To Mary and the world's children

A Marie et aux enfants du monde

A Maria e ai bambini del monde.

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ACKNOWLEDGEMENTS

I am very grateful to all those people who, for many years, have contributed to diffuse and stimulate my interest in those fields of science which I think are essential to human needs: Nuclear Physics and Nuclear Technology.

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INTRODUCTION

<< Those who fall in love with practice without science are like a sailor who enters a ship without helm or compass and who never can be certain whither he is going. >>

L. da Vinci

I started this thesis with the unique idea of treating fission product behavior in CANDU reactors. Suddenly I realized that fission product poisoning of thermal nuclear reactors requires a clear understanding of nuclear reactions, cross section, fission process, and neutronics.

Based on these requirements, the steps in this thesis are arranged so as to present the material with a growing complexity level. It begins with basic elements of power reactors and is extended to include the treatment of neutron density, neutron transport equation, its simplifications, and the derivation of the one-group diffusion equation which is heavily detailed, including its applications to various reactor geometries.

In conclusion, this thesis give for the first time, a detailed and systematic description of the time dependent differential equations governing the fission product behavior in thermal reactors.

The differential equations are solved in detail for various nuclear reactor conditions (startup, shutdown) and the solutions are tabulated in Tables V-1 and V-2 for reactor startup and reactor shutdown conditions respectively.

Computer programs have been written for each individual solution given in Tables V-1 and V-2 and also for reactivity changes following reactor shutdown.

The flux levels, cross sections, decay constants and yields used for the calculations are given in Appendix C.

Some of the data used in this thesis is the same as the parameters used for calculations in modern CANDU power reactors (Slowpoke, Darlington for example). These programs are now available for simulating fission product behavior and they can be used for more extensive analysis.

CHAPTER I

I. NUCLEAR FISSION REACTORS

I.1 INTRODUCTION

The term nuclear reactor will be used in this context to refer to devices in which controlled nuclear fission chain reactions can be maintained. (This restricted definition may offend that segment of the nuclear community involved in nuclear fusion research, but since even a prototype nuclear fusion reactor seems several years down the road, no confusion should result.) In such a device, neutrons are used to induce nuclear fission reactions in heavy nuclei. These nuclei fission into lighter nuclei (fission products), accompanied by the release of energy (some 200 MeV per event) plus several additional neutrons. These fission neutrons can then be utilized to induce still further fission reactions, thereby inducing a chain of fission events. In a very narrow sense then, a nuclear reactor is simply a sufficiently large mass of appropriately fissile material (e.g., ^{235}U or ^{239}Pu) in which such a controlled fission chain reaction can be sustained. Indeed a small sphere of ^{235}U metal slightly over 8 cm in radius could support such a chain reaction and hence would be classified as a nuclear reactor.

However a modern power reactor is a considerably more complex device. It must not only contain a lattice of very refined and fabricated nuclear fuel, but must as well provide for cooling this fuel during the course of the chain reaction as fission energy is released, while maintaining the fuel in a very precise geometrical arrangement with

appropriate structural materials. Furthermore some mechanism must be provided to control the chain reaction, shield the surroundings of the reactor from the intense nuclear radiation generated during the fission reactions, and provide for replacing nuclear fuel assemblies when the fission chain reaction has depleted their concentration of fissile nuclei. If the reactor is to produce power in a useful fashion, it must also be designed to operate both economically and safely.

Nuclear reactors have been used for over 30 years in a variety of applications. They are particularly valuable tools for nuclear research since they produce copious amounts of nuclear radiation, primarily in the form of neutrons and gamma rays. Such radiation can be used to probe the microscopic structure and dynamics of matter (neutron or gamma spectroscopy).

The radiation produced by reactors can also be used to transmute nuclei into artificial isotopes that can then be used, for example, as radioactive tracers in industrial or medical applications. Reactors can use the same scheme to produce nuclear fuel from nonfissile materials. For example, ^{238}U can be irradiated by neutrons in a reactor and transmuted into the nuclear fuel ^{239}Pu. This is the process utilized to "breed" fuel in the fast breeder reactors currently being developed for commercial application in the next decade.

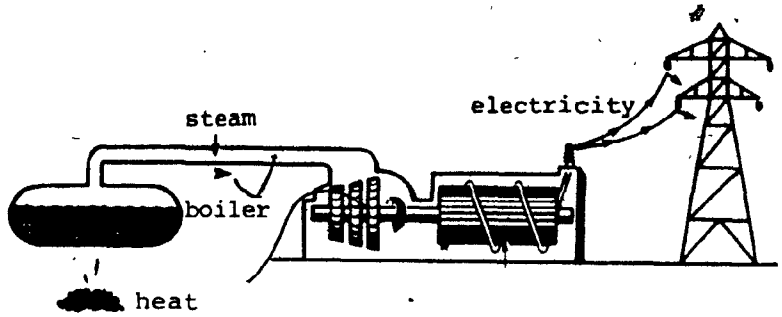
Small, compact reactors have been used for propulsion in submarines, ships aircraft, and rocket vehicles. Indeed the present generation

of light water reactors used in nuclear power plants are little more than the very big younger brothers of the propulsion reactors used in nuclear submarines. Reactors can also be utilized as small, compact sources of long-term power, such as in remote polar research stations or in orbiting satellites.

Yet by far the most significant application of nuclear fission reactors is in large, central station power plants. A nuclear power plant is actually very similar to a fossil-fueled power plant, except that it replaces the coal or oil-fired boiler by a nuclear reactor, which generates heat by sustaining a fission chain reaction in a suitable lattice of fuel material. Of course, there are some dramatic differences between a nuclear reactor and, say, a coal-fired boiler. However the useful quantity produced by each is high temperature, high pressure steam that can then be used to run turbogenerators and produce electricity. At the center of a modern nuclear plant is the nuclear supply system (NSSS), composed of the nuclear reactor, its associated coolant piping and pumps, and the heat exchangers ("steam generators") in which water is turned into steam. The remainder of the power plant is rather conventional. Refer to Figure I.1 for typical fossil and nuclear power plants.

Yet we must not let the apparent similarities between nuclear and fossil-fueled power plants overshadow the very significant differences between the two systems. For example, in a nuclear plant sufficient fuel must be inserted into the reactor core to allow operation for very long periods of time (typically one year). The nuclear fuel cycle itself is extremely complex, involving fuel refining, fabrication, reprocessing after utilization in the reactor, and eventually the disposal of radioactive fuel

CONVENTIONAL POWER PLANT



CANDU NUCLEAR POWER PLANT

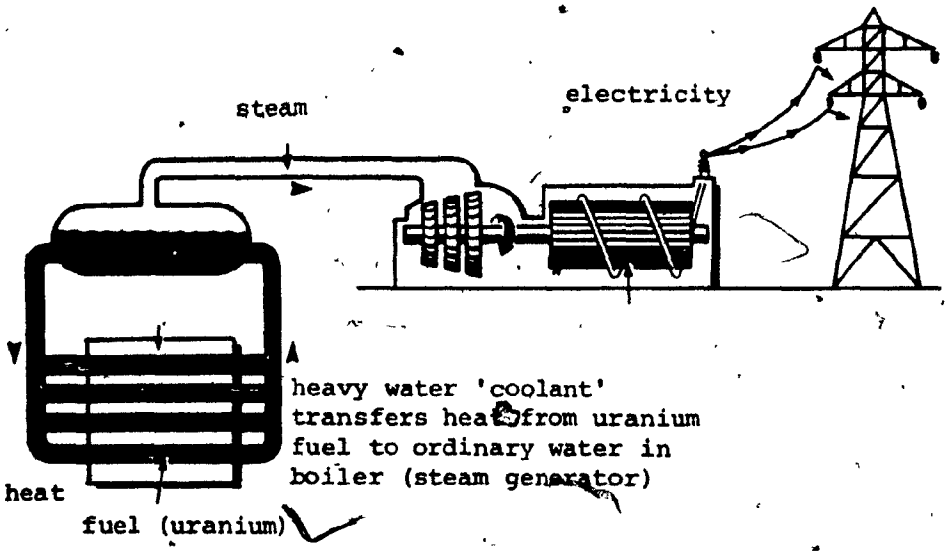


Fig. I.1 Comparison of Fossil and Nuclear Power Plants.

wastes. The safety aspects of nuclear plants are also quite different, since one must be concerned with avoiding possible radiological hazards. Furthermore the licensing required by a nuclear plant before construction or operation demands a level of sophisticated analysis totally alien to fossil-fueled plant design.

Therefore even through the NSSS contributes only a relatively modest fraction of the total capital cost of a nuclear power plant (presently about 20%), it is of central concern since it not only dictates the detailed design of the remainder of the plant, but also the procedures required in plant construction and operation. Furthermore it is the low fuel costs of the NSSS that are responsible for the economic advantages presently enjoyed by nuclear power generation.

I.2 CLASSIFICATION OF REACTORS

Nuclear reactors can be classified in various ways depending on neutron energy, fuel, heat removal, purposes, fuel and moderator arrangement and materials used in the reactor components.

Refer to Table I.1 for more detailed reactor classification.

TABLE I.1

CLASSIFICATION OF REACTORS

Energy of neutrons that produce fission

- fast
- intermediate (or epithermal)
- thermal

Nuclear fuel

- natural U (0.7% U^{235})
- slightly enriched U (1-2% U^{235})
- highly enriched U (≈90% U^{235})
- Pu^{239}
- U^{233}

Method of heat removal, by circulation of

- coolant only
- fuel mixed with coolant
- moderator-coolant
- fuel, moderator, and coolant

Purpose

- research
- prototype
- propulsion
- heat source
- electric power generation
- isotope production (fissionable or for industrial use)

TABLE I.1 (Cont'd)

Arrangement of fuel and moderator

- heterogeneous
- homogeneous

Materials used in the following reactor components

- moderator
- coolant
- structural materials
- reflector
- shield

If one formed all the possible combinations of reactor features according to the classification in Table I.1, about 1000 reactor types would be found. Many would not be feasible at all. Others would be inordinately expensive. Some of the types that have been operated or show the most promise are now described briefly, to assist in orientation.

I.2.1 THERMAL AND FAST REACTORS

The thermal reactors are those devices in which reactions are maintained by neutrons with characteristic energies comparable to the energy of thermal vibration of atoms comprising the reactor core.

Those devices in which the average neutron energy is more characteristic of the much higher energy neutrons released in a nuclear fission reaction are called fast reactors.

I.2.2 LWR (BWR AND PWR)

In the United States, and indeed throughout the world, the most popular of the present generation of reactors, the light water reactor (LWR) uses ordinary water as a coolant. Such reactors operate at very high pressures (approximately 70-150 bar) in order to achieve high operating temperatures while maintaining the water in its liquid phase. If the water is allowed to boil in the core, the reactor is referred to as a boiling water reactor (BWR), while if the system pressure is kept sufficiently high to prevent bulk boiling (155 bar), the reactor is known as a pressurized water reactor (PWR). Such reactors have benefited from a well-developed technology and performance experience achieved in the nuclear submarine program.

I.2.3 "MAGNOX" REACTORS

These are graphite moderated, CO_2 gas-cooled reactors fuelled with natural uranium metal clad with a magnesium alloy called Magnox. They have derived their generic name from this latter feature.

Figure I.2 shows a schematic arrangement of one version of this reactor. This type was pioneered by the British and French and was a natural outgrowth of the early air-cooled, graphite-moderated research and plutonium production reactors.

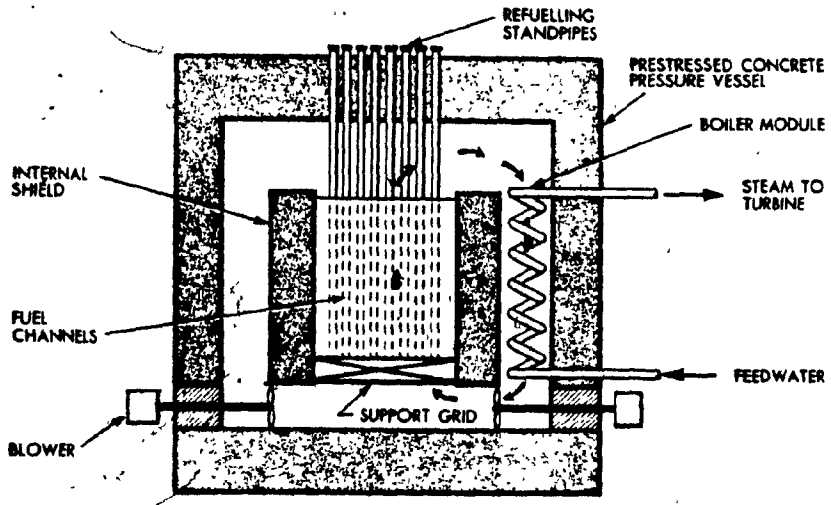


Fig. I.2 Gas Cooled Reactor.

The use of uranium metal fuel and the Magnox cladding, leads to relatively low turbine steam conditions, limiting the station overall efficiency to 30 per cent.

As is typical of all natural uranium power reactors, the Magnox reactors are fuelled on-load. This is because large quantities of excess reactivity, in the form of additional U-235, is not "built into" the new fuel.

The availability record of the Magnox reactors has proven to be good. On-load refuelling helps in this regard. Nevertheless, their high capital cost and modest achievable fuel utilization has led to the discontinuation of construction of further reactors of this type.

I.2.4 ADVANCED GAS COOLED REACTOR (AGCR)

The AGCR has been developed in the U.K. as a successor to the Magnox line of reactors. Several are now under construction.

The most significant change is that the fuel is clad in stainless steel. This permits rather higher fuel temperatures leading to conventional fossil fuel steam conditions (2400 psi, 1025°F). The fuel is in the form of a cluster of small diameter rods, permitting relatively high power levels to be achieved. This reduces the size of the reactor core compared with the Magnox reactors, but, because of these fuel changes, the AGCR requires some fuel enrichment.

I.2.5 HIGH TEMPERATURE GAS COOLED REACTOR (HTGCR)

This type represents the next evolutionary step in the Magnox-AGR line of gas-cooled, graphite-moderated reactors. It is being developed by Gulf General Atomic in the U.S. and by the West Germans and the British.

The HTGCR differs from the AGR in two major respects. The first is the use of helium as the coolant in place of CO_2. This permits even higher coolant temperatures without inducing a chemical reaction with the graphite moderator. The second relates to the fuel.

The fuel is fully enriched (93 per cent) U-235 mixed with thorium. Thorium absorbs neutrons and is converted, after a radioactive decay chain to U-233 which is fissile.

I.2.6 FAST BREEDER REACTOR (FBR)

All present commercial power reactors utilize neutrons at "thermal" speeds (i.e. the neutrons are slowed down by a moderator to maintain the fission process). It is also possible to sustain a chain reaction with "fast" neutrons provided the fuel is highly enriched with fissile material such as U-235 or Pu-239.

Under certain conditions the destruction of one fissile atom (by neutron absorption) can produce a little more than two neutrons, and only one is required to maintain the chain reaction. The surplus neutrons can be absorbed by a "fertile" material such as U-238 for conversion to fissile Pu-239. In fact it is possible to "breed" more fissile material than is used up in the capture fission process if few neutrons are wasted.

This possibility of breeding is very attractive as a means of extracting more energy from uranium. Less than one per cent of natural uranium is fissile so that if a substantial part of the other 99 per cent can also

be used then the world's energy resources would be considerably extended.

One practical form of this concept is in the development of the Liquid Metal Fast Breeder Reactor shown in Figure I.3. The reactor core consists of a closely packed array of highly enriched U-235 or Pu-239 oxide rods clad in a high temperature resistant material. The core is surrounded on all sides by a "breeder" blanket of fertile U-238 rods. No moderator is used. The excess neutrons from the fission process are absorbed by the blanket. Both the core and blanket are cooled by a circuit of liquid sodium. The hot sodium passes through a heat exchanger where the heat is transferred to an intermediate sodium circuit. This in turn is cooled by another heat exchanger where water is converted into steam to drive the turbine.

Fast Breeder reactor prototype stations are in operation in Britain and France and the USSR, but commercial versions are some years away.

