

VOLTERRA INTEGRAL OPERATOR WITH RADIAL ACTION ON THE HILBERT
SPACE $A^2(D)$ OF ANALYTIC FUNCTIONS ON THE UNIT DISC



DOMENICO E. MANZO

A THESIS

in

THE DEPARTMENT

of

MATHEMATICS

Presented in Partial Fulfillment of the Requirements
for the degree of Master of Science at
Concordia University
Montreal, Quebec, Canada

March 1982

© DOMENICO E. MANZO, 1982

ABSTRACT

VOLTERRA INTEGRAL OPERATOR WITH RADIAL ACTION ON THE HILBERT
SPACE $A^2(D)$ OF ANALYTIC FUNCTIONS ON THE UNIT DISC

DOMENICO E. MANZO

This master's degree thesis examines the feasibility of extending recently established results on radially acting linear integral operators on $H^p(\Pi_+)$, where Π_+ is the upper half plane of C , to radially acting Volterra-type (linear as well as non-linear) Integral Operators on a Hilbert space of analytic functions on the open unit disc D of C , which has hitherto not been dealt with.

It is initially proved that for linear operators of this type, $A^2(D)$ is the suitable space. Properties of $A^2(D)$ are demonstrated which guarantee that the set $(VK_2)(D)$ of kernels $K(r, r', \theta)$ defining these linear operators on $A^2(D)$ constitutes a Banach-Algebra. Inherent properties of iterates of $(VK_2)(D)$ -kernels allow the construction of Fredholm-Resolvent kernels via Neumann Series and thus, $A^2(D)$ -Volterra Integral Equation of second kind may be solved and approximately solved.

For the related Hammerstein as well as general type non linear radial $A^2(D)$ -Volterra Integral Equations admissibility conditions are defined and the following results are derived: first, existence of an $A^2(D)$ -convergent iteration scheme; second, this iteration scheme converges to an $A^2(D)$ -solution; and third, these $A^2(D)$ are unique.

The three different types of radial $A^2(D)$ -Volterra Integral Equations are accompanied by concrete examples which are solved.

ACKNOWLEDGMENTS

"E buon per me se la vita intera mi
frutterà di meritare un sasso che
portò scritto non muto bandiera"

G. Giusti

The author takes this opportunity to thank his sister Mrs. Rina Agostinelli for typing this thesis. He wishes to express his admiration for her perseverance as well as to thank his brother-in-law Paolo for his understanding patience. Further, the author expresses his appreciation of his thesis director's (Attila B. Von Keviczky) collaboration, guidance and constructive suggestions which brought about the realization of this thesis.

TABLE OF CONTENTS

	Page
ABSTRACT	i
ACKNOWLEDGMENTS	ii
CHAPTER I. INTRODUCTION	1
1.1 Description of Space $A^2(D)$	1
1.2 Origin of Problem	3
1.3 Objectives of this Thesis	5
CHAPTER II. BASIC CONCEPTS	9
2.1 The Space $H^2(D)$	9
2.2 The Space $A^2(D)$	10
2.3 The Relation between $H^2(D)$ and $A^2(D)$	12
2.4 The Relation between $A^2(D)$ and $H^2(D)$	19
CHAPTER III. THE HILBERT SPACE $A^2(D)$	22
3.1 An Equivalent Hilbert Space Norm for $A^2(D)$	22
3.2 A Linear Hilbert Space Homeomorphism	23
3.3 Properties of $A^2(D)$ -Functions	24
CHAPTER IV. RADIALLY ACTING VOLTERRA OPERATORS	30
4.1 The θ -Parameter Family of L_2 -Kernels	31
4.2 Examples of $(VK_2)(D)$ -Kernels	32
4.3 $(VK_2)(D)$ as a Normed Algebra	35
4.4 Completeness of $(VK_2)(D)$	39
CHAPTER V. THE FREDHOLM-RESOLVENT OF $(VK_2)(D)$ -Kernels	45
5.1 Elementary Estimates for $(VK_2)(D)$ -Kernels	46

	Page
5.2 Estimates for Iterates of $(VK_2)(D)$ -Kernels	48
5.3 The Neumann Series of $(VK_2)(D)$ -Kernels	53
5.4 The Fredholm-Resolvent Kernel	56
5.5 Solutions of Radial $A^2(D)$ -Volterra Integral Equations of Second Kind	58
5.6 Example of Radial $A^2(D)$ -Volterra Integral Equation of Second Kind	62
 CHAPTER VI NON-LINEAR RADIAL VOLTERRA $A^2(D)$ -INTEGRAL EQUATIONS OF GENERAL TYPE	 65
6.1 $L_2[a,b]$ -Volterra Integral Equations of General Type	65
6.2 Radially Modified Lipschitz Condition for Non-Linear Radial $A^2(D)$ -Volterra Integral Equation of General Type	67
6.3 Convergence Scheme for Non-Linear Radial $A^2(D)$ -Volterra Integral Equations of General Type	68
6.4 Existence of Solution of Non-Linear Radial $A^2(D)$ -Volterra Integral Equation of General Type	73
6.5 Uniqueness of Solution of Non-Linear Radial $A^2(D)$ -Volterra Integral Equation of General Type	76
6.6 Example of Non-Linear Radial $A^2(D)$ -Volterra Integral Equation of General Type	80
 CHAPTER VII NON-LINEAR RADIAL VOLTERRA $A^2(D)$ -INTEGRAL EQUATION OF HAMMERSTEIN TYPE	 85
7.1 $L_2[a,b]$ -Volterra Integral Equation of Hammerstein Type	85
7.2 Radially Modified Lipschitz-Condition for Non-Linear Radial $A^2(D)$ -Volterra Integral Equation of Hammerstein Type ..	86
7.3 Convergence Scheme for Non-Linear Radial $A^2(D)$ -Volterra Integral Equation of Hammerstein Type	87
7.4 Existence and Uniqueness of Solution of Non-Linear $A^2(D)$ -Volterra Integral Equation of Hammerstein Type	91

	Page
7.5 A 1-Admissibility Condition	95
7.6 An Example of Non-Linear Radial $A^2(D)$ -Volterra Integral Equation of Hammerstein Type	97
CHAPTER VIII CONCLUSION	101
8.1 Summary of Results of this Thesis	101
8.2 Suggestions for further Research	103
BIBLIOGRAPHY	105

CHAPTER I
INTRODUCTION

1.1 DESCRIPTION OF SPACE $A^2(D)$.

On the open unit disc $D = \{z : |z| < 1\}$ of the set \mathbb{C} of complex numbers, we define the analytic function space $A^2(D)$. $A^2(D)$ is the totality of analytic functions f on D satisfying

$$(1.1) \quad ||f||_{A^2(D)} = \left(\iint_D |f(x+iy)|^2 dx dy \right)^{1/2} < \infty.$$

Each $A^2(D)$ function f possesses a Taylor-Series representation

$$(1.2) \quad f(z) = \sum_{n=0}^{\infty} a_n(f) z^n$$

$$\left(a_n(f) = (2\pi i)^{-1} \oint_{|\xi|=R} f(\xi) \xi^{-n-1} d\xi; \quad (n \geq 0, 0 < R < 1) \right),$$

for which, after changing the variables of integration (x,y) to polar coordinates $(x = r\cos\theta, y = r\sin\theta; 0 \leq r < 1, 0 \leq \theta \leq 2\pi)$

$$(1.3) \quad \iint_{|z| \leq r} |f(x+iy)|^2 dx dy = \int_0^r \int_0^{2\pi} |f(se^{i\theta})|^2 s ds d\theta =$$

$$(2\pi) \sum_{n=0}^{\infty} |a_n(f)|^2 (2n+2)^{-1} r^{2n+2} \leq (||f||_{A^2(D)})^2 \quad (0 \leq r < 1)$$

holds. Letting $r \rightarrow 1^-$ in (1.3) yields

$$(1.4) \quad \pi \sum_{n=0}^{\infty} (n+1)^{-1} |a_n(f)|^2 = (||f||_{A^2(D)})^2 \quad (f \in A^2(D)).$$

Consequently, the norm $\|\cdot\|_{A^2(D)}$ is induced by the inner product

$$(1.5) \quad \langle f, g \rangle_{A^2(D)} = \iint_D f(x+iy)\overline{g(x+iy)} dx dy = \pi \sum_{n=0}^{\infty} (n+1)^{-1} a_n(f) \overline{a_n(g)}$$

$$(f(z) = \sum_{n=0}^{\infty} a_n(f) z^n, g(z) = \sum_{n=0}^{\infty} a_n(g) z^n \in A^2(D)).$$

Thus, this inner product possesses an integral as well as a series representation.

By reason of the fact that the Hilbert space

$$(1.6) \quad \ell^2((n+1)^{-1}; N) = \left\{ (z_n)_{n=0}^{\infty} : z_n \in \mathbb{C} (n \geq 0), \sum_{n=0}^{\infty} (n+1)^{-1} |z_n|^2 < \infty \right\}$$

is endowed with the inner product

$$(1.7) \quad \langle z, w \rangle_{\ell^2((n+1)^{-1}; N)} = \pi \sum_{n=0}^{\infty} (n+1)^{-1} z_n \overline{w_n}$$

$$(z = (z_n)_{n=0}^{\infty}, w = (w_n)_{n=0}^{\infty}),$$

we may conclude from (1.5) and the power series representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$

$\in A^2(D)$ whenever $\sum_{n=0}^{\infty} (n+1)^{-1} |a_n|^2 < \infty$, that $A^2(D)$, equipped with the inner

product $\langle \cdot, \cdot \rangle_{A^2(D)}$, is a Hilbert space - i.e. $A^2(D)$ is complete with respect to

the norm $\|\cdot\|_{A^2(D)}$.

1.2. ORIGIN OF PROBLEM.

This thesis deals primarily with linear radially acting $A^2(D)$ - Volterra Integral Operators $K : A^2(D) \rightarrow A^2(D)$, with

$$(1.8) \quad (Kf)(re^{i\theta}) = \int_0^r K(r, r', \theta) f(r'e^{i\theta}) e^{i\theta} dr'$$

a.e. in r on $[0, 1]$ ($0 < \theta < 2\pi$) induced by a θ -parameter family of Lebesgue measurable kernels $K(r, r', \theta)$ (in the variables (r, r')) on $[0, 1] \times [0, 1]$ ($0 < \theta < 2\pi$) with uniformly bounded "double-norms" [10, p. 177] - i.e.

$$(1.9) \quad \|K\|_{S(2)} = \sup_{0 \leq \theta < 2\pi} \left(\int_0^1 \int_0^r |K(r, r', \theta)|^2 dr' dr \right)^{1/2} < \infty.$$

Radially acting linear integral operators on the Banach spaces $H^p(\Pi_+)$ of analytic functions f on the upper half plane Π_+ of C , satisfying

$$(1.10) \quad \|f\|_{H^p(\Pi_+)} = \sup_{y > 0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx \right)^{1/p} < \infty,$$

have been studied by Classine Van Winter [9] in the Hilbert space $H^2(\Pi_+)$ and in the Banach space $H^p(\Pi_+)$ ($1 < p < \infty$), which space is not a Hilbert space for $p \neq 2$, by the author's thesis director Attila B. Von Keviczky [3]. The latter utilizes the fact that $H^p(\Pi_+)$ ($1 < p < \infty$) has an equivalent "radial norm"

$$(1.11) \quad \|f\|_{SL_p(0, \infty)} = \sup_{0 < \theta < \pi} \left(\int_0^{\infty} |f(re^{i\theta})|^p dr \right)^{1/p} (f \in H^p(\Pi_+))$$

with norm-equivalence given by

$$(1.12) \quad 2^{-1} \|f\|_{H^p(\Pi_+)} \leq \|f\|_{SL_p(0,\infty)} \leq \sec(2^{-1}\pi p^{-1}) \|f\|_{H^p(\Pi_+)}$$

[4]. This norm-equivalence allows one to investigate radially acting integral operators on the Banach space $H^p(\Pi_+)$.

In the Hilbert space $H^2(\Pi_+)$, Classine Van Winter [9] dealt with linear radially acting integral operators $K : H^2(\Pi_+) \rightarrow H^2(\Pi_+)$ with

$$(1.13) \quad (Kf)(r e^{i\theta}) = \int_0^\infty K(r, r', \theta) f(r' e^{i\theta}) e^{i\theta} dr,$$

a.e. on $(0, \infty)$ ($0 < \theta < \pi$), where

$$(1.14) \quad K(r, r', \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(s, t) (r e^{i\theta})^{-\frac{1}{2} + is} (r' e^{i\theta})^{-\frac{1}{2} - it} dt ds$$

$$\left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K(s, t)|^2 (1 + e^{2\pi(s-t)}) ds dt < \infty \right).$$

Her kernel representation (1.14) as well as her results were derived from a Melin-Transform version of the Paley-Wiener Theorem for $H^2(\Pi_+)$ -functions. In the second situation, namely that of $H^p(\Pi_+)$, the author's thesis director considered radially acting linear integral operators $K : H^p(\Pi_+) \rightarrow H^p(\Pi_+)$ ($1 < p < \infty$), whose action is defined by (1.13), but with kernels $K(r, r', \theta)$ possessing uniformly bounded "double-norms" - i.e.

$$(1.15) \quad \frac{\|K\|}{S(p,p')} = \sup_{0 < \theta < \pi} \left\{ \int_0^\infty \left[\int_0^\infty |K(r,r',\theta)|^{p'} dr' \right]^{p/p'} dr \right\}^{1/p'} < \infty,$$

where $p^{-1} + p'^{-1} = 1$.

Both of these cases exhibit the common advantage that the upper-half plane Π_+ admits the underlying interpretation of being a θ -parameter family of rays $R_\theta = \{re^{i\theta} : r > 0\} (0 < \theta < \pi)$ possessing the following property: $R_\theta \rightarrow R_0$ and R_π as $\theta \rightarrow 0^+$ and π^- respectively and $\partial\Pi_+ = R_\pi \cup \{0\} \cup R_0$. This geometric property guarantees the existence of boundary value kernels $K(r,r',0)$ and $K(r,r',\pi)$, which describe $K : H^p(\Pi_+) \rightarrow H^p(\Pi_+)$ by means of the closed $L_p(R)$ -subspace $H^p(\Pi_+)_{(+)}$, and the operators $K_0, K_\pi : L_p(0, \infty) \rightarrow L_p(0, \infty)$ with

$$(1.16) \quad (K_\psi f)(\cdot) = \int_0^\infty K(\cdot, r', \psi) f(r') e^{i\psi} dr' \quad (\psi = 0, \pi)$$

a.e. on $(0, \infty)$. $H^p(\Pi_+)_{(+)}$ denotes the totality of $L_p(R)$ -"boundary value functions" of elements of $H^p(\Pi_+)$. Thus the development of trace-class kernels for $H^p(\Pi_+)$ is achieved [3] ($1 < p < \infty$); in particular, the construction of Hilbert-Schmidt Operators on the Hilbert space $H^2(\Pi_+)$.

1.3 OBJECTIVES OF THIS THESIS.

In the case of a radially acting linear integral operator $K : A^2(D) \rightarrow A^2(D)$ with action (1.8), we are dealing with a space of analytic functions of the open unit disc D . This disc is a θ -parameter ($0 \leq \theta < 2\pi$) of rays emanating from the origin; however, these rays do not approach ∂D in any angular manner as was

the case for \mathbb{M}_+ . Moreover, D is bounded, whereas \mathbb{M}_+ was not. Therefore, any investigation of radially acting integral operators on $A^2(D)$ cannot be accomplished through radially acting boundary integral operators (induced by boundary kernels), simply because these do not exist.

The primary objective of this master's degree thesis is to examine the quasi-nilpotency [2,p.49] of the linear radially acting $A^2(D)$ -"Volterra Integral Operator" $K : A^2(D) \rightarrow A^2(D)$ defined by (1.8). This requires answering the following question affirmatively. Is $\lim_{n \rightarrow \infty} (\|K^n\|_{S(2)})^{1/n} = 0$? Or equivalently, does the Neumann-Series

$$(1.17) \quad \sum_{n=0}^{\infty} \lambda^n K^{n+1}(r, r', \theta)$$

$$(K^{n+1}(r, r', \theta) = \int_{r'}^r K^n(r, r'', \theta) K(r'', r', \theta) e^{i\theta} dr'' \text{ if } r' < r \text{ and } 0 \text{ if } r' \geq r \\ (0 \leq r, r' \leq 1; n \geq 1))$$

of the kernel $K(r, r', \theta)$ represent the Fredholm-Resolvent Kernel $H_\lambda(K)(r, r', \theta)$ of the $A^2(D)$ -"Volterra Integral Operator" determined by $K(r, r', \theta)$ for all $\lambda \in \mathbb{C}$?

The Fredholm-Resolvent Kernel $H_\lambda(K)(r, r', \theta)$ must satisfy the Fredholm-Resolvent Equation [6,p.17]:

$$(1.18) \quad \lambda \int_{r'}^r H_\lambda(K)(r, r'', \theta) K(r'', r', \theta) e^{i\theta} dr'' =$$

$$\lambda \int_{r'}^r K(r, r'', \theta) H_\lambda(K)(r'', r', \theta) e^{i\theta} dr'' =$$

$$H_\lambda(K)(r, r', \theta) = K(r, r', \theta) \quad (r' < r; 0 \leq r, r' < 1)$$

a.e. in (r, r') on $[0, 1] \times [0, 1]$ ($0 \leq \theta < 2\pi$) for all $\lambda \in C$ provided $H_\lambda(K)(r, r', \theta)$ exists. If we assume that the answer to the question preceding (1.18) is affirmative, then the unknown $A^2(D)$ -function f in the linear radial $A^2(D)$ - "Volterra Integral Equation"

$$(1.19) \quad f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r K(r, r', \theta) f(r'e^{i\theta}) e^{i\theta} dr'$$

a.e. in r on $[0, 1]$ ($0 \leq \theta < 2\pi$), has the unique solution

$$(1.20) \quad f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r H_\lambda(K)(r, r', \theta) g(r'e^{i\theta}) e^{i\theta} dr' = \\ g(re^{i\theta}) + \sum_{n=0}^{\infty} \lambda^{n+1} \int_0^r K^{n+1}(r, r', \theta) f(r'e^{i\theta}) e^{i\theta} dr'$$

a.e. in r on $[0, 1]$ ($0 \leq \theta < 2\pi$) for all $\lambda \in C$. To achieve our objective, we shall examine the following topics:

- 1)... Topological aspects of $A^2(D)$;
- 2)... Algebraic properties of the kernels $K(r, r', \theta)$ - i.e. the totality of these kernels constitutes a Banach-Algebra; and
- 3)... Estimates on $|K^{n+1}(r, r', \theta)|$ guaranteeing $\limsup_{n \rightarrow \infty} (|||K^n|||_{S(2)})^{1/n} = 0$.

Since non-linear Hammerstein Integral Equations arise out of L_2 -kernels, we shall also examine non-linear radial $A^2(D)$ -"Volterra Integral

"Equations" of the Hammerstein-type

$$(1.21) \quad f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r K(r, r', \theta) F(r'e^{i\theta}; f(r'e^{i\theta})) e^{i\theta} dr' \quad (g \in A^2(D))$$

a.e. in r on $[0, 1]$ ($0 \leq \theta < 2\pi$) as well as the general type

$$(1.22) \quad f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r F(re^{i\theta}, r'e^{i\theta}; f(r'e^{i\theta})) e^{i\theta} dr' \quad (g \in A^2(D))$$

for all $re^{i\theta} \in D$, where $f \in A^2(D)$ is sought after and the kernel $K(r, r', \theta)$ satisfies (1.15) and defines $K: A^2(D) \rightarrow A^2(D)$ ((1.13)).

CHAPTER II
BASIC CONCEPTS

The Hilbert space $A^2(D)$ is related to the following two Banach spaces $H^2(D)$ and $L^2(D)$; $H^2(D)$ is a Hilbert space whereas $L^2(D)$ is not.

2.1 THE SPACE $H^2(D)$.

Definition 2.1. $H^2(D)$ is the Hilbert space of analytic functions f on the unit disc D with

$$(2.1) \quad \|f\|_{H^2(D)} = \sup_{0 \leq r < 1} \left\{ (2\pi)^{-1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right\}^{1/2} < \infty.$$

In fact, for each $f \in H^2(D)$, the expression $\left\{ (2\pi)^{-1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right\}^{1/2}$

is an increasing function of the variable r on $[0,1]$ [5, Th. 17.6, p.360].

Further, every $H^2(D)$ -function f possesses an a.e. unique $L_2[0,2\pi]$ -boundary-

value-function $f(e^{i\theta})$ - i.e. $(2\pi)^{-1} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta < \infty$ such that

$$(2.2) \quad \lim_{r \rightarrow 1^-} \left\{ (2\pi)^{-1} \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^2 d\theta \right\}^{1/2} = 0;$$

in particular,

$$(2.3) \quad \|f\|_{H^2(D)} = \left\{ (2\pi)^{-1} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta \right\}^{1/2} \quad \{f \in H^2(D)\}.$$

Hence, $\| \cdot \|_{H^2(D)}$ satisfies the parallelogram law, and consequently, $H^2(D)$

is a Hilbert space [8, Satz 1.6, p.17] with inner product

$$(2.4) \quad \langle f, g \rangle_{H^2(D)} = (2\pi)^{-1} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta,$$

where $f(e^{i\theta})$ and $g(e^{i\theta})$ are the a.e. unique boundary value functions of the $H^2(D)$ -functions f and g respectively.

On the other hand, it would seem appropriate to consider a norm defined by the supremum of $L_2[0,1]$ -norms taken along the segments $\{re^{i\theta} : 0 < r < 1\}$ (of the ray $R_\theta = \{re^{i\theta} : r > 0\}$, $0 \leq \theta < 2\pi$) for examining linear radially acting Volterra Integral Operators on a Hilbert space of analytic functions of the unit disc D , analogous to what was done by C. Van Winter [9] and the author's thesis director [3]. We consider for this purpose the following Banach space of analytic functions.

