

Two Scott Models for the λ -Calculus

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ABSTRACT

TWO SCOTT MODELS FOR THE λ -CALCULUS

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At first glance a model of the λ -calculus appears to require a set X in which its own function space $X \rightarrow X$ can be embedded, and this requirement appears to contradict Cantor's theorem. This difficulty was overcome by D.S. Scott in his 1969 construction of D_∞ by restricting $X \rightarrow X$ to the set of continuous functions on X (provided with a suitable topology). Since 1969, a number of other constructions have been presented, including one based on the theory of domains in [16]. In this thesis, the construction of D_∞ and of the domain-theoretic model D are studied, and it is proved that D_∞ is an extensional λ -model, while D is non-extensional.

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INTRODUCTION

The λ -calculus is a theory about functions as rules, rather than as graphs. A function is regarded as a process of going from argument to value, a process coded by a definition. When functions are treated as rules, there is no need for an indication of a domain and range; thus the λ -calculus is a type-free structure where the objects of study are at the same time functions and arguments. In particular a function can be applied to itself. For the usual notion of function this is impossible since applying a function to itself violates the rules of ordinary set theory.

The λ -calculus and its related theory of combinatory logic were initiated around 1930 by Church, Schönfinkel and Curry respectively. Later on, based also on works by Kleene and Turing, a fundamental result emerged: the general recursive functions are exactly the λ -definable functions as are the Turing computable functions. By the analysis of Turing it follows that in spite of its very simple syntax, the λ -calculus is strong enough to describe all mechanically computable functions. Therefore the λ -calculus can be viewed as a paradigmatic programming language.

Because of the similarities of λ -calculus and some programming languages, ideas for the semantics of the former may be applied to the latter. In particular, the problem of the denotational semantics of programming languages appears in a pure form in the λ -calculus. The first model for the λ -calculus, D_∞ , was constructed by Scott in 1969, and since then about half a dozen other models have been discovered.

Summary:

In Chapter I we give an introduction to the theory of λ -calculus and define the notion of a λ -model. Chapter II gives the background we need for the construction of the D_∞ model. Chapter III gives the background we need for the construction of the domain-theoretic model.

In Chapter IV we construct the D_∞ model and the newer domain-theoretic model of [16], prove that they are models of the λ -calculus, and prove that D_∞ is extensional while the domain-theoretic model is not.

CHAPTER I

The material of this chapter is derived from Hindley and Seldin [10].

Section 1. The λ -calculus

In everyday differential calculus, an expression such as ' $x-y$ ' can be considered as defining either a function f of x or a function g of y . One convenient way to distinguish these two functions, suggested by the logician Alonzo Church in the 1920's, is to introduce a symbol, say ' λ ', and define

$$f = \lambda x. x-y, \quad g = \lambda y. x-y.$$

This gives a systematic way of constructing, for each expression involving ' x ', a notation for the corresponding function of x , and similarly for ' y ', etc. This is the starting point of the λ -calculus.

The above notation can be extended to functions of more than one variable. For example to the expression ' $x-y$ ' correspond two functions h, k of two variables defined by

$$h(x, y) = x-y, \quad k(y, x) = x-y.$$

However, we can avoid the need for a special notation for functions of several variables by using functions whose values are not numbers but functions. For example, instead of using the two-place function h above, we could represent h by the function h^* defined by

$$h^* = \lambda x. (\lambda y. x-y).$$

For this reason we shall only need a λ -notation for functions of one variable.

Begin by assuming that there is an infinite sequence of symbols called variables.

Definition 1.1.1

The set of λ -terms is defined inductively as follows:

(i) All variables are λ -terms (called atoms)

(ii) If X and Y are λ -terms, then (XY) is a λ -term

(iii) If Y is a λ -term and x is a variable, then $(\lambda x.Y)$ is a λ -term,

Notation

Letters ' x ', ' y ', ' z ', ' u ', ' v ', ' w ' will denote variables. Capital letters will denote arbitrary λ -terms. Parentheses will be omitted in such a way that, for example ' WXY ' denotes the λ -term $((WX)Y)$ and ' $\lambda x.XY$ ' denotes $(\lambda x.(XY))$. Also ' $\lambda x_1 \dots x_n.Y$ ' will be used for the λ -term $(\lambda x_1.(\lambda x_2.(\dots(\lambda x_n.Y)\dots)))$. Syntactic identity of terms will be denoted by ' \equiv '. It is assumed of course that if $XY \equiv UV$ then $X \equiv U$ and $Y \equiv V$, and if $\lambda x.Y \equiv \lambda u.V$ then $x \equiv u$ and $Y \equiv V$. It is also assumed that the three classes of terms do not intersect.

Interpretation

In general each λ -term is intended to represent a one-place function, whose values and arguments might themselves be functions. The variables represent arbitrary (one-place) functions, and (XY) represents the result of applying the function X to the argument Y . A term $(\lambda x.Y)$ stands for the function whose value at an argument A is calculated by substituting A for x in Y .

Definition 1.1.2

The length of a term X is the total number of atoms in X .

Definition 1.1.3

The relation X occurs in Y is defined by induction on the length of Y , as follows:

(i) X occurs in X

- (ii) If X occurs in U or in V , then X occurs in (UV)
- (iii) If X occurs in U or $X \equiv y$, then X occurs in $(\lambda y.U)$.

Definition 1.1.4

An occurrence of a variable x in a term Y is bound iff it is in a part of Y with the form $\lambda x.Z$; otherwise it is free. A variable x with at least one free occurrence in Y is called a free variable of Y ; the set of all such variables is called $FV(Y)$. A closed term is a term without any free variables.

Definition 1.1.5

For any M, N, x define $[N/x]M$ to be the result of substituting N for every free occurrence of x in M , and changing bound variables to avoid clashes. The precise definition is by induction on the length of M , as follows:

- (i) $[N/x]x \equiv N$
- (ii) $[N/x]y \equiv y$ for all atoms $y \neq x$
- (iii) $[N/x](PQ) \equiv ([N/x]P)([N/x]Q)$
- (iv) $[N/x](\lambda x.P) \equiv \lambda x.P$
- (v) $[N/x](\lambda y.P) \equiv \lambda y.[N/x]P$ if $y \neq x$, and
 $y \notin FV(N)$ or $x \notin FV(P)$
- (vi) $[N/x](\lambda y.P) \equiv \lambda z.[N/x][z/y]P$ if $y \neq x$ and
 $y \in FV(N)$ and $x \in FV(P)$. (z is the first variable
 $\notin FV(NP)$.)

Definition 1.1.6

A change of bound variables in a term X is the replacement of a part of X whose form is $\lambda x.N$ with x not bound in N , by $\lambda y.[y/x]N$ for any y which is neither free nor bound in N . X is congruent to Y iff Y is the result of applying a series of

changes of bound variables to X .

Congruence is symmetric, reflexive, and transitive. Congruent terms have identical interpretations and play identical roles in any application of λ -calculus.

Definition 1.1.7

X β -contracts to Y iff Y is the result of replacing a part of X of the form $(\lambda x.M)N$ by $[N/x]M$. X η -contracts to Y iff Y is the result of replacing a part of X of the form $\lambda x.Mx$, where $x \notin FV(M)$, by M .

X β -reduces to Y (or $X \geq_{\beta} Y$) iff Y is obtained from X by a finite (perhaps empty) series of β -contractions and changes of bound variables. X η -reduces to Y (or $X \geq_{\eta} Y$) iff Y is obtained from X by a finite (perhaps empty) series of β -contractions, η -contractions, and changes of bound variables.

Any term of the form $(\lambda x.M)N$ is called a β -redex, and any term of the form $\lambda x.Mx$ with $x \notin FV(M)$ is called an η -redex.

Definition 1.1.8

A term X which contains no β -redexes is said to be in β -normal form. If a term U β -reduces to an X in β -normal form, then X is called a β -normal form of U .

A term X which contains no η -redexes is said to be in η -normal form. If a term U η -reduces to an X in η -normal form, then X is called an η -normal form of U .

Theorem 1.1.9 (Church-Rosser Theorem for β -reduction)

If $U \geq_{\beta} X$ and $U \geq_{\beta} Y$, then there exists Z such that $X \geq_{\beta} Z$ and $Y \geq_{\beta} Z$.

Corollary 1.1.10

If U has β -normal forms X and Y , then X is congruent to Y .

Theorem 1.1.11 (Church-Rosser Theorem for η -reduction)

If $U \geq_{\eta} X$ and $U \geq_{\eta} Y$, then there exists Z such that $X \geq_{\eta} Z$ and $Y \geq_{\eta} Z$.

Corollary 1.1.12

If U has η -normal forms X and Y , then X is congruent to Y .

Definition 1.1.13

X is β -equal to Y (or $X =_{\beta} Y$) iff Y is obtained from X by a finite (perhaps empty) series of β -reductions and reversed β -reductions.

X is η -equal to Y (or $X =_{\eta} Y$) iff Y is obtained from X by a finite (perhaps empty) series of η -reductions and reversed η -reductions.

In other words $=_{\beta}$ is the equivalence relation generated by \geq_{β} and $=_{\eta}$ is the equivalence relation generated by \geq_{η} .

Definition 1.1.14 (The formal theory λ)

λ is a formal theory in the usual sense (e.g. Mendelson [14] ch. 1, § 4). It has formulae, axioms and rules.

The formulae are all the expressions $X = Y$ where X and Y are λ -terms.

The axioms are:

- (α) $\lambda x.N = \lambda y.[y/x]N$ if $y \notin FV(N)$ and x, y are not bound in N
- (β) $(\lambda x.M)N = [N/x]M$
- (ρ) $M = M$

The rules are:

- (μ) $X = X' \Rightarrow ZX = ZX'$
- (ν) $X = X' \Rightarrow XZ = X'Z$
- (ξ) $X = X' \Rightarrow \lambda x.X = \lambda x.X'$
- (τ) $X = Y \text{ and } Y = Z \Rightarrow X = Z$
- (σ) $X' = Y \Rightarrow Y = X$

Definition 1.1.15

The formal theory λ is said to be extensional if it has the extra rule:

- (η) $\lambda x.Mx = M$ if $x \notin FV(M)$.

Proposition 1.1.16 (see Curry et al [5] p. 92)

The rule (ξ) and the rule (η) taken together are equivalent to the following rule:

- (ζ) If $x \notin FV(MN)$ and if $Mx = Nx$, then $M = N$.

Proposition 1.1.17

$X =_{\beta} Y$ iff $X = Y$ is provable in the non-extensional theory.

$X =_{\eta} Y$ iff $X = Y$ is provable in the extensional theory.

Theorem 1.1.18 (Church-Rosser Theorem for β -equality)

If $X =_{\beta} Y$ then there is a Z such that $X \rightarrow_{\beta} Z$ and $Y \rightarrow_{\beta} Z$.

Corollary 1.1.19

If $X =_{\beta} Y$ and Y is in β -normal form, then $X \rightarrow_{\beta} Y$.

Corollary 1.1.20

If $X =_{\beta} Y$, then either X and Y do not have β -normal forms, or X and Y both have the same β -normal form.

Corollary 1.1.21

Two β -equal terms in β -normal form must be congruent.

Theorem 1.1.22 (Church-Rosser Theorem for η -equality)

If $X =_{\eta} Y$, then there is a Z such that $X \geq_{\eta} Z$ and $Y \geq_{\eta} Z$.

Corollary 1.1.23

If $X =_{\eta} Y$ and Y is in η -normal form, then $X \geq_{\eta} Y$.

Corollary 1.1.24

If $X =_{\eta} Y$, then either X and Y do not have η -normal forms, or X and Y both have the same η -normal form.

Corollary 1.1.25

Two η -equal terms in η -normal form must be congruent.

Section 2. Models for the λ -calculus

A model for the λ -calculus should give interpretations to λ -terms such that provably equal terms are interpreted as the same object. This section will present two equivalent definitions of λ -model.

Notation

'Vars' will denote the class of all variables.

Given a set D , 'a', 'b', 'c', 'd' will denote arbitrary members of D (not formal variables, which are denoted by 'x', 'y' as usual).

Given a mapping \circ from D^2 into D , expressions such as $((a \circ b) \circ c)$ will be shortened to $a \circ b \circ c$ (the convention of association to the left).

Definition 1.2.1

If D is a set, any mapping $\rho : \text{Vars} \rightarrow D$ will be called a valuation. For $d \in D$ and $x \in \text{Vars}$, the valuation ρ' which is the same as ρ except that $\rho'(x) = d$, will be called

$$[d/x]\rho.$$

(If $\rho(x) = d$, then $[d/x]\rho = \rho$.)

Definition 1.2.2

An applicative structure is a pair $\langle D, \circ \rangle$ where D is a set with at least two members, and \circ is any mapping from D^2 into D .

Definition 1.2.3

A combinatory algebra is a quadruple $\langle D, \circ, K, S \rangle$ where $\langle D, \circ \rangle$ is an applicative structure, and $K, S \in D$ such that

$$(i) \quad K \circ a \circ b = a \quad \text{for all } a, b \in D,$$

$$(ii) \quad S \circ a \circ b \circ c = a \circ c \circ (b \circ c) \quad \text{for all } a, b, c \in D.$$

Definition 1.2.4

A λ -model is a triple $\langle D, \circ, [[]] \rangle$ where $\langle D, \circ \rangle$ is an

applicative structure, and $[[\]]$ is a mapping which assigns, to each λ -term M and each valuation ρ , a member $[[M]] \rho \in D$, such that

- (i) $[[x]] \rho = \rho(x)$;
- (ii) $[[PQ]] \rho = [[P]] \rho \circ [[Q]] \rho$;
- (iii) $[[\lambda x.P]] \rho \circ d = [[P]] [d/x] \rho$ for all $d \in D$;
- (iv) $[[M]] \rho = [[M]] \sigma$ if $\sigma(x) = \rho(x)$ for all $x \in FV(M)$;
- (v) $[[\lambda x.M]] \rho = [[\lambda y.[y/x]M]] \rho$ if y does not occur in M ;
- (vi) if $[[M]] [d/x] \rho \leq [[N]] [d/x] \rho$ for all $d \in D$, then
 $[[\lambda x.M]] \rho = [[\lambda x.N]] \rho$.

Definition 1.2.5

A λ -model $\langle D, \circ, [[\]] \rangle$ is extensional iff for all $a, b \in D$
 $a \circ d = b \circ d$ for all $d \in D$ implies $a = b$. Note that in an extensional
 λ -model, (5), and hence (n), is true. Therefore an extensional
 λ -model is a model of the extensional λ -calculus.

Definition 1.2.6

A Scott-Meyer λ -model is a quintuple $\langle D, \circ, K, S, \ell \rangle$ where
 $\langle D, \circ, K, S \rangle$ is a combinatory algebra, and $\ell \in D$ such that for all
 $d_1, d_2 \in D$ we have:

- (i) $\ell \circ d_1 \circ d_2 = d_1 \circ d_2$
- (ii) if $d_1 \circ d = d_2 \circ d$ for all $d \in D$, then $\ell \circ d_1 = \ell \circ d_2$.

Theorem 1.2.7

Every λ -model $\langle D, \circ, [[\]] \rangle$ in the sense of 1.2.4 contains
 K, S and ℓ such that $\langle D, \circ, K, S, \ell \rangle$ is a Scott-Meyer λ -model.

Theorem 1.2.8

In every Scott-Meyer λ -model $\langle D, \circ, K, S, \ell \rangle$, $[[\]]$ can be
defined so that $\langle D, \circ, [[\]] \rangle$ is a λ -model in the sense of 1.2.4.

Remark 1.2.9

By Theorems 1.2.7 and 1.2.8, the two definitions of λ -model in this section are essentially equivalent.

Remark

As Scott has emphasized in Scott [17], the λ -calculus considered in this chapter can be viewed as the special case of the typed λ -calculus in which there is only one type. (For a definition of the typed λ -calculus see Barendregt [1].) Moreover every reflexive object of a Cartesian closed category is a model of the (type-free) λ -calculus.

(An object U is reflexive if there exists a pair of maps $i: (U \rightarrow U) \rightarrow U$ and $j: U \rightarrow (U \rightarrow U)$ where $j \circ i = \text{id}_{U \rightarrow U}$.) Note that any Cartesian closed category is a model of a typed λ -calculus.

(For a brief analysis of the subject see Scott [17] and Lambek [12].)

CHAPTER II

The material of this chapter is derived from Scott [15].

Section 0. Background

The purpose of this section is to introduce some of the fundamental concepts and definitions which will be used throughout.

Definition 2.0.1

Let (X, τ) be a topological space and let $Y \subseteq X$. Let $\tau_Y = \{Y \cap U \mid U \in \tau\}$. The space (Y, τ_Y) is called a subspace of (X, τ) .

Definition 2.0.2

A topological space is called T_0 if for each pair of distinct points, at least one has a neighborhood not containing the other.

All spaces in this thesis are T_0 -spaces.

Definition 2.0.3

The topological space consisting of a two-point set $X = \{\perp, \top\}$ with the topology $\tau = \{\emptyset, \{\top\}, X\}$ is called the Sierpinski space and is denoted by \mathbb{O} .

The Sierpinski space is a T_0 -space.

Definition 2.0.4

Let $\{X_i \mid i \in I\}$ be any family of topological spaces. For each $i \in I$, let τ_i be the topology for X_i . The cartesian product topology τ_P in $\prod_{i \in I} X_i$ is that having for a subbasis all sets of the form $p_i^{-1}(U_i)$ where $U_i \in \tau_i$, $p_i: \prod_{i \in I} X_i \rightarrow X_i$ is the projection onto the i th coordinate, and $i \in I$.

The basic open sets are all those sets of the form $\prod_{i \in I} A_i$ where $A_i \in \tau_i$ and the set $\{i \in I \mid A_i \neq X_i\}$ is finite.

Definition 2.0.5

Let $(X, \tau_X), (Y, \tau_Y)$ be topological spaces and let $x_0 \in X$. A

map $f: X \rightarrow Y$ is continuous at x_0 if for each neighborhood V of $f(x_0)$ in Y there exists a neighborhood U of x_0 in X such that $f(U) \subseteq V$.

Definition 2.0.6

Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f: X \rightarrow Y$ is called continuous if the inverse image of each set open in Y is open in X .

Definition 2.0.7

For topological spaces (X, τ_X) and (Y, τ_Y) we let $[X \rightarrow Y]$ be the space of all continuous functions $f: X \rightarrow Y$ endowed with the product topology, sometimes called the topology of pointwise convergence. This topology has as a subbase sets of the form:

$$F(x, U) = \{f \in [X \rightarrow Y] \mid f(x) \in U\},$$

where $x \in X$ and $U \in \tau_Y$.

Definition 2.0.8

Let X, Y be topological spaces. A continuous bijective map $f: X \rightarrow Y$ such that $f^{-1}: Y \rightarrow X$ is also continuous, is called a homeomorphism.

Two spaces X, Y are homeomorphic, written $X \cong Y$, if there is a homeomorphism $f: X \rightarrow Y$.

Theorem 2.0.9 ([7])

Let $f: X \rightarrow Y$ and $g: Y \rightarrow X$ be continuous and such that both $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$. Then f is a homeomorphism, and in fact, $g = f^{-1}$.

Definition 2.0.10

If Z is any space and $f: X \rightarrow Z$ is a map satisfying $X \cong f(X) \subseteq Z$, then f is called an embedding map of X into Z , and

X is said to be embedded in Z .

Definition 2.0.11

Let (X, τ) be a topological space and $A \subseteq X$. A is a retract of X if the identity map $\text{id}_A: A \rightarrow A$ is extendable to a continuous $j: X \rightarrow A$; such an extension is called a retraction. Equivalently, A is a retract of X if there exists a continuous $j: X \rightarrow A$ such that $j(a) = a$ for each $a \in A$.

Definition 2.0.12

A binary relation ' \leq ' in a set A is called a preorder if it is reflexive and transitive. A set together with a definite preorder is called a preordered set.

Definition 2.0.13

A directed set is a preordered set (I, \leq) with the following property: For each $i, j \in I$ there exists a $k \in I$ such that $i \leq k$ and $j \leq k$.

All directed sets in this thesis are considered to be non-empty.

Definition 2.0.14

Let (I, \leq) be a directed set and X any set. Any function $x: I \rightarrow X$ is called a net in X . We write $x(i) = x_i$ for all $i \in I$.

Definition 2.0.15

Let (I, \leq) be a directed set and (X, τ) a topological space. We say that a net $x: I \rightarrow X$ converges to $y \in X$ iff whenever $U \in \tau$ and $y \in U$, then for some $i \in I$ we have $x_j \in U$ for all $j \geq i$.

Proposition 2.0.16 ([8])

Let f be a function from a space (X, τ_X) to a space (Y, τ_Y) . Then f is continuous iff for every net $\{x_i | i \in I\}$ in X which converges to $x \in X$, the net $\{f(x_i) | i \in I\}$ in Y converges to $f(x)$.

Definition 2.0.17

Let X and Y be two sets and $A \subseteq X$. Given an $f: X \rightarrow Y$, the map f considered only on A is called the restriction of f to A and is written $f|_A$. In the inverse direction, if $g: A \rightarrow Y$ is a given map, a map $G: X \rightarrow Y$ satisfying $G|_A = g$ is called an extension of g over X relative to Y .

Section 1. Injective spaces

Definition 2.1.1

A T_0 -space D is injective iff for arbitrary spaces X and Y if $X \subseteq Y$ as a subspace, then every continuous function $f: X \rightarrow D$ can be extended to a continuous function $\bar{f}: Y \rightarrow D$.

Proposition 2.1.2

The Sierpinski space $\mathbb{0}$ is injective.

Proof:

Let X, Y be spaces such that X is a subspace of Y , and let $f: X \rightarrow \mathbb{0}$ be continuous. Then $U = f^{-1}(\{\top\})$ is open in X , and there exists V open in Y such that $U = V \cap X$. Define $g: Y \rightarrow \mathbb{0}$ to be the characteristic function of V (that is $g^{-1}(\{\top\}) = V$). Then $g: Y \rightarrow \mathbb{0}$ is continuous and $g|_X = f$. Hence $\mathbb{0}$ is injective. \square

Proposition 2.1.3

The Cartesian product of any number of injective spaces is injective under the product topology.

Proof:

Let $\{D_i | i \in I\}$ be a family of injective spaces, and $D = \prod_{i \in I} D_i$. Let X, Y be spaces such that X is a subspace of Y and $f: X \rightarrow D$ be continuous. Then each $f_i = p_i \circ f: X \rightarrow D_i$ can be extended to a continuous function $\bar{f}_i: Y \rightarrow D_i$. Thus $\bar{f} = (\bar{f}_1, \bar{f}_2, \dots): Y \rightarrow D$ is a continuous extension of f . Hence D is injective. \square

Proposition 2.1.4

A retract of an injective space is injective.

Proof:

Consider an injective space D , and let D' be a retract of D . There exists a continuous map $j: D \rightarrow D'$ such that $j(d) = d$ for all

$d \in D'$. Let X, Y be spaces such that X is a subspace of Y , and let $f: X \rightarrow D'$ be continuous. Thus $f: X \rightarrow D$ continuous, and let $\bar{f}: Y \rightarrow D$ be a continuous extension of f . Then $j \circ \bar{f}: Y \rightarrow D'$ is continuous and $(j \circ \bar{f})|_X = f$. Hence D' is injective. \square

Theorem 2.1.5 ([3], p. 484)

Every T_0 -space can be embedded in an injective space; in fact, in a Cartesian power of the two-element space $\mathbb{0}$.

Corollary 2.1.6

The injective spaces are exactly the retracts of the Cartesian powers of $\mathbb{0}$.

Proof:

Let D be an injective space. Then by 2.1.5 D is homeomorphic to a subspace of a power of $\mathbb{0}$. Since D is injective the identity function on the subspace to itself can be extended to the whole of the power of $\mathbb{0}$ providing the required retraction map.

Conversely if D is a retract of a power of $\mathbb{0}$, then by 2.1.4 D is injective. \square

Corollary 2.1.7

A space is injective iff it is a retract of every space of which it is a subspace.

Proof:

Let D be an injective space and suppose $D \subseteq D'$ as a subspace. Since D is injective the identity map on D to itself can be extended to D' providing the required retraction map.

Conversely let D be a space which is a retract of every space of which it is a subspace. Then by 2.1.5 D is a retract of a power of $\mathbb{0}$.

Hence D is injective. \square

Section 2. Partially ordered sets

Definition 2.2.1

A binary relation ' \leq ' in a set A is called a partial ordering if it is reflexive, antisymmetric, and transitive. A set together with a definite partial ordering is called a partially ordered set (poset).

Definition 2.2.2

Let (P, \leq) be a poset and let $X \subseteq P$.

(i) A least element of X is an element $a \in X$ such that $a \leq x$ for all $x \in X$.

(ii) A greatest element of X is an element $b \in X$ such that $x \leq b$ for all $x \in X$.

(iii) An element $c \in X$ is maximal if $c \leq x$ implies $c = x$ for all $x \in X$.

(iv) An element $d \in X$ is minimal if $x \leq d$ implies $d = x$ for all $x \in X$.

Definition 2.2.3

Let (P, \leq) be a poset and let $X \subseteq P$.

(i) An upper bound of X is an element $a \in P$ such that $x \leq a$ for all $x \in X$.

(ii) A lower bound of X is an element $b \in P$ such that $b \leq x$ for all $x \in X$.

(iii) An upper bound a of X is the least upper bound (lub or sup or \sqcup) of X if for any upper bound c of X we have $a \leq c$.

(iv) A lower bound b of X is the greatest lower bound (glb or inf or \sqcap) of X if for any lower bound c of X we have $c \leq b$.

Proposition 2.2.4 ([2])

A directed set is a poset in which any two elements, and hence any

finite subset, has an upper bound in the set.

Proposition 2.2.5 ([18])

Let (P, \leq) be a poset and let $A, B \subseteq P$. Then

$$(1) A \subseteq B \Rightarrow \sqcup A \leq \sqcup B \text{ and } \sqcap B \leq \sqcap A$$

$$(2) \sqcup \{z \in P \mid z \leq y\} = y \text{ and } \sqcap \{z \in P \mid y \leq z\} = y,$$

for all $y \in P$.

$$(3) \text{ If } a \text{ is an upper bound of } A \text{ and } a \in A, \text{ then } a = \sqcup A.$$

$$(4) \sqcup \{a\} = a \text{ for all } a \in P.$$

(5) Notation: If x and y are any elements of P we write $x \sqcup y$ for $\sqcup \{x, y\}$, and $x \sqcap y$ for $\sqcap \{x, y\}$.

$$(6) \text{ Whenever } x \sqcup y \text{ exists, } x \leq x \sqcup y \text{ and } y \leq x \sqcup y.$$

$$(7) \text{ Whenever } x \sqcup y \text{ exists, } x \sqcup y = y \text{ iff } x \leq y.$$

$$(8) \text{ Suppose } A_i \subseteq P \text{ and } \sqcup A_i, \sqcap A_i \text{ exist for all } i \in I. \text{ Then}$$

$$\sqcup \left(\bigcup_{i \in I} A_i \right) = \sqcup \{ \sqcup A_i \mid i \in I \} \text{ and}$$

$$\sqcap \left(\bigcup_{i \in I} A_i \right) = \sqcap \{ \sqcap A_i \mid i \in I \}.$$

$$(9) \text{ Convention: } \sqcup \emptyset = \perp, \text{ provided that } P \text{ has a least element. Also}$$

$$\sqcap \emptyset = \top, \text{ provided that } P \text{ has a greatest element.}$$

Proposition 2.2.6

Let $(P, \leq), (P', \leq')$ be posets and $A^* \subseteq P \times P' = P^*$. Consider the projections A and A' of A^* into P and P' respectively. Then, if the necessary lubs exist, $\sqcup A^* = (\sqcup A, \sqcup A')$.