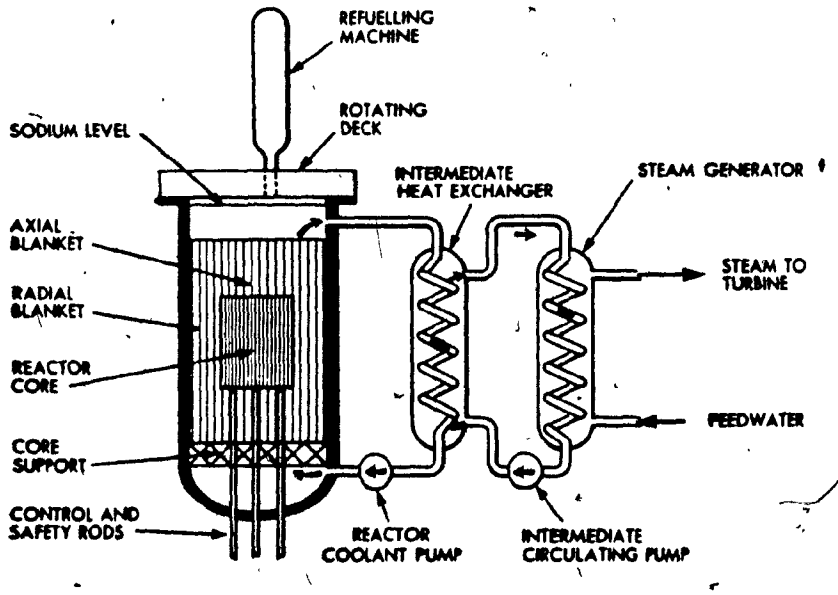


Fig. I.3 Fast Breeder Reactor.

I.2.7 CANDU

The Canadian program like most others, had its origin in the Second World War when Canada was assigned the task of developing the heavy water moderated reactor system as a method of plutonium production. With the end of hostilities all weapon activities ceased but this experience has made Canada the centre of world scientific knowledge and technology in heavy water moderated reactors.

It was natural in the early 1950's when a prototype power station was being considered in Canada that the heavy water systems should be preferred. The system chosen exploited the merits of heavy water as an extremely low neutron-absorption moderating material. The use of pressure tubes in place of pressure vessels for the primary coolant matched the national manufacturing capability, and the use of natural uranium as fuel allowed the direct use of Canada's national resources of uranium.

The reactor system is known as CANDU (CANada Deuterium Uranium) indicating the use of deuterium or heavy water as a moderator and natural uranium as a fuel. There are a number of alternative coolants available, the one brought to full commercial exploitation being the CANDU-PHW (Pressurized Heavy Water), with the heavy water coolant at a maximum pressure of about 11.5 MPa and maximum temperature of 285°C. Canada is also experimenting with light water and organics as coolant.

In the CANDU-PHW the calandria or reactor vessel is a cylindrical tank laid on its side with its end faces forming vertical plates. This tank

is filled with low temperature, atmospheric pressure heavy water moderator. Several hundred tubes (called pressure tubes) penetrate the tank and contain the uranium fuel. A fluid, called a coolant, is pumped past the uranium fuel within the pressure tubes, and the heat of fission is transferred to the coolant. The coolant flows on to the boilers or steam generators where it gives up its heat to ordinary or light water to produce steam. Figures I.4 to I.7 show some CANDU reactor characteristics.

Present CANDU reactors use natural uranium fuel. The fuel bundle is simple (see Figure I.8), the fuel cycle is simple, leading to a very low fuel cost of about 1 mill/kWh with no plutonium credit, and one that will stay low relative to other fuel cycles. Fuel fabrication is simple and fabrication facilities are inexpensive. Natural uranium is available from many sources, allowing diversity of supply.

CANDU reactors are fuelled on-power. This was originally introduced because it improves the neutron economy, but it is also very valuable in allowing rapid discharge of any failed fuel, a safety advantage. It also leads to high availability of the reactor system.

The magnitude of Canada's contribution to world development of nuclear power can be appreciated by the fact that Ontario Hydro's Pickering Generating Station is one of the largest operating nuclear power stations in the world. The Pickering Station has an installed capacity of 2,160,000 kilowatts. (Ontario Hydro production is ∿17% nuclear.)

The Pickering station has operated in a highly creditable manner since it was started up, particularly during the winter peak periods when reliability is most important. It is one of the world's most productive

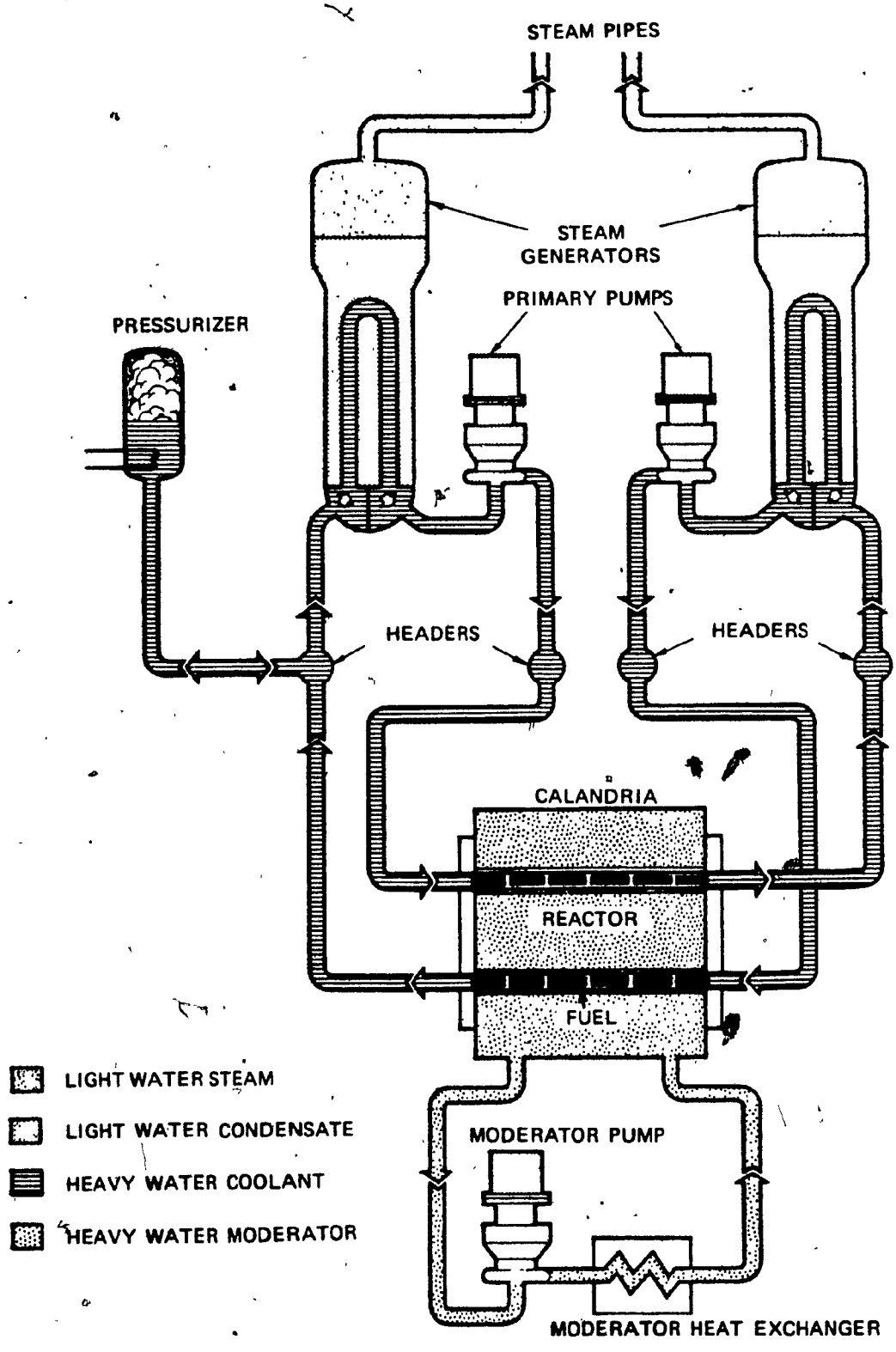
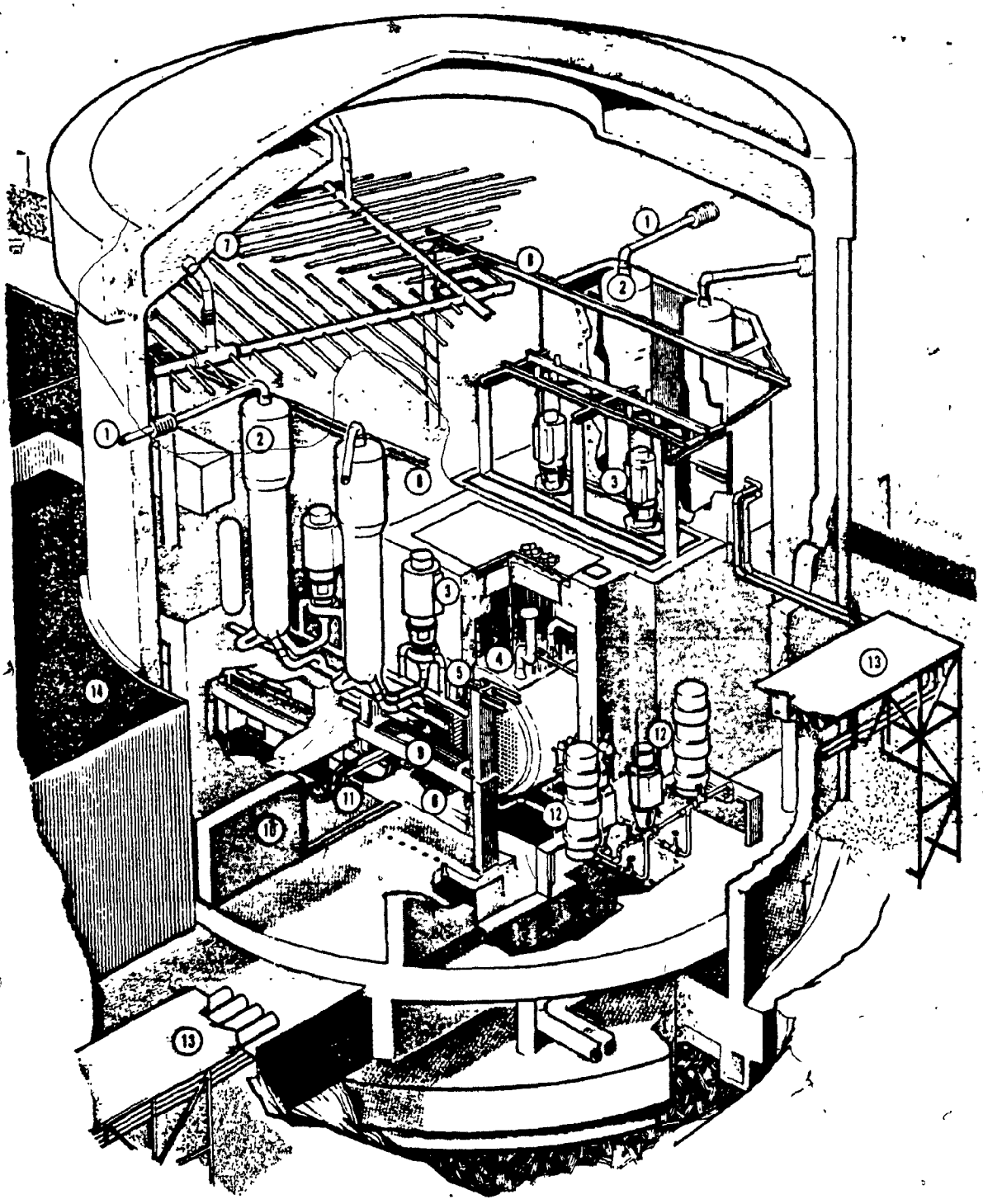
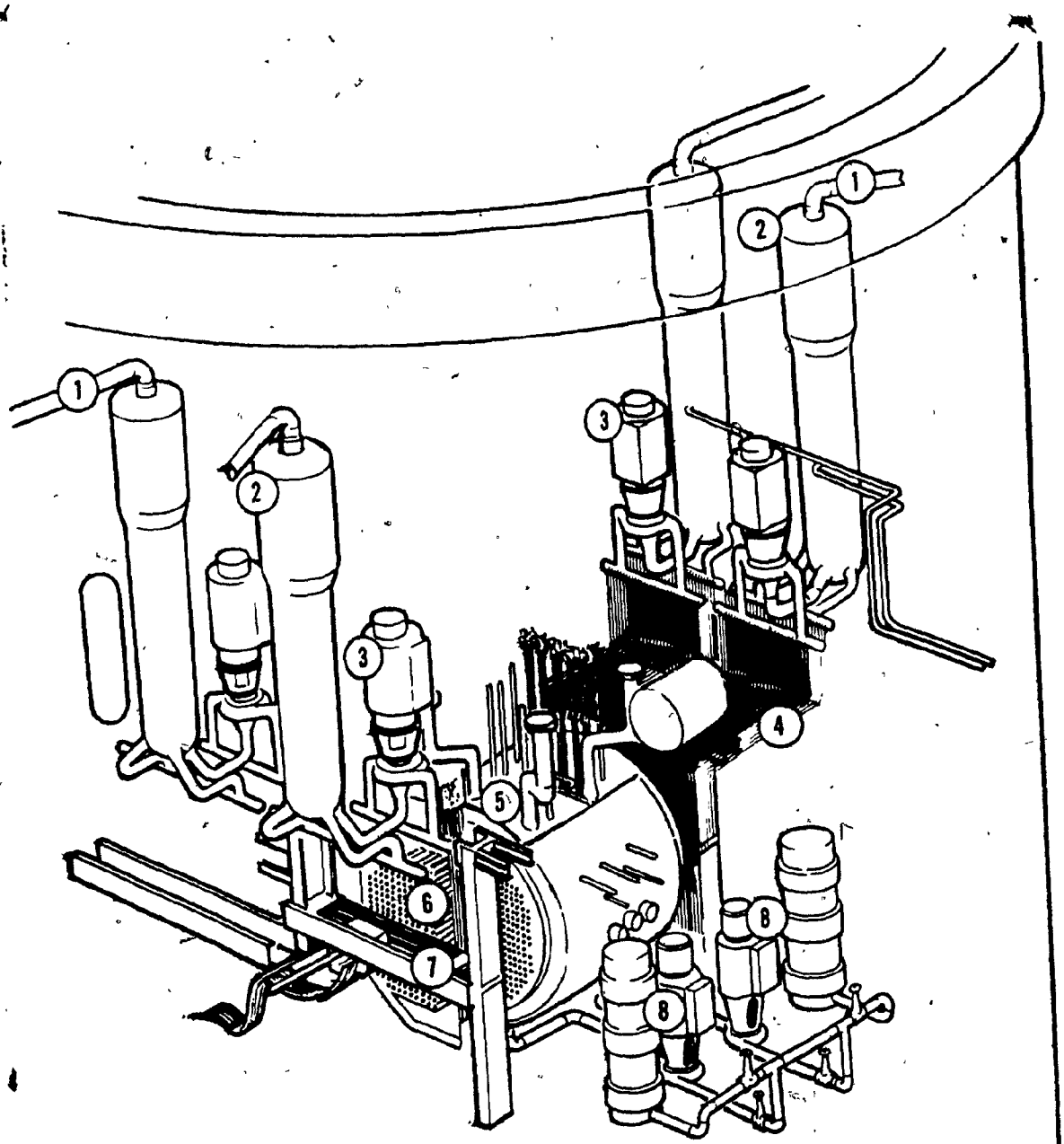


Fig. I.4 CANDU Reactor Simplified Flow Diagram.



