CALCULATION OF BARYON MASSES USING THE DYNAMICAL GROUP SU (n+1)

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ABSTRACT

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Masses of the charmed and uncharmed spin $\frac{1}{2}^+$ baryon isoplets are calculated using the dynamical group SU(n + 1). The predicted spectrum agrees with experiment to within 1.7%. Sum rules are obtained and the masses of the 20_m baryons are predicted.

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1: INTRODUCTION

The investigation of the elementary constituents of matter has always been a major preoccupation of physicists. Through the study of cosmic radiation and, later, with the aid of high energy particle accelerators, experimenters discovered a rich spectrum of particle states and resonances. The theory of unitary symmetry and the subsequent development of the quark model contributed significantly to our understanding of this complex spectrum.

Currently, virtually all successful theories on the nature and behavior of elementary particles include unitary symmetry as the underlying principle. The purpose of this thesis is to employ the method of dynamical groups based on unitary symmetry to calculate baryon masses. The particular calculations employed in this thesis are usually referred to as flavour-breaking calculations, as the symmetry of the multiplet is broken by the quark flavour rather than by some external mechanism (e.g. angular momentum).

In the procedure outlined in this thesis, the internal symmetry group of a baryon multiplet is imbedded into a larger group corresponding to the interaction Hamiltonian H'. This breaks the symmetry, resulting in non-degenerate masses for the members of that multiplet.

It will be demonstrated that not only does the dynamical group method predict the mass spectrum of the baryons, but also provides an intuitive picture of the quark model.

The theory of unitary symmetry and the quark model is presented in sections 2 and 3. The dynamical group method is developed in section 4. Section 5 contains a brief note on the problem of diagonalizing the mass matrix. The calculation of the baryon masses is discussed in sections 6 and 7. Final conclusions are presented in section 8.

2: CLASSIFICATION OF HADRON STATES

The elementary particles are classified on the basis of two criteria:

1) the spin statistics they obey and

2) the interactions which affect them.

Those particles which obey Fermi-Dirac statistics are called fermions and those particles which follow Bose-Einstein statistics are called bosons.

Particles which are affected by the strong nuclear interactions are called hadrons. Strongly interacting fermions are called baryons and strongly interacting bosons are called mesons.

Hadron states are also characterized by a set of quantum numbers: SPIN "s", CHARM "C", BARYON NUMBER "B", HYPERCHARGE "Y", ISOSPIN "I" and the third component of ISOSPIN "I_3".

Baryons are assigned baryon number B = 1 and antibaryons B = -1. This reflects the fact that in a closed system, the total number of baryons is conserved. Mesons and their corresponding anti-particles are assigned B = 0 since in any reaction mesons may be produced in any number.

For baryons ($B = 1$, $s = 1/2, 3/2 \ldots$) the following symbols are used:

N indicates a state with $I = 1/2$ and $Y = 1$

Σ indicates a state with $I = 1$ and $Y = 0$

Λ indicates a state with $I = 0$ and $Y = 0$

Ξ indicates a state with $I = 1/2$ and $Y = -1$

Ω indicates a state with $I = 0$ and $Y = -2$

In this thesis we shall be concerned with baryons exclusively. A list of the baryon states is given in Table 1.

3: UNITARY SYMMETRY

Isospin

Since the discovery that the nucleus of the atom appears to be composed of two constituents, the neutron and proton, physicists have attempted to understand how these particles interact with each other. Noting that the nuclear force seemed to be charge independent, Heisenberg[1] suggested that the neutron and proton are actually different charge states of the same particle state. By introducing a further internal degree of freedom, isospin, he theorized that the third component of isospin corresponds to the charge states (in analogy to ordinary spin).

Although the symmetry is exact for the strong interaction, it is broken by the electromagnetic interaction resulting in the slightly larger mass of the neutron.

Mathematically, the Pauli matrices τ_i are used to represent the isospin states:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (1)$$

As the list of elementary particles grew, new additions were found to fit the isospin scheme. The new particles appeared in "families" (called isoplets), differing only in charge and slightly in mass (see Table 1).

It was later recognized that the mathematics used to explain the properties of the particle states through isospin was the same as that used in the description of Lie groups. The Pauli matrices τ_i are generators of the Lie group SU(2), the group of 2 X 2 unitary, unimodular matrices.

It was this knowledge that led Gell-Mann[2,3] and others to postulate that the group-theoretical scheme could be extended further than isospin.

SU(3)

Under SU(2), families of particles (isoplets) are considered degenerate under isospin. Similarly, Gell-Mann proposed that larger groups of particles (families of isoplets) were degenerate under what he termed unitary spin SU(3). This extension to SU(3) requires the introduction of a new quantum number, hypercharge = Y. Y is defined by the Gell-Mann[4] - Nishijima[5] relation

$$Y = 2Q - I_3 \qquad (2a)$$

The hypercharge Y can also be defined by the relation

$$Y = S + B \qquad (2b)$$

The quantum number "S", strangeness, is assigned phenomenalogically from the selection rules for the weak interaction.

The SU(3) symmetry, however, is more badly broken than that of SU(2). Isoplet mass splitting is of the order of a few MeV or about 1% of the particle mass. Under SU(3), however, the splitting is of the order of 100 MeV or more, a very appreciable mass difference.

The generators (listed in Table 2) obey the Lie algebra

$$\left[\lambda_k, \lambda_n\right] = 2i \sum_v F_{knv} \lambda_v \qquad (3)$$

with the structure constants F_{knv} listed in Table 3. We associate the generators of SU(3) with the quantum numbers of the particle states as follows:

$$I_3 = 1/2\,\lambda_3 \qquad (4)$$

$$Y = \sqrt{1/3}\,\lambda_8 \qquad (5)$$

A major consequence of unitary symmetry was the development of the quark model which will be explained in the next section.

The Quark Model

As the elementary particle list grew, it became apparent that not all of these particles should be considered fundamental. The first model to include the strange particle states (that is, particles with non-zero strangeness) was that of Sakata[6], an extension of an earlier model of Fermi and Yang[7].

The Sakata model proposed a fundamental triplet composed of the neutron, proton, and lambda, with all subsequent particles constructed from the fundamental triplet.

Mesons would be formed by a triplet anti-triplet pair $(q,\bar{q})$ to conserve baryon number. Baryons would be formed from two triplets and an anti-triplet $(q,q,\bar{q})$, again to conserve baryon number.

Although the Sakata model provides a good description of mesons, it does not work in the case of the baryons. Particle states are predicted that are not observed by experiment (e.g. states with $S = +1$).

Despite this shortcoming, the Sakata model is still attractive due to the economy of the basic triplet. It is this desirable simplicity which led Gell-Mann[8] and Zwieg[9] to propose the quark model.

The quark model begins with a basic triplet of quarks (u,d,s) with each type of quark (u,d,s) referred to as flavours. Each has baryon number $B = 1/3$, and a non-integral charge in accordance with equation 2.

The quantum numbers of the u,d,s quarks are listed in Table 4. In the unitary symmetry scheme the quarks correspond to the fundamental triplet (3) representation of SU(3). Anti-quarks correspond to the $\bar{3}$ representation.

Mesons are formed from quark-anti-quark pairs. Taking the Kronecker product:

$$3 \otimes \bar{3} = 1 \oplus 8 \qquad (6)$$

yielding the SU(3) octet structure for the Mesons (see Figure 1).

Baryons are formed by 3 quarks. Again taking the Kronecker product:

$$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10 \qquad (7)$$

yielding only the observed octet and decuplet (see Figures 2, 3). The quark wave functions of the baryons under examination are given in Table 5.

SU(4) and Charm

With the discovery of the new heavy meson and the J/ψ in 1974, it became necessary to further enlarge the scheme of unitary symmetry. An enlargement to SU(4) with a new quantum number, charm, C, had already been proposed by several authors[10]. One theoretical argument required that the number of quarks equal the number of leptons. Since there were four leptons (e, ν_e, μ, ν_μ), there should be a fourth quark.

The generators of SU(4) are listed in Table 6. Again the generators obey the Lie algebra:

$$[\lambda_k, \lambda_n] = 2i \sum_v F_{knv} \lambda_v \tag{8}$$

with the structure constant F_{knv} listed in Table 7. We can equate the quantum number C with the generators of SU(4):

$$C = 1/4 \; (\lambda_0 - \sqrt{6}\,\lambda_{15}) \tag{9}$$

We re-define the Gell-Mann Nishijima formula (equation 2a) and the quantum number for the hypercharge as follows:

$$\begin{aligned} Y &= S + B - C \\ Q &= I_3 + 1/2\,(B + S + C) \end{aligned} \tag{10}$$

Therefore, to classify the baryons and mesons in SU(4), four quantum nunbers are used: charm, hypercharge, total isospin, and the third component of isospin. The same symbols ($N, \Lambda, \Sigma, \ldots$) are used as in SU(3), indicating isospin and hypercharge, with a superscript indicating the charge state and a subscript indicating charm. For example, the symbol Λ_c^+ signifies a lambda particle ($I = 0$, $Y = 0$) with $C = 1$ and $Q = 1$.

We now have four quarks: the u,d,s quarks as in SU(3) and a new quark c corresponding to charm (see Table 4).

Again, we take the Kronecker product for the baryons (see appendix 1):

$$4 \otimes 4 \otimes 4 = 20_s \oplus 20_m \oplus 20_m \oplus \bar{4} \qquad (11)$$

Thus in terms of the quark model we expect the baryons to occur in multiplets 20_m (mixed symmetry) and 20_s (totally symmetric) and anti- 4 plet.