Proof:

$$A^* \subseteq A \times A' \text{ therefore } \sqcup A^* \leq^* \sqcup (A \times A') = (\sqcup A, \sqcup A'). \text{ So } (\sqcup A, \sqcup A')$$

is an upper bound of A^* . Must show that it is the least upper bound.

Suppose (a, b) is another upper bound of A^* . Then $(x, y) \leq^* (a, b)$

for all $(x, y) \in A^*$. Thus $x \leq a$ for all $x \in A$ and $y \leq' b$ for all

$y \in A'$. Therefore $\sqcup A \leq a$ and $\sqcup A' \leq b$. So $(\sqcup A, \sqcup A') \leq^* (a, b)$.

Hence $\sqcup A^* = (\sqcup A, \sqcup A')$. \square

Definition 2.2.7

A poset D is a lattice if $x \sqcup y$ and $x \sqcap y$ exist for all $x, y \in D$.

Proposition 2.2.8 ([18])

Every non-empty finite subset of a lattice D has both a lub and a glb in D .

Proposition 2.2.9 ([2])

The direct product of any two lattices is a lattice.

Definition 2.2.10

A sublattice of a lattice D is a subset D' of D such that $x \in D', y \in D'$ imply $x \sqcup y \in D'$ and $x \sqcap y \in D'$.

Definition 2.2.11

A lattice D is said to be complete if every subset of D has a lub and a glb in D .

Proposition 2.2.12 ([9])

If D is a poset and every subset of D has a glb (or a lub) in D , then D is a complete lattice.

Proposition 2.2.13 ([2])

Any non-empty complete lattice contains a least element \perp (bottom), and a greatest element \top (top).

Proposition 2.2.14 ([2])

Any finite lattice is a complete lattice.

Proposition 2.2.15 ([4])

The product of any number of complete lattices is a complete lattice.

Theorem 2.2.16 ([2])

Let D be a complete lattice and f a monotonic function on D into D . Then $f(x) = x$ for some $x \in D$.

Definition 2.2.17

A sup-semilattice is a poset D in which every nonempty finite subset has a lub.

An inf-semilattice is a poset D in which every nonempty finite subset has a glb.

Proposition 2.2.18

Let D be a sup-semilattice with \perp . If every directed subset of D has a lub in D , then D is a complete lattice.

Proof:

Let $X \subseteq D$. Then every nonempty finite subset of X has a lub in D .

Also $\perp \in D$. Thus every finite subset of X has a lub in D .

Let $F = \{\cup A \mid A \text{ is a finite subset of } X\}$. F is obviously directed.

Therefore $\cup F \in D$. We have $X \subseteq F$. Thus any upper bound of F is also an upper bound of X . But an upper bound of X is also an upper bound of A for all $A \subseteq X$ and therefore is an upper bound of F . So X and F have the same set of upper bounds. Hence $\cup X = \cup F \in D$. So D is a complete lattice. \square

Definition 2.2.19

Let (X, τ) be a T_0 -space. Define a binary relation ' \sqsubseteq ' in X as follows:

$x \sqsubseteq y$ iff whenever $x \in U$ and $U \in \tau$ then $y \in U$,

for all $x, y \in X$.

This relation is obviously reflexive and transitive, and the condition that it be antisymmetric is exactly equivalent to the T_0 -axiom. Thus

' \sqsubseteq ' is a partial ordering on X , and it is called the induced partial ordering on X .

Proposition 2.2.20

Let X, Y be T_0 -spaces, and consider the function space $[X \rightarrow Y]$. Then the induced partial ordering on $[X \rightarrow Y]$ is such that:

$$f \sqsubseteq g \text{ iff } f(x) \sqsubseteq g(x) \text{ for all } x \in X,$$

where $f, g \in [X \rightarrow Y]$.

Proof:

Let $f, g \in [X \rightarrow Y]$.

Suppose $f \sqsubseteq g$, and let $x \in X$. Let V be open in Y with $f(x) \in V$.

Then the set $F(x, V) = \{f \in [X \rightarrow Y] \mid f(x) \in V\}$ is open in $[X \rightarrow Y]$, and

$f \in F(x, V)$. Then $g \in F(x, V)$. Thus $g(x) \in V$. Hence $f(x) \sqsubseteq g(x)$.

Conversely suppose $f(x) \sqsubseteq g(x)$ for all $x \in X$, and let $F(x, V)$ be open in $[X \rightarrow Y]$ with $f \in F(x, V)$. Then V is open in Y and

$f(x) \in V$. Then $g(x) \in V$. Thus $g \in F(x, V)$. Hence $f \sqsubseteq g$. \square

Definition 2.2.21

Let (X, \sqsubseteq) be a poset. Define $U \subseteq X$ to be open iff it satisfies the conditions:

(i) whenever $x \in U$ and $x \sqsubseteq y$ then $y \in U$

(ii) whenever $S \subseteq X$ is directed, $\sqcup S$ exists, and $\sqcup S \in U$,

then $S \cap U \neq \emptyset$.

The family $\{U \subseteq X \mid U \text{ is open}\}$ is obviously a topology on X , called the induced topology (τ_1) on X .

Proposition 2.2.22

Let (X, \sqsubseteq) be a poset, and consider the induced topology on X .

For any $y \in X$, the set $\{x \in X \mid x \sqsubseteq y\}$ is open.

Proof:

Let $y \in X$ and let $U = \{x \in X \mid x \not\sqsubseteq y\}$.

(i) Suppose $z \in U$ and $z \sqsubseteq w$. Then $z \not\sqsubseteq y$ and $z \sqsubseteq w$. Therefore $w \not\sqsubseteq y$. Hence $w \in U$.

(ii) Suppose $S \subseteq X$ is directed, $\sqcup S$ exists and $\sqcup S \in U$. Then $\sqcup S \not\sqsubseteq y$. Thus $z \not\sqsubseteq y$ for some $z \in S$. So $z \in U$ for some $z \in S$.

Hence $S \cap U \neq \emptyset$.

Therefore U is open. □

Proposition 2.2.23

Let (X, \sqsubseteq) be a poset. Then (X, τ_I) is a T_0 -space.

Proof:

Let $x, y \in X$ such that $x \neq y$. We cannot have both $x \sqsubseteq y$ and $y \sqsubseteq x$.

If $x \not\sqsubseteq y$, then the set $U = \{z \in X \mid z \not\sqsubseteq y\}$ is open, $x \in U$, and $y \notin U$.

Similarly if $y \not\sqsubseteq x$, then the set $V = \{z \in X \mid z \not\sqsubseteq x\}$ is open, $y \in V$, and $x \notin V$.

So at least one of x, y has a neighborhood not containing the other.

Hence (X, τ_I) is T_0 . □

Proposition 2.2.24

Let (X, \sqsubseteq) be a poset, and consider the induced topology on X .

Let $S \subseteq X$ be directed. Then S is obviously a net in X , and S converges to $\sqcup S$ if this lub exists.

Proof:

Let U be open in X with $\sqcup S \in U$. Then $S \cap U \neq \emptyset$. So there exists

$x' \in S$ such that $x' \in U$. Thus, there exists $x' \in S$ such that $x \in U$

for all $x \in X$ with $x' \sqsubseteq x$. Hence S converges to $\sqcup S$. □

Definition 2.2.25

Let (I, \leq) be a directed set and let (X, \sqsubseteq) be a poset. A

net $x: I \rightarrow X$ is monotone if $i \leq j$ implies $x_i \sqsubseteq x_j$ for all $i, j \in I$.

Proposition 2.2.26

In a poset (X, \sqsubseteq) with the induced topology, a monotone net $x: I \rightarrow X$ with a lub converges to an element $y \in X$ iff $y \sqsubseteq \bigcup \{x_i \mid i \in I\}$.

Proof:

Suppose x converges to $y \in X$, and suppose that $y \not\sqsubseteq \bigcup \{x_i \mid i \in I\}$.

Then the set $U = \{z \in X \mid z \not\sqsubseteq \bigcup \{x_i \mid i \in I\}\}$ is open and $y \in U$. Thus for some $i \in I$ we have $x_j \in U$ for all $j \geq i$. So for some $i \in I$ we have $x_j \not\sqsubseteq \bigcup \{x_i \mid i \in I\}$ for all $j \geq i$. Contradiction. Hence $y \sqsubseteq \bigcup \{x_i \mid i \in I\}$.

Conversely suppose $y \sqsubseteq \bigcup \{x_i \mid i \in I\}$, and let U be open with $y \in U$.

Then $\bigcup \{x_i \mid i \in I\} \in U$. But $\{x_i \mid i \in I\}$ is directed because $x: I \rightarrow X$ is a monotone net. Therefore $\{x_i \mid i \in I\} \cap U \neq \emptyset$. Thus $x_i \in U$ for some $i \in I$. So, since $x: I \rightarrow X$ is monotone we have that there exists $i \in I$ such that $x_j \in U$ for all $j \geq i$. Hence, x converges to y .

Proposition 2.2.27

In T_0 -spaces continuous functions are always monotonic.

Proof:

Let $(X, \tau_X), (Y, \tau_Y)$ be T_0 -spaces and let $f: X \rightarrow Y$ be continuous.

Consider the induced partial orderings on X and Y .

Let $x, y \in X$ such that $x \sqsubseteq y$, and let $V \in \tau_Y$ with $f(x) \in V$. Then $f^{-1}(V) \in \tau_X$ and $x \in f^{-1}(V)$. Thus $y \in f^{-1}(V)$. Therefore $f(y) \in V$.

Thus $f(x) \sqsubseteq f(y)$. □

Proposition 2.2.28

If D and D' are complete lattices with their induced topologies, then a function $f: D \rightarrow D'$ is continuous iff for all directed subsets

$S \subseteq D$:

$$f(\sqcup S) = \sqcup \{f(x) \mid x \in S\}.$$

Proof:

Suppose $f: D \rightarrow D'$ is continuous, and let $S \subseteq D$ be directed. Then S converges to $\sqcup S$, and $\{f(x) \mid x \in S\}$ converges to $f(\sqcup S)$. Thus, by

2.2.25

$$f(\sqcup S) \sqsubseteq \sqcup \{f(x) \mid x \in S\} \quad (1)$$

Also, since f is monotonic we have $f(x) \sqsubseteq f(\sqcup S)$ for all $x \in S$.

Therefore

$$\sqcup \{f(x) \mid x \in S\} \sqsubseteq f(\sqcup S) \quad (2)$$

$$(1), (2) \Rightarrow f(\sqcup S) = \sqcup \{f(x) \mid x \in S\}.$$

Conversely suppose $f(\sqcup S) = \sqcup \{f(x) \mid x \in S\}$ for all directed subsets $S \subseteq D$, and let $x, y \in D$ with $x \sqsubseteq y$. Then

$$f(y) = f(x \sqcup y) = \sqcup \{f(x), f(y)\} = f(x) \sqcup f(y). \text{ So } f(x) \sqsubseteq f(y).$$

Hence f is monotonic.

Now let $U' \subseteq D'$ be open and consider the set $U = f^{-1}(U') \subseteq D$. We have:

(i) If $x \in U$ and $x \sqsubseteq y$ then $f(x) \in U'$ and $f(x) \sqsubseteq f(y)$. Thus $f(y) \in U'$. So $y \in U$.

(ii) If $S \subseteq D$ is directed and $\sqcup S \in U$ then $f(\sqcup S) \in U'$. Thus $\sqcup \{f(x) \mid x \in S\} \in U'$. So $f(S) \cap U' \neq \emptyset$. Hence $S \cap U \neq \emptyset$.

Therefore U is open. Hence $f: D \rightarrow D'$ is continuous. \square

Proposition 2.2.29

With functions from complete lattices to complete lattices, a function of several variables is continuous in the variables jointly iff it is continuous in the variables separately.

Proof:

It will be sufficient to discuss functions of two variables. Let

D, D', D'' be complete lattices and let $f: D \times D' \rightarrow D''$ be continuous on the product lattice $D \times D'$. Then clearly

for all $d' \in D'$ the function $d \mapsto f(d, d'): D \rightarrow D''$ is continuous, and for all $d \in D$ the function $d' \mapsto f(d, d'): D' \rightarrow D''$ is continuous.

Hence f is continuous in each variable separately.

To check the converse suppose that $f: D \times D' \rightarrow D''$ is a map where the separate continuity holds as follows:

$$f(\sqcup S, y) = \sqcup \{f(x, y) \mid x \in S\}, \text{ and}$$

$$f(x, \sqcup S') = \sqcup \{f(x, y) \mid y \in S'\},$$

where $S \subseteq D$ and $S' \subseteq D'$ are directed and $x \in D$ and $y \in D'$. Let now

$S^* \subseteq D \times D'$ be directed in the product. The projection of S^* to

$S \subseteq D$ and S^* to $S' \subseteq D'$ produces directed subsets of D and D' ,

and $\sqcup S^* = (\sqcup S, \sqcup S')$. Thus, by assumption

$$\begin{aligned} f(\sqcup S^*) &= f(\sqcup S, \sqcup S') = \sqcup \{f(\sqcup S, y) \mid y \in S'\} = \sqcup \{\sqcup \{f(x, y) \mid x \in S\} \mid y \in S'\} \\ &= \sqcup \{f(x, y) \mid x \in S, y \in S'\}. \end{aligned}$$

But since S^* is directed, $x \in S$ and $y \in S'$ implies $x \sqsubseteq u$ and $y \sqsubseteq v$ for $(u, v) \in S^*$. Thus, by monotonicity of f we have:

$$f(\sqcup S^*) = \sqcup \{f(u, v) \mid (u, v) \in S^*\}$$

which gives the joint continuity. \square

Proposition 2.2.30

Let $\{D_i \mid i \in I\}$ be a family of complete lattices and D be a complete lattice. Then the lattice operation $\sqcup: \prod_{i \in I} D_i \rightarrow D$ is continuous.

Proof:

Since D_i are complete lattices we have by 2.2.15 that $\prod_{i \in I} D_i$ is a complete lattice. Let $S^* \subseteq \prod_{i \in I} D_i$ be directed in the product. Then

$$\sqcup(\sqcup S^*) = \sqcup S^* = \sqcup\{\sqcup\{x\} \mid x \in S^*\}.$$

Hence \sqcup is continuous.

□

Section 3. Continuous lattices

Definition 2.3.1

Let D be a complete lattice with its induced topology. For all $x, y \in D$ define:

$$x < y \text{ iff } y \in \text{Int}\{w \in D \mid x \sqsubseteq w\}.$$

Proposition 2.3.2

Let D be a complete lattice with its induced topology. Then for all $x, y, z \in D$ we have:

- (i) $\perp < x$;
- (ii) $x < z$ and $y < z$ imply $x \sqcup y < z$;
- (iii) $x < y \sqsubseteq z$ implies $x < z$;
- (iv) $x \sqsubseteq y < z$ implies $x < z$;
- (v) $x < y$ implies $x \sqsubseteq y$;
- (vi) $x < x$ iff $\{w \in D \mid x \sqsubseteq w\}$ is open;
- (vii) if $S \subseteq D$ is directed, then

$$x < \sqcup S \text{ iff } x < y \text{ for some } y \in S.$$

Proof:

Let $x, y, z \in D$. Then

- (i) $x \in D \Rightarrow x \in \text{Int } D \Rightarrow x \in \text{Int}\{w \in D \mid \perp \sqsubseteq w\} \Rightarrow \perp < x$.
- (ii) $x < z$ and $y < z \Rightarrow z \in \text{Int}\{w \in D \mid x \sqsubseteq w\}$ and $z \in \text{Int}\{w \in D \mid y \sqsubseteq w\} \Rightarrow$
 $z \in \text{Int}\{w \in D \mid x \sqsubseteq w\} \cap \text{Int}\{w \in D \mid y \sqsubseteq w\} \Rightarrow$
 $z \in \text{Int}\{w \in D \mid x \sqsubseteq w \text{ and } y \sqsubseteq w\} \Rightarrow$
 $z \in \text{Int}\{w \in D \mid x \sqcup y \sqsubseteq w\} \Rightarrow x \sqcup y < z$.
- (iii) $x < y \sqsubseteq z \Rightarrow y \in \text{Int}\{w \in D \mid x \sqsubseteq w\}$ and $y \sqsubseteq z \Rightarrow$
 $z \in \text{Int}\{w \in D \mid x \sqsubseteq w\} \Rightarrow x < z$.

(iv) $x \sqsubseteq y < z \Rightarrow x \sqsubseteq y$ and $z \in \text{Int}\{w \in D \mid y \sqsubseteq w\} \Rightarrow$
 $\{w \in D \mid y \sqsubseteq w\} \subseteq \{w \in D \mid x \sqsubseteq w\}$ and $z \in \text{Int}\{w \in D \mid y \sqsubseteq w\} \Rightarrow$
 $\text{Int}\{w \in D \mid y \sqsubseteq w\} \subseteq \text{Int}\{w \in D \mid x \sqsubseteq w\}$ and
 $z \in \text{Int}\{w \in D \mid y \sqsubseteq w\} \Rightarrow z \in \text{Int}\{w \in D \mid x \sqsubseteq w\} \Rightarrow x < z$.

(v) $x < y \Rightarrow y \in \text{Int}\{w \in D \mid x \sqsubseteq w\} \subseteq \{w \in D \mid x \sqsubseteq w\} \Rightarrow$
 $y \in \{w \in D \mid x \sqsubseteq w\} \Rightarrow x \sqsubseteq y$.

(vi) Suppose $x < x$, and $z \in \{w \in D \mid x \sqsubseteq w\}$. Then $x < x$ and
 $x \sqsubseteq z \Rightarrow x < z \Rightarrow z \in \text{Int}\{w \in D \mid x \sqsubseteq w\}$. Thus
 $\{w \in D \mid x \sqsubseteq w\} \subseteq \text{Int}\{w \in D \mid x \sqsubseteq w\} \Rightarrow \{w \in D \mid x \sqsubseteq w\} = \text{Int}\{w \in D \mid x \sqsubseteq w\}$. Hence
 $\{w \in D \mid x \sqsubseteq w\}$ is open.

Conversely suppose that the set $\{w \in D \mid x \sqsubseteq w\}$ is open. Then
 $\{w \in D \mid x \sqsubseteq w\} = \text{Int}\{w \in D \mid x \sqsubseteq w\}$. Therefore $x \in \text{Int}\{w \in D \mid x \sqsubseteq w\}$.
Hence $x < x$.

(vii) Suppose $S \subseteq D$ is directed. Then
 $x < \bigcup S \iff \bigcup S \in \text{Int}\{w \in D \mid x \sqsubseteq w\} \iff$
 $S \cap \text{Int}\{w \in D \mid x \sqsubseteq w\} \neq \emptyset \iff y \in \text{Int}\{w \in D \mid x \sqsubseteq w\}$ for some $y \in S \iff x < y$
for some $y \in S$. \square

Proposition 2.3.3

Let D be a complete lattice with its induced topology. Then for
each $x \in D$ the set $\{w \in D \mid w < x\}$ is directed.

Proof:

Let $x \in D$, and let $S = \{w \in D \mid w < x\}$. Then $s, t \in S \Rightarrow s < x$ and
 $t < x \Rightarrow s \sqcup t < x \Rightarrow s \sqcup t \in S$. Hence S is directed. \square

Definition 2.3.4

A continuous lattice is a complete lattice D in which for every
 $y \in D$ we have:

$$y = \sqcup \{w \in D \mid w < y\}.$$

Proposition 2.3.5

Let D be a continuous lattice under its induced topology. Then the sets of the form $\{y \in D \mid x < y\}$ where $x \in D$ form a basis for the open sets of D .

Proof:

Let $x \in D$. Then $\{y \in D \mid x < y\} = \text{Int}\{w \in D \mid x \sqsubseteq w\}$. Thus $\{y \in D \mid x < y\}$ is open.

Let now U be open with $z \in U$. But $z = \sqcup \{w \in D \mid w < z\}$ and $\{w \in D \mid w < z\}$ is directed. Therefore $\{w \in D \mid w < z\} \cap U \neq \emptyset$. So $w_0 < z$ for some $w_0 \in U$.

Since U is open we have

$$\{y \in D \mid w_0 \sqsubseteq y\} \subseteq U \Rightarrow \text{Int}\{y \in D \mid w_0 \sqsubseteq y\} \subseteq U \Rightarrow \{y \in D \mid w_0 < y\} \subseteq U.$$

Also $z \in \{y \in D \mid w_0 < y\}$. Hence the family $\{\{y \in D \mid x < y\} \mid x \in D\}$ is a basis for the induced topology on D . □

Proposition 2.3.6

A complete lattice D is continuous iff for every $y \in D$ we have:

$$y = \sqcup \{\sqcap U \mid y \in U\},$$

where U ranges over the open subsets of D .

Proof:

Suppose D is a continuous lattice under its induced topology τ_I , and let $y \in D$. Let also $V_x = \{z \in D \mid x < z\}$ be a basic open set of D with $y \in V_x$. Then

$$\begin{aligned} V_x &\subseteq \{z \in D \mid x \sqsubseteq z\} \Rightarrow \sqcap \{z \in D \mid x \sqsubseteq z\} \subseteq \sqcap V_x \subseteq y \\ &\Rightarrow x \sqsubseteq \sqcap V_x \subseteq y \Rightarrow \sqcup \{x \mid x < y\} \subseteq \sqcup \{\sqcap V_x \mid x < y\} \subseteq y \\ &\Rightarrow y \subseteq \sqcup \{\sqcap V_x \mid y \in V_x\} \subseteq y. \text{ Thus } y = \sqcup \{\sqcap V \mid y \in V\} \text{ where } V \text{ ranges} \\ &\text{over the basic open subsets of } D. \text{ So } y = \sqcup \{\sqcap U \mid y \in U\} \text{ where } U \end{aligned}$$

ranges over the open subsets of D .

For the converse let $y \in D$. Then $y = \sqcup \{ \sqcap U \mid y \in U \text{ and } U \text{ is open} \}$.

But whenever U is open we have

$$y \in U \iff y \in \text{Int } U \iff y \in \text{Int} \{ z \in D \mid \sqcap U \subseteq z \} \iff \sqcap U < y.$$

So $y = \sqcup \{ w \in D \mid w < y \}$. Hence D is a continuous lattice. \square

Definition 2.3.7

In any complete lattice D define the principal limit of a net $x: I \rightarrow D$ by the formula

$$\lim \langle x_i \mid i \in I \rangle = \sqcup \{ \sqcap \{ x_j \mid j \geq i \} \mid i \in I \}.$$

Proposition 2.3.8

Let D be a continuous lattice and let $x: I \rightarrow D$ be a net in X . Then x converges to $y \in D$ iff

$$y \subseteq \lim \langle x_i \mid i \in I \rangle.$$

Proof:

Suppose x converges to $y \in D$, and let U be open in D with $y \in U$. Then for some $i \in I$ we have $x_j \in U$ for all $j \geq i$. Thus $\sqcap U \subseteq \sqcap \{ x_j \mid j \geq i \}$. Therefore

$$\sqcup \{ \sqcap U \mid y \in U \} \subseteq \sqcup \{ \sqcap \{ x_j \mid j \geq i \} \mid i \in I \} = y \subseteq \lim \langle x_i \mid i \in I \rangle.$$

For the converse suppose that $y \subseteq \lim \langle x_i \mid i \in I \rangle$, and let U be open in D with $y \in U$. Then

$$\begin{aligned} \lim \langle x_i \mid i \in I \rangle \in U &\Rightarrow \sqcup \{ \sqcap \{ x_j \mid j \geq i \} \mid i \in I \} \in U \\ &\Rightarrow \{ \sqcap \{ x_j \mid j \geq i \} \mid i \in I \} \cap U \neq \emptyset \Rightarrow \sqcap \{ x_j \mid j \geq i \} \in U \text{ for some} \end{aligned}$$

$i \in I \Rightarrow$ for some $i \in I$ we have $x_j \in U$ for all $j \geq i$. Hence x converges to y . \square

We have shown that in continuous lattices the two notions of convergence are the same. Moreover if the two notions of convergence coincide for a complete lattice D , then D is a continuous lattice.

Proposition 2.3.9

Let (D, τ) be a T_0 -space which becomes a complete lattice under its induced partial ordering. Consider now the induced topology τ_I on D . If $\tau \subseteq \tau_I$ and for all $y \in D$ $y = \sqcup \{\sqcap V | y \in V, V \in \tau\}$, then $\tau = \tau_I$ and D is a continuous lattice.

Proof:

Let $U \in \tau_I$ and $y \in U$. Then

$y = \sqcup \{\sqcap V | y \in V, V \in \tau\} \in U \Rightarrow \{\sqcap V | y \in V, V \in \tau\} \cap U \neq \emptyset \Rightarrow \sqcap V \in U$ for some $V \in \tau$ with $y \in V$. But $V \subseteq U$ follows, and so U is a union of given open sets and it is itself open in the given topology.

Thus $\tau_I \subseteq \tau$. Hence $\tau_I = \tau$, and for all $y \in D$ we have:

$$y = \sqcup \{\sqcap U | y \in U, U \in \tau_I\}.$$

Hence D is a continuous lattice. □

Proposition 2.3.10

A finite lattice is a continuous lattice.

Proof:

Let D be a finite lattice. Then, by 2.2.14 D is complete. Let $y \in D$, and consider the set $U = \{w \in D | y \sqsubseteq w\}$. Then

(i) $x \in U$ and $x \sqsubseteq z \Rightarrow y \sqsubseteq x$ and $x \sqsubseteq z \Rightarrow y \sqsubseteq z \Rightarrow z \in U$

(ii) if $S \subseteq D$ is directed and $\sqcup S \in U$, then $\sqcup S \in S$ because S is finite and $\sqcup S \in U \Rightarrow \sqcup S \in S \cap U \Rightarrow S \cap U \neq \emptyset$.

Thus $U = \{w \in D | y \sqsubseteq w\}$ is open, and so $y < y$. Therefore

$y = \sqcup \{x \in D | x < y\}$. Hence D is a continuous lattice. □

Proposition 2.3.11

The Cartesian product of any number of continuous lattices is a continuous lattice with the induced topology agreeing with the product

topology.