1 MAIN STEAM SUPPLY PIPING
2 STEAM GENERATORS
3 MAIN PRIMARY SYSTEM PUMPS
4 CALANDRIA ASSEMBLY
5 FEEDERS
6 FUEL CHANNEL ASSEMBLY
7 DOUSING WATER SUPPLY
8 CRANE RAILS
9 FUELLING MACHINE
10 FUELLING MACHINE DOOR
11 CATENARY
12 MODERATOR CIRCULATION SYSTEM
13 PIPE BRIDGE
14 SERVICE BUILDING

Fig. 1.5 600 MW(e) Reactor Building Cutaway



1 MAIN STEAM SUPPLY PIPING
2 STEAM GENERATORS
3 MAIN PRIMARY SYSTEM PUMPS
4 FEEDERS
5 CALANDRIA ASSEMBLY
6 FUEL CHANNEL ASSEMBLY
7 FUELLING MACHINE BRIDGE
8 MODERATOR CIRCULATION SYSTEM

Fig. 1.6 CANDU P.H.W. Reactor

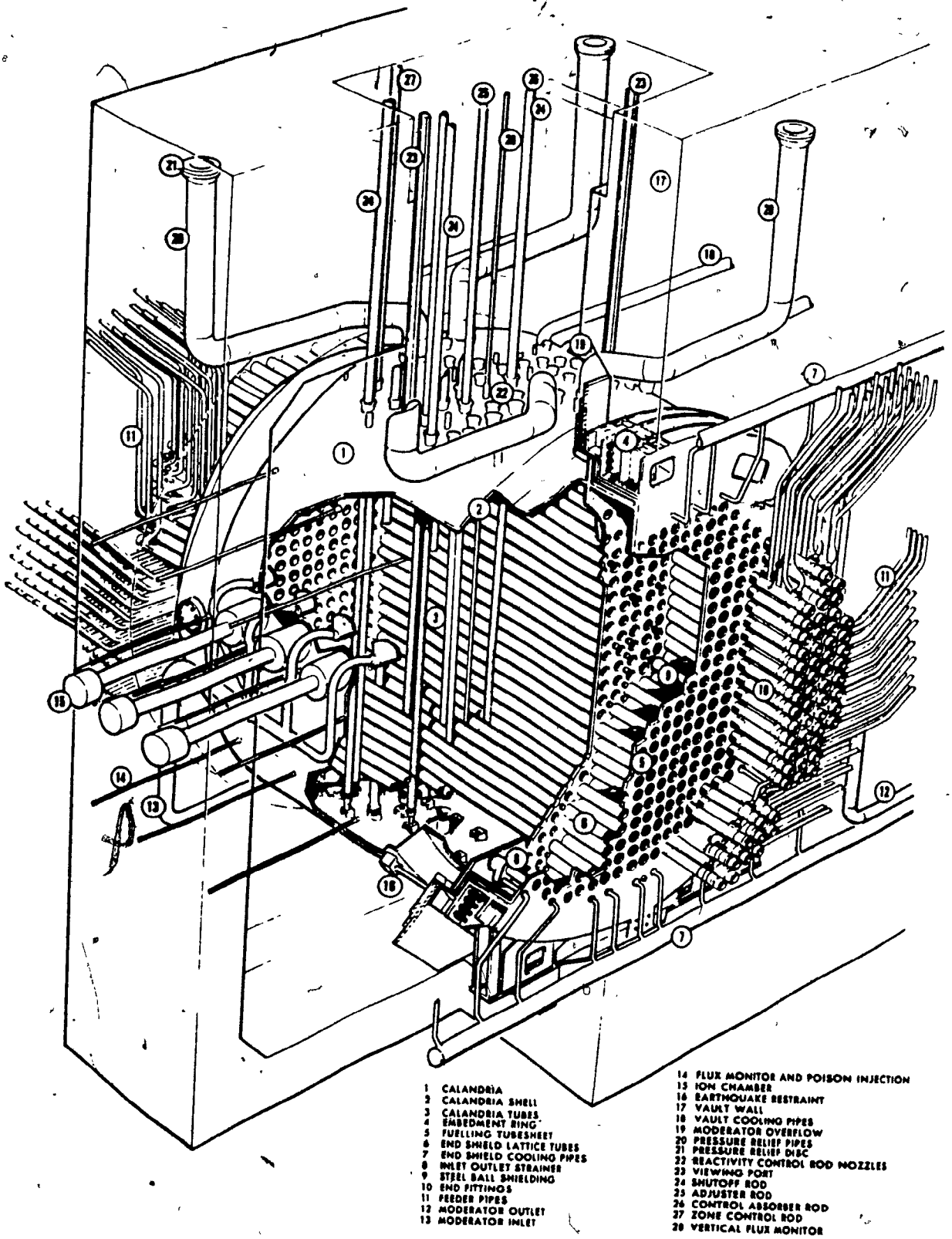


Fig. 1.7 Reactor Assembly

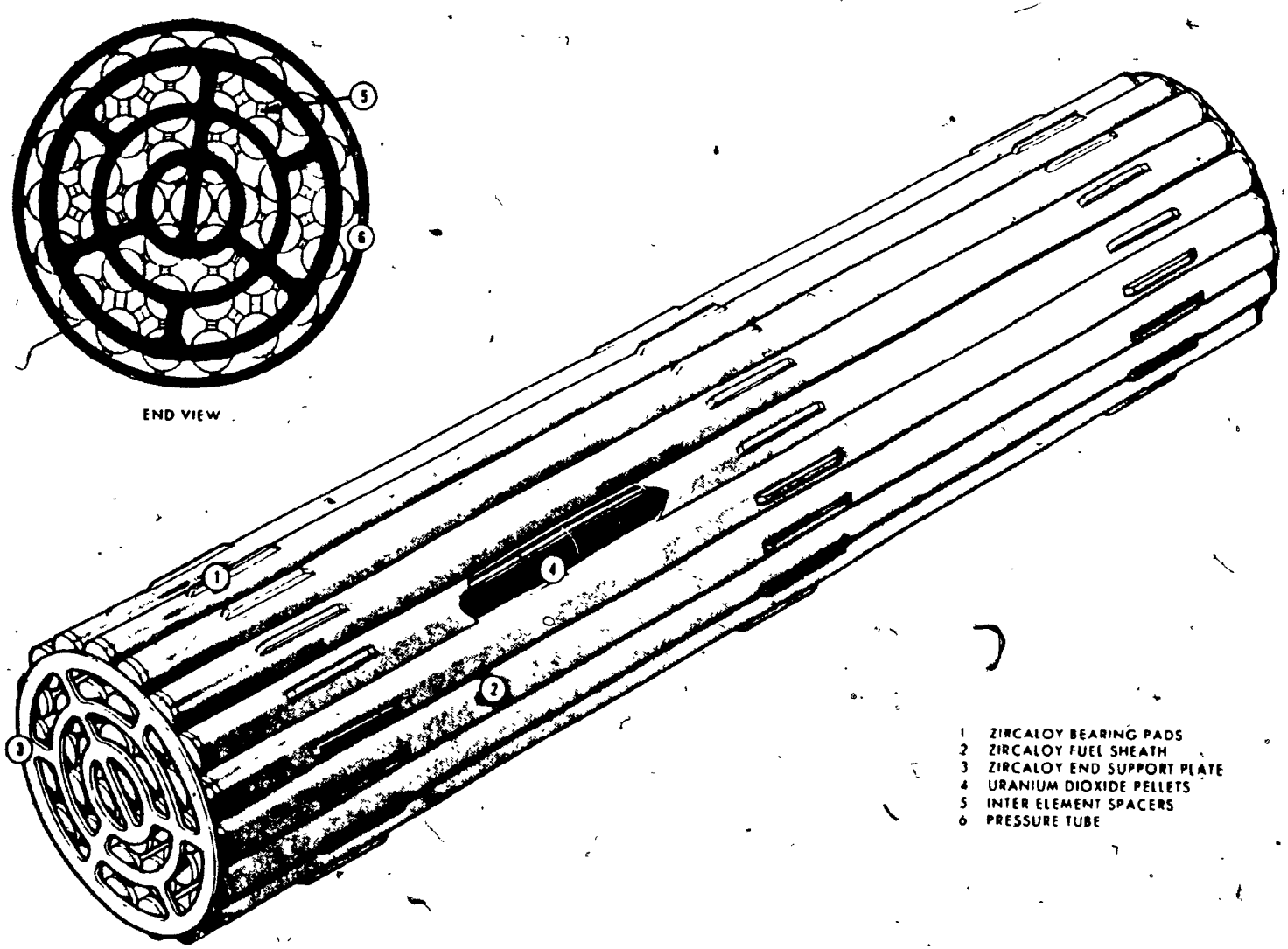


FIGURE I.8 37 ELEMENT FUEL BUNDLE

in terms of electrical output illustrating the capability of the CANDU system and the effectiveness of on-power refuelling.

The performance of Pickering Generating Station encouraged Ontario Hydro to build even larger Nuclear Power Generating Station. Next multi-unit will be Bruce Generating Station with an installed capacity of 3,000,000 kilowatts. Larger multi-unit stations are considered.

The in-service performance of the CANDU fuel has been excellent. Evolution of design and improvements in the fuel management schedules have already resulted in a defect rate of less than 0.1 per cent.

In conclusion, the CANDU system is one which offers a number of advantages for Canada and some other countries. It is a proven, commercially successful system. The use of natural uranium, which is readily available in many countries of the world, permits self sufficiency in fuel supply.

The pressure tube concept results in a reactor which can be manufactured in Canada, and in other countries with similar industrial manufacturing capabilities. For countries just entering the nuclear power field, only modest investments in industrial plant and equipment would be needed to begin indigenous manufacture. The system is also consistent with the economic structure of Canadian provincial utilities. No large investments in fuel enrichment plants are necessary. Capital investment in heavy water plants would be necessary if a large program, such as that of Ontario Hydro, were undertaken but these plants cost very much less than enrichment plants

and can be built economically in much smaller modules thus allowing the utility to spread capital investments over a large period of time.

Overseas, Argentina and Korea have decided to buy CANDU power reactors and other nations are showing intense interest.

I.3 MODERATOR

The neutrons that are produced by the fission processes come off at a very high velocity. At these high speeds, the probability of colliding with a U-235 atom and causing fission is very small. Hence, a material called a moderator is used to slow down the neutrons so that they will have a better chance of causing fission.

The first reactor to be built used a graphite moderator. This was the Fermi reactor built in the United States in 1942. It consisted of a stack of graphite blocks 32 feet long by 30 feet wide by 21 feet high, with six-pound lumps of uranium on 8 1/4 inch centres throughout the assembly.

Other materials commonly used as a moderator include light water and heavy water. Figure I.9 shows the moderator slowing down the neutrons by a series of collisions. The material used as a moderator:

1) Must not absorb many neutrons, since they are required to collide ultimately with U-235 atoms to cause further fission
2) must be light in mass so that a very few collisions are required to slow the neutrons down to the required velocity
3) must have a high probability of collision with a neutron

There are a number of materials which have these characteristics to varying degrees. Their overall performance as moderators can be stated numerically by their "moderating ratio". This is a calculated value which takes into account the relevant characteristics (II.6.2.3).

Some common moderating materials and their moderating ratios are shown in Table I.2. (Refer to later sections for moderating ratios.)

TABLE I.2

MODERATOR	MODERATING RATIO
Light Water	62
Graphite	170
Heavy Water (Deuterium Oxide)	2000

To summarize the choices facing a power reactor designer, there is first the basic choice of natural or enriched uranium. If natural uranium is chosen, than a moderator of high efficiency such as graphite or heavy water is necessary.

Whatever type of reactor is used the final product is steam to drive the turbine generator. From the main steam line on, the nuclear power station is the same as any conventional coal or oil burning power plant as shown in Figure I.1.

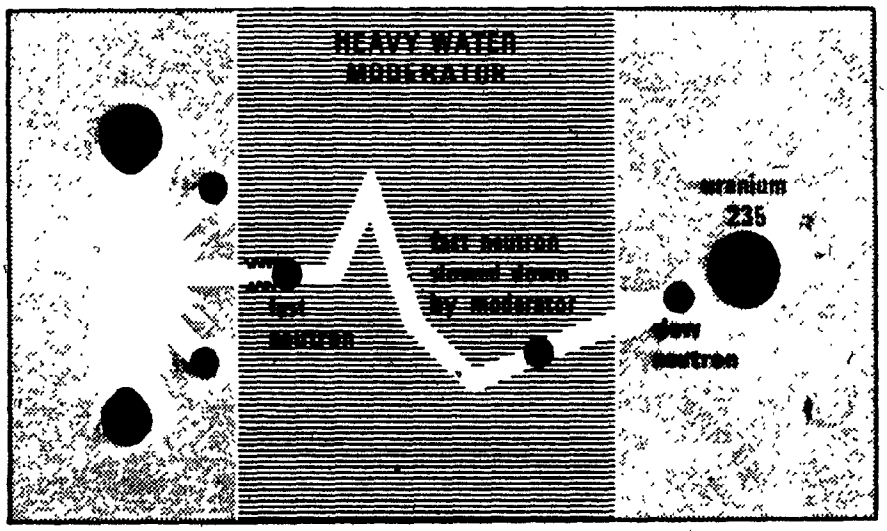


Fig. I.9 Moderation of Neutron Speed.

I.4 THE NUCLEAR STEAM SUPPLY SYSTEM (NSSS) AND NUCLEAR REACTOR CORE TERMINOLOGY

The NSSS consists essentially of three major components: (a) a nuclear reactor supplying the fission heat energy, (b) several primary coolant loops and primary coolant pumps that circulate a coolant through the nuclear reactor to extract the fission heat energy, and (c) heat exchangers or steam generators that use the heated primary coolant to turn feedwater into steam.

Several very simplified diagrams of NSSS components are given in Figure I.7 for most of American type reactors and in Figure I.1 for CANDU type reactors.

At the heart of the NSSS is the nuclear reactor core. Far from being just a relatively simple "pile" of fuel, a la Fermi, a modern power reactor is an enormously complicated system designed to operate under the most severe conditions of temperature, pressure and intense radiation.

To introduce the general components of a typical power reactor, we will consider the specific example of a modern, Canadian designed type reactor: The CANDU as illustrated in Figure 8 on page 13.

Basic components forming the CANDU reactors are listed below.

I.5 INTRODUCTORY CONCEPTS OF NUCLEAR POWER REACTOR ANALYSIS

(1) Fuel: Any fissionable material. This can be either fissile material such as ^{233}U, ^{235}U, ^{239}Pu, or ^{241}Pu or fissionable material such as ^{232}Th, ^{238}U, or ^{240}Pu. Most modern power reactors utilize this fuel in a ceramic form - either as an oxide such as UO_2, a carbide such as UC, or a nitride, UN.

(2) Fuel element: The smallest sealed unit of fuel. In a CANDU reactor the fuel element is a metal tube containing ceramic pellets of fuel (such as UO_2).

(3) Fuel assembly or bundle: The smallest unit combining fuel elements into an assembly.

(4) Moderator: Material of low mass number which is inserted into the reactor to slow down or moderate neutrons via scattering collisions. Typical moderators include light water, heavy water, graphite, and beryllium.

(5) Coolant: A fluid which circulates through the reactor removing fission heat. The coolant can be either liquid, such as water or sodium, or gaseous, such as helium or carbon dioxide.

(6) Coolant channel: One of the many channels through which coolant flows in the fuel lattice.

(7) Structure: The geometry and integrity of the reactor core is maintained by structural elements such as support plates, spacer grids, or the metallic tubes used to clad the fuel in some reactor designs.

(8) Control elements: Absorbing material inserted into the reactor to control core multiplication. Although most commonly regarded as movable rods of absorber, control elements may also consist of fixed absorbers or absorbing materials dissolved in the coolant. Common absorbing materials include boron, cadmium, gadolinium, and hafnium.

(9) Reactor core: The total array of fuel, moderator, and control elements.

(10) Reflector: A material characterized by a low absorption cross section used to surround the core in order to reflect or scatter leaking neutrons back into the core.

(11) Shielding: The reactor is an intense source of radiation. Not only must operating personnel and the public be shielded from this radiation, but reactor components must as well be protected. Hence absorbing material is introduced to attenuate both neutron and gamma radiation. Thermal shielding is used to attenuate the emergent core radiation to levels that do not result in significant heat generation and hence damage in reactor components. Biological shielding reduces the radiation still further to acceptable levels for operating personnel.

(12) Support structure: The support plates that serve to maintain the core geometry.

(13) Reactor pressure vessel: The high pressure containment for reactor and associated primary coolant system.

It is also useful to introduce at this point several quantities which are used to describe reactor performance. The units in which these quantities are usually expressed are denoted in brackets.

(1) Reactor thermal power (MWt): The total heat produced in the reactor core.

(2) Plant electrical output (MWe): Net electrical power generated by the plant.

