Unequal Mass Quarkonium Spectra in a Consistent Quark Model with Fine and Hyperfine Interactions

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A Thesis

in

The Department

of

Physics

Presented in Partial Fulfillment of the Requirements
for the degree of Master of Science at
Concordia University
Montreal, Quebec, Canada

March 1984

ABSTRACT

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Masses of all the low-lying mesons, with orbital angular momentum ≤ 2, in the uc, dc, sc, ub, sb, cb, and db systems are calculated using a consistent quark phenomenological model with hyperfine and spin-orbit interactions. Seven of these low-lying meson states are experimentally verified. Two of these are input parameters, the other five, (the F, D^{*0}, D^{*+}, B^- and B^0 mesons), are used to test the accuracy of the model. The results are in excellent agreement with experiment.

ACKNOWLEDGEMENT

I would like to thank Dr. C. S. Kalman for suggesting the problem of this thesis and for his supervision, guidance, and above all, patience during the period of preparation of this thesis.

Special thanks also go to my mother, Mrs. E. D'Souza, for typing this thesis on very short notice, and, the Physics Department for providing financial assistance.

TABLE OF CONTENTS

INTRODUCTION

The universe is not a simple system. The great diversity of the forms which matter takes is self-evident. An understanding of the structure of matter can only be attained by constructing physical models. At a first level, we find that we can organize matter into the various chemical elements, or equivalently, atoms.

The atoms, once thought to be indivisible, are found to have an internal structure. All atoms are combinations of the three particles: the ELECTRON, PROTON and NEUTRON. The neutron and proton bind together to form the nucleus of the atom. The electrons then bind with the nucleus to form the atom. A chemical reaction is an interaction of the electrons of two or more atoms. Thus, the rules for the interaction of electrons give the laws of chemistry. A theory of interactions of electrons exists; it is called quantum electrodynamics. For a more complete picture of the atom we need an understanding of the nucleus.

It is found that the protons and neutrons which form the nucleus are themselves made up of more elementary parts called QUARKS. An understanding of the nucleus then requires a theory of the interactions of quarks. Such a theory is partially complete. It is called quantum chromodynamics (QCD) and is similar in many respects to quantum electrodynamics (QED)

There are 6 types of quarks (called flavors) and, the electron is a member of a family of 6 similar particles called LEPTONS. Therefore, we can think of matter as being formed of interacting quarks and leptons. It is necessary to consider what these interactions are. There are four types of interactions known. They are the GRAVITATIONAL, ELECTRIC, WEAK and COLOR forces.

Corresponding to each force we introduce one or more "exchange quanta". These exchange quanta are emitted or absorbed by the quarks and leptons. The absorption or emission of exchange quanta is the means by which a force is transmitted (Fig.1). For each emission or absorption of exchange quanta, a quantity of momentum is transferred between the particles. We define force as the momentum transfer per unit time.

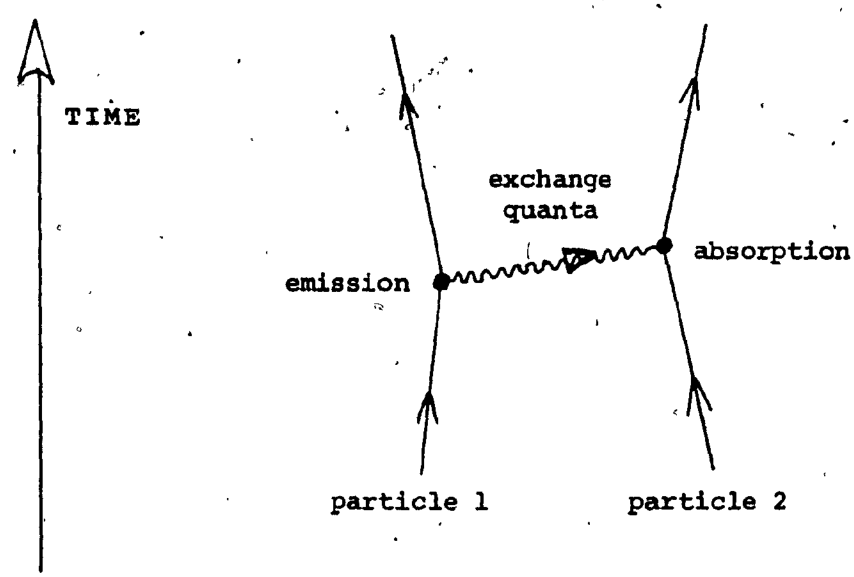

FIG.1

Particle 1 exerts a force on Particle 2

In summary, there are particles and there are exchange quanta. Particles exert forces (binding) on each other through emission and absorption of their exchange quanta. Each force has its own exchange quanta (Table 1).

FORCE	EXCHANGE QUANTA
GRAVITATIONAL	GRAVITON
ELECTRIC	PHOTON
COLOR	GLUON
WEAK	W^+, W^-, Z^0

Table 1. The four forces of nature.