2.2 THE SPACE $A^2(D)$.

Definition 2.2. $A^2(D)$ is the Banach space of analytic functions f on the unit disc D satisfying

$$(2.5) \quad \|f\|_{SL_2[0,1]} = \sup_{0 \leq \theta < 2\pi} \left(\int_0^1 |f(re^{i\theta})|^2 dr \right)^{1/2} < \infty.$$

We remark, $A^2(D)$ is no longer a Hilbert space, because the parallelogram

law is violated. This is demonstrated by the following counter-example. Let $f(z) = z$ and $g(z) = e^{i\psi}$ (where ψ is a fixed angle, $0 \leq \psi < 2\pi$). Obviously, f and $g \in A^2(D)$ and

$$(2.6) \quad \|f\|_{SL_2[0,1]} = \sup_{0 \leq \theta < 2\pi} \left(\int_0^1 r^2 dr \right)^{1/2} = (3)^{-\frac{1}{2}},$$

$$\|g\|_{SL_2[0,1]} = \sup_{0 \leq \theta < 2\pi} \left(\int_0^1 dr \right)^{1/2} = 1.$$

Prior to calculating the $A^2(D)$ -norms of $f+g$ and $f-g$, we note:

$$(f+g)(re^{i\theta}) = (r\cos\theta + \cos\psi) + i(r\sin\theta + \sin\psi) \text{ and } (f-g)(re^{i\theta}) = (r\cos\theta - \cos\psi) + i(r\sin\theta - \sin\psi). \text{ Thus,}$$

$$(2.7) \quad \|f+g\|_{SL_2[0,1]} = \sup_{0 \leq \theta < 2\pi} \left(\int_0^1 [r^2 + 1 + 2r\cos(\theta - \psi)] dr \right)^{1/2} = \\ \sup_{0 \leq \theta < 2\pi} (3^{-1} + 1 + \cos(\theta - \psi))^{1/2} = (7)^{\frac{1}{2}}(3)^{-\frac{1}{2}}$$

and

$$(2.8) \quad \|f-g\|_{SL_2[0,1]} = \sup_{0 \leq \theta < 2\pi} \left(\int_0^1 [r^2 + 1 - 2r\cos(\theta - \psi)] dr \right)^{1/2} = \\ \sup_{0 \leq \theta < 2\pi} (3^{-1} + 1 - \cos(\theta - \psi))^{1/2} = (7)^{\frac{1}{2}}(3)^{-\frac{1}{2}}.$$

The norm $\|\cdot\|_{SL_2[0,1]}$ cannot be induced by an inner product, because

$$(2.9) \quad \left(\|f+g\|_{SL_2[0,1]} \right)^2 + \left(\|f-g\|_{SL_2[0,1]} \right)^2 =$$

$$(7)(3)^{-1} + (7)(3)^{-1} = (14)(3)^{-1},$$

$$(2.10) \quad 2 \left(\|f\|_{SL_2[0,1]} \right)^2 + 2 \left(\|g\|_{SL_2[0,1]} \right)^2 =$$

$$(2)(3)^{-1} + (2)(1)^{-1} = (8)(3)^{-1}.$$

Nevertheless, the norm $\|\cdot\|_{SL_2[0,1]}$ is determined by the norm of the Hilbert space $L_2[0,1]$.

As purely algebraic objects (topological considerations disregarded), the spaces $H^2(D)$, $A^2(D)$ and $A^2(D)$ are interrelated by ascending strict inclusion and appropriate descending norm estimates. We arrive at this conclusion via the subsequent lemmas. For the sake of notational convenience, we introduce the following convention : $f(e^{i\theta})$ always denotes the a.e. unique boundary value function of the analytic function f of the unit disc D , whenever such a boundary value function exists.

2.3 THE RELATION BETWEEN $H^2(D)$ AND $A^2(D)$.

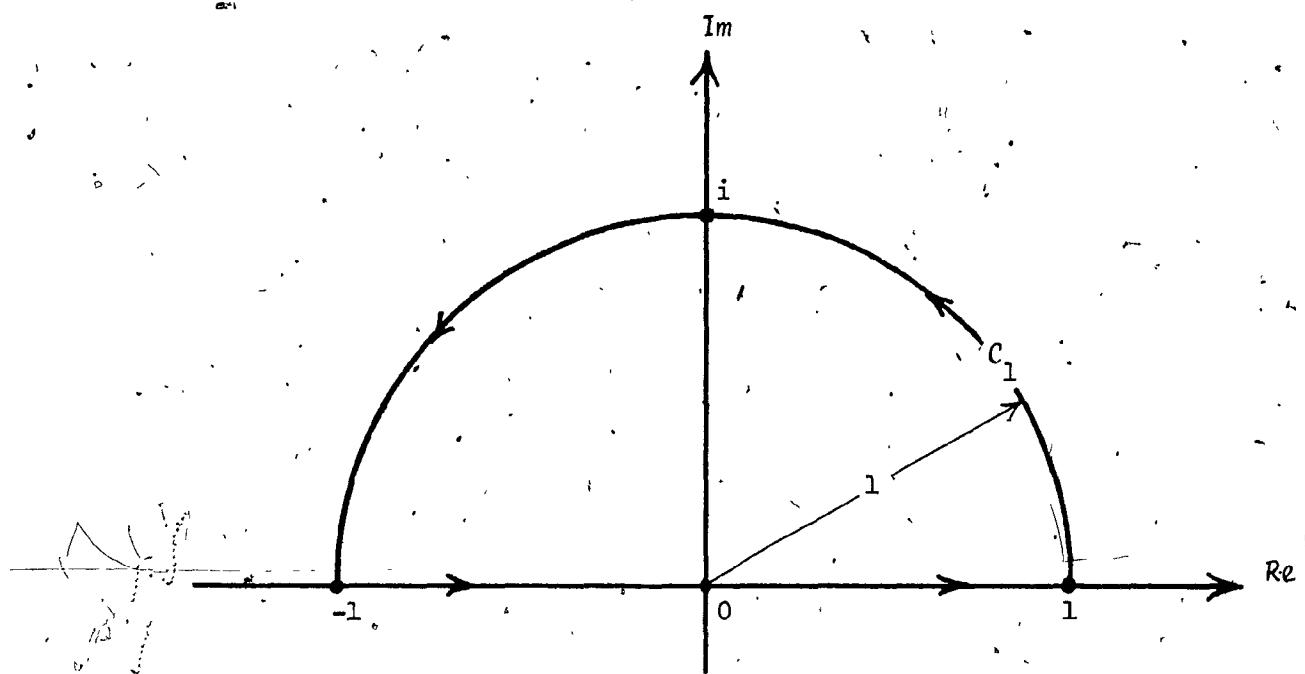
Lemma 2.3. If $f \in H^2(D)$ and $(\operatorname{Im} f)(x) = 0$ for all x ($-1 < x < 1$), then

$$(2.11) \quad \int_{-1}^1 |f(x)|^2 dx \leq 2^{-1} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta.$$

Proof. $f \in H^2(D)$ and $(\operatorname{Im} f)'(x) = 0$ ($-1 < x < 1$) imply

$$(2.12) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \left(\sum_{n=0}^{\infty} |a_n|^2 < \infty ; a_n \in \mathbb{R} (n \geq 0) \right)$$

Integrating over the contour



yields via the Cauchy-Integral Theorem [1, p.115]

$$(2.13) \quad \int_{-1}^1 (f(x))^2 dx + i \int_0^\pi (f(e^{i\theta}))^2 e^{i\theta} d\theta = 0$$

However, the mapping property $f: (-1, 1) \rightarrow \mathbb{R}$ implies $(f(x))^2 = |f(x)|^2$ and hence,

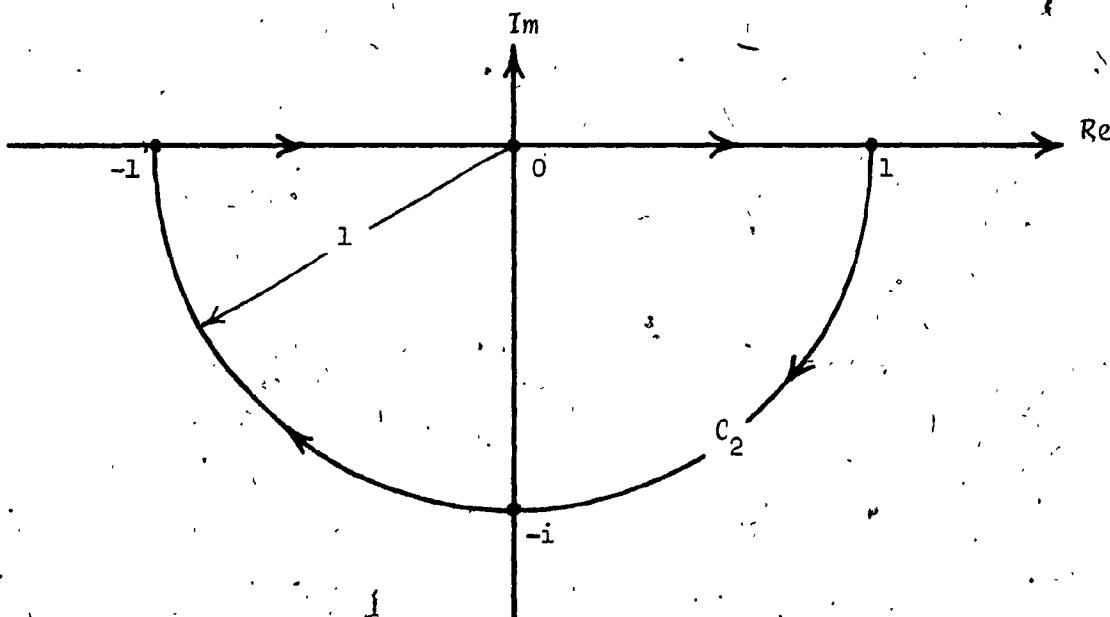
$$(2.14) \quad \int_{-1}^1 |f(x)|^2 = -i \int_0^\pi (f(e^{i\theta}) e^{i\theta/2})^2 d\theta =$$

$$\begin{aligned}
 -i \int_0^{\pi} \left\{ \operatorname{Re}((f(e^{i\theta})e^{i\theta/2})^2) + i \operatorname{Im}((f(e^{i\theta})e^{i\theta/2})^2) \right\} d\theta = \\
 \int_0^{\pi} \operatorname{Im}((f(e^{i\theta})e^{i\theta/2})^2) d\theta - i \int_0^{\pi} \operatorname{Re}((f(e^{i\theta})e^{i\theta/2})^2) d\theta = \\
 \int_0^{\pi} \operatorname{Im}((f(e^{i\theta})e^{i\theta/2})^2) d\theta \leq \int_0^{\pi} |f(e^{i\theta})e^{i\theta/2}|^2 d\theta = \int_0^{\pi} |f(e^{i\theta})|^2 d\theta.
 \end{aligned}$$

In (2.14) we utilized that $\operatorname{Im}((f(e^{i\theta})e^{i\theta/2})^2)$ and $\operatorname{Re}((f(e^{i\theta})e^{i\theta/2})^2)$ are real valued functions, whose respective integrals (on the interval $[0, \pi]$) are also real. In consequence thereof,

$$(2.15) \int_{-1}^1 |f(x)|^2 dx \leq \int_0^{\pi} |f(e^{i\theta})|^2 d\theta.$$

By means of the contour



we obtain

$$(2.16) \int_{-1}^1 |f(x)|^2 dx = i \int_{-\pi}^{2\pi} (f(e^{i\theta}) e^{i\theta/2})^2 d\theta$$

and by resorting to the immediately preceding arguments, modified for the contour C_2 , we arrive at

$$(2.17) \int_{-1}^1 |f(x)|^2 dx \leq \int_{-\pi}^{2\pi} |f(e^{i\theta})|^2 d\theta.$$

Adding inequalities (2.15) and (2.17) completes the proof of this theorem. Q.E.D.

Lemma 2.4. If $f \in H^2(D)$, then

$$(2.18) \int_{-1}^1 |f(xe^{i\phi})|^2 dx \leq 2^{-1} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta \quad (0 \leq \phi < 2\pi).$$

Proof. Since f is an $H^2(D)$ -function, we may write

$$(2.19) f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \left(\sum_{n=0}^{\infty} |a_n|^2 < \infty \right),$$

where $a_n = \alpha_n + i\beta_n$ ($\alpha_n, \beta_n \in \mathbb{R}; n \geq 0$). On account of $(e^{i\phi})^n =$

$\cos(n\phi) + i\sin(n\phi)$ and

$$(2.20) (\alpha_n \cos(n\phi) - \beta_n \sin(n\phi))^2, (\alpha_n \sin(n\phi) + \beta_n \cos(n\phi))^2 \leq (\alpha_n^2 + \beta_n^2)^2 =$$

$$|a_n|^2 \quad (n \geq 0),$$

We know that the $H^2(D)$ -functions

$$(2.21) \quad g_\phi(z) = \sum_{n=0}^{\infty} (\alpha_n \cos(n\phi) - \beta_n \sin(n\phi)) z^n,$$

$$h_\phi(z) = \sum_{n=0}^{\infty} (\alpha_n \sin(n\phi) + \beta_n \cos(n\phi)) z^n$$

have real values provided z is restricted to the interval $(-1,1)$ of \mathbb{R} and

$$(2.22) \quad f(e^{i\phi} z) = g_\phi(z) + i h_\phi(z) \quad (z \in D).$$

Applying Lemma 2.4 to the $H^2(D)$ -functions g_ϕ and h_ϕ separately justifies

$$(2.23) \quad \int_{-1}^1 (|g_\phi(x)|^2 + |h_\phi(x)|^2) dx \leq 2^{-1} \int_0^{2\pi} (|g_\phi(e^{i\theta})|^2 + |h_\phi(e^{i\theta})|^2) d\theta.$$

We note $f(e^{i\phi} x) = g_\phi(x) + i h_\phi(x)$ ($-1 < x < 1$), where $g_\phi(x), h_\phi(x) \in \mathbb{R}$
 $(-1 < x < 1)$,

$$(2.24) \quad g_\phi(e^{i\theta}) = \sum_{n=0}^{\infty} (\alpha_n \cos(n\phi) - \beta_n \sin(n\phi)) e^{in\theta},$$

$$h_\phi(e^{i\theta}) = \sum_{n=0}^{\infty} (\alpha_n \sin(n\phi) + \beta_n \cos(n\phi)) e^{in\theta}$$

a.e. $in\theta \in [0, 2\pi]$ and

$$(2.25) \int_0^{2\pi} |g_\phi(e^{i\theta})|^2 d\theta = 2\pi \sum_{n=0}^{\infty} (\alpha_n \cos(n\phi) - \beta_n \sin(n\phi))^2,$$

$$\int_0^{2\pi} |h_\phi(e^{i\theta})|^2 d\theta = 2\pi \sum_{n=0}^{\infty} (\alpha_n \sin(n\phi) + \beta_n \cos(n\phi))^2.$$

Relations (2.24) and (2.25) together yield

$$(2.26) \int_{-1}^1 (|g_\phi(x)|^2 + |h_\phi(x)|^2) dx = \int_{-1}^1 |f(xe^{i\theta})|^2 dx$$

and

$$(2.27) \int_0^{2\pi} (|g_\phi(e^{i\theta})|^2 + |h_\phi(e^{i\theta})|^2) d\theta = 2\pi \sum_{n=0}^{\infty} ((\alpha_n)^2 + (\beta_n)^2) = \\ 2\pi \sum_{n=0}^{\infty} |a_n|^2 = \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta.$$

Combining (2.27) with (2.23) confirms (2.18). Q.E.D.

Theorem 2.5. $H^2(D) \subset A^2(D)$ and

$$(2.28) \|f\|_{SL_2^1[0,1]} \leq 2^{-1/2} \|f\|_{H^2(D)} \quad (f \in H^2(D)).$$

Proof. Since $\int_0^1 |f(re^{i\theta})|^2 dr < \int_{-1}^1 |f(xe^{i\theta})|^2 d\theta$, Lemma 2.5 implies

(2.28). Q.E.D.

$H^2(D)$ being a proper submanifold of $A^2(D)$ is demonstrated by the analytic function f :

$$(2.29) \quad f_0(z) = \sum_{n=0}^{\infty} z^{2^n} \quad (z \in D).$$

$f_0(z) \in A^2(D)$, because

$$(2.30) \quad \int_0^1 |f_0(re^{i\theta})|^2 dr = \sum_{n,m=0}^{\infty} (2^n + 2^m + 1)^{-1} \exp(i(2^n - 2^m)\theta) \quad (0 \leq \theta < 2\pi)$$

and

$$(2.31) \quad \sup_{0 \leq \theta < 2\pi} \left(\int_0^1 |f_0(re^{i\theta})|^2 dr \right)^{1/2} = \left(\sum_{n,m=0}^{\infty} (2^n + 2^m + 1)^{-1} \right)^{1/2} \leq \\ \sum_{n,m=0}^{\infty} (2^n + 2^m)^{-1} \leq \sum_{n,m=0}^{\infty} 2^{-1} 2^{-n/2} 2^{-m/2} = \\ 2^{-1} \left(\sum_{n=0}^{\infty} 2^{-n/2} \right) \left(\sum_{m=0}^{\infty} 2^{-m/2} \right) = 2^{-1} (1 - 2^{-1/2})^{-2} = (2^{1/2} - 1)^{-2},$$

where the inequality $(a^2 + b^2)^{-1} \leq \frac{1}{2}a^{-1}b^{-1}$ (derived from $2ab \leq a^2 + b^2$) justifies the third step. On the other hand, $f_0 \notin H^2(D)$, because

$$(2.32) \quad (2\pi)^{-1} \int_0^{2\pi} |f_0(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} r^{2^n}$$

implies $\lim_{r \rightarrow 1^-} \left((2\pi)^{-1} \int_0^{2\pi} |f_0(re^{i\theta})|^2 d\theta \right)^{1/2} = \infty$. Therefore, $f_0 \in A^2(D) \sim H^2(D)$ -i.e.
 $H^2(D) \subsetneq A^2(D)$. Q.E.D.

This completes half of the strictly ascending inclusion relationship for $H^2(D)$, $A^2(D)$ and $A^2(D)$ with corresponding decreasing norm inequalities. The

next theorem deals with $A^2(D)$ and $A^2(\bar{D})$.

2.4 THE RELATION BETWEEN $A^2(D)$ AND $A^2(\bar{D})$.

Theorem 2.6. $A^2(D) \subset A^2(\bar{D})$ and

$$(2.33) \quad \|f\|_{A^2(D)} \leq (2\pi)^{1/2} \|f\|_{SL_2[0,1]} \quad (f \in A^2(D)).$$

Proof. We consider an arbitrary $A^2(D)$ -function f , change the variables

of integration in $\iint_D |f(x+iy)|^2 dx dy$ to polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$)

and derive via the Fubini-Tonelli Theorem

$$(2.34) \quad \left(\|f\|_{A^2(D)} \right)^2 = \int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^2 r dr \right) d\theta \leq \\ \int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^2 dr \right) d\theta \leq 2\pi (\|f\|_{SL_2[0,1]})^2.$$

This verifies (2.33) and $A^2(D) \subset A^2(\bar{D})$.

Prior to showing $A^2(\bar{D}) \not\subset A^2(D)$, we remark that if

$$(2.35) \quad g(z) = \sum_{n=0}^{\infty} a_n z^n \quad (a_n > 0 \quad (n \geq 0)),$$

then

$$(2.36) \quad \sup_{0 \leq \theta < 2\pi} \int_0^1 |g(re^{i\theta})|^2 dr = \sup_{0 \leq \theta < 2\pi} \left| \sum_{n,m=0}^{\infty} a_n a_m (n+m+1)^{-1} e^{i(n-m)\theta} \right| = \\ \sum_{n,m=0}^{\infty} a_n a_m (n+m+1)^{-1}.$$

For a μ ($2^{-1} < \mu < 1$) the analytic function of the unit disc

$$(2.37) \quad g_0(z) = \sum_{n=0}^{\infty} (\ln(n+2))^{-\mu} z^n \quad (a_n = (\ln(n+2))^{-\mu} \quad (n \geq 0))$$

has the following two properties. First, $g_0 \in A^2(D)$, because

$$(2.38) \quad \left(\|g_0\|_{A^2(D)} \right)^2 = \int_0^1 \left(\int_0^{2\pi} |g_0(re^{i\theta})|^2 d\theta \right) r dr = \\ 2^{-1} \sum_{n=0}^{\infty} (\ln(n+2))^{-2\mu} (n+1)^{-1} \leq \sum_{n=0}^{\infty} (\ln(n+2))^{-2\mu} (n+2)^{-1} \leq \\ \int_0^{\infty} (\ln(x+2))^{-2\mu} (x+2)^{-1} dx = (2\mu-1)^{-1} (\ln(2))^{1-2\mu} < \infty.$$

Second, $g_0 \notin A^2(D)$, because (2.36) ($g = g_0$, $a_n = a_n(g_0) = (\ln(n+2))^{-\mu} > 0$ for all $n \geq 0$) yields

$$(2.39) \quad \left(\|g_0\|_{SL_2[0,1]} \right)^2 = \sum_{n,m=0}^{\infty} (\ln(n+2))^{-\mu} (n+m+1)^{-1} (\ln(m+1))^{-\mu} \geq \\ \sum_{n,m=1}^{\infty} (\ln(n+2))^{-\mu} (n+m+1)^{-1} (\ln(m+1))^{-\mu} \geq \\ \sum_{n=1}^{\infty} (\ln(n+2))^{-\mu} \left\{ \sum_{m=1}^{\infty} (n+m+1)^{-1} (\ln(n+m+1))^{-\mu} \right\} = \infty,$$

since the sum in the curly brackets is bounded below by

$$\int_2^\infty (n+x+1)^{-1} (\ln(n+x+1))^{-\mu} dx = \infty. \text{ Therefore, } g_0 \in A^2(D) \sim A^2(D). \text{ Q.E.D.}$$

We combine Theorems 2.5 and 2.6 and summarize:

$$(2.40) \quad (2\pi)^{-1/2} \|f\|_{A^2(D)} \leq \|f\|_{SL_2[0,1]} \leq 2^{-1/2} \|f\|_{H^2(D)} \quad (f \in H^2(D)).$$

Moreover, since $H^2(D)$ is dense in $A^2(D)$ (polynomial-functions belong to $H^2(D)$, $A^2(D)$ and $A^2(D)$) and $A^2(D)$ is not a Hilbert space, neither of the spaces $H^2(D)$ and $A^2(D)$ is suitable for investigating linear radially acting Volterra Integral Operators on a Hilbert space of analytic functions on D . $A^2(D)$ is for this reason the appropriate Hilbert space for linear radially acting Volterra Integral Operators ((1.8)).