Proof:

Let $\{D_i | i \in I\}$ be a family of continuous lattices. The product $D^* = \prod_{i \in I} D_i$ is a complete lattice by 2.2.15, and has its induced topology τ_I . Suppose $y \in D^*$ and let $i \in I$. Then $y_i \in D_i$. Since D_i is a continuous lattice

$$y_i = \sqcup \{x \in D_i | x < y_i\}.$$

For $x \in D_i$, let $[x]^i \in D^*$ be defined by:

$$[x]^i_j = \begin{cases} x & \text{if } i = j \\ \perp & \text{if } i \neq j \end{cases}.$$

Then since D_i is continuous we have:

$$[y_i]^i = \sqcup \{[x]^i | x < y_i\}$$

and

$$y = \sqcup \{[y_i]^i | i \in I\}.$$

Considering $V = \{u \in D_i | x < u\}$, which is a basic open subset of D_i , we have:

$$\begin{aligned} [x]^i &\subseteq z \text{ for all } z \in D^* \text{ with } z_i \in V \Rightarrow \\ [x]^i &\subseteq \cap \{z \in D^* | z_i \in V\} \Rightarrow \\ \sqcup \{[x]^i | x < y_i\} &\subseteq \sqcup \{\cap \{z | z_i \in V\} | y_i \in V\} \Rightarrow \\ [y_i]^i &\subseteq \cap \{z | z_i \in V\} | y_i \in V \Rightarrow \\ \sqcup \{[y_i]^i | i \in I\} &\subseteq \sqcup \{\cap \{z | z_i \in V\} | y_i \in V, i \in I\} \Rightarrow \\ y &\subseteq \cap \{z | z_i \in V\} | y_i \in V, i \in I. \end{aligned} \quad (1)$$

Also

$$\cap \{\cap \{z | z_i \in V\} | y_i \in V, i \in I\} \subseteq y. \quad (2)$$

$$(1), (2) \Rightarrow y = \cap \{\cap \{z | z_i \in V\} | y_i \in V, i \in I\}.$$

But the sets $\{z | z_i \in V\} = p_i^{-1}(V)$ are open in the product sense, and so

$$y = \bigcup \{ \bigcap U \mid y \in U \},$$

where U ranges over the members of the product topology τ_p on D^* .

Let now $U \subseteq D^*$ be a basic open set of τ_p . Then $U = \prod_{i \in I} A_i$, where A_i is open in D_i and the set $\{i \in I \mid A_i \neq D_i\}$ is finite, and we have:

(i) $(x_1, x_2, \dots) \in U$ and $(x_1, x_2, \dots) \sqsubseteq (y_1, y_2, \dots) \Rightarrow$
 $x_i \in A_i$ and $x_i \sqsubseteq y_i$ for all $i \in I \Rightarrow$
 $y_i \in A_i$ for all $i \in I \Rightarrow (y_1, y_2, \dots) \in U$

(ii) if $S^* \subseteq D^*$ is directed and

$\bigcup S^* = (\bigcup S_1, \bigcup S_2, \dots) \in U$, then $\bigcup S_i \in A_i$ for all
 $i \in I \Rightarrow S_i \cap A_i \neq \emptyset$ for all $i \in I \Rightarrow S^* \cap U \neq \emptyset$.

Thus $U \in \tau_I$. So τ_I contains all the basic open subsets of τ_p .

Therefore $\tau_p \subseteq \tau_I$. Hence by 2.3.9, $\tau_p = \tau_I$ and D^* is a continuous lattice. \square

Lemma 2.3.12

Every monotonic function on a complete lattice into itself has a least fixed point.

Proof:

Let D be a complete lattice and $f: D \rightarrow D$ monotonic. Then by 2.2.16

we have that the set $M = \{x \in D \mid f(x) = x\}$ is not empty. Let

$N = \{x \in D \mid f(x) \sqsubseteq x\}$ and let $a = \bigcap N$. Then $f(a) \sqsubseteq f(x)$ for all

$x \in N \Rightarrow f(a) \sqsubseteq x$ for all $x \in N \Rightarrow f(a)$ is a lower bound of

$N \Rightarrow f(a) \sqsubseteq a \Rightarrow f(f(a)) \sqsubseteq f(a) \Rightarrow f(a) \in N \Rightarrow f(a) = a \Rightarrow a \in M$.

Also $M \subseteq N \Rightarrow \bigcap N \sqsubseteq \bigcap M \Rightarrow a \sqsubseteq \bigcap M$.

Thus a is the least element of M . \square

Proposition 2.3.13

A retract of a continuous lattice is a continuous lattice with the

subspace topology agreeing with the induced topology.

Proof:

Let D' be a continuous lattice and let $D \subseteq D'$ be a subspace which is a retract. Then there exists a continuous $j: D' \rightarrow D$ such that

$j(x) = x$ for all $x \in D$.

Suppose $x, y \in D$. Let $z' = x \sqcup' y \in D'$ and define $z = j(z') \in D$.

Now $x \sqsubseteq z'$ and $y \sqsubseteq z'$ and j is monotonic, so $j(x) \sqsubseteq j(z')$ and

$j(y) \sqsubseteq j(z') \Rightarrow x \sqsubseteq z$ and $y \sqsubseteq z$. Suppose $x \sqsubseteq w$ and $y \sqsubseteq w$ with

$w \in D$. Then in D' we have $x \sqcup' y \sqsubseteq w \Rightarrow z' \sqsubseteq w \Rightarrow j(z') \sqsubseteq j(w) \Rightarrow$

$z \sqsubseteq w$. Hence $z = x \sqcup y$ in D .

Also j is monotonic and D is the set of fixed points of j . So by

2.3.12 D has a least element \perp . Thus D is a sup-semilattice

with \perp . By 2.2.18 to show that D is a complete lattice we need to

show that every directed subset of D has a lub in D .

Let $S \subseteq D$ be directed. Then $S \subseteq D'$ is directed and $\sqcup' S \in D'$. So

S converges to $\sqcup' S$ in D' ; and by 2.0.16, $\{j(x) | x \in S\}$ converges

to $j(\sqcup' S)$ in D . Moreover $\{j(x) | x \in S\}$ is a monotone net. So,

by 2.2.26

$$j(\sqcup' S) \sqsubseteq \sqcup \{j(x) | x \in S\}. \quad (1)$$

Also $x \sqsubseteq \sqcup' S$ for all $x \in S \Rightarrow j(x) \sqsubseteq j(\sqcup' S)$ for all $x \in S \Rightarrow$

$$\sqcup \{j(x) | x \in S\} \sqsubseteq j(\sqcup' S). \quad (2)$$

$$(1), (2) \Rightarrow j(\sqcup' S) = \sqcup \{j(x) | x \in S\}$$

$$= \sqcup S.$$

Hence D is a complete lattice. We must still show that D is continuous.

Suppose $y \in D$. In D' we have:

$$y = \bigcup \{x \in D' \mid x < y\}.$$

But, since $\{x \in D' \mid x < y\} \subseteq D'$ is directed and $j: D' \rightarrow D$ is continuous we have

$$j(y) = y = \bigcup \{j(x) \mid x < y, x \in D'\}.$$

Note that for each $x \in D'$ the sets $V = \{z \in D \mid x < z\}$ are open in D and we have:

$$\begin{aligned} x \subseteq z \text{ for all } z \in V &\Rightarrow j(x) \subseteq j(z) \text{ for all } z \in V \Rightarrow j(x) \subseteq z \text{ for all } \\ z \in V &\Rightarrow j(x) \subseteq \bigcap V \Rightarrow \bigcup \{j(x) \mid x < y, x \in D'\} \subseteq \bigcup \{\bigcap V \mid y \in V\} \\ y &\subseteq \bigcup \{\bigcap V \mid y \in V\}; \end{aligned}$$

also $\bigcup \{\bigcap V \mid y \in V\} \subseteq y$. Therefore

$$y = \bigcup \{\bigcap V \mid y \in V\},$$

where V ranges over the members of the subspace topology on D .

Let now $V = \{z \in D \mid x < z\}$ where $x \in D'$ be a basic open set of the subspace topology on D . Then $V = U \cap D$ for some U open in D' , and we have:

$$\begin{aligned} (i) \quad w \in V \text{ and } w \subseteq u \text{ and } u \in D &\Rightarrow w \in U \cap D \text{ and } w \subseteq u \text{ and } u \in D \\ &\Rightarrow w \in U \text{ and } w \subseteq u \text{ and } u \in D \Rightarrow u \in U \text{ and } u \in D \Rightarrow u \in U \cap D \Rightarrow u \in V \end{aligned}$$

$$\begin{aligned} (ii) \text{ if } S \subseteq D \text{ is directed and } \bigcup S = j(\bigcup S) \in V, \text{ then there} \\ \text{exists } W \subseteq D' \text{ open such that } \bigcup S \in W \text{ and } j(W) \subseteq V \Rightarrow S \cap W \neq \emptyset \\ \text{and } j(W) \subseteq V \Rightarrow S \cap V \neq \emptyset. \end{aligned}$$

Thus V belongs to the induced topology on D . So the induced topology on D contains all the basic open subsets of the subspace topology. Therefore it contains the subspace topology. Hence by 2.3.9 the two topologies on D coincide and D is a continuous lattice. \square

Proposition 2.3.14

Every continuous lattice is an injective space under its induced

topology.

Proof:

Let D be a continuous lattice with its induced topology, and let $X \subseteq Y$ be two T_0 -spaces in the subspace relation. Suppose $f: X \rightarrow D$ is continuous, and define $\bar{f}: Y \rightarrow D$ by the formula:

$$\bar{f}(y) = \sqcup \{ \sqcap \{ f(x) \mid x \in X \cap U \} \mid y \in U \},$$

where U ranges over the open subsets of Y . We need to show that \bar{f} extends f and that it is continuous.

First the continuity: Let $y \in Y$, and let $V = \{ z \in D \mid d < z \}$ be open in D with $\bar{f}(y) \in V$.

Since D is continuous $\bar{f}(y) = \sqcup \{ w \in D \mid w < \bar{f}(y) \}$, and the set $\{ w \in D \mid w < \bar{f}(y) \}$ is directed. Therefore $\{ w \in D \mid w < \bar{f}(y) \} \cap V \neq \emptyset \Rightarrow d' \in \{ w \in D \mid w < \bar{f}(y) \}$ for some $d' \in V \Rightarrow$ there exists $d' \in D$ such that $d < d' < \bar{f}(y)$.

Now $d' < \bar{f}(y) \Rightarrow \bar{f}(y) \in \text{Int} \{ z \in D \mid d' \sqsubseteq z \} \Rightarrow$

$$\sqcup \{ \sqcap \{ f(x) \mid x \in X \cap U \} \mid y \in U \} \in \text{Int} \{ z \in D \mid d' \sqsubseteq z \}$$

$\Rightarrow \sqcap \{ f(x) \mid x \in X \cap U \} \in \text{Int} \{ z \in D \mid d' \sqsubseteq z \}$ for some U open in Y with

$y \in U \Rightarrow d' \sqsubseteq \sqcap \{ f(x) \mid x \in X \cap U \}$ for some U open in Y with

$y \in U \Rightarrow d' \sqsubseteq \bar{f}(y')$ for all $y' \in U$ by virtue of the definition of \bar{f} .

Therefore $d < \bar{f}(y')$ for all $y' \in U$. Then U is open in Y , $y \in U$,

and $\bar{f}(U) \subseteq V$. Hence \bar{f} is continuous.

Next the extension property: Let $x' \in X$. Then

$\sqcap \{ f(x) \mid x \in X \cap U \} \sqsubseteq f(x')$ for all U open in Y with $x' \in U$. Thus

$$\bar{f}(x') \sqsubseteq f(x'). \quad (1)$$

Let now $d \in D$ be such that $d < f(x')$. Then

$f(x') \in V = \{ z \in D \mid d < z \}$ and V is open in D . Thus, by the continuity of f there exists U open in Y such that $x' \in X \cap U$ and

$f(X \cap U) \subseteq V$. Then

$f(x'') \in V$ for all $x'' \in X \cap U \Rightarrow d < f(x'')$ for all $x'' \in X \cap U \Rightarrow d \subseteq f(x'')$

for all $x'' \in X \cap U \Rightarrow$

$$d \subseteq \bigcup \{ \bigcap \{ f(x'') \mid x'' \in X \cap U \} \mid x' \in U \} = \bar{f}(x').$$

So $d < f(x')$ always implies $d \subseteq \bar{f}(x')$. Therefore

$$\bigcup \{ d \in D \mid d < f(x') \} \subseteq \bar{f}(x'). \text{ Thus}$$

$$f(x') \subseteq \bar{f}(x'). \quad (2)$$

(1), (2) $\Rightarrow \bar{f}(x') = f(x')$. Hence \bar{f} extends f . \square

Theorem 2.3.15

The injective spaces are exactly the continuous lattices.

Proof:

Let D be an injective space. Then, by 2.1.6 D is a retract of a power of \mathcal{O} . But \mathcal{O} is a finite lattice ($\perp \subseteq \top$), and so D is a continuous lattice under its induced topology. On the other hand a continuous lattice is an injective space, by 2.3.14. \square

Theorem 2.3.16

If D and D' are continuous lattices, then so is $[D \rightarrow D']$ under the induced partial ordering with the induced topology agreeing with the product topology.

Proof:

Let $f, g \in [D \rightarrow D']$. Since by 2.2.30 the lattice operation \sqcup on D' is continuous, then the composition $f \sqcup g$, defined by

$$(f \sqcup g)(x) = f(x) \sqcup g(x),$$

for all $x \in D$, is also continuous and represents the lub of $\{f, g\}$ in $[D \rightarrow D']$.

Also, the constant function with value $\perp \in D'$ is obviously continuous and is the least element of $[D \rightarrow D']$.

So $[D \rightarrow D']$ is a sup-semilattice with \sqcup . By 2.2.18 to show that $[D \rightarrow D']$ is a complete lattice we need to show that every directed subset of $[D \rightarrow D']$ has a lub in $[D \rightarrow D']$.

Let $S \subseteq [D \rightarrow D']$ be directed. Define a function from D into D' by the equation:

$$(\sqcup S)(x) = \sqcup \{f(x) \mid f \in S\},$$

for all $x \in D$. If we can show that $\sqcup S$ is continuous, then being in $[D \rightarrow D']$ it has to be the lub of S . Consider $V \subseteq D'$, an open subset, and take the inverse image $(\sqcup S)^{-1}(V) = \{x \mid (\sqcup S)(x) \in V\}$. Then we have

$$\{x \mid (\sqcup S)(x) \in V\} = \cup \{\{x \mid f(x) \in V\} \mid f \in S\}$$

because:

$$x \in \{x \mid (\sqcup S)(x) \in V\} \iff (\sqcup S)(x) \in V \iff$$

$$\sqcup \{f(x) \mid f \in S\} \in V \iff f(x) \in V \text{ for some } f \in S \iff$$

$$x \in \cup \{\{x \mid f(x) \in V\} \mid f \in S\}.$$

But the set $\{x \mid f(x) \in V\}$ is open in D , since f is continuous, for all $f \in S$. Thus $\cup \{\{x \mid f(x) \in V\} \mid f \in S\}$ is open in D . Therefore $(\sqcup S)^{-1}(V)$ is open in D . Hence $\sqcup S$ is continuous.

So $[D \rightarrow D']$ is a complete lattice. We must still show that $[D \rightarrow D']$ is continuous.

For $e \in D$ and $e' \in D'$ we define the continuous function $\vec{e}[e, e']$

by:

$$\vec{e}[e, e'](x) = \begin{cases} e' & \text{if } e < x \\ \perp & \text{if not,} \end{cases}$$

for all $x \in D$. Then for all $f \in [D \rightarrow D']$ we have:

$$f = \sqcup \{\vec{e}[e, e'] \mid e' < f(e)\}. \quad (1)$$

Proof of (1):

Let $f \in [D \rightarrow D']$ and $x \in D$. Then

$$(\sqcup \{ \vec{e}[e, e'] \mid e' < f(e) \})(x) = \sqcup \{ e' \mid \exists e < x \text{ with } e' < f(e) \} \text{ and} \\ \{ e' \mid \exists e < x \text{ with } e' < f(e) \} \subseteq \{ e' \mid e' < f(x) \}.$$

Let now $e' \in \{ e' \mid e' < f(x) \}$. Then

$f(x) \in \{ z \in D' \mid e' < z \} \Rightarrow$ there exists V open in D' with $x \in V$ such that $f(V) \subseteq \{ z \in D' \mid e' < z \} \Rightarrow$ there exists $d \in D$ such that $x \in \{ y \in D \mid d < y \}$ and $f(\{ y \in D \mid d < y \}) \subseteq \{ z \in D' \mid e' < z \}$.

Since D is continuous we can also find $e \in D$ such that $d < e < x$.

So there exists $e \in D$ such that $e < x$ and $e' < f(e)$. Thus

$e' \in \{ e' \mid \exists e < x \text{ with } e' < f(e) \}$. Therefore

$$\{ e' \mid \exists e < x \text{ with } e' < f(e) \} = \{ e' \mid e' < f(x) \} \Rightarrow$$

$$\sqcup \{ e' \mid \exists e < x \text{ with } e' < f(e) \} = \sqcup \{ e' \mid e' < f(x) \} = f(x). \quad \square$$

Moreover for all $g \in [D \rightarrow D']$ we have:

$$e' < g(e) \Rightarrow \vec{e}[e, e'] \sqsubseteq g. \quad (2)$$

Proof of (2):

Let $g \in [D \rightarrow D']$ and let $x \in D$. Then we have:

$$(i) \text{ if } e < x \text{ then } \vec{e}[e, e'](x) = e' < g(e) \Rightarrow \\ e' \sqsubseteq g(e) \sqsubseteq g(x)$$

$$(ii) \text{ if } e \not< x \text{ then } \vec{e}[e, e'](x) = \perp \sqsubseteq g(x).$$

Hence $\vec{e}[e, e'] \sqsubseteq g$. □

Let now $V = \{ f \in [D \rightarrow D'] \mid e' < f(e) \}$. Then $V = F(e, \{ z \in D' \mid e' < z \})$ is a subbasic open set of the product topology on $[D \rightarrow D']$ and we have:

$$\vec{e}[e, e'] \sqsubseteq f \text{ for all } f \in V \Rightarrow$$

$$\vec{e}[e, e'] \sqsubseteq \cap V \Rightarrow$$

$\bigcup \{ \vec{e}[e, e'] \mid e' < f(e) \} \subseteq \bigcup \{ \bigcap V \mid f \in V \} \Rightarrow f \subseteq \bigcup \{ \bigcap V \mid f \in V \}$. Therefore

$$f = \bigcup \{ \bigcap V \mid f \in V \},$$

where V ranges over the subbasic open sets of the product topology on $[D \rightarrow D']$.

Let now $F(e, U)$ be a subbasic open set of the product topology on $[D \rightarrow D']$. Then we have:

(i) $f \in F(e, U)$ and $f \subseteq g \Rightarrow f(e) \in U$ and

$f(e) \subseteq g(e) \Rightarrow g(e) \in U \Rightarrow g \in F(e, U)$

(ii) if $S \subseteq [D \rightarrow D']$ is directed and $\bigcup S \in F(e, U)$ then

$(\bigcup S)(e) \in U \Rightarrow \bigcup \{ f(e) \mid f \in S \} \in U \Rightarrow f(e) \in U$ for some $f \in S$

$\Rightarrow f \in F(e, U)$ for some $f \in S \Rightarrow F(e, U) \cap S \neq \emptyset$.

Thus $F(e, U)$ belongs to the induced topology on $[D \rightarrow D']$. So the induced topology on $[D \rightarrow D']$ contains the product topology. Hence by 2.3.9 the two topologies on $[D \rightarrow D']$ coincide and $[D \rightarrow D']$ is a continuous lattice. \square

Corollary 2.3.17

For continuous lattices D and D' , the evaluation map $\text{eval}: [D \rightarrow D'] \times D \rightarrow D'$ such that

$$\text{eval}(f, x) = f(x);$$

for all $f \in [D \rightarrow D']$ and $x \in D$ is continuous.

Proof:

With f fixed, this is obviously continuous. With x fixed, we proved the continuity above in view of 2.2.28. Hence eval is jointly continuous. \square

Proposition 2.3.18

If an expression $E(x, y, z, \dots)$ is continuous in all its vari-

variables x, y, z, \dots with values in D' as x ranges in D , then the expression

$$\lambda x: D . E(x, y, z, \dots)$$

with values in $[D \rightarrow D']$ is continuous in the remaining variables y, z, \dots .

Proof:

Let the variable y , say range over D'' and let $S \subseteq D''$ be a directed subset. Then

$$\begin{aligned} \lambda x: D . E(x, \sqcup S, z, \dots) &= \lambda x: D . \sqcup \{E(x, y, z, \dots) \mid y \in S\} \\ &= \sqcup \{\lambda x: D . E(x, y, z, \dots) \mid y \in S\}, \end{aligned}$$

because the lubs of functions are computed pointwise. \square

These uses of λ -notation should not be confused with that of the ordinary λ -calculus. The notation is part of the meta-language, and is used as a notation for functions, where the variable after the λ is the argument and the expression after the $.$ is the value (as a function of the argument).

Proposition 2.3.19

For continuous lattices D, D' , and D'' , the map of functional abstraction

$$\text{lambda}: [[D \times D'] \rightarrow D''] \rightarrow [D \rightarrow [D' \rightarrow D'']] \text{ such that}$$

$$\text{lambda}(f)(x)(y) = f(x, y),$$

for all $f \in [[D \times D'] \rightarrow D'']$ and $x \in D$ and $y \in D'$ is continuous.

Proof:

We write:

$$\text{lambda} = \lambda f: [[D \times D'] \rightarrow D''] . \lambda x: D . \lambda y: D' . f(x, y),$$

and because $f(x, y)$ is continuous in f, x , and y , the conclusion follows. \square

These results give only some examples of the number of continuous functions that are available on continuous lattices. As another fundamental example we have that composition $f \circ g$ of functions (on continuous lattices) is continuous in the two function variables, where we write

$$(f \circ g)(x) = f(g(x)).$$

Definition 2.3.20

A continuous lattice D is said to be a projection of a continuous lattice D' iff there is a pair of continuous maps

$$i: D \rightarrow D' \quad \text{and} \quad j: D' \rightarrow D$$

such that

$$j \circ i = \text{id}_D \quad \text{and} \quad i \circ j \sqsubseteq \text{id}_{D'}.$$

Proposition 2.3.21

Suppose the two pairs of maps

$$i_n: D_n \rightarrow D'_n \quad \text{and} \quad j_n: D'_n \rightarrow D_n$$

for $n = 0, 1$ make D_n a projection of D'_n . Then $[D_0 \rightarrow D_1]$ is also a projection of $[D'_0 \rightarrow D'_1]$ by means of the pair of maps:

$$\tilde{i}(f) = i_1 \circ f \circ j_0, \quad \text{and}$$

$$\tilde{j}(f') = j_1 \circ f' \circ i_0,$$

where $f \in [D_0 \rightarrow D_1]$ and $f' \in [D'_0 \rightarrow D'_1]$.

Proof:

$[D_0 \rightarrow D_1]$ and $[D'_0 \rightarrow D'_1]$ are continuous lattices by 2.3.16, and for

the pair of continuous maps $\tilde{i}: [D_0 \rightarrow D_1] \rightarrow [D'_0 \rightarrow D'_1]$ and

$\tilde{j}: [D'_0 \rightarrow D'_1] \rightarrow [D_0 \rightarrow D_1]$ we have:

$$(\tilde{j} \circ \tilde{i})(f) = \tilde{j}(\tilde{i}(f)) = j_1 \circ (i_1 \circ f \circ j_0) = j_1 \circ i_1 \circ f \circ j_0 \circ i_0 = f, \quad \text{for all}$$

$f \in [D_0 \rightarrow D_1]$; and

$$(\bar{f} \circ j)(f') = \bar{f}(j(f')) = \bar{f}(j_1 \circ f' \circ i_0) =$$

$$i_1 \circ j_1 \circ f' \circ i_0 \circ j_0 \subseteq f', \text{ for all } f' \in [D'_0 \rightarrow D'_1].$$

$$\text{Hence } \bar{f} \circ \bar{f} = \text{id}_{[D_0 \rightarrow D_1]} \text{ and } \bar{f} \circ j \subseteq \text{id}_{[D'_0 \rightarrow D'_1]}.$$

□

Proposition 2.3.22

If D is a continuous lattice and $e: X \rightarrow Y$ a subspace embedding, then for each continuous $f: X \rightarrow D$, the function $\bar{f}: Y \rightarrow D$ given by the formula:

$$\bar{f}(y) = \sqcup \{ \sqcap \{ f(x) \mid e(x) \in U \} \mid y \in U \},$$

where U ranges over the open subsets of Y and x over X , is the maximal extension of f to a function in the poset $[Y \rightarrow D]$.

Proof:

Let $f: X \rightarrow D$ be continuous. In 2.3.14 we have proved that $\bar{f}: Y \rightarrow D$ is a continuous extension of f . In other words \bar{f} is a solution to the equation

$$f = \bar{f} \circ e.$$

Must show that \bar{f} is the maximal solution. So let f' be any other, and let $y \in Y$. Then

$$\begin{aligned} f'(y) &= \sqcup \{ \sqcap f'(U) \mid y \in U, U \subseteq Y \text{ open} \} \\ &\subseteq \sqcup \{ \sqcap f'(e(X) \cap U) \mid y \in U, U \subseteq Y \text{ open} \} \\ &= \sqcup \{ \sqcap \{ f'(z) \mid z \in e(X) \cap U \} \mid y \in U, U \subseteq Y \text{ open} \} \\ &= \sqcup \{ \sqcap \{ f'(e(x)) \mid e(x) \in U \} \mid y \in U, U \subseteq Y \text{ open} \} \\ &= \sqcup \{ \sqcap \{ f(x) \mid e(x) \in U \} \mid y \in U, U \subseteq Y \text{ open} \} \\ &= \bar{f}(y). \end{aligned}$$

Hence $f' \subseteq \bar{f}$.

□

Corollary 2.3.23

Let D be a continuous lattice and $e: X \rightarrow Y$ a subspace embedding.

Then:

$$(i) f \sqsubseteq g \Rightarrow \bar{f} \sqsubseteq \bar{g} \quad \text{for all } f, g \in [X \rightarrow D]$$

(ii) if $f \in [X \rightarrow D]$ and $h \in [Y \rightarrow D]$ are such that $h \circ e \sqsubseteq f$, then $h \sqsubseteq \bar{f}$.

Proof:

(i) Let $f, g \in [X \rightarrow D]$ and $y \in Y$. Then

$$f \sqsubseteq g \Rightarrow f(x) \sqsubseteq g(x) \quad \text{for all } x \in X \Rightarrow$$

$$\bigcup \{ \bigcap \{ f(x) \mid e(x) \in U \} \mid y \in U \} \sqsubseteq \bigcup \{ \bigcap \{ g(x) \mid e(x) \in U \} \mid y \in U \} \Rightarrow$$

$$\bar{f}(y) \sqsubseteq \bar{g}(y).$$

Thus $\bar{f} \sqsubseteq \bar{g}$.