Quarks can interact via all four forces whereas the leptons do not interact via the color force. For each quark and lepton there are corresponding anti-matter particles (or antiquarks and antileptons). Anti-matter states are similar to the ordinary particle states except that they are rotated in isospin space, hypercharge space and baryon number space. The quarks, leptons, antiquarks, antileptons and their exchange quanta constitute all the known matter.

The subject of this thesis is quark - antiquark bound states. Before the material is presented the subject of quarks will be presented briefly. Table 2 shows the quarks, leptons and their quantum numbers.

QUARKS	B[a]	C[b]	S[c]	Q[d]
Up (u)	1/3	0	0	2/3
Down (d)	1/3	0	0	-1/3
Strange (s)	1/3	0	-1	-1/3
Charm (c)	1/3	1	0	2/3
Bottom (b)	1/3	0	0	-1/3
Top (t)	1/3	0	0	2/3
LEPTONS				
Electron (e^-)	0	0	0	-1
e^- neutrino	0	0	0	0
Muon (μ)	0	0	0	-1
μ neutrino	0	0	0	0
Tau (τ)	0	0	0	-1
τ neutrino	0	0	0	0

Table 2. Quarks, Leptons and their quantum numbers

[a] Baryon number

[b] Charm quantum number

[c] Strangeness quantum number

[d] Electric charge in fractions of the electron charge.

CHAPTER 1.

1.1 Quarks

Particles formed with quarks are strongly interacting, that is, they interact through the color force. These particles are called HADRONS. Hadrons come in at least two families, the BARYONS and the MESONS. Baryons are made of three quarks and mesons are quark - antiquark pairs. Because quarks are spin ½ particles the baryons are half-integer spin particles (fermions) and the mesons are integer spin particles (bosons). The neutron and proton, for example, are baryons. The neutron is a ddu combination and the proton is a uud combination of quarks. Adding the quantum numbers of these quarks from Table 2 gives the quantum numbers of the neutron and proton; except for spin which must be combined by the quantum mechanical rules for addition of angular momenta. The pion ($\pi^{0\pm}$) is an example of a meson. It is thought to be the exchange quanta of the "nuclear force" (a residual color force) between nucleons (proton or neutron). The neutral pion π^0, for example, is a $u\bar{d}$ combination.

Quarks were originally a theoretical construction used as a method of classifying the large number of hadrons being discovered in the debris of high-energy collisions. It was noticed by Gell-Mann[17] and Ne'eman[18] that it was possible to group all the hadrons into families. The mesons could form families of one and eight particles and the baryons formed families of one, eight and ten particles. The particles are classified according to their quantum numbers. At that time all the hadrons fit into these families. These families or groups of particles are representations of the symmetry group SU(3). The basic triplet or fundamental representation of the SU(3) group was labelled u, d, and s (or quarks). By assigning the quantum numbers shown in Table 2 to this triplet it was possible to generate all the then known hadrons. However, the idea of quarks solely as theoretical objects began to change. Quarks became more than a classification scheme. In deep-inelastic scattering of leptons on nucleons it was expected that the inelastic cross-section would be modulated by a form factor and would decrease with increasing momentum transfer. However, the cross-sections were much larger and there was no scale parameter. This indicated the presence of small scattering centers within the nucleon[19]. These scattering centers are called partons, (a name coined by R.P. Feynman), and may be identified with the quarks.

1.2 Color

A possible combination of quarks is three s quarks (the Ω^- baryons). Since these quarks are identical, their quantum numbers are identical. Then, these three s quarks in a relative S-wave orbital state constitutes a violation of the Pauli exclusion principle for fermions. The way out of this difficulty was either to modify the Fermi-Dirac statistics for quarks or to modify the quarks themselves by introducing another quantum number which would differ for the three quarks in the baryon.

The method suggested by O.W. Greenberg is to assume that the quarks obey a para-fermi statistics of order 3. That is, for quarks we would allow three identical fermions to be in the same state. The other approach suggested by M.Y. Han, Y. Nambu and independently by A. Tavkhelidze and Y. Miyamoto was to assign each quark an additional quantum number which could assume three possible values. If each quark in the baryon had a different one of these numbers, then, the Pauli principle was saved. This additional quantum number is called COLOR. The introduction of color has the effect of tripling the number of quarks and thus predicts the existence of new particles. These new baryons are not seen. However, if we make the assumption that color cannot be observed then the number of baryons remain the same. Also, the effect of colorless hadrons is to make the para-statistics and the color hypotheses equivalent.

Each quark must have a different color within a hadron. For convenience, the colors can be referred to as red, yellow and blue. The anti-colors can then be referred to as anti-red, anti-yellow and anti-blue. The baryons will have one quark of each color and will therefore be colorless or white. A meson will have two quarks one with some color and the antiquark with the corresponding anti-color. The effect of color, however, was more than saving the Pauli principle. The color quantum in QCD is the analogy of the electric charge quantum in QED. That is, color is ultimately responsible for the force which binds quarks. In this sense we can understand why leptons do not experience the color force - leptons do not have a color quantum.