CHAPTER III

THE HILBERT SPACE $A^2(D)$

The norm determined by the inner product $\langle \cdot, \cdot \rangle_{A^2(D)}$ ((1.5)) is impaired by the presence of r in formula (2.34). Consequently, the norm $\| \cdot \|_{A^2(D)}$ is awkward for investigating linear radially acting Volterra Integral Operators on the Hilbert space $A^2(D)$.

3.1 AN EQUIVALENT HILBERT SPACE NORM FOR $A^2(D)$.

We circumvent this obstacle by introducing the more effective alternative norm

$$(3.1) \quad \|f\| = \left(\int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta dr \right)^{1/2} = \left(2\pi \sum_{n=0}^{\infty} (2n+1)^{-1} |a_n(f)|^2 \right)^{1/2}$$
$$(f \in A^2(D); f(z) = \sum_{n=0}^{\infty} a_n(f) z^n \quad (z \in D)).$$

Theorem 3.1. The topology on $A^2(D)$ determined by $\| \cdot \|$ is equivalent to the topology on $A^2(D)$ determined by $\| \cdot \|_{A^2(D)}$. Further, this equivalence is given by

$$(3.2) \quad \|f\|_{A^2(D)} \leq \|f\| \leq 2^{1/2} \|f\|_{A^2(D)} \quad (f \in A^2(D)).$$

Proof. (3.2) follows directly from

$$(3.3) \quad 2\pi \sum_{n=0}^{\infty} (2n+2)^{-1} |a_n(f)|^2 \leq 2\pi \sum_{n=0}^{\infty} (2n+1)^{-1} |a_n(f)|^2$$

$$2(2\pi) \sum_{n=0}^{\infty} (2n+2)^{-1} |a_n(f)|^2 \quad (f \in A^2(D)).$$

The terms in (3.3) are series expressions of the squares of the respective norms appearing in (3.2); hence, (3.3), follows.

(3.3) guarantees that the open sphere $\{g \in A^2(D): ||g-f|| < \rho\}$ (of the metric topology generated by $||\cdot||$) is contained in the open sphere $\{g \in A^2(D): ||f-g||_{A^2(D)} < \rho\}$ (of the metric topology generated by $||\cdot||_{A^2(D)}$) for all $f \in A^2(D)$ and $\rho > 0$. Correspondingly, the open sphere $\{g \in A^2(D): ||g-f||_{A^2(D)} < \rho\}$ (of the metric topology generated by $||\cdot||_{A^2(D)}$) is contained in the open sphere $\{g \in A^2(D): ||g-f|| < 2^{1/2}\rho\}$ (of the metric topology generated by $||\cdot||$) for all $f \in A^2(D)$ and $\rho > 0$. Hence, these two topologies on $A^2(D)$ are equivalent. Q.E.D.

3.2 A LINEAR HILBERT SPACE HOMEOMORPHISM.

Norm $||\cdot||$ is generated by the inner product

$$(3.4) \quad (f, g) = \int_0^{2\pi} \int_0^1 f(re^{i\theta}) \overline{g(re^{i\theta})} dr d\theta = 2\pi \sum_{n=0}^{\infty} (2n+1)^{-1} a_n(f) \overline{a_n(g)}$$

$$(f(z) = \sum_{n=0}^{\infty} a_n(f) z^n, g(z) = \sum_{n=0}^{\infty} a_n(g) z^n),$$

and $A^2(D)$ endowed with this inner product ((3.4)) is a Hilbert space. To

distinguish between $A^2(D)$ as a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ or (\cdot, \cdot) , we introduce the following definition.

Definition 3.2. We write $\langle\!\langle H, H \rangle\!\rangle$ if H is a Hilbert space with inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

By means of this convention, it is easy to see that the linear transformation $\Phi: \langle\!\langle A^2(D), A^2(D) \rangle\!\rangle \rightarrow (A^2(D), A^2(D))$ with

$$(3.5) \quad \Phi(f) = f \quad (f \in A^2(D))$$

$$(\Phi(f), \Phi(g)) = 2\pi \sum_{n=0}^{\infty} (2n+1)^{-1} a_n(f) \overline{a_n(g)} \quad (f, g \in A^2(D))$$

defines a Hilbert space homeomorphism.

Theorem 3.3. If $\{f_n\}_{n=0}^{\infty} \subset A^2(D)$ and $f \in A^2(D)$, then

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{A^2(D)} = 0 \text{ if and only if } \lim_{n \rightarrow \infty} \|f_n - f\| = 0.$$

Proof. (3.5) defines a Hilbert space homeomorphism. Q.E.D.

3.3 PROPERTIES OF $A^2(D)$ -FUNCTIONS.

Theorem 3.4. $H^2(D)$ and $A^2(D)$ are dense in $\langle\!\langle A^2(D), A^2(D) \rangle\!\rangle$ and $(A^2(D), A^2(D))$.

Proof. For $f \in A^2(D)$ and $\epsilon > 0$ there exists $N(\epsilon)$ such that

$$(3.6) \quad 2\pi \sum_{n=N(\epsilon)+1}^{\infty} (2n+1)^{-1} |a_n(f)|^2 < \epsilon^2.$$

Via (3.2) we may write

$$(3.7) \quad \|f - \sum_{n=0}^{N(\epsilon)} a_n(f)(\cdot)^n\|_{A^2(D)}, \quad \|f - \sum_{n=0}^{N(\epsilon)} a_n(f)(\cdot)^n\| < \epsilon;$$

therefore, $\sum_{n=0}^{N(\epsilon)} a_n(f)(\cdot)^n \in H^2(D)$ and $A^2(D)$. Q.E.D.

Theorem 3.5. If $f \in A^2(D)$, then

$$(3.8) \quad \int_0^1 (1-r)|f(re^{i\phi})|^2 dr \leq \pi \sum_{n=0}^{\infty} (2n+1)^{-1} |a_n(f)|^2 \quad (0 \leq \phi < 2\pi).$$

Proof. We define

$$(3.9) \quad f_r(\cdot) = f(r(\cdot)) \quad (f \in A^2(D), 0 \leq r \leq 1)$$

and derive from Lemma 2.5 the property

$$(3.10) \quad \int_{-1}^1 |f_r(xe^{i\phi})|^2 dx \leq 1/2 \int_0^{2\pi} |f_r(e^{i\theta})|^2 d\theta \quad (0 \leq \phi < 2\pi).$$

Replacing $f_r(xe^{i\phi})$ with $f(rx e^{i\phi})$ and x with $r^{-1}x$ in (3.10) yields

$$(3.11) \quad r^{-1} \int_{-r}^r |f(xe^{i\phi})|^2 dx \leq 1/2 \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta.$$

Further, inserting the series representation $f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n(f) r^n e^{in\theta}$

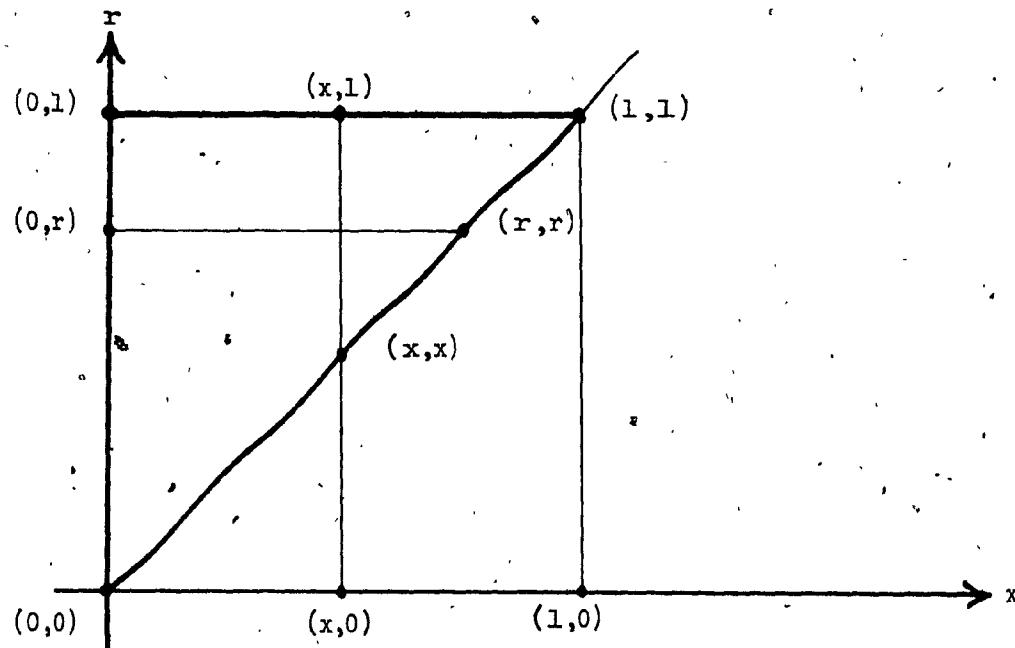
$(0 < r < 1)$ into the second integral of (3.11) leads to

$$(3.12) \quad r^{-1} \int_0^r |f(xe^{i\phi})|^2 dx \leq \pi \sum_{n=0}^{\infty} |a_n(f)|^2 r^{2n} \quad (0 < r < 1).$$

We multiply (3.12) by $r\psi(r)$ ($\psi \in L_\infty[0,1]$) and integrate both sides of (3.12) with respect to r on $[0,1]$, which gives us

$$(3.13) \quad \int_0^1 |\psi(r)| \left(\int_0^r |f(xe^{i\phi})|^2 dx \right) dr \leq \sum_{n=0}^{\infty} |a_n(f)|^2 \int_0^1 |\psi(r)| r^{2n+1} dr \quad (\psi \in L_\infty[0,1]).$$

By changing the order of integration in the first expression (3.13),



we obtain

$$(3.14) \int_0^1 |\psi(r)| \left(\int_0^r |f(xe^{i\phi})|^2 dx \right) dr = \int_0^1 \left(\int_x^1 |\psi(r)| |f(xe^{i\phi})|^2 dr \right) dx = \\ \int_0^1 \left(\int_x^1 |\psi(r)| dr \right) |f(xe^{i\phi})|^2 dx \quad (\psi \in L_\infty[0,1]).$$

To each of the expressions $\int_0^1 |\psi(r)| r^{2n+1} dr = \int_0^1 (r^{n+1/2}) (|\psi(r)| r^{n+1/2}) dr$

($n \geq 0$) in (3.13) we apply the Cauchy-Schwarz Inequality and conclude therefrom

$$(3.15) \int_0^1 |\psi(r)| r^{2n+1} dr \leq (2n+2)^{-1/2} \left(\int_0^1 |\psi(r)|^2 r^{2n+1} dr \right)^{1/2} \quad (n \geq 0),$$

where we used $\int_0^1 r^{2n+1} dr = (2n+2)^{-1}$. Applying the Cauchy-Schwarz Inequality

to the second integral of (3.15), namely $\int_0^1 |\psi(r)|^2 r^{2n+1} dr = \int_0^1 (r^{n+1/2}) \times$

$(|\psi(r)|^2 r^{n+1/2}) dr$, yields

$$(3.16) \int_0^1 |\psi(r)| r^{2n+1} dr \leq (2n+2)^{-(2^{-1} + 2^{-2})} \left(\int_0^1 |\psi(r)|^2 r^{2n+1} dr \right)^{2^{-2}};$$

in particular (by iteration)

$$(3.17) \int_0^1 |\psi(r)| r^{2n+1} dr \leq (2n+2)^{-(2^{-1} + 2^{-2} + 2^{-3} + \dots + 2^{-m})} \times$$

$$\left(\int_0^1 |\psi(r)|^{2^m} r^{2n+1} dr \right)^{2^{-m}} \quad (m \geq 0).$$

If $m \rightarrow \infty$ in the second expression of (3.17), then $\sum_{m=1}^{\infty} 2^{-m} = 1$ implies

$$(3.18) \quad \int_0^1 |\psi(r)| r^{2n+1} dr \leq (2n+2)^{-1} \lim_{m \rightarrow \infty} \left(\int_0^1 |\psi(r)|^{2^m} r^{2n+1} dr \right)^{2^{-m}} = \\ (2n+2)^{-1} \|\psi\|_{\infty},$$

since $r^{2n+1} dr = dv_n$ defines a finite Borel measure on $[0,1]$ for each $n \geq 0$; thus,

$$(3.19) \quad \int_0^1 |\psi(r)| r^{2n+1} dr \leq (2n+2)^{-1} \|\psi\|_{\infty} \quad (n \geq 0).$$

In consequence of (3.14) and (3.19), (3.13) lets us conclude

$$(3.20) \quad \int_0^1 \left(\int_x^1 |\psi(r)| dr \right) |f(xe^{i\phi})|^2 dx \leq \left[2^{-1} \pi \sum_{n=0}^{\infty} (n+1)^{-1} |a_n(f)|^2 \right] \|\psi\|_{\infty} \\ (\psi \in L_{\infty}[0,1]).$$

This means

$$(3.21) \quad \int_0^1 \left(\int_x^1 (\|\psi\|_{\infty})^{-1} |\psi(r)| dr \right) |f(xe^{i\phi})|^2 dx \leq$$

$$(2^{-1}\pi) \sum_{n=0}^{\infty} (n+1)^{-1} |a_n(f)|^2 \quad (\psi \in L_\infty[0,1], \|\psi\|_\infty > 0),$$

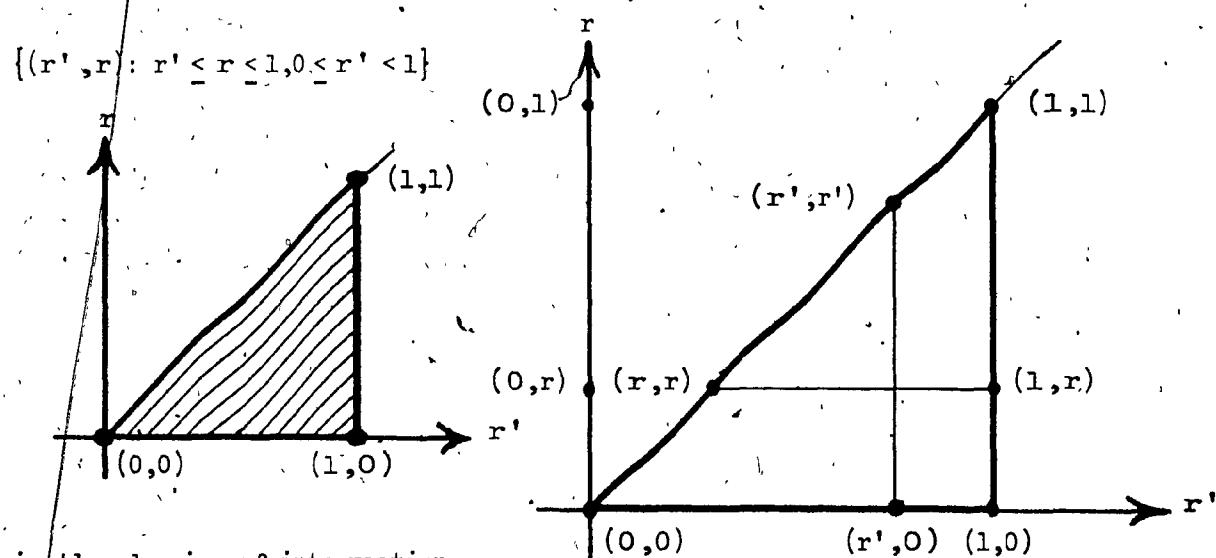
where the maximum value in the first expression of (3.21) is attained when $|\psi(r)|$ is constant a.e. on $[0,1]$; hence, (3.8) follows. Q.E.D.

CHAPTER IV
RADIALLY ACTING VOLterra OPERATORS

We begin this chapter with the concept of linear radially acting $A^2(D)$ -Volterra Integral Operators. Thereafter, we demonstrate that the collection of kernels, which define these linear radially acting $A^2(D)$ -Volterra Integral Operators, constitutes a Banach-Algebra. This shall in part be achieved by frequent use of the Cauchy-Schwarz Inequality

$$(4.1) \quad \left| \iint_{\Delta} u(r, r') v(r, r') dr dr' \right| \leq \left(\iint_{\Delta} |u(r, r')|^2 dr dr' \right)^{1/2} \left(\iint_{\Delta} |v(r, r')|^2 dr dr' \right)^{1/2}$$

for all $L_2(\Delta)$ -functions u and v , where $\Delta = \{(r', r) : 0 \leq r' \leq r, 0 \leq r < 1\} =$



4.1 THE θ -PARAMETER FAMILY OF L_2 -KERNELS.

Let $K(r, r', \theta)$ denote a θ -parameter family of Lebesgue measurable functions of the variables (r, r') on $[0, 1] \times [0, 1]$ ($0 \leq \theta < 2\pi$) such that $K(r, r', \theta) = 0$ whenever $0 \leq r < r' < 1$ ($0 \leq \theta < 2\pi$) and

$$(4.2) \quad |||K|||_{S(2)} = \sup_{0 \leq \theta < 2\pi} \left\{ \int_0^1 \int_0^r |K(r, r', \theta)|^2 dr dr' \right\}^{1/2} < \infty.$$

Then the function of the variables (r, θ)

$$(4.3) \quad (Kf)(r, \theta) = \int_0^1 K(r, r', \theta) f(r' e^{i\theta}) e^{i\theta} dr' \quad (f \in A^2(D))$$

defines an $L_2([0, 1] \times [0, 2\pi])$ -function with

$$(4.4) \quad \left(\int_0^1 \int_0^{2\pi} |(Kf)(r, \theta)|^2 dr d\theta \right)^{1/2} \leq |||K|||_{S(2)} ||f|| \quad (f \in A^2(D)).$$

By utilizing the Fubini-Tonelli Theorem and the Cauchy-Schwarz Inequality for $L_2[0, 1]$ -functions, we write

$$(4.5) \quad \begin{aligned} \int_0^1 \int_0^{2\pi} |(Kf)(r, \theta)|^2 dr d\theta &= \int_0^1 \int_0^{2\pi} \left| \int_0^r K(r, r', \theta) f(r' e^{i\theta}) e^{i\theta} dr' \right|^2 dr d\theta \leq \\ &\leq \int_0^{2\pi} \int_0^1 \left(\int_0^r |K(r, r', \theta)|^2 dr' \right) \left(\int_0^r |f(r' e^{i\theta})|^2 dr' \right) dr d\theta \leq \\ &\leq \int_0^{2\pi} \left[\int_0^1 \int_0^r |K(r, r', \theta)|^2 dr' dr \right] \left(\int_0^1 |f(r' e^{i\theta})|^2 dr' \right) dr \leq \end{aligned}$$

$$\int_0^{2\pi} \left(\frac{\|K\|}{S(2)} \right)^2 \left\{ \int_0^1 |f(r'e^{i\theta})|^2 dr \right\} d\theta =$$

$$\left(\frac{\|K\|}{S(2)} \right)^2 \int_0^{2\pi} \int_0^1 |f(r'e^{i\theta})|^2 dr d\theta = \left(\frac{\|K\|}{S(2)} \right)^2 (\|f\|)^2;$$

hence, (4.4) is valid and in consequence thereof, we define the following linear manifold.

Definition 4.1. $(VK_2)(D)$ denotes the totality of Lebesgue-measurable kernels $K(r, r', \theta)$ in variables (r, r') on Δ for each θ ($0 \leq \theta < 2\pi$) with uniformly bounded "double-norms"

$$(4.6) \quad \|K\|_{S(2)} = \sup_{0 \leq \theta < 2\pi} \left\{ \int_0^1 \int_0^r |K(r, r', \theta)|^2 dr dr' \right\}^{1/2} < \infty,$$

which act collectively as a bounded radial Volterra-Integral Operator $K: A^2(D) \rightarrow A^2(D)$ with

$$(4.7) \quad (Kf)(re^{i\theta}) = \int_0^r K(r, r', \theta) f(r'e^{i\theta}) e^{i\theta} dr' \quad (\|Kf\| \leq \|K\|_{S(2)} \|f\|)$$

a.e. in r' on $[0, 1]$ ($0 \leq \theta < 2\pi$).

4.2 EXAMPLES OF $(VK_2)(D)$ -KERNELS.

$(VK_2)(D)$ is not empty, since it contains kernels of the following type. For any pair of $A^2(D)$ -functions a and b , we define

$$(4.8) \quad (a \otimes b)(r, r', \theta) = a(r e^{i\theta}) b(r' e^{i\theta}) \quad (0 \leq r' \leq r < 1) \text{ and } 0 \quad (0 \leq r < r' < 1)$$

The (VK_2) (D)-norm of $(a \otimes b)(r, r', \theta)$ is finite- i.e.

$$(4.9) \quad |||a \otimes b|||_{S(2)} \leq \|a\|_{SL_2[0,1]} \|b\|_{SL_2[0,1]},$$

which is a direct consequence of $\sup_{0 \leq \theta < 2\pi} \left(\int_0^1 \int_0^r |a(r e^{i\theta})|^2 |b(r' e^{i\theta})|^2 dr dr' \right)^{1/2} \leq$

$$\left[\sup_{0 \leq \theta < 2\pi} \left(\int_0^1 |a(r e^{i\theta})|^2 dr \right)^{1/2} \right] \left[\sup_{0 \leq \theta < 2\pi} \left(\int_0^1 |b(r' e^{i\theta})|^2 dr' \right)^{1/2} \right].$$

Further, for each $A^2(D)$ -function f , the functions $a(z)f(z)$, $\int_0^z b(\xi)f(\xi)d\xi$

(Morera's Theorem) and $a(z) \int_0^z b(\xi)f(\xi)d\xi$ are analytic functions of the variable

z on D . Thus, $(a \otimes b)(f)(z) = a(z) \int_0^z b(\xi)f(\xi)d\xi$ is analytic on D and

$$(4.10) \quad ||(a \otimes b)(f)||_{A^2(D)} =$$

$$\left(\int_0^1 \int_0^{2\pi} |a(r e^{i\theta})|^2 \left| \int_0^r b(r' e^{i\theta}) f(r' e^{i\theta}) e^{i\theta} dr' \right|^2 r d\theta dr \right)^{1/2} \leq$$

$$\left(\int_0^1 \int_0^{2\pi} |a(r e^{i\theta})|^2 \left[r \int_0^r |b(r' e^{i\theta})|^2 dr' \right] \left[\int_0^r |f(r' e^{i\theta})|^2 dr' \right] d\theta dr \right)^{1/2} \leq$$

$$\|b\|_{SL_2[0,1]} \left(\int_0^{2\pi} \int_0^1 |a(re^{i\theta})|^2 \left[\int_0^r |f(r'e^{i\theta})|^2 dr' \right] dr d\theta \right)^{1/2} =$$

$$\|b\|_{SL_2[0,1]} \left(\int_0^{2\pi} \left\{ \int_0^1 |f(r'e^{i\theta})|^2 \left[\int_{r'}^1 |a(re^{i\theta})|^2 dr \right] dr' \right\} d\theta \right)^{1/2}$$

$$\|a\|_{SL_2[0,1]} \|b\|_{SL_2[0,1]} \left(\int_0^{2\pi} \int_0^1 |f(r'e^{i\theta})|^2 dr' d\theta \right)^{1/2} =$$

$$\|a\|_{SL_2[0,1]} \|b\|_{SL_2[0,1]} \|f\| \leq$$

$$2^{1/2} \|a\|_{SL_2[0,1]} \|b\|_{SL_2[0,1]} \|f\|_{A^2(D)}$$

where $r \int_0^r |b(r'e^{i\theta})|^2 dr'$, $\int_{r'}^1 |a(re^{i\theta})|^2 dr$ and $\|f\|$ were replaced by

$(\|b\|_{SL_2[0,1]})^2$, $(\|a\|_{SL_2[0,1]})^2$ and $2^{1/2} \|f\|_{A^2(D)}$ ((3.2)) respectively.