(ii) Let $f \in [X \rightarrow D]$ and $h \in [Y \rightarrow D]$ such that $h \circ e \sqsubseteq f$. Then, $\overline{h \circ e} \sqsubseteq \bar{f}$. But h is an extension of $h \circ e$ and $\overline{h \circ e}$ is the maximal extension of $h \circ e$ in $[Y \rightarrow D]$. Thus $h \sqsubseteq \overline{h \circ e} \sqsubseteq \bar{f}$. \square

Lemma 2.3.24

Suppose the continuous lattice D is a projection of the continuous lattice D' via the pair of continuous maps i, j . Let $e: X \rightarrow Y$ be a subspace embedding and $f: X \rightarrow D, g: X \rightarrow D'$ continuous. If f and g are extended to $\bar{f}: Y \rightarrow D$ and $\bar{g}: Y \rightarrow D'$ as in 2.3.22, and if $f = j \circ g$, then $\bar{f} = j \circ \bar{g}$.

Proof:

We have:

$$f = \bar{f} \circ e \quad \text{and} \quad g = \bar{g} \circ e.$$

Now $f = j \circ g \Rightarrow f = j \circ \bar{g} \circ e$. Thus

$$j \circ \bar{g} \sqsubseteq \bar{f} \quad (1)$$

because \bar{f} is the maximal extension of f .

Also $i \circ \bar{f} \circ e = i \circ f = i \circ j \circ g \sqsubseteq g$. Thus, by 2.3.23(ii)

$$i \circ \bar{f} \sqsubseteq \bar{g} \Rightarrow j \circ i \circ \bar{f} \sqsubseteq j \circ \bar{g} \Rightarrow$$

$$\bar{f} \sqsubseteq j \circ \bar{g}$$

(2)

$$(1), (2) \Rightarrow \bar{f} = j \circ \bar{g}.$$

□

Proposition 2.3.25

Every continuous lattice D is a projection of its function space

$[D \rightarrow D]$.

Proof:

Consider the following pair of mappings $\text{con}: D \rightarrow [D \rightarrow D]$ and

$\text{min}: [D \rightarrow D] \rightarrow D$ where

$$\text{con}(x)(y) = x \quad \text{and}$$

$$\text{min}(f) = f(\perp),$$

for all $x, y \in D$ and $f \in [D \rightarrow D]$. They are obviously continuous,

and we have:

$$(\text{min} \circ \text{con})(x) = \text{min}(\text{con}(x)) = \text{con}(x)(\perp) = x, \quad \text{for all } x \in D. \quad \text{And}$$

$$(\text{con} \circ \text{min})(f) = \text{con}(\text{min}(f)) = \text{con}(f(\perp)) \sqsubseteq f, \quad \text{for all } f \in [D \rightarrow D].$$

Hence D is a projection of $[D \rightarrow D]$.

□

CHAPTER III

The material of this chapter is derived from Scott [16]

Section 1. Information systems

Definition 3.1.1

An information system A is a structure $(D_A, \Delta_A, \text{Con}_A, \vdash_A)$

where:

D_A is a set;

Δ_A is a distinguished member of D_A ;

Con_A is a set of finite subsets of D_A ;

satisfying the axioms:

(i) If $u \subseteq v$ and $v \in \text{Con}_A$ then $u \in \text{Con}_A$

(ii) If $X \in D_A$ then $\{X\} \in \text{Con}_A$

(iii) If $u \in \text{Con}_A$ and $u \vdash_A X$ then $u \cup \{X\} \in \text{Con}_A$,

for all finite subsets $u, v \subseteq D_A$;

\vdash_A is a binary relation between members of Con_A and members of D_A satisfying the axioms:

(iv) $u \vdash_A \Delta_A$

(v) If $X \in u$ then $u \vdash_A X$

(vi) If $v \vdash_A Y$ for all $Y \in u$ and $u \vdash_A X$ then $v \vdash_A X$,

for all $u, v \in \text{Con}_A$, and all $X \in D_A$.

The notion of an information system is introduced in order to construct the elements of a domain. An element can often be determined by a selection of its properties. Therefore, we can think of an information system as a set of propositions that can be made about possible elements of the desired domain. More precisely:

D_A is the set of data objects or propositions. We think of the members of D_A as consisting of finite data objects some of which are more in-

formative than others. It is of course possible to introduce information systems where the data objects are infinite sets but relative to D_A they are finite in the sense that they can be specified by a finite amount of information about D_A .

Δ_A is the least informative member of D_A . Note that every element of the domain to be constructed contains Δ_A because Δ_A provides zero information.

Con_A is a set of finite subsets of D_A . A finite set of data objects u belongs to Con_A if the "propositions" in u can all be applied to the same element at the same time. Not every finite subset of D_A belongs to Con_A since not any combination of data objects will describe a possible element of the desired domain. Finally $u \subseteq D_A$ is called consistent if every finite subset of u belongs to Con_A .

\vdash_A is the entailment relation for objects. For $u \in \text{Con}_A$ and $X \in D_A$ the meaning of $u \vdash_A X$ can be expressed as "whenever all the propositions in u are true of an element then so is X true of that element".

Definition 3.1.2

Let A be an information system. For all $u, v \in \text{Con}_A$ we write $u \vdash_A v$ to mean $u \vdash_A X$ for all $X \in v$.

Proposition 3.1.3

For all $u, v, w, u', v' \in \text{Con}_A$ we have:

- (i) $\phi \vdash_A \{\Delta_A\}$
- (ii) $u \vdash_A v$ implies $u \cup v \in \text{Con}_A$
- (iii) $u \vdash_A u$
- (iv) $u \vdash_A v$ and $v \vdash_A w$ imply $u \vdash_A w$
- (v) $u' \supseteq u$, $u \vdash_A v$, and $v \supseteq v'$ imply $u' \vdash_A v'$

(vi) $u \vdash_A v$ and $u \vdash_A v'$ imply $u \vdash_A v \cup v'$.

Proof:

(i) $\phi \in \text{Con}_A$ by 3.1.1(i) and $\phi \vdash_A \Delta_A$ by 3.1.1(iv). Therefore

$\phi \vdash_A \{\Delta_A\}$.

(ii) $u \vdash_A v \Rightarrow u \vdash_A X$ for all $X \in v \Rightarrow u \cup \{X\} \in \text{Con}_A$ for all

$X \in v \Rightarrow u \cup v \in \text{Con}_A$.

(iii) $u \vdash_A X$ for all $X \in u \Rightarrow u \vdash_A u$

(iv) $u \vdash_A v$ and $v \vdash_A w \Rightarrow u \vdash_A X$ for all $X \in v$ and $v \vdash_A Y$ for all

$Y \in w \Rightarrow u \vdash_A Y$ for all $Y \in w \Rightarrow u \vdash_A w$

(v) $u' \supseteq u$ and $u \vdash_A v$ and $v \supseteq v' \Rightarrow u' \vdash_A Y$ for all $Y \in u \subseteq u'$ and

$u \vdash_A v$ and $v \vdash_A X$ for all $X \in v' \subseteq v \Rightarrow u' \vdash_A u$ and

$u \vdash_A v$ and $v \vdash_A v' \Rightarrow u' \vdash_A v$ and $v \vdash_A v' \Rightarrow u' \vdash_A v'$.

(vi) $u \vdash_A v$ and $u \vdash_A v' \Rightarrow u \vdash_A X$ for all $X \in v$ and $u \vdash_A X$ for

all $X \in v' \Rightarrow u \vdash_A X$ for $X \in v \cup v' \Rightarrow u \vdash_A v \cup v'$. \square

Example 3.1.4

Consider the structure $N = (D_N, \Delta_N, \text{Con}_N, \vdash_N)$ where:

$D_N = \{(n, m) \mid n \leq m \text{ and } n, m \text{ are non-negative integers}\}$, and any data object $(n, m) \in D_N$ stands for the proposition $n \leq x \leq m$ where x is an element yet to be determined.

$\Delta_N = (0, \infty)$.

Con_N is defined by saying that $u \in \text{Con}_N$ iff there is an integer satisfying all the propositions in u , where $u \subseteq D_N$ finite. It is obvious then that Con_N does not contain all the finite elements of D_N ; for example the set $\{(1, 3), (4, 7)\}$ does not belong to Con_N because there is no integer x satisfying $1 \leq x \leq 3$ and $4 \leq x \leq 7$.

\vdash_N is defined by saying that $u \vdash_N X$ iff whenever an integer satisfies

all the propositions in u then it satisfies X , for all $u \in \text{Con}_N$ and $X \in D_N$.

Then $N = (D_N, \Delta_N, \text{Con}_N, \vdash_N)$ is an information system.

Proof:

(i) Let $v \in \text{Con}_N$ and $u \subseteq v$. There is an integer x satisfying all the propositions in v . Then x satisfies all the propositions in u . Therefore $u \in \text{Con}_N$.

(ii) Let $(n, m) \in D_N$. Then for the integer n we have $n \leq n \leq m$. Thus $\{(n, m)\} \in \text{Con}_N$.

(iii) Suppose $u \vdash_N (n, m)$ where $u \in \text{Con}_N$ and $(n, m) \in D_N$. There is an integer x satisfying all the propositions in u . Then x satisfies also (n, m) . Thus x satisfies all the propositions in $u \cup \{(n, m)\}$. Therefore $u \cup \{(n, m)\} \in \text{Con}_N$.

(iv) Let $u \in \text{Con}_N$ and let x be an integer which satisfies all the propositions in u . Then obviously $0 \leq x \leq \infty$. So x satisfies Δ_N . Therefore $u \vdash_N \Delta_N$.

(v) Let $u \in \text{Con}_N$ and $(n, m) \in u$. They obviously whenever an integer satisfies all the propositions in u it also satisfies (n, m) . Thus $u \vdash_N (n, m)$.

(vi) Let $u, v \in \text{Con}_N$, and suppose that $v \vdash_N (n, m)$ for all $(n, m) \in u$ and $u \vdash_N (n', m')$. Let x be an integer which satisfies all the propositions in v . Then x satisfies (n, m) for all $(n, m) \in u$. So x satisfies all the propositions in u . Then x must also satisfy (n', m') . Therefore $v \vdash_N (n', m')$. □

Example 3.1.5 (for Predicate Logic see Dalen [6])

Consider the structure $L = (D_L, \Delta_L, \text{Con}_L, \vdash_L)$ where:

$D_L = \{\phi \mid \phi \text{ is a consistent sentence of Predicate Logic}\}$

$\Delta_L = \forall x (x = x)$

$Con_L = \{u \mid u \text{ is a finite consistent set of sentences of Predicate Logic}\}$, and

\vdash_L means derivability in Predicate Logic.

Then $L = (D_L, \Delta_L, Con_L, \vdash_L)$ is an information system.

Proof:

(i) Let $u \in Con_L$ and $v \subseteq u$. Then u is a finite consistent set of sentences. I.e. $u \not\vdash \perp$. Then $v \not\vdash \perp$ (because $v \vdash \perp$ would imply $u \vdash \perp$). Therefore $v \in Con_L$.

(ii) Let $\phi \in D_L$. Then $\{\phi\}$ is a finite consistent set. Therefore $\{\phi\} \in Con_L$.

(iii) Let $u \in Con_L$ and $u \vdash_L \phi$. Suppose that $u \cup \{\phi\}$ is inconsistent. Then $u \vdash \neg \phi$. But $u \vdash \phi$. Thus u is inconsistent; contradiction. Therefore $u \cup \{\phi\}$ is consistent and, since $u \in Con_L$, we have $u \cup \{\phi\} \in Con_L$.

(iv) Let $u \in Con_L$. Then $u \vdash \forall x (x = x)$. Therefore $u \vdash_L \Delta_L$.

(v) Let $u \in Con_L$ and $\phi \in u$. Then $u \vdash \phi$. Therefore $u \vdash_L \phi$.

(vi) Let $u, v \in Con_L$ and $v \vdash_L \phi$ for all $\phi \in u$ and $u \vdash_L \psi$. Then $v \vdash \phi$ for all $\phi \in u$ and $u \vdash \psi$. So $v \vdash \psi$. Therefore $v \vdash_L \psi$. \square

Section 2. Domains

Let $A = (D_A, \Delta_A, \text{Con}_A, \vdash_A)$ be an information system. The members of D_A (the data objects) are meant to be propositions about the desired elements, therefore we have to assume that D_A contains enough objects to distinguish between distinct elements. Formally, we can write of two elements x and y :

$x = y$ iff all $X \in D_A$ which are true of x are also true of y , and conversely.

The elements can be identified with the sets of propositions true of them.

So if x is an element,

$$x = \{X \in D_A \mid X \text{ is true of } x\}.$$

Definition 3.2.1

The elements of an information system A are those subsets x of D_A where:

- (i) all finite subsets of x are in Con_A
- (ii) whenever $u \subseteq x$ and $u \vdash_A X$, then $X \in x$.

The set of all elements of an information system A is called the domain determined by A and is denoted by $|A|$.

Since the members of $|A|$ are introduced as sets the set-theoretical inclusion relation can be applied. Intuitively, for $x, y \in |A|$, $x \subseteq y$ means that every proposition (among the ones given by A) true of x is also true of y . We often read " $x \subseteq y$ " as " x approximates y ". Obviously every element of an information system A contains Δ_A as a member because the least informative proposition is true of all elements.

Proposition 3.2.2

Let A be an information system and let $\perp_A = \{X \in D_A \mid \{\Delta_A\} \vdash_A X\}$.
Then $\perp_A \in |A|$.
Moreover, \perp_A is contained in all other elements.

Proof:

(i) Let $u \subseteq \perp_A$ be finite. Then $\{\Delta_A\} \vdash_A X$ for all $X \in u \Rightarrow \{\Delta_A\} \vdash_A u \Rightarrow \{\Delta_A\} \cup u \in \text{Con}_A \Rightarrow u \in \text{Con}_A$.

(ii) Let $u \subseteq \perp_A$ and $u \vdash_A X$. Then $\{\Delta_A\} \vdash_A u$ and $u \vdash_A X \Rightarrow \{\Delta_A\} \vdash_A X \Rightarrow X \in \perp_A$.

Hence $\perp_A \in |A|$.

Let now $X \in \perp_A$. Then $\{\Delta_A\} \vdash_A X$, and since Δ_A is true of all elements so is X true of all elements. Thus $X \in x$ for all $x \in |A|$.
Hence $\perp_A \subseteq x$ for all $x \in |A|$. \square

Definition 3.2.3

Given an information system A the element $\perp_A = \{X \in D_A \mid \{\Delta_A\} \vdash_A X\}$ is called the bottom element of the domain $|A|$.

Moreover if there exists an element $\top_A \in |A|$ such that $x \subseteq \top_A$ for all $x \in |A|$ then \top_A is called the top element of the domain $|A|$.

Every domain has a bottom, but not every domain has a top element.

Proposition 3.2.4

Let A be an information system. \top_A exists iff all finite subsets of D_A are in Con_A , in which case, as a set, $\top_A = D_A$.

Proof:

Suppose $|A|$ has a top element \top_A and let $u \subseteq D_A$ be finite. Then $u \subseteq \top_A$. Therefore $u \in \text{Con}_A$.

Conversely if all finite subsets of D_A are in Con_A then $D_A \in |A|$.

and obviously D_A is the top element of $|A|$. □

Definition 3.2.5

Let A be an information system. An element that is not included in any strictly larger element in the domain is called a total element. Any other element is called a partial element.

Proposition 3.2.6

Let A be an information system. If \top_A exists it is the unique total element of $|A|$ and conversely.

Proof:

If \top_A exists then obviously \top_A is the unique total element of $|A|$. For the converse suppose $y \in |A|$ is the unique total element of $|A|$. Then $x \subseteq y$ for all $x \in |A|$. Hence $y = \top_A$. □

Proposition 3.2.7

Let A be an information system. Suppose we have a sequence of elements such that

$$x_0 \subseteq x_1 \subseteq \dots \subseteq x_n \subseteq x_{n+1} \subseteq \dots$$

Then $y = \bigcup_{n=0}^{\infty} x_n$ is also an element of A .

Proof:

y is a subset of D_A and

(i) If $u \subseteq y$ finite then $u \subseteq x_n$ for some n since the sequence is increasing. Therefore $u \in \text{Con}_A$.

(ii) If $u \subseteq y$ and $u \vdash_A X$ then $u \subseteq x_n$ for some n and $u \vdash_A X$.

Then $X \in x_n$ for the same n . Therefore $X \in y$. Hence $y \in |A|$. □

Definition 3.2.8

The finite elements of an information system A are all those sets of the form $\bar{u} = \{X \in D_A \mid u \vdash_A X\}$ - where $u \in \text{Con}_A$.

Proposition 3.2.9

Let A be an information system. Then for all $x \in |A|$ we have:

$$x = \bigcup \{ \bar{u} \mid u \in \text{Con}_A \text{ and } u \subseteq x \}$$

Proof:

Let $x \in |A|$.

If $X \in x$ then $\{X\} \in \text{Con}_A$, $\{X\} \subseteq x$, and $X \in \bar{\{X\}}$. So

$X \in \bigcup \{ \bar{u} \mid u \in \text{Con}_A \text{ and } u \subseteq x \}$. Therefore $x \subseteq \bigcup \{ \bar{u} \mid u \in \text{Con}_A \text{ and } u \subseteq x \}$.

If $X \in \bigcup \{ \bar{u} \mid u \in \text{Con}_A \text{ and } u \subseteq x \}$ then $X \in \bar{u}$ for some $u \in \text{Con}_A$ with $u \subseteq x$. $X \in \bar{u} \Rightarrow u \vdash_A X$ for some $u \in \text{Con}_A$ with $u \subseteq x \Rightarrow X \in x$. Therefore $\bigcup \{ \bar{u} \mid u \in \text{Con}_A \text{ and } u \subseteq x \} \subseteq x$. Hence $x = \bigcup \{ \bar{u} \mid u \in \text{Con}_A \text{ and } u \subseteq x \}$. \square

Intuitively the meaning of this result is that every element of the domain is the limit of its finite approximations.

Return now to the examples of information systems given above to see what the elements are.

Example 3.1.4

We have seen that Con_N does not contain all the finite subsets of D_N . Therefore there is no top element. The elements of N are all the sets of the form

$$\{(n, q) \mid n \leq m \leq p \leq q\}$$

where $m \leq p$ are given and also $q = \infty$ is allowed. If $m = p$ we get a total element $\{(n, q) \mid n \leq m = m \leq q\}$ which corresponds to the non-negative integer m .

Otherwise we get an element $\{(n, q) \mid n \leq m < p \leq q\}$ which is a partial element since there is always a larger element say,

$$\{(n, q) \mid n \leq m < p \leq q\} \subset \{(n, q) \mid n \leq m + 1 \leq p \leq q\}.$$

Example 3.1.5

The elements of L are the theories, because if T is a theory then:

- (i) $T \subseteq D_L$,
- (ii) $T \vdash \varphi \Rightarrow \varphi \in T$, and
- (iii) T is consistent.

Remark 3.2.10

Let A be an information system. We have seen that the set-theoretical inclusion relation can be applied to elements and the domain $|A|$ becomes a poset under inclusion.

Let now $x, y \in |A|$. Then $x \cap y \in |A|$ because $x \cap y \subseteq D_A$ and

- (i) If $u \subseteq x \cap y$ is finite $\Rightarrow u \subseteq x$ is finite $\Rightarrow u \in \text{Con}_A$
- (ii) If $u \subseteq x \cap y$ and $u \vdash_A X \Rightarrow u \subseteq x$ and $u \subseteq y$ and $u \vdash_A X \Rightarrow X \in x$ and $X \in y \Rightarrow X \in x \cap y$.

So, according to definition 2.2.17, $|A|$ is an inf-semilattice.

Consider now any nonempty subfamily of $|A|$. The set-theoretical intersection of all the elements in the subfamily is again an element of the domain. Hence every nonempty subset of $|A|$ has an inf in $|A|$. Moreover if \top_A exists then $|A|$ is a complete lattice. (see Gierz, et al [9] p. 8 Proposition 2.2 (ii)).

The domain $|A|$ can be also regarded as a topological space. For each $u \in \text{Con}_A$ we define a corresponding neighborhood of $|A|$ by the equation:

$$[u]_A = \{x \in |A| \mid u \subseteq x\}.$$

The neighborhoods of an element x are all those sets $[u]_A$ where $u \subseteq x$.

Let now $x, y \in |A|$ and suppose that x and y have exactly the same

neighborhoods. Since $x = \bigcup \{\bar{u} \mid u \in \text{Con}_A \text{ and } u \subseteq x\}$ and $y = \bigcup \{\bar{v} \mid v \in \text{Con}_A \text{ and } v \subseteq y\}$ we have that $x = y$.

Hence $|A|$ is a T_0 -space.

Section 3. Approximable mappings between domains

Once the notion of a domain has been defined, the next major issue is how are the different domains to be related one to another. In the theory of domains, where the approximations to the elements are all we can ever know at one time, an appropriate mapping between them is a mapping that proceeds by approximation.

Definition 3.3.1

Let A and B be two information systems. An approximable mapping $f: A \rightarrow B$ is a binary relation between the sets Con_A and Con_B such that:

- (i) $\emptyset f \emptyset$
- (ii) $u f v$ and $u f v' \Rightarrow u f (v \cup v')$
- (iii) $u' \vdash_A u$, $u f v$, and $v \vdash_B v' \Rightarrow u' f v'$.

We say that A is the source of f and B is the target.

Note 3.3.2

$u f v$ iff $u f \{Y\}$ for all $Y \in v$.

In other words an approximable mapping is completely determined by the relation set up between consistent sets on the left and single data objects on the right.

Intuitively the relationship $u f v$ is an input/output passage which can be read informally as: "if you are willing to give at least u amount of information about the argument, then the mapping f is willing to give at least v amount of information about the value". Of course to get the full effect of f it is necessary to take all the v 's related to the given u and that way any approximable mapping $f: A \rightarrow B$ naturally defines a function f between the elements of A and B .

Proposition 3.3.3

Given information systems A and B an approximable mapping $f: A \rightarrow B$ always determines a function $f: |A| \rightarrow |B|$ between domains by virtue of the formula:

$$\begin{aligned} f(x) &= \{Y \in D_B \mid u f \{Y\} \text{ for some } u \subseteq x\} \\ &= \bigcup \{v \in \text{Con}_B \mid u f v \text{ for some } u \subseteq x\}, \end{aligned}$$

for all $x \in |A|$.

Conversely the function uniquely determines the original relation by the equivalence:

$$u f v \text{ iff } \bar{v} \subseteq f(\bar{u}).$$

Proof:

To prove that the range of f is $|B|$ must show that $f(x) \in |B|$ for all $x \in |A|$.

So let $x \in |A|$ and consider $f(x)$. Then,

(i) If $w = \{Y_1, \dots, Y_k\} \subseteq f(x) \Rightarrow Y_i \in f(x)$ for all $i \in \{1, \dots, k\} \Rightarrow u_i f \{Y_i\}$ for some $u_i \subseteq x$, for all $i \in \{1, \dots, k\} \Rightarrow \bigcup_{i=1}^k u_i f \{Y_i\}$ for all $i \in \{1, \dots, k\}$ by 3.3.1(iii) $\Rightarrow \bigcup_{i=1}^k u_i f w \Rightarrow w \in \text{Con}_B$.

(ii) If $w \subseteq f(x)$ and $w \vdash_A Y$ then $u f w$ for some $u \subseteq x$ and by 3.3.1(iii) we have $u f \{Y\}$ for some $u \subseteq x$. Thus $Y \in f(x)$.

Hence $f(x) \in |B|$.

For the proof of the equivalence,

suppose $u f v$ and let $Y \in \bar{v}$. Then $v \vdash_B Y$ and by 3.3.1(iii) we have $u f \{Y\}$. Then $Y \in f(\bar{u})$. Therefore $\bar{v} \subseteq f(\bar{u})$.

Conversely suppose $\bar{v} \subseteq f(\bar{u})$ and let $Y \in v$. Then

$Y \in \bar{v} \Rightarrow Y \in f(\bar{u}) \Rightarrow w f \{Y\}$ for some $w \subseteq \bar{u} \Rightarrow w f \{Y\}$ for some $w \in \text{Con}_A$

with $u \vdash_A w \Rightarrow uf\{Y\}$ by 3.3.1(iii). Therefore ufv . \square

Proposition 3.3.4

Let f, g be two approximable mappings between the information systems A and B . Then:

- (i) $x \subseteq y$ in $|A|$ always implies $f(x) \subseteq f(y)$ in $|B|$
- (ii) $f = g$ iff $f(x) = g(x)$ for all $x \in |A|$
- (iii) $f \subseteq g$ iff $f(x) \subseteq g(x)$ for all $x \in |A|$.

Proof:

(i) Let $x, y \in |A|$ with $x \subseteq y$. Then,
 $Y \in f(x) \Rightarrow uf\{Y\}$ for some $u \subseteq x \Rightarrow uf\{Y\}$ for some $u \subseteq y \Rightarrow Y \in f(y)$.
 Therefore $f(x) \subseteq f(y)$.

(ii) Suppose $f = g$ as relations, and let $x \in |A|$. Then,

$$\begin{aligned} f(x) &= \{Y \in D_B \mid uf\{Y\} \text{ for some } u \subseteq x\} \\ &= \{Y \in D_B \mid ug\{Y\} \text{ for some } u \subseteq x\} \\ &= g(x) \end{aligned}$$

Conversely suppose $f(x) = g(x)$ for all $x \in |A|$, and let $u \in \text{Con}_A$, $v \in \text{Con}_B$. Then,

$$ufv \iff \bar{v} \subseteq f(\bar{u}) = g(\bar{u}) \iff ug v$$

Thus f and g are identical as relations.

(iii) Suppose $f \subseteq g$ as relations, and let $x \in |A|$. Then,

$Y \in f(x) \Rightarrow uf\{Y\}$ for some $u \subseteq x \Rightarrow ug\{Y\}$ for some $u \subseteq x \Rightarrow Y \in g(x)$.
 Therefore $f(x) \subseteq g(x)$.

Conversely suppose $f(x) \subseteq g(x)$ for all $x \in |A|$, and let $u \in \text{Con}_A$, $v \in \text{Con}_B$. Then,

$$ufv \iff \bar{v} \subseteq f(\bar{u}) \subseteq g(\bar{u}) \iff ug v$$

Thus $f \subseteq g$ as relations. \square

Proposition 3.3.5

Let A be an information system. Then the following formula defines an approximable mapping $I_A: A \rightarrow A$:

$$u I_A v \text{ iff } u \vdash_A v,$$

for all $u, v \in \text{Con}_A$. And we have:

$$I_A(x) = x, \text{ for all } x \in |A|.$$

Proof:

We have for all $u, v, u', v' \in \text{Con}_A$:

$$(i) \quad \phi I_A \phi$$

$$(ii) \quad u I_A v \text{ and } u I_A v' \Rightarrow u \vdash_A v \text{ and } u \vdash_A v' \\ \Rightarrow u \vdash_A v \cup v' \Rightarrow u I_A (v \cup v').$$

$$(iii) \quad u' \vdash_A u, u I_A v, \text{ and } v \vdash_A v' \Rightarrow u' \vdash_A u, \\ u \vdash_A v, \text{ and } v \vdash_A v' \Rightarrow u' \vdash_A v \text{ and } v \vdash_A v' \Rightarrow u' \vdash_A v' \Rightarrow u' I_A v'.$$

So $I_A: A \rightarrow A$ is an approximable mapping.