The 20_m-plet decomposes to SU(3) multiplets

$$20_m \supset 8 \oplus 6 \oplus \bar{3} \oplus 3 \qquad (12)$$

The 8 of the 20_m-plet is simply the basic octet, with $C = 0$. The 6 and $\bar{3}$ have $C = 1$, and the 3 has $C = 2$.

The quark content of the 20_m states are listed in Table 5, and the states are listed diagrammatically in Figure 4.

Again the symmetry under SU(4) is not exact. Indeed, it is more badly broken than that of SU(3), since the mass of the lightest charmed baryon is more than twice that of the uncharmed lambda.

4: THE FINITE DIMENSIONAL REPRESENTATION AND THE MASS OPERATOR

Gel'fand and Graev[11] give a method of determining the irreducible representation of the Lie algebra L^n of a finite-dimensional group G^n, $G_n \supset G^{n-1} \ldots G^1$. A basis for L^n is the set of n-dimensional matrices e_{ik} with elements $(e_{ik})_{mn} = \delta_{im}\ \delta_{kn}$.

The algebra L^n is defined by the commutation relations

$$\begin{aligned} \left[e_{ik}, e_{kp}\right] &= e_{ip} && \text{for } i \neq p \\ \left[e_{ik}, e_{ki}\right] &= e_{ii} - e_{kk} && \\ \left[e_{ij}, e_{kp}\right] &= 0 && \text{for } i \neq p,\ j \neq k \end{aligned} \tag{13}$$

Specifying a representation of the algebra L^n is equivalent to giving a set of operators E_{kl} satisfying the commutation relations

$$\begin{aligned} \left[E_{ik}, E_{kp}\right] &= E_{ip} && \text{for } i \neq p \\ \left[E_{ik}, E_{ki}\right] &= E_{ii} - E_{kk} && \\ \left[E_{ij}, E_{kp}\right] &= 0 && \text{for } i \neq p,\ j \neq k \end{aligned} \tag{14}$$

If only the operators E_{kk}, $E_{k,k-1}$, $E_{k-1,k}$ are considered the following commutation relations are obtained:

$$\begin{aligned} &\left[E_{ii}, E_{kk}\right] = 0 \\ &\left[E_{ii}, E_{k,k-1}\right] = \left[E_{ii}, E_{k-1,k}\right] = 0 \text{ for } i \quad k-1 \\ &\left[E_{ii}, E_{i,i-1}\right] = \left[E_{i,i-1}, E_{i-1,i-1}\right] = E_{i,i-1} \\ &\left[E_{i-1,i-1}, E_{i-1,i}\right] = \left[E_{i-1,i}, E_{ii}\right] = E_{i-1,i} \\ &\left[E_{i,i-1}, E_{i-1,i}\right] = E_{ii} - E_{i-1,i-1} \\ &\left[E_{i,i-1}, E_{k-1,k}\right] = 0 \qquad \text{for } k \neq i \\ &\left[E_{i,i-1}, E_{k,k-1}\right] = \left[E_{i-1,i}, E_{k-1,k}\right] = 0 \quad \text{for } k \neq i \pm 1 \end{aligned} \tag{15}$$

If the operators E_{kk}, $E_{k,k-1}$, $E_{k-1,k}$ are given any operator E_{kj} may be constructed by use of the relations:

$$\begin{aligned} E_{k,k-p} &= \left[E_{k,k-1}, E_{k-1,k-p}\right] \\ E_{k-p,k-1} &= \left[E_{k-p,k-1}, E_{k-1,k}\right] \end{aligned} \qquad \text{where } j = k-p \qquad (16)$$

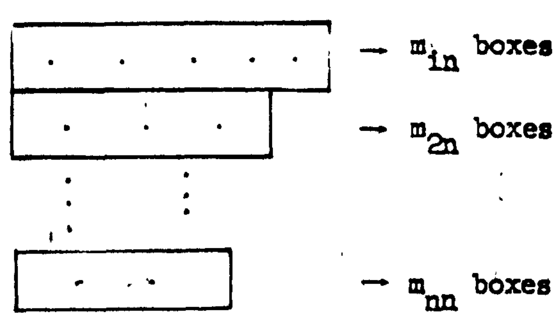
Now every finite-dimensional irreducible representation of the Lie algebra L^n is specified by a set of n integers $m_{1n} > m_{in} \geq m_{nn}$. As an orthonormal basis (a set of orthogonal one-dimensional subspaces) we use all possible triangular arrays referred to as the Gel'fand - Cetlin basis:

$$\xi(m) = \begin{pmatrix} m_{1n} \quad m_{2m} \cdots\cdots\cdots\cdots\cdots \quad m_{nn} \\ m_{1,n-1} \; m_{2,n-1} \cdots\cdots \; m_{n-1,n-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ m_{11} \end{pmatrix} \qquad (17)$$

where the m_{ij} are integers satisfying the triangular inequality

$$m_{ij} \geq m_{i,j-1} \geq m_{i+1,j} \qquad (18)$$

These arrays correspond to the Young tableau for the algebra L^n of the group U(n)



(19)

In these arrays the top row is fixed, labeling the representation. For example, for SU(3), SU(4) $\supset$SU(3) and n = 4 where n is the dimension of the representation. Therefore the basis element is

$$\xi(m) = \begin{pmatrix} m_{14} & & m_{24} & & m_{34} & & m_{44} \\ & m_{13} & & m_{23} & & m_{33} & \\ & & m_{12} & & m_{22} & & \\ & & & m_{11} & & & \end{pmatrix} \tag{20}$$

and m_{44} is set to zero to preserve unimodularity.

Then the top row corresponds to Young's tableau for SU(4); the next row to Young's tableau U(3), a subgroup of SU(4); the third to that of U(2) $\subset$ U(3); and the bottom row to that of U(1) $\subset$ U(2).

Because every Lie algebra L^n is specified by the set of parameters $m_{1n} > m_{in} > m_{nn}$ i=1,n then a particular SU(3) multiplet is specified by the elements m_{13}, m_{23}, m_{33}.

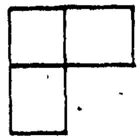
For an SU(3) octet representation the basis element would be:

$$\xi(m) = \begin{pmatrix} m_{14} & & m_{24} & & m_{34} & & 0 \\ & m_{13} & & m_{13}-1 & & m_{13}-2 & \\ & & m_{12} & & m_{22} & & \\ & & & m_{11} & & & \end{pmatrix} \tag{21}$$

with the two bottom rows accepting all admissible values by the triangular inequality (equation 18).

The lowest possible value for m_{13} is $m_{13} = 2$, as $m_{13}-2 \geq 0$ (equation 18).

Therefore the Young's tableau would be:



→ 2 boxes	m_{13}	$= 2$	
→ 1 box	$m_{13}-1$	$= 1$	(22)
0 boxes	$m_{13}-2$	$= 0$	

To recapitulate, we have a large scale symmetry scheme SU(n)

$$SU(n) \supset U(n-1) \ldots U(4) \supset U(3) \supset U(2) \supset U(1) \qquad (23)$$

Therefore:

U(4) multiplets are specified by fixing the fourth row

U(3) multiplets are specified by fixing the third row

U(2) multiplets (isoplets) are specified by fixing the second row

U(1) the particle states, are specified by fixing the first row, the parameter m_{11}.

The quantum numbers of a particular state of SU(4) can be expressed in terms of the elements of the array m_{ij} as follows:

$$\begin{aligned}
C &= m_{14} + m_{24} + m_{34} - (m_{13} + m_{23} + m_{33}) \\
Y &= m_{12} + m_{22} - 2\ (m_{13} + m_{23} + m_{33}) \quad /3 \\
I &= (m_{12} - m_{22}) \quad /2 \\
I_3 &= m_{11} - (m_{12} + m_{22}) \quad /2 \\
Q &= I_3 + Y/2 + 2/3C
\end{aligned} \qquad (24)$$

The operators E_{kk}, $E_{k,k-1}$, $E_{k-1,k}$ are defined as follows:[11]

1) $E_{kk}\ \xi(m) = (r_k - r_{k-1})\ \xi(m)$ (25)

where $\xi(m)$ is a Gel'fand - Cetlin array and $r_k = m_{1k} + \ldots m_{kk}$ for $k = 1,n$; $r_0 = 0$

2) $E_{k,k-1}\ \xi(m) = a^{1}_{k-1}\ \xi(m^{1}_{k-1}) + \ldots a^{k-1}_{k-1}\ \xi(m^{k-1}_{k-1})$ (26)

where $\xi(m^{i}_{k-1})$ is identical to the array $\xi(m)$ except that the the element $m_{i,k-1}$ is replaced by $m_{i,k-1}-1$ and

$$a^{j}_{k-1} = \left(\frac{\prod_{i=1}^{K} (m_{ik}-m_{j,k-1}-i+j+1) \prod_{i=1}^{K} (m_{i,k-2}-m_{j,k-1}-i+j)}{\prod_{i\neq j}^{k-1} (m_{i,k-1}-m_{j,k-1}-i-j+1)\ (m_{i,k-1}-m_{j,k-1}-i+j)} \right)^{-1/2} \quad (27)$$

3) $E_{k-1,k}\ \xi(m) = b^{1}_{k-1}\ \xi(m^{1}_{k-1}) \ldots b^{k-1}_{k-1}\xi(m^{k-1}_{k-1})$ (28)

where $\xi(m^{i}_{k-1})$ is the array $\xi(m)$ but with the element $m_{i,k-1}$ replaced by $m_{i,k-1}+1$ and

$$b^{j}_{k-1} = \left(\frac{\prod_{i=1}^{K} (m_{ik}-m_{j,k-1}-i+j) \prod_{i=1}^{K-2} (m_{i,k-2}-m_{j,k-1}-i+j-1)}{\prod_{i\neq j}^{K-1} (m_{i,k-1}-m_{j,k-1}-i+j)\ (m_{i,k-1}-m_{j,k-1}-i+j-1)} \right)^{1/2} \quad (29)$$

The Mass Operator

Consider a Hamiltonian H of the form[12,13]

$$H = H_{o}^{part} + H_{o}^{field} + H' \quad (30)$$

where H_{o}^{part} is the free particle hamiltonian, an operator whose eigenfunctions are the particle states before and after the emission of some field quanta. Similarly, the free field hamiltonian H_{o}^{field} is an operator corresponding to the quantized field after the emission of some field quanta and H' is the interaction hamiltonian.