(3) Net plant efficiency (%): $\frac{\text{Plant electrical output}}{\text{Reactor thermal power}}$

(4) Plant capacity factor (%): $\frac{\text{Total energy generated over time period}}{\text{(Plant rating) X (time)}}$

(5) Plant load factor (%): $\frac{\text{Average plant electrical power level}}{\text{Peak power level}}$

(6) Plant availability factor (%): $\frac{\text{Integrated electrical energy output capacity}}{\text{Total rated energy capacity for period}}$

(7) Core power density (kW/liter): $\frac{\text{Reactor thermal power}}{\text{Total core volume}}$

(8) Linear power density (kW/m): Thermal heat generated per unit length of coolant channel.

(9) Specific power (kW/kg): $\frac{\text{Reactor thermal power}}{\text{Total mass of fissionable material}}$

(10) Fuel loading (kg): Total mass of fissionable material.

(11) Fuel burnup (Megawatt-days)metric ton uranium = MWD/TU):

$$\frac{\text{Energy generated in fuel during core residence}}{\text{Total mass of fuel}}$$

(12) Fuel residence time: $\frac{\text{Fuel burnup}}{\text{(Specific power) X (capacity factor)}}$

The foregoing are the more common terms used in characterizing nuclear plant performance. We will introduce other more specific concepts and terminology later as we develop the more detailed theory of nuclear reactors.

CHAPTER II

REACTOR PHYSICS: NUCLEAR REACTIONS

II.1 INTRODUCTION

When a neutron does get within range of the nuclear force of a nucleus, it is pulled into that nucleus very abruptly. To put it another way, the potential energy it had by virtue of being outside the range of the nuclear force is suddenly transformed into kinetic energy when it enters the nucleus. Thus the neutron suddenly picks up great speed and then, by "collisions" with the other neutrons and protons making up the nucleus, increases the average kinetic energy of all the particles in the nucleus. The nucleus thus gets energized to an unstable excited state. Because of the presence of the extra neutron it is said to be a compound nucleus.

Excited states of a nucleus can persist anywhere from 10^{-14} seconds to years depending on the nucleus and the state in question. Yet even the short time (10^{-14}sec) is around a hundred million times longer than the time required for a neutron to make one trip across the nucleus. Thus it is proper to think of the excited nucleus as being in a well-defined state.

When the excited nucleus does give up its excess energy, it can do so in a great variety of ways. Most of the nuclear reaction types that occur in a nuclear power reactor are presented in following examples.

II.1.1 ELASTIC SCATTERING

A neutron is emitted and the nucleus returns to its initial ground state. The emitted neutron need not be the same one that originally struck the nucleus; but, nevertheless, the elastic-scattering process is in an important sense analogous to the collision of one billiard ball with another. The lifetime of the excited compound nucleus is so short (10^{-12} sec) that, for the purposes of reactor calculations, it may be neglected and the process of elastic scattering may be analyzed as a billiard-ball collision.

This type of elastic scattering, called resonance scattering and involving the formation of a compound nucleus, is actually very rare. A far more common form is potential scattering, in which the impinging neutron does behave exactly like one billiard ball striking another. It interacts with the nucleus as a whole without entering and forming a compound nucleus. Reaction times can thus be on the order of 10^{-22} seconds.

Thus, whether the elastic scattering is anomalous resonance scattering or the more common potential scattering, we can picture the original neutron as striking the nucleus, imparting some of its momentum and kinetic energy to the nucleus, and then moving off in a direction different from its original flight path. Elastic scattering is an extremely important process for nuclear reactors since it is one of the chief mechanisms by which the high-energy neutrons born in fission lose their kinetic energy. We shall analyze the process in detail later in this Chapter.

II.1.2 INELASTIC SCATTERING

A neutron is emitted from the compound nucleus but the nucleus still remains in an excited state. Again the process is effectively instantaneous (10^{-12} sec) with respect to the time-scale of processes important to reactor physics. Thus we can still picture the initially incident neutron as being scattered by the nucleus and losing an unusually large amount of energy in the process.

II.1.3 CHARGED-PARTICLE EMISSION

The excited compound nucleus becomes de-exited by emitting a charged particle (proton, deuteron, α particle - occasionally an electron). From the viewpoint of the free-neutron population in the system, this results in the loss of a neutron and hence is equivalent to a neutron capture.

II.1.4 NEUTRON CAPTURE

The compound nucleus de-excites itself by emitting a γ ray, a high-energy photon or quantum of electromagnetic energy. The E of a γ ray, (as is the case for all photons) is related to its frequency ν by $E = h\nu$, where h (Planck's constant) $= 6.6 \times 10^{-27}$ erg-sec. Typical frequencies for γ rays are around 2×10^{20} Hz, whereas, those for the photons of visible light are around 6×10^{4} Hz.

II.2. INTERACTION OF NEUTRONS WITH MATTER

The operation of a nuclear reactor depends fundamentally on the way in which neutrons interact with atomic nuclei. It is necessary, therefore, to consider the nature of these interactions in some detail.

Neutrons interact with nuclei in a variety of ways. For instance, if the nucleus is unchanged in either isotopic composition or internal energy after interacting with a neutron, the process is called elastic scattering. On the other hand, if the nucleus, still unchanged in composition, is left in an excited state, the process is called inelastic scattering. The symbols (n, n) and (n, n') are often used to denote these processes. In referring to these interactions it is common to say that the incident neutron has been "scattered," elastically or inelastically, as the case may be, because a neutron reappears after the interaction. However, this term is somewhat misleading, since the emerging neutron may not be the same neutron that originally struck the nucleus.

Neutrons disappear in a reactor as the result of absorption reactions, the most important of which is the (n, γ) reaction. This process is also known as radiative capture, since one of the products of the reaction is γ-radiation. Neutrons also disappear in charged-particle reactions such as the (n, p) or (n, α) reactions. Occasionally, two or more neutrons are emitted when a nucleus is struck by a high-energy neutron. The processes involved here are of the (n, 2n) or (n, 3n)

type. A closely related process is the (n, pn) reaction, which also occurs with highly energetic incident neutrons. Finally, when a neutron collides with certain heavy nuclei, the nucleus splits into two large fragments with the release of considerable energy. This, of course, is the fission process, which will be discussed later.

We introduce here a general notation for a nuclear reaction in which a particle or nucleus x strikes a nucleus X and results in the emission of particle y, leaving nucleus Y behind.

$$x + X \longrightarrow Y + y \qquad \text{(II.1)}$$

This is often abbreviated as

$$X(x,y)Y \qquad \text{(II.2)}$$

in which we write first the symbol for the struck nucleus, in parentheses, the incoming and outgoing particles, respectively, and after the parentheses, the residual nucleus.

In one way or another, most of these interactions must be taken into account in the design of a nuclear reactor. Before considering the specific interactions, however, it is necessary to set up a framework with which these interactions can be discussed quantitatively.

II.2.1 EXOTHERMIC AND ENDOTHERMIC REACTIONS

Nuclear reactions are generally accompanied by either the absorption or emission of energy. One can calculate the energy released by (or required for) a given nuclear reaction by using the important result from the theory of relativity:

$$E = mc^2 \tag{II.3}$$

where c is the speed of light and m is the mass converted into energy in a reaction. The appropriate quantity to use for the variable m that appears in this formula is the mass difference between the interacting particles before and after the collision.

For the reaction a(b,c)d we would calculate the reaction energy as

$$Q = \left[(M_a + M_b) - (M_c + M_d)\right] c^2 \tag{II.4}$$

If $Q > 0$, then we say the reaction is exothermic, which corresponds to a release of energy in the reaction. If $Q < 0$, then the reaction is said to be endothermic, and energy must be supplied to the colliding nuclei in order to stimulate the reaction to occur. Obviously, nuclear fission is an example of an exothermic reaction.

II.2.2 RADIOACTIVE DECAY

Certain nuclei are unstable in the sense that they may spontaneously undergo a transformation into a different nuclide, usually accompanied by the emission of energetic particles. Such a spontaneous nuclear transformation is referred to as radioactive decay. The three most common types of radioactive decay found in naturally occurring nuclides include alpha decay, in which the nucleus emits a helium nucleus $^{4}_{2}He$; beta decay, which corresponds to the conversion of a neutron in the nucleus into a proton, generally accompanied by the emission of an electron and a neutrino; and gamma decay, the transition of a nucleus from one excited state to a lower excited state with the accompanying emission of a proton. However other types of radioactive decay are possible in a nuclear reactor since many unstable nuclides are produced in fission which do not occur in nature. For example, certain nuclei such as $^{87}_{36}Kr$ may decay by emitting a neutron. (We will later find that this particular type of decay process is extremely important for reactor operation.)

The fundamental law describing radioactive decay is based on the experimental observation that the probability that a nucleus will decay in a given time interval is essentially constant, independent of the age of the nucleus or its environment, dependent only on the type of the nucleus itself. Hence the time rate of change of the number of original nuclei of a given type must be proportional to the number of nuclei present at that time. Let us call the proportionality constant λ. Then if $N(t)$ is the number of original nuclei left at time t, we find

$$-\frac{dN}{dt} = \lambda N(t) \qquad \text{(II.5)}$$

Here λ is referred to as the radioactive decay constant characteristic of the nucleus and has units of inverse time. If we initially have N_0 nuclei present, then at any later time t the number of nuclei present will be given by an exponential law:

$$N(t) = N_0 e^{-\lambda t} \qquad \text{(II.6)}$$

From this time behavior, it is apparent that the probability that a given nucleus will decay in a time interval t to t + dt is just

$$p(t)dt = \lambda e^{-\lambda t} dt \qquad \text{(II.7)}$$

Since radioactive decay is a statistical phenomenon, we cannon predict with any certainty precisely when a given nucleus will decay. However we can calculate the mean lifetime $\bar{t}$ of the nucleus before decay using our expression for p(t) from Eq. (II.7).

$$\bar{t} \equiv \int_0^\infty dt\, tp(t) = \lambda \int_0^\infty dt\, te^{-\lambda t} = \frac{1}{\lambda} \qquad \text{(II.8)}$$

Hence on the average a given nucleus will decay after a time $1/\lambda$.

A closely related quantity is the length of time necessary for half of the original number of nuclei present to decay away. Such a time $T_{1/2}$ is referred to as radioactive half-life for the nucleus and can be calculated from its definition by noting

$$N(T_{1/2}) = N_0/2 = N_0 e^{-\lambda T_{1/2}} \qquad \text{(II.9)}$$

or

$$T_{1/2} = \frac{Ln2}{\lambda} = \frac{0.693}{\lambda} \qquad \text{(II.10)}$$

It is common practice to tabulate such radioactive half-lives of various unstable nuclei in preference to their mean life time $\bar{t}$ or decay constant λ.

Yet another definition of some importance is that of the activity characterizing a sample of radioactive material. This quantity is simply the total number of disintegrations occuring per second $\lambda N(t)$. Activity is usually measured in units of curies, where one curie (Ci) is defined to be that quantity of radioactive nuclei for which the number of disintegrations per second is 3.70×10^{10}. (This is roughly the activity of 1 g of radium.)

Actually it is more common and far more useful to regard the dependent variable N(t) as the atomic number density (#/cm^3) of the nuclide of interest rather than the total number of nuclei present in the sample. We will adhere to this practice in our subsequent discussion.

Most radioactive decay processes are somewhat more complicated than those described by Eq. (II.5). For example, the decaying nuclide may itself be produced by some type of source, say, R(t) nuclei/cm^3.sec. Then the nuclide balance equation becomes

$$\frac{dN}{dt} = -\lambda N(t) + R(t) \qquad \text{(II.11)}$$

II.2.3 THE DECAY OF EXCITED STATES

It has been found that the fundamental law of natural radioactivity, i.e., the probability per unit time that a system decays is a constant, also applies to the spontaneous decay of nuclei in excited states. It is customary, however, in discussing the decay of an excited state to express the decay constant λ in terms of a new quantity, Γ, called the level width, which is defined by the relation

$$\Gamma = h\lambda \text{, h is the Planck's Constant.} \qquad \text{(II.12)}$$

Since h has the units of energy times time, and λ has units of inverse time, it is evident that Γ has units of energy. In other words, Γ is the decay constant of an excited state expressed in energy units.

The level width can be used instead of the usual decay constant to describe the decay of nuclei from excited states. If, for example, there are n_0 nuclei in a certain excited state at $t = 0$, t sec later there will be

$$n = n_0 e^{-\Gamma t/h} \qquad \text{(II.13)}$$

nuclei left in this state. Formulas such as Eq. (II.13) are not often useful, however since the decay of excited states usually occurs so quickly that the time dependence of the process is not easily observed.

From Eqs. (II.8) and (II.12) it follows that the mean-life of a state of

width Γ is given by

$$\bar{t} = 1/\lambda = h/\Gamma \qquad \text{(II.14)}$$

Thus a state of large width tends to be short-lived; a state of small width is long-lived.

The widths of a great many excited states have been measured. For instance, the first virtual state of U^{239} is at 6.67 eV above the virtual energy and has a width of 27 millivolts (i.e., 0.027 eV). The mean-life of this state is therefore about

$$6.58 \times 10^{-16}/0.027 = 2.4 \times 10^{-14} \text{ sec}$$

Lifetimes as short as this cannot be measured by ordinary methods, and this state appears to decay (in this case primarily by γ-ray emission) as soon as it is formed.

The decay of a nucleus from an excited state can frequently occur in a number of ways. If the nucleus is in a bound state, however, nucleon emission cannot occur, and, with few exceptions, (the first excited state of O^{16} at 6.06 MeV, for example) decays by emitting an electron-positron pair; decay by β-ray emission is also possible from certain long-lived isomeric states - the nucleus decays by the emission of γ-rays. On the other hand, if the nucleus is in a virtual state, then one or more nucleons, in addition to γ-rays, may be emitted, depending on the energy of the state.

The probability per unit time of each mode of decay of an excited state is described in terms of a partial width characteristic of each process. For instance, the partial width for γ-ray emission, Γ_γ, which is also known as the radiation width, is the probability per unit time (expressed in energy units) that the excited nucleus decays by γ-ray emission. Similarly, Γ_n, the neutron width, giving the total decay probability is the sum of the probabilities for all possible processes, the total width is the sum of the partial widths:

$$\Gamma = \Gamma_\gamma + \Gamma_n \ldots \qquad \text{(II.15)}$$

The relative probability that an excited state decays by a given mode is evidently the ratio of the partial width of the particular mode to the total width. For example, the relative probability that a state decays by γ-ray emission is Γ_γ/Γ, that it decays by neutron emission in Γ_n/Γ, and so on.

II.3 MICROSCOPIC CROSS SECTIONS

The probability that a neutron-nuclear reaction will occur is characterized by a quantity called a nuclear cross section. Let us first define this quantity operationally by considering a beam of neutrons, all traveling with the same speed and direction, which is incident normally upon and uniformly across the face of a target of material. If the target is sufficiently thin (say, one atomic layer thick), then no nuclei in the target will be shielded by other nuclei from the incident neutron beam (see Figure II.3). In this case we would expect that the rate of neutron-nuclear reactions in the target will be proprotional to both the incident neutron beam intensity I (in units of number of neutrons/cm^2.sec) and the number of target atoms per unit area N_A (#/cm^2). If we call the constant of proportionality σ, we can write the rate at which reactions occur per unit area on the target as

$$\text{Rate} \equiv \underset{\left[\frac{\#}{cm^2.sec}\right]}{R} = \underset{[cm^2]}{\sigma} \underset{\left[\frac{\#}{cm^2.sec}\right]}{I} \underset{\left[\frac{\#}{cm^2}\right]}{N_A} \qquad \text{(II.16)}$$

We have indicated the units of each of these quantities since they imply that the proportionality factor σ must have the units of an area.

If the incident neutrons and target nuclei could be visualized as classical particles, σ would quite naturally correspond to the cross sectional area presented by each of the target nuclei to the beam. Hence σ is known as the microscopic cross section characterizing the probability of a neutron-nuclear reaction for the nucleus. We might continue

to think of σ as the effective cross sectional area presented by the nucleus to the beam of incident neutrons. Since the nuclear radius is roughly 10^{-12}cm, the geometrical cross sectional area of the nucleus is roughly 10^{-24} cm^2. Hence we might expect that nuclear cross sections are of the order of 10^{-24} cm^2. In fact microscopic cross sections are usually measured in units of this size, called barns (b). However this geometrical interpretation of a nuclear cross section can frequently be misleading since σ can be much larger (or smaller) than the geometrical cross section of the nucleus due to resonance effects which, in turn, are a consequence of the quantum mechanical nature of the neutron and the nucleus. For example, the absorption cross section of $^{135}_{54}Xe$ for slow neutrons is almost one million times larger than its geometrical cross section.

