The Stochastic Approach

to

Future Shares

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Abstract

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The traditional deterministic definitions of actuarial future values and Ramsay's stochastic definitions are discussed. A modification of Ramsay's definitions is proposed and an application to life insurance reserves is suggested.

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TO

MY WIFE,

MY DAUGHTERS

AND

MY PARENTS.

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Introduction

In life contingencies there are many situations in which actuaries face the problem of determining the expected future share per survivor of an initial group, each of whose members makes payments into a fund. This problem is especially important in the case of long term life insurance and pension funds.

In the theory of stochastic life contingencies as presented by Bowers et. al. [1] and Gerber [3], definitions of future accumulated values are not stochastic.

Some authors have proposed different approaches. Martin-Löf [8] presents a model that takes into account stochastic interest rates and the random fluctuations in the collective. Frees [2] considers different ways to define random net premiums and life insurance reserves. In this work we are interested in the point of view of Ramsay [11], who proposes an alternative stochastic definition of future shares taking into account the size of the initial group and its random evolution. By two examples, he shows that when the size of the initial group is large enough, the deterministic traditional definitions are quite appropriate.

In Chapter one of this thesis we present both approaches, the deterministic and Ramsay's stochastic formulations of future shares.

In Chapter two some properties of the first inverse moment of a positive binomial variate are presented and Ramsay's definitions of future shares are discussed.

In Chapter three we present some modifications of Ramsay's stochastic future values under the principle of equivalence. These new definitions are used in Chapter four to define retrospective random losses. The thesis ends with some numerical results and conclusions.

Chapter 1

Actuarial Future Shares

This chapter presents our notation and assumptions. Two problems are discussed to which both the deterministic solution and Ramsay's stochastic solution are presented.

1.1 Assumptions and notation

This section describes some assumptions and the notation from Bowers et. al. [1] that will be used throughout this work.

(i) Time is assumed discrete. (x) means a person of age x, ${}_tp_x$ the probability that (x) survives t years and ${}_tq_x = 1 - {}_tp_x$ that (x) dies within t years.

(ii) A group of $n = 1, 2, \ldots$ mutually independent lives is considered, all of age fixed, $x = 0, 1, \ldots, w - 1$ where w is the span of the future lifetime of (0).

(iii) The curtate future lifetime of (x), usually denoted by K, is assumed to have the same distribution for all (x) in the group. Its probability function is ${}_{k|}q_x = {}_kp_x\, q_{x+k}$, for $k = 0, 1, \ldots, w - x - 1$. When future lifetime is considered continuous then it is denoted by T. Its probability density function is given by $g(t) = ({}_tq_x)' = {}_tp_x\,\mu_{x+t}$, where $\mu_x = -s'(x)/s(x)$ is the force of mortality and s is the survival function of (0) given by $s(x) = {}_xp_0$.

(iv) The annual rate of interest is denoted by i and is assumed to be fixed for all times. The discount factor is $v = 1/(1+i)$, the rate of interest in advance is $d = 1 - v$ and the force of interest is $\delta = \ln(1+i)$.

1.2 Two problems of life contingencies

The two life contingencies problems considered by Ramsay [11] are presented here. The traditional actuarial solutions to these two problems are described in the next section. In Section 1.4 Ramsay's stochastic solutions are presented and in Chapters 2 and 3 they are discussed and reformulated.

Problem 1 : Each member of a group of n independent lives age x makes a single payment of 1 into a fund at time 0. At time t the accumulated value of the fund is divided equally among the survivors, if any. Assuming the fund earns interest at rate i per annum, calculate the expected share of each survivor.

Problem 2 : Each member of a group of n independent lives age x deposits an amount of 1 into a fund at the start of each year as long as he/she is alive. At time t, the accumulated value of the fund is divided equally among the survivors, if any. Assuming the fund earns interest at rate i per annum, calculate the expected share of each survivor.

1.3 The traditional deterministic approach

There is no book on life contingencies that presents stochastic definitions of future values or future shares. In the modern stochastic theory of life contingencies solutions to Problems 1 and 2 are found by considering a single policy (x) and its future random lifetime. In this way, present values are defined stochastically, but future values are defined only as functions of actuarial present values and so they are deterministic (e.g. Bowers et. al. [1]).

Definition 1.1 *Pure endowment.* Consider a unit payment due at the end of t years provided that (x) survives. This benefit is called a t-year pure endowment of 1 with respect to (x). Its present value is defined as the random variable:

$$Z = \begin{cases} 0 & \text{if } T \leq t, \\ v^t & \text{if } T > t. \end{cases}$$

The expected value of Z, $E[Z]$, is called the net single premium (NSP) or the actuarial present value of the benefit. It is given by:

$$E[Z] = A_{\substack{\ \ 1 \\ x:\overline{t|}}} = {}_tE_x = v^t\, {}_tp_x\,.$$

Definition 1.2 *Actuarial accumulated value.* The actuarial accumulated value at the end of t years of 1 contributed at age x is defined as the amount ${}_ts_x$ such that its actuarial present value is 1. That is, $({}_tE_x){}_ts_x = 1$, or

$${}_ts_x = \frac{1}{{}_tE_x} = \frac{(1+i)^t}{{}_tp_x}.$$

Let l_x be the initial size of the group. Since the number of survivors after t years, denoted by $\mathcal{L}(x+t)$, is a binomial random variable with parameters $(l_x, {}_tp_x)$ then the expected number of survivors at time t is $E[\mathcal{L}(x+t)] = l_x\, {}_tp_x$. Therefore,

$$\begin{aligned} {}_ts_x &= \frac{l_x(1+i)^t}{E[\mathcal{L}(x+t)]}, \\ &= \frac{\text{Future accumulated value of the } l_x \text{ payments of 1 at time 0}}{\text{Future expected number of survivors from the initial group } l_x}, \\ &= \text{Expected future share per survivor.} \end{aligned}$$

In this sense ${}_ts_x$ becomes the traditional deterministic solution to Problem 1. That is, the expected future share per survivor at time t is given by ${}_ts_x$ (see Bowers et. al. [1], pp. 101, 140 and 146).

Definition 1.3 *Temporary life annuity.* A t-year temporary life annuity of 1 payable at the beginning of each year while (x) survives has a present value defined as:

$$Y = \begin{cases} \ddot{a}_{\overline{K+1|}} & \text{if } K < t, \\ \ddot{a}_{\overline{t|}} & \text{if } K \geq t. \end{cases} \tag{1.1}$$

The actuarial present value or NSP of this annuity, denoted by $\ddot{a}_{x:\overline{t|}}$, is given by:

$$E[Y] = \ddot{a}_{x:\overline{t|}} = \sum_{k=0}^{t-1} {}_kE_x = \sum_{k=0}^{t-1} v^k\ {}_kp_x\,. \tag{1.2}$$

Definition 1.4 *Term life insurance.* The present value of a t-year term life insurance of 1 payable at the end of the year of death of (x) is defined as:

$$Z = \begin{cases} v^{K+1} & \text{if } K < t, \\ 0 & \text{if } K \geq t. \end{cases} \tag{1.3}$$

The NSP of this benefit, denoted by $A_{\overset{1}{x}:\overline{t|}}$, is given by:

$$E[Z] = A_{\overset{1}{x}:\overline{t|}} = \sum_{k=0}^{t-1} v^{k+1}\, {}_kp_x q_{x+k}\,. \tag{1.4}$$

Definition 1.5 *Endowment life insurance.* The present value of a t-year endowment life insurance of 1 payable at the end of the year of death of (x) or at t, whichever occurs first, is defined as:

$$Z = \begin{cases} v^{K+1} & \text{if } K < t, \\ v^t & \text{if } K \geq t. \end{cases} \tag{1.5}$$

The NSP, equals to $E(Z)$, is denoted by $A_{x:\overline{t}|}$, and is given by:

$$E[Z] = A_{x:\overline{t}|} = A^{\;1}_{x:\overline{t}|} + {}_tE_x \; . \tag{1.6}$$

The variance of Z is easily seen to be:

$$V(Z) = {}^2A_{x:\overline{t}|} - A^2_{x:\overline{t}|}, \tag{1.7}$$

where ${}^2A_{x:\overline{t}|}$ is the NSP of a t-year endowment insurance computed at a force of interest of 2δ.

Since (1.1) can be written as $Y = (1 - Z)/d$, where Z is given by (1.5), then using (1.7) the variance of this random variable is found to be:

$$V[Y] = \frac{{}^2A_{x:\overline{t}|} - A^2_{x:\overline{t}|}}{d^2}. \tag{1.8}$$

Definition 1.6 *Actuarial accumulated value of a temporary life annuity.* The actuarial accumulated value at the end of the term of a t-year temporary life annuity of 1 per annum, payable while (x) survives, is denoted by $\ddot{s}_{x:\overline{t}|}$ and is defined as the amount such that its actuarial present value equals $\ddot{a}_{x:\overline{t}|}$. In other words, $\ddot{s}_{x:\overline{t}|}\, {}_tE_x = \ddot{a}_{x:\overline{t}|}$, or

$$\ddot{s}_{x:\overline{t}|} = \frac{\ddot{a}_{x:\overline{t}|}}{{}_tE_x} = (1+i)^t \frac{\ddot{a}_{x:\overline{t}|}}{{}_tp_x}.$$

Assuming that mortality occurs according to the life table then ${}_tp_x = l_{x+t}/l_x$. Using (1.2) we get:

$$\begin{aligned} \ddot{s}_{x:\overline{t}|} &= \frac{\sum_{k=0}^{t-1} l_{x+k}(1+i)^{t-k}}{l_{x+t}}, \\ &= \frac{\text{Actuarial accumulated value of all payments}}{\text{Expected number of survivors}}, \\ &= \text{Expected future share per survivor.} \end{aligned}$$

This is why $\ddot{s}_{x:\overline{t}|}$ is seen as the traditional deterministic solution of Problem 2, i.e., the future share per survivor at time t is given by $\ddot{s}_{x:\overline{t}|}$ (see Bowers et. al. [1], pp. 144-146).

1.4 Ramsay's stochastic approach

We present here Ramsay's stochastic definitions of future shares for Problems 1 and 2 and some preliminary results needed to this (c.f. [11]).

1.4.1 Preliminary results

Definition 1.7 *The positive binomial distribution.* Let X be a binomial random variable with parameters (n,p), i.e., its probability function is given by

$$f_X(m) \quad = \quad P_{nm}(p) \; = \; \binom{n}{m} p^m q^{n-m}, \qquad m = 0, 1, \ldots, n. \tag{1.9}$$

Then a random variable Y is called a positive binomial distribution with parameters (n,p) if $f_Y(m) = f_{X|X>0}(m)$, i.e.,

$$f_Y(m) = \frac{P_{nm}(p)}{1-q^n}, \qquad m = 1, 2, \ldots, n.$$

In this case its k^{th} inverse moment is denoted by $b_k(n,p) = E[Y^{-k}]$ (c.f. [6]).

Definition 1.8 For $n \geq 1$ and $0 \leq p \leq 1$ we define the polynomials:

$$B_n(p) \quad = \quad \sum_{m=1}^{n} \frac{n}{m} P_{nm}(p), \tag{1.10}$$

$$Q_n(p) \quad = \quad \sum_{m=1}^{n} \frac{n^2}{m^2} P_{nm}(p). \tag{1.11}$$

The polynomials B_n and Q_n have various applications in actuarial mathematics and statistics. In a more general interpretation they can be identified, respectively, as the Bernstein polynomials of the functions

$$b(x) \quad = \quad \begin{cases} \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0, \end{cases} \qquad \text{and} \qquad q(x) \quad = \quad \begin{cases} \frac{1}{x^2} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases} \tag{1.12}$$

As a second application, let N be a binomial random variable with parameters (n,p) and define X as

$$X \quad = \quad \begin{cases} \frac{n}{N} & \text{if } N = 1, 2, \ldots, n, \\ 0 & \text{if } N = 0. \end{cases} \tag{1.13}$$

Then

$$B_n(p) = E[X], \quad Q_n(p) = E[X^2] \quad \text{and} \quad V[X] = Q_n(p) - B_n^2(p). \tag{1.14}$$

A third application of the polynomials B_n and Q_n is as a representation of $b_1(n,p)$ and $b_2(n,p)$, respectively, up to a factor. In fact, we see that

$$B_n(p) = n(1-q^n)b_1(n,p), \tag{1.15}$$

$$Q_n(p) = n^2(1-q^n)b_2(n,p). \tag{1.16}$$

Using (1.10) and (1.11) one can compute $B_n(p)$ and $Q_n(p)$ directly for any finite n. Unfortunately, for large n, the binomial probabilities are curbersome to find. In view of this difficulty, recursive formulas are used to accelerate the computations. Alternatively, some approximations are also used.

Grab and Savage [5] provided the recursive equation:

$$b_1(n+1,p) = \frac{q(1-q^n)}{(1-q^{n+1})}b_1(n,p) + \frac{1}{n+1}, \quad n = 0, 1, \ldots \tag{1.17}$$

with $b_1(0,p) = 0$. Hence (1.15), together with (1.17), can be used to compute $B_n(p)$ recursively.

Govindarajulu [4] provided the following recursive relationship:

$$b_2(n+1,p) = \frac{q(1-q^n)}{(1-q^{n+1})}\left[\frac{b_1(n,p)}{n+1} + b_2(n,p)\right] + \frac{1}{(n+1)^2}, \quad n = 0, 1 \ldots \tag{1.18}$$

with $b_2(0,p) = 0$. Then (1.17) and (1.18), together with (1.15) and (1.16), can be used to compute $B_n(p)$ and $Q_n(p)$ jointly and recursively.

To approximate $B_n(p)$ and $Q_n(p)$ the following formula from Mendenhall and Lehman [10] can be used:

$$b_k(n,p) \approx \frac{1}{n^k}\prod_{r=1}^{k}\frac{a+b-r}{a-r}, \tag{1.19}$$

where $a = (n-1)p$ and $b = (n-1)q$. Other approximations to $b_k(n,p)$ are found in [12] and [13].

When $k = 1$ the expression in (1.19) reduces to $b_1(n,p) \approx \frac{(n-2)}{n(a-1)}$ for $p > \frac{1}{n-1}$. Replacing this in (1.15) yields

$$B_n(p) \approx \frac{(1-q^n)(n-2)}{a-1}. \tag{1.20}$$

For large values of n this approximation becomes fairly accurate and may be used to replace (1.10) and (1.17).

Replacing (1.19) in (1.16) when $k = 2$ we obtain

$$Q_n(p) \approx \frac{(1-q^n)(n-2)(n-3)}{(a-1)(a-2)}.$$

For large values of n this approximation can be used to estimate $Q_n(p)$ instead of equations (1.11) and (1.18).

1.4.2 Stochastic future shares

Ramsay [11] defines the random future shares for Problems 1 and 2 as follows:

Definition 1.9 *Random future shares.* Let S_1 and S_2 be the actual share accumulated in the fund for each survivor at time t, in Problems 1 and 2, respectively. If there are no survivors then $S_1 = 0$ and $S_2 = 0$. Therefore,

$$S_1 = \begin{cases} \dfrac{n(1+i)^t}{N} & \text{if } N = 1,2,\ldots,n, \\ 0 & \text{if } N = 0, \end{cases} \tag{1.21}$$

and

$$S_2 = \begin{cases} \dfrac{\left(N\ddot{a}_{\overline{t|}} + \sum_{i=1}^{D} \ddot{a}_{\overline{K_i+1|}}\right)(1+i)^t}{N} & \text{if } N = 1,2,\ldots,n, \\ 0 & \text{if } N = 0, \end{cases} \tag{1.22}$$

where

$N =$ number of survivors at age $x+t$ from the group of n lives age x,

$D = n - N$ is the number of deaths between ages x and $x+t$,

$K_i =$ curtate future life time for the i^{th} life who died.

By assumption, the n lives are mutually independent so N has a binomial distribution with parameters $(n, {}_tp_x)$. From (1.21) and (1.22), both $E[S_1]$ and $E[S_2]$ depend on $b_1(n, {}_tp_x)$. It should be pointed out that, in addition to depending on N, S_2 also depends on the random variables K_i, for $i = 1, 2, \ldots, D$, so expectations must be taken over K_i as well.

Proposition 1.1 *(Ramsay [11])* The expectations of S_1 and S_2 defined by (1.21) and (1.22) are given by

$$E[S_1] = (1+i)^t B_n({}_tp_x), \tag{1.23}$$

$$E[S_2] = \ddot{s}_{\overline{t}|}(1 - {}_tq_x^n) + [B_n({}_tp_x) - (1 - {}_tq_x^n)]\left(\ddot{s}_{x:\overline{t}|} - \ddot{s}_{\overline{t}|}\right)\frac{{}_tp_x}{{}_tq_x}, \tag{1.24}$$

where $B_n({}_tp_x)$ is defined in (1.10).

Proof. The proof of (1.23) is immediate. For (1.24) using (1.9) we have:

$$\begin{aligned} E[S_2] &= E[E(S_2|N)] = \sum_{m=1}^{n} E[S_2|N=m]P_{nm}({}_tp_x), \\ &= (1+i)^t \sum_{m=1}^{n}\left[\frac{m\ddot{a}_{\overline{t}|} + \sum_{i=1}^{n-m} E(\ddot{a}_{\overline{K_i+1}|}|K_i < t)}{m}\right] P_{nm}({}_tp_x). \end{aligned}$$

But

$$E[\ddot{a}_{\overline{K_i+1}|}|K_i < t] = \sum_{k=0}^{t-1}\frac{{}_{k|}q_x}{{}_tq_x}\ddot{a}_{\overline{k+1}|} = \frac{\ddot{a}_{x:\overline{t}|} - {}_tp_x\,\ddot{a}_{\overline{t}|}}{{}_tq_x}. \tag{1.25}$$

Then

$$\begin{aligned} E[S_2] &= (1+i)^t \sum_{m=1}^{n}\left[\frac{m\ddot{a}_{\overline{t}|} + (n-m)\left(\ddot{a}_{x:\overline{t}|} - {}_tp_x\,\ddot{a}_{\overline{t}|}\right)/{}_tq_x}{m}\right] P_{nm}({}_tp_x), \\ &= (1+i)^t \sum_{m=1}^{n}\left[\ddot{a}_{\overline{t}|} + \frac{(n-m)}{m}\frac{\left(\ddot{a}_{x:\overline{t}|} - {}_tp_x\,\ddot{a}_{\overline{t}|}\right)}{{}_tq_x}\right] P_{nm}({}_tp_x), \\ &= \sum_{m=1}^{n}\left[\ddot{s}_{\overline{t}|} + \left(\frac{n}{m} - 1\right)\left(\ddot{s}_{x:\overline{t}|} - \ddot{s}_{\overline{t}|}\right)\frac{{}_tp_x}{{}_tq_x}\right] P_{nm}({}_tp_x), \\ &= \ddot{s}_{\overline{t}|}(1 - {}_tq_x^n) + [B_n({}_tp_x) - (1 - {}_tq_x^n)]\left(\ddot{s}_{x:\overline{t}|} - \ddot{s}_{\overline{t}|}\right)\frac{{}_tp_x}{{}_tq_x}. \end{aligned}$$

$\diamond$

To compare both, stochastic and deterministic formulas of future shares, Ramsay defines the following ratios:

$$\rho_1 = \frac{E[S_1]}{{}_t s_x} = B_n({}_t p_x)\, {}_t p_x, \tag{1.26}$$

$$\rho_2 = \frac{E[S_2]}{\ddot{s}_{x:\overline{t}|}}. \tag{1.27}$$

For numerical evaluation of ρ_1 and ρ_2 he makes the following assumptions:

(i) The same mortality law used by Bowers et. al. [1], p. 78, i.e.,

$$\mu_x = A + Bc^x, \tag{1.28}$$

where $A = 0.0007$, $B = 0.00005$ and $c = 10^{0.04}$.