The change in the order of integration in the third step of (4.10) is justified by the diagram appearing after (4.1), which describes the domain of integration in the (r', r) plane. Therefore, $(a \otimes b)(f) \in A^2(D)$ ($f \in A^2(D)$) and

$$(4.11) \quad \|(a \otimes b)(f)\|_{A^2(D)} \leq 2^{1/2} \|a\|_{SL_2[0,1]} \|b\|_{SL_2[0,1]} \|f\|_{A^2(D)} \quad (f \in A^2(D))$$

holds - i.e., $(a \otimes b)(r, r', \theta) \in (VK_2)(D)$. Also, all finite sums

$\sum_{k=1}^n (a_k \otimes b_k)(r, r', \theta)$ ($a_k, b_k \in A^2(D)$ for $1 \leq k \leq n$) belong to $(VK_2)(D)$ and (4.9)

implies

$$(4.12) \quad \left\| \sum_{k=1}^n a_k \otimes b_k \right\|_{S(2)} \leq \sum_{k=1}^n \|a_k\|_{SL_2[0,1]} \|b_k\|_{SL_2[0,1]}$$

4.3 $(VK_2)(D)$ AS A NORMED ALGEBRA.

The construction of a $(VK_2)(D)$ -kernel $(a \otimes b)(r, r', \theta)$ with either a or $b \in A^2(D) \sim A^2(D)$ is doubtful because of function g_0 ((2.37)).

(4.4) entails that every $(VK_2)(D)$ -kernel defines (via (4.7)) a bounded operator on $A^2(D)$. Inherent in action (4.7) is the fact that the algebraic structure of $(VK_2)(D)$ is directly determined by the algebraic structure of the Banach-Algebra of bounded operators on $A^2(D)$. Hence, the linear combination $(\alpha K + \beta L): A^2(D) \rightarrow A^2(D)$ ($\alpha, \beta \in \mathbb{C}$), and the product $(KL): A^2(D) \rightarrow A^2(D)$, where $K: A^2(D) \rightarrow A^2(D)$ and $L: A^2(D) \rightarrow A^2(D)$ are determined by the $(VK_2)(D)$ -kernels $K(r, r', \theta)$ and $L(r, r', \theta)$ ((4.7)) respectively, are again linear radially acting Volterra-Integral Operators on the Hilbert space $A^2(D)$. Their actions on $A^2(D)$ are

$$(4.13) \quad (\alpha K + \beta L)(f)(re^{i\theta}) = \int_0^r (\alpha K + \beta L)(r, r', \theta) f(r'e^{i\theta}) e^{i\theta} dr'$$

a.e. in r on $[0, 1]$ ($0 \leq \theta < 2\pi$) and

$$(4.14) \quad (KL)(f)(re^{i\theta}) = \int_0^r (KL)(r, r', \theta) f(r'e^{i\theta}) e^{i\theta} dr'$$

a.e. in r on $[0, 1]$ ($0 \leq \theta < 2\pi$) with corresponding $(VK_2)(D)$ -kernels

$$(4.15) \quad (\alpha K + \beta L)(r, r', \theta) = \alpha K(r, r', \theta) + \beta L(r, r', \theta) \quad (\alpha, \beta \in C)$$

and

$$(4.16) \quad (KL)(r, r', \theta) = \int_{r'}^r K(r, r'', \theta) L(r'', r', \theta) e^{i\theta} dr'' \quad (0 \leq r' < r < 1) \text{ and}$$

$$0 \quad (0 \leq r < r' < 1)$$

respectively.

Lemma 4.2. $(VK_2)(D)$ endowed with the norm $\| \cdot \|_{S(2)}$ is a normed algebra. If $K(r, r', \theta), L(r, r', \theta) \in (VK_2)(D)$, then

$$(4.17) \quad \| \alpha K + \beta L \|_{S(2)} \leq |\alpha| \|K\|_{S(2)} + |\beta| \|L\|_{S(2)} \quad (\alpha, \beta \in C)$$

and

$$(4.18) \quad \|KL\|_{S(2)} \leq \|K\|_{S(2)} \|L\|_{S(2)}$$

Proof. For all $\alpha, \beta \in C$ and θ ($0 \leq \theta < 2\pi$), Lemma 4.1 implies

$$(4.19) \quad \left[\int_0^1 \int_0^r |\alpha K(r, r', \theta) + \beta L(r, r', \theta)|^2 dr dr' \right]^{1/2} \leq \\ |\alpha| \left(\int_0^1 \int_0^r |K(r, r', \theta)|^2 dr dr' \right)^{1/2} + |\beta| \left(\int_0^1 \int_0^r |L(r, r', \theta)|^2 dr dr' \right)^{1/2}$$

Taking the supremum with respect to θ ($0 \leq \theta < 2\pi$) on both sides of (4.19) yields (4.17).

For the product kernel $(KL)(r, r', \theta)$ ((4.16)), the Cauchy-Schwarz Inequality (4.1) allows us to write

$$(4.20) \quad \left[\int_0^1 \int_0^r |(KL)(r, r', \theta)|^2 dr dr' \right]^{1/2} \leq \\ \left(\int_0^1 \int_0^r \left[\int_{r'}^r |K(r, r'', \theta)| |L(r'', r', \theta)| dr'' \right]^2 dr dr' \right)^{1/2} \leq \\ \left(\int_0^1 \int_0^r \left[\int_0^r |K(r, r'', \theta)|^2 dr'' \right] \left[\int_{r'}^1 |L(r'', r', \theta)|^2 dr'' \right] dr dr' \right)^{1/2} = \\ \left(\int_0^1 \int_0^r |K(r, r'', \theta)|^2 dr'' dr \right)^{1/2} \left(\int_0^1 \int_{r'}^1 |L(r'', r', \theta)|^2 dr'' dr' \right)^{1/2} \leq$$

$$|||K|||_{S(2)} |||L|||_{S(2)}$$

where the Fubini-Tonelli Theorem and the diagram appearing after (4.1) justify

$$\int_0^1 \int_0^1 |L(r'', r', \theta)|^2 dr'' dr' = \int_0^1 \int_0^r |L(r'', r', \theta)|^2 dr' dr''. \text{ Taking the supremum}$$

with respect to θ ($0 \leq \theta < 2\pi$) of the first expression of (4.20) confirms (4.17).

We now demonstrate the radial action of (KL)-i.e. (KL): $A^2(D) \rightarrow A^2(D)$. (KL)(r, r', θ) ((4.16)) determines a linear radially acting Volterra Integral Operator on $A^2(D)$, because

$$(4.21) \quad (KL)(f)(re^{i\theta}) = \int_0^r (KL)(r, r', \theta) f(r'e^{i\theta}) e^{i\theta} dr' =$$

$$= \int_0^r \left[\int_{r'}^r K(r, r'', \theta) L(r'', r', \theta) e^{i\theta} dr'' \right] f(r'e^{i\theta}) e^{i\theta} dr' =$$

$$= \int_0^r K(r, r'', \theta) \left[\int_0^{r''} L(r'', r', \theta) f(r'e^{i\theta}) e^{i\theta} dr' \right] e^{i\theta} dr'' =$$

$$= \int_0^r K(r, r'', \theta) (\Delta f)(r''e^{i\theta}) e^{i\theta} dr'' \quad (f \in A^2(D))$$

a.e. in r on $[0,1]$ ($0 \leq \theta < 2\pi$). The change of the order of integration in the third step of (4.21) is justified by the Fubini-Tonelli Theorem and the diagram appearing after (4.1). Thus,

$$(4.22) \quad (KL)(f)' re^{i\theta} = \int_0^r K(r, r', \theta) (\Delta f)(r'e^{i\theta}) e^{i\theta} dr' \quad (f \in A^2(D))$$

a.e. in r on $[0,1)$ ($0 \leq \theta < 2\pi$).

The other norm property, namely $\|K\|_{S(2)} = 0$ if and only if

$K(r, r', \theta) = 0$ a.e. in Δ ($0 \leq \theta < 2\pi$), is obvious. Q.E.D.

4.4 COMPLETENESS OF $(VK_2)(D)$.

Lemma 4.3. $(VK_2)(D)$ is complete with respect to $\| \cdot \|_{S(2)}$.

Proof. Let $\{K_n(r, r', \theta)\}_{n=1}^{\infty}$ be a Cauchy-Sequence of $(VK_2)(D)$ -

kernels-i.e., to every $\epsilon > 0$ corresponds a $N(\epsilon)$ such that $n, m \geq N(\epsilon)$ implies

$$(4.23) \quad \left(\int_0^1 \int_0^r |K_n(r, r', \theta) - K_m(r, r', \theta)|^2 dr dr' \right)^{1/2} \leq \|K_n - K_m\|_{S(2)} < \epsilon$$

$(0 \leq \theta < 2\pi)$.

Thus, $\{K_n(r, r', \theta)\}_{n=1}^{\infty}$ is a Cauchy-Sequence of $L_2(\Delta)$ for each θ ($0 \leq \theta < 2\pi$).

The completeness of $L_2(\Delta)$ guarantees the existence of a unique $K(r, r', \theta) \in L_2(\Delta)$ ($0 \leq \theta < 2\pi$) satisfying:

$$(4.24) \quad \left(\int_0^1 \int_0^r |K(r, r', \theta)|^2 dr dr' \right)^{1/2} < \infty \quad (0 \leq \theta < 2\pi),$$



$$(4.25) \lim_{n \rightarrow \infty} K_n(r, r', \theta) = K(r, r', \theta)$$

for almost all $(r, r') \in \Delta$ ($0 \leq \theta < 2\pi$) and

$$(4.26) \lim_{n \rightarrow \infty} \left\{ \int_0^1 \int_0^r |K(r, r', \theta) - K_n(r, r', \theta)|^2 dr dr' \right\}^{1/2} = 0.$$

We derive by means of Fatou's Lemma from (4.26) that

$$(4.27) \left\{ \int_0^1 \int_0^r |K(r, r', \theta) - K_n(r, r', \theta)|^2 dr' dr \right\}^{1/2} =$$

$$\left\{ \int_0^1 \int_0^r \left[\liminf_{m \rightarrow \infty} |K_m(r, r', \theta) - K_n(r, r', \theta)|^2 \right] dr dr' \right\}^{1/2} \leq$$

$$\liminf_{m \rightarrow \infty} \left\{ \int_0^1 \int_0^r |K_m(r, r', \theta) - K_n(r, r', \theta)|^2 dr dr' \right\}^{1/2} \leq$$

$$\liminf_{m \rightarrow \infty} |||K_m - K_n|||_{S(2)} \leq \varepsilon \quad (0 \leq \theta < 2\pi).$$

Consequently, the $L_2(\Delta)$ -norm of $K(r, r', \theta) - K_n(r, r', \theta)$ is uniformly bounded

(with respect to θ) by ε ($n \geq N(\varepsilon)$) and hence

$$(4.28) |||K|||_{S(2)} \leq |||K_n|||_{S(2)} + \varepsilon \quad (n \geq N(\varepsilon)).$$

In order to prove the radial action (4.7) of the limit kernel

$K(r, r', \theta)$, we have to return to series representation (3.1) of $A^2(D)$ -functions.

Thus, $(K_n f)(z)$ has Taylor-Series representation

$$(4.29) \quad (K_n f)(z) = \sum_{m=0}^{\infty} a_m (K_n f) z^m \quad (0 \leq |z| < 1, n \geq 1).$$

$$(4.23) \text{ implies via (4.4) and (4.7) that } \left\{ (K_n f)(z) = \sum_{m=0}^{\infty} a_m (K_n f) z^m \right\}_{n=1}^{\infty}$$

is a Cauchy-Sequence in $A^2(D)$ with respect to $\|\cdot\|$ ((3.1) and (3.2)). Since

$$\ell_2((2n+1)^{-1}; N) = \left\{ (a_n)_{n=0}^{\infty} : a_n \in C (n \geq 0), \sum_{n=0}^{\infty} (2n+1)^{-1} |a_n|^2 < \infty \right\}$$

is a Hilbert space, (3.1) guarantees the existence of a unique $\ell_2((2n+1)^{-1}; N)$ -element $\{a_n\}_{n=0}^{\infty}$ such that

$$(4.30) \quad \lim_{n \rightarrow \infty} \sum_{m=0}^{\infty} (2m+1)^{-1} |a_m (K_n f) - a_m|^2 = 0$$

and

$$(4.31) \quad g(z) = \sum_{m=0}^{\infty} a_m z^m = \lim_{n \rightarrow \infty} \sum_{m=0}^{\infty} a_m (K_n f) z^m \quad (|z| < 1),$$

where $g \in A^2(D)$. Further, limit (4.31) is attained uniformly on $\{z : |z| \leq R\}$

$(0 \leq R < 1)$. Thus, (4.29) and (4.31) entail

$$(4.32) \quad g(re^{i\theta}) = \lim_{n \rightarrow \infty} (K_n f)(re^{i\theta}) = \sum_{m=0}^{\infty} a_m r^m e^{im\theta}$$

uniformly in (r, θ) on $[0, R] \times [0, 2\pi]$.

Theorem 3.5 applied to $(K_n f)(z) - g(z)$ implies

$$(4.33) \quad \int_0^1 (1-r) |(K_n f)(re^{i\theta}) - g(re^{i\theta})|^2 dr \leq \sum_{m=0}^{\infty} (2m+1)^{-1} |a_m(K_n f) - a_m|^2 \quad (0 \leq \theta < 2\pi),$$

where $a_m(g) = a_m$ ($m \geq 0$). Out of $(1-R) \leq (1-r)$ ($0 \leq r \leq R$) follows

$$\int_0^R (1-R) |(K_n f)(re^{i\theta}) - g(re^{i\theta})|^2 dr \leq \int_0^1 (1-r) |(K_n f)(re^{i\theta}) - g(re^{i\theta})|^2 dr$$

$(0 \leq R < 1)$ and with the help of (4.29), we write

$$(4.34) \quad (1-R) \int_0^R |(K_n f)(re^{i\theta}) - g(re^{i\theta})|^2 dr \leq \sum_{m=0}^{\infty} (2m+1)^{-1} |a_m(K_n f) - a_m|^2 \quad (0 \leq R < 1, 0 \leq \theta < 2\pi).$$

On the other hand, from (4.6) it is clear that

$$(4.35) \quad \left(\iint_{0,0}^{R,r} |K_n(r, r', \theta) - K(r, r', \theta)|^2 dr' dr \right)^{1/2} \leq |||K_n - K|||_{S(2)}$$

$$(0 \leq R \leq 1, 0 \leq \theta < 2\pi)$$

holds. (4.35), via the Cauchy-Schwarz Inequality applied to

$$\left| \int_0^r \left[K_n(r, r', \theta) - K(r, r', \theta) \right] f(r' e^{i\theta}) e^{i\theta} dr' \right|, \text{ yields}$$

$$(4.36) \quad \left\| (K_n f)(\cdot e^{i\theta}) - \int_0^r K(\cdot, r', \theta) f(r' e^{i\theta}) e^{i\theta} dr' \right\|_{L_2[0, R]} =$$

$$\left(\int_0^R \left| \int_0^r \left[K_n(r, r', \theta) - K(r, r', \theta) \right] f(r' e^{i\theta}) e^{i\theta} dr' \right|^2 dr \right)^{1/2} \leq$$

$$\left(\int_0^R \left[\int_0^r |K_n(r, r', \theta) - K(r, r', \theta)|^2 dr' \right] \left[\int_0^r |f(r' e^{i\theta})|^2 dr' \right] dr \right)^{1/2} =$$

$$\left(\int_0^R \int_0^r |K_n(r, r', \theta) - K(r, r', \theta)|^2 dr' dr \right)^{1/2} \|f(\cdot e^{i\theta})\|_{L_2[0, R]} \leq$$

$$\|K_n - K\|_{S(2)} \|f(\cdot e^{i\theta})\|_{L_2[0, R]} \quad (0 \leq R < 1, 0 \leq \theta < 2\pi);$$

in particular,

$$(4.37) \quad \left\| (K_n f)(\cdot e^{i\theta}) - \int_0^r K(\cdot, r', \theta) f(r' e^{i\theta}) e^{i\theta} dr' \right\|_{L_2[0, R]} \leq$$

$$\|K_n - K\|_{S(2)} \|f(\cdot e^{i\theta})\|_{L_2[0, R]} \quad (0 \leq R < 1, 0 \leq \theta < \pi).$$

We note that $\|f(\cdot e^{i\theta})\|_{L_2[0,R]} < \infty$, $\|g(\cdot e^{i\theta})\|_{L_2[0,R]} < \infty$

($0 \leq R < 1$, $0 \leq \theta < 2\pi$). (4.30), the fact that $(K_n f)(\cdot e^{i\theta})$ converges to $g(\cdot e^{i\theta})$ uniformly on $[0,R]$ ((4.32)), (4.37) and $\lim_{n \rightarrow \infty} \|K_n - K\|_{S(2)} = 0$

applied to (4.34) imply

$$(4.38) \quad \int_0^R \left| \int_0^r K(r,r',\theta) f(r'e^{i\theta}) e^{i\theta} dr' - g(re^{i\theta}) \right|^2 dr = 0 \quad (0 \leq R < 1, 0 \leq \theta < 2\pi).$$

Therefore, $g(re^{i\theta}) = \int_0^r K(r,r',\theta) f(r'e^{i\theta}) e^{i\theta} dr'$ a.e. in r on $[0,1]$ ($0 \leq \theta < 2\pi$) -

i.e. $K : A^2(D) \rightarrow A^2(D)$. Q.E.D.

Lemmas 4.2 and 4.3 combined allow us to formulate the following.

Theorem 4.4. $(VK_2)(D)$ is a Banach-algebra.

CHAPTER V

THE FREDHOLM-RESOLVENT OF $(VK_2)(D)$ -KERNELS

This chapter deals with the Fredholm-Resolvent Kernel of $(VK_2)(D)$ -kernels by means of the Neumann-Series. We shall demonstrate herein that

every $(VK_2)(D)$ -kernel $K(r, r', \theta)$ is quasi-nilpotent i.e.

$$\lim_{n \rightarrow \infty} \left(\frac{\|K^n\|}{S(2)} \right)^{1/n} = 0 \quad [2, p.49]. \text{ This shall allow us to write}$$

the Fredholm-Resolvent Kernel $H_\lambda(K)(r, r', \theta)$ of the (VK_2) -kernel $K(r, r', \theta)$ in

$$\text{the form } H_\lambda(K)(r, r', \theta) = \sum_{n=0}^{\infty} \lambda^n K^{n+1}(r, r', \theta) \text{ for all } \lambda \in C \ (\lambda \neq 0);$$

specifically, the set of regular values of the $(VK_2)(D)$ -kernel $K(r, r', \theta)$ is

the set $C \sim \{0\}$. Moreover, the Fredholm-Resolvent Kernel $H_\lambda(K)(r, r', \theta)$ enables us to solve the linear radial $A^2(D)$ -Volterra Integral Equation

$$(5.1) \quad f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r K(r, r', \theta) f(r'e^{i\theta}) e^{i\theta} dr' \quad (g \in A^2(D))$$

a.e. in r on $[0,1]$ ($0 \leq \theta < 2\pi$), where $f \in A^2(D)$ is unknown and $K(r,r',\theta) \in (\mathcal{V}K_2)(D)$.

Every $(\mathcal{V}K_2)(D)$ -kernel $K(r,r',\theta)$ and $L_2[0,1]$ -function f , defines a θ -parameter family of $L_2[0,1]$ -functions

$$(5.2) \quad (Kf)(r,\theta) = \int_0^r K(r,r',\theta)f(r')e^{i\theta} dr' \quad (0 \leq r < 1, 0 \leq \theta < 2\pi),$$

for which the Cauchy-Schwarz Inequality yields $| (Kf)(re^{i\theta}) | \leq$

$$\left[\int_0^r |K(r,r',\theta)|^2 dr' \right]^{1/2} \|f\|_{L_2[0,1]} ; \text{ in particular,}$$

$$(5.3) \quad \|(Kf)(\cdot, \theta)\|_{L_2[0,1]} \leq \|K\|_{S(2)} \|f\|_{L_2[0,1]} \quad (0 \leq \theta < 2\pi).$$

5.1 ELEMENTARY ESTIMATES FOR $(\mathcal{V}K_2)(D)$ -KERNELS.

Lemma 5.1. If $K(r,r',\theta)$, $L(r,r',\theta) \in (\mathcal{V}K_2)(D)$ and $f \in L_2[0,1]$, then

$$(5.4) \quad (KL)(f)(r,\theta) = \int_0^r K(r,r',\theta)(Lf)(r',\theta)e^{i\theta} dr' \quad (0 \leq r < 1, 0 \leq \theta < 2\pi).$$

Proof. Via the diagram appearing after (4.1) and the Fubini-Tonelli Theorem we change the order of integration in

$$(5.5) \quad (KL)(f)(r, \theta) = \int_0^r (KL)(r, r', \theta) f(r') e^{i\theta} dr' = \\ \int_0^r \left[\int_{r'}^r K(r, r'', \theta) L(r'', r', \theta) e^{i\theta} dr'' \right] f(r') e^{i\theta} dr',$$

which leads to

$$(5.6) \quad (KL)(f)(r, \theta) = \int_0^r K(r, r'', \theta) \left[\int_0^{r''} L(r'', r', \theta) f(r') e^{i\theta} dr' \right] e^{i\theta} dr'' = \\ \int_0^r K(r, r', \theta) (Lf)(r', \theta) e^{i\theta} dr'$$

and thereby completes this proof. Q.E.D.