Let now $x \in |A|$. Then,

$$\begin{aligned} I_A(x) &= \bigcup \{v \in \text{Con}_A \mid u I_A v \text{ for some } u \subseteq x\} \\ &= \bigcup \{v \in \text{Con}_A \mid u \vdash_A v \text{ for some } u \subseteq x\} \\ &= \bigcup \{\bar{u} \mid u \in \text{Con}_A \text{ and } u \subseteq x\} \\ &= x \end{aligned}$$

□

Proposition 3.3.6

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two approximable mappings between the information systems A, B and C . Then the following formula defines an approximable mapping $g \circ f: A \rightarrow C$:

$$u (g \circ f) w \text{ iff } u f v \text{ and } v g w \text{ for some } v \in \text{Con}_B,$$

for all $u \in \text{Con}_A$ and $w \in \text{Con}_C$. And we have:

$$(g \circ f)(x) = g(f(x)), \text{ for all } x \in |A|.$$

Proof:

We have for all $u, u' \in \text{Con}_A$ and $w, w' \in \text{Con}_C$:

$$(i) \quad \phi(g \circ f) \phi$$

(ii) $u(g \circ f)w$ and $u(g \circ f)w' \Rightarrow ufv$ and $v g w$ for some $v \in \text{Con}_B$ and ufv' and $v' g w'$ for some $v' \in \text{Con}_B \Rightarrow uf(v \cup v')$ and $(v \cup v') g w$ and $(v \cup v') g w'$ for some $v, v' \in \text{Con}_B \Rightarrow uf(v \cup v')$ and $(v \cup v') g(w \cup w')$ for some $v, v' \in \text{Con}_B \Rightarrow u(g \circ f)(w \cup w')$.

(iii) $u' \vdash_A u$, $u(g \circ f)w$ and $w \vdash_C w' \Rightarrow u' \vdash_A u$, ufv and $v g w$ for some $v \in \text{Con}_B$, and $w \vdash_C w' \Rightarrow u' f v$ and $v g w'$ for some $v \in \text{Con}_B \Rightarrow u'(g \circ f)w'$.

So $g \circ f: A \rightarrow C$ is an approximable mapping. Let now $x \in |A|$. Then,

$$\begin{aligned} (g \circ f)(x) &= \bigcup \{w \in \text{Con}_C \mid u(g \circ f)w \text{ for some } u \subseteq x\} \\ &= \bigcup \{w \in \text{Con}_C \mid ufv \text{ and } v g w \text{ for some } v \in \text{Con}_B \text{ and some } u \subseteq x\} \\ &= \bigcup \{w \in \text{Con}_C \mid v g w \text{ for some } v \subseteq f(x)\} \\ &= g(f(x)). \end{aligned}$$

Proposition 3.3.7

Given information systems A, B, C and D , and given a fixed element $b \in |B|$, then there is a unique approximable mapping

$(\text{const } b)_{A,B}: A \rightarrow B$ such that:

$$(i) \quad (\text{const } b)_{A,B}(x) = b,$$

for all $x \in |A|$. Moreover we have:

$$(ii) \quad (f \circ (\text{const } b)_{A,B})(x) = (\text{const } f(b))_{A,C}(x),$$

for all $x \in |A|$ and for all approximable mappings $f: B \rightarrow C$; and

$$(iii) \quad ((\text{const } b)_{A,B} \circ g)(x) = (\text{const } b)_{D,B}(x),$$

for all $x \in |D|$ and for all approximable mappings $g: D \rightarrow A$.

Subscripts will be omitted when they are clear from the context.

Proof:

We define $(\text{const } b)_{A,B}: A \rightarrow B$ by the formula:

$$u (\text{const } b)_{A,B} v \text{ iff } v \subseteq b,$$

for all $u \in \text{Con}_A$ and $v \in \text{Con}_B$. Then $(\text{const } b)_{A,B}: A \rightarrow B$ is obviously an approximable mapping, and

(i) for all $x \in |A|$:

$$\begin{aligned} (\text{const } b)_{A,B}(x) &= \{Y \in D_B \mid u (\text{const } b)_{A,B} \{Y\} \text{ for some } u \subseteq x\} \\ &= \{Y \in D_B \mid \{Y\} \subseteq b\} \\ &= b. \end{aligned}$$

Also if $(\text{const}' b)_{A,B}: A \rightarrow B$ is another approximable mapping such that $(\text{const}' b)_{A,B}(x) = b$ for all $x \in |A|$, then for each $x \in |A|$ we have:

$$(\text{const}' b)_{A,B}(x) = b = (\text{const } b)_{A,B}(x).$$

So $(\text{const}' b)_{A,B} = (\text{const } b)_{A,B}$; therefore $(\text{const } b)_{A,B}: A \rightarrow B$ is unique.

(ii) Let $f: B \rightarrow C$ be an approximable mapping and let $x \in |A|$.

We have:

$$\begin{aligned} (f \circ (\text{const } b)_{A,B})(x) &= f((\text{const } b)_{A,B}(x)) \\ &= f(b) \\ &= (\text{const } f(b))_{A,C}(x). \end{aligned}$$

(iii) Let $g: D \rightarrow A$ be an approximable mapping and let $x \in |D|$.

We have:

$$\begin{aligned} ((\text{const } b)_{A,B} \circ g)(x) &= (\text{const } b)_{A,B}(g(x)) \\ &= b = (\text{const } b)_{D,B}(x). \end{aligned}$$

□

Remark 3.3.8

Let $f: A \rightarrow B$ be an approximable mapping between the information systems A and B . If $|A|$ and $|B|$ are regarded as topological spaces then $f: |A| \rightarrow |B|$ is continuous on the finite elements of $|A|$ because:

Let $\bar{u} \in |A|$ where $u \in \text{Con}_A$ and let $[v]_B = \{y \in |B| \mid v \subseteq y\}$ be a neighborhood of $f(\bar{u})$ in $|B|$. Then,

$$v \subseteq f(\bar{u}), \quad (1)$$

Take $[u]_A = \{x \in |A| \mid u \subseteq x\}$, which is a neighborhood of \bar{u} in $|A|$ since $u \subseteq \bar{u}$, and let $y \in f([u]_A)$. Then $y = f(x)$ for some $x \in [u]_A$. And we have:

$$u \subseteq x \Rightarrow \bar{u} \subseteq x \Rightarrow f(\bar{u}) \subseteq f(x) \Rightarrow f(\bar{u}) \subseteq y \stackrel{(1)}{\Rightarrow} v \subseteq y \Rightarrow y \in [v]_B.$$

Hence $f([u]_A) \subseteq [v]_B$. □

Moreover every infinite element of $|A|$ is the directed union of its finite approximations, and $f: |A| \rightarrow |B|$ preserves directed unions.

Thus we can finally conclude that $f: |A| \rightarrow |B|$ is continuous.

Hence the notion of an approximable mapping is the same as that of a continuous mapping when domains are viewed topologically.

Section 4. New domain constructions

We have defined in this chapter the general notion of a domain and we have seen how the different domains are to be related one to another. The next basic topic is how to construct new domains given old ones. We shall give three basic constructs: the product, the sum and the function space constructions.

Given the information systems A and B we shall construct the product domain directly from the given data objects and prove that it is just the one expected when we look at the elements.

Definition 3.4.1

Let A and B be two information systems. By $A \times B$, the product system, we understand the system where:

$$(i) \quad D_{A \times B} = \{(X, \Delta_B) \mid X \in D_A\} \cup \{(\Delta_A, Y) \mid Y \in D_B\};$$

$$(ii) \quad \Delta_{A \times B} = (\Delta_A, \Delta_B);$$

for all $u \subseteq D_{A \times B}$ we let:

$$\text{fst } u = \{X \in D_A \mid (X, \Delta_B) \in u\}$$

$$\text{snd } u = \{Y \in D_B \mid (\Delta_A, Y) \in u\}; \text{ then}$$

$$(iii) \quad u \in \text{Con}_{A \times B} \text{ iff } \text{fst } u \in \text{Con}_A \text{ and } \text{snd } u \in \text{Con}_B,$$

where u is any finite subset of $D_{A \times B}$;

$$(iv) \quad u \vdash_{A \times B} (X', \Delta_B) \text{ iff } \text{fst } u \vdash_A X' \text{ and}$$

$$u \vdash_{A \times B} (\Delta_A, Y') \text{ iff } \text{snd } u \vdash_B Y',$$

where $u \in \text{Con}_{A \times B}$.

Lemma 3.4.2

For all $u, v \subseteq D_{A \times B}$ we have:

$$u \subseteq v \text{ iff } \text{fst } u \subseteq \text{fst } v \text{ and } \text{snd } u \subseteq \text{snd } v.$$

Proof:

Suppose $u \subseteq v$. Then

$x \in \text{fst } u \Rightarrow x \in D_A$ and $(x, \Delta_B) \in u \Rightarrow x \in D_A$ and $(x, \Delta_B) \in v \Rightarrow x \in \text{fst } v$.

And, $y \in \text{snd } u \Rightarrow y \in D_B$ and $(\Delta_A, y) \in u \Rightarrow y \in D_B$ and

$(\Delta_A, y) \in v \Rightarrow y \in \text{snd } v$.

Hence $\text{fst } u \subseteq \text{fst } v$ and $\text{snd } u \subseteq \text{snd } v$.

Conversely suppose $\text{fst } u \subseteq \text{fst } v$ and $\text{snd } u \subseteq \text{snd } v$. Then:

a) $(x, \Delta_B) \in u \Rightarrow x \in \text{fst } u \Rightarrow x \in \text{fst } v \Rightarrow (x, \Delta_B) \in v$

b) $(\Delta_A, y) \in u \Rightarrow y \in \text{snd } u \Rightarrow y \in \text{snd } v \Rightarrow (\Delta_A, y) \in v$.

Hence $u \subseteq v$ □

Proposition 3.4.3

Let A, B and C be information systems. Then

(I) $A \times B$ is also an information system.

(II) There exist mappings

$$\text{fst}: A \times B \rightarrow A \quad \text{and} \quad \text{snd}: A \times B \rightarrow B,$$

such that, for approximable mappings

$$f: C \rightarrow A \quad \text{and} \quad g: C \rightarrow B,$$

there is one and only one approximable mapping $\langle f, g \rangle: C \rightarrow A \times B$

such that

$$(a) \text{fst} \circ \langle f, g \rangle = f \quad \text{and} \quad (b) \text{snd} \circ \langle f, g \rangle = g$$

Proof:

(I) Must show that $A \times B$ satisfies the (vi) axioms of definition 3.1.1.

(i) $u \subseteq v$ and $v \in \text{Con}_{A \times B} \Rightarrow \text{fst } u \subseteq \text{fst } v$ and $\text{snd } u \subseteq \text{snd } v$
and $\text{fst } v \in \text{Con}_A$ and $\text{snd } v \in \text{Con}_B \Rightarrow \text{fst } u \in \text{Con}_A$ and
 $\text{snd } u \in \text{Con}_B \Rightarrow u \in \text{Con}_{A \times B}$.

(ii) If $(X, \Delta_B) \in D_{A \times B}$ then $\text{fst} \{(X, \Delta_B)\} = \{X\} \in \text{Con}_A$

and $\text{snd} \{(X, \Delta_B)\} = \begin{cases} \phi \in \text{Con}_B & \text{if } X \neq \Delta_A \\ \{\Delta_B\} \in \text{Con}_B & \text{if } X = \Delta_A \end{cases}$ so $\{(X, \Delta_B)\} \in \text{Con}_{A \times B}$.

Similarly if $(\Delta_A, Y) \in D_{A \times B}$ then $\{(\Delta_A, Y)\} \in \text{Con}_{A \times B}$.

(iii) $u \in \text{Con}_{A \times B}$ and $u \vdash_{A \times B} (X', \Delta_B) \Rightarrow \text{fst } u \in \text{Con}_A$ and
 $\text{snd } u \in \text{Con}_B$ and $\text{fst } u \vdash_A X' \Rightarrow \text{fst } u \cup \{X'\} \in \text{Con}_A$ and
 $\text{snd } u \in \text{Con}_B \Rightarrow \text{fst } (u \cup \{(X', \Delta_B)\}) \in \text{Con}_A$ and
 $\text{snd } (u \cup \{(X', \Delta_B)\}) \in \text{Con}_B \Rightarrow u \cup \{(X', \Delta_B)\} \in \text{Con}_{A \times B}$.

Similarly, $u \in \text{Con}_{A \times B}$ and $u \vdash_{A \times B} (\Delta_A, Y') \Rightarrow u \cup \{(\Delta_A, Y')\} \in \text{Con}_{A \times B}$.

(iv) $u \in \text{Con}_{A \times B} \Rightarrow \text{fst } u \in \text{Con}_A$ and $\text{snd } u \in \text{Con}_B \Rightarrow \text{fst } u \vdash_A \Delta_A$
and $\text{snd } u \vdash_B \Delta_B \Rightarrow u \vdash_{A \times B} (\Delta_A, \Delta_B)$.

(v) $u \in \text{Con}_{A \times B}$ and $(X, \Delta_B) \in u \Rightarrow \text{fst } u \in \text{Con}_A$ and
 $X \in \text{fst } u \Rightarrow \text{fst } u \vdash_A X \Rightarrow u \vdash_{A \times B} (X, \Delta_B)$.

Similarly $u \in \text{Con}_{A \times B}$ and $(\Delta_A, Y) \in u \Rightarrow u \vdash_{A \times B} (\Delta_A, Y)$.

(vi) $v \vdash_{A \times B} (X, \Delta_B)$ for all $(X, \Delta_B) \in u$ and
 $u \vdash_{A \times B} (X', \Delta_B) \Rightarrow \text{fst } v \vdash_A X$ for all $X \in \text{fst } u$ and
 $\text{fst } u \vdash_A X' \Rightarrow \text{fst } v \vdash_A X' \Rightarrow v \vdash_{A \times B} (X', \Delta_B)$;
Similarly, $v \vdash_{A \times B} (\Delta_A, Y)$ for all $(\Delta_A, Y) \in u$ and
 $u \vdash_{A \times B} (\Delta_A, Y') \Rightarrow v \vdash_{A \times B} (\Delta_A, Y')$.

(II) We define $\text{fst}: A \times B \rightarrow A$ and $\text{snd}: A \times B \rightarrow B$ by the formulae:

$u \text{ fst } v$ iff $\text{fst } u \vdash_A v$ and

$u \text{ snd } w$ iff $\text{snd } u \vdash_B w$,

for all $u \in \text{Con}_{A \times B}$, $v \in \text{Con}_A$, and $w \in \text{Con}_B$.

Let now $f: C \rightarrow A$ and $g: C \rightarrow B$ be approximable mappings; we define

$\langle f, g \rangle : C \rightarrow A \times B$ by the formula:

$s < f, g > u$ iff $s f (fst u)$ and $s g (snd u)$,

for all $s \in Con_C$ and $u \in Con_{A \times B}$.

Then $fst: A \times B \rightarrow A$, $snd: A \times B \rightarrow B$, and $< f, g >: C \rightarrow A \times B$ are approximable mappings, and we have:

(a) For each $x \in |C|$:

$Y \in (fst \circ < f, g >)(x) \Rightarrow s (fst \circ < f, g > \{Y\})$ for some

$s \subseteq x \Rightarrow s < f, g > u$ and $u fst \{Y\}$ for some $u \in Con_{A \times B}$ and

$s \subseteq x \Rightarrow s f (fst u)$ and $fst u \vdash_A \{Y\}$ for some $u \in Con_{A \times B}$ and

$s \subseteq x \Rightarrow s f \{Y\}$ for some $s \subseteq x \Rightarrow Y \in f(x)$, thus

$(fst \circ < f, g >)(x) \subseteq f(x)$; also

$Y \in f(x) \Rightarrow s f \{Y\}$ for some $s \subseteq x \Rightarrow s f (fst \{(Y, \Delta_B)\})$ and

$s g (snd \{(Y, \Delta_B)\})$ and $fst \{(Y, \Delta_B)\} \vdash_A \{Y\}$ for some

$s \subseteq x \Rightarrow s (fst \circ < f, g > \{Y\})$ for some $s \subseteq x \Rightarrow Y \in (fst \circ < f, g >)(x)$,

thus $f(x) \subseteq (fst \circ < f, g >)(x)$; so, $(fst \circ < f, g >)(x) = f(x)$.

Hence $fst \circ < f, g > = f$.

(b) Similarly $snd \circ < f, g > = g$.

lemma

If z and z' are two elements of the product system $A \times B$ such that $fst z = fst z'$ and $snd z = snd z'$, then $z = z'$.

Proof:

$(x, \Delta_B) \in z \iff x \in fst z \iff x \in fst z' \iff (x, \Delta_B) \in z'$, and

$(\Delta_A, Y) \in z \iff Y \in snd z \iff Y \in snd z' \iff (\Delta_A, Y) \in z'$.

Hence $z = z'$. □

Now if $< f, g >': C \rightarrow A \times B$ is another approximable mapping satisfying conditions (a) and (b) then we have:

for all $x \in |C|$:

$\text{fst}(\langle f, g \rangle'(x)) = f(x) = \text{fst}(\langle f, g \rangle(x))$, and
 $\text{snd}(\langle f, g \rangle'(x)) = g(x) = \text{snd}(\langle f, g \rangle(x))$; thus, according to the
 previous lemma, $\langle f, g \rangle'(x) = \langle f, g \rangle(x)$.

Hence $\langle f, g \rangle' = \langle f, g \rangle$ and therefore $\langle f, g \rangle: C \rightarrow A \times B$ is
 unique. □

Definition 3.4.4

Let A and B be information systems. For elements $x \in |A|$
 and $y \in |B|$ we define

$$(x, y) = \langle \text{const}(x), \text{const}(y) \rangle (\perp_C) ,$$

for any convenient fixed information system C . *

Proposition 3.4.5

Let A and B be information systems. For all $x \in |A|$ and
 $y \in |B|$ we have:

$$(i) \quad (x, y) = \{(X, \Delta_B) | X \in x\} \cup \{(\Delta_A, Y) | Y \in y\} \in |A \times B|$$

$$(ii) \quad \text{fst}(x, y) = x , \quad \text{snd}(x, y) = y .$$

For all $z \in |A \times B|$ we have:

$$(iii) \quad z = (\text{fst } z, \text{snd } z) .$$

Given also an information system C we have that for all approximable
 mappings $f: C \rightarrow A$, $g: C \rightarrow B$, and for all $t \in |C|$:

$$(iv) \quad \langle f, g \rangle (t) = (f(t), g(t)) .$$

Proof:

Let $x \in |A|$ and $y \in |B|$. Then for some convenient fixed information
 system C we have:

$$\begin{aligned} (i) \quad (x, y) &= \langle \text{const}(x), \text{const}(y) \rangle (\perp_C) \\ &= \{(X, \Delta_B) \in D_{A \times B} | u \langle \text{const}(x), \text{const}(y) \rangle \{(X, \Delta_B)\} \text{ for some } u \subseteq \perp_C\} \\ &\cup \{(\Delta_A, Y) \in D_{A \times B} | u \langle \text{const}(x), \text{const}(y) \rangle \{(\Delta_A, Y)\} \text{ for some } u \subseteq \perp_C\} \end{aligned}$$

$$\begin{aligned}
 &= \{(X, \Delta_B) \in D_{A \times B} \mid u \text{ const } (x) \text{ (fst } \{(X, \Delta_B)\}) \text{ and} \\
 &\quad u \text{ const } (y) \text{ (snd } \{(X, \Delta_B)\}) \text{ for some } u \subseteq \perp_C\} \\
 &\cup \{(\Delta_A, Y) \in D_{A \times B} \mid u \text{ const } (x) \text{ (fst } \{(\Delta_A, Y)\}) \text{ and} \\
 &\quad u \text{ const } (y) \text{ (snd } \{(\Delta_A, Y)\}) \text{ for some } u \subseteq \perp_C\} \\
 &= \{(X, \Delta_B) \in D_{A \times B} \mid \{X\} \subseteq x\} \cup \{(\Delta_A, Y) \in D_{A \times B} \mid \{Y\} \subseteq y\} \\
 &= \{(X, \Delta_B) \mid X \in x\} \cup \{(\Delta_A, Y) \mid Y \in y\} \in |A \times B|.
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad \text{fst}(x, y) &= \text{fst}(\{(X, \Delta_B) \mid X \in x\} \cup \{(\Delta_A, Y) \mid Y \in y\}) \\
 &= x,
 \end{aligned}$$

$$\begin{aligned}
 \text{snd}(x, y) &= \text{snd}(\{(X, \Delta_B) \mid X \in x\} \cup \{(\Delta_A, Y) \mid Y \in y\}) \\
 &= y.
 \end{aligned}$$

(iii) Let $z \in |A \times B|$. Then $\text{fst } z \in |A|$, $\text{snd } z \in |B|$, and

$$\begin{aligned}
 (\text{fst } z, \text{snd } z) &= \{(X, \Delta_B) \mid X \in \text{fst } z\} \cup \{(\Delta_A, Y) \mid Y \in \text{snd } z\} \\
 &= z.
 \end{aligned}$$

(iv) Let $f: C \rightarrow A$ and $g: C \rightarrow B$ be approximable mappings, and let $t \in |C|$. Then

$$\begin{aligned}
 \langle f, g \rangle(t) &= \{(X, \Delta_B) \in D_{A \times B} \mid s \langle f, g \rangle \{(X, \Delta_B)\} \text{ for some } s \subseteq t\} \\
 &\cup \{(\Delta_A, Y) \in D_{A \times B} \mid s \langle f, g \rangle \{(\Delta_A, Y)\} \text{ for some } s \subseteq t\} \\
 &= \{(X, \Delta_B) \mid X \in D_A \text{ and } s f \{X\} \text{ for some } s \subseteq t\} \\
 &\cup \{(\Delta_A, Y) \mid Y \in D_B \text{ and } s g \{Y\} \text{ for some } s \subseteq t\} \\
 &= \{(X, \Delta_B) \mid X \in f(t)\} \cup \{(\Delta_A, Y) \mid Y \in g(t)\} \\
 &= (f(t), g(t)). \quad \square
 \end{aligned}$$

So, given information systems A and B , the domain $|A \times B|$ is in an one-one correspondence with the set-theoretical product of the domains $|A|$ and $|B|$.

Now, from the information systems A and B we shall construct the sum system $A + B$ whose elements divide into disjoint copies of

those of A and B plus an extra element, the bottom of $|A + B|$.

Definition 3.4.6

Let A and B be two information systems. By $A + B$, the separated sum system, we understand the system where, after choosing some convenient object Δ belonging neither to D_A nor to D_B , we have:

$$(i) \quad D_{A+B} = \{(X, \Delta) \mid X \in D_A\} \cup \{(\Delta, Y) \mid Y \in D_B\} \cup \{(\Delta, \Delta)\};$$

$$(ii) \quad \Delta_{A+B} = (\Delta, \Delta);$$

for all $u \subseteq D_{A+B}$ we let:

$$\text{left } u = \{X \in D_A \mid (X, \Delta) \in u\}$$

$$\text{right } u = \{Y \in D_B \mid (\Delta, Y) \in u\}; \text{ then}$$

$$(iii) \quad u \in \text{Con}_{A+B} \text{ iff either } \text{left } u \in \text{Con}_A \text{ and } \text{right } u = \emptyset \\ \text{or } \text{left } u = \emptyset \text{ and } \text{right } u \in \text{Con}_B,$$

where u is any finite subset of D_{A+B} ;

$$(iv) \quad u \vdash_{A+B} (X', \Delta) \text{ iff } \text{left } u \neq \emptyset \text{ and } \text{left } u \vdash_A X'$$

$$u \vdash_{A+B} (\Delta, Y') \text{ iff } \text{right } u \neq \emptyset \text{ and } \text{right } u \vdash_B Y'$$

$$u \vdash_{A+B} (\Delta, \Delta) \text{ always,}$$

where $u \in \text{Con}_{A+B}$.

Proposition 3.4.7

Let A, B and C be information systems. Then

(I) $A + B$ is also an information system

(II) There exist mappings

$$\text{inl}: A \rightarrow A + B \text{ and } \text{inr}: B \rightarrow A + B$$

such that, for approximable mappings

$$f: A \rightarrow C \text{ and } g: B \rightarrow C,$$

there is one and only one approximable mapping $[f, g]: A + B \rightarrow C$

such that

a) $[f, g] \circ \text{inl} = f$, (b) $[f, g] \circ \text{inr} = g$, and (c) $[f, g](\perp_{A+B}) = \perp_C$.

Proof:

(I) Must show that $A + B$ satisfies the (vi) axioms of definition 3.1.1.

(i) $u \subseteq v$ and $v \in \text{Con}_{A+B} \Rightarrow \text{left } u \subseteq \text{left } v$ and $\text{right } u \subseteq \text{right } v$, and

$$\begin{cases} \text{either } \text{left } v \in \text{Con}_A \text{ and } \text{right } v = \phi \\ \text{or } \text{left } v = \phi \text{ and } \text{right } v \in \text{Con}_B \end{cases}$$

$\Rightarrow \begin{cases} \text{either } \text{left } u \in \text{Con}_A \text{ and } \text{right } u = \phi \\ \text{or } \text{left } u = \phi \text{ and } \text{right } u \in \text{Con}_B \end{cases} \Rightarrow u \in \text{Con}_{A+B}.$

(ii) If $(X, \Delta) \in D_{A+B}$ then $\text{left } \{(X, \Delta)\} = \{X\} \in \text{Con}_A$ and $\text{right } \{(X, \Delta)\} = \phi$ therefore $\{(X, \Delta)\} \in \text{Con}_{A+B}$.

Similarly if $(\Delta, Y) \in D_{A+B}$ then $\{(\Delta, Y)\} \in \text{Con}_{A+B}$.

And $\{(\Delta, \Delta)\} \in \text{Con}_{A+B}$.

(iii) $u \in \text{Con}_{A+B}$ and $u \vdash_{A+B} (X', \Delta) \Rightarrow \text{left } u \in \text{Con}_A$ and $\text{right } u = \phi$ and $\text{left } u \vdash_A X' \Rightarrow \text{left } u \cup \{X'\} \in \text{Con}_A$ and $\text{right } u = \phi \Rightarrow \text{left } (u \cup \{(X', \Delta)\}) \in \text{Con}_A$ and $\text{right } (u \cup \{(X', \Delta)\}) = \phi \Rightarrow u \cup \{(X', \Delta)\} \in \text{Con}_{A+B}$.

Similarly $u \in \text{Con}_{A+B}$ and $u \vdash_{A+B} (\Delta, Y') \Rightarrow u \cup \{(\Delta, Y')\} \in \text{Con}_{A+B}$.

And $u \in \text{Con}_{A+B} \Rightarrow u \cup \{(\Delta, \Delta)\} \in \text{Con}_{A+B}$.

(iv) $u \in \text{Con}_{A+B} \Rightarrow u \vdash_{A+B} (\Delta, \Delta)$.