Following Kalman[12,13], we suggest the form of H' be

$$H' = \sum_{i \leq j} a_{ij} M_{ij} \tag{31}$$

where a_{ij} are creation operators for mesons and M_{ij} are meson coupling matrices.

For SU(n) and the quark model the operator M_{ij} may be represented as a product of the n-plet and the anti n-plet quark representations, since the mesonic states correspond to a quark anti-quark pair.

Let q_k and $\bar{q}_k$, k = 1,....n be elements of that representation and A_{ij} i,j, = 1,....n be the generators of SU(n). Then:

$$\left.\begin{aligned}
&[A_{ij}, A_{km}] = \delta_{im} A_{kj} - \delta_{jk} A_{im} && i,j,k,m = 1,\ldots.n \\
&\sum_{i=1}^{n} A_{ii} = 0 && i = 1,\ldots.n \\
&[A_{ij}, q_k] = \delta_{ik} q_j && i,j,k = 1,\ldots.n \\
&[A_{ij}, \bar{q}_k] = -\delta_{jk} \bar{q}_i && i,j,k = 1,\ldots.n
\end{aligned}\right\} \tag{32}$$

To close the algebra the following dynamical postulate can be made:

$$[q_i, \bar{q}_j] = \theta(\delta_{ij} A_{n+1,n+1} - A_{ji}) \quad i,j = 1,\ldots.n \tag{33}$$

where $\theta = +1$ corresponds to the Lie algebra of SU(n+1),

$\theta = -1$ corresponds to the Lie algebra of SU(1,n), and

$\theta = 0$ corresponds to the Lie algebra of $T_K \otimes SU(n)$

(where K = 2n + 1)

and $A_{n+1,n+1}$ is a diagonal generator needed to complete the algebra of SU(n+1), SU(1,n).

The group $T_k \otimes SU(n)$ cannot be considered as its use will not predict elementary particle transitions.[12]

For our purpose (baryon mass spectroscopy) either SU(1,n) or SU(n+1) will result in equivalent results.[13] In this thesis we shall consider SU(n+1) only.

In accordance with the hypothesis of unitary symmetry and equations 31 - 33 Kalman[13] suggests a mass formula:

$$M(m) = C_0 + \langle \xi(m) | \sum q_i \bar{q}_j c_i c_j | \xi(m) \rangle \tag{34}$$

where C_0 is the degenerate eigenvalue of the free hamiltonian and the term

$$\langle \xi(m) | \quad \sum q_i \bar{q}_j c_i c_j \quad | \xi(m) \rangle$$

is due to the interaction hamiltonian H', representing the symmetry breaking mechanism.

Since the internal symmetry groups for the baryons is SU(n), then the dynamical group will be SU(n+1). The particle states $\xi(m)$ are represented by the Gel'fand - Cetlin basis for SU(n+1). The operators q_i, $\bar{q}_j$ are the operators of the Gel'fand Cetlin basis:

$$\begin{aligned} q_i &= E_{i,n+1} & i &= 1,\ldots.n \\ \bar{q}_j &= E_{n+1,j} & j &= 1,\ldots.n \end{aligned} \tag{35}$$

Specifically, with SU(3) as the internal symmetry group

$$E_{4,3} \rightarrow \bar{s} \qquad E_{3,4} \rightarrow s$$
$$E_{4,2} \rightarrow \bar{u} \qquad E_{2,4} \rightarrow u \qquad (36a)$$
$$E_{4,1} \rightarrow \bar{d} \qquad E_{1,4} \rightarrow d$$

and with SU(4) as the internal symmetry group

$$E_{5,4} \rightarrow \bar{c} \qquad E_{4,5} \rightarrow c$$
$$E_{5,3} \rightarrow \bar{s} \qquad E_{3,5} \rightarrow s$$
$$E_{5,2} \rightarrow \bar{u} \qquad E_{2,5} \rightarrow u \qquad (36b)$$
$$E_{5,1} \rightarrow \bar{d} \qquad E_{1,5} \rightarrow d$$

The constants $C_i C_j$ are associated with each quark, anti-quark operator q_i, $\bar{q}_j$.

For SU(4) the first order perturbation mass formula corresponding to equation 34 is

$$M(m) = C_0 + \tilde{C}_4 \langle \xi(m) | \; c\,\bar{c} \; | \xi(m) \rangle + \tilde{C}_3 \langle \xi(m) | \; s\,\bar{s} \; | \xi(m) \rangle + \tilde{C}_2 \langle \xi(m) | \; u\,\bar{u} \; | \xi(m) \rangle + \tilde{C}_1 \langle \xi(m) | \; d\,\bar{d} \; | \xi(m) \rangle \qquad (37)$$

and for SU(3) the corresponding formula is

$$M(m) = C_0 + \tilde{C}_3 \langle \xi(m) | \; s\,\bar{s} \; | \xi(m) \rangle + \tilde{C}_2 \langle \xi(m) | \; u\,\bar{u} \; | \xi(m) \rangle + \tilde{C}_1 \langle \xi(m) | \; d\,\bar{d} \; | \xi(m) \rangle \qquad (38)$$

The success which Kalman[14-20] experienced using these mass formulae motivates us to consider the full mass matrix (equation 34) with element

$$M_{ij} = \langle \xi_i(m) | \; C_0 + \sum_{k,p}^{n} C_k C_p \; q_k \bar{q}_p \; | \xi_j(m) \rangle \qquad (39)$$

where the $\xi_i(m)$, $\xi_j(m)$ are SU(n+1) Gel'fand arrays representing the states of a given SU(n) multiplet, and C_0, C_k, C_p, q_k, $\overline{q}_p$ are as previously defined. It is clear that the mass formulae equations 37 and 38 are simply the diagonal elements of this matrix.

5: MATRIX DIAGONALIZATION

The mass matrix element is of the form

$$M_{ij} = \langle \xi_i | \; C_0 + \sum_{k,p=1}^{n} C_k C_p \, q_k \bar{q}_p \; | \xi_j \rangle \tag{40}$$

The problem becomes one of fitting values for the constants C_0, C_k, C_p that will yield eigenvalues corresponding to known particle masses, thereby making good theoretical predictions for the as yet unobserved (higher-energy) states.

The eigenvalue equation leads us to a matrix $M' = M - \lambda I$ with elements

$$M'_{ij} = \langle \xi_i | \, C_0 + \sum_{k,p=1}^{n} C_k C_p \, q_k \bar{q}_p - \lambda \quad | \xi_j \rangle \tag{41}$$

We note C_0 occurs only along the matrix diagonal as

$$\langle \xi_i | \quad C_0 \, | \xi_j \rangle = C_0 \, \delta_{ij} \tag{42}$$

the basis, ξ_i, ξ_j being orthonormal.

Therefore we re-write equation 41 using

$$\lambda' = \lambda - C_0 \tag{43}$$

and the matrix element becomes

$$M'_{ij} = \langle \xi_i | - \lambda' \quad + \sum_{k,p=1}^{n} C_k C_p \, q_k \bar{q}_p \; | \xi_j \rangle \tag{44}$$

Now we factor one of the constants C_v out of the matrix by

1) setting $C_i = X_i C_v$
   with $X_i = 1$ if $i = v$ (45a)

2) we write $\lambda'' = \lambda' / C_v^2 = (\lambda - C_0) / C_v^2$ (45b)

Therefore

$$M'_{ij} = C_v^2 M''_{ij} = C_v^2 \langle \xi_i | -\lambda'' + \sum_{k,p=1}^{n} X_k X_p q_k \bar{q}_p | \xi_j \rangle \quad (46)$$

The matrix $\| M''_{ij} \|$ explicitly contains two less constants (C_0 , C_v) than the matrix $\| M'_{ij} \|$. This facilitates fitting the eigenvalues to the baryon mass spectrum.

The matrix may be further reduced by setting the constants associated with the up and down quarks (C_2 , C_1) equal. This sets the action of the up and down quark operators equal and results in the elimination of isoplet mass splitting.[15,18] We can then reduce the dimension of the mass matrix to that of the number of isoplets in the multiplet. For example, in the case of the baryon octet, the matrix is reduced from an 8 X 8 array (the number of particle states) to a 4 X 4 array (the number of isoplets).