In addition to the solution of the fermion statistics problem, there is other evidence for the existence of color. The ratio of cross-sections:

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+ \mu^-)}$$

becomes larger by a factor of 3 due to the introduction of color, bringing it into reasonably good agreement with experiment. The lifetime of the neutral pion is calculated with color to be 7.87 eV compared to the experimental value of 7.85 ± 0.5 eV. In the hadronic production of lepton pairs via the Drell-Yan process, the predicted cross-section is one-third as large without color. Also, the branching ratio for decay of the τ lepton into electrons + neutrinos, muons + neutrinos or hadrons is 1:1:1 without color and 1:1:3 with color. This is in good agreement with experiment. Color also provides an explanation of why states such as qqqq (which would exhibit color overall) have not been observed - since all observed hadrons must be colorless by hypothesis.

1.3 Confinement

Free quarks have never been observed experimentally. There are three possible explanations. (1) Quarks do not exist, (2) quarks are permanently confined, (3) we have not yet reached energies necessary to ionize them. The first is unlikely as the proton and neutron exhibit an internal structure.[19] It is found that multiprong hadronic events have a "jet" structure, that is, at high center of mass energy the fragments are emitted with small momentum transverse to the direction in which the partons were produced. Also, the form of quarkonium spectra indicates the presence of constituents;[21] quarkonium has energy levels that are approximately equally spaced - indicative of an internal structure.

The second and third possibilities are likely candidates. Any successful quark model must therefore have confinement implicit in its structure. The analytical derivation of confinement as a direct consequence of QCD is mathematically very difficult if at all possible. It has not yet been accomplished. Numerical calculations, by Rebbi[15], seem to indicate that it is contained in the theory.

Since quarks come in different colors, then, postulating that only colorless states are observable immediately confines the quarks. Confining quarks using the colorless hypothesis implies that these confining forces depend on color.

CHAPTER 2

2.1 Phenomenological Models

The general structure for phenomenological quark models is based on quantum chromodynamics in regions of high energy probes (short wavelength) where the strong coupling constant α_s becomes very small. In this energy region the quark potential can be thought of as a Coulomb-like potential arising from one - gluon exchange at short range. The QCD Hamiltonian will be of the same form as that in QED (De Rujula, Georgi, Glashow[7]). At long range, a scalar confining potential is used. The one - gluon exchange term introduces hyperfine and spin-orbit interactions similar to those of the one - photon exchange term of QED. The scalar confining potential also causes spin-orbit splitting. The Hamiltonian for a qqq or $q\bar{q}$ state is (De Rujula, Georgi, Glashow[7]):

$$H = L(\vec{r}_1, \vec{r}_2, \ldots) + \sum_i (m_i + p_i^2/2m_i + \ldots)$$

$$+ \sum_{i>j} (\alpha Q_i Q_j + K\alpha_s) S_{ij}$$

Here, $L(\vec{r}_i)$ is the interaction responsible for long range binding; $\vec{r}_i$, $\vec{p}_i$, m_i and Q_i are the position, momentum, mass and charge of the i^{th} quark. The factor K comes from the averaging of the color vectors and is -4/3 for mesons and -2/3 for baryons. The factor S_{ij} is a two-body

Coulombic interaction of the form:

$$S_{ij} = \frac{1}{r} - \frac{1}{2m_i m_j}\left[\frac{\vec{p}_i\cdot\vec{p}_j}{r} + \frac{\vec{r}\cdot(\vec{r}\cdot\vec{p}_i)\vec{p}_j}{r^3}\right]$$

$$- \frac{\pi\delta^3(\vec{r})}{2}\left[\frac{1}{m_i^2} + \frac{1}{m_j^2} + \frac{16\vec{S}_i\cdot\vec{S}_j}{3m_i m_j}\right] - \frac{1}{2r^3}\left\{\frac{\vec{r}\times\vec{p}_i\cdot\vec{S}_i}{m_i^2}\right.$$

$$- \frac{\vec{r}\times\vec{p}_j\cdot\vec{S}_j}{m_j^2} + \frac{1}{m_i m_j}\left[2\vec{r}\times\vec{p}_i\cdot\vec{S}_j - 2\vec{r}\times\vec{p}_j\cdot\vec{S}_i - 2\vec{S}_i\cdot\vec{S}_j\right.$$

$$\left.\left. + \frac{6(\vec{S}_i\cdot\vec{r})(\vec{S}_j\cdot\vec{r})}{r^2}\right]\right\} + \ldots$$