(ii) Ages $x = 20, 30, 40, 50, 60$ and $x + t = 65$, so that lives must survive to age 65 in order to receive survival benefits.

(iii) A constant annual interest rate $i = 6\%$, since ρ_1 does not depend on i and ρ_2 appears to be fairly insensitive to interest rates changes.

(iv) Values of $n = 5, 10, 20, 30, 40, 50, 100, 200, 300, 400, 500$.

Ramsay computes the exact values of $B_n({}_t p_x)$ and also uses (1.20). Here we present only the exact results.

Effectively, as Ramsay points out, for large values of n, say $n > 20$, it appears that there is no need to use exact formulas since, in those cases, the traditional actuarial approach provides a very good approximation [see Tables 1.1 and 1.2]. But it is clear that for small values of n the traditional approach and Ramsay's stochastic approach are quite different.

Numerical results in Tables 1.1 and 1.2 seem to indicate that $\rho_1 > 1$ and $\rho_2 > 1$. Ramsay argues for ρ_1 that the result is due to Jensen's inequality [see Corollary 2.1]. For ρ_2 he simply points out that this makes sense from the point of view of risk and return, but provides no proof of the inequality in general. In both cases the inequalities turn out to be true only por large values of $n\, {}_t p_x$, but not in general. This point is further discussed in Chapter 2.

Table 1.1: ρ_1 when (x) survives to age 65

n	$x = 20$	$x = 30$	$x = 40$	$x = 50$	$x = 60$
5	1.08105	1.07628	1.06849	1.05344	1.02311
10	1.03317	1.03117	1.02798	1.02197	1.00986
20	1.01503	1.01416	1.01277	1.01011	1.00461
30	1.00973	1.00918	1.00828	1.00657	1.00301
40	1.00720	1.00679	1.00613	1.00487	1.00224
50	1.00571	1.00539	1.00487	1.00387	1.00178
100	1.00281	1.00265	1.00240	1.00191	1.00088
200	1.00139	1.00132	1.00119	1.00095	1.00044
300	1.00093	1.00087	1.00079	1.00063	1.00029
400	1.00069	1.00066	1.00059	1.00047	1.00022
500	1.00055	1.00052	1.00047	1.00038	1.00017

Table 1.2: ρ_2 when (x) survives to age 65

n	$x = 20$	$x = 30$	$x = 40$	$x = 50$	$x = 60$
5	1.07305	1.06573	1.05502	1.03878	1.01547
10	1.02991	1.02688	1.02250	1.01596	1.00661
20	1.01355	1.01221	1.01027	1.00734	1.00309
30	1.00878	1.00792	1.00666	1.00477	1.00202
40	1.00649	1.00586	1.00493	1.00354	1.00150
50	1.00515	1.00465	1.00391	1.00281	1.00119
100	1.00253	1.00229	1.00193	1.00139	1.00059
200	1.00126	1.00113	1.00096	1.00069	1.00029
300	1.00084	1.00075	1.00064	1.00046	1.00019
400	1.00063	1.00057	1.00048	1.00034	1.00015
500	1.00050	1.00045	1.00038	1.00027	1.00012

Chapter 2

Further Results on Ramsay's Definitions

Some additional properties of the polynomials B_n and Q_n, given by (1.10) and (1.11), are shown here. Ramsay's definitions of random future shares presented in Chapter 1 are also further discussed.

2.1 Preliminary technical results

Lemma 2.1 *(Jensen's inequality)* Let X be a random variable in (a, b) with $\mu = E[X] < \infty$. Let f be a function such that $f''(x) > 0$, for all $x \in (a, b)$. Then:

$$E[f(X)] \geq f(\mu).$$

Proof. For each $x \in (a, b)$ there exists a ζ_x between μ and x such that

$$f(x) = f(\mu) + f'(\mu)(x - \mu) + \frac{1}{2} f''(\zeta_x)(x - \mu)^2 \geq f(\mu) + f'(\mu)(x - \mu).$$

Then $f(X) \geq f(\mu) + f'(\mu)(X - \mu)$. Thus, $E[f(X)] \geq f(\mu) + 0 = f(\mu)$. $\diamond$

Corollary 2.1 Let X be a positive random variable such that $E[X] < \infty$. Then:

$$E\left[\frac{1}{X}\right] \geq \frac{1}{E[X]}.$$

Proof. Apply Lemma 2.1 to $f(x) = 1/x$ for $x > 0$. $\diamond$

Lemma 2.2 Let X be a random variable such that $X \geq 0$, $0 < \rho = Pr[X = 0] < 1$ and $V[X] < \infty$. For any pair of real numbers a and b define the random variable

$$\begin{aligned} Y &= X + (a-b)I_{[X>0]} + b, \\ &= \begin{cases} X + a & \text{if } X > 0, \\ b & \text{if } X = 0, \end{cases} \end{aligned}$$

where I is an indicator function. Then $V[Y]$, as a function of a and b, is minimal when $a = b - E[X]/(1-\rho)$. In this case $E[Y] = b$ and $V[Y] = V[X] - E[X]^2\rho/(1-\rho)$.

Proof. The variance of Y is given by

$$\begin{aligned} V[Y] &= V[X] + V[(a-b)I_{[X>0]}] + 2Cov[X, (a-b)I_{(X>0)}], \\ &= V[X] + \rho(1-\rho)(a-b)^2 + 2\rho E[X](a-b). \end{aligned} \tag{2.1}$$

Then $V[Y]$ is seen to be a second degree polynomial in $(a-b)$, with coefficient $\rho(1-\rho) > 0$. This means that $V[Y]$ reaches a minimun when $a - b = -2\rho E[X]/[2\rho(1-\rho)] = -E[X]/(1-\rho)$, i.e., $a = b - E[X]/(1-\rho)$. Replacing this expression in (2.1) we get $V[Y] = V[X] - E[X]^2\rho/(1-\rho)$. Finally $E[Y] = E[X] + (a-b)(1-\rho) + b = E[X] - E[X] + b = b$. ◇

Lemma 2.3 Let K be the curtate future life time of (x) and t an integer. Then

$$a_1 = E[\ddot{a}_{\overline{K+1}|}|K < t] = \frac{\ddot{a}_{x:\overline{t}|} - {}_tp_x\, \ddot{a}_{\overline{t}|}}{{}_tq_x}, \tag{2.2}$$

$$v_1 = V[\ddot{a}_{\overline{K+1}|}|K < t] = \frac{1}{{}_tq_x}\left[\frac{{}^2A_{x:\overline{t}|} - A^2_{x:\overline{t}|}}{d^2} - \left(\ddot{a}_{x:\overline{t}|} - \ddot{a}_{\overline{t}|}\right)^2 \frac{{}_tp_x}{{}_tq_x}\right]. \tag{2.3}$$

Proof. (2.2) was proved in (1.25). To prove (2.3) we consider the random variable Y given in (1.1). We see that

$$\begin{aligned} \sum_{k=0}^{t-1} {}_{k|}q_x \ddot{a}^2_{\overline{k+1}|} &= E[Y^2] - {}_tp_x\, \ddot{a}^2_{\overline{t}|} = V[Y] + E[Y]^2 - {}_tp_x\, \ddot{a}^2_{\overline{t}|}, \\ &= \frac{{}^2A_{x:\overline{t}|} - A^2_{x:\overline{t}|}}{d^2} + \ddot{a}^2_{x:\overline{t}|} - {}_tp_x\, \ddot{a}^2_{\overline{t}|}, \text{ from (1.8) and (1.2).} \end{aligned}$$

Therefore,

$$E[\ddot{a}^2_{\overline{K+1}|}|K<t] \;=\; \sum_{k=0}^{t-1} {}_{k|}q_x \ddot{a}^2_{\overline{k+1}|}/{}_tq_x \;=\; \left[\frac{{}^2A_{x:\overline{t}|}-A^2_{x:\overline{t}|}}{d^2}+\ddot{a}^2_{x:\overline{t}|}-{}_tp_x\,\ddot{a}^2_{\overline{t}|}\right]/\,{}_tq_x\,.$$

This and (2.2) yield

$$\begin{aligned}
v_1 &= E[\ddot{a}^2_{\overline{K+1}|}|K<t]-E[\ddot{a}_{\overline{K+1}|}|K<t]^2,\\
&= \left[\frac{{}^2A_{x:\overline{t}|}-A^2_{x:\overline{t}|}}{d^2}+\ddot{a}^2_{x:\overline{t}|}-{}_tp_x\,\ddot{a}^2_{\overline{t}|}\right]/\,{}_tq_x-\left(\frac{\ddot{a}_{x:\overline{t}|}-{}_tp_x\,\ddot{a}_{\overline{t}|}}{{}_tq_x}\right)^2,\\
&= \frac{1}{{}_tq_x}\left[\frac{{}^2A_{x:\overline{t}|}-A^2_{x:\overline{t}|}}{d^2}+\ddot{a}^2_{x:\overline{t}|}-{}_tp_x\,\ddot{a}^2_{\overline{t}|}-\frac{\left(\ddot{a}_{x:\overline{t}|}-{}_tp_x\,\ddot{a}_{\overline{t}|}\right)^2}{{}_tq_x}\right],\\
&= \frac{1}{{}_tq_x}\left[\frac{{}^2A_{x:\overline{t}|}-A^2_{x:\overline{t}|}}{d^2}-\left(\ddot{a}_{x:\overline{t}|}-\ddot{a}_{\overline{t}|}\right)^2\frac{{}_tp_x}{{}_tq_x}\right].
\end{aligned}$$

◇

Lemma 2.4 Let $0 \le p \le 1$, $n \ge 1$ and $P_{nm}(p)$ as in (1.9). Then

$$0 \le z \le \frac{3}{2}\sqrt{np(1-p)} \;\Rightarrow\; \sum_{|m-np|\ge 2z\sqrt{np(1-p)}} P_{nm}(p) \le 2e^{-z^2}.$$

Proof. It can be found in Lorentz [7], pp. 18-19. ◇

Corollary 2.2 Let $0 < a < b < 1$ and $0 < \delta \le \min\{a(1-a), b(1-b)\}$. Then

$$\sum_{|\frac{m}{n}-p|\ge\delta} P_{nm}(p) \le 2e^{-n\delta^2}, \quad \text{for all } p \in [a,b].$$

Proof. For any $p \in [a,b]$ the inequalities $\delta \le 3p(1-p)$ and $4p(1-p) \le 1$ hold. Now

$$\left|\frac{m}{n}-p\right| \ge \delta \Leftrightarrow |m-np| \ge \delta n = \frac{2\delta\sqrt{n}}{2\sqrt{p(1-p)}}\sqrt{np(1-p)} = 2z\sqrt{np(1-p)},$$

where $z = \dfrac{\delta\sqrt{n}}{2\sqrt{p(1-p)}}$. Since $0 \le z \le \dfrac{3p(1-p)\sqrt{n}}{2\sqrt{p(1-p)}} = \dfrac{3}{2}\sqrt{np(1-p)}$, then Lemma 2.4 yields

$$\begin{aligned}
\sum_{|\frac{m}{n}-p|\ge\delta} P_{nm}(p) &= \sum_{|m-np|\ge 2z\sqrt{np(1-p)}} P_{nm}(p),\\
&\le 2e^{-z^2} = 2e^{\frac{-n\delta^2}{4p(1-p)}} \le 2e^{-n\delta^2}.
\end{aligned}$$

◇

Lemma 2.5 Let $n \geq 2$ be an integer.

(i) If $0 < p \leq 1$ then $(1+np)(1-p)^n < 1$.

(ii) If $n \geq 3$ and $2/n \leq p \leq 1$ then $[2(n-1)p+1](1-p)^{n-1} < 1$.

(iii) $r > 0$ and $0 \leq p \leq 1 \Rightarrow p^r(1-p)^n \leq \left(\frac{r}{n+r}\right)^r \left(1 - \frac{r}{n+r}\right)^n < \left(\frac{r}{n}\right)^r e^{-r}$.

(iv) If $0 < p \leq 1$ then $\left[\frac{n(n-1)}{2}p^2 + (n-1)p + 1\right](1-p)^{n-1} < 1$.

(v) $0 < z < 1 \Rightarrow \frac{z}{\sqrt{1-z}} + \ln(1-z) > 0$. Also, for $n \geq 1$ and $0 < x < n$:

$$\frac{x}{n-x} + \ln\left(1 - \frac{x}{n}\right) > 0. \tag{2.4}$$

(vi) For $0 < q < 1$, $p = 1-q$ and $k = 1, 2, \cdots$,

$$q^{-k} \geq 1 + k\left(\frac{p}{q}\right) + \frac{1}{2}k(k-1)\left(\frac{p}{q}\right)^2 \geq 1 + k\left(\frac{p}{q}\right). \tag{2.5}$$

Proof. In (i)-(iv) we may assume if necessary that $p < 1$ since if $p = 1$ all these inequalities are trivialy true.

(i) If $f(p) = (1+np)(1-p)^n$ for $0 \leq p < 1$, then $f'(p) < 0$. Hence $f(p) < f(0) = 1$.
(ii) Let $f(p) = [2(n-1)p+1](1-p)^{n-1}$. Then $f'(p) < 0$ for $\frac{1}{2n} < p \leq 1$. Since $1/(2n) < 2/n$ then $f(p) \leq f(2/n) = \left(\frac{5n-4}{n-2}\right)\left(1 - \frac{2}{n}\right)^n$, for $2/n \leq p \leq 1$. From this we see that $f(p) < 1$ for $n = 3, 4$. If $n \geq 5$ then $f(p) < \left(\frac{5(5)-4}{5-2}\right)e^{-2} < 1$.
(iii) The first inequality is due to the fact that the function $u(p) = p^r(1-p)^n$, for $0 \leq p \leq 1$, has a global maximum at $p = r/(n+r)$. The second inequality follows since the function $h(x) = (1-1/x)^x$ is increasing for $x \geq 1$ with $\lim_{x\to\infty} h(x) = e^{-1}$.
(iv) Let $h(p) = \left[\frac{n(n-1)}{2}p^2 + (n-1)p + 1\right](1-p)^{n-1}$ for $0 < p < 1$. Then

$$\begin{aligned} h'(p) &= (n-1)(1-p)^{n-2}\left\{(np+1)(1-p) - \left[\frac{n(n-1)p^2}{2} + (n-1)p + 1\right]\right\}, \\ &= (n-1)(1-p)^{n-2}\left[np + 1 - np^2 - p - \frac{n(n-1)p^2}{2} - np + p - 1\right], \\ &= -(n-1)(1-p)^{n-2}\left(n + \frac{n(n-1)}{2}\right)p^2 < 0. \end{aligned}$$

This yields $h(p) < h(0) = 1$ for $0 < p \leq 1$.

(v) The function $f(z) = \frac{z}{\sqrt{1-z}} + \ln(1-z)$ for $0 \leq z < 1$ is such that $f'(z) > 0$ for $0 < z < 1$. Therefore, $f(z) > f(0) = 0$. This proves the first inequality. Using this with $z = x/n$, and since $1 - z < \sqrt{1-z}$ for $0 < z < 1$, we conclude that (2.4) is also true (for more general results see [9]; p. 272; theorem 3.6.15).

(vi) It follows from the identity $q^{-k} = (1+p/q)^k = \sum_{j=0}^{k} \binom{k}{j} \left(\frac{p}{q}\right)^j$. ◇

Corollary 2.3 (i) $K_n(x) = \left(1 - \frac{x}{n}\right)^{1/x}$ is decreasing in $]0, n]$ for $n \geq 1$.

(ii) $F_n(x) = \left[\frac{(n-x)(n-x-1)}{n(n-1)}\right]^{1/x}$ is decreasing in $]0, n-1]$ for $n \geq 2$.

Proof. **(i)** If $R_n(x) = \ln[K_n(x)]$ for $0 < x < n$, then from (2.4)

$$R_n'(x) = -\frac{\frac{x}{n-x} + \ln\left(1 - \frac{x}{n}\right)}{x^2} < 0.$$

(ii) It is true since $F_n(x) = K_n(x)K_{n-1}(x)$. ◇

2.2 Some properties of the B_n and Q_n polynomials

In this section we present some properties of the polynomials B_n and Q_n defined in (1.10) and (1.11). They are related with the first two inverse moments of a positive binomial variate according to (1.15) and (1.16).

2.2.1 Recursive equations for B_n and Q_n

Proposition 2.1 The polynomials B_n, given by (1.10), satisfy the recursive equation:

$$B_{n+1}(p) = q\left(\frac{n+1}{n}\right) B_n(p) + 1 - q^{n+1}, \quad n = 1, 2, \ldots \tag{2.6}$$

Proof. From (1.10) and the identity $\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}$, for $n \geq 1$, $m = 1, \ldots, n$:

$$B_{n+1}(p) = \sum_{m=1}^{n+1} \frac{(n+1)}{m} \binom{n+1}{m} p^m q^{n+1-m},$$

$$
\begin{aligned}
&= p^{n+1} + \sum_{m=1}^{n} \frac{(n+1)}{m} \binom{n+1}{m} p^m q^{n+1-m}, \\
&= p^{n+1} + \sum_{m=1}^{n} \frac{(n+1)}{m} \left[\binom{n}{m} + \binom{n}{m-1} \right] p^m q^{n+1-m}, \\
&= p^{n+1} + q \left(\frac{n+1}{n} \right) \sum_{m=1}^{n} \frac{n}{m} \binom{n}{m} p^m q^{n-m} + \sum_{m=1}^{n} \frac{(n+1)}{m} \binom{n}{m-1} p^m q^{n+1-m}, \\
&= q \left(\frac{n+1}{n} \right) B_n(p) + \sum_{m=1}^{n+1} \binom{n+1}{m} p^m q^{n+1-m}, \\
&= q \left(\frac{n+1}{n} \right) B_n(p) + 1 - q^{n+1}. \qquad \diamondsuit
\end{aligned}
$$

Note that this proposition also follows from (1.15) when used with (1.17). The formula in (2.6), together with $B_1(p) = p$, can be used to compute $B_n(p)$ recursively.

Proposition 2.2 The polynomials Q_n, given by (1.11),satisfy the recursive equation:

$$
Q_{n+1}(p) = q \left(\frac{n+1}{n} \right)^2 Q_n(p) + B_{n+1}(p), \quad n = 1, 2, \ldots
$$

where B_n is given in (1.10).