(v) $u \in \text{Con}_{A+B}$ and $(X, \Delta) \in u \Rightarrow \text{left } u \in \text{Con}_A$ and $X \in \text{left } u \Rightarrow \text{left } u \neq \phi$ and $\text{left } u \vdash_A X \Rightarrow u \vdash_{A+B} (X, \Delta)$.

Similarly, $u \in \text{Con}_{A+B}$ and $(\Delta, Y) \in u \Rightarrow u \vdash_{A+B} (\Delta, Y)$.

And $u \vdash_{A+B} (\Delta, \Delta)$ for all $u \in \text{Con}_{A+B}$.

(vi) $v \vdash_{A+B} (X, \Delta)$ for all $(X, \Delta) \in u$ and
 $u \vdash_{A+B} (X', \Delta) \Rightarrow \text{left } v \neq \emptyset$ and $\text{left } v \vdash_A X$ for all $X \in \text{left } u$ and
 $\text{left } u \neq \emptyset$ and $\text{left } u \vdash_A X' \Rightarrow \text{left } v \neq \emptyset$ and
 $\text{left } v \vdash_A X' \Rightarrow v \vdash_{A+B} (X', \Delta)$.

Similarly $v \vdash_{A+B} (\Delta, Y)$ for all $(\Delta, Y) \in u$ and
 $u \vdash_{A+B} (\Delta, Y') \Rightarrow v \vdash_{A+B} (\Delta, Y')$.

(II) We define $\text{inl}: A \rightarrow A + B$ and $\text{inr}: B \rightarrow A + B$ by the formulae:

$v \text{ inl } u$ iff $\{(X, \Delta) | X \in v\} \vdash_{A+B} u$ and

$w \text{ inr } u$ iff $\{(\Delta, Y) | Y \in w\} \vdash_{A+B} u$,

for all $u \in \text{Con}_{A+B}$, $v \in \text{Con}_A$, and $w \in \text{Con}_B$.

Let now $f: A \rightarrow C$ and $g: B \rightarrow C$ be approximable mappings; we define
 $[f, g]: A + B \rightarrow C$ by the formula:

$u [f, g] s$ iff either $\vdash_C s$, or $\text{left } u \neq \emptyset$ and $\text{left } u f s$
 or $\text{right } u \neq \emptyset$ and $\text{right } u g s$,

for all $s \in \text{Con}_C$ and $u \in \text{Con}_{A+B}$.

Then $\text{inl}: A \rightarrow A + B$, $\text{inr}: B \rightarrow A + B$, and $[f, g]: A + B \rightarrow C$ are
 approximable mappings, and we have:

a) For each $x \in |A|$:

$Z \in ([f, g] \circ \text{inl})(x) \Rightarrow v ([f, g] \circ \text{inl}) \{Z\}$ for some $v \subseteq x \Rightarrow v \text{ inl } u$
 and $u [f, g] \{Z\}$ for some $u \in \text{Con}_{A+B}$ and $v \subseteq x \Rightarrow \{(X, \Delta) | X \in v\} \vdash_{A+B} u$
 and $u [f, g] \{Z\}$ for some $u \in \text{Con}_{A+B}$ and

$v \subseteq x \Rightarrow v \vdash_A \text{left } u$ and $\left\{ \begin{array}{l} \text{either } \vdash_C \{Z\} \\ \text{or } \text{left } u \neq \emptyset \text{ and } \text{left } u f \{Z\} \end{array} \right.$

for some $u \in \text{Con}_{A+B}$ and $v \subseteq x \Rightarrow v f \{Z\}$ for some $v \subseteq x \Rightarrow Z \in f(x)$,

thus $([f, g] \circ \text{inl})(x) \subseteq f(x)$; also

$Z \in f(x) \Rightarrow v f \{Z\}$ for some $v \subseteq x \Rightarrow v \text{ inl } \{(X, \Delta) | X \in v\}$ and $\{(X, \Delta) | X \in v\} [f, g] \{Z\}$ for some $v \subseteq x \Rightarrow v ([f, g] \circ \text{inl}) \{Z\}$ for some $v \subseteq x \Rightarrow Z \in ([f, g] \circ \text{inl})(x)$, thus $f(x) \subseteq ([f, g] \circ \text{inl})(x)$; so $([f, g] \circ \text{inl})(x) = f(x)$.

Hence $[f, g] \circ \text{inl} = f$.

(b) Similarly $[f, g] \circ \text{inr} = g$.

(c) $[f, g] (\perp_{A+B}) = \{Z \in D_C \mid u [f, g] \{Z\} \text{ for some } u \subseteq \perp_{A+B}\}$
 $= \{Z \in D_C \mid \vdash_C \{Z\}\}$
 $= \perp_C$.

Lemma

The elements of $A + B$, apart from \perp_{A+B} , are just the elements in the ranges of inl and inr .

Proof:

It is sufficient to prove the lemma for all the finite elements of $A + B$. So, let $u \in \text{Con}_{A+B}$ and $u \neq \perp_{A+B}$. Then:

(i) If $\text{lft } u \in \text{Con}_A$ and $\text{lft } u \neq \phi$ and $\text{rht } u = \phi$, then

$$\begin{aligned} \bar{u} &= \{W \in D_{A+B} \mid u \vdash_{A+B} W\} \\ &= \{W \in D_{A+B} \mid \{(X, \Delta) \mid X \in \text{lft } u\} \vdash_{A+B} W\} \\ &= \{W \in D_{A+B} \mid \text{lft } u \text{ inl } \{W\}\} \\ &= \text{inl}(\overline{\text{lft } u}). \end{aligned}$$

(ii) Similarly if $\text{lft } u = \phi$ and $\text{rht } u \in \text{Con}_B$ and $\text{rht } u \neq \phi$ then $\bar{u} = \text{inr}(\overline{\text{rht } u})$. □

Now, let $[f, g]': A + B \rightarrow C$ be another approximable mapping satisfying conditions (a), (b), (c). Then:

$$[f, g]'(\text{inl}(x)) = f(x) = [f, g](\text{inl}(x)),$$

for all $x \in |A|$;

$$[f, g]'(\text{inr}(y)) = g(y) = [f, g](\text{inr}(y)),$$

for all $y \in |B|$; and

$$[f, g]'(\perp_{A+B}) = \perp_C = [f, g](\perp_{A+B}).$$

So, according to the previous lemma, $[f, g]' = [f, g]$ and therefore

$[f, g]: A + B \rightarrow C$ is unique. □

Remark 3.4.8

There is a trivial product of no terms, 1 , called the unit type or domain. It is such that $D_1 = \{\Delta_1\}$, and that equation determines it up to isomorphism. The domain 1 has but one element, namely \perp_1 .

Moreover, if A is an information system, all approximable mappings

$f: 1 \rightarrow A$ are constant and there is but one approximable mapping

$f: A \rightarrow 1$, namely $f = 0 = \text{const}(\perp_1)$.

The domain $\text{BOOL} = 1 + 1$ has two elements true and false, such that

any mapping on BOOL is uniquely determined by its action on true,

false and \perp_{BOOL} , and the values on the first two elements may be arbitrarily chosen.

The construction that makes the whole theory of domains work so smoothly is the function space construct: Given information systems A and B we construct the function space $A \rightarrow B$ and prove that the approximable mappings from A to B are exactly the elements of the domain $|A \rightarrow B|$.

Definition 3.4.9

Let A and B be two information systems. By $A \rightarrow B$, the function space, we understand the system where:

$$(1) \quad D_{A \rightarrow B} = \{(u, v) \mid u \in \text{Con}_A \text{ and } v \in \text{Con}_B\};$$

(ii) $\Delta_{A \rightarrow B} = (\phi, \phi)$; and where,

for all n and all $w = \{(u_0, v_0), \dots, (u_{n-1}, v_{n-1})\}$, we have:

(iii) $w \in \text{Con}_{A \rightarrow B}$ iff whenever $I \subseteq \{0, \dots, n-1\}$ and

$\bigcup \{u_i | i \in I\} \in \text{Con}_A$, then $\bigcup \{v_i | i \in I\} \in \text{Con}_B$;

(iv) $w \vdash_{A \rightarrow B} (u', v')$ iff $\bigcup \{v_i | u' \vdash_A u_i\} \vdash_B v'$,

for all $u' \in \text{Con}_A$ and $v' \in \text{Con}_B$.

Proposition 3.4.10

Let $w = \{(u_0, v_0), \dots, (u_{n-1}, v_{n-1})\} \in \text{Con}_{A \rightarrow B}$. Then the set $\{(u', v') | \bigcup \{v_i | u' \vdash_A u_i\} \vdash_B v'\} = \bar{w}$ where $u' \in \text{Con}_A$ and $v' \in \text{Con}_B$, defines an approximable mapping from A to B .

Proof:

$u' \vdash_A \bigcup \{u_i | u' \vdash_A u_i\} \Rightarrow \bigcup \{u_i | u' \vdash_A u_i\} \in \text{Con}_A \Rightarrow \bigcup \{v_i | u' \vdash_A u_i\} \in \text{Con}_B$, since $w \in \text{Con}_{A \rightarrow B}$. So \bar{w} is well defined, and we have:

(i) $(\phi, \phi) \in \bar{w}$;

(ii) $(u, v) \in \bar{w}$ and $(u, v') \in \bar{w} \Rightarrow \bigcup \{v_i | u \vdash_A u_i\} \vdash_B v$ and

$\bigcup \{v_i | u \vdash_A u_i\} \vdash_B v' \Rightarrow \bigcup \{v_i | u \vdash_A u_i\} \vdash_B v \cup v' \Rightarrow (u, v \cup v') \in \bar{w}$;

(iii) $u' \vdash_A u$, $(u, v) \in \bar{w}$, and $v \vdash_B v' \Rightarrow u' \vdash_A u$,

$\bigcup \{v_i | u \vdash_A u_i\} \vdash_B v$, and $v \vdash_B v' \Rightarrow \bigcup \{v_i | u' \vdash_A u_i\} \vdash_B v$

and $v \vdash_B v' \Rightarrow \bigcup \{v_i | u' \vdash_A u_i\} \vdash_B v' \Rightarrow (u', v') \in \bar{w}$.

So \bar{w} is an approximable mapping from A to B . □

Proposition 3.4.11

Let A, B and C be information systems. Then

(I) $A \rightarrow B$ is also an information system, and the approximable mappings

$f: A \rightarrow B$ are exactly the elements $f \in |A \rightarrow B|$.

(II) There exists an approximable mapping $\text{apply}: (B \rightarrow C) \times B \rightarrow C$

such that

$\text{apply}(g, y) = g(y)$,

for all $g: B \rightarrow C$ and $y \in |B|$.

(III) For all approximable mappings $h: A \times B \rightarrow C$ there is one and only one approximable mapping $\text{curry } h: A \rightarrow (B \rightarrow C)$ such that

$$h = \text{apply} \circ \langle (\text{curry } h) \circ \text{fst}, \text{snd} \rangle.$$

Proof:

(I) Must show that $A \rightarrow B$ satisfies the (vi) axioms of definition 3.1.1.

(i) Let $w = \{(u_0, v_0), \dots, (u_{n-1}, v_{n-1})\} \in \text{Con}_{A \rightarrow B}$ and $w' = \{(u'_0, v'_0), \dots, (u'_{m-1}, v'_{m-1})\} \subseteq w$. Suppose $I \subseteq \{0, \dots, m-1\}$ and $\bigcup \{u'_i | i \in I\} \in \text{Con}_A$. Then $\bigcup \{u'_i | i \in I\} = \bigcup \{u_j | j \in J\} \in \text{Con}_A$, where $J \subseteq \{0, \dots, n-1\}$. Then $\bigcup \{v_j | j \in J\} \in \text{Con}_B$. Therefore $\bigcup \{v'_i | i \in I\} \in \text{Con}_B$. Hence $w' \in \text{Con}_{A \rightarrow B}$.

(ii) Let $(u, v) \in D_{A \rightarrow B}$. Then $u \in \text{Con}_A$ and $v \in \text{Con}_B$. Therefore $\{(u, v)\} \in \text{Con}_{A \rightarrow B}$.

(iii) Let $w = \{(u_0, v_0), \dots, (u_{n-1}, v_{n-1})\} \in \text{Con}_{A \rightarrow B}$ and $w \vdash_{A \rightarrow B} (u', v')$. Suppose $I \subseteq \{0, \dots, n-1\}$ and $\bigcup \{u_i | i \in I\} \cup u' \in \text{Con}_A$. Then $\bigcup \{u_i | i \in I\} \in \text{Con}_A$. Thus $\bigcup \{v_i | i \in I\} \in \text{Con}_B$. But $\bigcup \{v_i | i \in I\} \vdash_B v'$. Thus $\bigcup \{v_i | i \in I\} \cup v' \in \text{Con}_B$. So $w \cup \{(u', v')\} \in \text{Con}_{A \rightarrow B}$.

(iv) $w \vdash_{A \rightarrow B} (\phi, \phi)$ for all $w \in \text{Con}_{A \rightarrow B}$.

(v) Let $w = \{(u_0, v_0), \dots, (u_{n-1}, v_{n-1})\} \in \text{Con}_{A \rightarrow B}$ and $(u, v) \in w$. Then $v \in \bigcup \{v_i | u_i \vdash_A u\}$. So $\bigcup \{v_i | u_i \vdash_A u\} \vdash_B v$. Therefore $w \vdash_{A \rightarrow B} (u, v)$.

(vi) Let $w = \{(u_0, v_0), \dots, (u_{n-1}, v_{n-1})\}$, $w' = \{(u'_0, v'_0), \dots, (u'_{m-1}, v'_{m-1})\} \in \text{Con}_{A \rightarrow B}$ and $(u, v) \in D_{A \rightarrow B}$. Suppose $w \vdash_{A \rightarrow B} (u'_j, v'_j)$ for all $j \in \{0, \dots, m-1\}$ and

$w' \vdash_{A \rightarrow B} (u, v)$. Then we have:

$\bigcup \{v_j | u_j' \vdash_A u_1'\} \vdash_B v_j'$ for all $j \in \{0, \dots, m-1\}$ and

$\bigcup \{v_j' | u \vdash_A u_j'\} \vdash_B v \Rightarrow \bigcup \{v_j | u_j' \vdash_A u_1' \text{ and } u \vdash_A u_j'\} \vdash_B v_j'$ for all

$v_j' \in \{v_j' | u \vdash_A u_j'\}$ and $\bigcup \{v_j' | u \vdash_A u_j'\} \vdash_B v \Rightarrow \bigcup \{v_j | u \vdash_A u_1'\} \vdash_B v_j'$ for

all $v_j' \in \{v_j' | u \vdash_A u_j'\}$ and $\bigcup \{v_j' | u \vdash_A u_j'\} \vdash_B v \Rightarrow \bigcup \{v_j | u \vdash_A u_1'\} \vdash_B v$.

Therefore $w \vdash_{A \rightarrow B} (u, v)$.

So $A \rightarrow B$ is an information system, and we shall show that the elements are exactly the approximable mappings $f: A \rightarrow B$.

Let an approximable mapping $f: A \rightarrow B$. Then, as a binary relation between Con_A and Con_B , $f \subseteq D_{A \rightarrow B}$ and we have:

(i) Let $w = \{(u_0, v_0), \dots, (u_{n-1}, v_{n-1})\} \subseteq f$. Suppose

$I \subseteq \{0, \dots, n-1\}$ and $\bigcup \{u_i | i \in I\} \in \text{Con}_A$. Then

$\bigcup \{u_i | i \in I\} f v_1$ for all $i \in I$. Thus

$\bigcup \{u_i | i \in I\} f \bigcup \{v_i | i \in I\}$. Therefore $\bigcup \{v_i | i \in I\} \in \text{Con}_B$.

So $w \in \text{Con}_{A \rightarrow B}$.

(ii) Let $w = \{(u_0, v_0), \dots, (u_{n-1}, v_{n-1})\} \subseteq f$ and $w \vdash_{A \rightarrow B} (u', v')$.

Then $u_i f v_1$ for all i and $\bigcup \{v_i | u' \vdash_A u_i'\} \vdash_B v'$.

So $u' f \bigcup \{v_i | u' \vdash_A u_i'\}$ and $\bigcup \{v_i | u' \vdash_A u_i'\} \vdash_B v'$. Thus $u' f v'$.

Therefore $(u', v') \in f$.

Hence $f \in |A \rightarrow B|$.

Conversely suppose $f \in |A \rightarrow B|$. Then $f \subseteq D_{A \rightarrow B}$ is a binary relation between Con_A and Con_B and we have:

(i) $(\phi, \phi) \in f$

(ii) $(u, v), (u, v') \in f \Rightarrow \{(u, v), (u, v')\} \vdash_{A \rightarrow B} (u, v \cup v')$

$\Rightarrow (u, v \cup v') \in f$

(iii) $u' \vdash_A u, (u, v) \in f, v \vdash_B v' \Rightarrow \{(u, v)\} \vdash_{A \rightarrow B} (u', v') \Rightarrow (u', v') \in f$.

Hence f is an approximable mapping from A to B .

(II) Let $g: B \rightarrow C$ and $y \in |B|$. We define $\text{apply}: (B \rightarrow C) \times B \rightarrow C$ by the formula:

$$(w, u') \text{ apply } v' \text{ iff } w \vdash_{B \rightarrow C} (u', v'),$$

for all $w \in \text{Con}_{B \rightarrow C}$, $u' \in \text{Con}_B$, and $v' \in \text{Con}_C$. Then

$\text{apply}: (B \rightarrow C) \times B \rightarrow C$ is an approximable mapping and we have:

$z \in \text{apply}(g, y) \Rightarrow (w, u') \text{ apply } \{z\}$ for some

$(w, u') \subseteq (g, y) \Rightarrow w \vdash_{B \rightarrow C} (u', \{z\})$ for some

$w \subseteq g$ and $u' \subseteq y \Rightarrow \{(u', \{z\})\} \subseteq g$ for some $u' \subseteq y \Rightarrow u' g \{z\}$

for some $u' \subseteq y \Rightarrow z \in g(y)$, thus, $\text{apply}(g, y) \subseteq g(y)$; also

$z \in g(y) \Rightarrow \{(u', \{z\})\} \subseteq g$ for some $u' \subseteq y \Rightarrow (\{(u', \{z\})\}, u') \text{ apply } \{z\}$

and $(\{(u', \{z\})\}, u') \subseteq (g, y) \Rightarrow z \in \text{apply}(g, y)$, thus $g(y) \subseteq \text{apply}(g, y)$.

Hence $\text{apply}(g, y) = g(y)$.

(III) Let $h: A \times B \rightarrow C$. We want to define an approximable mapping $\text{curry } h: A \rightarrow (B \rightarrow C)$ such that

$$h = \text{apply} \circ \langle (\text{curry } h) \circ \text{fst}, \text{snd} \rangle \quad (1)$$

But (1) is equivalent to

$$h(x, y) = (\text{curry } h)(x)(y), \quad (2)$$

for all $x \in |A|$ and $y \in |B|$ because:

$$\begin{aligned} h(x, y) &= (\text{apply} \circ \langle (\text{curry } h) \circ \text{fst}, \text{snd} \rangle)(x, y) \\ &= \text{apply}(\langle (\text{curry } h) \circ \text{fst}, \text{snd} \rangle(x, y)) \\ &= \text{apply}(((\text{curry } h) \circ \text{fst})(x, y), \text{snd}(x, y)) \\ &= \text{apply}((\text{curry } h)(x), y) \\ &= (\text{curry } h)(x)(y). \end{aligned}$$

We define $\text{curry } h: A \rightarrow (B \rightarrow C)$ by the formula:

$$u(\text{curry } h) \{(v, w)\} \text{ iff } (u, v) h w ,$$

for all $u \in \text{Con}_A$, $v \in \text{Con}_B$, and $w \in \text{Con}_C$.

Then $\text{curry } h: A \rightarrow (B \rightarrow C)$ is an approximable mapping and for all $x \in |A|$, $y \in |B|$ we have:

$$\begin{aligned} (\text{curry } h)(x)(y) &= \{Z \in D_C \mid v((\text{curry } h)(x))\{Z\} \text{ for some } v \subseteq y\} \\ &= \{Z \in D_C \mid u(\text{curry } h)\{(v, \{Z\})\} \text{ for some } u \subseteq x \\ &\quad \text{and } v \subseteq y\} \\ &= \{Z \in D_C \mid (u, v) h \{Z\} \text{ for some } (u, v) \subseteq (x, y)\} \\ &= h(x, y) . \end{aligned}$$

So (2), and equivalently (1), is proved.

Now, let $\text{curry}' h: A \rightarrow (B \rightarrow C)$ be another approximable mapping satisfying (1) and equivalently (2). Then for all $x \in |A|$, $y \in |B|$ we have:

$$\begin{aligned} (\text{curry}' h)(x)(y) &= h(x, y) \\ &= (\text{curry } h)(x)(y) . \end{aligned}$$

So $\text{curry}' h = \text{curry } h$ and therefore $\text{curry } h: A \rightarrow (B \rightarrow C)$ is unique. □

The elements of the domain $|A \rightarrow B|$ are exactly the approximable mappings $f: A \rightarrow B$. An element of the form $\{(u, v)\}$ where $u \in \text{Con}_A$ and $v \in \text{Con}_B$ corresponds to the constant function $\text{const } v: A \rightarrow B$.

Moreover the elements of the form $\bar{w} = \{(u', v') \mid w \vdash_{A \rightarrow B} (u', v')\}$ where $w \in \text{Con}_{A \rightarrow B}$ are the finite elements of the domain and they correspond to the simple functions from A to B .

In general, elements of domains are the limits of their finite approximations. But the approximable mappings form the elements of a domain themselves. It follows that the approximable mappings can be

approximated by simple functions, and that is why we call them "approximable".

In the above discussion we have already combined the function-space construction with other domains by means of products. We can also iterate the arrow domain constructor with itself as much as we like, and this is how the so-called higher types are formed. There are many mappings between these spaces that can be defined in terms of the simple notions we have been working with. Some basic higher-type operators are defined in the following propositions.

Proposition 3.4.12

Given information systems A and B there is a unique approximable mapping $\text{const}_{A,B}: B \rightarrow (A \rightarrow B)$ such that

$$\text{const}_{A,B}(b) = (\text{const } b)_{A,B} ,$$

for all $b \in |B|$.

Proof:

Define $\text{const}_{A,B}: B \rightarrow (A \rightarrow B)$ as follows:

$$v' \text{ const}_{A,B} \{(u, v)\} \text{ iff } v \subseteq \bar{v}' ,$$

for all $v, v' \in \text{Con}_B$ and $u \in \text{Con}_A$.

Then $\text{const}_{A,B}: B \rightarrow (A \rightarrow B)$ is clearly an approximable mapping, and for each $b \in |B|$ we have:

$$\begin{aligned} \text{const}_{A,B}(b) &= \{(u, v) \in D_{A \rightarrow B} \mid v' \text{ const}_{A,B} \{(u, v)\} \text{ for some } v' \subseteq b\} \\ &= \{(u, v) \in D_{A \rightarrow B} \mid v \subseteq \bar{v}' \text{ for some } v' \subseteq b\} \\ &= \{(u, v) \in D_{A \rightarrow B} \mid v \subseteq b\} \\ &= \{(u, v) \in D_{A \rightarrow B} \mid u (\text{const } b)_{A,B} v\} \\ &= (\text{const } b)_{A,B} . \end{aligned}$$

Finally $\text{const}_{A,B}: B \rightarrow (A \rightarrow B)$ is obviously unique. \square

Proposition 3.4.13

Given information systems A, B and C there is a unique approximable mapping $\text{pair}: (C \rightarrow A) \times (C \rightarrow B) \rightarrow (C \rightarrow (A \times B))$ such that

$$\text{pair}(f, g) = \langle f, g \rangle,$$

for all $f \in |C \rightarrow A|$ and $g \in |C \rightarrow B|$.

Proof:

Define $\text{pair}: (C \rightarrow A) \times (C \rightarrow B) \rightarrow (C \rightarrow (A \times B))$ as follows:

$$v^* \text{ pair } \{(w, u)\} \text{ iff } (w, \text{fst } u) \subseteq \overline{\text{fst } v^*} \text{ and}$$

$$(w, \text{snd } u) \subseteq \overline{\text{snd } v^*},$$

for all $v^* \in \text{Con}_{(C \rightarrow A) \times (C \rightarrow B)}$, $w \in \text{Con}_C$ and $u \in \text{Con}_{A \times B}$.

Then $\text{pair}: (C \rightarrow A) \times (C \rightarrow B) \rightarrow (C \rightarrow (A \times B))$ is clearly an approximable mapping and for each $f \in |C \rightarrow A|$ and $g \in |C \rightarrow B|$ we have:

$$\begin{aligned} \text{pair}(f, g) &= \{(w, u) \in D_{C \rightarrow (A \times B)} \mid v^* \text{ pair } \{(w, u)\} \text{ for some } v^* \subseteq (f, g)\} \\ &= \{(w, u) \in D_{C \rightarrow (A \times B)} \mid (w, \text{fst } u) \subseteq \overline{\text{fst } v^*} \\ &\quad \text{and } (w, \text{snd } u) \subseteq \overline{\text{snd } v^*} \text{ for some } v^* \subseteq (f, g)\} \\ &= \{(w, u) \in D_{C \rightarrow (A \times B)} \mid (w, \text{fst } u) \subseteq f \text{ and } (w, \text{snd } u) \subseteq g\} \\ &= \{(w, u) \in D_{C \rightarrow (A \times B)} \mid w \subseteq \langle f, g \rangle\} \\ &= \langle f, g \rangle. \end{aligned}$$

Finally, $\text{pair}: (C \rightarrow A) \times (C \rightarrow B) \rightarrow (C \rightarrow (A \times B))$ is obviously unique. \square

Proposition 3.4.14

Given information systems A, B and C there is a unique approximable mapping $\text{comp}: (B \rightarrow C) \times (A \rightarrow B) \rightarrow (A \rightarrow C)$ such that

$$\text{comp}(g, f) = g \circ f,$$

for all $g \in |B \rightarrow C|$ and $f \in |A \rightarrow B|$.

Proof:

Define $\text{comp}: (B \rightarrow C) \times (A \rightarrow B) \rightarrow (A \rightarrow C)$ as follows:

$v^* \text{ comp } \{(u, w)\}$ iff $(v, w) \subseteq \overline{\text{fst } v^*}$ and

$(u, v) \subseteq \overline{\text{snd } v^*}$ for some $v \in \text{Con}_B$,

for all $v^* \in \text{Con}_{(B \rightarrow C) \times (A \rightarrow B)}$, $u \in \text{Con}_A$ and $w \in \text{Con}_C$.