The eigenvalues r_i resulting from these "reduced" mass matrices are associated with the mass eigenvalues as follows:

$$\begin{aligned} \lambda''_i &= (\lambda_i - C_0) / C_v^2 = r_i \\ \lambda_i &= C_0 + C_v^2 r_i \end{aligned} \quad (47)$$

6: CALCULATION OF UNCHARMED BARYON MASS

As a first example of the use of the Gel'fand basis and the mass operator described in section 4, we will examine the octet of the uncharmed baryons. This case was studied by Kalman[14,15] in the first order in several papers.

We begin by describing the Gel'fand representation for the octet. In SU(4) the basis is in 10 parameters:

$$\xi(m) = \begin{pmatrix} m_{14} & & m_{24} & & m_{34} & & m_{44} \\ & m_{13} & & m_{23} & & m_{33} & \\ & & m_{12} & & m_{22} & & \\ & & & m_{11} & & & \end{pmatrix} \tag{48}$$

For unimodularity we set $m_{44} = 0$. The third row is fixed to label an octet representation: $m_{23} = m_{13} - 1$; $m_{33} = m_{13} - 2$.

The parameters in the bottom rows are set in accordance with equations 18 and 24 and are listed in Table 8 with the particle states to which they correspond.

We now form the mass matrix according to

$$M_{ij} = \langle \xi_i | \, c_0 + \sum_{k,p=1}^{3} (c_k c_p) \, E_{k4} \, E_{4p} \, | \xi_j \rangle \tag{49}$$

The operators E_{k4} , E_{4p} are the quark anti-quark operators defined in equations 35 and 36. The elements of the matrix are of the form

$$M_{ij} = c_0 \, \delta_{ij} + {}^{\circ}F_{ij} \, (c_1, c_2, c_3, \alpha, \beta, \gamma) \tag{50}$$

where the term F_{ij} is a simple arithmetic expression in terms of the constants C_1, C_2, C_3, and the parameters α, β, γ. These parameters are the coefficients defined in equations 26 to 29, resulting from the operators E_{k4} E_{4p} with

$$\begin{aligned}
\alpha &= (m_{14}-m_{13}-1)\ (m_{13}-m_{24})\ (m_{13}-m_{34}+1)\ (m_{13}-2) \\
\beta &= (m_{14}-m_{13}+3)\ (m_{24}-m_{13}+2)\ (m_{13}-m_{34}-1)\ (m_{13}) \\
\gamma &= (m_{14}-m_{13}+5)\ (m_{24}-m_{13}+4)\ (m_{34}-m_{13}+3)\ (m_{13}-2)
\end{aligned} \qquad (51)$$

Owing to the triangular inequality (equation 18), there are only four possible assignments for m_{24}, m_{34} :

$$\begin{aligned}
&a)\ m_{24} = m_{13} && m_{34} = m_{13}-1 \\
&b)\ m_{24} = m_{13} && m_{34} = m_{13}-2 \\
&c)\ m_{24} = m_{13}-1 && m_{34} = m_{13}-1 \\
&d)\ m_{24} = m_{13}-1 && m_{34} = m_{13}-2
\end{aligned} \qquad (52)$$

Of these, only cases a) and b) are suitable for representing the baryons.[21] Case c) provides a description of mixed states of baryons and anti-baryons. Case d) is suitable for a description of mesons.

In the first order mass calculation case b) results in non-positive integer solutions for m_{13}, m_{14}.[12] This violates the triangular inequality equation 18 as $m_{44} = 0$. For this reason only case a) will be examined in this thesis.

Using case a) $\alpha = \beta = 0$, resulting in only one parameter $\gamma = 8(m_{14}-m_{13}+5)\ (m_{13}-2)$. The constant C_i is then written $D_i = C_i \gamma$. The elements of the matrix are listed in Table 9.

Following the method described in section 5 we define

$$\lambda_i'' = (\lambda_i - c_0) / D_3^{\,2}$$
$$X_1 = D_1 / D_3 \qquad (53)$$
$$X_2 = D_2 / D_3$$

Also as in section 5, isoplet mass splitting is eliminated by setting $X = X_1 = X_2$. The matrix is consequently reduced from an 8 X 8 array to a 4 X 4 array (figure 5).

The matrix is diagonalized numerically in terms of the free parameter X, and the eigenvalue λ'' is fit to the average mass of each isoplet N, Λ, Σ, Ξ.

The best fit occurs for X = .217 with

$$r_1 = 4.6898$$
$$r_2 = 6.8303 \qquad (54)$$
$$r_3 = 7.9893$$
$$r_4 = 9.9537$$

and $D_3^{\,2} = 66.6$ $\qquad c_0 = 642.5$

The list of predicted mass values, the experimentally measured masses, and the % error are given in Table 10.

The mass formula is expressed as

$$M_1 = \lambda_1 = C_0 + D_3^{2}\lambda_1 \qquad (55)$$

Kalman developed a similar mass formula [14]

$$M(\alpha) = C_0 - Aa\ B(\xi(\alpha)) \qquad (56)$$

where A is a constant, a is a term dependent on the parameters of the first row of the Gel'fand - Cetlin array and

$$\begin{aligned} B(\xi(\Xi)) &= 3 \\ B(\xi(\Sigma)) &= 4 \\ B(\xi(\Lambda)) &= 6 \\ B(\xi(N)) &= 8 \end{aligned} \qquad (57)$$

are the terms $B(\xi(\alpha))$ calculated by considering the action of $E_{34}\ E_{43}$ (s 8) on the Gel'fand - Cetlin arrays.

From equations 56 and 57 two mass formulae were developed:

$$(M(N) + M(\Sigma))/2 = M(\Lambda) \qquad (58a)$$

$$(M(\Lambda) + 2M(\Xi))/3 = M(\Sigma) \qquad (58b)$$

which are satisfied by experiment to within 5 %, and, when combined, result in the Gell-Mann - Okubo[3,6] mass formula

$$1/2(M(N) + M(\Xi)) = 3/4(M(\Lambda)) + 1/4(M(\Sigma)) \qquad (59)$$

Using equations 54 and 55 a mass relation is developed (corresponding to equation 56)

$$M(\Lambda) = (1.29747\ M(\Sigma) + .702531\ M(N))/2 \qquad (60a)$$

which is satisfied to within 1 % by experiment.

Similarly, we can develop another mass relation (corresponding to equation 57)

$$M(\Sigma) = (1.88679\ M(\Lambda) + 1.1132\ M(\Xi))/3 \qquad (60b)$$

which is satisfied by experiment to within .22 % error.

7: CALCULATION OF CHARMED BARYON MASS

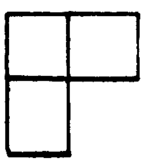
As shown previously the SU(4) baryon multiplet 20_m is composed of the basic octet, triplet, anti-triplet, and sextet.

The Gel'fand basis is of the form

$$\xi(m) = \begin{pmatrix} m_{15} & & m_{25} & & m_{35} & & m_{45} & & m_{55} \\ & m_{14} & & m_{24} & & m_{34} & & m_{44} & \\ & & m_{13} & & m_{23} & & m_{33} & & \\ & & & m_{12} & & m_{22} & & & \\ & & & & m_{11} & & & & \end{pmatrix} \qquad (61)$$

Here 55 = 0 to preserve unimodularity.

The 20_m is represented by the Young tableau



$\rightarrow m_{1n} =$ 2 boxes

$\rightarrow m_{2n} =$ 1 box

$\rightarrow m_{3n} =$ 0 boxes

$\vdots$

$\rightarrow m_{nn} =$ -0 boxes

and the Lie algebra is specified by the integers $m_{1n} = m_{2n}+1 = m_{3n}+2 = \ldots m_{nn}+2$. Therefore the 20_m is labeled by setting $m_{24} = m_{14}-1$, $m_{34} = m_{14}-2$, $m_{44} = m_{14}-2$. The remaining parameters are fixed in accordance with equations 18 and 24 to label the particle states (Table 11).

The elements of the mass matrix are calculated using

$$M_{ij} = \langle \xi_i \mid C_0 + \sum_{k,p=1}^{4} C_k C_p E_{k5} E_{5p} \mid \xi_j \rangle \qquad (62)$$

The M_{ij} have the form

$$M_{ij} = C_0 \, \delta_{ij} + F_{ij} \, (C_1, C_2, C_3, C_4, \alpha, \beta, \gamma) \qquad (63)$$

Again the term F_{ij} is a simple arithmetic expression with

$$\left.\begin{aligned} \alpha &= +(m_{15}-m_{14}+1)\,(m_{25}-m_{14})\,(m_{35}-m_{14}-1)\,(m_{14}+3) \\ \beta &= (m_{15}-m_{14}+3)\,(m_{25}-m_{14}+2)\,(m_{35}-m_{14}+1)\,(m_{14}+1) \\ \gamma &= (m_{15}-m_{14}+6)\,(m_{25}-m_{14}+5)\,(m_{35}-m_{14}+4)\,(m_{14}-2) \end{aligned}\right\} \qquad (64)$$

The parameters of the top row m_{15}, m_{25}, m_{35}, m_{45} can only be given the following four assignments (due to equation 18):

$$\begin{aligned} &a)\quad m_{25} = m_{14} && m_{35} = m_{14}-1 && m_{45} = m_{14}-2 \\ &b)\quad m_{25} = m_{14}-1 && m_{35} = m_{14}-1 && m_{45} = m_{14}-2 \\ &c)\quad m_{25} = m_{14} && m_{35} = m_{14}-2 && m_{45} = m_{14}-2 \\ &d)\quad m_{25} = m_{14}-1 && m_{35} = m_{14}-2 && m_{45} = m_{14}-2 \end{aligned} \qquad (65)$$

In the first order calculation,[16] cases b), c), and d) result in essentially non-positive integer solutions for m_{15} and m_{14}. This violates the inequality equation 18 as $m_{55} = 0$. Therefore the conditions of case a) are applied in this thesis.