Proof. Using the relation $\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}$ we have

$$
\begin{aligned}
Q_{n+1}(p) &= \sum_{m=1}^{n+1} \frac{(n+1)^2}{m^2} \binom{n+1}{m} p^m q^{n+1-m}, \\
&= p^{n+1} + \sum_{m=1}^{n} \frac{(n+1)^2}{m^2} \left[\binom{n}{m} + \binom{n}{m-1} \right] p^m q^{n+1-m}, \\
&= p^{n+1} + \sum_{m=1}^{n} \frac{(n+1)^2}{m^2} \binom{n}{m} p^m q^{n+1-m} + \sum_{m=1}^{n} \frac{(n+1)^2}{m^2} \binom{n}{m-1} p^m q^{n+1-m}, \\
&= q \left(\frac{n+1}{n} \right)^2 \sum_{m=1}^{n} \frac{n^2}{m^2} \binom{n}{m} p^m q^{n-m} + \sum_{m=1}^{n} \frac{n+1}{m} \binom{n+1}{m} p^m q^{n+1-m} + p^{n+1}, \\
&= q \left(\frac{n+1}{n} \right)^2 Q_n(p) + \sum_{m=1}^{n+1} \frac{n+1}{m} \binom{n+1}{m} p^m q^{n+1-m}, \\
&= q \left(\frac{n+1}{n} \right)^2 Q_n(p) + B_{n+1}(p). \qquad \diamondsuit
\end{aligned}
$$

This proposition also follows from (1.16) when used with (1.18). It together with (2.6) and $B_1(p) = Q_1(p) = p$, can be used to compute $B_n(p)$ and $Q_n(p)$ jointly and recursively.

2.2.2 An exact formula for B_n

Proposition 2.3 The polynomials B_n can be written as powers of $q = 1 - p$:

$$B_n(p) = \sum_{m=0}^{n-1} \frac{n}{n-m} q^m - q^n \sum_{m=1}^{n} \frac{n}{m}.$$

Proof. Using $p^m = (1-q)^m = \sum_{k=0}^{m} \binom{m}{k} (-q)^k$ in (1.10) we have

$$\begin{aligned} B_n(p) &= \sum_{m=1}^{n} \frac{n}{m} \binom{n}{m} (1-q)^m q^{n-m}, \\ &= \sum_{m=1}^{n} \frac{n}{m} \binom{n}{m} \left[\sum_{k=0}^{m} \binom{m}{k} (-q)^k \right] q^{n-m}, \\ &= \sum_{m=1}^{n} \sum_{k=0}^{m} \frac{n}{m} \binom{n}{m} \binom{m}{k} (-1)^k q^{n-m+k}. \end{aligned}$$

Rearranging the sums according to terms with common power of q yields

$$B_n(p) = \sum_{m=1}^{n-1} \underbrace{\left[\sum_{k=0}^{m} \frac{n(-1)^k}{n-m+k} \binom{n}{n-m+k} \binom{n-m+k}{k} \right]}_{a_m} q^m + \underbrace{\left[\sum_{k=1}^{n} \frac{n}{k} \binom{n}{k} (-1)^k \right]}_{a_n} q^n.$$

We need to show that $a_m = \frac{n}{n-m}$ and $a_n = -\sum_{m=1}^{n} \frac{n}{m}$. To this

$$\begin{aligned} a_m &= n \binom{n}{m} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{n-m+k}, \\ &= (-1)^{n-m} n \binom{n}{m} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{n-m+k}}{n-m+k}, \\ &= (-1)^{n-m} n \binom{n}{m} \sum_{k=0}^{m} \binom{m}{k} \int_0^{-1} t^{n-m+k-1} dt, \\ &= (-1)^{n-m} n \binom{n}{m} \int_0^{-1} \left[\sum_{k=0}^{m} \binom{m}{k} t^{n-m+k-1} \right] dt, \\ &= (-1)^{n-m} n \binom{n}{m} \int_0^{-1} \left[t^{n-m-1} \sum_{k=0}^{m} \binom{m}{k} t^k \right] dt, \\ &= (-1)^{n-m} n \binom{n}{m} \int_0^{-1} t^{n-m-1} (1+t)^m dt, \\ &= (-1)^{2(n-m)} n \binom{n}{m} \int_0^{1} z^{n-m-1} (1-z)^m dz, \quad \text{change of variable } z = -t, \\ &= n \binom{n}{m} \frac{\Gamma(n-m)\Gamma(m+1)}{\Gamma(n+1)} \int_0^1 \frac{\Gamma(n+1)}{\Gamma(n-m)\Gamma(m+1)} z^{n-m-1} (1-z)^{m+1-1} dz, \end{aligned}$$

$$
\begin{aligned}
&= n\binom{n}{m}\frac{\Gamma(n-m)\Gamma(m+1)}{\Gamma(n+1)}(1), \quad \text{since the latter has a beta distribution,}\\
&= \frac{n}{n-m}, \quad \text{because } \Gamma(r+1) = r! \text{ for all integer } r.
\end{aligned}
$$

For $n \geq 2$ we have:

$$
\begin{aligned}
a_n &= \sum_{k=0}^{n-1} \frac{n}{k+1}\binom{n}{k+1}(-1)^{k+1},\\
&= \sum_{k=0}^{n-2} \frac{n}{k+1}\left[\binom{n-1}{k+1} + \binom{n-1}{k}\right](-1)^{k+1} + (-1)^n,\\
&= \sum_{k=0}^{n-2} \frac{n}{k+1}\binom{n-1}{k+1}(-1)^{k+1} + \sum_{k=0}^{n-2} \frac{n}{k+1}\binom{n-1}{k}(-1)^{k+1} + (-1)^n,\\
&= \sum_{k=1}^{n-1} \frac{n}{k}\binom{n-1}{k}(-1)^{k} - \sum_{k=0}^{n-1} \frac{n}{k+1}\binom{n-1}{k}(-1)^{k} + (-1)^{n-1} + (-1)^n,\\
&= \frac{n}{n-1}\sum_{k=1}^{n-1} \frac{n-1}{k}\binom{n-1}{k}(-1)^{k} + \sum_{k=0}^{n-1}\binom{n}{k+1}(-1)^{k+1},\\
&= \left(\frac{n}{n-1}\right)a_{n-1} + (1-1)^n - 1,\\
&= \left(\frac{n}{n-1}\right)a_{n-1} - 1.
\end{aligned}
$$

Hence, $a_n = \left(\frac{n}{n-1}\right)a_{n-1} - 1$. This recursive relation and $a_1 = -1$ imply $a_n = -\sum_{k=1}^{n} \frac{n}{k}$, for all $n \geq 1$. ◇

Formula in Proposition 2.3 is the counterpart of the one cited without proof by Stephan [12] for $b_1(n,p)$. This also can be proved using (2.6) and used to compute $B_n(p)$ easily.

2.2.3 Limit properties of the B_n and Q_n polynomials

As pointed out in Subsection 1.4.1, the polynomials B_n and Q_n are the Bernstein polynomials of the functions given in (1.12). But since these functions are not bounded at 0 then we can not ensure that their Bernstein polynomials converge uniformly in $[0,1]$ nor in $]0,1]$. But they do in any closed interval $[a,1]$ as stated in the following propositions.

Proposition 2.4 For $0 < a < 1$, the polynomials B_n given in (1.10) satisfy:

$$\lim_{n\to\infty} B_n(p) = \frac{1}{p}, \text{ uniformly in } [a,1],$$
$$\lim_{n\to\infty} B_n(p) = \frac{1}{p}, \text{ point-wise in }]0,1]. \tag{2.7}$$

Proof. For simplicity suppose that $a < 1/2$. Let $1/2 < b < 1$ and $\epsilon > 0$. Denote by $c = a/2$ and $d = (1+b)/2$. Since the function $f(p) = 1/p$ is uniformly continuous in $[c,d]$ there exists a $\delta_0 > 0$ such that

$$\left|\frac{1}{x} - \frac{1}{y}\right| < \epsilon/3, \text{ for all } x,y \in [c,d] \text{ and } |x-y| < \delta_0. \tag{2.8}$$

Let $\delta < \min\{\delta_0, c(1-c), d(1-d)\}$. This makes sure that if $p \in [a,b]$ and $|x-p| < \delta$ then $x,p \in [c,d]$ and (2.8) holds for x and p. Also, since $\delta < \min\{a(1-a), b(1-b)\}$ then the inequality of Corollary 2.2 holds in $[a,b]$. Besides, there exists n_0 such that for all $n \geq n_0$

$$\frac{q^n}{p} < \epsilon/3, \text{ for all } p \in [a,b], \tag{2.9}$$
$$2ne^{-n\delta^2} < \epsilon/3. \tag{2.10}$$

Let $p \in [a,b]$ and $n \geq n_0$. It is clear that

$$\left|\frac{1}{m/n} - \frac{1}{p}\right| \leq n, \text{ for all } m = 1,2,...,n. \tag{2.11}$$

Then

$$\begin{aligned}
\left|B_n(p) - \frac{1}{p}\right| &= \left|B_n(p) - \frac{1}{p} + \frac{q^n}{p} - \frac{q^n}{p}\right|, \\
&= \left|B_n(p) - \frac{1}{p}(1-q^n) - \frac{q^n}{p}\right|, \\
&\leq \left|B_n(p) - \frac{1}{p}(1-q^n)\right| + \frac{q^n}{p}, \text{ by the triangular inequality,} \\
&< \left|\sum_{m=1}^{n} \frac{n}{m} P_{nm}(p) - \sum_{m=1}^{n} \frac{1}{p} P_{nm}(p)\right| + \epsilon/3, \text{ from (1.10) and (2.9),} \\
&= \left|\sum_{m=1}^{n} \left[\frac{1}{m/n} - \frac{1}{p}\right] P_{nm}(p)\right| + \epsilon/3,
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{m=1}^{n} \left| \frac{1}{m/n} - \frac{1}{p} \right| P_{nm}(p) + \epsilon/3, \text{ by the triangular inequality,} \\
&= \sum_{|\frac{m}{n}-p|<\delta} \left| \frac{1}{m/n} - \frac{1}{p} \right| P_{nm}(p) + \sum_{|\frac{m}{n}-p|\geq\delta} \left| \frac{1}{m/n} - \frac{1}{p} \right| P_{nm}(p) + \epsilon/3.
\end{aligned}
$$

For the first term:

$$
\begin{aligned}
\sum_{|\frac{m}{n}-p|<\delta} \left| \frac{1}{m/n} - \frac{1}{p} \right| P_{nm}(p) &\leq (\epsilon/3) \sum_{|\frac{m}{n}-p|<\delta} P_{nm}(p), \quad \text{by (2.8)}, \\
&\leq (\epsilon/3) \sum_{m=0}^{n} P_{nm}(p) \quad = \quad \epsilon/3.
\end{aligned}
$$

For the second term:

$$
\begin{aligned}
\sum_{|\frac{m}{n}-p|\geq\delta} \left| \frac{1}{m/n} - \frac{1}{p} \right| P_{nm}(p) &\leq n \sum_{|\frac{m}{n}-p|\geq\delta} P_{nm}(p), \quad \text{by (2.11)}, \\
&\leq 2ne^{-n\delta^2}, \quad \text{by Corollary 2.2}, \\
&< \epsilon/3, \quad \text{by (2.10)}.
\end{aligned}
$$

Therefore,

$$
\left| B_n(p) - \frac{1}{p} \right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \quad \text{for all } p \in [a,b],\ n > n_0.
$$

Hence, $B_n(p) \longrightarrow \frac{1}{p}$ uniformly in $[a,b]$. Since this holds for any b as close as possible to 1 and $B_n(1) = 1$ continuously, then it holds in $[a,1]$ as well and so $B_n(p) \longrightarrow \frac{1}{p}$ uniformly in $[a,1]$. This also implies (2.7). ◇

We present here an alternative proof of Proposition 2.4.

Proof. Let $0 < \epsilon < a$. There exists n_1 such that for all $n > n_1$:

$$
\begin{cases}
(1-a)^n < \epsilon a, \\
n\left[\ln(n) + 1\right](1-a)^n < \epsilon, \\
\left(\frac{k}{n-k}\right)\left(\frac{1-a}{1-\epsilon}\right)^k < \epsilon^2, \quad k = 0, 1, ..., n-1.
\end{cases}
\tag{2.12}
$$

From Proposition 2.3

$$
B_n(p) \quad = \quad \sum_{k=0}^{n-1} \left(1 + \frac{k}{n-k}\right) q^k - q^n \sum_{k=1}^{n} \frac{n}{k},
$$

$$
\begin{aligned}
&= \frac{1-q^n}{p} + \sum_{k=0}^{n-1} \frac{k}{n-k} q^k - q^n \sum_{k=1}^{n} \frac{n}{k}, \qquad (2.13)\\
&= \frac{1}{p} - \frac{q^n}{p} + \sum_{k=0}^{n-1} \frac{k}{n-k} q^k - q^n \sum_{k=1}^{n} \frac{n}{k}.
\end{aligned}
$$

Then for $a \leq p \leq 1$ and $n > n_1$,

$$
\begin{aligned}
\left| B_n(p) - \frac{1}{p} \right| &= \left| -\frac{q^n}{p} + \sum_{k=0}^{n-1} \frac{k}{n-k} q^k - q^n \sum_{k=1}^{n} \frac{n}{k} \right|, \\
&\leq \frac{q^n}{p} + \sum_{k=0}^{n-1} \frac{k}{n-k} q^k + q^n \sum_{k=1}^{n} \frac{n}{k}, \text{ by the triangular inequality,} \\
&\leq \frac{(1-a)^n}{a} + \sum_{k=0}^{n-1} \frac{k}{n-k} (1-a)^k + (1-a)^n \sum_{k=1}^{n} \frac{n}{k}, \\
&\leq \frac{(1-a)^n}{a} + \sum_{k=0}^{n-1} \left[\frac{k}{n-k} \left(\frac{1-a}{1-\epsilon} \right)^k (1-\epsilon)^k \right] + n[\ln(n)+1](1-a)^n, \\
&\leq \epsilon + \epsilon^2 \sum_{k=0}^{n-1} (1-\epsilon)^k + \epsilon, \text{ from (2.12),} \\
&= 2\epsilon + \epsilon^2 \left[\frac{1-(1-\epsilon)^n}{\epsilon} \right] < 3\epsilon.
\end{aligned}
$$

◇

Proposition 2.5 For $0 < a < 1$, the polynomials Q_n defined in (1.11) satisfy

$$
\begin{aligned}
\lim_{n\to\infty} Q_n(p) &= \frac{1}{p^2}, \text{ uniformly in } [a, 1], \\
\lim_{n\to\infty} Q_n(p) &= \frac{1}{p^2}, \text{ point-wise in }]0, 1]. \qquad (2.14)
\end{aligned}
$$

Proof. It can be proved in a similar way to that of Proposition 2.4. ◇

2.2.4 Further properties of the B_n polynomials

By studying the first and second derivatives of B_n we find some inequalities as well as some increase and convexity properties for these polynomials.

Corollary 2.4 For $n \geq 1$ the following inequality holds

$$
B_n(p) \geq \frac{(1-q^n)^2}{p}.
$$

Proof. It follows from (1.15) and Corollary 2.1. ◇

Proposition 2.6 For each $n \geq 2$ there exists a unique $0 < \alpha_n < 1$ such that

(i) $B_n(\alpha_n) = \dfrac{1-(1-\alpha_n)^n}{\alpha_n}$.

(ii) $B_n(p) < B_n(\alpha_n)$, $\forall\, p \in [0,1], p \neq \alpha_n$.

(iii) B_n increases in $[0,\alpha_n]$ and decreases in $]\alpha_n, 1]$.

(iv) $B_n(p) < \dfrac{1-q^n}{p}$, $\forall\, p \in]0,\alpha_n[$ and $B_n(p) > \dfrac{1-q^n}{p}$, $\forall\, p \in]\alpha_n, 1[$.

(v) The following inequalities hold for $n \geq 3$:

$$\frac{1}{n} \leq \alpha_n \leq u_n = \frac{\sqrt{3(n-1)(7n-11)} - 3(n-1)}{\sqrt{3(n-1)(7n-11)} + (n-1)(n-5)} \leq \frac{2}{n+1}. \quad (2.15)$$

(vi) $\{\alpha_n\}$ is decreasing.

(vii) $\lim_{n\to\infty} \alpha_n = 0$.

Proof. For $0 < p < 1$:

$$\begin{aligned} B_n'(p) &= n\left[\sum_{k=0}^{n-1} \frac{q^k}{n-k} - q^n \sum_{k=1}^{n} \frac{1}{k}\right]', \text{ from Proposition 2.3,} \\ &= n\left[\sum_{k=0}^{n-1} \frac{-kq^{k-1}}{n-k} + nq^{n-1} \sum_{k=1}^{n} \frac{1}{k}\right], \text{ since } q = 1-p, \quad (2.16) \\ &= -\frac{n}{q}\left[B_n(p) - \frac{1-q^n}{p}\right], \text{ by (2.13),} \\ &= -nq^{n-1} f(p), \end{aligned}$$

where

$$\begin{aligned} f(p) &= \sum_{k=0}^{n-1} \frac{k}{n-k} q^{-(n-k)} - \sum_{k=1}^{n} \frac{n}{k}, \quad (2.17) \\ &= \frac{B_n(p) - \frac{1-q^n}{p}}{q^n}. \quad (2.18) \end{aligned}$$

This function is continuous, strictly increasing in $]0,1[$, $\lim_{p\to 0^+} f(p) = -n < 0$ and

$$\lim_{p\to 1^-} f(p) = +\infty. \quad (2.19)$$

Thus, by the mean value theorem there exists a unique $\alpha_n \in]0,1[$ such that

$$f(p) < f(\alpha_n) = 0 < f(t), \tag{2.20}$$

for all $p \in]0, \alpha_n[$ and all $t \in]\alpha_n, 1[$. From (2.20) and (2.18) the proof of (i)-(iv) is complete.

(v) We may write (2.17) as

$$f(p) = \sum_{k=1}^{n} \left[\frac{(n-k)q^{-k} - n}{k} \right]. \tag{2.21}$$

Then we get $B_n'(p) > 0$ if $(n-k)q^{-k} - n < 0$, for all $k = 1, ..., n$. From Corollary 2.3 - (i), we get $q > \max_{\{k=1,\ldots,n\}} \left(\frac{n-k}{n}\right)^{1/k} = 1 - 1/n$. This yields $p < 1/n$ and so $1/n \le \alpha_n$, which proves the first inequality of (2.15). Use of (2.5) in (2.21) yields

$$\begin{aligned} f(p) &\ge \sum_{k=1}^{n} \left\{ \frac{(n-k)\left[1 + k\left(\frac{p}{q}\right) + \frac{1}{2}k(k-1)\left(\frac{p}{q}\right)^2\right] - n}{k} \right\}, \\ &\ge \sum_{k=1}^{n} \left\{ \frac{(n-k)\left[1 + k\left(\frac{p}{q}\right)\right] - n}{k} \right\}, \end{aligned}$$

or

$$\begin{aligned} f(p) &\ge \underbrace{n\left[\frac{1}{12}(n-1)(n-2)\left(\frac{p}{q}\right)^2 + \frac{1}{2}(n-1)\left(\frac{p}{q}\right) - 1\right]}_{A_2}, \qquad (2.22) \\ &\ge \underbrace{n\left[\frac{1}{2}(n-1)\left(\frac{p}{q}\right) - 1\right]}_{A_1}. \end{aligned}$$

Solving $A_2 > 0$ for $n \ge 3$ we get $f(p) > 0$, i.e., $B'(p) < 0$, for

$$p > u_n = \frac{\sqrt{3(n-1)(7n-11)} - 3(n-1)}{\sqrt{3(n-1)(7n-11)} + (n-1)(n-5)}.$$

This proves the second inequality of (2.15). Now solving $A_1 > 0$ for $n \ge 3$ we get $B'(p) < 0$ for $p > \frac{2}{n+1}$. This means that $\alpha_n \le \frac{2}{n+1}$. The last inequality of (2.15) follows since $A_2 \ge A_1$.