Then $\text{comp}: (B \rightarrow C) \times (A \rightarrow B) \rightarrow (A \rightarrow C)$ is an approximable mapping

and for each $g \in |B \rightarrow C|$ and $f \in |A \rightarrow B|$ we have:

$$\begin{aligned} \text{comp}(g, f) &= \{(u, w) \in D_{A \rightarrow C} \mid v^* \text{ comp } \{(u, w)\} \text{ for some } v^* \subseteq (g, f)\} \\ &= \{(u, w) \in D_{A \rightarrow C} \mid (v, w) \subseteq \overline{\text{fst } v^*} \text{ and} \\ &\quad (u, v) \subseteq \overline{\text{snd } v^*} \text{ for some } v \in \text{Con}_B \text{ and } v^* \subseteq (g, f)\} \\ &= \{(u, w) \in D_{A \rightarrow C} \mid v g w \text{ and } u f v \text{ for some } v \in \text{Con}_B\} \\ &= \{(u, w) \in D_{A \rightarrow C} \mid u (g \circ f) w\} \\ &= g \circ f. \end{aligned}$$

Finally $\text{comp}: (B \rightarrow C) \times (A \rightarrow B) \rightarrow (A \rightarrow C)$ is obviously unique. \square

Remark 3.4.15 (for Cartesian closed categories see Lambek [13])

Consider the category C of domains (objects) and their approximable mappings (arrows) where composition on arrows is defined in 3.3.6 and for each object $|A|$ the identity on $|A|$ is defined in 3.3.5. Then,

(i) the domain 1 , (see remark 3.4.8), is a terminal object in C ;

(ii) by 3.4.1 and 3.4.3 every two objects have a product;

(iii) by 3.4.9 and 3.4.11 for every two objects $|A|$ and $|B|$, there is a power $|A \rightarrow B|$.

Thus C is a Cartesian closed category and therefore it is a model of the typed λ -calculus. Moreover the typed atomic combinators can be defined as follows:

- (i) I_A
- (ii) $K_{A,B} = \text{const}_{B,A}$
- (iii) $S_{A,B,C} = \text{curry } (((\text{curry comp}) (\text{apply})) \circ \text{pair})$

CHAPTER IV

The material of this chapter is derived primarily from Scott [15] and Scott [16].

Section 1. A lattice-theoretic model for the λ -calculus

Preliminaries 4.1.0 (see Dugundji [7])

Suppose that the sequence $\langle X_n, j_n \rangle_{n=0}^{\infty}$ is an inverse system of T_0 -spaces X_n and continuous maps $j_n: X_{n+1} \rightarrow X_n$. The inverse limit space X_{∞} of the system consists of exactly those infinite sequences

$$x = \langle x_n \rangle_{n=0}^{\infty},$$

where for each n we have:

$$x_n \in X_n \text{ and } j_n(x_{n+1}) = x_n.$$

The space X_{∞} , which is a subspace of the product space $\prod_n X_n$, is given the product topology, and the maps $j_{\infty n}: X_{\infty} \rightarrow X_n$ such that

$$j_{\infty n}(x) = x_n$$

are of course continuous and satisfy the recursion equation:

$$j_{\infty n} = j_n \circ j_{\infty(n+1)}.$$

Moreover for all $x, y \in X_{\infty}$ we have

$$x = y \text{ iff } j_{\infty n}(x) = j_{\infty n}(y) \text{ for all } n.$$

Given also a T_0 -space Y and a system of continuous maps $f_n: Y \rightarrow X_n$ where for each n

$$f_n = j_n \circ f_{n+1},$$

we can define $f_{\infty}: Y \rightarrow X_{\infty}$ by the equation

$$f_{\infty}(y) = \langle f_n(y) \rangle_{n=0}^{\infty},$$

for all $y \in Y$; whence

$$f_n = j_{\infty n} \circ f_{\infty}.$$

Proposition 4.1.1

Let $\langle D_n, j_n \rangle_{n=0}^{\infty}$ be an inverse system of continuous lattices where each $j_n: D_{n+1} \rightarrow D_n$ is a projection. Then the inverse limit space D_{∞} is also a continuous lattice.

Proof:

By 2.3.15 it is sufficient to show that D_{∞} as a T_0 -space is injective.

So suppose X, Y are T_0 -spaces with $X \subseteq Y$ as a subspace, and

$f_{\infty}: X \rightarrow D_{\infty}$ is continuous. Define $f_n: X \rightarrow D_n$ by the equation

$$f_n = j_{\infty n} \circ f_{\infty},$$

and let $\bar{f}_n: Y \rightarrow D_n$ be the maximal extension of f_n according to 2.3.22.

But

$$\begin{aligned} f_n &= j_{\infty n} \circ f_{\infty} \\ &= j_n \circ j_{\infty(n+1)} \circ f_{\infty} \\ &= j_n \circ f_{n+1}. \end{aligned}$$

Therefore $\bar{f}_n = j_n \circ \bar{f}_{n+1}$ by 2.3.24.

Then $\bar{f}_{\infty}: Y \rightarrow D_{\infty}$, defined by the equation

$$\bar{f}_{\infty}(y) = \langle \bar{f}_n(y) \rangle_{n=0}^{\infty}$$

for all $y \in Y$, is a continuous extension of $f_{\infty}: X \rightarrow D_{\infty}$. Hence D_{∞} is injective. □

Note 4.1.2

$x \subseteq y$ iff $x_n \subseteq y_n$ for all n ,

for all $x, y \in D_{\infty}$.

Proposition 4.1.3

Let $\langle D_n, j_n \rangle_{n=0}^{\infty}$ be an inverse system of continuous lattices where each $j_n: D_{n+1} \rightarrow D_n$ is a projection. Then the maps $j_{\infty n}: D_{\infty} \rightarrow D_n$ are projections.

Proof:

The projections $j_n: D_{n+1} \rightarrow D_n$, as we know, have their uniquely determined inverses $i_n: D_n \rightarrow D_{n+1}$. We can define $i_{n^\infty}: D_n \rightarrow D_\infty$ by the equation:

$$i_{n^\infty}(x) = \langle y_m \rangle_{m=0}^\infty$$

where

$$y_m = \begin{cases} j_m(y_{m+1}) & \text{if } m < n \\ x & \text{if } m = n \\ i_{m-1}(y_{m-1}) & \text{if } m > n \end{cases},$$

for all $x \in D_n$. Then we have:

(i) $(j_{n^\infty} \circ i_{n^\infty})(x) = j_{n^\infty}(i_{n^\infty}(x)) = j_{n^\infty}(\langle y_m \rangle_{m=0}^\infty) = y_n = x$, for all $x \in D_n$

(ii) $(i_{n^\infty} \circ j_{n^\infty})(y) = i_{n^\infty}(j_{n^\infty}(y)) = i_{n^\infty}(y_n) \subseteq y$ by 4.1.2, for all $y \in D_\infty$.

Thus $j_{n^\infty} \circ i_{n^\infty} = \text{id}_{D_n}$ and $i_{n^\infty} \circ j_{n^\infty} \subseteq \text{id}_{D_\infty}$.

Hence i_{n^∞} and j_{n^∞} form a projection. □

The maps $i_n: D_n \rightarrow D_\infty$ satisfy the recursion equation

$$i_{n^\infty} = i_{(n+1)^\infty} \circ i_n$$

and for all $x \in D_\infty$ we have:

$$x = \bigsqcup_{n=0}^\infty i_{n^\infty}(x_n).$$

Moreover we have:

$$i_{n^\infty}(x_n) = (i_{(n+1)^\infty} \circ i_n \circ j_n)(x_{n+1}) \subseteq i_{(n+1)^\infty}(x_{n+1}).$$

Thus $x = \bigsqcup_{n=0}^\infty i_{n^\infty}(x_n)$ is a monotone lub, and so we can say each $x \in D_\infty$ is the limit of its projections x_n . In fact, from what we know about projections, x_n is the best possible approximation to x in the space D_n .

Definition 4.1.4 (Construction of the λ -model D_∞)

Let $D = D_0$ be a given non-trivial continuous lattice. Consider the continuous lattice $[D_0 \rightarrow D_0] = D_1$ and let i_0, j_0 be a pair of continuous maps that makes D_0 a projection of D_1 . Define recursively:

$$D_{n+1} = [D_n \rightarrow D_n]$$

and introduce the pairs i_{n+1}, j_{n+1} making D_{n+1} a projection of D_{n+2} by the method of 2.3.21. Specifically we shall have for $x \in D_{n+1}$ and $x' \in D_{n+2}$:

$$i_{n+1}(x) = i_n \circ x \circ j_n,$$

$$j_{n+1}(x') = j_n \circ x' \circ i_n.$$

Then the sequence $\langle D_n, j_n \rangle_{n=0}^\infty$ is an inverse system of continuous lattices and define D_∞ to be the inverse limit space of the system.

Proposition 4.1.5

The inverse limit D_∞ of the recursively defined sequence $\langle D_n, j_n \rangle_{n=0}^\infty$ of function spaces is not only a continuous lattice but it is also homeomorphic to its own function space $[D_\infty \rightarrow D_\infty]$.

Proof:

D_∞ is a continuous lattice by 4.1.1.

Define $i_\infty: D_\infty \rightarrow [D_\infty \rightarrow D_\infty]$ and $j_\infty: [D_\infty \rightarrow D_\infty] \rightarrow D_\infty$ by the formulae:

$$i_\infty(x) = \bigcup_{n=0}^\infty (i_{n+1} \circ x \circ j_{n+1}),$$

for all $x \in D_\infty$; and

$$j_\infty(f) = \bigcup_{n=0}^\infty i_{(n+1)\infty} (j_{\infty n} \circ f \circ i_{n\infty}),$$

for all $f \in [D_\infty \rightarrow D_\infty]$.

Then i_∞, j_∞ are continuous maps since the composition of continuous functions is continuous, and the lattice operation \sqcup is continuous.

Let $x \in D_\infty$. Since all the lubs are monotone we have:

$$\begin{aligned}
 (j_{\infty} \circ i_{\infty})(x) &= \bigcup_{n=0}^{\infty} i_{(n+1)\infty} (j_{\infty n} \circ i_{n\infty} \circ x_{n+1} \circ j_{\infty n} \circ i_{n\infty}) \\
 &= \bigcup_{n=0}^{\infty} i_{(n+1)\infty} (x_{n+1}) \\
 &= x.
 \end{aligned}$$

Hence $j_{\infty} \circ i_{\infty} = \text{id}_{D_{\infty}}$.

Lemma

• Suppose for each n we have $u_{(n+1)} \in D_{n+1}$.

Let $u = \bigcup_{n=0}^{\infty} i_{(n+1)\infty} (u_{(n+1)})$. If

$$j_{n+1}(u_{(n+2)}) = u_{(n+1)},$$

for each n , then

$$j_{\infty(n+1)}(u) = u_{(n+1)}.$$

Proof:

For each n we have:

$$\begin{aligned}
 i_{(n+1)\infty}(u_{(n+1)}) &= (i_{(n+2)\infty} \circ i_{n+1})(u_{(n+1)}) = \\
 &= (i_{(n+2)\infty} \circ i_{n+1} \circ j_{n+1})(u_{(n+2)}) \sqsubseteq i_{(n+2)\infty}(u_{(n+2)}).
 \end{aligned}$$

Thus the lub defining u is monotone. Also $j_{\infty(n+1)}$ is continuous for each n . So

$$\begin{aligned}
 j_{\infty(n+1)}(u) &= j_{\infty(n+1)}\left(\bigcup_{m=0}^{\infty} i_{(m+1)\infty}(u_{(m+1)})\right) \\
 &= \bigcup_{m=0}^{\infty} j_{\infty(n+1)}(i_{(m+1)\infty}(u_{(m+1)})).
 \end{aligned}$$

Therefore it is sufficient to prove that

$$j_{\infty(n+1)}(i_{(m+1)\infty}(u_{(m+1)})) = u_{n+1}, \quad (1)$$

for all $m \geq n$. The proof is by induction on $m - n$.

If $m = n$ then

$$j_{\infty(n+1)}(i_{(n+1)\infty}(u_{(n+1)})) = u_{(n+1)}.$$

Suppose (1) is true for $m = n + k$; then

$$j_{\infty(n+2)}(i_{(n+k+2)\infty}(u_{(n+k+2)})) = u_{(n+2)}$$

$$(j_{n+1} \circ j_{\infty(n+2)})(i_{(n+k+2)\infty}(u_{(n+k+2)})) = j_{n+1}(u_{(n+2)})$$

$$j_{\infty(n+1)}(i_{(n+k+2)\infty}(u_{(n+k+2)})) = u_{(n+1)}$$

so (1) is true for $m = n + k + 1$. Hence (1) is true for all $m \geq n$. \square

Let $f \in [D_{\infty} \rightarrow D_{\infty}]$. Then

$$\begin{aligned} (i_{\infty} \circ j_{\infty})(f) &= \bigcup_{n=0}^{\infty} (i_{n\infty} \circ j_{\infty(n+1)}(j_{\infty}(f)) \circ j_{\infty n}) \\ &= \bigcup_{n=0}^{\infty} (i_{n\infty} \circ j_{\infty(n+1)}(\bigcup_{m=0}^{\infty} i_{(m+1)\infty}(j_{\infty m} \circ f \circ i_{m\infty})) \circ j_{\infty n}) \\ &= \bigcup_{n=0}^{\infty} (i_{n\infty} \circ j_{\infty n} \circ f \circ i_{n\infty} \circ j_{\infty n}) \text{ by the previous lemma} \\ &= \bigcup_{n=0}^{\infty} (i_{n\infty} \circ j_{\infty n}) \circ f \circ \bigcup_{n=0}^{\infty} (i_{n\infty} \circ j_{\infty n}) \text{ by the continuity of } f \\ &= f \text{ because } \bigcup_{n=0}^{\infty} (i_{n\infty} \circ j_{\infty n}) = id_{D_{\infty}} \end{aligned}$$

Hence $i_{\infty} \circ j_{\infty} = id_{[D_{\infty} \rightarrow D_{\infty}]}$.

So $D_{\infty} = [D_{\infty} \rightarrow D_{\infty}]$. \square

Definition 4.1.6 (D_{∞} - application)

Define a binary operation application ' \cdot ' on D_{∞} by the equation:

$$x(y) = x \cdot y = \bigcup_{n=0}^{\infty} i_{n\infty}(x_{n+1}(y_n)),$$

for all $x, y \in D_{\infty}$. Application ' \cdot ' on D_{∞} is well defined because $D_{\infty} = [D_{\infty} \rightarrow D_{\infty}]$.

Proposition 4.1.7

For all $x, y \in D_{\infty}$ we have

$$x = y \iff x \cdot z = y \cdot z \text{ for all } z \in D_{\infty}.$$

Proof:

Let $x, y \in D_\infty$.

Suppose $x = y$ and let $z \in D_\infty$. Then

$$x \cdot z = \bigcup_{n=0}^{\infty} i_{n\infty}(x_{n+1}(z_n)) = \bigcup_{n=0}^{\infty} i_{n\infty}(y_{n+1}(z_n)) = y \cdot z.$$

Conversely suppose $x \cdot z = y \cdot z$ for all $z \in D_\infty$. Then

$$(x \cdot z)_n = (y \cdot z)_n \text{ for all } n \text{ and for all } z \in D_\infty.$$

Thus $x_{n+1}(z_n) = y_{n+1}(z_n)$ for all n and for all $z \in D_\infty$. Therefore

$$x_{n+1} = y_{n+1} \text{ for all } n. \text{ Hence } x = y. \quad \square$$

Parentheses will be omitted by association to the left as in λ -calculus.

Proposition 4.1.8

There exist $K, S \in D_\infty$ so that $\langle D_\infty, \cdot, K, S \rangle$ is a combinatory algebra.

Proof:

Define $K \in D_\infty$ by the sequence:

$$K_0 = j_0(K_1) = \perp$$

$$K_1 = j_1(K_2) = \lambda x: D_0. x$$

$$K_{n+2} = \lambda x: D_{n+1}. \lambda y: D_n. j_n(x) \text{ for } n \geq 0.$$

Define $S \in D_\infty$ by the sequence:

$$S_0 = j_0(S_1) = \perp$$

$$S_1 = j_1(S_2) = \lambda x: D_0. x$$

$$S_2 = j_2(S_3) = \lambda x: D_1. \lambda y: D_0. x(\perp)$$

$$S_{n+3} = \lambda x: D_{n+2}. \lambda y: D_{n+1}. \lambda z: D_n. x(i_n(\tilde{z}))(y(z)), \text{ for } n \geq 0.$$

Then for all $x, y, z \in D_\infty$ we have:

$$\begin{aligned}
 (i) \quad K \cdot x \cdot y &= \bigcup_{n=0}^{\infty} i_{n\infty} (K_{n+2}(x_{n+1})(y_n)) \\
 &= \bigcup_{n=0}^{\infty} i_{n\infty} (j_n(x_{n+1})) \\
 &= \bigcup_{n=0}^{\infty} i_{n\infty} (x_n) \\
 &= x
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad S \cdot x \cdot y \cdot z &= \bigcup_{n=0}^{\infty} i_{n\infty} (S_{n+3}(x_{n+2})(y_{n+1})(z_n)) \\
 &= \bigcup_{n=0}^{\infty} i_{n\infty} (x_{n+2}(i_n(z_n))(y_{n+1}(z_n))) \\
 &= x \cdot z \cdot (y \cdot z)
 \end{aligned}$$

Thus, according to 1.2.3, $\langle D_{\infty}, \cdot, K, S \rangle$ is a combinatory algebra. \square

The λ -notation is part of the meta-language as in 2.3.18.

Proposition 4.1.9

There exists $l \in D_{\infty}$ such that

$$(i) \quad l \cdot x \cdot y = x \cdot y \quad \text{and}$$

$$(ii) \quad \text{if } x \cdot z = y \cdot z \text{ for all } z \in D_{\infty} \text{ then } l \cdot x = l \cdot y,$$

for all $x, y \in D_{\infty}$.

Proof:

Define

$$l = S \cdot (S \cdot (K \cdot S) \cdot K) \cdot (K \cdot (S \cdot K \cdot K))$$

Then $l \in D_{\infty}$, and for all $x, y \in D_{\infty}$ we have:

$$\begin{aligned}
 (i) \quad l \cdot x \cdot y &= S \cdot (S \cdot (K \cdot S) \cdot K) \cdot (K \cdot (S \cdot K \cdot K)) \cdot x \cdot y \\
 &= S \cdot (K \cdot S) \cdot K \cdot x \cdot (K \cdot (S \cdot K \cdot K) \cdot x) \cdot y \\
 &= K \cdot S \cdot x \cdot (K \cdot x) \cdot (S \cdot K \cdot K) \cdot y \\
 &= S \cdot (K \cdot x) \cdot (S \cdot K \cdot K) \cdot y \\
 &= K \cdot x \cdot y \cdot (S \cdot K \cdot K \cdot y) \\
 &= x \cdot (K \cdot y \cdot (K \cdot y)) \\
 &= x \cdot y
 \end{aligned}$$

(ii) suppose $x \cdot z = y \cdot z$ for all $z \in D_\infty$; then

$$(l \cdot x) \cdot z = x \cdot z = y \cdot z = (l \cdot y) \cdot z,$$

for all $z \in D_\infty$. Hence by 4.1.7 $l \cdot x = l \cdot y$. □

Proposition 4.1.10

$\langle D_\infty, \cdot, K, S, l \rangle$ is a Scott-Meyer λ -model.

Proof:

By 4.1.8 and 4.1.9 it is obvious that the quintuple $\langle D_\infty, \cdot, K, S, l \rangle$ satisfies definition 1.2.6.

Now, according to theorem 1.2.8, $[[\]]$ can be defined in $\langle D_\infty, \cdot, K, S, l \rangle$ so that $\langle D_\infty, \cdot, [[\]] \rangle$ is a λ -model in the sense of 1.2.4.

Proposition 4.1.11

$\langle D_\infty, \cdot, [[\]] \rangle$ is an extensional λ -model.

Proof:

Obvious by 4.1.7. □

Section 2. A domain - theoretic model for the λ -calculus

Definition 4.2.1

Let A be an information system. Take an object Δ outside D_A and let $\Delta_D = (\Delta, \Delta)$. Define the structure $D = (D_D, \Delta_D, \text{Con}_D, \vdash_D)$ inductively as follows:

- (1) $\Delta_D \in D_D$;
- (2) $(X, \Delta) \in D_D$ whenever $X \in D_A$;
- (3) $(\Delta, (u, v)) \in D_D$ whenever $u, v \in \text{Con}_D$;
- (4) $\phi \in \text{Con}_D$;
- (5) $u \cup \{\Delta_D\} \in \text{Con}_D$ whenever $u \in \text{Con}_D$;
- (6) $\{(X, \Delta) \mid X \in w\} \in \text{Con}_D$ whenever $w \in \text{Con}_A$;
- (7) $\{(\Delta, (u_0, v_0)), \dots, (\Delta, (u_{n-1}, v_{n-1}))\} \in \text{Con}_D$ provided $u_i, v_i \in \text{Con}_D$ for all $i < n$ and whenever $I \subseteq \{0, \dots, n-1\}$ and $U\{u_i \mid i \in I\} \in \text{Con}_D$, then $U\{v_i \mid i \in I\} \in \text{Con}_D$;
- (8) $u \vdash_D \Delta_D$ always;
- (9) $u \cup \{\Delta_D\} \vdash_D Y$ whenever $u \vdash_D Y$;
- (10) $\{(X, \Delta) \mid X \in w\} \vdash_D (W, \Delta)$ whenever $w \vdash_A W$;
- (11) $\{(\Delta, (u_0, v_0)), \dots, (\Delta, (u_{n-1}, v_{n-1}))\} \vdash_D (\Delta, (u', v'))$ whenever $U\{v_i \mid u'_i \vdash_D u_i\} \vdash_D v'$ and the set on the left is in Con_D .

Proposition 4.2.2

Let A be an information system. Then D is also an information system.

Proof:

Must show that consistency and entailment for D satisfy the axioms of 3.1.1. The proof will be aided by noting that the cases in (4) -

(7) and (8) - (11) are disjoint - except for a trivial overlap between

(8) and (9).

(i) Must show that $u \in \text{Con}_D$ whenever $u \subseteq v \in \text{Con}_D$.

(a) If $v = \emptyset$ it is obvious

(b) If $v = v' \cup \{\Delta_D\}$ where $v' \in \text{Con}_D$ and $u' \in \text{Con}_D$ whenever $u' \subseteq v'$ is the induction hypothesis, then $u \subseteq v \Rightarrow$ either $u \subseteq v'$ or $u = u' \cup \{\Delta_D\}$ for some $u' \subseteq v' \Rightarrow u \in \text{Con}_D$

(c) If $v = \{(X, \Delta) | X \in w\}$ where $w \in \text{Con}_A$, then $u \subseteq v \Rightarrow u = \{(X, \Delta) | X \in w'\}$ for some $w' \subseteq w \Rightarrow u = \{(X, \Delta) | X \in w'\}$ for some $w' \in \text{Con}_A \Rightarrow u \in \text{Con}_D$

(d) If $v = \{(\Delta, (u_0, v_0)), \dots, (\Delta, (u_{n-1}, v_{n-1}))\}$ where $u_i, v_i \in \text{Con}_D$ for all $i < n$ and whenever $I \subseteq \{0, \dots, n-1\}$ and $\bigcup \{u_i | i \in I\} \in \text{Con}_D$ then $\bigcup \{v_i | i \in I\} \in \text{Con}_D$, then $u \subseteq v \Rightarrow u$, without loss of generality, is of the form

$u = \{(\Delta, (u_0, v_0)), \dots, (\Delta, (u_{k-1}, v_{k-1}))\}$ where $k \leq n$, and we have:

$u_i, v_i \in \text{Con}_D$ for all $i < k$ and whenever $I \subseteq \{0, \dots, k-1\}$ and $\bigcup \{u_i | i \in I\} \in \text{Con}_D$ then $\bigcup \{v_i | i \in I\} \in \text{Con}_D$. Thus $u \in \text{Con}_D$.

(ii) Must show that $\{Z\} \in \text{Con}_D$ whenever $Z \in D_D$.

(a) $\{\Delta_D\} \in \text{Con}_D$ by 4.2.1(4) and 4.2.1(5)

(b) If $(X, \Delta) \in D_D$ where $X \in D_A$, then $\{(X, \Delta)\} = \{(X, \Delta) | X \in \{X\}\}$ and $\{X\} \in \text{Con}_A$, so by 4.2.1(6), $\{(X, \Delta)\} \in \text{Con}_D$

(c) If $(\Delta, (u, v)) \in D_D$, where $u, v \in \text{Con}_D$, then by 4.2.1(7), $\{(\Delta, (u, v))\} \in \text{Con}_D$.

(iii) Must show that $u \cup \{Z\} \in \text{Con}_D$ whenever $u \vdash_D Z$.

(a) If $u \vdash_D \Delta_D$ then by 4.2.1(5), $u \cup \{\Delta_D\} \in \text{Con}_A$

(b) If $u \cup \{\Delta_D\} \vdash_D Y$ where $u \vdash_D Y$ and $u \cup \{Y\} \in \text{Con}_A$ is the

induction hypothesis, then by 4.2.1(5), $u \cup \{Y\} \cup \{\Delta_D\} \in \text{Con}_D$

(c) If $\{(X, \Delta) | X \in w\} \vdash_D (W, \Delta)$ where $w \vdash_D W$, then $w \cup \{W\} \in \text{Con}_A$, so by 4.2.1(6) $\{(X, \Delta) | X \in w\} \cup \{(W, \Delta)\} = \{(X, \Delta) | X \in w \cup \{W\}\} \in \text{Con}_D$

(d) If $\{(\Delta, (u_0, v_0)), \dots, (\Delta, (u_{n-1}, v_{n-1}))\} \vdash_D (\Delta, (u', v'))$ where $\cup \{v_i | u_i \vdash_D u_i\} \vdash_D v'$ and the set on the left is in Con_D , then $\cup \{v_i | u_i \vdash_D u_i\} \cup \{v'\} \in \text{Con}_D$ by the induction hypothesis, and for the set $\{(\Delta, (u_0, v_0)), \dots, (\Delta, (u_{n-1}, v_{n-1}))\} \cup \{(\Delta, (u', v'))\} = \{(\Delta, (u_0, v_0)), \dots, (\Delta, (u_{n-1}, v_{n-1})), (\Delta, (u_n, v_n))\}$ where $(u_n, v_n) = (u', v')$, we have:

$u_i, v_i \in \text{Con}_D$ for all $i \leq n$ and whenever $I \subseteq \{0, \dots, n\}$ and $\cup \{u_i | i \in I\} \in \text{Con}_D$ then $\cup \{v_i | i \in I\} \in \text{Con}_D$ so by 4.2.1(7)

$\{(\Delta, (u_0, v_0)), \dots, (\Delta, (u_{n-1}, v_{n-1}))\} \cup \{(\Delta, (u', v'))\} \in \text{Con}_D$.

(iv) $u \vdash_D \Delta_D$ for all $u \in \text{Con}_D$ by 4.2.1(8).

(v) For all $u \in \text{Con}_D$ must show $u \vdash_D Z$ whenever $Z \in u$.