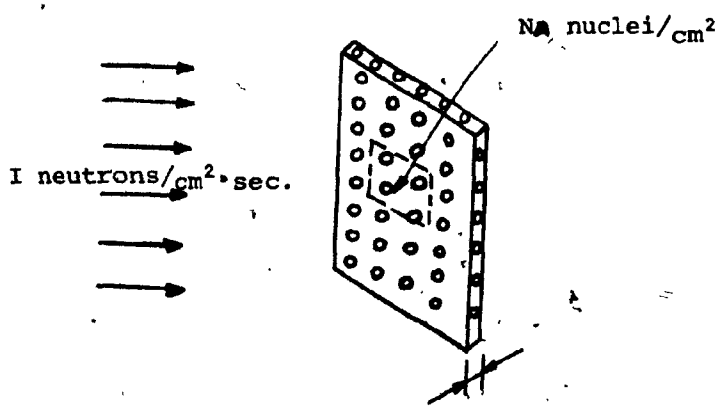


Fig. II.1 Monoenergetic Neutron Beam Incident Normally Upon a Thin Target.

We can give a slightly more formal definition of the microscopic cross section by rearranging Eq. (II.16) to write

$$\sigma = \frac{\text{Number of reactions/nucleus/sec}}{\text{Number of incident neutrons/cm}^2\text{/sec}} = \frac{(R/N_A)}{I} \qquad \text{(II.17)}$$

In this sense, then, if the target has a total cross sectional area a all of which is uniformly exposed to the incident beam, then

$$\frac{\sigma}{a} = \text{Probability per nucleus that a neutron in the beam will interact with it} \qquad \text{(II.18)}$$

Thus far we have been discussing the concept of a nuclear cross section in a rather abstract sense without actually specifying the type of reaction we have in mind. Actually such cross sections can be used to characterize any type of nuclear reaction. We can define a microscopic cross section for each type of neutron-nuclear reaction and each type of nuclide. For example, the appropriate cross sections characterizing the three types of reactions we discussed earlier, fission, radiative capture, and scattering, are denoted by σ_f, σ_γ, and σ_e in which the target nucleus remains in its ground state, and inelastic scattering σ_{in} in which the target nucleus is left in an excited state. Since cross sections are related to probabilities of various types of reactions, it is apparent that the scattering cross section is

$$\sigma_s = \sigma_e + \sigma_{in} \qquad \text{(II.19)}$$

In a similar sense we can define the absorption cross section characterizing those events in which a nucleus absorbs a neutron. There are a number of possible types of absorption reactions including fission, radiative capture, (n, α) reactions, and so on. (Actually one could argue that fission is not really an absorption reaction since several neutrons are created in the fission reaction. It has become customary, however, to treat fission as an absorption event and then add back in the fission neutrons released in the reaction at another point, as we will see later.) Finally, we can introduce the concept of the total cross section σ, characterizing the probability that any type of neutron-nuclear reaction will occur. Obviously

$$\sigma_t = \sigma_s + \sigma_a = \sigma_{in} + \sigma_f + \sigma_\gamma + \sigma_{n\alpha} + \ldots \qquad \text{(II.20)}$$

A schematic diagram[9] of the heirarchy of cross sections along with their conventional notation is shown in Figure II.2. Notice that in general one would define the absorption cross section to characterize any event other than scattering

$$\sigma_a = \sigma_t - \sigma_s \qquad \text{(II.21)}$$

In a similar fashion, one occasionally defines a nonelastic cross section as any event other than elastic scattering

$$\sigma_{ne} = \sigma_t - \sigma_e \qquad \text{(II.22)}$$

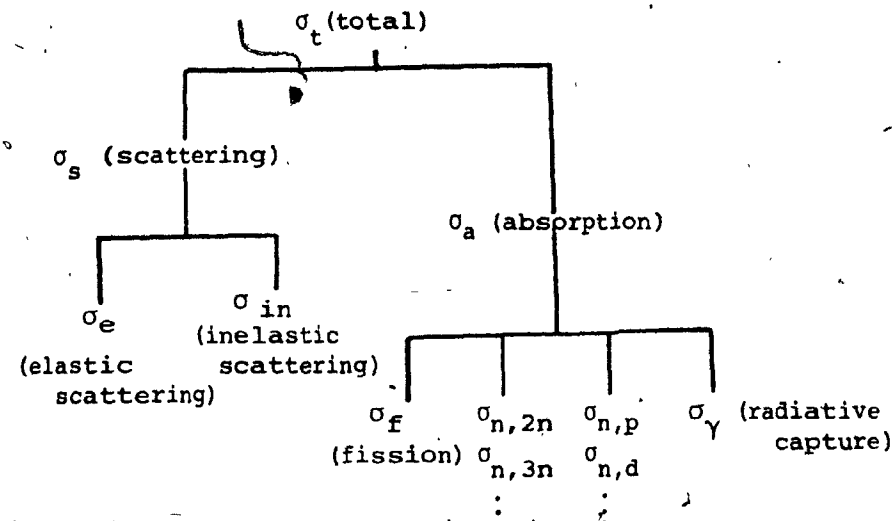


Fig. II.2 Neutron Cross Section Heirarchy.

Thus far we have defined the concept of a microscopic cross section by considering a beam of neutrons of identical speeds incident normally upon the surface of a target. However it is certainly conceivable that such cross sections will vary, depending on the incident neutron speed (or energy) and direction. Indeed if the microscopic cross section for various incident neutron energies is measured, a very strong energy dependence of the cross section is found. The dependence of neutron cross sections on the incident beam angle is usually much weaker and can almost always be ignored in nuclear reactor applications.

II.4 MACROSCOPIC CROSS SECTIONS

Thus far we have considered a beam of neutrons incident upon a very thin target. This was done to insure that each nucleus in the target would be exposed to the same beam intensity. If the target were thicker, the nuclei

deeper within the target would tend to be shielded from the incident beam by the nuclei nearer the surface since interactions remove neutrons from the beam. To account for such finite thickness effects, let us now consider a neutron beam incident upon the surface of a target of arbitrary thickness as indicated schematically in Figure II.3. We will derive an equation for the "virgin" beam intensity I(x) at any point x in the target. By virgin beam we are referring to that portion of the neutrons in the beam that have not interacted with target nuclei. Consider a differential thickness of target between x and x + dx. Then since dx is infinitesimally thin, we know that the results from our study of thin targets can be used to calculate the rate at which neutrons suffer interactions in dx per cm^2. If we recognize that the number of target nuclei per cm^2 in dx is given by $dN_A = Ndx$, where N is the number density of nuclei in the target, then the total reaction rate per unit area in dx is just

$$dR = \sigma_t I dN_A = \sigma_t I N dx \qquad \text{(II.23)}$$

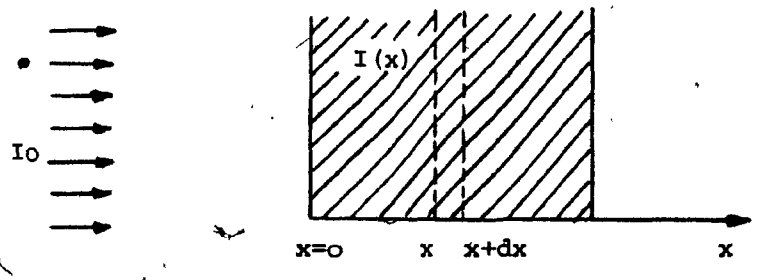


Fig. II.3 Attenuation of Neutron Beam Incident Normally on a Thick Target.

Notice that, consistent with our prescription that any type of interaction will deflower an incident neutron, we have utilized the total microscopic cross section σ_t in computing dR.

We can now equate this reaction rate to the decrease in beam intensity between x and x + dx

$$-dI(x) = -\left[I(x+dx) - I(x)\right] = \sigma_t I N dx \tag{II.24}$$

Dividing by dx we find a differential equation for the beam intensity I(x)

$$\frac{dI}{dx} = -N\sigma_t I(x) \tag{II.25}$$

If we solve this equation subject to an incident beam intensity of I_0 at x = 0, we find exponential attenuation of the incident beam of the form

$$I(x) = I_0 \exp\ (-N\sigma_t x) \tag{II.26}$$

The product of the atomic number density N and the microscopic cross section σ_t that appears in the exponential term arises so frequently in nuclear reactor studies that it has become customary to denote it by a special symbol:

$$\Sigma_t \equiv N\sigma_t = \left[\#/cm^3\right]\left[cm^2\right] = \left[cm^{-1}\right] \tag{II.27}$$

One refers to Σ_t as the total macroscopic cross section characterizing the target material. The term "macroscopic" arises from the recognition that Σ_t characterizes the probability of neutron interaction in a macroscopic chunk of material (the target), whereas the microscopic cross section characterizes the probability of interaction with only a single nucleus.

It should be noted that Σ_t is not really a "cross section" at all, however, since its units are inverse length. A more appropriate interpratation can be achieved by reexamining Eq. II.24) and noting that the fractional change in beam intensity occuring over a distance dx is just given by

$$\frac{\left(\frac{-dI}{I}\right)}{dx} = \Sigma_t \qquad \text{(II.28)}$$

Hence, it is natural to interpret Σ_t as the probability per unit path length traveled that the neutron will undergo a reaction with a nucleus in the sample. In this sense then

$\exp(-\Sigma_t x) \equiv$ probability that a neutron moves a distance dx (any interaction);

$\Sigma_t \exp(-\Sigma_t x)dx \equiv$ probability that a neutron has its first interaction in dx

$\equiv p(x)dx$

With this interaction probability, we can calculate the average distance a neutron travels before interacting with a nucleus in the sample.

$$\bar{x} \equiv \int_0^\infty dx \; x p \quad (x) = \Sigma_t \int_0^\infty dx \; x \exp(-\Sigma_t x) = \frac{1}{\Sigma_t} \tag{II.29}$$

It is customary to refer to this distance as the neutron mean free path since it essentially measures the average distance a neutron is likely to stream freely before colliding with a nucleus.

The reader has probably noticed the similarity of this analysis to our earlier treatment of radioactive decay. The spatial attentuation of a neutron beam passing through a sample of material and the temporal decay of a sample of radioactive nuclei are similar types of statistical phenomenon is which the probability of an event occuring that removes a neutron or nucleus from the original sample depends only on the number of neutrons or nuclei present at the position or time of interest. It should be stressed that both the mean free path and the mean lifetime for decay are very much average quantities. There will be statistical fluctuations about these mean values.

If we recall that Σ_t is the probability per unit path length that a neutron will undergo a reaction, while the neutron speed v is the distance traveled by the neutron in a unit time, then evidently

$$v\Sigma_t = \left[\frac{cm}{sec}\right]\left[cm^{-1}\right] = \left[sec^{-1}\right] = \text{Frequency with which reactions occur for one traveling neutron.} \tag{II.30}$$

This quantity is usually referred to as the collision frequency for the neutron in the sample. Its reciprocal, $\left[v\Sigma_t\right]^{-1}$, is therefore interpretable as the mean time between neutron reactions.

Thus far our discussion has been restricted to total macroscopic cross sections that characterize the probability that a neutron will undergo any type of reaction. We can generalize this concept by formally defining the macroscopic cross section for any specific reaction as just the microscopic cross section for the reaction of interest multiplied by the number density N characterizing the material of interest. For example, the macroscopic fission cross section would be defined as

$$\Sigma_f \equiv N\sigma_f \qquad \text{(II.31)}$$

In a similar fashion we can define

$$\Sigma_a \equiv N\sigma_a \quad , \quad \Sigma_s \equiv N\sigma_s \qquad \text{(II.32)}$$

Notice also that in analogy with Eq. II.20

$$\Sigma_t = \Sigma_a + \Sigma_s \qquad \text{(II.33)}$$

It should be stressed that while one can formally define such macroscopic cross sections for specific reactions, our earlier discussion of neutron penetration into a thick target applies only to the total macroscopic cross section Σ_t. We could not extend this discussion,

for example, to the calculation of the probability of neutron penetration to a depth x prior to absorption by merely replacing Σ_t in Eq. (II.26) by Σ_a since it may be possible for the neutron to undergo a number of scattering reactions before finally suffering an absorption reaction.

II.5 CROSS SECTIONS OF MIXTURES AND MOLECULES

The concept of a macroscopic cross section can also be generalized to homogeneous mixtures of different nuclides. For example, if we have a homogeneous mixture of three different species of nuclide, X, Y, and Z, with respective atomic number densities N_x, N_y and N_z, then the total macroscopic cross section characterizing the mixture is given by

$$\Sigma_t = N_x \sigma_t^x + N_y \sigma_t^y + N_z \sigma_t^z \qquad \text{(II.34)}$$

where σ_t^x is the microscopic total cross section for nuclide X, and so on. It should be noted that such a prescription for determining the macroscopic cross section for a mixture arises quite naturally from our interpretation of such cross sections as probabilities of reactions.

As we mentioned earlier, all neutron-nuclear reaction cross sections (fission, radiative capture, scattering, etc.) depend to some degree on the energy of the incident neutron. If we denote the neutron energy by E, we acknowledge this dependence by including a functional dependence on E in the microscopic cross section $\sigma(E)$ and hence by inference also

in the macroscopic cross section $\Sigma(E)$.

However, the macroscopic cross section can depend on additional variables as well. For example, suppose that the target material does not have a uniform composition. Then the number density N will depend on the position r in the sample, and hence the macroscopic cross sections themselves will be space-dependent. In a similar manner, the number densities might depend on time - suppose, for example, that the nuclide of interest was unstable such that its number density was decaying as a function of time. Therefore in the most general case we would write

$$\Sigma(r,E,t) = N(r,t)\sigma(E) \qquad \text{(II.35)}$$

to indicate the explicit dependence of the macroscopic cross section on neutron energy E, position r, and time t.

In summary then, nuclear cross sections can be used to characterize the probability of various types of neutron-nuclear reactions occuring. They obviously will be a very basic ingredient in any study of fission chain reactions. The determination of such cross sections is the task of the nuclear physicist and involves both experimental measurement and theoretical calculations. The enormous amount of cross section information required for nuclear reactor analysis is gathered by numerous nuclear research centers throughout the world. These cross section data are compiled, evaluated, and then organized into data sets to be used by nuclear engineers. We will return in a later section to discuss

in further detail such nuclear cross section data sets. For convenience, however, we have included in Appendix C a table of some of the more important cross sections characteristic of "thermal" neutrons - that is, of neutrons whose energies are comparable to the thermal energy of atoms in a reactor core at room temperature, E = 0.025 eV - which serves to illustrate typical orders of magnitudes of these quantities.

II.6 GENERALIZATIONS OF THE CONCEPT OF THE CROSS SECTION

II.6.1 DIFFERENTIAL SCATTERING CROSS SECTIONS

Neutron cross sections provide a quantitative measure of the probability that various types of neutron-nuclear reactions will occur. For example, we have introduced $\sigma_a(E)$ to characterize the probability that a neutron with kinetic energy E incident upon a nucleus will be absorbed. Similar cross sections have been introduced to describe reactions such as scattering.

It is frequently useful to introduce a generalization of the concept of a neutron cross section characterizing the scattering reaction. In such reactions the incident neutron will usually experience a change in both direction of motion and energy in the scattering event. (Just imagine a billiard-ball collision). The microscopic scattering cross section will describe the probability that such a scattering collision occurs. However it provides no information about the change in neutron direction or energy that occurs in such a collision. This latter information is very important in certain types of reactor studies. To characterize it, we must introduce the concept of the

differential scattering cross section.

First we must introduce variables that characterize the motion of the incident neutron. The natural choice would be the neutron velocity v. Then the cross section we wish to define would describe the probability that a neutron incident with a velocity v would be scattered by a nucleus to a new velocity v.

However, in reactor analysis it will be more convenient to describe the neutron motion with slightly different variables. We will essentially decompose the neutron velocity vector into two components, one variable characterizing the neutron speed and a second variable for the neutron direction of motion. We use the kinetic energy of the neutron $E = \frac{1}{2}mv^2$ instead of the neutron speed itself. Then to specify the direction of neutron motion, we introduce a unit vector in the direction of the neutron velocity vector v:

$$\hat{\Omega} = v/|v| = \hat{e}_x \sin\theta\cos\phi + \hat{e}_y \sin\theta\sin\phi + \hat{e}_z \cos\theta \qquad \text{(II.36)}$$

where we have chosen to represent this direction unit vector in spherical velocity-space coordinates (θ,ϕ) (see Fig. II.4). Notice that to describe the incident neutron velocity, we now specify both its energy E and its direction $\hat{\Omega}$.