Definition 5.2. We assign to every $(VK_2)(D)$ -kernel $K(r, r', \theta)$ the auxiliary θ -parameter family of $L_2[0, 1]$ -functions

$$(5.7) \quad k_1(u, \theta) = \left[\int_0^u |K(u, r', \theta)|^2 dr' \right]^{1/2}, \\ k_2(u, \theta) = \left[\int_u^1 |K(r, u, \theta)|^2 dr \right]^{1/2} \quad (0 \leq \theta < 2\pi).$$

These auxiliary $L_2[0, 1]$ -functions $k_i(\cdot, \theta)$ ($i=1, 2$) of the $(VK_2)(D)$ -kernel $K(r, r', \theta)$ satisfy

$$(5.8) \quad \|k_i(\cdot, \theta)\|_{L_2[0,1]} = \left[\int_0^1 \int_0^r |K(r, r', \theta)|^2 dr' dr \right]^{1/2} \leq \|K\|_{S(2)}$$

$(i=1,2; 0 \leq \theta \leq 2\pi).$

5.2 ESTIMATES FOR ITERATES OF $(VK_2)(D)$ -KERNELS.

Lemma 5.3. If $K(r, r', \theta) \in (VK_2)(D)$ and $f \in L_2[0,1]$, then

$$(5.9) \quad |(K^n f)(r, \theta)| \leq k_1(r, \theta) [(n-1)!]^{-1/2} \left[\int_0^r k_1^2(r', \theta) dr' \right]^{(n-1)/2}$$

$\|f\|_{L_2[0,1]} \quad (0 \leq r < 1, 0 \leq \theta < 2\pi).$

Proof. We proceed via induction on n . The inequality between (5.2) and (5.3) confirms (5.9) ($n=1$). If we assume (5.9) to be valid for $n=m$, then Lemma 5.1 and the Cauchy-Schwarz Inequality lead to

$$(5.10) \quad |(K^{m+1} f)(r, \theta)| = \left| \int_0^r K(r, r', \theta) (K^m f)(r', \theta) e^{i\theta} dr' \right| \leq \\ k_1(r, \theta) \left[\int_0^r |(K^m f)(r', \theta)|^2 dr' \right]^{1/2} \quad (0 \leq r < 1, 0 \leq \theta < 2\pi).$$

However, the integral expression in the third term of (5.10) may be estimated,

by means of (5.9) ($n = m$), as follows:

$$(5.11) \quad \left[\int_0^r |(K^m f)(r', \theta)|^2 dr' \right]^{1/2} \leq$$

$$\left\{ \int_0^r [(m-1)!]^{-1} \left[\int_0^{r'} k_1^2(r'', \theta) dr'' \right]^{m-1} k_1^2(r', \theta) dr' \right\}^{1/2} \|f\|_{L_2[0,1]} =$$

$$\left\{ [m!]^{-1} \left[\int_0^r k_1^2(r', \theta) dr' \right]^m \right\}^{1/2} \|f\|_{L_2[0,1]} \quad (0 \leq r < 1, 0 \leq \theta < 2\pi),$$

where the last step is a consequence of the "chain rule" for integrals. Replacing the integral expression in the third term of (5.10) by the last term of (5.11) yields

$$(5.12) \quad |(K^{m+1} f)(r, \theta)| \leq$$

$$k_1(r, \theta) [m!]^{-1/2} \left[\int_0^r k_1^2(r', \theta) dr' \right]^{m/2} \|f\|_{L_2[0,1]}$$

$$(0 \leq r < 1, 0 \leq \theta < 2\pi),$$

which is the same as (5.9) ($n = m+1$). Q.E.D.

Corollary 5.4. If $K(r, r', \theta) \in (VK_2)(D)$, then

$$(5.13) \quad |K^{n+1}(r, r', \theta)| \leq$$

$$k_1(r, \theta) [(n-1)!]^{-1/2} \left[\int_0^r k_1^2(r', \theta) dr' \right]^{(n-1)/2} k_2(r', \theta)$$

$$(0 \leq r, r' < 1; 0 \leq \theta < 2\pi; n \geq 1).$$

Proof. We momentarily hold r' and θ fixed and set $f(\cdot) = K(\cdot, r', \theta)$.

For this $L_2[0,1]$ -function $f(\cdot)$ (otherwise $\|f(\cdot)\|_{L_2[0,1]} = \infty$), we have:

$$(5.14) \quad (K^n f)(r, \theta) = \int_0^r K^n(r, r'', \theta) K(r'', r', \theta) e^{i\theta} dr'' = K^{n+1}(r, r', \theta) \quad (n \geq 1).$$

$$\text{and } \|f\|_{L_2[0,1]} = k_2(r', \theta), \text{ because } \int_0^r K^n(r, r'', \theta) K(r'', r', \theta) e^{i\theta} dr'' =$$

$$\int_{r'}^r K^n(r, r'', \theta) K(r'', r', \theta) e^{i\theta} dr'' \text{ or } 0 \text{ (according as } r' \leq r \text{ or } r < r') \text{ and}$$

$$\|f\|_{L_2[0,1]} = \left[\int_{r'}^1 |K(r, r', \theta)|^2 dr \right]^{1/2}. \text{ Substituting the } L_2[0,1] \text{-function}$$

$f(\cdot)$ into (5.9) establishes (5.13). Q.E.D.

Lemma 5.5. If $K(r, r', \theta) \in (VK_2)(D)$, then

$$(5.15) \quad \left[\int_0^r |K^{n+1}(r, r', \theta)|^2 dr' \right]^{1/2} \leq [(n-1)!]^{-1/2} (\|K\|_{S(2)})^n k_1(r, \theta)$$

$(0 \leq r < 1, 0 \leq \theta < 2\pi)$

and

$$(5.16) \quad \left[\int_{r'}^1 |K^{n+1}(r, r', \theta)|^2 dr' \right]^{1/2} \leq [n!]^{-1/2} (\|K\|_{S(2)})^n k_2(r', \theta)$$

$(0 \leq r' < 1, 0 \leq \theta < 2\pi).$

Proof. To prove (5.15) we increase the upper limit r of the integral expression

expression $\int_0^r k_1^2(r', \theta) dr'$ to 1 and write (via (5.8) ($i=1$))

$$\int_0^r k_1^2(r', \theta) dr' \leq (\|K\|_{S(2)})^2 \quad (0 \leq \theta < 2\pi), \text{ which we apply to (5.13). Thus,}$$

$$\text{we obtain } |K^{n+1}(r, r', \theta)| \leq k_1(r, \theta) [(n-1)!]^{-1/2} (\|K\|_{S(2)})^{n+1} k_2(r', \theta)$$

$(0 \leq r, r' < 1; 0 \leq \theta < 2\pi; n \geq 1)$, which leads to

$$(5.17) \quad \left[\int_0^r |K^{n+1}(r, r', \theta)|^2 dr' \right]^{1/2} \leq$$

$$[(n-1)!]^{-1/2} (\|K\|_{S(2)})^{n-1} \left[\int_0^r k_1^2(r', \theta) dr' \right]^{1/2} k_1(r, \theta)$$

$(0 \leq r < 1, 0 \leq \theta < 2\pi).$

In (5.17) we may replace $\int_0^r k_1^2(r', \theta) dr'$ by $(\|K\|_{S(2)})^2$, because of (5.8)

$(i=2)$, and thereby we have proved (5.15).

For the proof of (5.16) we decrease the lower limit r' of the integral in (5.16) to 0 and use (5.13) as well as the "chain rule" for integration to demonstrate

$$(5.18) \quad \left[\int_{r'}^1 |K^{n+1}(r, r', \theta)|^2 dr' \right]^{1/2} \leq$$

$$\left[\int_0^1 [(n-1)!]^{-1} \left[\int_0^r k_1^2(r'', \theta) dr'' \right]^{n-1} k_1^2(r, \theta) dr \right]^{1/2} k_2(r', \theta) \leq$$

$$[n!]^{-1/2} \left[\int_0^1 k_1^2(r'', \theta) dr'' \right]^{n/2} k_2(r', \theta) \leq$$

$$[n!]^{-1/2} (\|K\|_{S(2)})^n k_2(r', \theta) \quad (0 \leq r' < 1, 0 \leq \theta < 2\pi; n \geq 1),$$

where the last step follows out of $\|k_2(\cdot, \theta)\|_{L_2[0,1]} \leq \|\|K\|\|_{S(2)}$

((5.8) ($i = 2$)). This terminates the proof of (5.16). Q.E.D.

Corollary 5.6. If $K(r, r', \theta) \in (\mathcal{V}K_2)(D)$, then

$$(5.19) \quad \|\|K^{n+1}\|\|_{S(2)} \leq [n!]^{-1/2} (\|\|K\|\|_{S(2)})^{n+1} \quad (n \geq 0).$$

Proof. The diagram appearing after (4.1) and the Fubini-Tonelli Theorem let us write

$$(5.20) \quad \left(\int_0^1 \left[\int_0^r |K^{n+1}(r, r', \theta)|^2 dr' \right] dr \right)^{1/2} = \\ \left(\int_0^1 \left[\int_{r'}^1 |K^{n+1}(r, r', \theta)|^2 dr \right] dr' \right)^{1/2} \quad (0 \leq \theta < 2\pi).$$

If we estimate the inside integral of the second term of (5.20) with (5.16), then (5.19) immediately follows. Q.E.D.

5.3 THE NEUMANN SERIES OF $(\mathcal{V}K_2)(D)$ -KERNELS.

For showing that the Neumann-Series $\sum_{n=0}^{\infty} \lambda^n K^{n+1}(r, r', \theta)$ of the

$(VK_2)(D)$ -kernels $K(r, r', \theta)$ converges in $(VK_2)(D)$ for all $\lambda \in C$, we shall need

the following entire function

$$(5.21) \quad \Gamma(z) = \sum_{n=0}^{\infty} [n!]^{-1/2} z^n \quad (z \in C).$$

Theorem 5.7. If $K(r, r', \theta) \in (VK_2)(D)$, then $H_{\lambda}(K)(r, r', \theta) \in (VK_2)(D)$

for all $\lambda \in C$, where

$$(5.22) \quad H_{\lambda}(K)(r, r', \theta) = \sum_{n=0}^{\infty} \lambda^n K^{n+1}(r, r', \theta) \quad (\lambda \in C).$$

Proof. We only need to show that the sequence of partial sums

$\{S_n(K; \lambda)(r, r', \theta)\}_{n=0}^{\infty}$ of the series (5.22) converges in $(VK_2)(D)$, where

$$(5.23) \quad S_n(K; \lambda)(r, r', \theta) = \sum_{j=0}^n \lambda^j K^{j+1}(r, r', \theta) \quad (n \geq 0).$$

If $q \geq 1$, then it is easy to see that

$$(5.24) \quad |||S_{n+q}(K; \lambda) - S_n(K; \lambda)|||_{S(2)} = |||\sum_{j=n+1}^{n+q} \lambda^j K^{j+1}|||_{S(2)} \leq$$

$$\sum_{j=n+1}^{n+q} |\lambda|^j |||K^{j+1}|||_{S(2)} \leq \sum_{j=n+1}^{n+q} [j!]^{-1/2} |\lambda|^j \left(|||K|||_{S(2)} |\lambda| \right)^{j+1} \leq$$

$$|||K|||_{S(2)} [(n+1)!]^{-1/2} \left(|||K|||_{S(2)} |\lambda| \right)^n r \left(|||K|||_{S(2)} |\lambda| \right)$$

$(n, q \geq 1; \lambda \in C)$

holds, where $[(n+1)(n+2)\dots(n+q)]^{-1/2} \leq [q!]^{-1/2}$ ($n \geq 0, q \geq 1$).

(5.24) implies

$$(5.25) \quad |||s_{n+q}(K; \lambda) - s_n(K; \lambda)|||_{S(2)} \leq$$

$$|||K|||_{S(2)} [(n+1)!]^{-1/2} \left(|||K|||_{S(2)} |\lambda| \right)^n r \left(|||K|||_{S(2)} |\lambda| \right)$$

$(n \geq 0; q \geq 1),$

which combined with $\lim_{n \rightarrow \infty} [(n+1)!]^{-1/2} \left(|||K|||_{S(2)} |\lambda| \right)^n = 0$ ($\lambda \in C$)

allows us to conclude that $H_\lambda(K)(r, r', \theta) = \lim_{n \rightarrow \infty} s_n(K; \lambda)(r, r', \theta)$ exists and

belongs to $(VK_2)(D)$. Q.E.D.

5.4 THE FREDHOLM-RESOLVENT KERNEL.

Theorem 5.8. If $K(r, r', \theta) \in (VK_2)(D)$, then $H_\lambda(K)(r, r', \theta) = \sum_{n=0}^{\infty} \lambda^n K^{n+1}(r, r', \theta)$ satisfies the Fredholm-Resolvent Equations

$$(5.26) \quad H_\lambda(K)(r, r', \theta) - K(r, r', \theta) = \lambda [KH_\lambda(K)](r, r', \theta) = \lambda [H_\lambda(K)K](r, r', \theta) \\ (\lambda \in \mathbb{C}, \lambda \neq 0).$$

Proof. Out of Lemma 4.2 ((4.18)) and (5.25) we infer

$$(5.27) \quad |||KS_{n+q-1}(K; \lambda) - KS_{n-1}(K; \lambda)|||_{S(2)},$$

$$|||S_{n+q-1}(K; \lambda)K - S_{n-1}(K; \lambda)K|||_{S(2)} \leq$$

$$\frac{(|||K|||_{S(2)})^2[n!]^{-1/2}}{(n+q-1)!} (|||K|||_{S(2)} |\lambda|)^{n-1} \Gamma(|||K|||_{S(2)} |\lambda|)$$

$$(n, q \geq 1).$$

This means that $\{[KS_{n-1}(K; \lambda)](r, r', \theta)\}_{n=1}^{\infty}$ and $\{[S_{n-1}(K; \lambda)K](r, r', \theta)\}_{n=1}^{\infty}$

are Cauchy-Sequences in the Banach-Algebra $(V\mathcal{K}_2)(D)$ (Theorem 4.4). By

letting $q \rightarrow \infty$ in (5.25) and (5.27), we obtain

$$(5.28) \quad \left\| KH_\lambda(K) - KS_{n-1}(K; \lambda) \right\|_{S(2)}, \quad \left\| H_\lambda(K)K - S_{n-1}(K; \lambda)K \right\|_{S(2)} \leq$$
$$\left(\left\| K \right\|_{S(2)} \right)^2 [n!]^{-1/2} \left(\left\| K \right\|_{S(2)} |\lambda| \right)^{n-1} r \left(\left\| K \right\|_{S(2)} |\lambda| \right)$$
$$(n > 1).$$

Consequently, $\lim_{n \rightarrow \infty} [KS_{n-1}(K; \lambda)](r, r', \theta) = [KH_\lambda(K)](r, r', \theta)$ and

$\lim_{n \rightarrow \infty} [S_{n-1}(K; \lambda)K](r, r', \theta) = [H_\lambda(K)K](r, r', \theta)$. After letting $n \rightarrow \infty$ in

$$(5.29) \quad S_n(K; \lambda)(r, r', \theta) - K(r, r', \theta) = \lambda [KS_{n-1}(K; \lambda)](r, r', \theta) =$$

$$\lambda [S_{n-1}(K; \lambda)K](r, r', \theta) \quad (\lambda \in C, \lambda \neq 0),$$

we arrive at the Fredholm-Resolvent Equations (5.26). Q.E.D.

5.5 SOLUTIONS OF RADIAL $A^2(D)$ - VOLTERRA INTEGRAL EQUATIONS OF SECOND KIND.

Theorem 5.8 guarantees that the Neumann-Series $H_\lambda(K)(r, r', \theta) = \sum_{n=0}^{\infty} \lambda^n K^{n+1}(r, r', \theta)$ represents the Fredholm-Resolvent Kernel of every

$K(r, r', \theta) \in (VK_2)(D)$ for all $\lambda \in C$. If we write Fredholm-Resolvent Equations

(5.26) in operator form, namely $H_\lambda(K) - K = \lambda [KH_\lambda(K)] = \lambda[H_\lambda(K)K]$ ($\lambda \neq 0$),

then $[I + \lambda H_\lambda(K)] [I - \lambda K] = [I - \lambda K] [I + \lambda H_\lambda(K)] = I$ ($\lambda \neq 0$).

Obviously, equation $f - \lambda Kf = g$ shall have the unique solution $f = [I + \lambda H_\lambda(K)]g =$

$g + \lambda H_\lambda(K)g$, which leads us to the following theorem.

Theorem 5.9. If $K(r, r', \theta) \in (VK_2)(D)$, then the linear radial $A^2(D)$ -Volterra Integral Equation

$$(5.30) \quad f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r K(r, r', \theta) f(r'e^{i\theta}) e^{i\theta} dr, \quad (g \in A^2(D), \lambda \neq 0)$$

a.e. in r on $[0, 1]$ ($0 \leq \theta < 2\pi$), where $f \in A^2(D)$ is unknown, possesses the unique solution

$$(5.31) \quad f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r H_\lambda(K)(r, r', \theta) g(r' e^{i\theta}) e^{i\theta} dr \quad (\lambda \neq 0)$$

a.e. in r on $[0,1]$ ($0 \leq \theta < 2\pi$) for all $A^2(D)$ -functions g .

Proof. The immediately preceding paragraph justifies this conclusion.

Q.E.D.

Corollary 5.10. If $K(r, r', \theta) \in (VK_2)(D)$, then the solution (5.31) of the linear radial $A^2(D)$ -Volterra Integral Equation (5.30) has the form

$$(5.32) \quad f(re^{i\theta}) = g(re^{i\theta}) + \sum_{n=0}^{\infty} \lambda^{n+1} (K^{n+1} g)(re^{i\theta})$$

a.e. in r on $[0,1]$ ($0 \leq \theta < 2\pi$), with estimates

$$(5.33) \quad |(K^{n+1} g)(re^{i\theta})| \leq [(n-1)!]^{-1/2} \left(\left\| K \right\|_{S(2)} \right)^n k_1(r, \theta) \times \\ \left[\int_0^1 |g(r' e^{i\theta})|^2 dr' \right]^{1/2} \quad \left\{ \begin{array}{l} 0 \leq r < 1, 0 \leq \theta < 2\pi; n \geq 1 \end{array} \right.$$

and

$$(5.34) \quad \|K^{n+1} g\| \leq [(n+1)!]^{-1/2} \left(\left\| K \right\|_{S(2)} \right)^{n+1} \|g\| \quad (n \geq 0).$$

Proof. The series representation of the solution (5.31) follows out of the Neumann-Series representation (5.22) of $H_\lambda(K)(r, r', \theta)$. (5.33)

is a direct consequence of $\left| \int_0^r K^{n+1}(r, r', \theta) g(r' e^{i\theta}) e^{i\theta} dr' \right|$

$\left[\int_0^r |K^{n+1}(r, r', \theta)|^2 dr' \right]^{1/2} \left[\int_0^1 |g(r' e^{i\theta})|^2 dr' \right]^{1/2}$ and Lemma 5.5 ((5.15)).

(5.34) follows out of Lemma 5.3, wherein $\left[\int_0^r k_1^2(r', \theta) dr' \right]^{1/2}$ and the $L_2[0, 1]$ -

function f are replaced by $\|K\|_{S(2)}$ ((5.8) ($i = 1$)) and $g(\cdot e^{i\theta})$ respectively.

Q.E.D.

As result of Corollary 5:10 (specifically (5.34)), we can estimate the error, which arises if we approximate the solution of the linear radial

$A^2(D)$ -Volterra Integral Equation with the $A^2(D)$ -functions

$$(5.35) \quad f_n(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r S_n(K; \lambda)(r, r', \theta) g(r' e^{i\theta}) e^{i\theta} dr'$$

a.e. in r on $[0, 1]$ ($0 \leq \theta < 2\pi$), where $S_n(K; \lambda)(r, r', \theta)$ is the n -th partial sum

of the Fredholm-Resolvent Kernel $H_\lambda(K)(r, r', \theta)$ written in the Neumann-Series

form (5.22), as follows:

$$(5.36) \quad ||f - f_n|| \leq [(n+1)!]^{-1/2} (|||K|||_{S(2)} |\lambda|)^{n+1} r (|||K|||_{S(2)} |\lambda|)$$

\int

$$(n \geq 0).$$

These error estimates follow from

$$(5.37) \quad |f(re^{i\theta}) - f_n(re^{i\theta})| \leq |\lambda| \left[\int_0^r |H_\lambda(K)(r, r', \theta) - S_n(K; \lambda)(r, r', \theta)|^2 dr' \right]^{1/2} \times$$

$$\left[\int_0^1 |g(r'e^{i\theta})|^2 dr' \right]^{1/2} \quad (0 \leq r < 1, 0 \leq \theta < 2\pi, n \geq 0),$$

which in turn imply

$$(5.38) \quad \int_0^1 |f(re^{i\theta}) - f_n(re^{i\theta})|^2 dr =$$

$$|\lambda|^2 \left[\int_0^1 \int_0^r |H_\lambda(K)(r, r', \theta) - S_n(K; \lambda)(r, r', \theta)|^2 dr' dr \right] \left[\int_0^1 |g(r'e^{i\theta})|^2 dr' \right] \leq$$

$$|\lambda|^2 (|||H_\lambda(K) - S_n(K; \lambda)|||_{S(2)})^2 \left[\int_0^1 |g(r'e^{i\theta})|^2 dr' \right] \quad (0 \leq \theta < 2\pi).$$

By letting $q \rightarrow \infty$ in (5.25), we infer

$$(5.39) \quad |||H_\lambda(K) - S_n(K; \lambda)|||_{S(2)} \leq$$
$$|||K|||_{S(2)} [(n+1)!]^{-1/2} (|||K|||_{S(2)} |\lambda|)^{n+1} r(|||K|||_{S(2)} |\lambda|) \quad (n \geq 0),$$

and this combined with (5.38) leads to

$$(5.40) \quad ||f - f_n||^2 = \int_0^{2\pi} \int_0^1 |f(re^{i\theta}) - f_n(re^{i\theta})|^2 dr d\theta \leq$$
$$[(n+1)!]^{-1} (|||K|||_{S(2)} |\lambda|)^{2(n+1)} [r(|||K|||_{S(2)} |\lambda|)]^2 \times$$
$$\int_0^{2\pi} \int_0^1 |g(re^{i\theta})|^2 dr d\theta \quad (n \geq 0).$$

If we replace $\int_0^{2\pi} \int_0^1 |g(re^{i\theta})|^2 dr d\theta$ by $||g||^2$ in (5.40), then the error estimates

(5.36) shall immediately follow.