(a) If $u = \emptyset$ it is obvious

(b) If $u = u' \cup \{\Delta_D\}$ where $u' \in \text{Con}_D$ and $u' \vdash_D Z$ whenever $Z \in u'$ is the induction hypothesis, then $Z \in u \Rightarrow$ either $Z \in u'$ or $Z = \{\Delta_D\} \Rightarrow u' \vdash_D Z$ or $Z = \{\Delta_D\} \Rightarrow u \vdash_D Z$ by 4.2.1(9) or 4.2.1(8)

(c) If $u = \{(X, \Delta) | X \in w\}$ where $w \in \text{Con}_A$, then

$(X, \Delta) \in u \Rightarrow X \in w \Rightarrow w \vdash_A X \Rightarrow u \vdash_D (X, \Delta)$ by 4.2.1(10)

(d) If $u = \{(\Delta, (u_0, v_0)), \dots, (\Delta, (u_{n-1}, v_{n-1}))\}$ where $u_i, v_i \in \text{Con}_D$ for all $i < n$ and whenever $I \subseteq \{0, \dots, n-1\}$ and $\cup \{u_i | i \in I\} \in \text{Con}_D$ then $\cup \{v_i | i \in I\} \in \text{Con}_D$, then

$(\Delta, (u', v')) \in u \Rightarrow v' \in \cup \{v_i | u_i \vdash_D u_i\} \Rightarrow \cup \{v_i | u_i \vdash_D u_i\} \vdash_D v'$ by the induction hypothesis, $\Rightarrow u \vdash_D (\Delta, (u', v'))$ by 4.2.1(11).

(vi) For all $u, v \in \text{Con}_D$ and $Z \in D_D$ must show that if $v \vdash_D Y$ for all $Y \in u$ and $u \vdash_D Z$ then $v \vdash_D Z$.

(a) If $Z = \Delta_D$ then $v \vdash_D Z$ for all $v \in \text{Con}_D$ by 4.2.1(8).

(b) If $Z = (X', \Delta)$ where $X' \in D_A$, then $v \vdash_D Y$ for all $Y \in u$ and $u \vdash_D (X', \Delta) \Rightarrow v \vdash_D Y$ for all $Y \in u$ and $u = \{(X, \Delta) \mid X \in w\} \vdash_D (X', \Delta) \Rightarrow v \vdash_D (X, \Delta)$ for all $(X, \Delta) \in u$ and $u = \{(X, \Delta) \mid X \in w\} \vdash_D (X', \Delta) \Rightarrow v = \{(X, \Delta) \mid X \in w'\} \vdash_D (X, \Delta)$ for all $(X, \Delta) \in u$ and $u = \{(X, \Delta) \mid X \in w\} \vdash_D (X', \Delta) \Rightarrow w' \vdash_A X$ for all $X \in w$ and $w \vdash_A X' \Rightarrow w' \vdash_A X' \Rightarrow v \vdash_D (X', \Delta)$.

(c) If $Z = (\Delta, (u^*, v^*))$ where $u^*, v^* \in \text{Con}_D$, then $v \vdash_D Y$ for all $Y \in u$ and $u \vdash_D (\Delta, (u^*, v^*)) \Rightarrow v \vdash_D Y$ for all $Y \in u$ and $u = \{(\Delta, (u_0, v_0)), \dots, (\Delta, (u_{n-1}, v_{n-1}))\} \vdash_D (\Delta, (u^*, v^*)) \Rightarrow v \vdash_D (\Delta, (u_1, v_1))$ for all $(\Delta, (u_1, v_1)) \in u$ and $u = \{(\Delta, (u_0, v_0)), \dots, (\Delta, (u_{n-1}, v_{n-1}))\} \vdash_D (\Delta, (u^*, v^*)) \Rightarrow v = \{(\Delta, (u'_0, v'_0)), \dots, (\Delta, (u'_{k-1}, v'_{k-1}))\} \vdash_D (\Delta, (u_1, v_1))$ for all $(\Delta, (u_1, v_1)) \in u$ and $u = \{(\Delta, (u_0, v_0)), \dots, (\Delta, (u_{n-1}, v_{n-1}))\} \vdash_D (\Delta, (u^*, v^*)) \Rightarrow \bigcup \{v'_j \mid u_1 \vdash_D u'_j\} \vdash_D v_1$ and the set on the left is in Con_D for all $i < n$, and $\bigcup \{v_i \mid u^* \vdash_D u_i\} \vdash_D v^*$ and the set on the left is in $\text{Con}_D \Rightarrow \bigcup \{v'_j \mid u^* \vdash_D u'_j\} \vdash_D v^*$ by the induction hypothesis $\Rightarrow v \vdash_D (\Delta, (u^*, v^*))$. □

Proposition 4.2.3

Given an information system A , let D be the information system defined by 4.2.1. Then $D \equiv A + (D + D)$.

Proof:

Take an object Δ' in neither D_A nor D_{D+D} and let

$\Delta_{A+(D \rightarrow D)} = (\Delta', \Delta')$. Define $\varphi: D_D \rightarrow D_{A+(D \rightarrow D)}$ by:

- (1) $\varphi(\Delta_D) = \Delta_{A+(D \rightarrow D)}$
- (ii) $\varphi((X, \Delta)) = (X, \Delta')$ whenever $X \in D_A$
- (iii) $\varphi((\Delta, (u, v))) = (\Delta', (u, v))$ whenever $u, v \in \text{Con}_D$.

Then, according to definitions 3.4.6, 3.4.9, and 4.2.1, we have that

$\varphi(D_D) \cong D_{A+(D \rightarrow D)}$. Also $\varphi(\text{Con}_D) = \text{Con}_{A+(D \rightarrow D)}$ because:

- (1) If $\phi \in \text{Con}_D$ then $\varphi(\phi) = \phi \in \text{Con}_{A+(D \rightarrow D)}$
- (2) If $u \cup \{\Delta_D\} \in \text{Con}_D$ where $u \in \text{Con}_D$ and $\varphi(u) \in \text{Con}_{A+(D \rightarrow D)}$ is the induction hypothesis, then $\varphi(u \cup \{\Delta_D\}) = \varphi(u) \cup \{\varphi(\Delta_D)\} = \varphi(u) \cup \{\Delta_{A+(D \rightarrow D)}\} \in \text{Con}_{A+(D \rightarrow D)}$
- (3) If $\{(X, \Delta) | X \in w\} \in \text{Con}_D$ where $w \in \text{Con}_A$, then $\varphi(\{(X, \Delta) | X \in w\}) = \{\varphi((X, \Delta)) | X \in w\} = \{(X, \Delta') | X \in w\} \in \text{Con}_{A+(D \rightarrow D)}$
- (4) If $\{(\Delta, (u_0, v_0)), \dots, (\Delta, (u_{n-1}, v_{n-1}))\} \in \text{Con}_D$ where $u_1, v_1 \in \text{Con}_D$ for all $i < n$ and whenever $I \subseteq \{0, \dots, n-1\}$ and $\bigcup \{u_i | i \in I\} \in \text{Con}_D$ then $\bigcup \{v_i | i \in I\} \in \text{Con}_D$, then $\varphi(\{(\Delta, (u_0, v_0)), \dots, (\Delta, (u_{n-1}, v_{n-1}))\}) = \{(\Delta', (u_0, v_0)), \dots, (\Delta', (u_{n-1}, v_{n-1}))\} \in \text{Con}_{A+(D \rightarrow D)}$.

Moreover $\varphi(\vdash_D) = \vdash_{A+(D \rightarrow D)}$ because:

- (5) If $u \vdash_D \Delta_D$ where $u \in \text{Con}_D$, then $\varphi(u) \in \text{Con}_{A+(D \rightarrow D)}$ and $\varphi(u) \vdash_{A+(D \rightarrow D)} \Delta_{A+(D \rightarrow D)}$; so $\varphi(u) \vdash_{A+(D \rightarrow D)} \varphi(\Delta_D)$
- (6) If $u \cup \{\Delta_D\} \vdash_D Y$ where $u \vdash_D Y$ and where $\varphi(u) \vdash_{A+(D \rightarrow D)} \varphi(Y)$ is the induction hypothesis, then $\varphi(u) \cup \{\Delta_{A+(D \rightarrow D)}\} \vdash_{A+(D \rightarrow D)} \varphi(Y)$; so $\varphi(u \cup \{\Delta_D\}) \vdash_{A+(D \rightarrow D)} \varphi(Y)$
- (7) If $\{(X, \Delta) | X \in w\} \vdash_D (W, \Delta)$ where $w \vdash_A W$, then $\{(X, \Delta') | X \in w\} \vdash_{A+(D \rightarrow D)} (W, \Delta')$; so $\varphi(\{(X, \Delta) | X \in w\}) \vdash_{A+(D \rightarrow D)} \varphi((W, \Delta))$
- (8) If $\{(\Delta, (u_0, v_0)), \dots, (\Delta, (u_{n-1}, v_{n-1}))\} \vdash_D (\Delta, (u', v'))$

where $\cup \{v_i | u' \vdash_D u_i\} \vdash_D v'$, then

$\{(\Delta', (u_0, v_0)), \dots, (\Delta', (u_{n-1}, v_{n-1}))\} \vdash_{A+(D \rightarrow D)} (\Delta', (u', v'))$; so
 $\varphi(\{(\Delta, (u_0, v_0)), \dots, (\Delta, (u_{n-1}, v_{n-1}))\}) \vdash_{A+(D \rightarrow D)} \varphi(\Delta, (u', v'))$.

Hence $D \approx A + (D \rightarrow D)$. □

Proposition 4.2.4

There exist approximable mappings $f: D \rightarrow A + (D \rightarrow D)$ and $f^{-1}: A + (D \rightarrow D) \rightarrow D$ such that

$$f \circ f^{-1} = I_{A+(D \rightarrow D)} \quad \text{and} \quad f^{-1} \circ f = I_D.$$

Proof:

Define $f: D \rightarrow A + (D \rightarrow D)$ and $f^{-1}: A + (D \rightarrow D) \rightarrow D$ by the formulae:

$$u.f.v \text{ iff } \varphi(u) \vdash_{A+(D \rightarrow D)} v \quad \text{and}$$

$$v.f^{-1}.u \text{ iff } v \vdash_{A+(D \rightarrow D)} \varphi(u),$$

for all $u \in \text{Con}_D$ and $v \in \text{Con}_{A+(D \rightarrow D)}$.

Then $f: D \rightarrow A + (D \rightarrow D)$ and $f^{-1}: A + (D \rightarrow D) \rightarrow D$ are clearly approximable mappings and we have:

(i) for all $y \in |A + (D \rightarrow D)|$

$$(f \circ f^{-1})(y) = \{Z \in D_{A+(D \rightarrow D)} | v.(f \circ f^{-1}).\{Z\} \text{ for some } v \subseteq y\}$$

$$= \{Z \in D_{A+(D \rightarrow D)} | v.f^{-1}.u \text{ and } u.f.\{Z\} \text{ for some}$$

$$u \in \text{Con}_D \text{ and } v \subseteq y\}$$

$$= \{Z \in D_{A+(D \rightarrow D)} | v \vdash_{A+(D \rightarrow D)} \varphi(u) \text{ and } \varphi(u) \vdash_{A+(D \rightarrow D)} \{Z\}$$

$$\text{for some } u \in \text{Con}_D \text{ and } v \subseteq y\}$$

$$= \{Z \in D_{A+(D \rightarrow D)} | v \vdash_{A+(D \rightarrow D)} \{Z\} \text{ for some } v \subseteq y\}$$

$$= \{Z \in D_{A+(D \rightarrow D)} | v.I_{A+(D \rightarrow D)}.\{Z\} \text{ for some } v \subseteq y\}$$

$$= I_{A+(D \rightarrow D)}(y).$$

Hence $f \circ f^{-1} = I_{A+(D \rightarrow D)}$. Also

(ii) for all $x \in |D|$

$$\begin{aligned}
 (f^{-1} \circ f)(x) &= \{Y \in D_D \mid u (f^{-1} \circ f) \{Y\} \text{ for some } u \subseteq x\} \\
 &= \{Y \in D_D \mid u f v \text{ and } v f^{-1} \{Y\} \text{ for some} \\
 &\quad v \in \text{Con}_{A+(D \rightarrow D)} \text{ and } u \subseteq x\} \\
 &= \{Y \in D_D \mid \varphi(u) \vdash_{A+(D \rightarrow D)} v \text{ and } v \vdash_{A+(D \rightarrow D)} \varphi(\{Y\}) \\
 &\quad \text{for some } v \in \text{Con}_{A+(D \rightarrow D)} \text{ and } u \subseteq x\} \\
 &= \{Y \in D_D \mid \varphi(u) \vdash_{A+(D \rightarrow D)} \varphi(\{Y\}) \text{ for some } u \subseteq x\} \\
 &= \{Y \in D_D \mid u \vdash_D \{Y\} \text{ for some } u \subseteq x\} \\
 &= \{Y \in D_D \mid u \vdash_D \{Y\} \text{ for some } u \subseteq x\} \\
 &= I_D(y) .
 \end{aligned}$$

Hence $f^{-1} \circ f = I_D$. □

Proposition 4.2.5

Let $A, B,$ and C be information systems. Then for all approximable mappings $h: A \rightarrow (B \rightarrow C)$ there is a unique approximable mapping $\text{uncurry } h: A \times B \rightarrow C$ such that

$$(\text{uncurry } h)(x, y) = h(x)(y) ,$$

for all $x \in |A|$ and $y \in |B|$.

Proof:

Let $h: A \rightarrow (B \rightarrow C)$ be an approximable mapping. Define

$\text{uncurry } h: A \times B \rightarrow C$ by the equation:

$$\text{uncurry } h = S_{A \times B, B, C} (h \circ \text{fst})(\text{snd}) ,$$

where $S_{A \times B, B, C}$ is given in 3.4.15. Then $\text{uncurry } h: A \times B \rightarrow C$ is

clearly an approximable mapping, and for all $x \in |A|, y \in |B|$ we have:

$$\begin{aligned}
 (\text{uncurry } h)(x,y) &= (S_{A \times B, B, C} (h \circ \text{fst})(\text{snd}))(x,y) \\
 &= (\text{curry } (((\text{curry comp})(\text{apply})) \circ \text{pair})(h \circ \text{fst})(\text{snd}))(x,y) \\
 &= (((\text{curry comp})(\text{apply})) \circ \text{pair})(h \circ \text{fst}, \text{snd})(x,y) \\
 &= ((\text{curry comp})(\text{apply})(\text{pair}(h \circ \text{fst}, \text{snd}))(x,y) \\
 &= ((\text{curry comp})(\text{apply})(\langle h \circ \text{fst}, \text{snd} \rangle))(x,y) \\
 &= (\text{comp}(\text{apply}, \langle h \circ \text{fst}, \text{snd} \rangle))(x,y) \\
 &= (\text{apply} \circ \langle h \circ \text{fst}, \text{snd} \rangle)(x,y) \\
 &= \text{apply}(\langle h \circ \text{fst}, \text{snd} \rangle(x,y)) \\
 &= \text{apply}((h \circ \text{fst})(x,y), \text{snd}(x,y)) \\
 &= \text{apply}(h(x), y) \\
 &= h(x)(y) .
 \end{aligned}$$

Finally $\text{uncurry } h: A \times B \rightarrow C$ is obviously unique. \square

Definition 4.2.6

Let D be the information system defined by 4.2.1, and let $f: D \rightarrow A + (D \rightarrow D)$, $f^{-1}: A + (D \rightarrow D) \rightarrow D$ be the approximable mappings defined in 4.2.4. Then $x \in |D|$ is said to be functional iff $x = f^{-1}(\text{inr}(z))$ for some $z \in |D \rightarrow D|$. Otherwise $x \in |D|$ is not functional.

If $x \in |D|$ is functional, let x^f be the element of $|D \rightarrow D|$ such that $x = f^{-1}(\text{inr}(x^f))$.

Proposition 4.2.7

There exists an approximable mapping $\text{apply}": D \rightarrow (D \rightarrow D)$ such that

$$\text{apply}''(x) = \begin{cases} x^f & \text{if } x \text{ is functional} \\ \text{const } \perp_D & \text{otherwise} \end{cases} ,$$

for all $x \in |D|$.

Proof:

Define $\text{apply}'' : D \rightarrow (D \rightarrow D)$ by the equation:

$$\text{apply}'' = [\text{const } (\text{const } \perp_D), (\text{curry apply})] \text{ of } .$$

Then $\text{apply}'' : D \rightarrow (D \rightarrow D)$ is clearly an approximable mapping, and for all $x \in |D|$ we have:

(i) if x is functional then

$$\begin{aligned} \text{apply}''(x) &= [\text{const } (\text{const } \perp_D), (\text{curry apply})](f(x)) \\ &= [\text{const } (\text{const } \perp_D), (\text{curry apply})](\text{inr}(x^f)) \\ &= ([\text{const } (\text{const } \perp_D), (\text{curry apply})] \circ \text{inr})(x^f) \\ &= (\text{curry apply})(x^f) \\ &= x^f \end{aligned}$$

(ii) if $f(x) = \text{inl}(z)$ for some $z \in |A|$ then

$$\begin{aligned} \text{apply}''(x) &= [\text{const } (\text{const } \perp_D), (\text{curry apply})](f(x)) \\ &= [\text{const } (\text{const } \perp_D), (\text{curry apply})](\text{inl}(z)) \\ &= ([\text{const } (\text{const } \perp_D), (\text{curry apply})] \circ \text{inl})(z) \\ &= \text{const } (\text{const } \perp_D)(z) \\ &= \text{const } \perp_D \end{aligned}$$

(iii) if $f(x) = \perp_{A+(D \rightarrow D)}$ then

$$\begin{aligned} \text{apply}''(x) &= [\text{const } (\text{const } \perp_D), (\text{curry apply})](f(x)) \\ &= [\text{const } (\text{const } \perp_D), (\text{curry apply})](\perp_{A+(D \rightarrow D)}) \\ &= \perp_{D \rightarrow D} \\ &= \text{const } \perp_D \end{aligned}$$

□

Proposition 4.2.8

There exists an approximable mapping $\text{apply}' = \cdot : D \times D \rightarrow D$ such that

$$\text{apply}^f(x, y) = x \cdot y = \begin{cases} \text{apply}(x^f, y) = x^f(y) & \text{if } x \text{ is functional} \\ \perp_D & \text{otherwise} \end{cases}$$

for all $x, y \in |D|$.

Proof:

Define $\text{apply}' = \cdot : D \times D \rightarrow D$ by the equation:

$$\text{apply}' = \text{uncurry apply}''$$

Then $\text{apply}': D \times D \rightarrow D$ is clearly an approximable mapping, and for all $x, y \in |D|$ we have:

(i) if x is functional

$$\begin{aligned} \text{apply}'(x, y) &= (\text{uncurry apply}'')(x, y) \\ &= \text{apply}''(x)(y) \\ &= x^f(y) \end{aligned}$$

(ii) if $f(x) = \text{inl}(z)$ for some $z \in |A|$ then

$$\begin{aligned} \text{apply}'(x, y) &= (\text{uncurry apply}'')(x, y) \\ &= \text{apply}''(x)(y) \\ &= (\text{const } \perp_D)(y) \\ &= \perp_D \end{aligned}$$

(iii) if $f(x) = \perp_{A+(D \rightarrow D)}$ then

$$\begin{aligned} \text{apply}'(x, y) &= (\text{uncurry apply}'')(x, y) \\ &= \text{apply}''(x)(y) \\ &= (\text{const } \perp_D)(y) \\ &= \perp_D \end{aligned}$$

Note that we omit parentheses by association to the left as in

λ -calculus.

Proposition 4.2.9

There exist $K, S \in |D|^D$ so that $\langle |D|, \cdot, K, S \rangle$ is a combinatory algebra.

Proof:

Define

$$K = f^{-1}(\text{inr}((f^{-1} \circ \text{inr}) \circ \text{const}_{D,D})).$$

Then clearly $K \in |D|$ and K is functional, and for all $x, y \in |D|$ we have:

$$\begin{aligned} K \cdot x \cdot y &= f^{-1}(\text{inr}((f^{-1} \circ \text{inr}) \circ \text{const}_{D,D})) \cdot x \cdot y \\ &= ((f^{-1} \circ \text{inr}) \circ \text{const}_{D,D})(x) \cdot y \\ &= f^{-1}(\text{inr}(\text{const}_{D,D}(x))) \cdot y \\ &= \text{const}_{D,D}(x)(y) \\ &= x. \end{aligned}$$

Now let $h_1: D \times D \rightarrow (D \rightarrow D \times D)$ be the approximable mapping defined by the equation:

$$h_1 = S^*((\text{curry pair}) \circ (\text{apply}'' \circ \text{fst}))(\text{apply}'' \circ \text{snd}),$$

where $S^* = S_{D \times D, D \rightarrow D, D \rightarrow D \times D}$, and let $h_2: D \rightarrow (D \rightarrow D)$ be the approximable mapping defined by the equation:

$$h_2 = \text{curry}((f^{-1} \circ \text{inr}) \circ (((\text{curry comp})(\text{apply}')) \circ h_1)).$$

Then define

$$S = f^{-1}(\text{inr}((f^{-1} \circ \text{inr}) \circ h_2)).$$

Then clearly $S \in |D|$ and S is functional, and for all $x, y, z \in |D|$ we have:

$$\begin{aligned} S \cdot x \cdot y \cdot z &= f^{-1}(\text{inr}((f^{-1} \circ \text{inr}) \circ h_2)) \cdot x \cdot y \cdot z \\ &= f^{-1}(\text{inr}(h_2(x))) \cdot y \cdot z \\ &= h_2(x)(y) \cdot z \end{aligned}$$

$$\begin{aligned}
 &= (\text{curry } ((f^{-1} \circ \text{inr}) \circ (((\text{curry comp})(\text{apply}')) \circ h_1))) (x)(y) \cdot z \\
 &= ((f^{-1} \circ \text{inr}) \circ (((\text{curry comp})(\text{apply}')) \circ h_1)) (x, y) \cdot z \\
 &= (f^{-1} \circ \text{inr}) (((\text{curry comp})(\text{apply}')) \circ h_1) (x, y) \cdot z \\
 &= f^{-1} (\text{inr} (((\text{curry comp})(\text{apply}')) (h_1(x, y)))) \cdot z \\
 &= (((\text{curry comp})(\text{apply}')) (h_1(x, y))) (z) \\
 &= (\text{comp}(\text{apply}', h_1(x, y))) (z) \\
 &= (\text{apply}' \circ h_1(x, y)) (z) \\
 &= \text{apply}' (h_1(x, y) (z)) \\
 &= \text{apply}' (S^* ((\text{curry pair}) \circ (\text{apply}'' \circ \text{fst})) (\text{apply}'' \circ \text{snd}) (x, y) (z)) \\
 &= \text{apply}' (((\text{curry pair}) \circ (\text{apply}'' \circ \text{fst})) (x, y)) ((\text{apply}'' \circ \text{snd}) (x, y)) (z) \\
 &= \text{apply}' ((\text{curry pair}) (\text{apply}''(x)) (\text{apply}''(y)) (z)) \\
 &= \text{apply}' (\langle \text{apply}''(x), \text{apply}''(y) \rangle (z)) \\
 &= \text{apply}' (\text{apply}''(x)(z), \text{apply}''(y)(z)) \\
 &= \text{apply}' (x \cdot z, y \cdot z) \\
 &= x \cdot z \cdot (y \cdot z) .
 \end{aligned}$$

Thus, according to 1.2.3, $\langle |D|, \cdot, K, S \rangle$ is a combinatory algebra. \square

Proposition 4.2.10

There exists $\ell \in |D|$ such that

(i) $\ell \cdot x \cdot y = x \cdot y$ and

(ii) if $x \cdot z = y \cdot z$ for all $z \in |D|$ then $\ell \cdot x = \ell \cdot y$,

for all $x, y \in |D|$.

Proof:

Define

$$\ell = S \cdot (S \cdot (K \cdot S) \cdot K) \cdot (K \cdot (S \cdot K \cdot K)) .$$

Then $\ell \in |D|$ and ℓ is functional. Also for each $d \in |D|$

$$\begin{aligned}
 \ell \cdot d &= S \cdot (S \cdot (K \cdot S) \cdot K) \cdot (K \cdot (S \cdot K \cdot K)) \cdot d \\
 &= (S \cdot (K \cdot S) \cdot K) \cdot d \cdot (K \cdot (S \cdot K \cdot K) \cdot d) \\
 &= S \cdot (K \cdot S) \cdot K \cdot d \cdot (S \cdot K \cdot K) \\
 &= K \cdot S \cdot d \cdot (K \cdot d) \cdot (S \cdot K \cdot K) \\
 &= S \cdot (K \cdot d) \cdot (S \cdot K \cdot K)
 \end{aligned}$$

is functional because

$S \cdot x \cdot y = h_2(x)(y) = f^{-1}(\text{inr}(((\text{curry comp})(\text{apply}'))(h_1(x,y))))$ is functional for all $x, y \in |D|$. Then for all $x, y \in |D|$ we have:

$$\begin{aligned}
 (i) \quad \ell \cdot x \cdot y &= S \cdot (K \cdot x) \cdot (S \cdot K \cdot K) \cdot y \\
 &= K \cdot x \cdot y \cdot (S \cdot K \cdot K \cdot y) \\
 &= x \cdot (S \cdot K \cdot K \cdot y) \\
 &= x \cdot (K \cdot y \cdot (K \cdot y)) \\
 &= x \cdot y
 \end{aligned}$$

(ii) suppose $x \cdot d = y \cdot d$ for all $d \in |D|$; since $\ell \cdot x$, $\ell \cdot y$ are functional we have:

$$(\ell \cdot x)^f(d) = \ell \cdot x \cdot d = x \cdot d = y \cdot d = \ell \cdot y \cdot d = (\ell \cdot y)^f(d),$$

for all $d \in |D|$; thus

$$(\ell \cdot x)^f = (\ell \cdot y)^f \quad \text{by 3.3.4(ii)}$$

so $\ell \cdot x = \ell \cdot y$. □

Proposition 4.2.11

$\langle |D|, \cdot, K, S; \ell \rangle$ is a Scott-Meyer λ -model.

Proof:

By 4.2.9 and 4.2.10 it is obvious that the quintuple $\langle |D|, \cdot, K, S, \ell \rangle$ satisfies definition 1.2.6. □

Now, according to theorem 1.2.8, $[[\quad]]$ can be defined in $\langle |D|, \cdot, K, S, \ell \rangle$ so that $\langle |D|, \cdot, [[\quad]]$ is a λ -model in the

sense of 1.2.4.

Proposition 4.2.12

$\langle |D|, \cdot, [[\] \rangle$ is not an extensional λ -model.

Proof:

Let $x \in |D|$ such that $f(x) = \text{inl}(z)$ for some $z \in |A|$. Then

$$x \cdot y = \perp_D = \perp_D \cdot y,$$

for all $y \in |D|$. But $x \neq \perp_D$. Thus $\langle |D|, \cdot, [[\] \rangle$ is not an extensional λ -model. \square

The domain-theoretic model fails to be extensional because $D \rightarrow D$ is not isomorphic to D' . If definition 4.2.1 were modified in the obvious way to delete all references to A , then $D \rightarrow D$ would be isomorphic to D and hence D would be an extensional λ -model by 3.3.4(11) (because every element of D would be "functional").

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