Isgur and Karl[3,4,5] have introduced a Hamiltonian using a harmonic oscillator confining potential consistent with the De Rujula, Georgi, Glashow hamiltonian. They described the spectra and decay couplings of the low-lying baryon states. However, a different parameter set was used for the odd and even parity states. The Isgur-Karl hamiltonian is

$$H = \sum_i m_i + H_o + H_{hyp}$$

where

$$H_o = \sum_i p_i^2/2m_i + \sum_{i<j} V^{ij}_{conf}$$

and

$$V^{ij}_{conf} = K\left[\tfrac{1}{2}kr^2_{ij} + U(r_{ij})\right]$$

$U(r_{ij})$ is a non-harmonic part of the potential containing a Coulomb-like piece at short range and deviations from the harmonic oscillator potential at large distances. K is a color factor equal to 4/3 for mesons and 2/3 for baryons. H_{hyp} is a hyperfine interaction. This Hamiltonian gives the correct magnitude and sign of the $\Lambda(5/2^-) - \Sigma(5/2^-)$ splitting. However, if the parameters obtained from the fit to positive-parity baryons are applied to these negative parity states the splitting will be reduced from 50 MeV to 15 MeV. Isgur and Karl[3,4,5] calculate the contribution of the non-harmonic term, $U(r_{ij})$, in the SU(3) limit: $m_s = m_u = m_d$. Kalman and Hall[1] have shown that by modifying the calculation of this term and taking into account the mass difference between the strange and up(down) quarks, it was possible to restore the $\Lambda(5/2^-) - \Sigma(5/2^-)$ splitting. Thus, Kalman and Hall[1] and Kalman[2] have shown that a consistent parameter set can be developed for all baryons. This consistent model was applied by Kalman, Hall and Misra[11] and Kalman and Misra[20] to baryonium. Kalman and Pfeffer[9,10] have applied the model successfully to charmed and bottom baryons. Recently, Kalman and Mukerji[6] applied the Isgur - Karl model to mesons and calculated all the low-lying states of the Ψ and Υ systems. Also, recent developments[12,13,14] seem to indicate that the same forces occur in both baryons and mesons.

2.2 Spin-Orbit Interactions

In their baryon model, Isgur and Karl[2,3,4] neglected the spin-orbit force completely. They suggested that the absence of the spin-orbit force in baryons is due to a cancellation of the part of this force arising from one - gluon exchange by the Thomas precession term arising from the harmonic confining potential. Schnitzer[8] considered this point in detail and notes that the total spin-orbit force depends on $\langle r \rangle_{hadron}$. Therefore, this force is absent for baryons, is weak for ordinary mesons and is stronger for charmonium. The spin-orbit interaction then must be included for mesons.

CHAPTER 3

3.1 The Meson Hamiltonian

The purpose of this thesis is to calculate the masses of mesons in low-lying states containing quarks of unequal mass (different flavors). Spin-orbit and hyperfine structure are taken into account. Seven such states have masses experimentally known, two of which are input parameters, leaving five states to verify the predictions of the model.

For mesons, the Hamiltonian used is of the form:

$$H = m_1 + m_2 + p_1^2/2m_1 + p_2^2/2m_2 - (p_1 + p_2)^2/2(m_1 + m_2) + (V_0 + H_{Hyp} + H_{so})\sum_\alpha \Lambda_1^{\alpha *} \Lambda_2^\alpha \quad (1)$$

where m_1 and m_2 are the quark masses and

$$V_0 = \tfrac{1}{2}k^* r^2 + U(r) \quad (2)$$

$$H_{hyp} = \frac{\alpha_s}{m_1 m_2}\left\{\frac{8\pi(\vec{S}_1\cdot\vec{S}_2)\delta^3(r)}{3} + \frac{1}{r^3}\left[\frac{3(\vec{S}_1\cdot\vec{r})(\vec{S}_2\cdot\vec{r})}{r^2} - \vec{S}_1\cdot\vec{S}_2\right]\right\} \quad (3)$$

$$H_{so} = \frac{\alpha_s}{r^3}\left\{\frac{\vec{S}_1\cdot\vec{r}\times\vec{p}_1}{m_1^2} - \frac{\vec{S}_2\cdot\vec{r}\times\vec{p}_2}{m_2^2} + \frac{2\vec{S}_1\cdot\vec{r}\times\vec{p}_2}{m_1m_2} - \frac{2\vec{S}_2\cdot\vec{r}\times\vec{p}_1}{m_1m_2}\right\}$$

$$- k^*\left\{\frac{\vec{S}_1\cdot\vec{r}\times\vec{p}_1}{m_1^2} - \frac{\vec{S}_2\cdot\vec{r}\times\vec{p}_2}{m_2^2}\right\} \qquad (4)$$

where r is the interquark distance and $\vec{p}_i$, $\vec{S}_i$ and Λ_i^α are the momenta, spins and color vectors of the quarks. Since a harmonic oscillator confining potential is used, the wave functions are (up to $n = 2$):

$$\Psi_{000} = \frac{\beta^{3/2}}{\pi^{3/4}} \exp(-\tfrac{1}{2}\beta^2 r^2) \qquad (5a)$$

$$\Psi_{11m} = \left(\frac{2}{3}\right)^{1/2} \frac{2\beta^{5/2}}{\pi^{1/4}} r \exp(-\tfrac{1}{2}\beta^2 r^2) Y_{1m}(\theta,\phi) \qquad (5b)$$

$$\Psi_{200} = \left(\frac{2}{3}\right)^{1/2} \frac{\beta^{7/2}}{\pi^{3/4}}\left(\frac{3}{2}\beta^{-2} - r^2\right) \exp(-\tfrac{1}{2}\beta^2 r^2) \qquad (5c)$$

$$\Psi_{22m} = \frac{4\beta^{7/2}}{\sqrt{15}\,\pi^{1/4}} r^2 \exp(-\tfrac{1}{2}\beta^2 r^2)\; Y_{2m}(\theta,\phi) \qquad (5d)$$

Here $\beta = \left[\frac{2km_1m_2}{(m_1+m_2)}\right]^{1/4}$ (6)

and

$$k = \frac{2}{3}k^* \tag{7}$$

This relates the spring constant k* to the k used in Kalman and Mukerji[6].