5.6 EXAMPLE OF RADIAL $A^2(D)$ -VOLTERRA INTEGRAL EQUATION OF SECOND KIND.

An example of a linear radial $A^2(D)$ -Volterra Integral Equation is

$$(5.41) \quad f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r \exp((r-r')e^{i\theta}) f(r'e^{i\theta}) e^{i\theta} dr' \quad (g \in A^2(D), \lambda \in \gamma)$$

a.e. in r on $[0,1]$ ($0 \leq \theta < 2\pi$), where $f \in A^2(D)$ is unknown. Equation (5.41).

is of the form (5.1) with $(VK_2)(D)$ -kernel

$$(5.42) \quad K(r, r', \theta) = \exp((r-r')e^{i\theta}) (r' \leq r) \text{ and } 0(r < r') \quad (0 \leq r, r' < 1; 0 \leq \theta < 2\pi),$$

$$\text{because } |||K|||_{S(2)} = \sup_{0 \leq \theta < 2\pi} \left[\int_0^1 \int_0^r \exp(2(r-r')\cos\theta) dr' dr \right]^{1/2} < (2^{-1}e)^{1/2},$$

$(Kf)(z) = e^z \int_0^z e^{-w} f(w) dw$ is an analytic function on D (Morera's Theorem) and

$$||Kf|| \leq (2^{-1}e)^{1/2} ||f|| \text{ for all } f \in A^2(D). \text{ This } (VK_2)(D)-\text{kernel } K(r, r', \theta)$$

has iterates

$$(5.43) \quad K^{n+1}(r, r', \theta) = [n!]^{-1} (r-r')^n e^{in\theta} \exp((r-r')e^{i\theta}) (r' \leq r) \text{ and } 0(r < r')$$

$$(0 \leq r, r' < 1; 0 \leq \theta < 2\pi; n > 0).$$

With the help of Theorems 5.7 and 5.8 the Fredholm-Resolvent Kernel of $(VK_2)(D)$ -

kernel (5.42) is

$$(5.44) \quad H_\lambda(K)(r, r', \theta) = \sum_{n=0}^{\infty} [n!]^{-1} \lambda^n (r - r')^n e^{in\theta} \exp((r - r')e^{i\theta}) \quad (r' \leq r) \text{ and}$$

$$0(r < r') \quad (0 \leq r, r' < 1; \quad 0 \leq \theta < 2\pi)$$

$$= \exp((1+\lambda)(r - r')e^{i\theta}) \quad (r' \leq r) \text{ and } 0(r < r') \quad (0 \leq r, r' < 1;$$

$$0 \leq \theta < 2\pi).$$

Therefore, the solution of equation (5.41)(Theorem 5.9) is

$$(5.45) \quad f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r \exp((1+\lambda)(r - r')e^{i\theta}) g(r'e^{i\theta}) e^{i\theta} dr'$$

for all $g \in A^2(D)$ and $\lambda \in C$ ($\lambda \neq 0$).

CHAPTER VI

NON-LINEAR RADIAL VOLTERRA $A^2(D)$ -INTEGRAL EQUATIONS OF GENERAL TYPE

This chapter treats the solutions of the non-linear radial $A^2(D)$ -Volterra Integral Equations of the unknown $A^2(D)$ -function f of the general type

$$(6.1) \quad f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r F(re^{i\theta}, r'e^{i\theta}; f(r'e^{i\theta})) e^{i\theta} dr, \quad (g \in A^2(D), \lambda \in C)$$

in r on $[0,1]$ ($0 \leq \theta \leq 2\pi$). We establish for this equation the following: first, that a prescribed "iteration scheme" converges; second, that the limit of this iteration scheme is a solution; and third, that the solutions of these radial integral equations are unique.

6.1 $L_2[a,b]$ -VOLTERRA INTEGRAL EQUATION OF GENERAL TYPE.

The method described by F.G. Tricomi [7, p.42-47] for solving the non-linear $L_2[a,b]$ -Volterra Integral Equation

$$(6.2) \quad x(s) = y(s) + \lambda \int_0^s F(s,t;x(t)) dt \quad (y \in L_2[a,b], \lambda \in C)$$

a.e. in s on $[a,b]$, where $x \in L_2[a,b]$ is unknown, requires $F(s,t;w)$ to satisfy the "Lipschitz-condition"

$$(6.3) \quad |F(s,t;w_1) - F(s,t;w_2)| \leq A(s,t)|w_1 - w_2|$$

for almost all $(s,t) \in [a,b] \times [a,b]$, where $A(s,t)$ is an L_2 -kernel on

$[a,b] \times [a,b]$ and $\int_a^s F(s,t;y(t))dt$ defines an $L_2[a,b]$ -function of the variable

s . Since we are dealing with the analytic function space $A^2(D)$ and our integra-

tion is along the straight line segment from 0 to $re^{i\theta}$, the function $F(z,w;\eta)$

appearing in (6.1) must be an entire function of the variable η . If we impose

upon $F(z,w;\eta)$ a "Lipschitz-condition" in variable η , namely

$$|F(z,w;\eta_1) - F(z,w;\eta_2)| \leq A(z,w)|\eta_1 - \eta_2| \quad (z,w \in D; \eta_1, \eta_2 \in C), \text{ then by}$$

Liouville's Theorem (on entire functions) $F(z,w;\eta) = a_0(z,w) + a_1(z,w)\eta$. This

reduces (6.1) to the linear radial $A^2(D)$ -Volterra Integral Equation (5.1), which

was extensively dealt with at the end of the previous chapter. Therefore, the

method utilizing a direct "Lipschitz-condition" [7,p.42-47] for the non-linear

radial $A^2(D)$ -Volterra Integral Equation (6.1) fails and hence, modification must

be introduced.

Further, we shall in this chapter as well as in the subsequent one

consistently use the well-established Minkowski Integral Inequality, which states for the pair of σ -finite measure spaces (X, M_X, μ_X) and (Y, M_Y, μ_Y)

the following. If $f: X \times Y \rightarrow C$ is $M_X \times M_Y$ -measurable, then

$$(6.4) \quad \left[\int_X \left[\int_Y |f(s, t)|^p d\mu_Y(t) \right]^{1/p} d\mu_X(s) \right]^{1/p} \leq \int_Y \left[\int_X |f(s, t)|^p d\mu_X(s) \right]^{1/p} d\mu_Y(t) \\ (1 < p < \infty).$$

Since $F(z, w; f(w))$ is dependent upon the two complex variables $(z, w) \in D \times D$, we introduce the following two definitions.

6.2 RADIALLY MODIFIED LIPSCHITZ CONDITION FOR NON-LINEAR RADIAL $A^2(D)$ - VOLTERRA INTEGRAL EQUATION OF GENERAL TYPE.

Definition 6.1. $H(D \times D)$ denotes the vector space of analytic functions of two complex variables (z, w) on $D \times D$.

Definition 6.2. The transformation $F: A^2(D) \rightarrow H(D \times D)$ with $F: f(w) \mapsto F(z, w; f(w))$ is said to be 2-admissible if F induces a radially acting Volterra Integral Operator of general-type $\Phi_F: A^2(D) \rightarrow A^2(D)$ with

$$(6.5) \quad \Phi_F(f)(re^{i\theta}) = \int_0^r F(re^{i\theta}, r'e^{i\theta}; f(r'e^{i\theta})) e^{i\theta} dr \quad (0 \leq r < 1, 0 \leq \theta < 2\pi)$$

and F satisfies the radially modified $A(r, r')$ -Lipschitz-condition

$$(6.6) \quad \left[\int_0^{2\pi} |F(re^{i\theta}, r'e^{i\theta}; g_2(r'e^{i\theta})) - F(re^{i\theta}, r'e^{i\theta}; g_1(r'e^{i\theta}))|^2 d\theta \right]^{1/2} \leq A(r, r') \left[\int_0^{2\pi} |g_2(r'e^{i\theta}) - g_1(r'e^{i\theta})|^2 d\theta \right]^{1/2} \quad (g_1, g_2 \in A^2(D))$$

for almost all $(r, r') \in [0, 1] \times [0, 1]$, where $A(r, r')$ is an L_2 -kernel on $[0, 1] \times [0, 1]$.

6.3 CONVERGENCE SCHEME FOR NON-LINEAR RADIAL $A^2(D)$ -VOLTERRA INTEGRAL EQUATIONS OF GENERAL TYPE.

Theorem 6.3. If $F: A^2(D) \rightarrow H(D \times D)$ with $F: f(w) \mapsto F(z, w; f(w))$

is 2-admissible with the radially modified $A(r, r')$ -Lipschitz-condition, then

the iteration scheme $f_0 = g$ and

$$(6.7) \quad f_{n+1}(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r F(re^{i\theta}, r'e^{i\theta}; f_n(r'e^{i\theta})) e^{i\theta} dr \quad (n \geq 0)$$

converges in $A^2(D)$ for all $\lambda \in C$.

Proof. Since $A(r, r')$ is a non-negative L_2 -kernel on $[0, 1] \times [0, 1]$, we let $A^2(r, r') = |A(r, r')|^2$ and define for all r ($0 \leq r < 1$)

$$(6.8) \quad A(r) = \left[\int_0^r A^2(r, r') dr' \right]^{1/2}, \quad A^2(r) = |A(r)|^2, \quad A = \left[\int_0^1 A^2(r) dr \right]^{1/2} < \infty.$$

We note $f_1 - f_0 = \lambda \phi_F(g)$ ((6.5) and (6.7)) and calculate

$$(6.9) \quad \left[\int_0^{2\pi} |f_1(re^{i\theta}) - f_0(re^{i\theta})|^2 d\theta \right]^{1/2} \leq |\lambda| \left[\int_0^{2\pi} |\phi_F(g)(re^{i\theta})|^2 d\theta \right]^{1/2}$$

$(0 \leq r < 1)$.

Further, we contend and shall prove inductively on n , that

$$(6.10) \quad \left[\int_0^{2\pi} |f_{n+1}(re^{i\theta}) - f_n(re^{i\theta})|^2 d\theta \right]^{1/2} \leq \\ ||\phi_F(g)|| |\lambda|^{n+1} [(n-1)!]^{-1/2} A(r) \left[\int_0^r A^2(u) du \right]^{(n-1)/2} \quad (0 \leq r < 1; n \geq 1).$$

We prove estimate (6.10) ($n=1$) by writing $f_2 - f_1 = \lambda [\phi_F(f_1) - \phi_F(f_0)]$ ((6.5))

and by resorting to the Minkowski Integral Inequality (6.4), the radially modified $A(r, r')$ -Lipschitz-condition (6.6), (6.9), the Cauchy-Schwarz Inequality and quantities (6.8). These justify

$$(6.11) \quad \left[\int_0^{2\pi} |f_2(re^{i\theta}) - f_1(re^{i\theta})|^2 d\theta \right]^{1/2} =$$

$$\begin{aligned}
 & |\lambda| \left[\int_0^{2\pi} \left| \int_0^r [F(re^{i\theta}, r'e^{i\theta}; f_1(r'e^{i\theta})) \right. \right. \\
 & \quad \left. \left. - F(re^{i\theta}, r'e^{i\theta}; f_0(r'e^{i\theta})) \right|^2 e^{i\theta} dr' \right|^{1/2} d\theta \leq \\
 & |\lambda| \int_0^r \left[\int_0^{2\pi} |F(re^{i\theta}, r'e^{i\theta}; f_1(r'e^{i\theta})) \right. \\
 & \quad \left. - F(re^{i\theta}, r'e^{i\theta}; f_0(r'e^{i\theta}))|^2 d\theta \right]^{1/2} dr' \leq \\
 & |\lambda| \int_0^r A(r, r') \left[\int_0^{2\pi} |f_1(r'e^{i\theta}) - f_0(r'e^{i\theta})|^2 d\theta \right]^{1/2} dr' \leq \\
 & |\lambda|^2 \int_0^r A(r, r') \left[\int_0^{2\pi} |\Phi_F(g)(r'e^{i\theta})|^2 d\theta \right]^{1/2} dr' \leq \\
 & |\lambda|^2 \left[\int_0^r A^2(r, r') dr' \right]^{1/2} \left[\int_0^r \int_0^{2\pi} |\Phi_F(g)(r'e^{i\theta})|^2 d\theta dr' \right]^{1/2} \leq \\
 & \|\Phi_F(g)\| |\lambda|^2 A(r).
 \end{aligned}$$

Hence, inequality (6.10) holds for $n=1$.

If (6.10) is valid for $n=m$, then $f_{m+2} - f_{m+1} =$

$\lambda [\Phi_F(f_{m+1}) - \Phi_F(f_m)]$ ((6.5)). By repeating the arguments used for (6.11),

we obtain

$$\begin{aligned}
 (6.12) \quad & \left[\int_0^{2\pi} |f_{m+2}(re^{i\theta}) - f_{m+1}(re^{i\theta})|^2 d\theta \right]^{1/2} = \\
 & |\lambda| \left[\int_0^{2\pi} \left| \int_0^r [F(re^{i\theta}, r'e^{i\theta}; f_{m+1}(r'e^{i\theta})) \right. \right. \\
 & \quad \left. \left. - F(re^{i\theta}, r'e^{i\theta}; f_m(r'e^{i\theta})) \right|^2 e^{i\theta} dr' \right|^{1/2} d\theta \leq \\
 & |\lambda| \int_0^r \left[\int_0^{2\pi} |F(re^{i\theta}, r'e^{i\theta}; f_{m+1}(r'e^{i\theta})) \right. \\
 & \quad \left. - F(re^{i\theta}, r'e^{i\theta}; f_m(r'e^{i\theta}))|^2 d\theta \right]^{1/2} dr' \leq \\
 & |\lambda| \int_0^r A(r, r') \left[\int_0^{2\pi} |f_{m+1}(r'e^{i\theta}) - f_m(r'e^{i\theta})|^2 d\theta \right]^{1/2} dr' \leq \\
 & ||\Phi_F(g)|| |\lambda|^{m+2} \int_0^r A(r, r') [(m-1)!]^{-1/2} A(r') \left[\int_0^{r'} A^2(u) du \right]^{(m-1)/2} dr' \leq \\
 & ||\Phi_F(g)|| |\lambda|^{m+2} \left[\int_0^r A^2(r, r') dr' \right]^{1/2} \times \\
 & \left(\int_0^r [(m-1)!]^{-1} \left[\int_0^{r'} A^2(u) du \right]^{m-1} A^2(r') dr' \right)^{1/2} = \\
 & ||\Phi_F(g)|| |\lambda|^{m+2} [m!]^{-1/2} A(r) \left[\int_0^r A^2(u) du \right]^{m/2} \quad (0 \leq r < 1),
 \end{aligned}$$

where we used (6.10) ($n=m$) in the third step; therefore, (6.10) is valid for
 $n = m + 1$.

We now integrate the squares of both sides of inequality (6.10) with respect to variable r on $[0,1]$ and thus (via (3.1))

$$(6.13) \quad \|f_{n+1} - f_n\| \leq \|\Phi_F(g)\| |\lambda| [n!]^{-1/2} (|\lambda|A)^n \quad (n \geq 0).$$

Utilizing this result ((6.13)) and the triangle inequality for the norm $\|\cdot\|$,

$$\text{we conclude } \|f_{n+q} - f_n\| \leq \sum_{j=0}^{q-1} \|f_{n+j+1} - f_{n+j}\| \leq$$

$$\sum_{j=0}^{\infty} \|\Phi_F(g)\| |\lambda| [(n+j)!]^{-1/2} (|\lambda|A)^{n+j} \leq$$

$$\|\Phi_F(g)\| |\lambda| [n!]^{-1/2} (|\lambda|A)^n \Gamma(|\lambda|A) \text{ i.e.}$$

$$(6.14) \quad \|f_{n+q} - f_n\| \leq \|\Phi_F(g)\| |\lambda| [n!]^{-1/2} (|\lambda|A)^n \Gamma(|\lambda|A) \quad (n, q \geq 0),$$

where Γ is the entire function defined by (5.21). Consequently, $\{f_n\}_{n=0}^{\infty}$ is

a Cauchy-sequence of $A^2(D)$ -functions with $\lim_{n \rightarrow \infty} f_n(z) = f(z)$ ($z \in D$) for some $f \in A^2(D)$ (Theorem 3.1). Q.E.D.

6.4 EXISTENCE OF SOLUTION OF NON-LINEAR RADIAL $A^2(D)$ -VOLTERRA INTEGRAL EQUATION OF GENERAL TYPE.

Theorem 6.4. Let $F: A^2(D) \rightarrow H(D \times D)$ with $F: f(w) \mapsto F(z, w; f(w))$ be 2-admissible, $g \in A^2(D)$ and $f = \lim_{n \rightarrow \infty} f_n$ ($f_0 = g$, $f_{n+1} = g + \lambda \Phi_F(f_n)$ $n \geq 0$)

be given. Then,

$$(6.15) \quad f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r F(re^{i\theta}, r'e^{i\theta}; f(r'e^{i\theta})) e^{i\theta} dr, \quad (re^{i\theta} \in D)$$

i.e. the iteration scheme of Theorem 6.3 converges to a solution of the non-linear radial $A^2(D)$ -Volterra Integral Equation of general-type.

Proof. (6.5) and (6.7) enable us to write

$$(6.16) \quad f - g - \lambda \Phi_F(f) = f - f_{n+1} - \lambda [\Phi_F(f) - \Phi_F(f_n)]$$

and the Minkowski Integral Inequality (6.4) ($p=2$) and (6.6) imply

$$\begin{aligned}
 (6.17) \quad & \left[\int_0^{2\pi} |\Phi_F(f)(re^{i\theta}) - \Phi_F(f_n)(re^{i\theta})|^2 d\theta \right]^{1/2} = \\
 & \left[\int_0^{2\pi} \left| \int_0^r [F(re^{i\theta}, r'e^{i\theta}; f(r'e^{i\theta})) \right. \right. \\
 & \quad \left. \left. - F(re^{i\theta}, r'e^{i\theta}; f_n(r'e^{i\theta})) \right|^2 e^{i\theta} dr' \right|^{1/2} d\theta \right]^{1/2} \leq \\
 & \int_0^r \left[\int_0^{2\pi} |F(re^{i\theta}, r'e^{i\theta}; f(r'e^{i\theta})) \right. \right. \\
 & \quad \left. \left. - F(re^{i\theta}, r'e^{i\theta}; f_n(r'e^{i\theta}))|^2 d\theta \right]^{1/2} dr' \leq \\
 & \int_0^r A(r, r') \left[\int_0^{2\pi} |f(r'e^{i\theta}) - f_n(r'e^{i\theta})|^2 d\theta \right]^{1/2} dr' \leq \\
 & A(r) ||f - f_n|| \quad (n \geq 0),
 \end{aligned}$$

where the last step follows out of the Cauchy-Schwarz Inequality. This means

$$(6.18) \quad \left[\int_0^{2\pi} |\Phi_F(f)(re^{i\theta}) - \Phi_F(f_n)(re^{i\theta})|^2 d\theta \right]^{1/2} \leq$$

$$||f - f_n|| A(r) \quad (0 \leq r < 1, n \geq 0),$$

which (via (6.16)) leads to

$$(6.19) \quad \left[\int_0^{2\pi} |f(re^{i\theta}) - g(re^{i\theta}) - \lambda \Phi_F(f)(re^{i\theta})|^2 d\theta \right]^{1/2} \leq \\ \left[\int_0^{2\pi} |f(re^{i\theta}) - f_{n+1}(re^{i\theta})|^2 d\theta \right]^{1/2} + |\lambda| \|f - f_n\| A(r) \\ (0 \leq r < 1, n \geq 0).$$

Because the second term of (6.19) is the sum of two $L_2[0,1]$ functions of the variable r , the triangle inequality for the $L_2[0,1]$ -norm yields

$$(6.20) \quad \|f - g - \lambda \Phi_F(f)\| = \\ \left(\int_0^1 \left[\int_0^{2\pi} |f(re^{i\theta}) - g(re^{i\theta}) - \lambda \Phi_F(f)(re^{i\theta})|^2 d\theta \right] dr \right)^{1/2} \leq \\ \|f - f_{n+1}\| + |\lambda| \|f - f_n\| A \quad (n \geq 0),$$

where the constant A is given by (6.8). Theorem 6.3 asserts $\lim_{n \rightarrow \infty} \|f - f_n\| = 0$;

hence, (6.20) implies $f = g + \lambda \Phi_F(f)$. Q.E.D.

6.5 UNIQUENESS OF SOLUTION OF NON-LINEAR RADIAL $A^2(D)$ -VOLTERRA INTEGRAL EQUATION OF GENERAL TYPE.

Theorem 6.5. If $F: A^2(D) \rightarrow H(D \times D)$ with $F: f(w) \mapsto F(z, w; f(w))$ is 2-admissible and $g \in A^2(D)$, then the non-linear radial $A^2(D)$ -Volterra Integral Equation of general type

$$(6.21) \quad f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r F(re^{i\theta}, r'e^{i\theta}; f(r'e^{i\theta})) e^{i\theta} dr, \quad (re^{i\theta} \in D)$$

possesses a unique solution in $A^2(D)$.

Proof. The existence of at least one solution of (6.21) is guaranteed by Theorem 6.4. Let $h \in A^2(D)$ be another solution of (6.21)-i.e.- $h = g + \lambda \Phi_F(h)$.