Using case a) $\alpha = \beta = 0$ and

$$\gamma = 15(m_{14}-2)\,(m_{15}-m_{14}+6)$$

The parameter γ is incorporated in the constant C_1 by setting $D_1 = C_1 \gamma$.

To eliminate isoplet mass splitting we set $C_1 = C_2$, reducing the matrix from a 20 X 20 array to an 11 X 11 array.[16] The elements of the reduced matrix are listed in Table 12.

Using the method described in section 5 we define

$$\begin{aligned} \lambda'' &= (\lambda - C_0) / D_4^2 \\ x &= D_3 / D_4 \\ y &= D_2 / D_4 = D_1 / D_4 \end{aligned} \tag{66}$$

The matrix is then diagonalized numerically in terms of the two parameters (x, y). The resulting eigenvalues λ'' are fit to the masses of the five known baryons in this multiplet.

We find the best fit to the experimental data occurs with the parameters x, y fixed, to x = .098 and y = .026. The eigenvalues for this fit are listed in Table 13. Again the eigenvalues are related to the particle masses by the relation

$$M_i = C_0 + D_4^2 r_i \tag{67}$$

The predicted mass spectrum is also listed in Table 13.

As with the case of the uncharmed baryons, we can use the eigenvalues to develop the following mass formulae:

$$2M(\Lambda) = 1.69565\, M(\Sigma) + .30435\, M(N) \tag{68}$$

$$3M(\Sigma) = .627\, M(\Xi) + 2.373\, M(\Lambda) \tag{69}$$

Equation 68 is satisfied to within 3.5 % and equation 69 to within 3 %.

8: CONCLUSION

In this thesis the dynamical group method was employed to calculate the baryon masses. Not only did this method yield a mass spectrum, but the discrete representation provides a clear picture of the quark model.

In the case of the group SU(4), which was used to represent the uncharmed $\frac{1}{2}^+$ baryon octet, the predicted mass spectrum agrees with experiment to within 1.7 %. Also, two mass sum rules were developed which are satisfied by experiment to within 1 %, a significant improvement over previous calculations. Therefore, the dynamical group method is a satisfactory representation of the SU(3) baryons.

However, when the dynamical group method is employed in representing SU(4), the results are not as encouraging. Although the predicted mass spectrum agrees well with the known mass spectrum, the relatively large masses predicted for the charmed baryons seem to be incorrect. Mass calculations using flavour independent quark potentials[23,24] have predicted much lower masses for charmed baryons.

In recent experimental work, Baltay et al[25] discovered a Σ_c state at 2426 MeV, the spin of which is not yet established. Since the spin $\frac{1}{2}$ Σ_c is predicted to have a mass of 3200 MeV in our model, the determination of the spin of the $\Sigma_c(2426)$ will be a crucial test of the validity of the model presented in this thesis.

APPENDIX 1

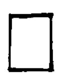
YOUNG'S TABLEAU

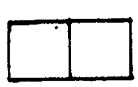
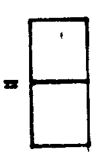
Young's tableau is a group-theoretical method used to compute the effect of permutations on a system of n identical interacting particles in different states.

Let us consider a system of two identical particles in states 1 or 2, with a total wave function $\psi(12)$. If there is no particular symmetry under the interchange of the two particles then we can construct a totally symmetric state ψ_s and a totally anti-symmetric state ψ_a as follows:

$$\psi_s = \psi(12) + \psi(21)$$
$$\psi_a = \psi(12) - \psi(21)$$
$$\psi(12) = P_{12} \psi(12)$$

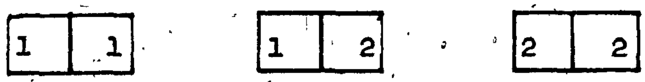
where P_{12} is the transposition of particle states 1 and 2.

If we use the method of Young's tableau each particle is represented by a box  , with a symbol in the box to represent the state which the particle is in. Boxes in a row represent a symmetric state; boxes in a column represent an anti-symmetric state:

$\psi_s =$  $\psi_a =$ 

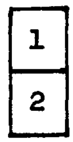
We can determine the number of states corresponding to each diagram by putting the symbols in the boxes corresponding to the states of the individual particle.

For ψ_s we get



which is a triplet.

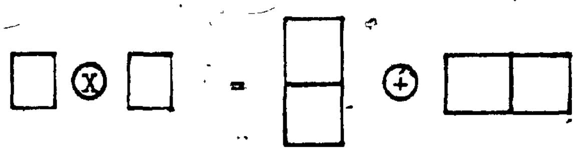
For ψ_a we get



which is a singlet.

This is known as the standard arrangement of Young's tableau: the possible states are numbered by positive integers such that the numbers do not decrease in proceeding from left to right in a row, and increase in proceeding from top to bottom in a column.

This system of two identical particles can be described as the the product



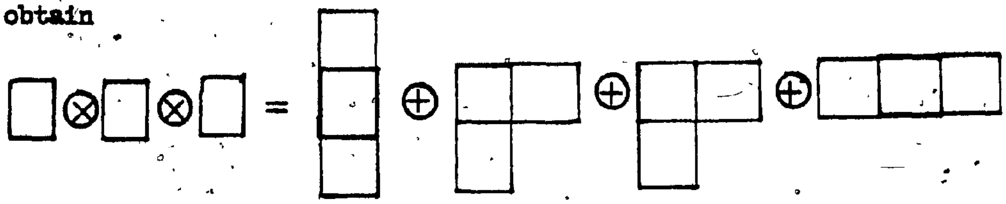
which is equivalent to the Kronecker product

$$2 \otimes 2 = 1 \oplus 3$$

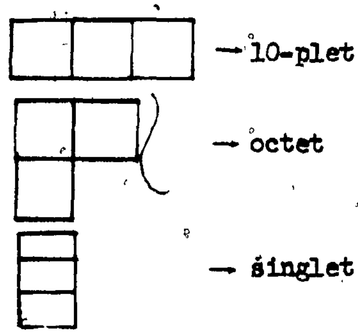
The dimension of a given tableau is determined by the standard arrangement.

Using the quark model, we have systems of three identical particles for baryons or particle anti-particle pairs for mesons. These particles can be in three states for SU(3), or four states for SU(4).

Looking at the products of three identical particles we obtain



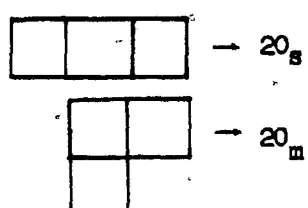
Each of the tableaux corresponds to a particular SU(n) multiplet for SU(3).



which is precisely the result of the Kronecker product

$$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$$

For SU(4)



again the equivalent of

$$4 \otimes 4 \otimes 4 = 20_s \oplus 20_m \oplus 20_m \oplus \bar{4}$$

TABLE 1 BARYON PROPERTIES

PARTICLE	S	C	I	I_3	Y	Q	MASS[a] (MeV)
P	0	0	1/2	1/2	1	+1	938.38
N	0	0	1/2	-1/2	1	0	939.57
Λ^0	-1	0	0	0	0	0	1115.6±0.1
Σ^-	-1	0	1	-1	0	-1	1189.4±0.1
Σ^0	-1	0	1	0	0	0	1192.5±0.1
Σ^+	-1	0	1	-1	0	+1	1197.5±0.1
Ξ^0	-2	0	1/2	1/2	-1	0	1314.9±0.7
Ξ^-	-2	0	1/2	-1/2	-1	-1	1321.3±0.2
Λ_c^+	0	1	0	0	0	+1	2257
Ξ_{ac}^0	-1	1	1/2	-1/2	-1	0	?
Ξ_{ac}^+	-1	1	1/2	+1/2	-1	+1	?
Σ_c^{++}	0	1	1	1	0	+2	?
Σ_c^+	0	1	1	0	0	+1	?
Σ_c^0	0	1	1	1	0	0	?
Ξ_{sc}^+	-1	1	1/2	1/2	-1	+1	?
Ξ_{sc}^0	-1	1	1/2	-1/2	-1	0	?
Ω_c^0	-2	1	0	0	-2	0	?
Ξ_{cc}^{++}	0	2	1/2	1/2	-1	+2	?
Ξ_{cc}^+	0	2	1/2	-1/2	-1	+1	?
Ω_{cc}^+	-1	2	0	0	-2	+1	?

[a] Particle Data Group, Phys. Lett. 75B, 1 (1978).

TABLE 2

Generators of SU(3)

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

TABLE 3

NON - ZERO STRUCTURE CONSTANTS[a]

K N V	F_{knv}
123	1
147	1/2
156	-1/2
246	1/2
257	1/2
345	1/2
367	-1/2
458	$\sqrt{3}/2$
678	$\sqrt{3}/2$

[a] The F_{knv} are antisymmetric under the permutation of any two indices.