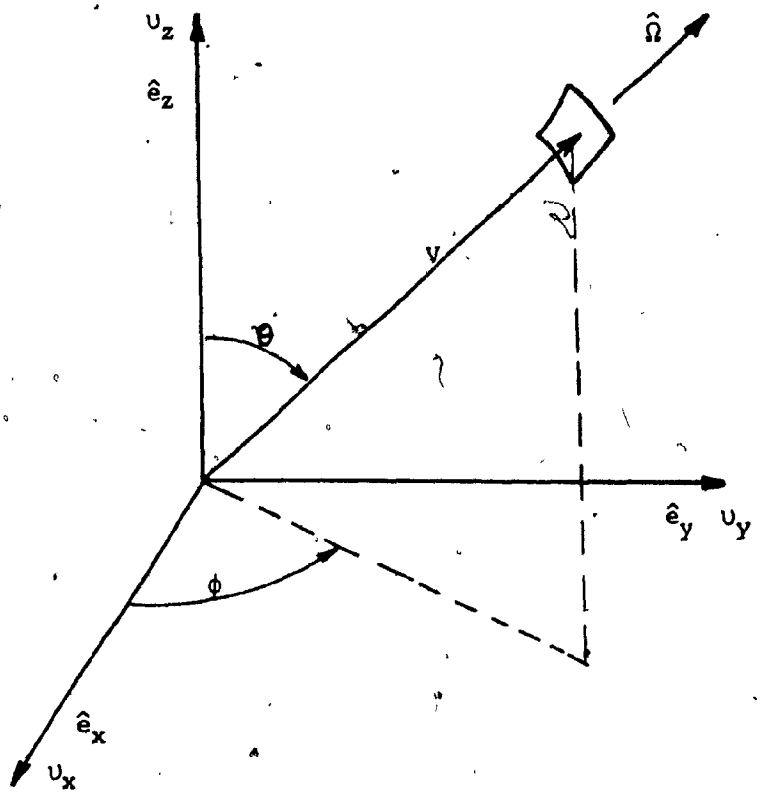


Fig. II.4 The Neutron Direction Unit Vector in Spherical coordinates.

At this point it is convenient to consider how one integrates over these variables (as we shall have cause to do later in this section). Suppose we wished to integrate over all possible neutron velocities. This integration could then be performed in either Cartesian or spherical velocity coordinates:

$$\int d^3\upsilon\, f(v) = \int_{-\infty}^{\infty} d\upsilon_x \int_{-\infty}^{\infty} d\upsilon_y \int_{-\infty}^{\infty} d\upsilon_z\; f(v) = \int_0^{\infty} d\upsilon\; \upsilon^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta f(v) \qquad (II.37)$$

However we have defined the unit vector in the direction of the velocity vector as $\hat{\Omega}$. Hence we can identify the angular portion of the integration in Eq. (II.37) as just the integration over all directions:

$$\int_{4\pi} d\Omega \equiv \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \tag{II.38}$$

In this sense we see that the differential $d\Omega$ corresponds to a differential solid angle (not a vector)

$$d\Omega = \sin\theta d\theta d\phi \tag{II.39}$$

One final modification is useful here. We will usually choose to work with the neutron energy E rather than the neutron speed. Hence rather than choosing to integrate functions of v over all neutron velocities, we will integrate functions of E and $\hat{\Omega}$ over all possible neutron energies and directions:

$$\int d^3v f(v) \rightarrow \int_0^{\infty} dE \int_{4\pi} d\Omega f(E,\hat{\Omega}) \tag{II.40}$$

So much for mathematical preliminaries. We will now proceed to introduce the concept of a cross section that characterizes the probability that a neutron is scattered from an initial energy E and direction of motion $\hat{\Omega}$ to a final energy E' and direction of motion $\hat{\Omega}'$. To make life simple, we will first do this for the situation in which we are only interested in the change in neutron energy in scattering. Imagine a beam of neutrons of incident intensity I, all of energy E, incident upon a thin target of surface atomic density N_A. Then the rate/cm^2 at which

neutrons will be scattered from their original energy E to a final energy E' in the range E' to E' + dE' is proportional to the beam intensity I, the target surface density N_A, and the differential range dE' of final energies. We will define the microscopic differential scattering cross section σ_s (E→E') as the appropriate proportionality parameter

$$\text{Rate/cm}^2 = \sigma_s(E \rightarrow E')dE'IN_A \qquad \text{(II.41)}$$

Hence we find that σ_s (E→E') characterizes the probability that a scattering collision changes the original neutron energy from E to E' in dE'. It is important to notice that this differential scattering cross section is a "distribution" in the sense that it is associated with a certain range of final energies, E' to E'+ dE'. Hence its dimensions are cm^2/eV.

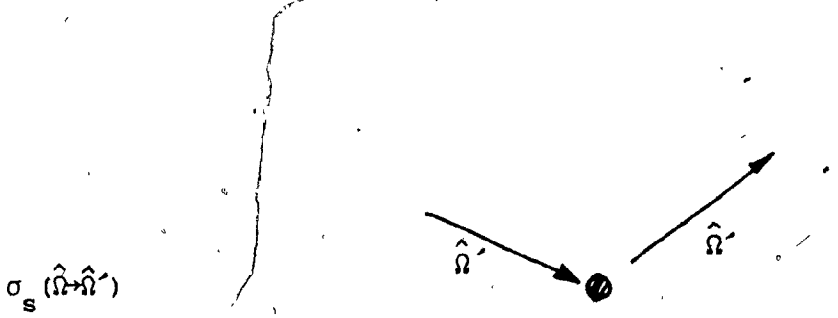
There is a very simple relationship between the differential scattering cross section $\sigma_s(E \rightarrow E')$ and our earlier definition of the microscopic scattering cross section $\sigma_s(E)$. If we recognize that the latter quantity is just related to the probability that a neutron of energy E will suffer a scattering collision, regardless of the final energy E' to which it is scattered, then it is apparent that $\sigma_s(E)$ is just the integral of the differential scattering cross section $\sigma_s(E \rightarrow E')$ over all final energies E'

$$\sigma_s(E) = \int_0^\infty dE'\ \sigma_s(E \rightarrow E') \qquad \text{(II.42)}$$

Of course this relationship explains the origin of the term "differential".

It should be mentioned that occasionally one encounters a somewhat different notation for the differential scattering cross section which may be written as $d\sigma/dE$. We find the convention of denoting differential cross sections by a multiple variable argument such as $(E \to E')$ to be more convenient for our purposes, and hence will use this notation throughout.

We can similarly introduce the concept of a differential scattering cross section describing the probability that a neutron scatters from an incident direction $\hat{\Omega}$ to a final direction $\hat{\Omega}'$ in a very similar manner:



Once again, $\sigma_s(\hat{\Omega} \to \hat{\Omega}')$ is related to our earlier microscopic scattering cross section by an integration over all final directions.

$$\sigma_s(\hat{\Omega}) = \int_{4\pi} \sigma_s(\hat{\Omega} \to \hat{\Omega}')\, d\Omega' = \sigma_s \qquad \text{(II.43)}$$

Two comments are useful here. First, it should be mentioned that the dependence of the scattering cross section $\sigma_s(\hat{\Omega})$ on the incident neutron direction is usually ignored. Indeed very few microscopic scattering cross sections in reactor applications depend on the incident neutron direction because the nuclei in any macroscopic sample are usually

randomly oriented, and thus any directional dependence averages out when averaged over all possible nuclear orientations. (One could imagine a sample in which most of the nuclei could be aligned--eg., a ferromagnetic material-- but such situations can safely be ignored in reactor analysis).

In this case, however, even though the differential scattering cross section $\sigma_s(\hat{\Omega}\rightarrow\hat{\Omega}')$ will not depend on the incident neutron direction, it will depend on the change in neutron direction. This is most conveniently expressed in terms of a functional dependence on the angle through which the incident neutron is scattered--the so-called scattering angle θ, or more conveniently, the cosine of this scattering angle, $\mu_0 \equiv \cos\theta$, which can be conveniently expressed as the dot product between the unit direction vectors $\mu_0 = \hat{\Omega}.\hat{\Omega}'$. (Again, the dependence on azimuthal angle ϕ does not arise for materials in which the nuclei have a random orientation.) One occasionally denotes this functional dependence by writing

$$\sigma_s(\hat{\Omega}\rightarrow\hat{\Omega}') = \sigma_s(\hat{\Omega}.\hat{\Omega}') = \sigma_s(\mu_0). \qquad \text{(II.44)}$$

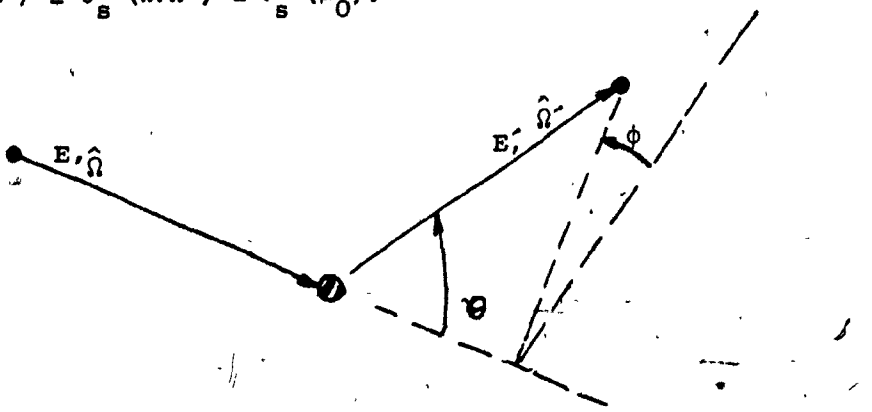
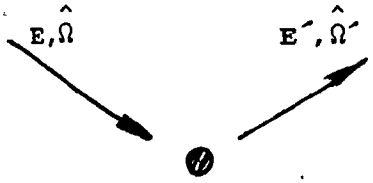


Fig. II.5 Definition of scattering angle θ

We will continue to use the somewhat more formal notation by writing $\sigma_s(\hat{\Omega}\to\hat{\Omega}')$, even though we know that this differential cross section usually depends only on μ_0.

Thus far we have developed the concept of differential scattering cross sections that characterize the probability of scattering from one energy to another or one direction to another. We can combine these concepts by defining a double differential scattering cross section that characterizes scattering from an incident energy E, direction $\hat{\Omega}$ to a final E' in dE' and $\hat{\Omega}'$ in $d\hat{\Omega}$

$$\sigma_s(E\to E', \hat{\Omega}\to\hat{\Omega}').$$



Again, alternative notations are occasionally used such as $\sigma_s(E,\hat{\Omega}\to E',\hat{\Omega}')$ or $d^2\sigma_s / dE\, d\Omega$.

We can again relate the double differential scattering cross section to the differential scattering cross section or the scattering cross section by integration over energy or angle:

$$\sigma_s(E\to E') = \int_{4\pi} d\Omega' \sigma_s(E\to E', \hat{\Omega}\to\hat{\Omega}'), \tag{II.45}$$

or

$$\sigma_s(E,\hat{\Omega}\to\hat{\Omega}') = \int_0^\infty dE' \sigma_s(E\to E', \hat{\Omega}\to\hat{\Omega}'), \tag{II.46}$$

$$\sigma_s(E) = \int_{4\pi} d\Omega' \int_0^\infty dE' \sigma_s(E\to E', \hat{\Omega}\to\hat{\Omega}') \tag{II.47}$$

The concept of a differential scattering cross section can also be applied to macroscopic cross sections by merely multiplying by the atomic number density N:

$$\Sigma_s(E\to E', \hat{\Omega}\to\hat{\Omega}') \equiv N\sigma_s(E\to E', \hat{\Omega}\to\hat{\Omega}') \tag{II.48}$$

$$\Sigma_s(E\to E') \equiv N\sigma_s(E\to E') \tag{II.49}$$

$$\Sigma_s(\hat{\Omega}\to\hat{\Omega}') \equiv N\sigma_s(\hat{\Omega}\to\hat{\Omega}') \tag{II.50}$$

Such differential cross sections are quite important in nuclear reactor analysis since they determine the manner in which neutrons move about in a reactor core, as well as the rate at which they leak out of the reactor. To prepare the way for the calculation of such cross sections, let us first decompose the differential scattering cross section into two factors:

$$\sigma_s(E\to E') = \sigma_s(E)P(E\to E'). \tag{II.51}$$

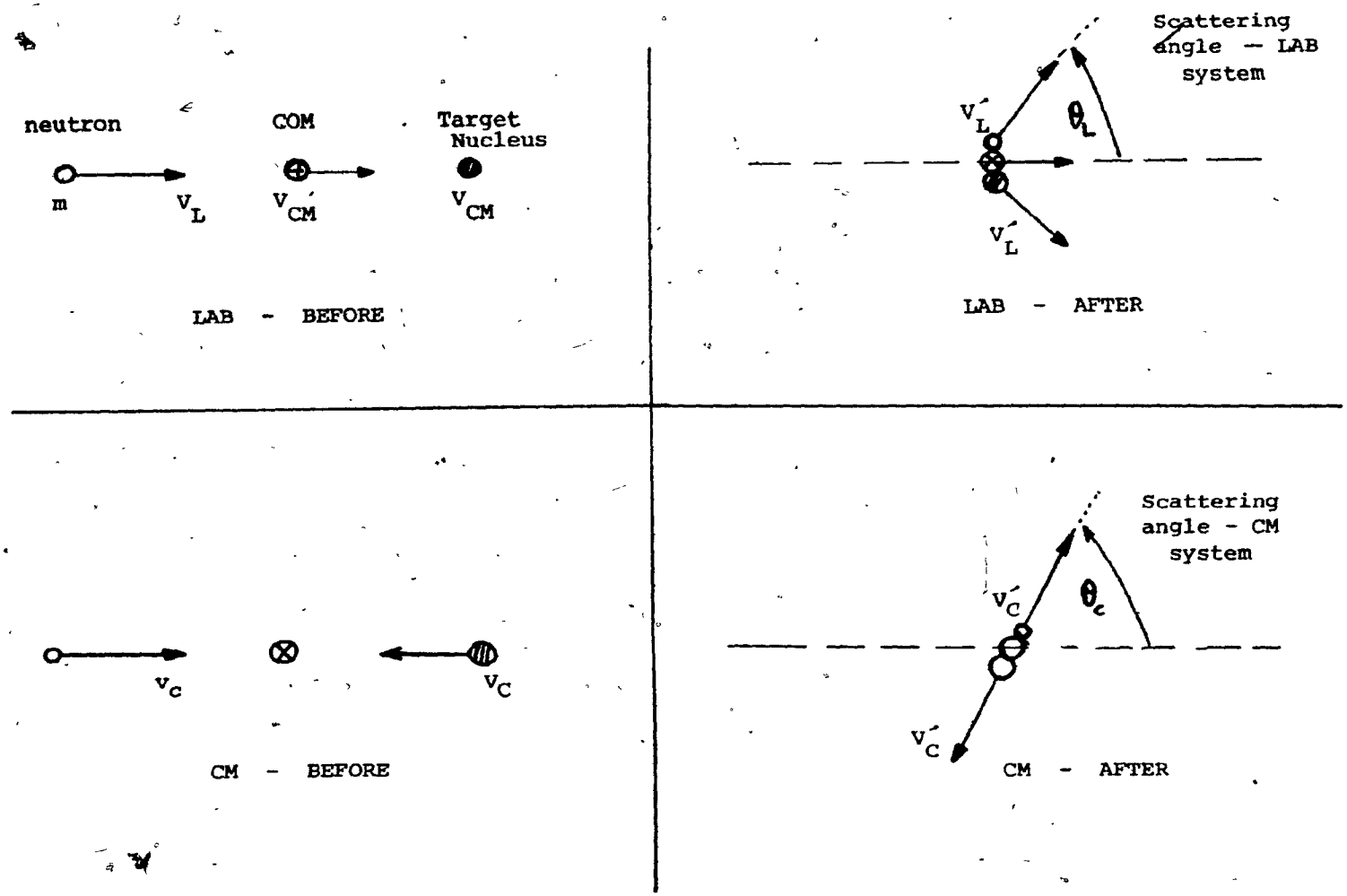
If we recall our earlier definition of the scattering cross section, $\sigma_s(E)$, then it becomes apparent that we can identify

$P(E \rightarrow E')dE'$ = Probability that a neutron scattering with initial energy E will emerge with a new energy E' in the range E' to $E' + dE'$.