The Minkowski Integral Inequality (6.4) ($p=2$), (6.6) and $f-h = \lambda [\Phi(f)-\Phi(h)]$

permits us to write

$$(6.22) \quad \left[\int_0^{2\pi} |f(re^{i\theta}) - h(re^{i\theta})|^2 d\theta \right]^{1/2} = \\ |\lambda| \left[\int_0^{2\pi} |\Phi_F(f)(re^{i\theta}) - \Phi_F(h)(re^{i\theta})|^2 d\theta \right]^{1/2} \leq$$

$$|\lambda| \int_0^r \left[\int_0^{2\pi} |F(re^{i\theta}, r'e^{i\theta}; f(r'e^{i\theta}))\right.$$

$$\left. - F(re^{i\theta}, r'e^{i\theta}; h(r'e^{i\theta}))|^2 d\theta \right]^{1/2} dr' \leq$$

$$|\lambda| \int_0^r A(r, r') \left[\int_0^{2\pi} |f(r'e^{i\theta}) - h(r'e^{i\theta})|^2 d\theta \right]^{1/2} dr' \quad (0 \leq r < 1);$$

in particular,

$$(6.23) \quad \left[\int_0^{2\pi} |f(re^{i\theta}) - h(re^{i\theta})|^2 d\theta \right]^{1/2} \leq$$

$$|\lambda| \int_0^r A(r, r') \left[\int_0^{2\pi} |f(r'e^{i\theta}) - h(r'e^{i\theta})|^2 d\theta \right]^{1/2} dr' \quad (0 \leq r < 1).$$

Applying inequality (6.23) recursively to itself leads to

$$(6.24) \quad \left[\int_0^{2\pi} |f(re^{i\theta}) - h(re^{i\theta})|^2 d\theta \right]^{1/2} \leq$$

$$|\lambda|^n \int_0^r A(r, r_1) \int_0^{r_1} A(r_1, r_2) \cdots \int_0^{r_{n-1}} A(r_{n-1}, r_n) \times$$

$$\left[\int_0^{2\pi} |f(r_n e^{i\theta}) - h(r_n e^{i\theta})|^2 d\theta \right]^{1/2} dr_n dr_{n-1} \cdots dr_2 dr_1 \leq$$

$$|\lambda|^n \int_0^r A(r, r_1) \int_0^{r_1} A(r_1, r_2) \cdots \int_0^{r_{n-2}} A(r_{n-2}, r_{n-1}) A(r_{n-1}) \times$$

$$||f-h|| dr_{n-1} dr_{n-2} \cdots dr_2 dr_1 \leq$$

$$|\lambda|^n ||f-h|| \int_0^r A(r, r_1) \int_0^{r_1} A(r_1, r_2) \cdots \int_0^{r_{n-3}} A(r_{n-3}, r_{n-2}) \times$$

$$\left[\left[\int_0^{r_{n-2}} A^2(u) du \right] A^2(r_{n-2}) \right]^{1/2} dr_{n-2} dr_{n-3} \cdots dr_2 dr_1 \leq$$

$$|\lambda|^n ||f-h|| \int_0^r A(r, r_1) \int_0^{r_1} A(r_1, r_2) \cdots \int_0^{r_{n-4}} A(r_{n-4}, r_{n-3}) \times$$

$$\left[[2!]^{-1} \left[\int_0^{r_{n-3}} A^2(u) du \right]^2 A^2(r_{n-3}) \right]^{1/2} dr_{n-3} dr_{n-4} \cdots dr_2 dr_1 \leq$$

$$\cdots \leq |\lambda|^n ||f-h|| \left[[(n-1)!]^{-1} \left[\int_0^r A^2(u) du \right]^{n-1} A^2(r) \right]^{1/2}$$

In the first step of (6.24) we used $\int_0^{r_{n-1}} A(r_{n-1}, r_n) \times$

$$\left[\int_0^{2\pi} |f(r_n e^{i\theta}) - h(r_n e^{i\theta})|^2 d\theta \right]^{1/2} dr_{n-1} \leq ||f-h|| A(r_{n-1}), \quad (0 \leq r_{n-1} < 1),$$

which follows out of the Cauchy-Schwarz Inequality. All the other steps of

$$(6.24) \text{ are justified by } \int_0^{r_{n-k-1}} A(r_{n-k-1}, r_{n-k}) [(k-1)!]^{-1} \times$$

$$[\int_0^{r_{n-k}} A^2(u) du]^{k-1} A^2(r_{n-k})^{1/2} dr_{n-k} \leq$$

$$[(k!)^{-1} [\int_0^{r_{n-k-1}} A^2(u) du]^k A^2(r_{n-k-1})]^{1/2} \quad (0 \leq r_{n-k-1} < 1, 1 \leq k \leq n),$$

which is an immediate consequence of the Cauchy-Schwarz Inequality and the "chain rule" for integrals. Hence,

$$(6.25) \left[\int_0^{2\pi} |f(re^{i\theta}) - h(re^{i\theta})|^2 d\theta \right]^{1/2} \leq$$

$$||f-h|| |\lambda|^n [(n-1)!]^{-1} [\int_0^r A^2(u) du]^{n-1} A^2(r)^{1/2} \quad (0 \leq r < 1, n \geq 1)$$

holds. (6.25), by means of (3.1) and the "chain rule" for integrals, implies

$$(6.26) ||f-h|| \leq ||f-h|| [n!]^{-1/2} (|\lambda|A)^n \quad (n \geq 0).$$

Letting $n \rightarrow \infty$ in (6.26) yields $f = h$. Q.E.D

6.6 EXAMPLE OF NON-LINEAR RADIAL $A^2(D)$ -VOLTERRA INTEGRAL EQUATION OF GENERAL TYPE

An example of a non-linear radial $A^2(D)$ -Volterra Integral Equation of general-type is

$$(6.27) \quad f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r \exp(rr'e^{i2\theta}) \times$$

$$\left[\sum_{n=0}^{\infty} |a_n(f)| [1 + |a_n(f)|]^{-1} r^{2n} r^n e^{i3n\theta} \right] e^{i\theta} dr, \quad (re^{i\theta} \in D),$$

where $g \in A^2(D)$ is given and f is the unknown $A^2(D)$ -function. For the Theorems 6.3, 6.4 and 6.5 to be applicable to equation (6.27), we must demonstrate that the rule of correspondence

$$(6.28) \quad F: f(w) \mapsto e^{zw} \sum_{n=0}^{\infty} |a_n(f)| [1 + |a_n(f)|]^{-1} z^{2n} w^n$$

$$(f(w) = \sum_{n=0}^{\infty} a_n(f) w^n \in A^2(D))$$

is 2-admissible. This requires showing: first, $F: A^2(D) \rightarrow H(D \times D)$; second,

$\Phi_F: A^2(D) \rightarrow A^2(D)$ ((6.5)); and third, F satisfies the radially modified

$A(r, r')$ -Lipschitz-condition (6.6).

The mapping property $F: A^2(D) \rightarrow H(D \times D)$ follows immediately from

$e^{zw}, \sum_{n=0}^{\infty} |a_n(f)| [1 + |a_n(f)|]^{-1} z^{2n} w^n \in H(D \times D)$ and the fact that $H(D \times D)$

is an algebra over C .

$\Phi_F: A^2(D) \rightarrow A^2(D)$ is a direct consequence of the following argument.

If $f \in A^2(D)$, then

$$(6.29) \quad |\Phi_F(f)(re^{i\theta})| \leq \int_0^r |F(re^{i\theta}, r'e^{i\theta}; f(r'e^{i\theta}))| dr' \leq$$

$$\sum_{n=0}^{\infty} |a_n(f)| [1 + |a_n(f)|]^{-1} \int_0^r \exp(rr' \cos(2\theta)) r^{2n} r'^n dr' \leq$$

$$2e \sum_{n=0}^{\infty} [(2n+1)^{-1/2} |a_n(f)|] [(2n+1)^{-1/2} r^{3n+1}] \leq$$

$$2e \|f\| \left[\sum_{n=0}^{\infty} (2n+1)^{-1} r^{6n+2} \right]^{1/2} \quad (0 \leq r < 1).$$

Consequently,

$$(6.30) \quad |\Phi_F(f)(re^{i\theta})|^2 \leq 4e^2 \|f\|^2 \sum_{n=0}^{\infty} (2n+1)^{-1} r^{6n+2} \quad (0 \leq r < 1, 0 \leq \theta < 2\pi),$$

wherefrom it is easy to see that

$$(6.31) \quad \|\Phi_F(f)\| = \left[\int_0^1 \int_0^{2\pi} |\Phi_F(f)(re^{i\theta})|^2 d\theta dr \right]^{1/2} \leq \\ 2(2\pi)^{1/2} e \|f\| \left[\sum_{n=0}^{\infty} (2n+1)^{-1} (6n+3)^{-1} \right]^{1/2} < \infty$$

holds; therefore, $\Phi_F: A^2(D) \rightarrow A^2(D)$.

The radially modified $A(r, r')$ -Lipschitz-condition (6.6) with a suitable L_2 -kernel $A(r, r')$ is demonstrated as follows:

$$(6.32) \quad \int_0^{2\pi} |F(re^{i\theta}, r'e^{i\theta}; g_2(r'e^{i\theta})) - F(re^{i\theta}, r'e^{i\theta}; g_1(r'e^{i\theta}))|^2 d\theta \leq \\ e^{2rr'} \int_0^{2\pi} \left| \sum_{n=0}^{\infty} [|a_n(g_1)| - |a_n(g_2)|] [1 + |a_n(g_2)|]^{-1} \times \right. \\ \left. [1 + |a_n(g_1)|]^{-1} r^{2n} r'^n e^{i3n\theta} \right|^2 d\theta =$$

$$2\pi e^{2rr'} \sum_{n=0}^{\infty} (|a_n(g_1)| - |a_n(g_2)|)^2 [1 + |a_n(g_2)|]^{-2} \times$$

$$[1 + |a_n(g_1)|]^{-2} r^{4n} r'^{2n} \leq$$

$$2\pi e^{2rr'} \sum_{n=0}^{\infty} |a_n(g_2 - g_1)|^2 r'^{2n} = e^{2rr'} \int_0^{2\pi} |g_2(r'e^{i\theta}) - g_1(r'e^{i\theta})|^2 d\theta,$$

where we used $\left| |a_n(g_1)| - |a_n(g_2)| \right| \leq |a_n(g_1) - a_n(g_2)| = |a_n(g_1 - g_2)|$

and $r^{4n} \leq 1 (0 \leq r < 1, n \geq 0)$. Clearly,

$$(6.33) \quad \left[\int_0^{2\pi} |F(re^{i\theta}, r'e^{i\theta}; g_2(r'e^{i\theta})) - F(re^{i\theta}, r'e^{i\theta}; g_1(r'e^{i\theta}))|^2 d\theta \right]^{1/2} \leq \\ e^{rr'} \left[\int_0^{2\pi} |g_2(r'e^{i\theta}) - g_1(r'e^{i\theta})|^2 d\theta \right]^{1/2} \quad (0 \leq r, r' < 1);$$

in particular, $F: A^2(D) \rightarrow H(D \times D)$ defined by (6.28) satisfies the radially

modified $A(r, r')$ -Lipschitz-condition (6.6) with L_2 -kernel $A(r, r') = e^{rr'}$.

We have proved that the map $F: A^2(D) \rightarrow H(D \times D)$ with action defined by (6.28) is 2-admissible. In consequence thereof, the non-linear radial

$A^2(D)$ -Volterra Integral Equation (6.27) possesses a unique $A^2(D)$ -solution

f for each $g \in A^2(D)$ and $\lambda \in C$, where $f = \lim_{n \rightarrow \infty} f_n$ ($f_0 = g, f_{n+1} = g + \lambda \Phi_F(f_n)$)

$(n \geq 0)$.

CHAPTER VII

NON-LINEAR RADIAL VOLTERRA $A^2(D)$ -INTEGRAL EQUATION OF HAMMERSTEIN TYPE

We examine in this chapter the non-linear radial $A^2(D)$ -Volterra Integral Equation of Hammerstein-type

$$(7.1) \quad f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r K(r, r', \theta) F(r'e^{i\theta}; f(r'e^{i\theta})) e^{i\theta} dr' \quad (g \in A^2(D))$$

a.e. in r on $[0,1]$ ($0 \leq \theta < 2\pi$) of the unknown $A^2(D)$ -function f , where $K(r, r', \theta) \in (VK_2)(D)$. This equation is considerably different from the non-linear radial Volterra $A^2(D)$ -Integral Equation (6.1) of general type, because of the presence of the $(VK_2)(D)$ -kernel $K(r, r', \theta)$, a different kind of $F(z; w)$ and the "a.e. in r on $[0,1]$ ($0 \leq \theta < 2\pi$)"-value of the solution.

7.1 $L_2[a, b]$ -VOLTERRA INTEGRAL EQUATION OF HAMMERSTEIN TYPE.

As in the equation of general-type (6.1), the method outlined by F.G. Tricomi [7, p. 198-201] for solving the non-linear $L_2[a, b]$ -Volterra Integral Equation of Hammerstein-type

$$(7.2) \quad x(s) = y(s) + \lambda \int_a^s K(s,t)F(t;x(t))dt \quad (y \in L_2[a,b]).$$

a.e. in s on $[a,b]$ (x being the unknown $L_2[a,b]$ -function) cannot be applied

to equation (7.1). The difficulty is caused by requiring $F(t,n)$ to satisfy

the Lipschitz-condition $|F(t;n_2) - F(t;n_1)| \leq a(t)|n_2 - n_1|$ for all $n_1, n_2 \in C$,

where $\|a\|_\infty < \infty$. Since $K(r,r',\theta) \in VK_2(D)$, the $F(z;f(z))$ occurring in

(7.1) must simultaneously define an $A^2(D)$ -function in z ($z \in D$) and satisfy

the Lipschitz-condition $|F(z;w_1) - F(z;w_2)| \leq a(z)|w_1 - w_2|$ ($w_1, w_2 \in C$).

Thus $F(z;w) = a_0(z) + a_1(z)w$, because $F(z;w)$ must be an entire function of w if

$\int_0^r K(r,r',\theta)F(r'e^{i\theta};f(r'e^{i\theta}))e^{i\theta}dr'$ is to define an $A^2(D)$ -function of $re^{i\theta}$. This

reduces equation (7.1) to the linear $A^2(D)$ -Volterra Integral Equation (5.1),

which has been discussed in Chapter V. Therefore, the idea of a Lipschitz-condition must be suitably modified.

7.2 RADIALLY MODIFIED LIPSCHITZ-CONDITION FOR NON-LINEAR RADIAL $A^2(D)$ -VOLTERRA-INTEGRAL EQUATION OF HAMMERSTEIN TYPE.

Definition 7.1. The transformation $F: A^2(D) \rightarrow A^2(D)$ with

F: $f(w) \mapsto F(w; f(w))$ is said to be 1-admissible for the $(VK_2)(D)$ -kernel

$K(r, r', \theta)$ if it satisfies the radially modified $A(r, r')$ -Lipschitz-condition

$$(7.3) \quad \left[\int_0^{2\pi} |K(r, r', \theta)|^2 |F(r'e^{i\theta}; g_2(r'e^{i\theta})) - F(r'e^{i\theta}; g_1(r'e^{i\theta}))|^2 d\theta \right]^{1/2} \leq$$

$$A(r, r') \left[\int_0^{2\pi} |g_2(r'e^{i\theta}) - g_1(r'e^{i\theta})|^2 d\theta \right]^{1/2} \quad (0 \leq r, r' < 1; 0 \leq \theta < 2\pi)$$

for all $g_1, g_2 \in A^2(D)$, where $A(r, r')$ is an L_2 -kernel on $[0, 1] \times [0, 1]$.

If $F: A^2(D) \rightarrow A^2(D)$ with $F: f(w) \mapsto F(w; f(w))$ is 1-admissible for the $(VK_2)(D)$ -kernel $K(r, r', \theta)$, then we define the non-linear $A^2(D)$ -Volterra

Integral Operator $K_F: A^2(D) \rightarrow A^2(D)$ with

$$(7.4) \quad K_F(f)(re^{i\theta}) = \int_0^r K(r, r', \theta) F(r'e^{i\theta}; f(r'e^{i\theta})) e^{i\theta} dr'$$

a.e. in r on $[0, 1]$ ($0 \leq \theta < 2\pi$).

7.3 CONVERGENCE SCHEME FOR NON-LINEAR RADIAL $A^2(D)$ -VOLTERRA INTEGRAL EQUATION OF HAMMERSTEIN TYPE.

Theorem 7.2. If $F: A^2(D) \rightarrow A^2(D)$ with $F: f(w) \mapsto F(w; f(w))$ is

L^1 -admissible for the $(VK_2)(D)$ -kernel $K(r, r', \theta)$ and $g \in A^2(D)$, then the iteration

scheme $f_0 = g$ and

$$(7.5) \quad f_{n+1}(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r K(r, r', \theta) F(r'e^{i\theta}; f_n(r'e^{i\theta})) e^{i\theta} dr' \quad (n \geq 0)$$

a.e. in r on $[0, 1]$ ($0 \leq \theta < 2\pi$) converges in $A^2(D)$ for all $\lambda \in C$.

Proof. Using the function $A(r)$ and the quantity A defined in (6.8), which was constructed by means of the L_2 -kernel $A(r, r')$, we write $f_1 - f_0 = \lambda K_F(g)$ and note

$$(7.6) \quad \left[\int_0^{2\pi} |f_1(re^{i\theta}) - f_0(re^{i\theta})|^2 d\theta \right]^{1/2} = |\lambda| \left[\int_0^{2\pi} |K_F(g)(re^{i\theta})|^2 d\theta \right]^{1/2} \quad (0 \leq r < 1).$$

As in the proof of Theorem 6.3, we contend and prove by induction on n , that

$$(7.7) \quad \left[\int_0^{2\pi} |f_{n+1}(re^{i\theta}) - f_n(re^{i\theta})|^2 d\theta \right]^{1/2} \leq \|K_F(g)\| |\lambda|^{n+1} [(n-1)!]^{-1/2} \left[\int_0^r A^2(u) du \right]^{(n-1)/2} \quad (0 \leq r < 1, n \geq 1).$$

The equation $f_2 - f_1 = \lambda [K_F(f_1) - K_F(f_0)]$ ((7.5)), the Minkowski Integral.

Inequality (6.4) ($p=2$), (7.4), (7.6) and the Cauchy-Schwarz Inequality, entail

$$\begin{aligned}
 (7.8) \quad & \left[\int_0^{2\pi} |f_2(re^{i\theta}) - f_1(re^{i\theta})|^2 d\theta \right]^{1/2} = \\
 & |\lambda| \left[\int_0^{2\pi} \left| \int_0^r K(r, r', \theta) [F(r'e^{i\theta}; f_1(r'e^{i\theta})) \right. \right. \\
 & \quad \left. \left. - F(r'e^{i\theta}; f_0(r'e^{i\theta})) \right|^2 e^{i\theta} dr' \right|^{1/2} d\theta \leq \\
 & |\lambda| \int_0^r \left[\int_0^{2\pi} |K(r, r', \theta)|^2 |F(r'e^{i\theta}; f_1(r'e^{i\theta})) \right. \\
 & \quad \left. - F(r'e^{i\theta}; f_0(r'e^{i\theta}))|^2 d\theta \right]^{1/2} dr' \leq \\
 & |\lambda| \int_0^r A(r, r') \left[\int_0^{2\pi} |f_1(r'e^{i\theta}) - f_0(r'e^{i\theta})|^2 d\theta \right]^{1/2} dr' \leq \\
 & |\lambda|^2 \left[\int_0^r A^2(r, r') dr' \right]^{1/2} \left[\int_0^r \int_0^{2\pi} |K_F(g)(r'e^{i\theta})|^2 d\theta dr' \right]^{1/2} \leq \\
 & ||K_F(g)|| |\lambda|^2 A(r) : (0 \leq r < 1),
 \end{aligned}$$

which proves (7.7) for $n \neq 1$. If we assume (7.7) to be valid for $n = m$, then

we derive out of $f_{m+2} - f_{m+1} = \lambda [K_F(f_{m+1}) - K_F(f_m)]$ ((7.5)) and the reasons justifying (7.8) that

$$\begin{aligned}
 (7.9) \quad & \left[\int_0^{2\pi} |f_{m+2}(re^{i\theta}) - f_{m+1}(re^{i\theta})|^2 d\theta \right]^{1/2} = \\
 & |\lambda| \left[\int_0^{2\pi} \left[\int_0^r |K(r, r', \theta)|^2 |F(r'e^{i\theta}; f_{m+1}(r'e^{i\theta})) \right. \right. \\
 & \quad \left. \left. - F(r'e^{i\theta}; f_m(r'e^{i\theta})) \right|^2 e^{i\theta} dr' \right]^{1/2} \leq \\
 & |\lambda| \int_0^r \left[\int_0^{2\pi} |K(r, r', \theta)|^2 |F(r'e^{i\theta}; f_{m+1}(r'e^{i\theta})) \right. \\
 & \quad \left. - F(r'e^{i\theta}; f_m(r'e^{i\theta}))|^2 d\theta \right]^{1/2} dr' \leq \\
 & ||K_F(g)|| |\lambda|^{m+2} \int_0^r A(r, r') [(m-1)!]^{-1/2} \left[\int_0^{r'} A^2(u) du \right]^{(m-1)/2} dr' \leq \\
 & ||K_F(g)|| |\lambda|^{m+2} [m!]^{-1/2} \left[\int_0^r A^2(u) du \right]^{m/2} A(r) \quad (0 \leq r < 1),
 \end{aligned}$$

where the last step follows by repeating the last two steps of (6.12) - i.e.

the Cauchy-Schwarz Inequality and the "chain rule" for integrals. This

demonstrates (7.7) for $n = m + 1$ and (7.7) holds for all $n \geq 1$.

The arguments in the last paragraph of the proof of Theorem 6.3

modified for (7.5) ($\Phi_F(g)$ replaced by $K_F(g)$) imply (via (7.7)) that

$$(7.10). \quad ||f_{n+q} - f_n|| \leq ||K_F(g)|| |\lambda| [n!]^{-1/2} (|\lambda| A)^n \Gamma(|\lambda| A) \quad (n, q \geq 0),$$

where Γ is given by (5.21); therefore, $\lim_{n \rightarrow \infty} f_n = f$ ($f \in A^2(D)$) exists. Q.E.D.

7.4 EXISTENCE AND UNIQUENESS OF SOLUTION OF NON-LINEAR $A^2(D)$ -VOLTERRA INTEGRAL EQUATION OF HAMMERSSTEIN TYPE

Theorem 7.3. Let $F: A^2(D) \rightarrow A^2(D)$ with $F: f(w) \mapsto F(w, f(w))$ be

λ -admissible for the $(VK_2)(D)$ -kernel $K(r, r', \theta)$, $g \in A^2(D)$ and $f = \lim_{n \rightarrow \infty} f_n$

$\{f_0 = g, f_{n+1} = g + \lambda K_F(f_n) \quad (n \geq 0)\}$ be given. Then,

$$(7.11) \quad f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r K(r, r', \theta) F(r'e^{i\theta}, f(r'e^{i\theta})) e^{i\theta} dr'$$

a.e. in r on $[0, 1]$ ($0 \leq \theta < 2\pi$) - i.e. the iteration scheme of Theorem 7.2

converges to a solution of the non-linear radial Integral Equation of Hammerstein type.