3.2 Zero-order Eigenvalues and An-Harmonic Perturbation

The harmonic oscillator energy eigenvalues are:

$$E_0 = (n + 3/2)\omega_Q \tag{8}$$

where

$$\omega_Q^2 = m_C\ (m_1 + m_2)\omega_C^2/2m_1m_2 \tag{9}$$

Here, m_C and ω_Q are the mass and oscillator frequency obtained by fitting to the charmonium system, by Kalman and Mukerji[6]. They found m_C = 2749.0 MeV and ω_C = 390.5 MeV.

The an-harmonic part of the potential, U(r), is not known analytically; its matrix elements are evaluated by first order perturbation theory. It is necessary to evaluate the following integrals (Kalman, Hall and Misra)[11]:

$$a(t) = \frac{\beta^3 t^{3/2}}{\pi^{3/2}} \int d^3r\ U(r)\exp(-t\beta^2 r^2) \tag{10a}$$

$$b(t) = \frac{\beta^5 t^{5/2}}{\pi^{3/2}} \int d^3r\, U(r) r^2 \exp(-t\beta^2 r^2) \qquad (10b)$$

$$c(t) = \frac{\beta^7 t^{7/2}}{\pi^{3/2}} \int d^3r\, U(r) r^4 \exp(-t\ \ r^2) \qquad (10c)$$

where

$$t = \left[\frac{2m_1 m_2}{m_C(m_1 + m_2)}\right]^{1/2} \qquad (11)$$

Here t is normalized so that for the charmonium sector t=1. The integrals a(1), b(1) and c(1) are fit from experimental data. Kalman and Mukerji[6] have found

a(1) = -3004.9 MeV, b(1) = -4430.3 MeV, and c(1) = -11349.9 MeV. The value of a(t), b(t), c(t) for unequal mass quarkonium systems are obtained by constructing quadratic approximations[1] of a(t), b(t), c(t):

$$a(t) \simeq A + Bt + Ct^2 \qquad (12a)$$

$$b(t) \simeq (3A + Bt - Ct^2)/2 \qquad (12b)$$

$$c(t) \simeq (15A + 3Bt - Ct^2)/4 \qquad (12c)$$

The values A, B and C are fixed by the parameters a(1), b(1), c(1). The contribution to the energy ignoring mixing, hyperfine and spin-orbit interactions follows from equations (1), (5), (8), (10) and is as follows:

$$E_0(S) = m_1 + m_2 + \frac{3}{2}\omega_Q + a(t) \tag{13a}$$

$$E_0(P) = m_1 + m_2 + \frac{5}{2}\omega_Q + \frac{2}{3}b(t) \tag{13b}$$

$$E_0(S') = m_1 + m_2 + \frac{7}{2}\omega_Q + \frac{3}{2}a(t) - 2b(t) + \frac{2}{3}c(t) \tag{13c}$$

$$E_0(D) = m_1 + m_2 + \frac{7}{2}\omega_Q + \frac{4}{15}c(t) \tag{13d}$$

The non-harmonic part of the potential, U(r), also has non-zero off diagonal matrix elements:

$$U_0 = \langle S \mid U \mid S'\rangle = \langle S' \mid U \mid S\rangle$$

$$= \left(\frac{3}{2}\right)^{1/2} a(t) - \left(\frac{2}{3}\right)^{1/2} b(t) \tag{14}$$

3.3 Fine and Hyperfine Structure

The one - gluon exchange terms give rise to color-magnetic forces similar to magnetic forces arising due to one - photon exchange. The hyperfine interaction contains two terms, a contact term operative only in S-wave states, and, a tensor term which has no contribution in S-wave states but mixes states which differ in angular momentum by 2 units.

Also introduced by the potential are spin-orbit couplings. There is no contribution from these terms for S-wave states. Thus, in S-wave states the only contribution is from the fermi contact terms. The Fermi-contact interaction gives the mixing matrix:

$$\begin{bmatrix} E_0(S) + \frac{2}{3}^{3/2} \delta_Q & U_0 + \frac{2}{\sqrt{3}} \delta_Q \\ U_0 + \frac{2}{\sqrt{3}} \delta_Q & E_0(S') + \sqrt{2} \delta_Q \end{bmatrix} \qquad (15)$$

for spin 1 particles, that is, $^3S_1$, $^3S_1'$ states. With $E_0(S)$, $E_0(S')$, U_0 given by equations (13a), (13c), (14) and

$$\delta_Q = \left[\frac{8m_C^5}{(m_1 + m_2)^3 m_1 m_2} \right]^{1/4} \delta_C \qquad (16)$$

where δ_C is a parameter from the charmonium sector fit by Kalman and Mukerji[6], they get $\delta_C = 21.63$ MeV.