TABLE 4

PROPERTIES OF QUARK

FLAVOUR	I	I_3	Y	S	C	Q	MASS [a]
u	1/2	1/2	1/3	0	0	2/3	336
d	1/2	-1/2	1/3	0	0	-1/3	339
s	0	0	-2/3	-1	0	-1/3	467
c	0	0	-2/3	0	1	2/3	1550

TABLE 5

BARYON	WAVE FUNCTION
P	(2uud -udu -duu) $/\sqrt{6}$
N	(udd -dud -2ddu $/\sqrt{6}$
Λ^0	(usd + sud -dsu -sdu) / 2
Σ^-	(2dds -dsd -sdd) $/\sqrt{6}$
Σ^0	(2uds + 2dus -usd -dsu -sud -sdu) $/\sqrt{12}$
Σ^+	(2uus -usu -suu) $/\sqrt{6}$
Ξ^0	(uss + sus -2ssu) $/\sqrt{6}$
Ξ^-	(dss + sds -2ssd) $/\sqrt{6}$
Λ_c^+	(ucd + cud -dcu -cdu) / 2
Ξ_{ac}^0	(dcs + cds - scd -csd) / 2
Ξ_{ac}^+	(ucs + cus -scu -csu) / 2

[a] These are estimated effective masses of bound quarks.

BARYON	WAVE FUNCTION
Σ_c^{++}	(2uuc -ucu -cuu) / $\sqrt{6}$
Σ_c^{+}	(2udc + 2duc -ucd -dcu -cud -cdu) / $\sqrt{12}$
Σ_c^{0}	(2ddc - dcd -cdd) / $\sqrt{6}$
Ξ_{sc}^{+}	(2usc + 2suc - ucs -scu - cus -csu) / $\sqrt{12}$
Ξ_{sc}^{0}	(2dsc + 2sdc -dcs -scd -cds -csd) / $\sqrt{12}$
Ω_c^{0}	(2ssc -scs -css) / $\sqrt{6}$
Ξ_{cc}^{++}	(ucc + cuc -2ccu) / $\sqrt{6}$
Ξ_{cc}^{+}	(dcc + cdc -2ccd) / $\sqrt{6}$
Ω_{cc}^{+}	(scc + csc -2ccs) / $\sqrt{6}$

NOTE: The subscript a or s indicates a state which is anti-symmetric or symmetric with respect to the quark indices.

TABLE 6

GENERATORS OF SU (4)

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_8 = 1/\sqrt{3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad \lambda_{15} = 1/\sqrt{6} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$\lambda_0 = I$$

TABLE 7

STRUCTURE CONSTANTS FOR SU(4) [a]

K	N	V	F_{knv}
1	2	3	1
1	4	7	1/2
1	5	6	-1/2
2	4	6	1/2
2	5	7	1/2
3	4	5	1/2
3	6	7	-1/2
4	5	8	$\sqrt{3}/2$
6	7	8	$\sqrt{3}/2$
1	9	12	1/2
1	10	11	-1/2
2	9	11	1/2
2	10	12	1/2
3	9	10	1/2
3	11	12	-1/2
4	9	14	1/2
4	10	13	-1/2
5	9	13	1/2
5	10	14	1/2
6	11	14	1/2
6	12	13	-1/2
7	11	13	1/2
7	12	14	1/2
8	9	10	$1/(2\sqrt{3})$
8	11	12	$1/(2\sqrt{3})$
8	13	14	$-1/\sqrt{3}$
9	10	15	$\sqrt{2/3}$
11	12	15	$\sqrt{2/3}$
13	14	15	$\sqrt{2/3}$

[a] The F_{knv} are antisymmetric under the exchange of any two indices.

TABLE 8

GEL'FAND ARRAYS OF SU(3)

The Gel'Fand pattern for an octet

$$\begin{pmatrix} m_{14} & & m_{24} & & m_{34} & & 0 \\ & m_{13} & & m_{13-1} & & m_{13-2} & \\ & & m_{12} & & m_{22} & & \\ & & & m_{11} & & & \end{pmatrix}$$

with the bottom 2 rows as follows:

$$N^{+} = \begin{pmatrix} m_{13} & & m_{13-1} \\ & m_{13} & \end{pmatrix} \qquad N^{0} = \begin{pmatrix} m_{13} & & m_{13-1} \\ & m_{13-1} & \end{pmatrix}$$

$$\Lambda^{0} = \begin{pmatrix} m_{13-1} & & m_{13-1} \\ & m_{13-1} & \end{pmatrix}$$

$$\Sigma^{+} = \begin{pmatrix} m_{13} & & m_{13-2} \\ & m_{13} & \end{pmatrix} \qquad \Sigma^{0} = \begin{pmatrix} m_{13} & & m_{13-2} \\ & m_{13-1} & \end{pmatrix} \qquad \Sigma^{-} = \begin{pmatrix} m_{13} & & m_{13-2} \\ & m_{13-2} & \end{pmatrix}$$

$$\Xi^{0} = \begin{pmatrix} m_{13-1} & & m_{13-2} \\ & m_{13-1} & \end{pmatrix} \qquad \Xi^{-} = \begin{pmatrix} m_{13-1} & & m_{13-2} \\ & m_{13-2} & \end{pmatrix}$$

TABLE 9

SU(4) MASS MATRIX ELEMENTS

$$M_{ij} = \langle \xi_i | H | \xi_j \rangle = M_{ji}$$

$$H = H_0 + H'$$

$$H' = \sum_{k,p=1}^{3} q_k \bar{q}_p C_k C_p$$

where

$$\xi_1 = N^+ \quad \xi_4 = \Sigma^+ \quad \xi_7 = \Xi^0$$
$$\xi_2 = N^0 \quad \xi_5 = \Sigma^0 \quad \xi_8 = \Xi^-$$
$$\xi_3 = \Lambda^0 \quad \xi_6 = \Sigma^-$$

$$D_i = C_i \gamma$$

$$M_{1,1} = C_0 + (D_1)^2 / 40 + (D_2)^2 / 30 + (D_3)^2 / 15$$

$$M_{1,2} = D_1 D_2 / 120$$

$$M_{1,3} = \sqrt{3/2}\, D_1 D_3 / 60$$

$$M_{1,4} = D_2 D_3 / 30$$

$$M_{1,5} = \sqrt{1/2}\, D_1 D_3 / 30$$

$$M_{1,6} = 0$$

$$M_{1,7} = 0$$

$$M_{1,8} = 0$$

$$M_{2,2} = C_0 + (D_1)^2 / 30 + (D_2)^2 / 40 + (D_3)^2 / 15$$

$$M_{2,3} = \sqrt{3/2}\, D_2 D_3 \;/\; 60$$

$$M_{2,4} = 0$$

$$M_{2,5} = \sqrt{1/2}\, D_2 D_3 \;/\; 30$$

$$M_{2,6} = D_1 D_3 \;/\; 30$$

$$M_{2,7} = 0$$

$$M_{2,8} = 0$$

$$M_{3,3} = C_0 + 3(D_1)^2 \;/\; 80 + 3(D_2)^2 \;/\; 80 + (D_3)^2 \;/\; 20$$

$$M_{3,4} = \sqrt{6}\, D_1 D_2 \;/\; 240$$

$$M_{3,5} = (D_1 + D_2)(3)^{1/2} \;/\; 240$$

$$M_{3,6} = \sqrt{6}\, D_1 D_2 \;/\; 240$$

$$M_{3,7} = \sqrt{3/2}\, D_2 D_3 \;/\; 40$$

$$M_{3,8} = \sqrt{3/2}\, D_1 D_3 \;/\; 40$$

$$M_{4,4} = C_0 + (D_1)^2 \;/\; 40 + (D_2)^2 \;/\; 15 + (D_3)^2 \;/\; 30$$

$$M_{4,5} = \sqrt{2}\, D_1 D_2 \;/\; 48$$

$$M_{4,6} = 0$$

$$M_{4,7} = D_1 D_3 \;/\; 120$$

$$M_{4,8} = 0$$

$$M_{5,5} = C_0 + 11(\,(D_1)^2 + (D_2)^2\,) \;/\; 240 + (D_3)^2 \;/\; 30$$

$$M_{5,6} = \sqrt{2}\, D_1 D_2 \,/\, 48$$

$$M_{5,7} = \sqrt{1/2}\, D_2 D_3 \,/\, 120$$

$$M_{5,8} = \sqrt{1/2}\, D_1 D_3 \,/\, 120$$

$$M_{6,6} = C_0 + (D_1)^2\, 15 + (D_2)^2 \,/\, 40 + (D_3)^2 \,/\, 30$$

$$M_{6,7} = 0$$

$$M_{6,8} = D_2 D_3 \,/\, 120$$

$$M_{7,7} = C_0 + (D_1)^2 \,/\, 30 + (D_2)^2 \,/\, 15 + (D_3)^2 \,/\, 40$$

$$M_{7,8} = D_1 D_2 \,/\, 30$$

$$M_{8,8} = C_0 + (D_1)^2 \,/\, 15 + (D_2)^2 \,/\, 30 + (D_3)^2 \,/\, 40$$

TABLE 10

SU(3) MASS SPECTRUM

PARTICLE	EXPERIMENTAL[a] MASS (MeV)	r_i	PREDICTED MASS (MeV)	% ERROR
N	938.3	4.6898	954.8	1.69
Λ	1115.6	6.8303	1097.4	1.63
Σ	1193.4	7.9893	1174.6	1.54
Ξ	1318.1	9.9537	1305.4	.95