(vi) By (iv) it is enough to prove that $B_{n+1}(\alpha_n) > \frac{1-(1-\alpha_n)^{n+1}}{\alpha_n}$. To this let $p = \alpha_n$ and $q = 1 - \alpha_n$. Then

$$\begin{aligned}
B_{n+1}(p) &= q\left(\frac{n+1}{n}\right) B_n(p) + 1 - q^{n+1}, \text{ by (2.6)}, \\
&= q\left(\frac{n+1}{n}\right)\left(\frac{1-q^n}{p}\right) + 1 - q^{n+1}, \text{ by (i)}, \\
&= \left(1+\frac{1}{n}\right)\left(\frac{q-q^{n+1}}{p}\right) + 1 - q^{n+1}, \\
&= \left(1+\frac{1}{n}\right)\left(\frac{1-q^{n+1}}{p} - 1\right) + 1 - q^{n+1}, \\
&= \frac{1-q^{n+1}}{p} - 1 + \frac{1}{n}\left(\frac{q-q^{n+1}}{p}\right) + 1 - q^{n+1}, \\
&= \frac{1-q^{n+1}}{p} + \frac{q}{np}\left[1 - q^n - npq^n\right], \\
&= \frac{1-q^{n+1}}{p} + \frac{q}{np}\left[1 - (np+1)q^n\right], \\
&> \frac{1-q^{n+1}}{p}, \text{ by Lemma 2.5 - (i)}.
\end{aligned}$$

(vii) It follows from (v). ◇

Table 2.1: Values of α_n, lower and upper bounds and $1.5/n$

n	$1/n$	α_n	u_n	$2/(n+1)$	$1.5/n$
5	0.200000	0.288677	0.292893	0.333333	0.300000
10	0.100000	0.147419	0.152067	0.181818	0.150000
20	0.050000	0.074439	0.077545	0.095238	0.075000
50	0.020000	0.029946	0.031394	0.039216	0.030000
100	0.010000	0.015001	0.015761	0.019802	0.015000
200	0.005000	0.007507	0.007897	0.009950	0.007500
500	0.002000	0.003005	0.003163	0.003992	0.003000
1000	0.001000	0.001503	0.001582	0.001998	0.001500
2000	0.000500	0.000751	0.000791	0.001000	0.000750
5000	0.000200	0.000301	0.000316	0.000400	0.000300

In our empirical study it was seen that $\alpha_n \approx 1.5/n$. It is shown in Table 2.1 together with α_n and its lower and upper bounds $1/n$, $2/(n+1)$ and u_n.

Proposition 2.7 For each $n \geq 2$ the equation

$$B_n(p) = \frac{1}{p}, \tag{2.23}$$

has a unique solution β_n in $]0, 1[$ such that

(i) $\alpha_n < \beta_n < 2.16/n$, where α_n is given by Proposition 2.6.

(ii) $B_n(p) < \frac{1}{p}$, $\forall\, p \in\,]0, \beta_n[$ and $B_n(p) > \frac{1}{p}$, $\forall\, p \in\,]\beta_n, 1[$.

(iii) $\{\beta_n\}$ is decreasing.

(iv) $\lim_{n\to\infty} \beta_n = 0$.

Proof. Since $\frac{1-q^n}{p} < \frac{1}{p}$ it follows from Proposition 2.6 that (2.23) has no solution in $]0, \alpha_n]$. Thus, (2.23) is equivalent to $g(p) = 0$, where

$$g(p) = f(p) - \frac{1}{p}, \tag{2.24}$$

for $p \in\ [\alpha_n, 1[$ and $f(p)$ defined by (2.17). This function g is continuous in $[\alpha_n, 1[$ and since $f(p)$ and $-1/p$ are strictly increasing then $g(p)$ is also strictly increasing. Besides from (2.20), $g(\alpha_n) = f(\alpha_n) - 1/\alpha_n = -1/\alpha_n < 0$ and

$$\lim_{p\to 1^-} g(p) \;=\; +\infty, \quad \text{by (2.24) and (2.19).} \tag{2.25}$$

Therefore, by the mean value theorem there exists a unique $\beta_n \in\,]\alpha_n, 1[$ such that

$$g(p) \;<\; g(\beta_n) = 0 < g(t), \tag{2.26}$$

for all $p \in\,]\alpha_n, \beta_n[$ and all $t \in\,]\beta_n, 1[$. This ends the proof of (i)-(ii), except the inequality $\beta_n \leq 2.16/n$. To prove this, we use the inequality (2.22) in (2.18) to get

$$B_n(p) \geq \frac{1-q^n}{p} + nq^n \left[\frac{1}{12}(n-1)(n-2)\left(\frac{p}{q}\right)^2 + \frac{1}{2}(n-1)\left(\frac{p}{q}\right) - 1\right]. \tag{2.27}$$

If $n = 2$ then $\beta_2 < 1 < 2.16/2$. Thus, from (2.27) it is sufficient to prove that for $n \geq 3$ and $p = 2.16/n$:

$$\frac{1-q^n}{p} + nq^n \left[\frac{1}{12}(n-1)(n-2)\left(\frac{p}{q}\right)^2 + \frac{1}{2}(n-1)\left(\frac{p}{q}\right) - 1\right] > \frac{1}{p}.$$

Equivalently, this can be written as

$$\frac{1}{12}n(n-1)(n-2)\left(\frac{p}{q}\right)^2+\frac{1}{2}n(n-1)\left(\frac{p}{q}\right)-n-\frac{1}{p}>0,$$

and finally, after replacing $p = 2.16/n$, as

$$\frac{0.006n(n+300.35)(n-2.12)}{(n-2.16)^2}>0,$$

which is obviously true for $n \geq 3$.

(iii) By (ii) it is enough to prove that $B_{n+1}(\alpha_n) > \frac{1}{\alpha_n}$. To prove this let $p = \alpha_n$ and $q = 1 - \alpha_n$. Then

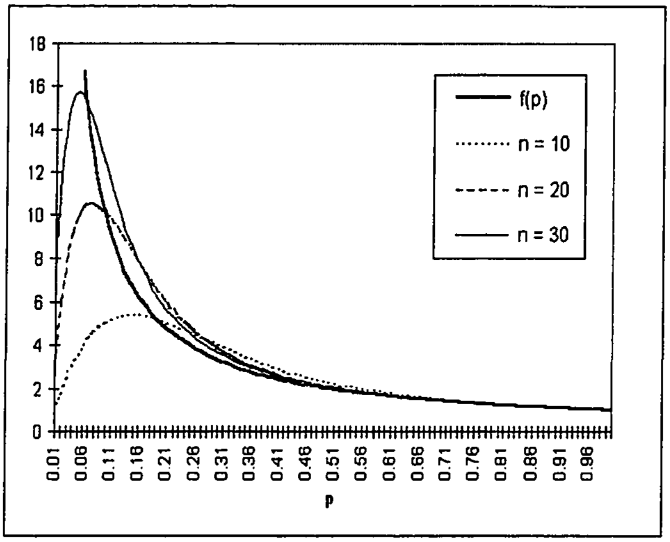
$$\begin{aligned}
B_{n+1}(p) &= q\left(\frac{n+1}{n}\right)B_n(p)+1-q^{n+1}, \text{ by (2.6)},\\
&= (1-p)\left(1+\frac{1}{n}\right)\frac{1}{p}+1-q^{n+1}, \text{ by (2.23)},\\
&= \frac{1}{p}+\frac{1}{np}-1-\frac{1}{n}+1-q^{n+1},\\
&= \frac{1}{p}+\frac{1-p-npq^{n+1}}{np},\\
&= \frac{1}{p}+\frac{q}{np}\left(1-npq^{n}\right),\\
&> \frac{1}{p}, \text{ by Lemma 2.5 - (iii) when } r=1.
\end{aligned}$$

(iv) It follows from (i). ◇

Table 2.2: Values of α_n, β_n and $2.16/n$

n	α_n	β_n	$2.16/n$
5	0.288677	0.364920	0.432000
10	0.147419	0.191705	0.216000
20	0.074439	0.098133	0.108000
50	0.029946	0.039797	0.043200
100	0.015001	0.019989	0.021600
200	0.007507	0.010017	0.010800
500	0.003005	0.004012	0.004320
1000	0.001503	0.002007	0.002160
2000	0.000751	0.001004	0.001080
5000	0.000301	0.000402	0.000432

Figure 2.1: $B_n(p)$ and $f(p) = 1/p$



In Table 2.2 we illustrate numerically Proposition 2.7 - (i). We see that for large values of n the upper bound $2.16/n$ can be taken as an approximation of β_n. Figure 2.1 illustrates Propositions 2.4, 2.6 and 2.7.

Proposition 2.8 For $n \geq 2$ and $r > 0$ the equation

$$B_n(p) = \frac{1}{p} + rq^n, \tag{2.28}$$

has a unique solution ${}_r\gamma_n$ in $]0,1[$ such that

(i) $\beta_n < {}_r\gamma_n$, where β_n is given by Proposition 2.7. If $0 < r \leq 3$, then ${}_r\gamma_n < 2.25/n$.

(ii) $B_n(p) < \dfrac{1}{p} + rq^n,\ \forall\, p \in\,]0, {}_r\gamma_n[$ and $B_n(p) > \dfrac{1}{p} + rq^n,\ \forall\, p \in\,]{}_r\gamma_n, 1[$.

(iii) $\{{}_r\gamma_n\}$ is decreasing as a function of n.

Proof. (i) - (ii) The inequality $\frac{1}{p} < \frac{1}{p} + rq^n$ and Proposition 2.7 show that (2.28) has no solution in $]0, \beta_n]$. Therefore, (2.28) can be written as $v(p) = 0$, where $v(p) = g(p) - r$, for $p \in\ [\beta_n, 1[$ and g is defined by (2.24). This function v is strictly increasing and continuous in $[\beta_n, 1[$. Besides, (2.26) implies that $v(\beta_n) = g(\beta_n) - r = -r < 0$, and from (2.25) it follows that $\lim_{p\to 1^-} v(p) = +\infty$. Then, by the mean value theorem there exists a unique ${}_r\gamma_n \in\ ]\beta_n, 1[$ such that $v(p) < v({}_r\gamma_n) = 0 < v(t)$, for all $p \in\ ]\beta_n, {}_r\gamma_n[$ and all $t \in\ ]_r\gamma_n, 1[$. To prove the last inequality of (i) it is enough to prove that $B_n(p) > \frac{1}{p} + 3q^n$, for $p = 2.25/n$ and $n \geq 3$. From (2.27) it is sufficient to prove that

$$\frac{1}{12}n(n-1)(n-2)\left(\frac{p}{q}\right)^2 + \frac{1}{2}n(n-1)\left(\frac{p}{q}\right) - n - \frac{1}{p} - 3 > 0.$$

Replacing $p = 2.25/n$ the latter becomes approximately

$$\frac{0.10(n-2.19)(n^2-11.69n+67.77)}{(n-2.25)^2} > 0,$$

which is true for $n \geq 3$.

(iii) For $p = {}_r\gamma_n$ and $q = 1 - {}_r\gamma_n$ we have

$$\begin{aligned} B_{n+1}(p) &= q\left(\frac{n+1}{n}\right)\left(\frac{1}{p} + rq^n\right) + 1 - q^{n+1}, \text{ by (2.6) and (2.28)}, \\ &= \frac{1}{p} + rq^{n+1} + \frac{q}{np}\left(1 + rpq^n - npq^n\right), \\ &> \frac{1}{p} + rq^{n+1} + \frac{q}{np}\left(1 - npq^n\right), \\ &> \frac{1}{p} + rq^{n+1}, \text{ by Lemma 2.5 -(iii)}. \end{aligned}$$

This and (ii) complete the proof. ◇

We observe that Propositions 2.7 and 2.8 are special cases of a more general result. In fact, it is clear that equation $B_n(p) = \frac{1}{p} + \psi(p)q^n$ has a unique solution in $]0, 1[$ for ψ a bounded, positive and decreasing function in $]0, 1]$.

As an illustration of Proposition 2.8 - (i) we present in Table 2.3 values of ${}_r\gamma_n$ for the special cases where r is given by

$$r = {}_tr_x = \left(\frac{\ddot{s}_{\overline{t}|}}{\ddot{s}_{x:\overline{t}|} - \ddot{s}_{\overline{t}|}}\right)\frac{{}_tq_x}{{}_tp_x} - 1, \tag{2.29}$$

and ${}_tp_x$ is derived using (1.28) for $x = 20, 30, 40, 50, 60$, $t = 65 - x$ and $i = 0.06$. For comparison we also include β_n and $2.25/n$. We note that since values ${}_tr_x$ are small then values of ${}_r\gamma_n$ are close to β_n.

Table 2.3: Values of β_n, $2.25/n$ and ${}_r\gamma_n$ for $r = {}_tr_x$

		$x=20$	$x=30$	$x=40$	$x=50$	$x=60$	
	${}_tp_x$	0.783335	0.792934	0.808959	0.841699	0.920114	
	${}_tr_x$	0.138972	0.201205	0.301175	0.447811	0.536277	
n	β_n	${}_r\gamma_n$					$2.25/n$
2	0.767592	0.774291	0.777179	0.781682	0.787994	0.791644	
5	0.364920	0.367624	0.368826	0.370744	0.373532	0.375198	0.450000
10	0.191705	0.192551	0.192928	0.193534	0.194419	0.194951	0.225000
20	0.098133	0.098367	0.098472	0.098640	0.098886	0.099034	0.112500
50	0.039797	0.039837	0.039854	0.039883	0.039925	0.039950	0.045000
100	0.019989	0.019999	0.020003	0.020011	0.020021	0.020028	0.022500
200	0.010017	0.010020	0.010021	0.010023	0.010025	0.010027	0.011250
300	0.006683	0.006684	0.006685	0.006685	0.006687	0.006687	0.007500
400	0.005014	0.005015	0.005015	0.005016	0.005016	0.005017	0.005625
500	0.004012	0.004013	0.004013	0.004013	0.004014	0.004014	0.004500

Proposition 2.9 For each $n \geq 3$ there exists a unique $0 < \delta_n < 1$ such that

(i) $B_n(\delta_n) = \dfrac{1}{\delta_n} + \dfrac{(1-\delta_n) - [2(n-1)\delta_n + 1](1-\delta_n)^n}{(n-1)\delta_n^2}$.

(ii) $$B_n(p) < \frac{1}{p} + \frac{q - [2(n-1)p+1]q^n}{(n-1)p^2}, \ \forall\, p \in \,]0, \delta_n[,$$
$$B_n(p) > \frac{1}{p} + \frac{q - [2(n-1)p+1]q^n}{(n-1)p^2}, \ \forall\, p \in \,]\delta_n, 1[.$$

(iii) B_n is concave in $]0, \delta_n]$ and convex in $]\delta_n, 1]$.

(iv) The following inequalities hold for $n \geq 4$:
$$\frac{2}{n} \leq \delta_n \leq v_n = \frac{\sqrt{4(n-2)(13n-35)} - 4(n-2)}{\sqrt{4(n-2)(13n-35)} + (n-2)(n-7)} \leq \frac{9}{2n+5}, \quad (2.30)$$
$$\beta_n < \delta_n. \quad (2.31)$$

(v) $\{\delta_n\}$ is decreasing.

(vi) $\lim_{n\to\infty} \delta_n = 0$.

Proof. We will prove that $B_n''(p)$ changes sign only once in $]0,1[$. This is not true for $n < 3$ since B_n is a polynomial of degree n. For $0 < p < 1$ we have from (2.16)
$$B_n''(p) = n\left[\sum_{k=0}^{n-1} \frac{k(k-1)q^{k-2}}{n-k} - n(n-1)q^{n-2}\sum_{k=1}^{n}\frac{1}{k}\right] = nq^{n-2}h(p), \quad (2.32)$$

where

$$h(p) = \sum_{k=0}^{n-1} \frac{k(k-1)}{n-k} q^{-(n-k)} - n(n-1) \sum_{k=1}^{n} \frac{1}{k}. \tag{2.33}$$

This function is continuous, strictly increasing in $]0,1[$, $\lim_{p\to 1^-} h(p) = +\infty$ and

$$\begin{aligned}
\lim_{p\to 0^+} h(p) &= \sum_{k=0}^{n-1} \frac{k(k-1)}{n-k} - n(n-1)\sum_{k=1}^{n}\frac{1}{k}, \\
&< \sum_{k=0}^{n-1} \frac{n(n-1)}{n-k} - n(n-1)\sum_{k=1}^{n}\frac{1}{k}, \\
&= 0.
\end{aligned}$$

Therefore, by the mean value theorem there exists a unique $\delta_n \in]0,1[$ such that $h(p) < h(\delta_n) = 0 < h(t)$, for all $p \in]0, \delta_n[$ and all $t \in]\delta_n, 1[$. To prove (i)-(iii) we still need to verify that for $0 < p < 1$,

$$B_n''(p) = 0 \Leftrightarrow B_n(p) = \frac{1}{p} + \frac{q - [2(n-1)p+1]q^n}{(n-1)p^2}. \tag{2.34}$$

From (2.32) we see that $B_n''(p) = 0$ if and only if

$$\begin{aligned}
0 &= \sum_{k=1}^{n} \frac{(n-k)(n-k-1)}{k} q^{n-k} - n(n-1)q^n \sum_{k=1}^{n}\frac{1}{k}, \\
&= \sum_{k=1}^{n} \frac{n(n-1) + (-2n+1)k + k^2}{k} q^{n-k} - n(n-1)q^n \sum_{k=1}^{n}\frac{1}{k}, \\
&= (n-1)\left[\sum_{k=1}^{n} \frac{n}{k} q^{n-k} - \sum_{k=1}^{n}\frac{n}{k}q^n\right] + \sum_{k=1}^{n} \frac{(-2n+1)k}{k} q^{n-k} + \sum_{k=1}^{n} k q^{n-k}, \\
&= (n-1)B_n(p) + (-2n+1)\frac{(1-q^n)}{p} + \sum_{k=0}^{n-1}(n-k)q^k, \\
&= (n-1)B_n(p) + (-2n+1)\frac{(1-q^n)}{p} + n\frac{(1-q^n)}{p} - q\frac{d}{dq}\left(\sum_{k=0}^{n-1} q^k\right), \\
&= (n-1)B_n(p) - (n-1)\frac{(1-q^n)}{p} - q\frac{d}{dq}\left(\frac{1-q^n}{1-q}\right), \\
&= (n-1)\left[B_n(p) - \frac{1}{p}\right] + (n-1)\frac{q^n}{p} - q\left(\frac{-npq^{n-1} + 1 - q^n}{p^2}\right), \\
&= (n-1)\left[B_n(p) - \frac{1}{p}\right] - \frac{-(n-1)pq^n - npq^n + q - q^{n+1}}{p^2},
\end{aligned}$$

$$
\begin{aligned}
&= (n-1)\left[B_n(p) - \frac{1}{p}\right] - \frac{q - [(n-1)p + np + q]q^n}{p^2}, \\
&= (n-1)\left[B_n(p) - \frac{1}{p}\right] - \frac{q - [2(n-1)p + 1]q^n}{p^2}.
\end{aligned}
$$

This yields (2.34).

