.169020	0.167691	0.054	0.236570	0.238310	0.180
				1	0.253942	0.243610	0.027	0.169020	0.167691	0.072	0.236570	0.238310	0.216
			1	-1	0.253942	0.461687	0.054	0.169020	0.421140	0.126	0.236570	0.563444	0.324
				0	0.253942	0.461687	0.027	0.169020	0.421140	0.072	0.236570	0.563444	0.216
				1	0.253942	0.461687	0.018	0.169020	0.421140	0.054	0.236570	0.563444	0.180
	200	1	-1	-1	0.012069	0.012276	0.008	0.025987	0.042379	0.024	0.080945	0.129556	0.080
				0	0.012069	0.012276	0.012	0.025987	0.042379	0.032	0.080945	0.129556	0.096
				1	0.012069	0.012276	0.024	0.025987	0.042379	0.056	0.080945	0.129556	0.144
			0	-1	0.012069	0.012164	0.012	0.025987	0.026251	0.032	0.080945	0.081420	0.096
				0	0.012069	0.012164	0.008	0.025987	0.026251	0.024	0.080945	0.081420	0.080
				1	0.012069	0.012164	0.012	0.025987	0.026251	0.032	0.080945	0.081420	0.096
			1	-1	0.012069	0.044340	0.024	0.025987	0.074411	0.056	0.080945	0.161572	0.144
				0	0.012069	0.044340	0.012	0.025987	0.074411	0.032	0.080945	0.161572	0.096
				1	0.012069	0.044340	0.008	0.025987	0.074411	0.024	0.080945	0.161572	0.080
		5	-1	-1	0.110306	0.044531	0.008	0.074542	0.026291	0.024	0.105084	0.089518	0.080
				0	0.110306	0.044531	0.012	0.074542	0.026291	0.032	0.105084	0.089518	0.096
				1	0.110306	0.044531	0.024	0.074542	0.026291	0.056	0.105084	0.089518	0.144
			0	-1	0.110306	0.108068	0.012	0.074542	0.074227	0.032	0.105084	0.105414	0.096
				0	0.110306	0.108068	0.008	0.074542	0.074227	0.024	0.105084	0.105414	0.080
				1	0.110306	0.108068	0.012	0.074542	0.074227	0.032	0.105084	0.105414	0.096
			1	-1	0.110306	0.204371	0.024	0.074542	0.186451	0.056	0.105084	0.249598	0.144
				0	0.110306	0.204371	0.012	0.074542	0.186451	0.032	0.105084	0.249598	0.096
				1	0.110306	0.204371	0.008	0.074542	0.186451	0.024	0.105084	0.249598	0.080

REFERENCES

1. Beale, E.M.L. (1962). Some uses of computers in operational research. Industrielle Organisation, 31, 51-52.
2. Brewer, K.R.W. (1963). Ratio estimation in finite population: some results deducible from the assumption of an underlying stochastic process. Australian Jour. Stat., 5, 93-105.
3. Cassel, C.M., Särndal, C.E. and Wretman, J.H. (1977). Foundations of Inference in Survey Sampling. John Wiley and Sons, N.Y.
4. Chaubey, Y.P., Dwivedi, T.D. and Singh, M. (1981). A note on the product estimator. Metron, 34(3,4), 142-145.
5. _______ (1984 a). A note on the optimality of the regression estimator. Biom. Journal, 26(4), 465-467.
6. _______ (1984 b). An efficiency comparison of product and ratio estimator. Commun. Statist. - Theor. Meth., 13(6), 699-709.
7. _______ (1985 a). On the exact variance of an unbiased product type estimator and its comparison with the sample mean. To appear in Metrika.
8. _______ (1985 b). Corrections to (1984 b). To appear in Comm. Statist. - Theory and Methods.
9. _______ (1985 c). On a dual to ratio estimator in double sampling. To appear in Journal of Organizational Behaviour and Statistics.
10. Cochran, W.G. (1939). The use of analysis of variance

in enumeration by sampling. Jour. Amer. Statist. Assoc. 36, 492-510.

11. _______ (1946). Relative accuracy of systematic and stratified random samples for a certain class of populations. Ann. Math. Statist., 17, 169-177.

12. _______ (1977). Sampling Techniques. Third Edition. John Wiley and Sons, N.Y.

13. Deming, D.E. and Stephan, S. (1941). On the interpretation of censuses as samples. Jour. Amer. Statist. Assoc., 36, 45-49.