For the spin 0 states, that is, $^1S_0$, $^1S_0'$, the mixing matrix has the form:

$$\begin{bmatrix} E_0(S) - 2^{3/2} \delta_Q & U_0 - 2\sqrt{3} \delta_Q \\ U_0 - 2\sqrt{3} \delta_Q & E_0(S') - 3\sqrt{2} \delta_Q \end{bmatrix} \qquad (17)$$

The Fermi contact term has no matrix elements for L=1 and L = 2 angular momentum states.

The tensor portion of the hyperfine interaction is evaluated by using the identity (Isgur and Karl[3,4,5]):

$$<LSJ|r^{-3}(3\vec{S}_1\cdot\hat{r}\ \vec{S}_2\cdot\hat{r} - \vec{S}_1\cdot\vec{S}_2)|L'S'J> =$$

$$(-1)^{J-L-S'}((2L+1)(2S+1))^{\frac{1}{2}}\ W(LL'SS';2J)$$

$$\times <L||\tfrac{1}{2}\sqrt{3}\ r^{-3}\hat{r}_+\hat{r}_+||L'> <S||\tfrac{1}{2}\sqrt{3}\ S_{1-}S_{2-}||S'> \quad (18)$$

Here, W is a Racah co-efficient and the last two factors are reduced matrix elements of the tensors whose elements are

$$<L|\ \tfrac{1}{2}\sqrt{3}\ r^{-3}\hat{r}_+\hat{r}_+|L'> \quad (19)$$

and

$$<S|\tfrac{1}{2}\ \sqrt{3}\ S_{1-}S_{2-}|S'> \quad (20)$$

Therefore, by applying equations (5b) and (5d) to equation (18) gives the following matrix elements:

$$\langle {}^1P_1 | H_{tensor} | {}^1P_1 \rangle = 0 \tag{21}$$

$$\langle {}^1D_2 | H_{tensor} | {}^1D_2 \rangle = 0 \tag{22}$$

$$\langle {}^3P_0 | H_{tensor} | {}^3P_0 \rangle = -\frac{4\sqrt{2}}{3}\delta_Q \tag{23}$$

$$\langle {}^3P_1 | H_{tensor} | {}^3P_1 \rangle = \frac{2\sqrt{2}}{3}\delta_Q \tag{24}$$

$$\langle {}^3P_2 | H_{tensor} | {}^3P_2 \rangle = -\frac{2\sqrt{2}}{15}\delta_Q \tag{25}$$

$$\langle {}^3D_1 | H_{tensor} | {}^3D_1 \rangle = -\frac{4\sqrt{2}}{15}\delta_Q \tag{26}$$

$$\langle {}^3D_2 | H_{tensor} | {}^3D_2 \rangle = \frac{4\sqrt{2}}{15}\delta_Q \tag{27}$$

$$\langle {}^3D_3 | H_{tensor} | {}^3D_3 \rangle = -\frac{8\sqrt{2}}{105}\delta_Q \tag{28}$$

The tensor hyperfine interaction also mixes the states ${}^3D_1$, ${}^3S_1$ and ${}^{'3}S_1$. The mixed elements are found from equations (5a), (5c) and (5d) into equation (18). These are:

$$\langle {}^3D_1 | H_{tensor} | {}^3S_1 \rangle = \frac{4}{\sqrt{15}} \delta_Q \tag{29}$$

$$\langle {}^3D_1 | H_{tensor} | {}^3S_1 \rangle = \frac{2}{3} \frac{\sqrt{2}}{\sqrt{5}} \delta_Q \tag{30}$$

A sample calculation of a tensor matrix element is found in Appendix (A).

The spin-orbit interaction has two terms. The first term, $H_{so(1GE)}$, arises from the one - gluon exchange color interaction. This term with chromo-magnetic interaction and Thomas precession correction is:

$$H_{so(1GE)} = \frac{4\alpha_s}{3r^3} \left[\frac{\vec{S}_1 \cdot \vec{r} \times \vec{p}_1}{m_1^2} - \frac{\vec{S}_2 \cdot \vec{r} \times \vec{p}_2}{m_2^2} + \frac{2\vec{S}_1 \cdot \vec{r} \times \vec{p}_2}{m_1 m_2} - \frac{2\vec{S}_2 \cdot \vec{r} \times \vec{p}_1}{m_1 m_2} \right] \tag{31}$$

This can be written as:

$$H_{so(1GE)} = \frac{4\alpha_s}{3r^3} \left[\frac{1}{4m_1^2} + \frac{1}{4m_2^2} + \frac{1}{m_1 m_2} \right] \vec{L} \cdot \vec{S} \tag{32}$$

where $\vec{S} = \vec{S}_1 + \vec{S}_2$ is the total quark intrinsic spin and $\vec{L}$ is the relative orbital angular momentum of the quarks.