We can explicitly calculate this quantity for the situation in which neutrons of moderate energies ($E < 1MeV$) scatter elastically via potential scattering from stationary nuclei of low mass number A.

II.6.2 KINEMATICS OF NEUTRON SCATTERING FROM STATIONARY NUCLEI

The kinematics of any two-body collision process is simplified very considerably when analyzed within the center-of-mass (CM) coordinate frame. We have sketched the collision event before and after the collision in both the LAB and CM coordinate frames in Figure II.6. Here, lower-case notation corresponds to the neutron and upper-case notation to the nucleus. The subscripts L and C refer to LAB or CM frames, respectively.



II-6 Definition of Collision Coordinates in LAB and CM System.

The velocity of the CM frame is defined by

$$v_{CM} = \frac{1}{(m+M)} (mv_L + MV_L) = \left(\frac{1}{1+A}\right) v_L \qquad \text{(II.52)}$$

where we have assumed that the initial nucleus velocity v_L is zero noting that the nucleus-neutron mass ratio M/m is essentially just the nuclear mass number A. If we note that the neutron and nucleus velocities in the CM frame are given by

$$v_C = v_L - v_{CM} = \frac{A}{A+1} v_L \qquad \text{(II.53)}$$

$$V_C = -v_{CM} = -\frac{1}{A+1} v_L \qquad \text{(II.54)}$$

then it is apparent that the total momentum in the CM frame is zero, as it must be.

We can relate the total kinetic energies in the LAB and CM frames by computing

$$\text{LAB:} \quad E_L = \tfrac{1}{2}mv^2 + \tfrac{1}{2}MV_L^2 \qquad \text{(II.55)}$$

$$\text{CM:} \quad E_C = \tfrac{1}{2}mv_C^2 + \tfrac{1}{2}MV_C^2 = \tfrac{1}{2}\mu v_L^2 \qquad \text{(II.56)}$$

where we have introduced the reduced mass $\mu \equiv mM/(m+M)$. Hence we find the important relation between the energy in the CM and the LAB frames as

$$E_C = \frac{M}{m + M} E_L = \frac{A}{1 + A} E_L \qquad \text{(II.57)}$$

In particular it should be noted that the total energy in the CM system is always less than that in the LAB system. The energy difference is taken up by the center of mass motion itself.

Using conservation of momentum and energy, it is easy for one to demonstrate that the magnitudes of the CM velocities do not change in the collosion:

$$v'_C = v_C = \frac{A}{A + 1} v_L \qquad \text{(II.58)}$$

$$V'_C = V_C = \frac{1}{A + 1} v_L \qquad \text{(II.59)}$$

only their velocity vectors are rotated through the CM scattering angle θ_C. This fact allows one to relate the scattering angles in the LAB and CM frames. Consider the vector diagram in Figure II.7 illustrating the velocities and scattering angles in these two frames.

If we note from this diagram that

$$v'_L \sin\theta_L = v'_C \sin\theta_C \qquad \text{(II.60)}$$

$$v'_L \cos\theta_L = v_{CM} + v'_C \cos\theta_C \qquad \text{(II.61)}$$

then we can relate the scattering angles in the CM and LAB frames by

$$\tan\theta_L = \frac{v'_C \sin\theta_C}{v_{CM} + v'_C \cos\theta_C} = \frac{\sin\theta_C}{\frac{1}{A} + \cos\theta_C} \qquad \text{(II.62)}$$

This relationship is particularly useful since cross sections are usually calculated in the CM frame, but are measured and used in the LAB frame. If we denote the differential scattering cross sections characterizing scattering through angles θ_L and θ_C in the LAB and CM frames, $\sigma_L(\theta_L)$ and $\sigma_{CM}(\theta_C)$ respectively, we can use

$$\sigma_L(\theta_L)\sin\theta_L d\theta_L = \sigma_{CM}(\theta_C)\sin\theta_C d\theta_C$$

to relate the LAB and CM differential scattering cross sections by

$$\sigma_L(\theta_L) = \sigma_{CM}(\theta_C) \frac{\left[\frac{1}{A^2} + \frac{2}{A}\cos\theta_C + 1\right]^{3/2}}{1 + \frac{1}{A}\cos\theta_C} \qquad \text{(II.63)}$$

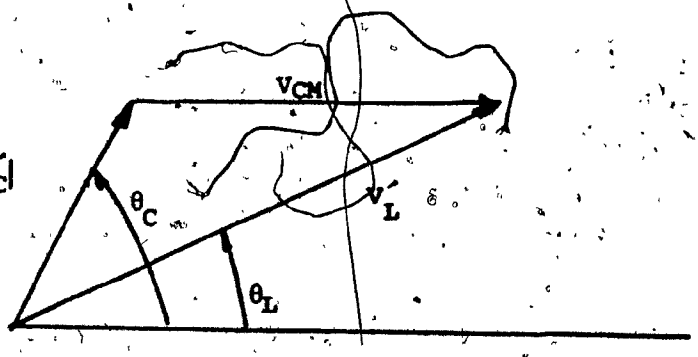


Fig. II.7 Relation Between the Scattering Angles in the LAB and CM Frames.

Returning to our vector diagram in Figure II.7 and using the law of cosines, we can find

$$\cos(180^\circ - \theta_c) = \frac{v_c^2 + v_{CM}^2 - v_L^2}{2 v_c v_{CM}} \qquad \text{(II.64)}$$

but using Eqs.(II.52) and (II.56), we can rewrite this as

$$\frac{1/2mv_L^2}{1/2mv_L^2} = \frac{E'}{E} = \frac{A^2 + 1 + 2A\cos\theta_c}{(A+1)^2} \qquad \text{(II.65)}$$

It is useful to introduce a parameter related to the nuclear mass number A

$$\alpha \equiv \left(\frac{A-1}{A+1}\right) \qquad \text{(II.66)}$$

Then we can write the final neutron energy after collision, $E_f \equiv E'$, in terms of the incident neutron energy, $E_i \equiv E$ as

$$E_f = \left[\frac{(1+\alpha) + (1-\alpha)\cos\theta_c}{2}\right] E_i \qquad \text{(II.67)}$$

Let us study this very important relation in more detail. First notice that it implies that the energy transfer from the neutron to the nucleus is directly related to the scattering angle in the CM frame. For example, if $\theta_c = 0$, then the neutron would lose no energy ($E_f = E_i$). This corres-

ponds, of course, to no collision at all (a "miss"). The maximum energy loss occurs in a backscattering collision in which $\theta_c = 180^\circ$. In this case, $E_f = \alpha E_i$. Hence the maximum energy that a neutron can lose in an elastic scattering collision with a stationary nucleus is $(1-\alpha)E_i$. For example, in scattering collisions with hydrogen nuclei (A = 1), the neutron could conceivably lose all of its energy, while in collision with a heavy nucleus such as ^{238}U it could lose at most 2% of its incident energy.

In summary, then, we have discoverd two very important facts. First, a neutron cannot gain energy in an elastic collision with a stationary nucleus (E_f is always less than E_i). Second, the neutron cannot emerge from an elastic scattering collision with an energy E_f less than αE_i.

We can now go one step further and actually calculate the scattering probability distribution, $P(E \to E')$, or in our present notation, $P(E_i \to E_f)$, for the case of elastic scattering from stationary nuclei. First we note from our preceding discussion that $P(E_i \to E_f)$ must vanish if the final energy E_f does not fall within the range $\alpha E_i < E_f < E_i$. To calculate $P(E_i \to E_f)$ in this range, we will utilize the relationship Eq. (II-6).

If we will recall that there is a one-to-one relationship between the neutron energy transfer and the scattering angle, we can infer that there must similarly be a relationship between the probability of the neutron experiencing a given energy transfer $E_i \to E_f$ and the probability that it will be scattered through a given scattering angle θ_c. Yet the probability of scattering through an angle θ_c into $d\theta_c$ about θ_c is just given by

$$P(\theta_c)\, 2\pi \sin\theta_c d\theta_c = \frac{\sigma_{CM}(\theta_c)}{\sigma_s}\, 2\pi \sin\theta_c d\theta_c \qquad \text{(II.68)}$$

Hence we can equate

$$P(E_i \to E_f) = - \frac{\sigma_{CM}(\theta_c)}{\sigma_s}\, 2\pi \sin\theta_c d\theta_c \qquad \text{(II.69)}$$

If we now differentiate Eq. (II.67)

$$dE_f = - \frac{E_i(1-\alpha)\, \sin\theta_c d\theta_c}{2} \qquad \text{(II.70)}$$

and substitute this into Eq.(II.69), we find the very important result

$$P(E_i \to E_f) = \frac{4\pi\sigma_{CM}(\theta c)}{(1-\alpha) E_i \sigma_s}, \quad \alpha E_i \leq E_f \leq E_i \qquad \text{(II.71)}$$

$$= 0, \qquad \text{otherwise}$$

To complete the determination of $P(E_i \to E_f)$, we still need to know the differential scattering cross section in the CM frame. This knowledge must come from both a consideration of quantum mechanics and the detailed nuclear physics of the interaction. Fortunately, however, we can avoid such considerations since the CM potential scattering cross sections characterizing neutrons of interest in reactor applications (with energie $E < 10$MeV) do not depend on θ_c for light nuclei (say $A < 12$). That is, the scattering in the CM frame is isotropic such that

$$\sigma_{CM}(\theta_c) = \frac{\sigma_s}{4\pi} \qquad \text{(II.72)}$$

Such behavior is known as "s-wave" scattering (a term which arises from quantum mechanics) and is the most common form of elastic scattering in nuclear reactors. For heavier nuclei, there will tend to be some mild angular dependence of $\sigma_{CM}(\theta_c)$, but since elastic scattering from such nuclei does not contribute appreciably to neutron energy loss in most nuclear reactor types, we will confine our attention here to s-wave elastic scattering from stationary nuclei. Then, using Eq. (II.72) in Eq. (II.71), we find that the scattering probability distribution for elastic s-wave scattering from stationary nuclei takes the form

$$P(E_i \to E_f) = \frac{1}{(1-\alpha)\,E_i}, \quad \alpha E_i \le E_f \le E_i \qquad \text{(II.73)}$$

$$= 0, \qquad \text{otherwise}$$

Notice in particular that the probability of scattering from an energy E_i to a final energy E_f is independent of the final energy E_f.

Using this probability distribution, we can compute the average final energy of a neutron suffering an elastic scattering collision as

$$\bar{E}_f \equiv \int_{\alpha E_i}^{E_i} dE_f E_f P(E_i \to E_f) = \left(\frac{1+\alpha}{2}\right) E_i \qquad \text{(II.74)}$$

Hence the average energy loss in such collisions is

$$\overline{\Delta E} = E_i - \bar{E}_f = \left(\frac{1-\alpha}{2}\right) E_i \qquad \text{(II.75)}$$

For example, neutrons suffering scattering collisions in hydrogen ($\alpha = 0$) will lose on the average half of their original energy in each collision. By way of contrast, in a scattering collision with a ^{238}U nucleus, they will lose on the average less that 1% of their original energy.

We can now write the differential scattering cross section characterizing elastic (s-wave) scattering from stationary nuclei by substituting Eq.(II.73) into Eq. (II.51)

$$\sigma_s(E_i \to E_f) = \frac{\sigma_s(E_i)}{(1-\alpha)E_i}, \quad \alpha E_i \leq E_f \leq E_i \qquad \text{(II.76)}$$

$$= 0, \qquad \text{otherwise}$$

This is about as far as we can go in determining the explicit form of $\sigma_s(E_i \to E_f)$ since the elastic scattering cross section $\sigma_s(E)$ itself depends on the details of the nuclear potential and is generally obtainable only by measurement. Fortunately $\sigma_s(E)$ characterizing potential scattering is only weakly dependent on energy and can frequently be taken as constant over a wide range of neutron energies.

Such elastic scattering plays a very important role in nuclear reactor behavior since it tends to slow the fast fission neutrons down to thermal energies. However inelastic scattering processes are also important, particularly in fast reactors where neutron moderation by light isotopes is minimized. Since kinetic energy is not a conserved quantity in an inelastic scattering collision (the nucleus is left in an excited state),

we can no longer use simple kinematical arguments to determine $P(E_i \rightarrow E_f)$ for such processes. Rather one must rely on measurements of the differential scattering cross section or on nuclear models. Although such models are useful for qualitative estimates of neutron scattering, most detailed reactor studies simply use the measured cross section in essentially tabular form for the various different energy transfer combinations $E_i \rightarrow E_f$.

One final comment should be made concerning the "other half" of a scattering event, namely the nuclear recoil. Although this is of little concern to the neutron economist, it is of very considerable concern to the reactor designer since the recoil energy of a nucleus suffering a collision with a fast neutron will be sufficient to rip it completely out of its crystalline lattice. To be more specific, the average recoil energy of a nucleus suffering an elastic scattering collision with a neutron is just $\frac{1}{2}(1 - \alpha) E_i$ (recall Eq. (II.75)). For fast neutrons, this recoil energy will be in the KeV to low MeV range. Hence the recoiling nucleus will not only be torn out of its own lattice position by the collision, but will possess sufficient recoil energy to dislocate other nuclei in the lattice, leading to significant radiation damage to the material. This is an extremely important process in materials exposed to the high radiation environment of a nuclear reactor core, and must be taken into account in nuclear reactor design.

II.6.2.1 THE AVERAGE LOGARITHMIC ENERGY DECREMENT

A very useful quantity in the study of the slowing down of neutrons is the average value of the decrease in the natural logarithm of the neutron energy per collision, or the average logarithmic energy decrement per collision. It is the average for all collisions of $\ln E_1 - \ln E_2$, i.e., of $\ln (E_1/E_2)$, where E_1 is the energy of the neutron before and E_2 is that after a collision. If this quantity is represented by the symbol ξ then, taking into consideration the equal probability of all values of $\cos \theta_c$ from -1 to $+1$, it follows that

$$\xi \equiv \overline{\ln \frac{E_1}{E_2}} = \frac{\int_{-1}^{1} \ln \frac{E_1}{E_2} d(\cos \theta_c)}{\int_{-1}^{1} d(\cos \theta_c)}, \qquad \text{(II.77)}$$

where the integration limits refer to $\cos \theta$. Upon substituting the expression for E_2/E_1 from equation (II.65), it is found that

$$\xi = 1 + \frac{(A-1)^2}{2A} \ln \frac{A-1}{A+1} \qquad \text{(II.78)}$$

or, using the definition of α from equation (II,66)

$$\xi = 1 + \frac{\alpha}{1-\alpha} \ln \alpha. \qquad \text{(II.79)}$$

For values of A in excess of 10, a good approximation to equation (II.79) may be written as

$$\xi \approx \frac{2}{A + \frac{2}{3}} \qquad \text{(II.80)}$$

Even for A = 2, the error of equation (II.80) is only 3.3 per cent.

It will be seen from equation (II.79) that the value of ξ is independent of the initial energy of the neutron, provided scattering is symmetrical in the CM system. In other words, in any collision with a given scattering nucleus, a neutron loses, on the average, the same fraction of the energy it had before collision. This fraction decreases with increasing mass of the nucleus. It may be repeated here that the foregoing results involve the assumption of an essentially stationary nucleus prior to collision; this is justified only if the vibrational energy of the scattering nucleus is small compared to the kinetic energy of the neutron. Such is the case for neutron energies in excess of the thermal value. The values of α and ξ for a number of elements, especially those of low mass number, are given in Table (II.1). The average number of collisions with nuclei of a particular moderator required to decrease the energy of fission neutron, with initial energy of, say, 2 Mev, to the thermal value at ordinary temperatures, i.e., 0.025 ev, is obtained upon dividing ln $(2 \times 10^6/0.025)$ by ξ for the given moderator; thus,

$$\text{Average number of collisions to thermalize (2 Mev to 0.025 ev)} = \frac{\ln \frac{2 \times 10^6}{0.025}}{\xi} \approx \frac{18.2}{\xi} . \qquad \text{(II.81)}$$

Some of the results obtained from this equation are included in Table II.1.