Proof. This proof follows the pattern of the proof of Theorem

6.4. By means of (7.6) we write

$$(7.12) \quad [f - g - \lambda K_F(f)](re^{i\theta}) = (f - f_{n+1})(re^{i\theta}) - \lambda [K_F(f) - K_F(f_n)](re^{i\theta})$$

$$(0 \leq r < 1, 0 \leq \theta < 2\pi, n \geq 0)$$

and resort to the Minkowski Integral Inequality (6.4) ($p=2$) and (7.3), in a manner similar to the way (6.6) was used for deriving (6.17), to arrive at

$$(7.13) \quad \left[\int_0^{2\pi} |K_F(f)(re^{i\theta}) - K_F(f_n)(re^{i\theta})|^2 d\theta \right]^{1/2} \leq ||f - f_n|| A(r)$$

$$(0 \leq r < 1, n \geq 0)$$

The triangle inequality for the $L_2[0, 2\pi]$ -norm applied to the three $L_2[0, 2\pi]$ -functions in (7.12) of the variable θ (r being momentarily fixed) implies

$$(7.14) \quad \left[\int_0^{2\pi} |f(re^{i\theta}) - g(re^{i\theta}) - \lambda K_F(f)(re^{i\theta})|^2 d\theta \right]^{1/2} \leq \\ \left[\int_0^{2\pi} |f(re^{i\theta}) - f_{n+1}(re^{i\theta})|^2 d\theta \right]^{1/2} + |\lambda| \|f - f_n\| A(r) \\ (0 \leq r < 1, n \geq 0)$$

wherein the three terms define $L_2[0,1]$ -functions of the variable r . Taking the $L_2[0,1]$ -norm of both sides of (7.14) leads to (via the triangle inequality for the $L_2[0,1]$ -norm)

$$(7.15) \quad \|f - g - K_F(f)\| \leq \|f - f_{n+1}\| + A|\lambda| \|f - f_n\| \quad (n \geq 0),$$

where the constant A is given by (6.8). Letting $n \rightarrow \infty$ in (7.15) implies

$$f = g + \lambda K_F(f). \quad \text{Q.E.D.}$$

Theorem 7.4. If $F: A^2(D) \rightarrow A^2(D)$ with $F: f(w) \mapsto F(w, f(w))$ is 1-admissible for the $(VK_2)(D)$ -kernel $K(r, r', \theta)$ and $g \in A^2(D)$, then the non-linear radial $A^2(D)$ -Volterra Integral Equation of Hammerstein-type

$$(7.16) \quad f(re^{i\theta}) = g(re^{i\theta}) + \lambda \int_0^r K_F(r, r', \theta) F(r'e^{i\theta}; f(r'e^{i\theta})) e^{i\theta} dr'$$

a.e. in r on $[0, 1]$, $(0 \leq \theta < 2\pi)$, possesses a unique $A^2(D)$ -solution.

Proof. The existence of at least one solution of (7.16) is guaranteed by Theorem 7.3; hence, let h be another $A^2(D)$ -solution of (7.16)-i.e.

$h = g + \lambda K_F(h)$: By using the Minkowski Integral Inequality (6.4) ($p=2$), (7.3) and $|f - h| = \lambda [K_F(f) - K_F(g)]$ we obtain

$$(7.17) \quad \begin{aligned} & \left[\int_0^{2\pi} |f(re^{i\theta}) - h(re^{i\theta})|^2 d\theta \right]^{1/2} \leq \\ & |\lambda| \left[\int_0^{2\pi} |K_F(f)(re^{i\theta}) - K_F(h)(re^{i\theta})|^2 d\theta \right]^{1/2} \leq \\ & |\lambda| \int_0^r \left[\int_0^{2\pi} |K(r, r', \theta)|^2 |F(r'e^{i\theta}; f(r'e^{i\theta))) \right. \\ & \quad \left. - F(r'e^{i\theta}; h(r'e^{i\theta}))|^2 d\theta \right]^{1/2} dr' \leq \\ & |\lambda| \int_0^r A(r, r') \left[\int_0^{2\pi} |f(r'e^{i\theta}) - h(r'e^{i\theta})|^2 d\theta \right]^{1/2} dr', \quad (0 \leq r < 1), \end{aligned}$$

which means

$$(7.18) \quad \left[\int_0^{2\pi} |f(re^{i\theta}) - h(re^{i\theta})|^2 d\theta \right]^{1/2} \leq \\ |\lambda| \int_0^r A(r, r') \left[\int_0^{2\pi} |f(r'e^{i\theta}) - h(r'e^{i\theta})|^2 d\theta \right]^{1/2} dr' \quad (0 \leq r < 1).$$

We apply the inequality (7.18) recursively to itself. Therefrom shall emerge inequalities identical to (6.24), (6.25) and (6.26). (ϕ_F replaced by K_F). This leads to $f = h$ provided we let $n \rightarrow \infty$ in (6.26). Q.E.D.

7.5 A 1-ADMISSIBILITY CONDITION.

We now state and prove a sufficient, although not necessary, condition for a $F: A^2(D) \rightarrow A^2(D)$ with $F: f(w) \mapsto F(w; f(w))$ to be 1-admissible for a $(VK_2)(D)$ -kernel $K(r, r', \theta)$.

Theorem 7.5. Let the $(VK_2)(D)$ -kernel $K(r, r', \theta)$ be Lebesgue-measurable in (r, r', θ) on $[0, 1] \times [0, 1] \times [0, 2\pi]$ and $a(r, \theta)$ be a non-negative Lebesgue-measurable function in (r, θ) on $[0, 1] \times [0, 2\pi]$ such that

$$(7.19) \quad |F(re^{i\theta}; g_2(re^{i\theta})) - F(re^{i\theta}; g_1(re^{i\theta}))| \leq a(r, \theta) |g_2(re^{i\theta}) - g_1(re^{i\theta})|$$

a.e. in (r, θ) on $[0,1] \times [0, 2\pi]$ for all $g_1, g_2 \in A^2(D)$ and

$$(7.20) \int_0^1 \int_0^r \left[\left\| K(r, r', \cdot) a(r', \cdot) \right\|_\infty \right]^2 dr dr' < \infty.$$

Then $F: A^2(D) \rightarrow A^2(D)$ is 1-admissible for the $(VK_2)(D)$ -kernel $K(r, r', \theta)$.

Proof. Since $F: A^2(D) \rightarrow A^2(D)$, we only need to find a L_2 -kernel

$A(r, r')$ on $[0,1] \times [0,1]$ such that F satisfies the radially modified $A(r, r')$ -Lipschitz-condition (7.3) for the given $(VK_2)(D)$ -kernel $K(r, r', \theta)$. (7.20)

guarantees that

$$(7.21) \quad A(r, r') = \left\| K(r, r', \cdot) a(r', \cdot) \right\|_\infty \quad (0 \leq r, r' \leq 1)$$

is an L_2 -Kernel on $[0,1] \times [0,1]$: By using the Hölder Integral Inequality for

$L_1[0, 2\pi]$ - and $L_\infty[0, 2\pi]$ -functions, namely $\left| \int_0^{2\pi} u(\theta) v(\theta) d\theta \right| \leq \|v\|_\infty \int_0^{2\pi} |u(\theta)| d\theta$,

We deduce out of (7.19) for all $g_1, g_2 \in A^2(D)$ that

$$\begin{aligned}
 (7.22) \quad & \left[\int_0^{2\pi} |K(r, r', \theta)|^2 |F(r'e^{i\theta}, g_2(r'e^{i\theta})) \right. \\
 & \quad \left. - F(r'e^{i\theta}, g_1(r'e^{i\theta}))|^2 d\theta \right]^{1/2} \leq \\
 & \left[\int_0^{2\pi} |K(r, r', \theta)a(r', \theta)|^2 |g_2(r'e^{i\theta}) - g_1(r'e^{i\theta})|^2 d\theta \right]^{1/2} \leq \\
 & A(r, r') \left[\int_0^{2\pi} |g_2(r'e^{i\theta}) - g_1(r'e^{i\theta})|^2 d\theta \right]^{1/2} \quad (0 \leq r, r' < 1),
 \end{aligned}$$

where in the last step we made use of (7.21). Hence, F satisfies the radially modified $A(r, r')$ -Lipschitz-condition (7.3). Q.E.D.

7.6 AN EXAMPLE OF NON-LINEAR RADIAL $A^2(D)$ -VOLTERRA INTEGRAL EQUATION OF HAMMERSTEIN TYPE.

An example of a non-linear radial $A^2(D)$ -Volterra Integral Equation of Hammerstein-type is :

$$\begin{aligned}
 (7.23) \quad f(re^{i\theta}) = & g(re^{i\theta}) + \lambda \int_0^r \sin(r^2 r'^3 e^{i5\theta}) \times \\
 & \left[\sum_{n=0}^{\infty} \tan^{-1}(|a_n(f)|) r'^{4n} e^{i4n\theta} \right] e^{i\theta} dr,
 \end{aligned}$$

$$(re^{i\theta} \in D),$$

where $g \in A^2(D)$ is given and $f(z) = \sum_{n=0}^{\infty} a_n(f) z^n$ is the $A^2(D)$ -function sought

after $\text{Sin}(r^2 r'^3 e^{i5\theta}) \in (VK_2)(D)$, because

$$(7.24) \quad |\text{Sin}(r^2 r'^3 e^{i5\theta})| \leq \exp(r^2 r'^3) \quad (0 \leq r, r' < 1; 0 \leq \theta < 2\pi).$$

For Theorems 7.2 and 7.4 to be applicable to equation (7.23), we must show.

that the rule of correspondence

$$(7.25) \quad F: f(w) \mapsto \sum_{n=0}^{\infty} \tan^{-1}(|a_n(f)|) w^{4n}, \quad (w \in D)$$

is 1-admissible for the $(VK_2)(D)$ -kernel $\text{Sin}(r^2 r'^3 e^{i5\theta})$. This requires: first,

that $F: A^2(D) \rightarrow A^2(D)$; and second, that F satisfy the radially modified

$A(r, r')$ -Lipschitz-condition (7.3) for some L_2 -kernel $A(r, r')$ on $[0,1] \times [0,1]$.

$F: A^2(D) \rightarrow A^2(D)$ ((7.25)), since $\sum_{n=0}^{\infty} \tan^{-1}(|a_n(f)|) w^{4n}$ is analytic

on D for all $f(w) = \sum_{n=0}^{\infty} a_n(f) w^{4n} \in A^2(D)$ ($|\tan^{-1}(|a_n(f)|)| < 2^{-1}\pi; n \geq 0$),

$f \in A^2(D)$. Furthermore, we calculate via (3.1)

$$(7.26) \quad \int_0^{1/2\pi} \int_0^1 |F(re^{i\theta}; f(re^{i\theta}))|^2 d\theta dr = \int_0^{1/2\pi} \int_0^1 \left| \sum_{n=0}^{\infty} \operatorname{Tan}^{-1}(|a_n(f)|) r^{4n} e^{i4n\theta} \right|^2 d\theta dr =$$

$$\int_0^1 \left[\sum_{n=0}^{\infty} |\operatorname{Tan}^{-1}(|a_n(f)|)|^2 r^{8n} \right] dr \leq \sum_{n=0}^{\infty} (8n+1)^{-1} |a_n(f)|^2 <$$

$$\sum_{n=0}^{\infty} (2n+1)^{-1} |a_n(f)|^2 = \|f\|^2 < \infty,$$

where in the last step we used $|\operatorname{Tan}^{-1}(x)| \leq |x| (x \in \mathbb{R})$. This implies that

$$F: A^2(D) \rightarrow A^2(D).$$

That $F: A^2(D) \rightarrow A^2(D)$ is 1-admissible for the $(VK_2)(D)$ -kernel

$\sin(r^2 r^3 e^{i5\theta})$ follows from the following considerations. If $g_1, g_2 \in A^2(D)$,

then

$$(7.27) \quad \left[\int_0^{2\pi} |K(r, r', \theta)|^2 |F(r'e^{i\theta}; g_2(r'e^{i\theta})) - F(r'e^{i\theta}; g_1(r'e^{i\theta}))|^2 d\theta \right]^{1/2} =$$

$$\left[\int_0^{2\pi} |\sin(r^2 r^3 e^{i5\theta})|^2 \left| \sum_{n=0}^{\infty} [\operatorname{Tan}^{-1}(|a_n(g_2)|) - \operatorname{Tan}^{-1}(|a_n(g_1)|)] r'^{4n} e^{i4n\theta} \right|^2 d\theta \right]^{1/2} \leq$$

$$\exp(r^2 r'^3) \left[\int_0^{2\pi} \sum_{n=0}^{\infty} [\tan^{-1}(|a_n(g_2)|)$$

$$- \tan^{-1}(|a_n(g_1)|)] r'^{4n} e^{i4n\theta} \left[\frac{d\theta}{r'} \right]^{1/2} =$$

$$\exp(r^2 r'^3) \left[\int_0^{2\pi} \sum_{n=0}^{\infty} [|\tan^{-1}(|a_n(g_2)|) - \tan^{-1}(|a_n(g_1)|)|^2 r'^{8n}] \right]^{1/2} \leq$$

$$\exp(r^2 r'^3) \left[\int_0^{2\pi} \sum_{n=0}^{\infty} |a_n(g_2 - g_1)|^2 r'^{2n} \right]^{1/2} =$$

$$\exp(r^2 r'^3) \left[\int_0^{2\pi} |g_2(r' e^{i\theta}) - g_1(r' e^{i\theta})|^2 d\theta \right]^{1/2},$$

where in the second step we used (7.24) and the fourth step follows out of

$$|\tan^{-1}(x_1) - \tan^{-1}(x_2)| \leq |x_1 - x_2| \quad (x_1, x_2 \in \mathbb{R}) \text{ and } a_n(g_2) - a_n(g_1) = a_n(g_2 - g_1).$$

(n > 0). Thus, $F: A^2(D) \rightarrow A^2(D)$ is 1-admissible for the $(V\mathcal{K}_2)(D)$ -kernel

$\sin(r^2 r'^3 e^{i5\theta})$ and satisfies the radially modified $A(r, r')$ -Lipschitz-condition

(7.3) with $A(r, r') = \exp(r^2 r'^3)$, where $\exp(r^2 r'^3)$ is clearly an L_2 -kernel on

$[0,1] \times [0,1]$.

CHAPTER VIII

CONCLUSION

We have proved in this master's degree thesis that for radially acting linear Volterra Integral Operators, induced by a θ -parameter family of kernels $K(r, r', \theta)$ ($0 \leq \theta < 2\pi$) with uniformly bounded double norms on $L_2[0,1]$ acting on an analytic function space of the unit disc D , the suitable space is $A^2(D)$. By deriving some inherent properties of the Hilbert space $A^2(D)$, in particular Theorem 3.5, we were able to demonstrate that the totality $(VK_2)(D)$ of these kernels $K(r, r', \theta)$ constitutes a Banach-Algebra. This depended upon Theorem 3.5, in absence of which it would not have been possible to show the topological closedness of $(VK_2)(D)$.

8.1 SUMMARY OF RESULTS OF THIS THESIS.

$(VK_2)(D)$ -kernels $K(r, r', \theta)$ are quasi-nilpotent - i.e.

$$\lim_{n \rightarrow \infty} \left[\left| \left| \left| K^n \right| \right|_{S(2)} \right]^{1/n} = 0 \text{ (Corollary 5.6)} - \text{ and hence, possesses Fredholm-}$$

Resolvent Kernels $H_\lambda(K)(r, r', \theta)$, which are representable by means of the

Neumann-Series for all $\lambda \in C$ ($\lambda \neq 0$). In other words, for every $(VK_2)(D)$ -kernel

$K(r, r', \theta)$, the Fredholm-Resolvent Kernel $H_\lambda(K)(r, r', \theta)$ - (Theorems 5.7 and 5.8)

is a $(VK_2)(D)$ -valued entire function of the complex variable λ . This enabled

us to solve linear radial $A^2(D)$ -Volterra Integral Equations (5.30) in the usual way as stated by Smithies [6, Th. 2.5.2, p. 29]. In general, all of the results for Fredholm-Resolvent Kernels of Volterra L_2 -kernels on $[a,b] \times [a,b]$, as described by Smithies [6, p. 31-35], may be carried over with some modifications to the Fredholm-Resolvent Kernel $H_\lambda(K)(r,r',\theta)$ of the $(VK_2)(D)$ -kernel $K(r,r',\theta)$.

For non-linear radial $A^2(D)$ -Volterra Integral Equations of general-type (6.1) and Hammerstein-type (7.1) some radial modifications must take place. Specifically, the well-established conditions on the $F(z,w;f(w))$ and $F(w;f(w))$ given by F.G. Tricomi [7, p. 42-47, p. 198-201], cannot hold for non-linear radial $A^2(D)$ -Volterra Integral Equations because $A^2(D)$ is an analytic function space. Therefore, we require the transformation $F(z,w;f(w))$ appearing in the general-type non-linear radial $A^2(D)$ -Volterra Integral Equation to be "2-admissible" (Definition 6.2), whereas the $F(w;f(w))$ occurring in the Hammerstein-type non-linear radial $A^2(D)$ -Volterra Integral Equation must be "1-admissible for the $(VK_2)(D)$ -kernel $K(r,r',\theta)$ " (Definition 7.1). Under these admissibility-conditions, the recursive iteration schemes (Theorems 6.3 and 7.2) converge to $A^2(D)$ -solutions, and these solutions are unique for both the general-type as well as the Hammerstein-type

non-linear radial $A^2(D)$ -Volterra Integral Equations.

In the case of the Hammerstein non-linear radial $A^2(D)$ -Volterra Integral Equation, the 1-admissibility depends upon the $(VK_2)(D)$ -kernel $K(r, r', \theta)$. Therefore, Theorem 7.5 gives a sufficient (although not necessary) condition, as to when a transformation $F: A^2(D) \rightarrow A^2(D)$ is 1-admissible for the $(VK_2)(D)$ -kernel $K(r, r', \theta)$. This depended upon being able to find a Lebesgue-measurable function $a(r', \theta)$ on $[0, 1] \times [0, 2\pi]$ such that $K(r, r', \theta) \times a(r', \theta)$ is Lebesgue-measurable in the three variables (r, r', θ) on $[0, 1] \times [0, 1] \times [0, 2\pi]$ and condition (7.21) is satisfied. This raises the following question. Is $K(r, r', \theta)$ Lebesgue-measurable in (r, r', θ) on $[0, 1] \times [0, 1] \times [0, 2\pi]$ for all $(VK_2)(D)$ -kernels $K(r, r', \theta)$?

8.2 SUGGESTIONS FOR FURTHER RESEARCH.

Hence, we suggest the following possible directions for further investigation of the topic of this thesis: first, the Lebesgue-measurability of $K(r, r', \theta)$ in variables (r, r', θ) on $[0, 1] \times [0, 1] \times [0, 2\pi]$ for $K(r, r', \theta) \in (VK_2)(D)$; second, the form these $(VK_2)(D)$ -kernels $K(r, r', \theta)$ take - i.e.

$$K(r, r', \theta) = \sum_{n, m=0}^{\infty} k_{n, m} (re^{i\theta})^n (r'e^{i\theta})^m \quad (k_{n, m} \text{ complex constants}) (?)$$

examining radial $A^2(D)$ -Volterra Integro-Differential Equations; and fourth,

developing these results in the Hilbert space $A^2(D^n)$ of n complex variables

(z_1, z_2, \dots, z_n) , where $D^n = \{(z_1, z_2, \dots, z_n) : z_k \in D, (1 \leq k \leq n)\}$ or in the Hilbert

space $A^2(D_n)$, where $D_n = \{\vec{z} = (z_1, z_2, \dots, z_n) : |\vec{z}|^2 = |z_1|^2 + |z_2|^2 + \dots + |z_n|^2 \leq 1\}$, with norms.

$$(8.1) \quad \|f\|_{A^2(D^n)} = \left[\int \int \cdots \int |f(x_1 + iy_1, x_2 + iy_2, \dots, x_n + iy_n)|^2 dx_1 dy_1 dx_2 dy_2 \cdots dx_n dy_n \right]^{1/2}$$

and

$$(8.2) \quad \|f\|_{A^2(D_n)} = \left[\int \int \cdots \int |f(x_1 + iy_1, x_2 + iy_2, \dots, x_n + iy_n)|^2 dx_1 dy_1 dx_2 dy_2 \cdots dx_n dy_n \right]^{1/2}$$

respectively.

BIBLIOGRAPHY

- [1] H. BEHNKE und F. SOMMER, Theorie der analytischen Funktionen einer komplexen Veränderlichen, Grundlehren der mathematischen Wissenschaften Band 77, Springer-Verlag, Berlin (1962).
- [2] K. JÖRGENS, Lineare Integraloperatoren, Mathematische Leitfaden, B.G. Teubner, Stuttgart, (1970).
- [3] A.B. VON KEVICZKY, Die Fredholm-Calemansche Auflösungs-Methode für radialwirkende lineare Integraloperatoren mit Cauchy-Integral darstellbaren Kernen in $H^p(\Pi_+)$, Math. Nachr. (105), (1982).
- [4] _____, Eine radiale Charakterisierung von $H^p(\Pi_+)$, Math. Nachr. (98), 257-268 (1980).
- [5] W. RUDIN, Real and Complex Analysis, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York. (1966).
- [6] F. SMITHIES, Integral Equations, Cambridge Tracts in Mathematics and Mathematical Physics Vol. 49, Cambridge University Press, London (1958).
- [7] F.G. TRICOMI, Integral Equations, Interscience Publishers, Inc., New York (1967).
- [8] J. WEIDMANN, Lineare Operatoren in Hilbert Räumen, B.G. Teubner, Stuttgart, (1976).
- [9] C. VAN WINTER, Fredholm Equations on a Hilbert Space of Analytic Functions, Trans. A.M.S. 39, 103-139 (1972).
- [10] A.C. ZAANEN, Linear Analysis, Biblioteca Mathematica Vol. 2, North-Holland Publishing Co.; Amsterdam, (1964).