The second term is due to the harmonic oscillator confining potential and from Thomas precession alone is:

$$H_{so(H.o)} = -\frac{1}{r}\frac{dV}{dr}\left(\frac{\vec{S}_1\cdot\vec{r}\times\vec{p}_1}{m_1^2} - \frac{\vec{S}_2\cdot\vec{r}\times\vec{p}_2}{m_2^2}\right) \qquad (33)$$

This can be written as

$$H_{so(H.o)} = -k\left(\frac{1}{m_1^2} + \frac{1}{m_2^2}\right)\vec{L}\cdot\vec{S} \qquad (34)$$

where

$$k = \frac{m_1 m_2 \omega_0^2}{2(m_1+m_2)} \qquad (35)$$

then, the total spin-orbit interaction is:

$$H_{so} = H_{so(1GE)} + H_{so(H.o)} \qquad (36)$$

Since the expectation value of $\vec{L}\cdot\vec{S}$ is :

$$\langle LSJM|L\cdot S|L'S'J'M'\rangle = \frac{1}{2}\left[J(J+1)-L(L+1)-S(S+1)\right]\delta_{JJ'}\delta_{LL'}\delta_{MM'} \qquad (37)$$

the spin-orbit contributions are as follows:

$$\langle {}^1P_1 | H_{so} | {}^1P_1 \rangle = 0 \tag{38}$$

$$\langle {}^1D_2 | H_{so} | {}^1D_2 \rangle = 0 \tag{39}$$

$$\langle {}^3P_0 | H_{so} | {}^3P_0 \rangle = -\frac{8\sqrt{2}}{3}\Delta_1 - 2\Delta_2 \tag{40}$$

$$\langle {}^3P_1 | H_{so} | {}^3P_1 \rangle = -\frac{4\sqrt{2}}{3}\Delta_1 - \Delta_2 \tag{41}$$

$$\langle {}^3P_2 | H_{so} | {}^3P_2 \rangle = \frac{4\sqrt{2}}{3}\Delta_1 + \Delta_2 \tag{42}$$

$$\langle {}^3D_1 | H_{so} | {}^3D_1 \rangle = -\frac{8\sqrt{2}}{5}\Delta_1 - 3\Delta_2 \tag{43}$$

$$\langle {}^3D_2 | H_{so} | {}^3D_2 \rangle = -\frac{8\sqrt{2}}{15}\Delta_1 - \Delta_2 \tag{44}$$

$$\langle {}^3D_3 | H_{so} | {}^3D_3 \rangle = \frac{16\sqrt{2}}{15}\Delta_1 + 2\Delta_2 \tag{45}$$

where :

$$\Delta_1 = \left[\frac{m_2}{4m_1} + \frac{m_1}{4m_2} + 1\right] \delta_Q \tag{46}$$

and

$$\Delta_2 = -\frac{m_1 m_2 \omega_Q^2}{(m_1+m_2)2} \left[\frac{1}{m_1^2} + \frac{1}{m_2^2}\right] \tag{47}$$

RESULTS AND CONCLUSIONS

The parameters obtained by Kalman and Mukerji[6] are as follows:

m_c	:	2749.0 MeV
m_b	:	6188.6 MeV
δ_c	:	21.63 MeV
ω_c	:	390.5 MeV
a(1)	:	-3004.9 MeV
b(1)	:	-4430.3 MeV
c(1)	:	-11349.9 MeV

The masses for the u, d and s quarks were obtained by fitting[16] to the D^o, D^+ and ϕ mesons respectively. The results are m_u = 1525.3 Mev, m_d = 1530.7 MeV, m_s = 1606.5 MeV. The masses for the uc, dc, sc, ub, db, sb and cb systems are calculated and are displayed in Tables I-VII. The only comparisons with experiment[16] are the F, D^{*o}, D^{*+}, B^- and B^o. It is seen that all the results are in very close agreement with experiment. The states not yet observed may be searched for by experimentalists in the energy regions indicated in the Tables I-VII. For the B^- and B^o mesons there is almost no error.

In conclusion, the Isgur - Karl shell model using a harmonic oscillator potential, fine and hyperfine interactions predicts the low-lying meson states (especially the heavier, less relativistic mesons) very well.

TABLE I. Masses of the uc system in GeV. Input parameter is underlined.

State		Calculation	Experiment
$^1S_0$	(D^0)	1.8647	<u>1.8647</u>
$^1S_0'$		2.6388	
$^3S_1$	(D^{*0})	1.9981	2.0072 ± .0021
$^3S_1'$		2.7883	
$^1P_1$		2.4492	
$^3P_0$		2.3286	
$^3P_1$		2.4460	
$^3P_2$		2.4753	
$^1D_2$		2.8552	
$^3D_1$		2.9141	
$^3D_2$		2.8892	
$^3D_3$		2.8067	

TABLE II. Masses of the dc system in Gev.
Input parameter is underlined.