(iv) We may write (2.33) as

$$
h(p) = \sum_{k=1}^{n} \frac{(n-k)(n-k-1)q^{-k} - n(n-1)}{k}. \tag{2.35}
$$

We can get $h(p) < 0$ and thus, $B_n''(p) < 0$, if the condition

$$
(n-k)(n-k-1)q^{-k} - n(n-1) < 0, \quad \text{fo all } k = 1, ..., n
$$

is added to (2.35). This condition is equivalent to

$$
q > \left[\frac{(n-k)(n-k-1)}{n(n-1)}\right]^{1/k}, \ k = 1, ..., n-2.
$$

From Corollary 2.3 - (ii), we obtain

$$
q > \max_{\{k=1,...,n\}} \left[\frac{(n-k)(n-k-1)}{n(n-1)}\right]^{1/k} = \frac{n-2}{n}
$$

or $p < 2/n$. This proves the left hand side inequality of (2.30). To prove the other inequalities in (2.30) we use the inequalities of (2.5) in (2.35) to get

$$
\begin{aligned}
h(p) &\geq \sum_{k=1}^{n} \left[\frac{(n-k)(n-k-1)\left[1 + k\left(\frac{p}{q}\right) + \frac{1}{2}k(k-1)\left(\frac{p}{q}\right)^2\right] - n(n-1)}{k}\right], \\
&\geq \sum_{k=1}^{n} \left[\frac{(n-k)(n-k-1)\left[1 + k\left(\frac{p}{q}\right)\right] - n(n-1)}{k}\right],
\end{aligned}
$$

or

$$
\begin{aligned}
h(p) &\geq n(n-1)\underbrace{\left[\frac{1}{24}(n-2)(n-3)\left(\frac{p}{q}\right)^2 + \frac{1}{3}(n-2)\left(\frac{p}{q}\right) - \frac{3}{2}\right]}_{D_2}, \\
&\geq n(n-1)\underbrace{\left[\frac{1}{3}(n-2)\left(\frac{p}{q}\right) - \frac{3}{2}\right]}_{D_1}.
\end{aligned}
$$

Solving $D_2 > 0$ for $n \geq 4$ we get $h(p) > 0$, i.e., $B''(p) > 0$, for

$$p > v_n = \frac{\sqrt{4(n-2)(13n-35)} - 4(n-2)}{\sqrt{4(n-2)(13n-35)} + (n-2)(n-7)}.$$

This proves the second inequality of (2.30). Now solving $D_1 > 0$ for $n \geq 4$ we get $B''(p) > 0$ for $p > \frac{9}{2n+5}$. This leads $\delta_n \leq \frac{9}{2n+5}$. The last inequality of (2.30) follows since $D_2 \geq D_1$. Inequality (2.31) follows from (i), the first inequality of (2.30), Lemma 2.5 - (ii) and Proposition 2.7 - (ii).

(v) From (ii) it is enough to prove that for $p = \delta_n$ and $q = 1 - \delta_n$

$$B_{n+1}(p) > \underbrace{\frac{1}{p} + \frac{q}{np^2}\left[1 - (2np+1)q^n\right)]}_{L_n(p)}.$$

To this, using (2.6) and (i) we have

$$\begin{aligned}
B_{n+1}(p) &= \left(\frac{n-1}{n} + \frac{2}{n}\right)\left\{\frac{q}{p} + \frac{q}{(n-1)p^2}\left[q - (2np+1)q^n + 2pq^n\right]\right\} + 1 - q^{n+1}, \\
&= \left(\frac{n-1}{n}\right)\left\{\frac{q}{p} + \frac{q}{(n-1)p^2}\left[q - (2np+1)q^n + 2pq^n\right]\right\} \\
&\quad + \frac{2q}{n}\left\{\frac{1}{p} + \frac{q}{(n-1)p^2}\left[1 - \{2(n-1)p+1\}q^{n-1}\right]\right\} + 1 - q^{n+1}, \\
&= \left(\frac{n-1}{n}\right)\frac{q}{p} + \frac{q}{np^2}\left[1 - p - (2np+1)q^n + 2pq^n\right] \\
&\quad + \frac{2q}{n}\left\{\frac{1}{p} + \frac{q}{(n-1)p^2}\left[1 - \{2(n-1)p+1\}q^{n-1}\right]\right\} + 1 - q^{n+1}, \\
&= L_n(p) - \frac{q}{np} + \frac{q}{np^2}(2pq^n - p) \\
&\quad + \frac{2q}{n}\left\{\frac{1}{p} + \frac{q}{(n-1)p^2}\left[1 - \{2(n-1)p+1\}q^{n-1}\right]\right\} - q^{n+1}, \\
&= L_n(p) + \frac{q}{np}\left\{(2-np)q^n + \frac{2q}{(n-1)p}\left[1 - \{2(n-1)p+1\}q^{n-1}\right]\right\}, \\
&= L_n(p) + \frac{q}{n(n-1)p^2}\left\{(n-1)(2-np)pq^n + 2q - 2[2(n-1)p+1]q^n\right\}, \\
&= L_n(p) + \frac{q}{n(n-1)p^2}\left\{2q + \left[(n-1)(2-np)p - 4np + 4p - 2\right]q^n\right\}, \\
&= L_n(p) + \frac{q^2}{n(n-1)p^2}\left\{2 + \left[-n(n-1)p^2 - 2(n-1)p - 2\right]q^{n-1}\right\},
\end{aligned}$$

$$
\begin{aligned}
&= L_n(p) + \frac{2q^2}{n(n-1)p^2}\left\{1 - \left[\frac{n(n-1)}{2}p^2 + (n-1)p + 1\right] q^{n-1}\right\}, \\
&> L_n(p), \text{ from Lemma 2.5 -(iv)}.
\end{aligned}
$$

(vi) It follows from (iv). ◇

Table 2.4: Values of δ_n, lower and upper bounds and $2.87/n$

n	$2/n$	β_n	δ_n	v_n	$9/(2n+5)$	$2.87/n$
5	0.400000	0.364920	0.530865	0.537525	0.600000	0.574000
10	0.200000	0.191705	0.277070	0.292359	0.360000	0.287000
20	0.100000	0.098133	0.141104	0.153010	0.200000	0.143500
50	0.040000	0.039797	0.057023	0.062978	0.085714	0.057400
100	0.020000	0.019989	0.028606	0.031797	0.043902	0.028700
200	0.010000	0.010017	0.014326	0.015977	0.022222	0.014350
500	0.004000	0.004012	0.005736	0.006410	0.008955	0.005740
1000	0.002000	0.002007	0.002869	0.003208	0.004489	0.002870
2000	0.001000	0.001004	0.001435	0.001605	0.002247	0.001435
5000	0.000400	0.000402	0.000574	0.000642	0.000900	0.000574

Proposition 2.9 - (iv) is illustrated numerically in Table 2.4. It is seen numerically that $2.87/n$ provides an adequate approximation for δ_n when n is large.

Proposition 2.10 Let $n \geq 2$. The function

$$
\varphi_n(p) = \begin{cases} \dfrac{n(1-q^n)}{(n-1)p} - B_n(p) & \text{if } 0 < p \leq 1, \\ \dfrac{n^2}{n-1} & \text{if } p = 0, \end{cases}
$$

has the following properties:

(i) For $n \geq 3$, φ_n is strictly decreasing in $[0, \delta_n[$ and strictly increasing in $]\delta_n, 1]$. Therefore, for all $p \in [0,1]$, $p \neq \delta_n$ we have

$$
\varphi_n(p) > \varphi_n(\delta_n) = \frac{1}{(n-1)\delta_n^2}\left\{2\delta_n - 1 + [(n-2)\delta_n + 1](1-\delta_n)^n\right\}. \quad (2.36)
$$

(ii) $\varphi_n(p) > 0$, for all $0 \leq p \leq 1$ and $2 \leq n \leq 5$.

(iii) $\varphi_n(p) > \left(\frac{nq}{n-1}\right)\varphi_{n-1}(p)$, for all $\frac{n}{2(n-1)} \leq p \leq 1$.

(iv) For $n \geq 6$, φ_n has two roots ϵ_n and ε_n in $]0,1[$ such that $\frac{1}{n-1} \leq \epsilon_n \leq \beta_n$ (the last inequality when $n \geq 10$) and $1/2 < \varepsilon_n < 7/12$, and

$$\begin{array}{rcccccccl} 0 &=& \varphi_n(\epsilon_n) &<& \varphi_n(p) &<& \varphi_n(0) &=& \frac{n^2}{n-1}, \quad \text{for all } 0 < p < \epsilon_n, \\ && \varphi_n(\delta_n) &<& \varphi_n(p) &<& \varphi_n(\varepsilon_n) &=& 0, \quad \text{for all } \epsilon_n < p < \varepsilon_n, \\ 0 &=& \varphi_n(\varepsilon_n) &<& \varphi_n(p) &<& \varphi_n(1) &=& \frac{1}{n-1}, \quad \text{for all } \varepsilon_n < p < 1. \end{array}$$

Proof. **(i)** We may write φ_n as

$$\varphi_n(p) = -\left(\frac{n}{n-1}\right) \sum_{k=1}^{n} \frac{(n-k-1)q^{n-k} - (n-1)q^n}{k}. \tag{2.37}$$

This and (2.32) yield $\varphi_n'(p) = \left(\frac{q}{n-1}\right) B_n''(p)$. Proposition 2.9 ends the proof.

(ii) It is easily seen that the result is true for $n = 2$. For $3 \leq n \leq 5$ we have $\delta_n > 1/2$ or $2\delta_n - 1 > 0$. Then from (2.36) we get $\varphi_n(p) \geq \varphi_n(\delta_n) > 0$, for all $0 \leq p \leq 1$.

(iii) The result is true for $p = 1$. Let $\frac{n}{2(n-1)} \leq p < 1$. Since the function $\frac{n-(n-1)p}{n(1-p)}$ is increasing in $]0,1[$, we get $\frac{n-(n-1)p}{nq} \geq \frac{n-1}{n-2}$. From this and (2.6)

$$\begin{aligned} \varphi_n(p) &= \frac{n(1-q^n)}{(n-1)p} - B_n(p), \\ &= \frac{n(1-q^n)}{(n-1)p} - \left\{ q\left[\frac{n}{n-1}\right] B_{n-1}(p) + 1 - q^n \right\}, \\ &= \frac{nq}{n-1}\left\{ \left[\frac{n-(n-1)p}{nq}\right]\left[\frac{1-q^n}{p}\right] - B_{n-1}(p) \right\}, \\ &> \frac{nq}{n-1}\left\{ \left[\frac{n-(n-1)p}{nq}\right]\left[\frac{1-q^{n-1}}{p}\right] - B_{n-1}(p) \right\}, \\ &\geq \frac{nq}{n-1}\left[\frac{(n-1)(1-q^{n-1})}{(n-2)p} - B_{n-1}(p)\right] = \frac{nq}{n-1}\varphi_{n-1}(p). \end{aligned} \tag{2.38}$$

(iv) Let $0 \leq p \leq \alpha_n$. From Proposition 2.6 - (iv) and since for these values of p, $\varphi_n(p) > \frac{1-q^n}{p} - B_n(p) \geq 0$, we get $\varphi_n(p) > 0$. For $n = 6$, by direct computation we can see that if $p > 0.52$ then $\varphi_6(p) > 0$. Since for $n \geq 7$ we have $\frac{n}{2(n-1)} \leq 7/12$, it follows from (iii) that $\varphi_n(p) > 0$, for $7/12 \leq p \leq 1$. Now take $\delta_n \leq p \leq 1/2$. Since in this interval φ_n is increasing we have $\varphi_n(p) \leq \varphi_n(1/2)$. From (2.38) and Proposition 2.9 - (ii), we get

$$\varphi_n(p) = \frac{nq}{n-1}\left\{ \left[\frac{n-(n-1)p}{nq}\right]\left[\frac{1-q^n}{p}\right] - B_{n-1}(p) \right\},$$

$$
\begin{aligned}
&< \frac{nq}{n-1}\left\{\left[\frac{n-(n-1)p}{nq}\right]\frac{1}{p}-\frac{1}{p}-\frac{q-[2(n-2)p+1]q^{n-1}}{(n-2)p^2}\right\},\\
&= \frac{nq}{(n-1)p}\left\{\left[\frac{n-(n-1)p}{nq}\right]-1-\frac{q-[2(n-2)p+1]q^{n-1}}{(n-2)p}\right\},\\
&= \frac{1}{(n-1)(n-2)p^2}\underbrace{\left\{-2p^2+n(2p-1)+n[2(n-2)p+1]q^n\right\}}_{H}.
\end{aligned}
$$

If $p = 1/2$ then $H = -\frac{1}{2} + \frac{n(n-1)}{2^n} < 0$ for $n \geq 6$. Hence $\varphi_n(1/2) < 0$ and so $\varphi_n(p) < 0$ for $\delta_n \leq p \leq 1/2$. Now, since $\varphi_n(0) > 0$, $\varphi_n(1/2) < 0$, $\varphi_n(1) > 0$ and φ_n is a continuous function with $\varphi_n'(p)$ vanishing only at δ_n, then φ_n has exactly two roots ϵ_n and ε_n in $]0,1[$ such that $\alpha_n < \epsilon_n < \delta_n$ and $1/2 < \varepsilon_n < 7/12$. We need to prove that $\frac{1}{n-1} \leq \epsilon_n \leq \beta_n$. The first inequality follows from (2.37). In fact from there it is enough to prove that for $p < \frac{1}{n-1}$

$$
\sum_{k=1}^{n} \frac{(n-k-1)q^{-k}-(n-1)}{k} < 0.
$$

This is true since by Corollary 2.3 - (i), $(n-k-1)q^{-k}-(n-1) < 0$ for $q > \frac{n-2}{n-1}$ or $p < \frac{1}{n-1}$. Now we have to prove that $\epsilon_n \leq \beta_n$ for $n \geq 10$. Let $\beta_n < p < 1 - n^{-1/n}$ (this is why we need $n \geq 10$). Then from Proposition 2.7 - (ii), we have

$$
\begin{aligned}
\varphi_n(p) &= \frac{n(1-q^n)}{(n-1)p} - B_n(p) < \frac{n(1-q^n)}{(n-1)p} - \frac{1}{p},\\
&= \frac{1}{(n-1)p}(1-nq^n) < 0, \quad \text{since } p < 1-n^{-1/n}.
\end{aligned}
$$

The inequalities are easily proved using ϵ_n, ε_n and (i). ◇

We observe that $\lim_{n\to\infty} \epsilon_n = 0$. Also, from Proposition 2.10 - (iii) it appears that ε_n is close to $\frac{n}{2(n-1)}$. In fact in our empirical studies it was seen that this expression provides a good approximation for ε_n. This is shown in Table 2.5 together with the values of ϵ_n and ε_n. We see that ε_n tends to $1/2$ as n increases.

In terms of B_n and $b_1(n,p)$ the inequalities of Proposition 2.10 - (iv) may be written as in the following corollary. For small values of np the inequalities of Propositions 2.6 and 2.7 and Corollary 2.4 are better so they have been included instead.

Table 2.5: Values of ϵ_n, ε_n, $1/(n-1)$ and $n/[2(n-1)]$

n	$1/(n-1)$	ϵ_n	ε_n	$n/[2(n-1)]$
6	0.200000	0.360000	0.513290	0.600000
10	0.111111	0.182082	0.558385	0.555556
20	0.052632	0.081842	0.528293	0.526316
50	0.020408	0.031047	0.510437	0.510204
100	0.010101	0.015270	0.505104	0.505051
200	0.005025	0.007574	0.502526	0.502513
500	0.002004	0.003015	0.501004	0.501002
1000	0.001001	0.001505	0.500501	0.500501
2000	0.000500	0.000752	0.500250	0.500250
5000	0.000200	0.000301	0.500100	0.500100

Corollary 2.5 Let $n \geq 2$.

(i) If $0 < p < \alpha_n$ then

$$\begin{aligned} \frac{(1-q^n)^2}{p} &< B_n(p) < \frac{1-q^n}{p}, \\ \frac{1-q^n}{np} &< b_1(n,p) < \frac{1}{np}. \end{aligned}$$

(ii) If $\alpha_n < p < \beta_n$ then

$$\begin{aligned} \frac{1-q^n}{p} &< B_n(p) < \frac{1}{p}, \\ \frac{1}{np} &< b_1(n,p) < \frac{1}{np(1-q^n)}. \end{aligned}$$

(iii) For $\epsilon_n < p < \varepsilon_n$ and $\varphi_n(\delta_n)$ given by (2.36),

$$\begin{aligned} \left(\frac{n}{n-1}\right)\left(\frac{1-q^n}{p}\right) &< B_n(p) < \left(\frac{n}{n-1}\right)\left(\frac{1-q^n}{p}\right) - \varphi_n(\delta_n), \\ \frac{1}{(n-1)p} &< b_1(n,p) < \frac{1}{(n-1)p} - \frac{\varphi_n(\delta_n)}{n(1-q^n)}. \end{aligned}$$

(iv) If $\varepsilon_n < p < 1$ then

$$\begin{aligned} \left(\frac{n}{n-1}\right)\left(\frac{1-q^n}{p}\right) - \frac{1}{n-1} &< B_n(p) < \left(\frac{n}{n-1}\right)\left(\frac{1-q^n}{p}\right), \\ \frac{1}{(n-1)p} - \frac{1}{n(n-1)(1-q^n)} &< b_1(n,p) < \frac{1}{(n-1)p}. \end{aligned}$$

Proof. Apply Corollary 2.4 and Propositions 2.6, 2.7 and 2.10. ◇

When n is large we can say that (i) and (ii) of Corollary 2.5 do not hold since in these cases β_n is almost null. Hence we can say that (iii) holds for almost the whole interval $]0, 1/2[$ and (iv) holds for almost the whole interval $]1/2, 1[$.

2.3 Results on Ramsay's definitions

In this section some inequalities for the ratios ρ_1 and ρ_2 from (1.26) and (1.27) are presented. The variances of Ramsay's random future shares are also derived.

2.3.1 Inequalities for ρ_1 and ρ_2

Some inequalities for ρ_1 and ρ_2 given by (1.26) and (1.27) are presented here. These inequalities could be improved in some intervals by using the inequalities of Corollary 2.5. It is also proved that the inequalities $\rho_1 > 1$ and $\rho_2 > 1$, conjectured by Ramsay, are not true in general.

Proposition 2.11 The ratios ρ_1 and ρ_2 satisfy:

(i) $(1 - {}_tq_x^n)^2 \leq \rho_1 < B_n({}_tp_x)$, for all $0 < {}_tp_x < 1$.

(ii) $(1 - {}_tq_x^n)^2 \leq \rho_2 \leq \dfrac{\ddot{s}_{\overline{t}|}}{\ddot{s}_{x:\overline{t}|}} B_n({}_tp_x) \leq B_n({}_tp_x)$, for all $0 < {}_tp_x < 1$.