$$(D_3)^2 = 66.6$$

$$C_0 = 642.5$$

[a] **The isoplet masses are determined as follows:**

$M(N) = (M(N^+) + M(N^0))/2$

$M(\Lambda) = M(\Lambda)$

$M(\Sigma) = (M(\Sigma^+) + M(\Sigma^-))/2$

$M(\Xi) = (M(\Xi^0) + M(\Xi^-))/2$

TABLE 11

GEL'FAND ARRAYS FOR SU(4), 20_m

The Gel'Fand pattern for the 20_m

$$\xi(m) = \begin{pmatrix} m_{15} & & m_{25} & & m_{35} & & m_{45} & & 0 \\ & m_{14} & & m_{14-1} & & m_{14-2} & & m_{14-2} & \\ & & m_{13} & & m_{23} & & m_{33} & & \\ & & & m_{12} & & m_{22} & & & \\ & & & & m_{11} & & & & \end{pmatrix}$$

for the octet the bottom 3 rows are

$$N^0 = \begin{pmatrix} m_{14} & & m_{14-1} & & m_{14-2} \\ & m_{14} & & m_{14-1} & \\ & & m_{14-1} & & \end{pmatrix} \quad N^+ = \begin{pmatrix} m_{14} & & m_{14-1} & & m_{14-2} \\ & m_{14} & & m_{14-1} & \\ & & m_{14} & & \end{pmatrix}$$

$$\Lambda^0 = \begin{pmatrix} m_{14} & & m_{14-1} & & m_{14-2} \\ & m_{14-1} & & m_{14-1} & \\ & & m_{14-1} & & \end{pmatrix}$$

$$\Sigma^- = \begin{pmatrix} m_{14} & & m_{14-1} & & m_{14-2} \\ & m_{14} & & m_{14-2} & \\ & & m_{14-2} & & \end{pmatrix} \quad \Sigma^0 = \begin{pmatrix} m_{14} & & m_{14-1} & & m_{14-2} \\ & m_{14} & & m_{14-2} & \\ & & m_{14-1} & & \end{pmatrix}$$

$$\Sigma^+ = \begin{pmatrix} m_{14} & & m_{14-1} & & m_{14-1} \\ & m_{14} & & m_{14-2} & \\ & & m_{14-1} & & \end{pmatrix} \quad \Xi^- = \begin{pmatrix} m_{14} & & m_{14-1} & & m_{14-2} \\ & m_{14-1} & & m_{14-2} & \\ & & m_{14-1} & & \end{pmatrix}$$

$$\Xi^{0} = \begin{pmatrix} m_{14} & m_{14-1} & m_{14-2} \\ & m_{14-1} & m_{14-2} \\ & & m_{14-1} \end{pmatrix}$$

for the anti-triplet

$$\Lambda_{c}^{+} = \begin{pmatrix} m_{14-1} & m_{14-1} & m_{14-2} \\ & m_{14-1} & m_{14-1} \\ & & m_{14-1} \end{pmatrix}$$

$$\Xi_{ca}^{0} = \begin{pmatrix} m_{14-1} & m_{14-1} & m_{14-2} \\ & m_{14-1} & m_{14-2} \\ & & m_{14-2} \end{pmatrix} \qquad \Xi_{ca}^{+} = \begin{pmatrix} m_{14-1} & m_{14-1} & m_{14-2} \\ & m_{14-1} & m_{14-2} \\ & & m_{14-1} \end{pmatrix}$$

for the sexplet

$$\Sigma_{c}^{0} = \begin{pmatrix} m_{14} & m_{14-2} & m_{14-2} \\ & m_{14} & m_{14-2} \\ & & m_{14-2} \end{pmatrix} \qquad \Sigma_{c}^{+} = \begin{pmatrix} m_{14} & m_{14-2} & m_{14-2} \\ & m_{14} & m_{14-2} \\ & & m_{14-1} \end{pmatrix}$$

$$\Sigma_{c}^{++} = \begin{pmatrix} m_{14} & m_{14-2} & m_{14-2} \\ & m_{14} & m_{14-2} \\ & & m_{14} \end{pmatrix} \qquad \Xi_{sc}^{0} = \begin{pmatrix} m_{14} & m_{14-2} & m_{14-2} \\ & m_{14-1} & m_{14-2} \\ & & m_{14-2} \end{pmatrix}$$

$$\Xi_{sc}^{+} = \begin{pmatrix} m_{14} & m_{14-2} & m_{14-2} \\ & m_{14-1} & m_{14-2} \\ & & m_{14-1} \end{pmatrix} \qquad \Omega_{c}^{0} = \begin{pmatrix} m_{14} & m_{14-2} & m_{14-2} \\ & m_{14-2} & m_{14-2} \\ & & m_{14-2} \end{pmatrix}$$

for the triplet

$$\Xi_{cc}^{+} = \begin{pmatrix} m_{14-1} & m_{14-2} & m_{14-2} \\ & m_{14-1} & m_{14-2} \\ & & m_{14-2} \end{pmatrix}$$

$$\Xi_{cc}^{++} = \begin{pmatrix} m_{14-1} & m_{14-2} & m_{14-2} \\ & m_{14-1} & m_{14-2} \\ & & m_{14-1} \end{pmatrix}$$

$$\Omega_{cc}^{+} = \begin{pmatrix} m_{14-1} & m_{14-2} & m_{14-2} \\ & m_{14-2} & m_{14-2} \\ & & m_{14-2} \end{pmatrix}$$

TABLE 12

SU(5) MASS MATRIX ELEMENTS

$$M_{ij} = \langle \xi_i^* | \; H \; | \xi_j \rangle = M_{ji}$$

$$H = H_Q + H'$$

$$H' = \sum_{k,p=1}^{4} q_k \bar{q}_p \, C_k C_p$$

where

$$\xi_1 = N \qquad \xi_4 = \Xi \qquad \xi_7 = \Sigma_c \qquad \xi_{10} = \Xi_{cc}$$
$$\xi_2 = \Lambda \qquad \xi_5 = \Lambda_c \qquad \xi_8 = \Xi_{sc} \qquad \xi_{11} = \Omega_{cc}$$
$$\xi_3 = \Sigma \qquad \xi_6 = \Xi_{ac} \qquad \xi_9 = \Omega_c$$