14. Draper, N. and Smith, H. (1966). Applied Regression Analysis. John Wiley and Sons, N.Y.

15. Durbin, J. (1959). A note on the application of Quenouille's method of bias reduction to the estimation of ratios. Biometrika, 46, 477-480.

16. Fuller, W.A. (1975). Regression analysis for sample survey. Sankhyā, Ser. C, 37, 117-132.

17. Goodman, L.A. (1960). On the exact variance of product. Jour. Amer. Stat. Assoc., 55, 708-713.

18. Hartley, H.O. and Ross, A. (1954). Unbiased ratio estimates. Nature, 174, 270-271.

19. Hutchison, M.C. (1971). A Monte Carlo comparison of some ratio estimators. Biometrika, 58, 313-321.

20. Johnson, N.L. and Katz, S. (1970). Continuous Univariate Distributions - I. John Wiley and Sons, N.Y.

21. Konijn, H.S. (1979). Model free evaluation of the bias

and the mean square error of the regression estimator. Sankhyā, Ser. C, 41, 69-75.

22. Lahiri, D.B. (1951). A method for sample selection providing unbiased ratio estimates. Bull. Int. Stat. Inst., 33, 2, 133-140.

23. Mickey, M.R. (1959). Some finite population unbiased ratio and regression estimators. Jour. Amer. Stat. Assoc., 54, 594-612.

24. Midzuno, H. (1952). On a sampling system with probability proportional to sum of sizes. Ann. Inst. Stat. Math., 3, 99-107.

25. Murthy, M.N. (1964). Product method of estimation. Sankhyā, Ser. A, 26, 69-74.

26. ________ (1967). Sampling: Theory and Methods. Statistical Publication Society, Calcutta, India.

27. Murthy, M.N. and Nanjamma, N.S. (1959). Almost unbiased ratio estimates based on interpretating subsamples. Sankhyā, 21, 381-392.

28. Nanjamma, N.S., Murthy, M.N. and Sethi, V.K. (1959). Some sampling systems providing unbiased ratio estimators. Sankhyā, 21, 299-314.

29. Neter, J. and Wasserman, W. (1974). Applied Linear Statistical Models. Homewood: Richard D. Irwin, Inc.

30. Nieto De Pascual, J. (1961). Unbiased ratio estimators in stratified sampling. Jour. Amer. Stat. Assoc., 56, 70-87.

31. Quenouille, M.H. (1956). Notes on bias in estimation. Biometrika, 43, 353-360.

32. Raj, D. (1968). Sampling Theory, McGraw-Hill, Inc.

33. Rao, J.N.K. (1967). The precision of Mickey's unbiased ratio estimator. Biometrika, 54, 321-324.

34. Rao, J.N.K. and Webster, J.T. (1966). On two methods of bias reduction in the estimation of ratios. Biometrika, 53, 571-577.

35. Rao, P.S.R.S. (1968). On three procedures of sampling from finite populations. Biometrika, 55, 438-441.

36. Rao, P.S.R.S. and Rao, J.N.K. (1971). Small sample results for ratio estimators. Biometrika, 58, 625-630.

37. Rao, T.J. (1966 a). On the variance of the ratio estimator for Midzuno-Sen sampling scheme. Metrika, 10, 89-91.

38. _______ (1966 b). On certain unbiased ratio estimators. Ann. Inst. Stat. Math., 18, 117-121.

39. _______ (1972). On the variance of the ratio estimator. Metrika, 18, 209-215.

40. _______ (1977). Estimating the variance of the ratio estimator for the Midzuno-Sen sampling scheme. Metrika, 24, 203-208.

41. Ray, S.K. and Sahai, Ajit. (1980). An efficient estimator using auxiliary information. Metrika, 27, 271-275.

42. Ray, S.K. and Sahai, Ashok. (1980). Efficient families of ratio and product-type estimators. Biometrika, 67, 211-215.

43. Ray, S.K., Sahai, A. and Sahai, A. (1974). A note on ratio and product-type estimators. Ann. Inst. Stat. Math., 31, 141-144.

44. Reddy, V.N. (1973). On ratio and product methods of estimation. Sankhyā, Ser. B, 35, 307-316.

45. ______ (1974). On a transformed ratio method of estimation. Sankhya, Ser. C, 36, 59-60.

46. Robson, D.S. (1957). Applications of multivariate polykays to the theory of unbiased ratio-type estimation. Jour. Amer. Stat. Assoc., 52, 511-522.