(iii) $\lim_{{}_tp_x \to 0^+} \rho_1 = 0, \quad \lim_{{}_tp_x \to 1^-} \rho_1 = 1, \quad$ and $\quad \lim_{n \to \infty} \rho_1 = 1.$

(iv) $\lim_{{}_tp_x \to 0^+} \rho_2 = 0, \quad \lim_{{}_tp_x \to 1^-} \rho_2 = 1, \quad$ and $\quad \lim_{n \to \infty} \rho_2 = 1.$

Proof. **(i)** It follows from Corollary 2.4 and from $\rho_1 = {}_tp_x\, B_n({}_tp_x) < B_n({}_tp_x)$.
(ii) It is easy to see that

$$\ddot{s}_{\overline{t}|} \leq \ddot{s}_{x:\overline{t}|} \leq \frac{\ddot{s}_{\overline{t}|}}{{}_tp_x}. \tag{2.39}$$

From Corollary 2.4 we have that $B_n({}_tp_x) \geq 1 - {}_tq_x^n$. Then

$$\rho_2 \;=\; \frac{\ddot{s}_{\overline{t}|}\,(1 - {}_tq_x^n) + [B_n({}_tp_x) - (1 - {}_tq_x^n)]\left(\ddot{s}_{x:\overline{t}|} - \ddot{s}_{\overline{t}|}\right)\frac{{}_tp_x}{{}_tq_x}}{\ddot{s}_{x:\overline{t}|}},$$

$$
\begin{aligned}
&\leq \frac{\ddot{s}_{\overline{t}|}(1-{}_tq_x^n) + [B_n({}_tp_x) - (1-{}_tq_x^n)]\left(\frac{\ddot{s}_{\overline{t}|}}{{}_tp_x} - \ddot{s}_{\overline{t}|}\right)\frac{{}_tp_x}{{}_tq_x}}{\ddot{s}_{x:\overline{t}|}}, \text{ by (2.39)},\\
&= \frac{\ddot{s}_{\overline{t}|}}{\ddot{s}_{x:\overline{t}|}}\left\{1 - {}_tq_x^n + [B_n({}_tp_x) - (1-{}_tq_x^n)]\left[\frac{1}{{}_tp_x} - 1\right]\frac{{}_tp_x}{{}_tq_x}\right\},\\
&= \frac{\ddot{s}_{\overline{t}|}}{\ddot{s}_{x:\overline{t}|}}[1 - {}_tq_x^n + B_n({}_tp_x) - (1-{}_tq_x^n)],\\
&= \frac{\ddot{s}_{\overline{t}|}}{\ddot{s}_{x:\overline{t}|}}B_n({}_tp_x) \;\leq\; B_n({}_tp_x), \text{ by (2.39)}.
\end{aligned}
$$

Again from Corollary 2.4,

$$
\begin{aligned}
\rho_2 &= \frac{\ddot{s}_{\overline{t}|}(1-{}_tq_x^n) + [B_n({}_tp_x) - (1-{}_tq_x^n)]\left(\ddot{s}_{x:\overline{t}|} - \ddot{s}_{\overline{t}|}\right)\frac{{}_tp_x}{{}_tq_x}}{\ddot{s}_{x:\overline{t}|}},\\
&\geq \frac{\ddot{s}_{\overline{t}|}(1-{}_tq_x^n) + \left[\frac{(1-{}_tq_x^n)^2}{{}_tp_x} - (1-{}_tq_x^n)\right]\left(\ddot{s}_{x:\overline{t}|} - \ddot{s}_{\overline{t}|}\right)\frac{{}_tp_x}{{}_tq_x}}{\ddot{s}_{x:\overline{t}|}},\\
&= \frac{(1-{}_tq_x^n)\left[\ddot{s}_{\overline{t}|} + (1-{}_tq_x^{n-1})\left(\ddot{s}_{x:\overline{t}|} - \ddot{s}_{\overline{t}|}\right)\right]}{\ddot{s}_{x:\overline{t}|}},\\
&= \frac{(1-{}_tq_x^n)\left[{}_tq_x^{n-1}\,\ddot{s}_{\overline{t}|} + (1-{}_tq_x^{n-1})\,\ddot{s}_{x:\overline{t}|}\right]}{\ddot{s}_{x:\overline{t}|}},\\
&= (1-{}_tq_x^n)\left[{}_tq_x^{n-1}\left(\frac{\ddot{s}_{\overline{t}|}}{\ddot{s}_{x:\overline{t}|}}\right) + 1 - {}_tq_x^{n-1}\right],\\
&\geq (1-{}_tq_x^n)\left[{}_tq_x^{n-1}({}_tp_x) + 1 - {}_tq_x^{n-1}\right], \text{ by (2.39)},\\
&= (1-{}_tq_x^n)^2.
\end{aligned}
$$

(iii) It follows from (i) and (2.7).

(iv) It follows from (ii) and (2.7). ◇

Proposition 2.12 For ρ_1 and ρ_2 we have:

(i) If $n = 1$ then $\rho_1 < 1$ and $\rho_2 < 1$, for all $0 < {}_tp_x < 1$.

(ii) For $n \geq 2$:

$$
\begin{aligned}
\rho_1 &< 1 \iff {}_tp_x < \beta_n,\\
\rho_1 &= 1 \iff {}_tp_x = \beta_n,\\
\rho_1 &> 1 \iff {}_tp_x > \beta_n.
\end{aligned}
$$

(iii) For $n \geq 2$ and a given ${}_tp_x$ consider r as in (2.29) and $\rho_2 = \rho_2({}_tp_x)$. Then

$$\begin{aligned} \rho_2 &< 1 \iff {}_tp_x < {}_r\gamma_n, \\ \rho_2 &= 1 \iff {}_tp_x = {}_r\gamma_n, \\ \rho_2 &> 1 \iff {}_tp_x > {}_r\gamma_n. \end{aligned}$$

Proof. **(i)** From (1.26) we see that $\rho_1 = {}_tp_x^2 < 1$ and from (1.27) and (2.39) it follows that $\rho_2 = \frac{{}_tp_x\, \ddot{s}_{\overline{t}|}}{\ddot{s}_{x:\overline{t}|}} < 1$.

(ii) From (1.26) it is seen that $\rho_1 = 1 \Leftrightarrow B_n({}_tp_x) = \frac{1}{{}_tp_x}$. Apply now Proposition 2.7.

(iii) For a given ${}_tp_x$ consider r as in (2.29). Let ε such that

$$B_n({}_tp_x) = \frac{1}{{}_tp_x} + (\varepsilon - 1)\, {}_tq_x^n .$$

Then from (1.27) we may write

$$E[S_2] - \ddot{s}_{x:\overline{t}|} = {}_tq_x^{n-1} \left[{}_tp_x\, \varepsilon(\ddot{s}_{x:\overline{t}|} - \ddot{s}_{\overline{t}|}) - {}_tq_x\, \ddot{s}_{\overline{t}|} \right] .$$

Therefore,

$$\rho_2 < 1 \Leftrightarrow E[S_2] - \ddot{s}_{x:\overline{t}|} < 0 \Leftrightarrow \varepsilon - 1 < r \Leftrightarrow B_n({}_tp_x) < \frac{1}{{}_tp_x} + r\, {}_tq_x^n, \qquad (2.40)$$

$$\rho_2 = 1 \Leftrightarrow E[S_2] - \ddot{s}_{x:\overline{t}|} = 0 \Leftrightarrow \varepsilon - 1 = r \Leftrightarrow B_n({}_tp_x) = \frac{1}{{}_tp_x} + r\, {}_tq_x^n, \qquad (2.41)$$

$$\rho_2 > 1 \Leftrightarrow E[S_2] - \ddot{s}_{x:\overline{t}|} > 0 \Leftrightarrow \varepsilon - 1 > r \Leftrightarrow B_n({}_tp_x) > \frac{1}{{}_tp_x} + r\, {}_tq_x^n . \qquad (2.42)$$

Since ${}_tp_x$ is fixed then r is a constant. Applying Proposition 2.8 to this fixed r there exists a unique ${}_r\gamma_n$ in $]0,1[$ such that

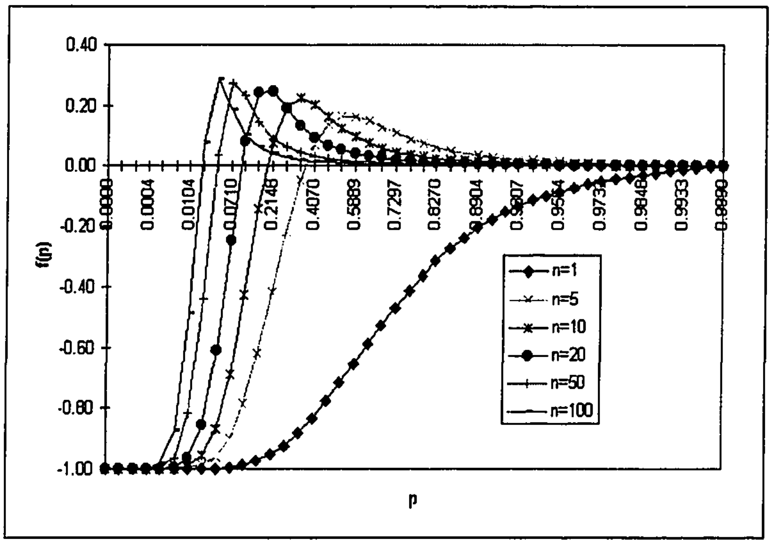
$$B_n(p) < \frac{1}{p} + rq^n \iff 0 < p < {}_r\gamma_n, \qquad (2.43)$$

$$B_n(p) = \frac{1}{p} + rq^n \iff p = {}_r\gamma_n, \qquad (2.44)$$

$$B_n(p) > \frac{1}{p} + rq^n \iff {}_r\gamma_n < p < 1. \qquad (2.45)$$

Now, since ${}_tp_x$ is given, the proof is ended by (2.43) - (2.40), (2.44) - (2.41) and (2.45) - (2.42). $\diamond$

Figure 2.2: $f(p) = \rho_1 - 1$



Since values ${}_r\gamma_n$ depend on ${}_tp_x$ and β_n does not depend on ${}_tp_x$ then (ii) and (iii) of Proposition 2.12 are very different from each other. In fact, (iii) is true for a given ${}_tp_x$ while (ii) is true for all ${}_tp_x$. The following corollary is clearer.

Corollary 2.6 (i) For $n \geq 2$:

$$\begin{aligned} \rho_1 &< 1, \quad \text{for all } 0 < {}_tp_x \leq 1/n, \\ \rho_1 &> 1, \quad \text{for all } 2.16/n \leq {}_tp_x < 1. \end{aligned}$$

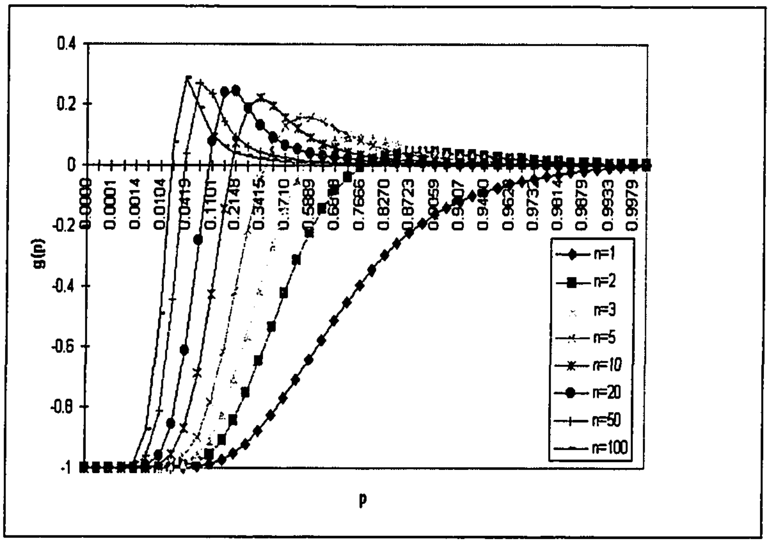
(ii) Let $n \geq 2$ and $r = {}_tr_x$ given by (2.29). Assume that $0 < r \leq 3$ for all $2.25/n \leq {}_tp_x < 1$. Then

$$\begin{aligned} \rho_2 &< 1, \quad \text{for all } 0 < {}_tp_x \leq \beta_n, \\ \rho_2 &< 1, \quad \text{for all } 0 < {}_tp_x \leq 1/n, \\ \rho_2 &> 1, \quad \text{for all } 2.25/n \leq {}_tp_x < 1. \end{aligned}$$

Proof. This follows from Propositions 2.12, 2.6, 2.7 and 2.8. ◇

The restriction $r \leq 3$ for all $2.25/n \leq {}_tp_x < 1$ in Corollary 2.6 - (ii) is not too

Figure 2.3: $g(p) = \rho_2 - 1$



strong as we may see from Table 2.3.

Proposition 2.12 and Corollary 2.6 make clear the fact that the inequalities $\rho_1 > 1$ and $\rho_2 > 1$ are not true in general. This is also illustrated in Figures 2.2 and 2.3.

2.3.2 Variances of S_1 and S_2

The variances of future random shares given by (1.21) and (1.22) are found here. They are important to estimate the risk involving the use of their means to pay future benefits. These formulas are also used to find the variances of the new random future shares defined in Chapter 3.

Proposition 2.13 The variances of S_1 and S_2 are found to be

(i) $V[S_1] = (1+i)^{2t}\left[Q_n({}_tp_x) - B_n^2({}_tp_x)\right]$.

(ii) $V[S_2] = E[S_2^2] - E[S_2]^2, \quad E[S_2^2] =$

$$(1+i)^{2t}\left\{\ddot{a}_{\overline{t}|}^2 B_n({}_tp_x) - \left(\ddot{a}_{\overline{t}|} - a_1\right)^2 \left[B_n({}_tp_x) - (1 - {}_tq_x^n)\right] + R_n({}_tp_x)\left(\frac{v_1}{n} + a_1^2\right)\right\},$$

where $B_n({}_tp_x)$ and $Q_n({}_tp_x)$ are defined in (1.10) and (1.11), a_1 and v_1 are given by (2.2) and (2.3), $E[S_2]$ by (1.24) and $R_n({}_tp_x) = Q_n({}_tp_x) - B_n({}_tp_x)$.

Proof. **(i)** From (1.21)

$$S_1^2 = \begin{cases} \dfrac{n^2(1+i)^{2t}}{N^2} & \text{if } N = 1, 2, ..., n, \\ 0 & \text{if } N = 0. \end{cases}$$

Then

$$E[S_1^2] = (1+i)^{2t} \sum_{m=1}^{n} \frac{n^2}{m} P_{nm}({}_tp_x) = (1+i)^{2t} Q_n({}_tp_x).$$

Using this and (1.23)

$$V[S_1] = E[S_1^2] - E[S_1]^2 = (1+i)^{2t} \left\{ Q_n({}_tp_x) - B_n^2({}_tp_x) \right\}.$$

(ii) From (1.22)

$$S_2^2 = \begin{cases} \dfrac{(1+i)^{2t} \left[N^2 \left(\ddot{a}_{\overline{t}|}^2 \right) + X + Y \right]}{N^2} & \text{if } N = 1, 2, ..., n, \\ 0 & \text{if } N = 0, \end{cases}$$

where $X = 2N\ddot{a}_{\overline{t}|} \sum_{i=1}^{D} \ddot{a}_{\overline{K_i+1}|}$ and $Y = \sum_{i=1}^{D} \sum_{j=1}^{D} \ddot{a}_{\overline{K_i+1}|} \ddot{a}_{\overline{K_j+1}|}$. Then

$$\begin{aligned} E[S_2^2] &= E[E(S_2^2|N)] = \sum_{m=1}^{n} E[S_2^2|N=m] P_{nm}({}_tp_x), \\ &= (1+i)^{2t} \sum_{m=1}^{n} \left(\frac{m^2 \ddot{a}_{\overline{t}|}^2 + E[X|N=m] + E[Y|N=m]}{m^2} \right) P_{nm}({}_tp_x). \end{aligned} \quad (2.46)$$

Using the fact that all lives are mutually independent we have

$$\begin{aligned} E[X|N=m] &= 2m\ddot{a}_{\overline{t}|}(n-m)a_1, \\ E[Y|N=m] &= \sum_{i=1}^{n-m} \sum_{j=1}^{n-m} E[\ddot{a}_{\overline{K_i+1}|} \ddot{a}_{\overline{K_j+1}|} | K_i < t,\ K_j < t]. \end{aligned} \quad (2.47)$$

Since

$$\begin{aligned} &E[\ddot{a}_{\overline{K_i+1}|} \ddot{a}_{\overline{K_j+1}|} | K_i < t,\ K_j < t] = \\ &Cov[\ddot{a}_{\overline{K_i+1}|}, \ddot{a}_{\overline{K_j+1}|} | K_i < t,\ K_j < t] + E[\ddot{a}_{\overline{K_i+1}|} | K_i < t] E[\ddot{a}_{\overline{K_j+1}|} | K_j < t] = \\ &\delta_{ij} v_1 + a_1^2, \end{aligned}$$

then

$$E[Y|N=m] \;=\; (n-m)v_1 + (n-m)^2 a_1^2. \tag{2.48}$$

Replacing (2.47) and (2.48) in (2.46) we get

$$\begin{aligned}
&v^{2t}E[S_2^2] - \ddot{a}_{\overline{t}|}^2(1 - {}_tq_x^n) = \\
&\quad \sum_{m=1}^{n}\left[\frac{2m\ddot{a}_{\overline{t}|}(n-m)a_1 + (n-m)v_1 + (n-m)^2 a_1^2}{m^2}\right] P_{nm}({}_tp_x) = \\
&\quad 2\ddot{a}_{\overline{t}|}a_1\left[B_n({}_tp_x) - (1 - {}_tq_x^n)\right] + \frac{v_1}{n}R_n({}_tp_x) + a_1^2\left\{R_n({}_tp_x) - \left[B_n({}_tp_x) - (1 - {}_tq_x^n)\right]\right\} = \\
&\quad 2\ddot{a}_{\overline{t}|}a_1\left[B_n({}_tp_x) - (1 - {}_tq_x^n)\right] - a_1^2\left[B_n({}_tp_x) - (1 - {}_tq_x^n)\right] + R_n({}_tp_x)\left(\frac{v_1}{n} + a_1^2\right) = \\
&\quad (2\ddot{a}_{\overline{t}|}a_1 - a_1^2)\left[B_n({}_tp_x) - (1 - {}_tq_x^n)\right] + R_n({}_tp_x)\left(\frac{v_1}{n} + a_1^2\right) = \\
&\quad \ddot{a}_{\overline{t}|}^2\left[B_n({}_tp_x) - (1 - {}_tq_x^n)\right] - (\ddot{a}_{\overline{t}|} - a_1)^2\left[B_n({}_tp_x) - (1 - {}_tq_x^n)\right] + R_n({}_tp_x)\left(\frac{v_1}{n} + a_1^2\right),
\end{aligned}$$

from where the result follows. ◇

As we see in the proof of Proposition 2.13, the notation used by Ramsay is not helpful to find expectations since we are dealing with various random variables at the same time. This complication is due to the use of $D = n - N$ random terms in the sum that defines S_2. The following proposition and its proof present a new derivation for the variance of S_2 based on conditional expectations.