$$D_i = C_i$$

$$M_{1,1} = C_0 + (D_4)^2 / 48 + (D_3)^2 / 48 + (D_2)^2 / 40$$

$$M_{1,2} = \sqrt{15/8}\, D_2 D_3 / 1200$$

$$M_{1,3} = 2\sqrt{6}\, D_2 D_3 / 1728$$

$$M_{1,4} = 0$$

$$M_{1,5} = \sqrt{3/2}\, D_2 D_4 / 240$$

$$M_{1,6} = 0$$

$$M_{1,7} = \sqrt{2/2}\, D_2 D_4 / 144$$

$$M_{1,8} = 0$$

$M_{1,9} = 0$

$M_{1,10} = 0$

$M_{1,11} = 0$

$M_{2,2} = C_0 + (D_4)^2 / 48 + (D_3)^2 / 60 + 7(D_2)^2 / 240$

$M_{2,3} = \sqrt{5/64}\, D_2 D_3 / 1200 + \sqrt{3/2}\, D_2 D_3 / 1728$

$M_{2,4} = \sqrt{15/32}\, D_2 D_3 / 1200 + 3\, D_2 D_3 / 1728$

$M_{2,5} = D_3 D_4 / 240$

$M_{2,6} = D_2 D_4 / 240$

$M_{2,7} = 0$

$M_{2,8} = \sqrt{3/4}\, D_4 D_2 / 144$

$M_{2,9} = 0$

$M_{2,10} = 0$

$M_{2,11} = 0$

$M_{3,3} = C_0 + (D_4)^2 / 48 + (D_3)^2 / 72 + 23\, (D_2)^2 / 720$

$M_{3,4} = \sqrt{45/32}\, D_2 D_3 / 1200 + 3\, D_2 D_3 / 1728$

$M_{3,5} = 0$

$M_{3,6} = \sqrt{3/4}\, D_2 D_2 / 240$

$M_{3,7} = D_3 D_4 / 144$

$M_{3,8} = D_2D_4 / 288$

$M_{3,9} = 0$

$M_{3,10} = 0$

$M_{3,11} = 0$

$M_{4,4} = C_0 + (D_4)^2 / 48 + (D_3)^2 / 90 + 5(D_2)^2 / 144$

$M_{4,5} = 0$

$M_{4,6} = \sqrt{3/2}\, D_4D_3 / 240$

$M_{4,7} = 0$

$M_{4,8} = \sqrt{1/2}\, D_3D_4 / 144$

$M_{4,9} = D_2D_4 / 288$

$M_{4,10} = 0$

$M_{4,11} = 0$

$M_{5,5} = C_0 + (D_4)^2 / 60 + (D_3)^2 / 48 + 7(D_2)^2 / 240$

$M_{5,6} = D_2D_3 / 360$

$M_{5,7} = 0$

$M_{5,8} = 0$

$M_{5,9} = 0$

$M_{5,10} = \sqrt{3/2}\, D_2 D_4 \;/\; 180$

$M_{5,11} = 0$

$M_{6,6} = C_0 + (D_4)^2 / 60 + 7(D_3)^2 / 480 + 17(D_2)^2 / 480$

$M_{6,7} = \sqrt{9/8}\, D_2 D_3 \;/\; 720$

$M_{6,8} = 0$

$M_{6,9} = \sqrt{9/4}\, D_2 D_3 \;/\; 720$

$M_{6,10} = \sqrt{3/2}\, D_3 D_4 \;/\; 180$

$M_{6,11} = \sqrt{3/2}\, D_2 D_4 \;/\; 180$

$M_{7,7} = C_0 + (D_4)^2 / 72 + (D_3)^2 / 48 + 23\,(D_2)^2 / 720$

$M_{7,8} = \sqrt{5/16}\, D_2 D_3 \;/\; 1800$

$M_{7,9} = 0$

$M_{7,10} = \sqrt{1/2}\, D_2 D_4 \;/\; 360$

$M_{7,11} = 0$

$M_{8,8} = C_0 + (D_4)^2 / 72 + 23(D_3)^2 / 1440 + 53(D_2)^2 / 1440$

$M_{8,9} = \sqrt{5/8}\, D_2 D_3 \;/\; 1800$

$M_{8,10} = \sqrt{1/2}\, D_3 D_4 \;/\; 180$

$M_{8,11} = \sqrt{1/2}\, D_2 D_4 \;/\; 180$

$$M_{9,9} = C_0 + (D_4)^2 / 72 + (D_3)^2 / 90 + (D_2)^2 / 24$$

$$M_{9,10} = 0$$

$$M_{9,11} = \sqrt{2/3}\, D_3 D_4 \ / 360$$

$$M_{10,10} = C_0 + (D_4)^2 / 90 + (D_3)^2 / 48 + 5(D_2)^2 / 144$$

$$M_{10,11} = 0$$

$$M_{11,11} = C_0 + (D_4)^2 / 90 + (D_3)^2 / 72 + (D_2)^2 / 24$$

TABLE 13

SU(4) MASS SPECTRUM

PARTICLE	EXPERIMENTAL[a] MASS (MeV)	r_i	PREDICTED MASS (MeV)	% ERROR
N	939.0	.02456	922.5	1.76
Λ	1115.6	.02378	1135.1	1.75
Σ	1193.4	.02364	1173.3	1.68
Ξ	1318.1	.02311	1317.7	0.03
Λ_c	2257.0	.01968	2252.8	0.32
Ξ_c	--	.01875	2506.3	--
Σ_c	--	.01620	3201.4	--
Ξ_c	--	.01613	3220.5	--
Ω_c	--	.01600	3255.9	--
Ξ_{cc}	--	.01348	3942.9	--
Ω_{cc}	--	.01259	4185.5	--

$$c_0 = 7621.56$$
$$B_4^2 = 272598.$$

[a] Isoplet masses are determined as in Table 10.

FIGURE 2

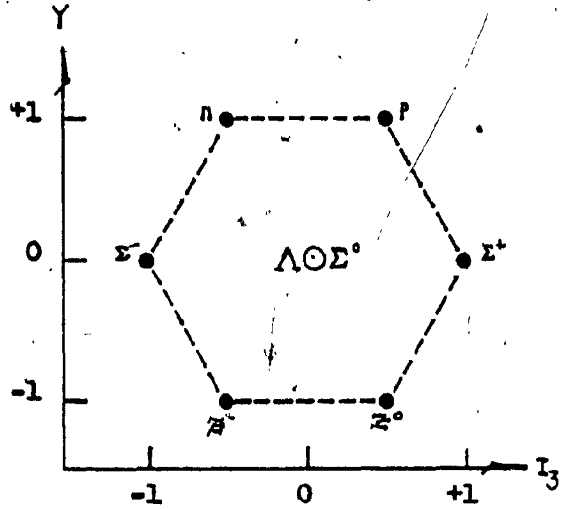


FIGURE 1

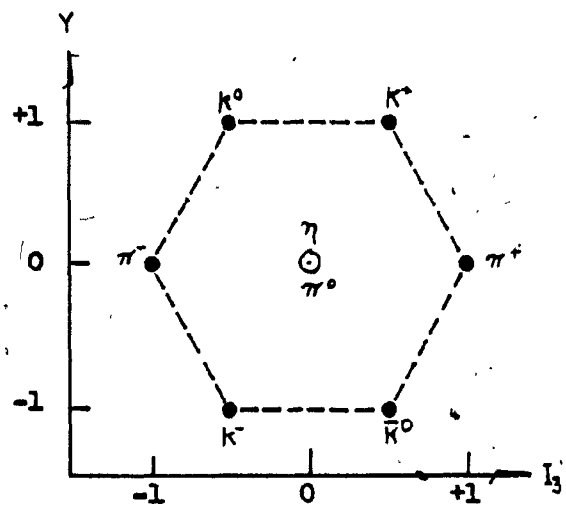


FIGURE 3

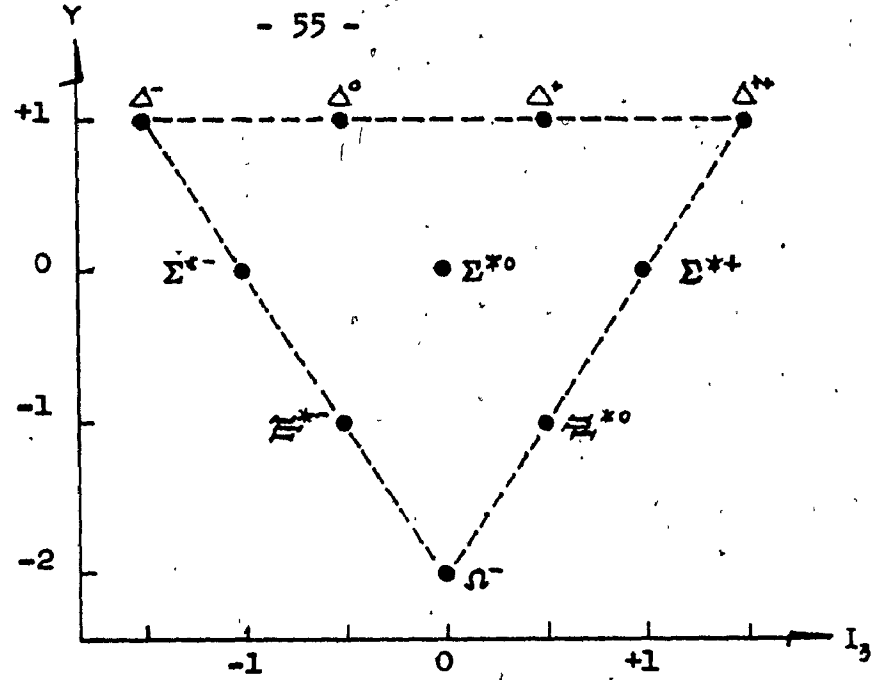


FIGURE 4

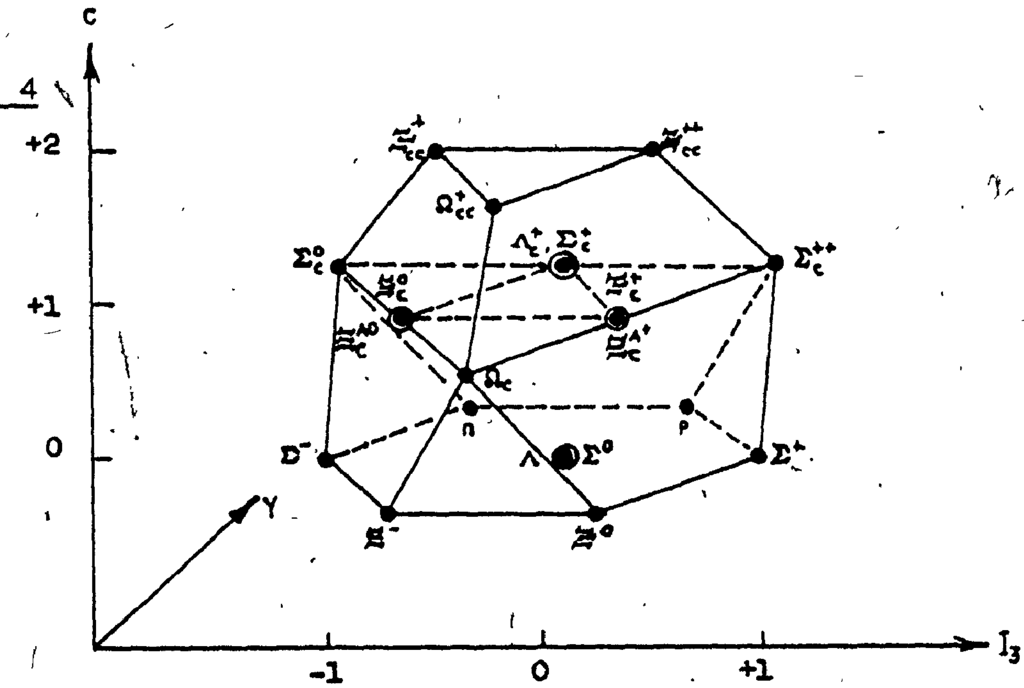


FIGURE 5

$$
\begin{pmatrix}
C_0' + 7X^2 + 8 & (8)^{1/2}X & (6)^{1/2}X & 0 \\
(8)^{1/2}X & C_0' + 11X^2 + 4 & (3)^{1/2}X^2 & X/(2)^{1/2} \\
(6)^{1/2}X & (3)^{1/2}X^2 & C_0' + 9X^2 + 6 & (27/2)^{1/2}X \\
0 & X/(2)^{1/2} & (27/2)^{1/2}X & C_0' + 12X^2 + 3
\end{pmatrix}
$$

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