47. Royall, R.M. (1970). On finite population sampling theory under certain linear regression models. Biometrika, 57, 377-387.

48. Royall, R.M. and Cumberland, W.G. (1981). The finite-population estimator and estimators of its variance - An empirical study. Jour. Amer. Stat. Assoc., 76, 924-930.

49. Royall, R.M. and Eberhardt, K.R. (1975). Variance estimates for the ratio estimator. Sankhyā, Ser. C, 37, 43-52.

50. Sahoo, L.N. and Swain, A.K.P.C. (1980). Unbiased ratio-cum product estimator. Sankhyā, Ser.C, 42, 56-62.

51. Särndal, C.E. (1984). Design consistent versus model dependent estimation for small domains. Jour. Amer. Stat. Assoc., 79, 624-631.

52. Sen, A.R. (1952). Present status of probability sampling

and its use in estimation of farm characteristics (abstract). Econometrica, 20, 103.

53. Sengupta, S. (1981). Jack-knifing the ratio and the product estimators in double sampling. Metrika, 28, 245-256.

54. Shah, S.M. and Shah, D.N. (1978). Ratio-cum-product estimators for estimating ratio (product) of two population parameters. Sankhyā, Ser. C, 40, 150-166.

55. ______ (1979). Unbiased product-type estimators. Gujarat Stat. Review, 6, 34-43.

56. Shukla, N.D. (1976). Almost unbiased product-type estimators. Metrika, 23, 127-133.

57. Singh, M.P. (1965). On the estimation of ratio and product of the population parameters. Sankhyā, Ser. B, 27, 321-328.

58. ______ (1967). Ratio-cum-product method of estimation. Metrika, 12, 34-42.

59. ______ (1969). Comparison of some ratio-cum-product estimators. Sankhyā, Ser. B, 31, 375-378.

60. Singh, P. and Srivastava, A.K. (1980). Sampling schemes providing unbiased regression estimators. Biometrika, 67, 205-209.

61. Singh, R. (1977). A note on optimum stratification for equal allocation with ratio and regression methods of estimation. Aust. Jour. Stat., 19, 96-104.

62. Singh, R. and Sukhatme, B.V. (1973). Optimum stratification with ratio and regression methods of estimation. Ann. Inst. Stat. Math., 25, 627-633.

63. Srivastava, S.K. (1966). Product estimator. Jour. Indian Stat. Assoc., 4, 29-37.

64. _______ (1967). An estimator using auxiliary information in sample survey. Calcutta Stat. Assoc. Bull., 16, 121-131.

65. Srivastava, V.K., Shukla, N.D. and Bhatnagar, S. (1981). Unbiased product estimators. Metrika, 28, 191-196.

66. Srivastava, V.K., Dwivedi, T.D., Chaubey, Y.P. and Bhatnagar, S. (1983). Finite sample properties of Beale's ratio estimator. Comm. Stat. Theory and Methods, 12 (15), 1795-1805.

67. Srivenkataramana, T. (1980). A dual to ratio estimator in sample surveys. Biometrika, 67, 199-204.

68. Srivenkataramana, T. and Tracy, D.S. (1979). On ratio and product methods of estimation in sampling. Statist. Neerlandica, 33, 37-49.

69. Sukhatme, P.V. and Sukhatme, B.V. (1970). Sampling Theory of Surveys with Applications. Second Edition. Iowa State University Press, Ames, Iowa.

70. Tamhane, A.C. (1978). Inference based on regression estimator in double sampling. Biometrika, 65, 419-427.

71. Tikkiwal, B.D. (1960). On the theory of classical regression and double sampling estimation. Jour. Royal

Stat. Society, Ser. B, 22, 131-138.

72. Tin, M. (1965). Comparison of some ratio estimators, Jour. Amer. Stat. Assoc., 60, 294-307.

73. Tweedie, M.C.K. (1957 a). Statistical properties of inverse Gaussian distributions, I, Ann. of Math. Stat., 28, 362-377.

74. _______ (1957 b). Ibid, II, Ann. of Math. Stat., 28, 696-705.

75. Watson, D.J. (1937). The estimation of leaf areas. Jour. Agri. Sci., 27, 474.

76. Williams, W.H. (1961). Generating unbiased ratio and regression estimators. Biometrics, 17, 267-274.

77. _______ (1963). The precision of some unbiased regression estimators. Biometrics, 19, 352-361.