Proposition 2.14 The variances of S_1 and S_2 are given by

(i) $V[S_1] = (1+i)^{2t}\left\{Q_n({}_tp_x) - B_n^2({}_tp_x)\right\}.$

(ii) $v^{2t}V[S_2] = a_1^2\left[Q_n({}_tp_x) - B_n^2({}_tp_x)\right] + 2\,{}_tq_x^n\, a_1\left(\ddot{a}_{\overline{t}|} - a_1\right)B_n({}_tp_x)$
$\quad + {}_tq_x^n\,(1 - {}_tq_x^n)\left(\ddot{a}_{\overline{t}|} - a_1\right)^2 + \left[Q_n({}_tp_x) - B_n({}_tp_x)\right]v_1/n.$

Proof. **(i)** We can write

$$S_1 \;=\; (1+i)^t X, \tag{2.49}$$

where X is given by (1.13) and apply (1.14).

(ii) We may write

$$
\begin{aligned}
S_2 &= (1+i)^t U, && (2.50)\\
U &= \begin{cases} \dfrac{Y}{N} & \text{if } N = 1, ..., n, \\ 0 & \text{if } N = 0, \end{cases} \\
Y &= \sum_{i=1}^{n} Y_i, && (2.51)\\
Y_i &= \begin{cases} \ddot{a}_{\overline{K_i+1}|} & \text{if } K_i < t, \\ \ddot{a}_{\overline{t}|} & \text{if } K_i \geq t, \end{cases} \\
&= a_{\overline{K_i+1}|} I_{[K_i<t]} + \ddot{a}_{\overline{t}|} I_{[K_i \geq t]}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E[Y] &= n\ddot{a}_{x:\overline{t}|}, \text{ by (1.2)}, && (2.52)\\
E[Y|N] &= (n-N)a_1 + N\ddot{a}_{\overline{t}|}, && (2.53)\\
V[Y|N] &= (n-N)v_1, && (2.54)\\
E[U|N] &= a_1 X + (\ddot{a}_{\overline{t}|} - a_1) I_{[N>0]}, \text{ by (2.53) and (1.13)}. && (2.55)
\end{aligned}
$$

Using (2.55), (2.1) and (1.14) we get

$$
V[E(U|N)] = a_1^2 \left[Q_n({}_tp_x) - B_n^2({}_tp_x)\right] + 2\, {}_tq_x^n\, a_1 \left(\ddot{a}_{\overline{t}|} - a_1\right) B_n({}_tp_x) + {}_tq_x^n \left(1 - {}_tq_x^n\right) \left(\ddot{a}_{\overline{t}|} - a_1\right)^2.
$$

Since (2.54) and (1.13) imply $V[U|N] = (X^2 - X)\, v_1/n$, then (1.14) yields

$$
E[V(U|N)] = [Q_n({}_tp_x) - B_n({}_tp_x)]\, v_1/n.
$$

Finally from (2.50) we have $v^{2t}V[S_2] = V[U] = V[E(U|N)] + E[V(U|N)]$. ◇

Corollary 2.7 For S_1 and S_2 we have

(i) $\lim_{n\to\infty} S_1 = {}_ts_x$. (ii) $\lim_{n\to\infty} S_2 = \ddot{s}_{x:\overline{t}|}$.

Proof. (i) Using (2.7) and (2.14) in Proposition 2.13 - (i) we get $\lim_{n\to\infty} V[S_1] = 0$ and from Proposition 2.11 - (iii) we see that $\lim_{n\to\infty} E[S_1] = {}_ts_x$.

(ii) Use of (2.7) and (2.14) in Proposition 2.13 - (ii) yields

$$\begin{aligned}
\lim_{n\to\infty} E[S_2^2] &= (1+i)^{2t}\left[\ddot{a}_{\overline{t}|}^2\frac{1}{{}_tp_x} - \left(\ddot{a}_{\overline{t}|} - a_1\right)^2\left(\frac{1}{{}_tp_x} - 1\right) + \left(\frac{1}{{}_tp_x^2} - \frac{1}{{}_tp_x}\right)a_1^2\right], \\
&= {}_ts_x^2\left[\ddot{a}_{\overline{t}|}^2\,{}_tp_x - \left(\ddot{a}_{\overline{t}|} - a_1\right)^2{}_tp_x\,{}_tq_x + a_1^2\,{}_tq_x\right], \\
&= {}_ts_x^2\left[\ddot{a}_{\overline{t}|}\,{}_tp_x + a_1\,{}_tq_x\right]^2 = {}_ts_x^2\,\ddot{a}_{x:\overline{t}|}^2 = \ddot{s}_{x:\overline{t}|}^2.
\end{aligned}$$

Then Propositions 2.11 - (iv) and 2.13 - (ii) imply $\lim_{n\to\infty} V[S_2] = 0$. ◇

Another advantage of Proposition 2.14 is the fact that the proof of Corollary 2.7 is straightforward since from there, $\lim_{n\to\infty} V[S_2] = 0$ is trivial.

Chapter 3

Other Approaches to Future Shares

The first part of this chapter is dedicated to random loss variables, their means and variances. In the second part a reformulation of future random shares is proposed under the principle of equivalence (c.f. Bowers et. al. [1], p. 169).

3.1 Random loss variables

In this section we define random losses for Problems 1 and 2, defined in Chapter 1, and find their expectations and variances. We also prove that under the equivalence principle Ramsay's stochastic definitions of future share are not unbiased.

Definition 3.1 *Random losses.* Let X_1 and X_2 be any random variables proposed as solution to Problems 1 and 2, respectively. Denote by $\pi_1 = \pi_1(X_1) = v^t E[X_1]$ and $\pi_2 = \pi_2(X_2) = v^t E[X_2]$. Therefore, if Y is defined as in (2.51), the random losses at time 0 are respectively given by

$$L_1(X_1) = N\pi_1 - n, \tag{3.1}$$

$$L_2(X_2) = N\pi_2 - Y. \tag{3.2}$$

Proposition 3.1 The expectations and variances of L_1 and L_2 are found to be

$$\begin{aligned} E[L_1(X_1)] &= n({}_tp_x\,\pi_1 - 1), \\ E[L_2[X_2)] &= n({}_tp_x\,\pi_2 - \ddot{a}_{x:\overline{t}|}), \\ V[L_1(X_1)] &= n\,{}_tp_x\,{}_tq_x\,\pi_1^2, \\ V[L_2(X_2)] &= n\,{}_tq_x\left[{}_tp_x(\pi_2 + a_1 - \ddot{a}_{\overline{t}|})^2 + v_1\right], \end{aligned}$$

where a_1 and v_1 are given by (2.2) and (2.3).

Proof. The first formula is true since $E[N] = n\,{}_tp_x$. The second equality is obtained from (2.52). The third comes from $V[N] = n\,{}_tp_x\,{}_tq_x$. Finally,

$$\begin{aligned} V[L_2(X_2)] &= V\{E[L_2(X_2)|N]\} + E\{V[L_2(X_2)|N]\}, \\ &= V[N\pi_2 - E(Y|N)] + E[V(Y|N)], \text{ from (3.2)}, \\ &= V\{N\pi_2 - [(n-N)a_1 + N\ddot{a}_{\overline{t}|}]\} + E[(n-N)v_1], \text{ by (2.53) and (2.54)}, \\ &= V[(\pi_2 + a_1 - \ddot{a}_{\overline{t}|})N] + E[(n-N)v_1], \\ &= n\,{}_tp_x\,{}_tq_x(\pi_2 + a_1 - \ddot{a}_{\overline{t}|})^2 + n\,{}_tq_x\,v_1, \\ &= n\,{}_tq_x\left[{}_tp_x(\pi_2 + a_1 - \ddot{a}_{\overline{t}|})^2 + v_1\right]. \end{aligned}$$

$\diamond$

Under the principle of equivalence we want unbiased random losses, i.e.,

$$E[L_1(X_1)] = E[L_2(X_2)] = 0.$$

From Proposition 3.1 this is equivalent to

$$E[X_1] = {}_ts_x, \tag{3.3}$$

$$E[X_2] = \ddot{s}_{x:\overline{t}|}. \tag{3.4}$$

Then solving Problems 1 and 2 under the principle of equivalence leads to find X_1 and X_2 such that (3.3) and (3.4) hold, respectively. The deterministic actuarial solutions ${}_ts_x$ and $\ddot{s}_{x:\overline{t}|}$ trivialy satisfy (3.3) and (3.4), respectively. In the meantime, Ramsay's stochastic solutions S_1 and S_2, in general do not satisfy conditions (3.3) and (3.4). In fact, from Proposition 2.12, the random losses $L_1(S_1)$ and $L_2(S_2)$ never are unbiased when $n = 1$. For $n \geq 2$, $L_1(S_1)$ is unbiased only if ${}_tp_x = \beta_n$, while $L_2(S_2)$ is unbiased only if ${}_tp_x = {}_r\gamma_n$. We keep in mind that these values tend to 0 as n increases.

Equations (3.3) and (3.4) imply that the actuarial present value of the future benefit equals 1 in Problem 1 and $\ddot{a}_{x:\overline{t}|}$ in Problem 2. Under Ramsay's definitions these actuarial present values are given respectively by ρ_1 and $\ddot{a}_{x:\overline{t}|}\rho_2$.

3.2 New definitions of stochastic future shares

We need to modify S_1 and S_2 so that the new random variables satisfy (3.3) and (3.4), respectively. Since there are many different ways to do this we present here two of them.

Definition 3.2 Define the random future share for Problem 1 as any of the following

$$S_{11} = S_1 + \left(\frac{{}_ts_x - E[S_1]}{1 - {}_tq_x^n}\right) I_{[N>0]}, \quad (3.5)$$

$$S_{12} = S_1 + {}_ts_x - \left(\frac{E[S_1]}{1 - {}_tq_x^n}\right) I_{[N>0]}. \quad (3.6)$$

For Problem 2 it is defined as any of the following

$$S_{21} = S_2 + \left(\frac{\ddot{s}_{x:\overline{t}|} - E[S_2]}{1 - {}_tq_x^n}\right) I_{[N>0]}, \quad (3.7)$$

$$S_{22} = S_2 + \ddot{s}_{x:\overline{t}|} - \left(\frac{E[S_2]}{1 - {}_tq_x^n}\right) I_{[N>0]}. \quad (3.8)$$

Observe that in (3.5) and (3.7) we added a correction term only when $N > 0$, while in (3.6) and (3.8) we added two terms, one when $N > 0$ and another when $N = 0$. The terms which provide minimal variance of the future random share are given in Lemma 2.2. We note that if $n = 1$ then $S_{12} = {}_ts_x$ and $S_{22} = \ddot{s}_{x:\overline{t}|}$, i.e., they are deterministic.

Proposition 3.2 Consider the random variables S_{11}, S_{12}, S_{21} and S_{22} given by (3.5), (3.6), (3.7) and (3.8), respectively. Then

(i) They have unbiased random losses, defined by (3.1) and (3.2).

(ii) Their variances are found to be

$$\begin{aligned}
V[S_{11}] &= V[S_1] + \left(\frac{{}_tq_x^n}{1 - {}_tq_x^n}\right)\left({}_ts_x^2 - E[S_1]^2\right), \\
V[S_{12}] &= V[S_1] - \left(\frac{{}_tq_x^n}{1 - {}_tq_x^n}\right) E[S_1]^2, \\
V[S_{21}] &= V[S_2] + \left(\frac{{}_tq_x^n}{1 - {}_tq_x^n}\right)\left(\ddot{s}_{x:\overline{t}|}^2 - E[S_2]^2\right), \\
V[S_{22}] &= V[S_2] - \left(\frac{{}_tq_x^n}{1 - {}_tq_x^n}\right) E[S_2]^2,
\end{aligned}$$

where $V[S_1]$ and $V[S_2]$ are given by Proposition 2.13 or 2.14, while $E[S_1]$ and $E[S_2]$ are as in (1.23) and (1.24).

(iii) The variances of the corresponding losses are given by

$$\begin{aligned}
V[L_1(S_{11})] &= V[L_1(S_{12})] = V[L_1({}_ts_x)] = \frac{n\,{}_tq_x}{{}_tp_x}, \\
V[L_2(S_{21})] &= V[L_2(S_{22})] = V[L_2(\ddot{s}_{x:\overline{t}|})] = n\frac{{}_tq_x}{{}_tp_x}\left[a_1^2 + {}_tp_x\, v_1\right],
\end{aligned}$$

where a_1 and v_1 are defined in (2.2) and (2.3).

(iv) We also have:

$$\begin{aligned}
V[S_1|N>0] &= V[S_{11}|N>0] = V[S_{12}|N>0] = \frac{E[S_1^2]}{1 - {}_tq_x^n} - \left(\frac{E[S_1]}{1 - {}_tq_x^n}\right)^2, \\
V[S_2|N>0] &= V[S_{21}|N>0] = V[S_{22}|N>0] = \frac{E[S_2^2]}{1 - {}_tq_x^n} - \left(\frac{E[S_2]}{1 - {}_tq_x^n}\right)^2, \\
E[S_1S_2] &= (1+i)^{2t}\left[a_1Q_n({}_tp_x) + (\ddot{a}_{\overline{t}|} - a_1)B_n({}_tp_x)\right], \\
Cov[S_1, S_2|N>0] &= Cov[S_{11}, S_{21}|N>0] = Cov[S_{12}, S_{22}|N>0], \\
&= \frac{E[S_1S_2]}{1 - {}_tq_x^n} - \frac{E[S_1]E[S_2]}{(1 - {}_tq_x^n)^2}.
\end{aligned}$$

Proof. **(i)** Clearly (3.3) holds for S_{11} and S_{12}, while (3.4) holds for S_{21} and S_{22}.
(ii) It follows from (2.1).
(iii) Here $\pi_1 = 1/\,{}_tp_x$, $\pi_2 = \ddot{a}_{x:\overline{t}|}/\,{}_tp_x$, and the result follows from Proposition 3.1.
(iv) We need to prove only the formula for $E[S_1S_2]$. To this

$$\begin{aligned}
E[S_1S_2] &= E\{E[S_1S_2|N]\}, \\
&= (1+i)^{2t}E\left\{XE[U|N]\right\}, \text{ by (2.49) and (2.50)}, \\
&= (1+i)^{2t}E\left\{X\left[a_1X + \left(\ddot{a}_{\overline{t}|} - a_1\right)I_{[N>0]}\right]\right\}, \text{ from (2.55)}, \\
&= (1+i)^{2t}E\left[a_1X^2 + \left(\ddot{a}_{\overline{t}|} - a_1\right)X\right], \\
&= (1+i)^{2t}\left[a_1Q_n({}_tp_x) + (\ddot{a}_{\overline{t}|} - a_1)B_n({}_tp_x)\right], \text{ by (1.14)}. \qquad \diamond
\end{aligned}$$

Variances of $L_1(S_1)$ and $L_2(S_2)$ can be computed from Proposition 3.1 with $\pi_1 = v^tE[S_1]$ and $\pi_2 = v^tE[S_2]$. We note that for ${}_tp_x$ large enough, say ${}_tp_x > \beta_n$ (see Proposition 2.7 and Corollary 2.6) for $L_1(S_1)$ and ${}_tp_x > 2.25/n$ for $L_2(S_2)$, these variances are larger than those given in Proposition 3.2 - (iii).

Chapter 4

An Application and Numerical Results

This chapter presents the deterministic approach of retrospective reserves in a particular case and also suggests a stochastic definition. The chapter ends with some numerical results and conclusions.

4.1 Reserves

In the modern theory of life contingencies there is no stochastic definition for retrospective net premium reserves. They are defined only as the mean of the stochastic prospective loss, then by definition both retrospective and prospective net reserves are the same.

In this section the results from Chapter 3 are extended to retrospective net reserves for a t-year pure endowment insurance. Other cases can be considered in a similar way. We start presenting the deterministic approach.

4.1.1 The actuarial deterministic approach

Let $P_{x:\overline{t}|} = P(A_{x:\overline{t}|})$ be the net level annual premium for a unit t-year endowment insurance for (x). The loss for this insurance is

$$L = Z - P_{x:\overline{t}|}Y,$$

where Y and Z are given by (1.1) and (1.5), respectively. Taking expectations and using the equivalence principle we get

$$P_{x:\overline{t}|} = \frac{A_{x:\overline{t}|}}{\ddot{a}_{x:\overline{t}|}} = \frac{1 - d\ddot{a}_{x:\overline{t}|}}{\ddot{a}_{x:\overline{t}|}}.$$

Since $Y = (1 - Z)/d$ then $L = \left(1 + \frac{P_{x:\overline{t}|}}{d}\right) Z - \frac{P_{x:\overline{t}|}}{d}$. Taking variance and using (1.7)

$$V[L] = \left(1 + \frac{P_{x:\overline{t}|}}{d}\right)^2 \left({}^2A_{x:\overline{t}|} - A^2_{x:\overline{t}|}\right).$$

Let $0 \leq h \leq t$ be an integer and J the curtate future life time of $(x + h)$. The p.f. of J is given by ${}_jp_{x+h}\, q_{x+h+j}$, for $j = 0, 1, \ldots, w - x - h - 1$. At time h the present value of future benefits for this insurance is

$$Z_1 = \begin{cases} v^{J+1} & \text{if } J < t - h, \\ v^{t-h} & \text{if } J \geq t - h. \end{cases}$$

Denote by $\ddot{a}_{\overline{0}|} = 0$. Then the present value of future contributions is $P_{x:\overline{t}|}Y_1$, where

$$Y_1 = \begin{cases} \ddot{a}_{\overline{J+1}|} & \text{if } J < t - h, \\ \ddot{a}_{\overline{t-h}|} & \text{if } J \geq t - h. \end{cases}$$

The difference

$${}_hL = Z_1 - P_{x:\overline{t}|}Y_1, \tag{4.1}$$

is called the prospective loss at time h for this insurance. Its mean, denoted by ${}_hV_{x:\overline{t}|}$ and given by

$${}_hV_{x:\overline{t}|} = \begin{cases} A_{x+h:\overline{t-h}|} - P_{x:\overline{t}|}\ddot{a}_{x+h:\overline{t-h}|} & \text{if } h < t, \\ 1 & \text{if } h = t, \end{cases} \tag{4.2}$$

is called the prospective net premium reserve at time h. At the beginning the reserve has been defined using the principle of equivalence, then ${}_0V_{x:\overline{t}|} = 0$.

Since $Y_1 = (1 - Z_1)/d$ then

$${}_hL = \left(1 + \frac{P_{x:\overline{t}|}}{d}\right) Z_1 - \frac{P_{x:\overline{t}|}}{d}.$$

From this and (1.7) we get

$$V[{}_hL] = \begin{cases} \left(1 + \frac{P_{x:\overline{t}|}}{d}\right)^2 \left({}^2A_{x+h:\overline{t-h}|} - A^2_{x+h:\overline{t-h}|}\right) & \text{if } h < t, \\ 0 & \text{if } h = t. \end{cases} \tag{4.3}$$

Formula in (4.2) can be written as

$${}_hV_{x:\overline{t}|} = P_{x:\overline{t}|}\ddot{s}_{x:\overline{h}|} - A^{\ 1}_{x:\overline{h}|}\, {}_hs_x\,. \tag{4.4}$$

The expression $A^{\ 1}_{x:\overline{h}|}\, {}_hs_x$ is usually denoted by ${}_hK_x$ and is called the accumulated cost of insurance. Formula in (4.4) is called the retrospective net premium reserve of the t-year endowment insurance. We observe that by definition the retrospective and the prospective net premium reserves are the same.