State		Calculation	Experiment
$^1S_0$	(D^+)	1.8694	1.8694
$^1S_0'$		2.6427	
$^3S_1$	(D^{*+})	2.0027	2.0101 ± .0007
$^3S_1'$		2.7914	
$^1P_1$		2.4535	
$^3P_0$		2.3327	
$^3P_1$		2.4501	
$^3P_2$		2.4780	
$^1D_2$		2.8588	
$^3D_1$		2.9171	
$^3D_2$		2.8926	
$^3D_3$		2.8107	

TABLE III. Mases of the sc system in GeV.

State	Calculation	Experiment
$^1S_0$ (F)	1.936	1.970± 5± 5
$^1S_0'$	2.692	
$^3S_1$	2.067	
$^3S_1'$	2.836	
$^1P_1$	2.514	
$^3P_0$	2.392	
$^3P_1$	2.509	
$^3P_2$	2.541	
$^1D_2$	2.911	
$^3D_1$	2.960	
$^3D_2$	2.941	
$^3D_3$	2.869	

TABLE IV. Masses of the ub system in Gev.

State	Calculation	Experiment
$^1S_0$ (B^-)	5.2728	5.2708 ± .0023 ± .002
$^1S_0'$	5.9244	
$^3S_1$	5.3471	
$^3S_1'$	5.9992	
$^1P_1$	5.7849	
$^3P_0$	5.7263	
$^3P_1$	5.7855	
$^3P_2$	5.7962	
$^1D_2$	6.1320	
$^3D_1$	6.1953	
$^3D_2$	6.1609	
$^3D_3$	6.0844	

TABLE V. Masses of the db system in GeV.

State	Calculation	Experiment
$^1S_0$ (B^0)	5.2772	5.2742±.0019±.0021
$^1S_0'$	5.9274	
$^3S_1$	5.3515	
$^3S_1'$	6.0020	
$^1P_1$	5.7890	
$^3P_0$	5.7302	
$^3P_1$	5.7895	
$^3P_2$	5.8005	
$^1D_2$	5.1354	
$^3D_1$	6.1980	
$^3D_2$	6.1641	
$^3D_3$	6.0883	

TABLE VI. Masses of the sb system in GeV.

State	Calculation	Experiment
$^1S_0$	5.389	
$^1S_0'$	5.970	
$^3S_1$	5.413	
$^3S_1'$	6.042	
$^1P_1$	5.848	
$^3P_0$	5.787	
$^3P_1$	5.846	
$^3P_2$	5.862	
$^3D_1$	6.238	
$^3D_2$	6.210	
$^3D_3$	6.144	
$^1D_2$	6.185	

TABLE VII. Masses of the cb system in GeV.

State	Calculation	Experiment
$^1S_0$	6.310	
$^1S_0'$	6.803	
$^3S_1$	6.392	
$^3S_1'$	6.835	
$^1P_1$	6.848	
$^3P_0$	6.780	
$^3P_1$	6.837	
$^3P_2$	6.867	
$^1D_2$	7.085	
$^3D_1$	7.084	
$^3D_2$	7.090	
$^3D_3$	7.081	

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21. In fact, if the relative spacing between the energy levels is ignored, the quarkonium spectrum is extremely similar to that of positronium.

APPENDIX A

Sample Calculation of a Tensor Matrix Element

The following is a procedure for evaluating $\langle {}^3P_0 | H_{tensor} | {}^3P_0 \rangle$. This is done easily by using equations (3) and (18).

The wave function needed is Ψ_{111} (equation 5b). This can be written as

$$\Psi_{111} = \frac{\beta^{5/2}}{\pi^{3/4}} \exp(-\tfrac{1}{2}\beta^2 r^2)\, r_+ \qquad \text{(A-1)}$$

where

$$r_+ = -\frac{1}{\sqrt{2}}(r_x + i r_y) \qquad \text{(A-2)}$$

the reduced matrix elements in equation (18) are evaluated using the Wigner-Eckart theorem. The results are

$$\langle \Psi_{111} || \frac{\sqrt{3}}{2}\, \hat{r}_+ \hat{r}_+ r^{-3} || \Psi_{111} \rangle = -\frac{8\beta^3}{\sqrt{45}\sqrt{\pi}} \qquad \text{(A-3)}$$

$$\langle \downarrow\downarrow || \frac{\sqrt{3}}{2}\, S_{1-} S_2 || \uparrow\uparrow \rangle = \frac{\sqrt{5}}{2} \qquad \text{(A-4)}$$

From equations (3) and (18), the total tensor term is:

Appendix A (continued)

$$\langle {}^3P_0 | H_{tensor} | {}^3P_0 \rangle = -\frac{4\alpha_s}{3m_1m_2}\frac{\sqrt{5}}{2}\frac{8\beta^3}{\sqrt{45}\sqrt{\pi}}$$

$$= -\frac{4\sqrt{2}}{3}\delta_Q \qquad \text{(A-5)}$$

where

$$\delta_Q = \frac{4\alpha_s\beta^3}{3\sqrt{2\pi}\, m_1m_2} \qquad \text{(A-6)}$$