According to the deterministic definition of future values, expression $P_{x:\overline{t}|}\,\ddot{s}_{x:\overline{h}|}$ is the actuarial accumulated value of an annuity of $P_{x:\overline{t}|}$ per year at the end of h years, while ${}_hK_x$ is the assessment against each of the l_{x+h} survivors to provide for the accumulated value of the death claims in the survivorship group between ages x and $x + h$. Thus, the reserve given by (4.4) can be interpreted as the difference between the net premiums, accumulated with interest and shared among the survivors at age $x + h$, and the accumulated cost of insurance (see Bowers et. al. [1], pp. 216-217).

4.1.2 A stochastic approach

Let π be the individual net level annual premium for a unit t-year endowment insurance for each member of n mutually independent lives age x. At time 0 the individual loss for this insurance can be defined to be

$$L(n) = \frac{Z - \pi Y}{n},$$

where $Z = \sum_{i=1}^{n} Z_i$, Z_i is defined as in (1.5) with $K = K_i$ and Y is given by (2.51). Since all lives are mutually independent then from (1.6), solving $E[L(n)] = 0$ we get $\pi = P_{x:\overline{t}|}$.

Now suppose that each member of the group is participating in an h-year term life insurance of 1 payable at the end of the year of death. Let π_h be the individual net single premium for it. The individual loss for this insurance can be defined as

$$L_1(n) = \frac{Z - n\pi_h}{n},$$

where $Z = \sum_{i=1}^{n} Z_i$ and Z_i is given by (1.3) with $K = K_i$ and $t = h$. Solving $E[L_1(n)] = 0$ we get $\pi_h = A_{\overset{1}{x}:\overline{h}|}$.

For a t-year endowment insurance we want to define a possible retrospective random loss at time h given that there is at least one survivor, that is, $N > 0$. As one of many possible definitions we propose the following. The individual retrospective loss at time $0 \leq h \leq t$ is denoted by ${}_hL(n)$ and is defined as the accumulated value of past premiums minus the accumulated value of past death benefits. For $h = 0$ it gives 0. For $1 \leq h \leq t$, from (3.7) we may interpret the expression $P_{x:\overline{t}|}S_{21}$ (here S_{21} is computed using h instead of t) as the future value at time h of the individual annuity $P_{x:\overline{t}|}$, with accumulation under interest and survivorship. Similarly, from (3.5), expression $A_{\overset{1}{x}:\overline{h}|}S_{11}$ (S_{11} computed using h) can be seen as the future value at time h of individual past death benefits up to time h (or the accumulated random cost of insurance). Therefore,

$${}_hL(n) = \begin{cases} P_{x:\overline{t}|}S_{21} - A_{\overset{1}{x}:\overline{h}|}S_{11} & \text{if } 0 < h \leq t, \\ 0 & \text{if } h = 0. \end{cases}$$

The retrospective net premium reserve at time h, denoted by ${}_hV_{x:\overline{t}|}(n)$, is given by

$${}_hV_{x:\overline{t}|}(n) = E[{}_hL(n)|N > 0] = \frac{{}_hV_{x:\overline{t}|}}{1 - {}_hq_x^n}.$$

We see that ${}_hV_{x:\overline{t}|}(n) > {}_hV_{x:\overline{t}|}$ for $h \geq 1$. Also ${}_hV_{x:\overline{t}|}(n) \approx {}_hV_{x:\overline{t}|}$ for large values of n. It is also clear that ${}_hL(1)$ is deterministic. Besides, we observe that when h is close to t our definition gives ${}_hV_{x:\overline{t}|}(n) > 1$. In particular when $h = t$ we get ${}_tV_{x:\overline{t}|}(n) = \frac{1}{1-{}_tq_x^n} > 1$. This property is not desirable since we would like that ${}_hV_{x:\overline{t}|}(n) \leq 1$. To fix this problem we can use in ${}_hL(n)$ the accumulation factors S_{12} and S_{22} instead of S_{11} and

S_{21}, i.e.,

$$ {}_hL(n) \;=\; \begin{cases} P_{x:\overline{t}|}S_{22} - A_{\stackrel{1}{x}:\overline{h}|}S_{12} & \text{if } 0 < h \leq t, \\ 0 & \text{if } h = 0. \end{cases} \tag{4.5} $$

Now since for $1 \leq h \leq t$ we have $E[S_{12}|N > 0] = {}_hs_x$ and $E[S_{22}|N > 0] = \ddot{s}_{x:\overline{h}|}$, then

$$ {}_hV_{x:\overline{t}|}(n) \;=\; E[{}_hL(n)|N > 0] = {}_hV_{x:\overline{t}|}. $$

Therefore, ${}_hV_{x:\overline{t}|}(n) = {}_hV_{x:\overline{t}|}$ for all $0 \leq h \leq t$.

Proposition 4.1 For ${}_hL(n)$ given in (4.5) we have $V[{}_0L(n)|N > 0] = 0$. If $1 \leq h \leq t$ then

$$ \begin{aligned} V[{}_hL(n)|N > 0] \;=\; & P^2_{x:\overline{t}|}\left[\frac{E[S_2^2]}{1 - {}_hq_x^n} - \left(\frac{E[S_2]}{1 - {}_hq_x^n}\right)^2\right] + A^2_{\stackrel{1}{x}:\overline{h}|}\left[\frac{E[S_1^2]}{1 - {}_hq_x^n} - \left(\frac{E[S_1]}{1 - {}_hq_x^n}\right)^2\right] \\ & - 2P_{x:\overline{t}|}A_{\stackrel{1}{x}:\overline{h}|}\left[\frac{E[S_1S_2]}{1 - {}_hq_x^n} - \frac{E[S_1]E[S_2]}{(1 - {}_hq_x^n)^2}\right]. \end{aligned} $$

Proof. It follows from (4.5) and Proposition 3.2 - (iv). ◇

There is a way similar to that given in (4.1) to define a prospective random loss such that the retrospective and the prospective net premium reserves are the same. In fact, for a t-year endowment insurance we denote by ${}_hPL(n)$ the average of the individual prospective random losses at time $0 \leq h \leq t$. Given that $N > 0$ we denote by J_i, $i = 1, 2, \cdots, N$, the curtate future life time of the i^{th} life and by ${}_hL_i$ the prospective loss given by (4.1) with $J = J_i$. Therefore, the prospective random loss is given by

$$ {}_hPL(n) = \begin{cases} \dfrac{\sum_{i=1}^{N} {}_hL_i}{N} & \text{if } N = 1, \cdots, n, \\ 0 & \text{if } N = 0. \end{cases} $$

The prospective net single premium is then

$$ E[{}_hPL(n)|N > 0] \;=\; E\{E[{}_hPL(n)|N,\, N > 0]\} \;=\; E[{}_hV_{x:\overline{t}|}] \;=\; {}_hV_{x:\overline{t}|}. $$

The variance of the prospective loss is

$$\begin{aligned} V[{}_hPL(n)|N>0] &= V\{E[{}_hPL(n)|N,\ N>0]\}+E\{V[{}_hPL(n)|N,\ N>0]\}, \\ &= V[{}_hV_{x:\overline{t}|}]+E\{V[{}_hL]/N|N>0\}, \text{ by (4.1) and (4.3)}, \\ &= V[{}_hL]b_1(n,{}_hp_x). \end{aligned} \tag{4.6}$$

We see that

$$\begin{aligned} \lim_{n\to\infty} V[{}_hPL(n)|N>0] &= 0, \\ V[{}_hPL(n)|N>0] &\le V[{}_hL], \\ V[{}_0PL(n)|N>0] &= V[{}_0L]/n, \\ V[{}_hPL(1)|N>0] &= V[{}_hL]. \end{aligned}$$

4.2 Numerical results

Here we present some numerical results using the formulas for variances of future shares and reserves found in previous sections and chapters. Ramsay's assumptions in page 12 are used in our computations when no other assumptions are specified.

4.2.1 Future shares

Tables 4.1 and 4.2 show the variances of the different future random shares defined in pages 10 and 51. The formulas are given in Propositions 2.14 and 3.2 - (ii). We can see that differences occur only for small values of $n\,{}_tp_x$, precisely when Ramsay's formulas differ from the deterministic formulas. The cases $n = 500$ and $n = 5000$ are presented to see how fast they decrease to 0 as n increases. There is a substantial difference between both tables. Variances in Table 4.1 are small and decrease quickly, but variances in Table 4.2 are large and decrease slowly. This is natural since in Problem 1 [see page 4] there is only a single payment, while payments in Problem 2 depend on the random variables K_i, for $i = 1, \ldots, n$, and hence involve more uncertainty, especially when t is large as seen in Table 4.2. When $n = 1$, the variances of S_{12} and S_{22} are null since in this case these expressions are deterministic.

Table 4.1: Variances of S_1, S_{11} and S_{12} when (x) survives to 65

n	$x=20$	$x=30$	$x=40$	$x=50$	$x=60$
1	32.1561	9.6997	2.8467	0.7653	0.1316
	85.4030	24.5363	6.6473	1.5247	0.1837
	0.0000	0.0000	0.0000	0.0000	0.0000
2	56.9967	17.1169	4.9970	1.3397	0.2393
	56.6153	16.9527	4.9332	1.3206	0.2379
	41.4067	12.7436	3.8670	1.1122	0.2243
5	49.3233	13.9120	3.6405	0.7672	0.0719
	49.2984	13.9064	3.6395	0.7671	0.0719
	49.1509	13.8706	3.6324	0.7663	0.0719
10	13.6042	3.8427	1.0137	0.2215	0.0243
	13.6042	3.8427	1.0137	0.2215	0.0243
	13.6041	3.8427	1.0137	0.2215	0.0243
500	0.1721	0.0494	0.0134	0.0031	0.0004
	0.1721	0.0494	0.0134	0.0031	0.0004
	0.1721	0.0494	0.0134	0.0031	0.0004

Table 4.2: Variances of S_2, S_{21} and S_{22} when (x) survives to 65

n	$x=20$	$x=30$	$x=40$	$x=50$	$x=60$
1	8630.98	2290.87	522.70	81.11	2.62
	21726.92	5399.97	1114.96	146.16	3.46
	0.00	0.00	0.00	0.00	0.00
2	13029.64	3219.19	660.65	88.27	2.56
	12978.18	3199.15	654.68	87.23	2.54
	9109.03	2272.81	475.84	67.25	2.29
5	10463.30	2361.35	417.97	42.54	0.70
	10457.62	2360.28	417.83	42.53	0.70
	10420.10	2352.41	416.63	42.45	0.70
10	2893.38	654.94	117.22	12.43	0.24
	2893.38	654.93	117.22	12.43	0.24
	2893.36	654.93	117.22	12.43	0.24
5000	3.65	0.84	0.15	0.02	0.00
	3.65	0.84	0.15	0.02	0.00
	3.65	0.84	0.15	0.02	0.00

In Tables 4.3 and 4.4 we present the variances of the random losses $L_1(S_1)$, $L_1({}_t s_x)$, $L_2(S_2)$ and $L_2(\ddot{s}_{x:\overline{t}|})$, defined in (3.1) and (3.2). The formulas are given in Proposition 3.1. As expected, variances increase as the size of the group increases. Also, when ${}_t s_x$ and $\ddot{s}_{x:\overline{t}|}$ (or any of S_{11}, S_{12}, S_{21}, S_{22}) are used, the variances are smaller, except for very small values of n. For large values of n they are asymptotically the same.

Table 4.3: Variances of $L_1(S_1)$ and $L_1({}_t s_x)$ when (x) survives to 65

n	$x = 20$	$x = 30$	$x = 40$	$x = 50$	$x = 60$
1	0.10	0.10	0.10	0.09	0.06
	0.28	0.26	0.24	0.19	0.09
2	0.57	0.54	0.50	0.41	0.19
	0.55	0.52	0.47	0.38	0.17
5	1.62	1.51	1.35	1.04	0.45
	1.38	1.31	1.18	0.94	0.43
20	5.70	5.37	4.84	3.84	1.75
	5.53	5.22	4.72	3.76	1.74
100	27.81	26.25	23.73	18.88	8.70
	27.66	26.11	23.62	18.81	8.68
200	55.47	52.37	47.34	37.69	17.38
	55.32	52.23	47.23	37.61	17.36

Table 4.4: Variances of $L_2(S_2)$ and $L_2(\ddot{s}_{x:\overline{t}|})$ when (x) survives to 65

n	$x = 20$	$x = 30$	$x = 40$	$x = 50$	$x = 60$
1	21.75	17.16	11.36	5.14	0.60
	59.05	44.74	27.51	10.65	0.85
2	119.80	91.63	57.14	22.67	1.84
	118.11	89.48	55.01	21.30	1.70
5	343.50	257.52	155.65	58.46	4.43
	295.27	223.70	137.53	53.24	4.26
20	1215.73	919.17	563.28	216.84	17.17
	1181.07	894.79	550.12	212.98	17.04
100	5937.56	4496.63	2762.88	1068.52	85.31
	5905.36	4473.93	2750.58	1064.89	85.18
200	11842.65	8970.36	5513.36	2133.38	170.49
	11810.72	8947.85	5501.17	2129.77	170.36

4.2.2 Reserves

The variance of ${}_hL(n)$ for a 45-year endowment insurance of 1 for (20), given in Proposition 4.1, is presented in Table 4.5. If $n = 1$ we see that $V[{}_hL(1)|N > 0] = 0$, showing that the variable is deterministic, as well as when $h = 0$. $V[{}_hL(n)|N > 0]$ increases with h and tends to 0 as n increases. Hence, for large n, ${}_hL(n)$ degenerates to the deterministic ${}_hV_{20:\overline{45}|}$.

Table 4.5: $V[{}_hL(n)]$ for a 45-year endowment insurance of 1 for (20)

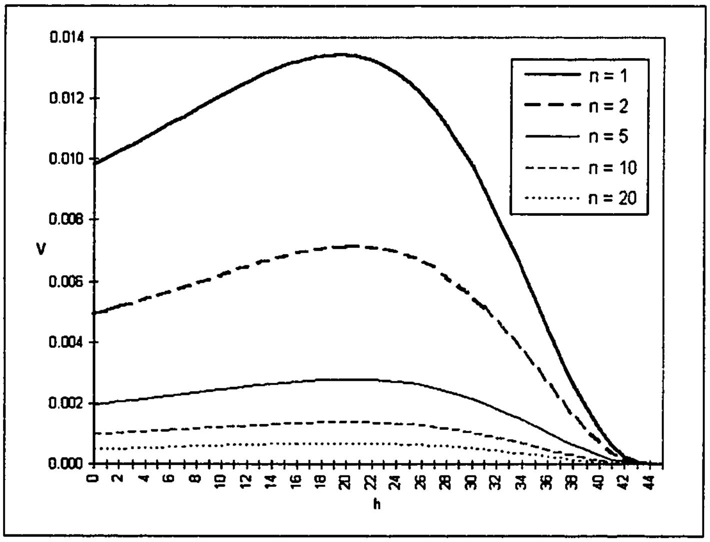
h	$n = 1$	$n = 2$	$n = 5$	$n = 10$	$n = 20$	$n = 50$
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
3	0.000000	0.000001	0.000000	0.000000	0.000000	0.000000
4	0.000000	0.000002	0.000000	0.000000	0.000000	0.000000
5	0.000000	0.000004	0.000001	0.000000	0.000000	0.000000
10	0.000000	0.000046	0.000008	0.000003	0.000001	0.000001
15	0.000000	0.000246	0.000045	0.000017	0.000008	0.000003
20	0.000000	0.000957	0.000190	0.000071	0.000032	0.000012
25	0.000000	0.003137	0.000706	0.000259	0.000114	0.000042
30	0.000000	0.009114	0.002483	0.000870	0.000378	0.000140
35	0.000000	0.023890	0.008701	0.002842	0.001213	0.000445
40	0.000000	0.056425	0.031576	0.009401	0.003882	0.001407
42	0.000000	0.077089	0.053592	0.015411	0.006234	0.002241
43	0.000000	0.089448	0.069982	0.019846	0.007927	0.002837
44	0.000000	0.103262	0.091482	0.025686	0.010109	0.003599
45	0.000000	0.118587	0.119646	0.033444	0.012936	0.004578

The variance of ${}_hPL(n)$ for a 45-year endowment insurance of 1 for (20), given by (4.6), is presented in Table 4.6. It equals $V[h_L]$ for $n = 1$ and tends to 0 as n increases. Hence, for large n, ${}_hPL(n)$ and ${}_hV_{x:\overline{t}|}$ are asymptotically the same. $V[{}_hPL(n)]$, for a fixed n, first increases and then decreases to 0. This is seen in Figure 4.1

Table 4.6: $V[{}_hPL(n)]$ for a 45-year endowment insurance of 1 for (20)

h	$n=1$	$n=2$	$n=5$	$n=10$	$n=20$	$n=50$
0	0.00984	0.00492	0.00197	0.00098	0.00049	0.00020
1	0.01005	0.00504	0.00201	0.00101	0.00050	0.00020
2	0.01027	0.00515	0.00206	0.00103	0.00051	0.00021
3	0.01049	0.00528	0.00211	0.00105	0.00053	0.00021
4	0.01072	0.00540	0.00215	0.00108	0.00054	0.00022
5	0.01095	0.00553	0.00220	0.00110	0.00055	0.00022
10	0.01212	0.00620	0.00246	0.00123	0.00061	0.00025
15	0.01309	0.00681	0.00269	0.00134	0.00067	0.00027
20	0.01342	0.00712	0.00280	0.00139	0.00069	0.00028
25	0.01250	0.00681	0.00266	0.00132	0.00066	0.00026
30	0.00979	0.00553	0.00215	0.00106	0.00053	0.00021
35	0.00541	0.00320	0.00124	0.00061	0.00030	0.00012
40	0.00117	0.00073	0.00029	0.00014	0.00007	0.00003
42	0.00026	0.00017	0.00007	0.00003	0.00002	0.00001
43	0.00006	0.00004	0.00002	0.00001	0.00000	0.00000
44	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
45	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Figure 4.1: $V[{}_hPL(n)]$ for a 45-year endowment insurance of 1 for (20)



4.3 Conclusions

The results on the polynomials B_n are important to study the ratios ρ_1 and ρ_2. They also can be used to investigate when some approximations for the first inverse moment of a positive binomial variate, found in [13], [10], [5] and [12], are appropriate.

Two different ways of fixing the bias problem of Ramsay's stochastics future shares under the principle of equivalence are proposed. S_{12} and S_{22} seem to be more convenient definitions. With these new definitions it is proved that the use of the deterministic formulas is adequate. These new random future shares can be used to estimate the inacuracy involving the use of ${}_ts_x$ and $\ddot{s}_{x:\overline{t}|}$.

We suggest a random retrospective loss variable that enables to estimate the inacuracy of ${}_hV_{x:\overline{t}|}$. It seems that the use of the deterministic ${}_hV_{x:\overline{t}|}$ is quite adequate. Under the proposed stochastic approach the retrospective and the prospective net single premium reserves are the same.

The stochastic life contingencies problems considered here focus on the definition of future shares with constant interest but random survivorship. Clearly, this research needs to be extended to the stochastic interest case, as proposed in the earlier works of Martin-Löf [8] and Frees [2].

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