TABLE II.1 SCATTERING PROPERTIES OF NUCLEI

Element	Mass No.	α	ξ	Collisions to Thermalize
Hydrogen.....	1	0	1.000	18
Deuterium.....	2	0.111	0.725	25
Helium.....	4	0.360	0.425	43
Beryllium.....	9	0.640	0.206	86
Carbon.....	12	0.716	0.158	114
Uranium.....	238	0.983	0.000838	2172

II.6.2.2 THE ξ FOR HETEROGENEOUS MODERATORS

If the moderator is not a single element, but a compound containing n nuclei, the effective (or mean) value of ξ is given by

$$\bar{\xi} = \frac{\sigma_{s1}\xi_1 + \sigma_{s2}\xi_2 + \cdots + \sigma_{sn}\xi_n}{\sigma_{s1} + \sigma_{s2} + \cdots + \sigma_{sn}}, \qquad \text{(II.82)}$$

where σ_s is the microscopic scattering cross section; the subscripts 1, 2, $\cdots$, n refer to the respective nuclei. For water, for example, containing two hydrogen nuclei and one oxygen nucleus, the appropriate expression for ξ_{H_2O} is

$$\bar{\xi}_{H_2O} = \frac{2\sigma_s(H) + \sigma_s(O)\xi(O)}{2\sigma_s(H) + \sigma_s(O)}, \qquad \text{(II.83)}$$

since ξ_H is unity. Similar expressions are applicable to mixtures of elements or compounds.

II.6.2.3 SLOWING DOWN POWER AND MODERATING RATIO

According to equation (II.81), the value of ξ is inversely proportional to the number of scattering collisions required to slow down a neutron from fission energy to thermal values. It is thus a partial measure of the moderating ability of the scattering material. A good moderator is one in which there is a considerable decrease of neutron energy per collision on the average and, hence, it is desirable that ξ is of little significance unless the probability of scattering, as indicated by the scattering cross section for neutrons with energy above thermal, is also large. The product $\xi\Sigma_s$, where Σ_s is the macroscopic scattering cross section of the moderator for epithermal neutrons, is called the slowing down power. For a compound, the slowing down power is given by

$$\xi\Sigma_s = N(\nu_1\sigma_{s1}\xi_1 + \nu_2\sigma_{s2}\xi_2 + \cdots \nu_i\sigma_{si}\xi_i + \cdots \nu_n\sigma_{sn}\xi_n), \quad \text{(II.84)}$$

where ν_i is the number of atoms of the ith kind in the molecule and N is the number of molecules per unit volume of the moderator. The slowing down power is a better measure than ξ alone of the efficiency of a moderator, for it is equal to ξ/λ_s, and so represents the average decrease in the logarithm of the neutron energy per cm of path. The slowing down powers of a number of materials consisting of (or containing) elements of low mass number are recorded in Table (II.2). (Cross sections are assumed to be constant in the energy range from 1 to 10^5 ev.

sponds to the first virtual state in O^{17}, has a total width of 41keV, giving a mean lifetime of 1.5×10^{-21} sec. Thus it is highly likely that the compound state in U^{239} decays at least to some extent by γ-ray emission, while the compound state in O^{17} must decay primarily by nucleon emission. The 443-keV resonance in O^{16} is clearly a scattering resonance, whereas the 6.67eV resonance in U^{238} is in part (and almost entirely, as it turns out) a capture resonance.

In view of the fact that the cross section for radiative capture is only large at low energy, it follows that it is primarily s-wave neutrons which are involved in this process. In this case, near an isolated resonance at the energy E_1, σ_γ is given by the Breit-Wigner one-level formula: (Ref.1,2)

$$\sigma_\gamma(E_c) = \pi\lambda\!\!\!^{-2}_1 g \left(\frac{E_1}{E_c}\right)^{\frac{1}{2}} \frac{\Gamma_n\Gamma_\gamma}{(E_c - E_1)^2 + \Gamma^2/4} \qquad \text{(II.86)}$$

Here, Γ_n and Γ_γ are the neutron and radiation widths, respectively, and the other symbols have the same meaning as in Eq.(II.85). Since capture resonances are found only in intermediate and heavy nuclei, the center-of-mass energy E_c is essentially equal to the laboratory energy E of the incident neutron. This is true, of course, provided the target nucleus is at rest in the laboratory system.

It should be observed from Eq.(II.86) that when $E(\simeq E_c)$ is very much less than E_1, σ_γ behaves as $1/\sqrt{E}$, that is, as $1/v$, where v is the speed of the neutron. The magnitude of σ_γ at low energy depends, however, upon whether or not there is a resonance in this energy region. If there is a low energy resonance, σ_γ is quite large and the $1/v$-portion of σ_γ is

short. This situation is illustrated in Figure II.9 for several important nuclides.

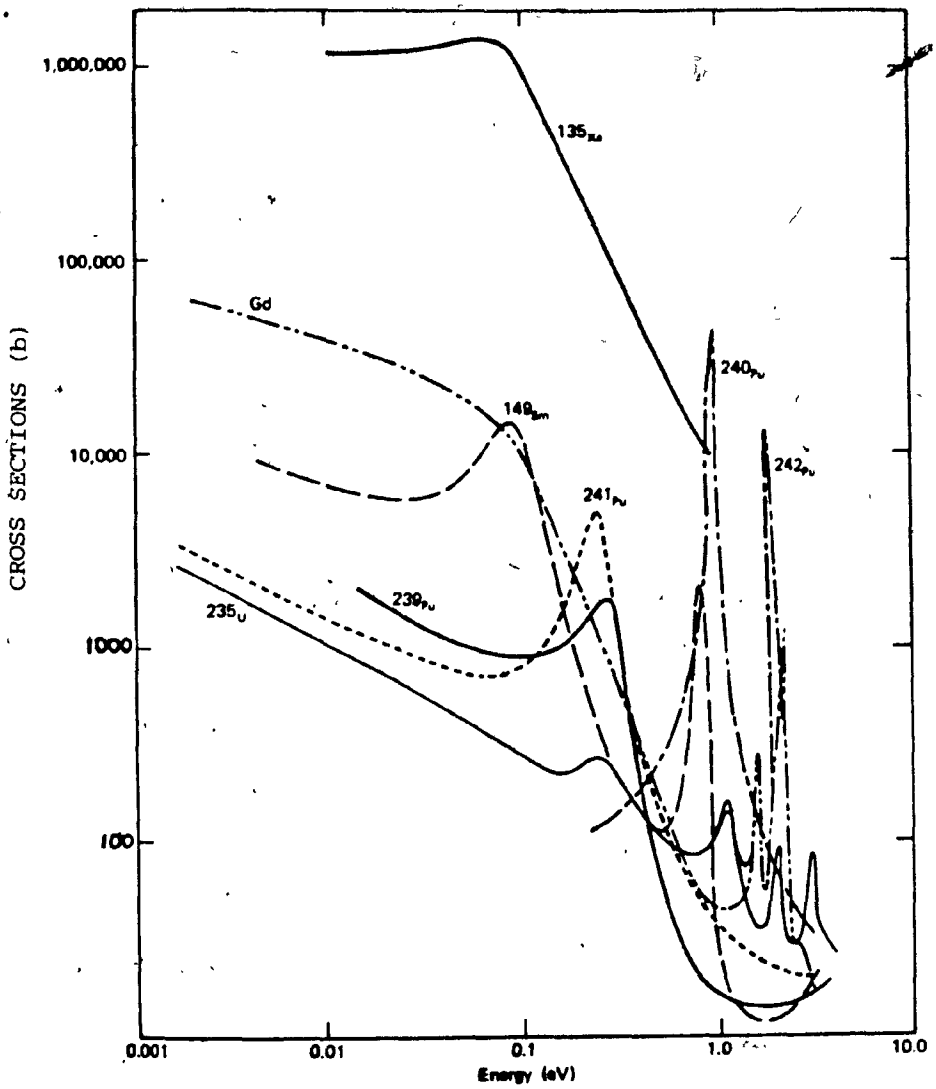


Fig. II.9 Low energy cross section behavior of several important nuclides.

CHARGED-PARTICLE REACTIONS

Neutrons also disappear as a result of charged-particle reactions, such as the (n,p) or (n,α) reactions. These reactions are usually endothermic and do not occur below a threshold energy. For a few light nuclei, however they are exothermic.

The most important exothermic reaction of this type is the $B^{10}(n,\alpha)\,Li^{7}$ reaction. The low-energy cross section for this reaction is very high, and for this reason boron is widely used to absorb slow neutrons. The total cross section of B^{10} at low energies, which is almost entirely due to the (n,α) reaction, is shown in Fig. (II,10). It will be noted that σ_t is 1/v over a wide range of energy.

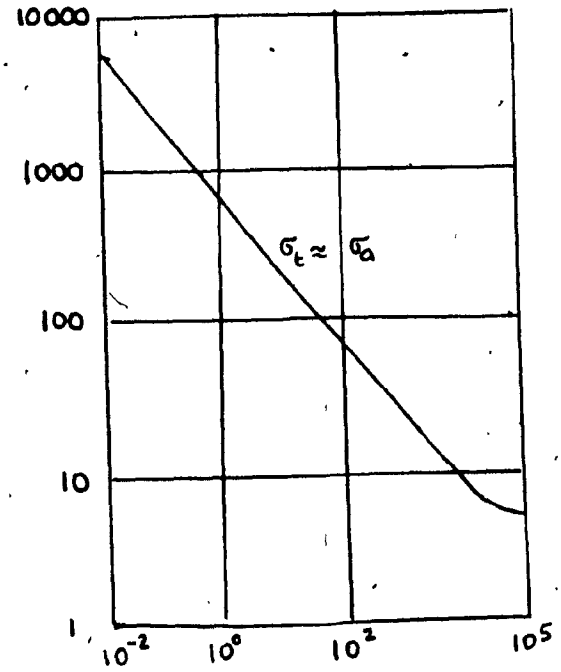


Figure II.10 The Total Cross Section of B^{10} from .01 to 10^5 eV

The reason for this is as follows. It can be shown (Ref. 2) that except in the vicinity of resonances the cross section of any reaction of the type (a,b) Y is given by the formula

$$\sigma_{a,b} = \left(\frac{E_b}{E_a}\right)^{\frac{1}{2}} H(E_a), \qquad \text{(II.87)}$$

where E_a and E_b are the kinetic energies of the incident and emergent

particles, respectively, and $H(E_a)$ is a slowly varying function of E_a. If the reaction is exothermic, E_b will usually exceed E_a by several MeV. Hence, at low energies of the incident neutron, i.e., eV, changes in E_a have almost no effect on E_b. The cross section therefore behaves essentially like $1/\sqrt{E_a}$, that is, like $1/v_a$.

A similar exothermic reaction which also shows a strong $1/v$ behavior is $Li^6(n,\alpha)H^3$. This reaction is used for the production of tritium, H^3.

Another important charged-particle reaction which occurs with low-energy neutrons is the $N^{14}(n,p)C^{14}$ reaction. Although the cross section is not particularly high (1.8 barns at 0.025eV), many neutrons emitted from nuclear explosions are absorbed in the atmosphere via this reaction and produce the long-lived and potentially dangerous β-emitter C^{14}.

Some endothermic charged-particle reactions are important in reactors even though their thresholds are high. In water reactors, for example, the $O^{16}(n,p)N^{16}$ reaction is the principal source of the radioactivity of the water (the N^{16} undergoes β-decay with a half-life of approximately 7 sec, which is accompanied by the emission of 6- to 7-MeV γ-rays), despite the fact that ordinarily only one neutron in several thousand has an energy greater than the 9-MeV threshold for this reaction.

With intermediate and heavy nuclei, the cross sections for charged-particle reactions are so small they cannot usually be measured. This is because the emitted charged particle must pass through a coulomb barrier in order to escape from the nucleus in much the same way that α-particles do when

emitted from radioactive nuclei. Except for light nuclei, this barrier is so high and the associated delay in charged-particle emission is so long that the compound system almost always decays by the emission of an elastic or inelastic neutron before a charged particle can be emitted.

II.7.3 NEUTRON PRODUCING REACTIONS

The principal source of neutrons in reactors is nuclear fission, and this reaction is discussed in detail in Section II.8. Neutrons are also produced in certain other reactions, however, though to a much smaller extent.

II.7.3.1 THE (n,2n) and (n,3n) REACTIONS

The (n,2n) reaction usually proceeds in two steps. First, the incident neutron is inelastically scattered by the target nucleus. Then, if the residual nucleus is left with an excitation energy above the binding energy of its least bound neutron, this neutron is free to escape from the system. According to the discussion on inelastic scattering, inelastically scattered neutrons usually leave most of their initial energy in the residual nucleus. This means that if the incident neutron has an energy above the threshold for the (n, 2n) reaction it is likely that a second neutron will appear. As a result, the (n,2n) cross section rises rapidly above its threshold at the expense of the inelastic cross section, since the bulk of the inelastic neutrons are now included as part of the (n,2n) reaction. This situation is illustrated in Fig.II.11, where the inelastic and (n,2n) cross sections are shown for U^{238}.

The Q-value of the (n,2n) reaction is equal, of course, to the binding energy of the "loosest" neutron in the target nucleus. The threshold energy in the laboratory system is then given by

$$E_t = \left(\frac{A+1}{A}\right) Q. \qquad \text{(II.88)}$$

Nuclei which contain a loosely bound neutron, thus have a low (n,2n) threshold. One of the most important examples of this kind is Be^9, whose (n,2n) threshold is only 1.8 MeV. Beryllium is often used in substantial quantities in reactors and when this is the case special attention must be given to this reaction. With most nuclei, however, the (n,2n) threshold is in the range from about 7 to 10 MeV.

The relationship of the (n,3n) reaction to the (n,2n) reaction is similar to that on the (n,2n) reaction to inelastic scattering. Thus a third neutron will be emitted provided the nucleus retains sufficient excitation energy after the emission of the second neutron in the (n,2n) reaction.

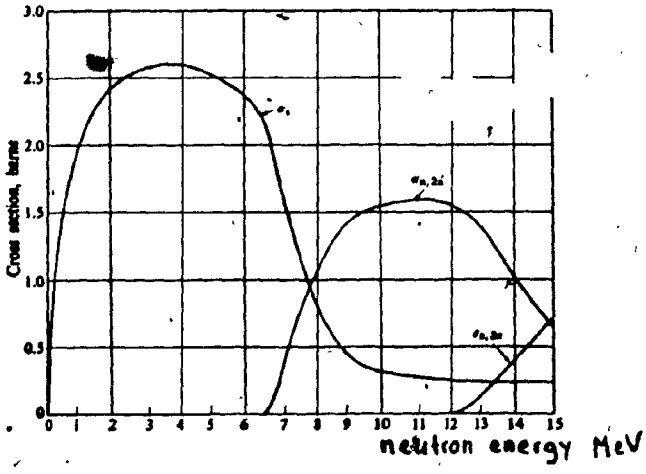


Fig. II.11 The inelastic, (n,2n) and (n,3n) cross sections of U^{238}. (Based on R.J. Howerton, "Semi-empirical Neutron Cross Sections 0.5-15 MeV," UCRL-5351, November 1958.)

The (n,3n) cross section therefore rises from the (n,3n) threshold at the expense of the (n,2n) cross section, as indicated in Fig. II.11. Ordinarily, however, the (n,3n) threshold is so high (it ranges from about 11 MeV to 30 MeV) that this reaction is not important in most reactor calculations.

In this connection, it may be mentioned that if a nucleus has a low (n,2n) threshold, it does not necessarily follow that its (n,3n) threshold will also be low. For instance, while the (n,2n) threshold of Be^9 is only 1.8 MeV, its (n,3n) threshold is 21 MeV. The origin of this disparity lies in the fact that although it may require only a small amount of energy to remove one neutron from a nucleus it may take considerably more energy to remove a second neutron.

II.7.3.2 THE (γ,n) REACTION

When a nuclear reactor is in operation, a great many energetic γ-rays are produced in its interior as the result of fission, from the decay of radioactive fission products, and from various neutron interactions, in particular, radiative capture and inelastic scattering. The more energetic of these γ-rays can produce neutrons by the (γ,n) reaction. This reaction is similar to the $(n,2n)$ reaction in that a neutron originally bound in the nucleus is ejected in the process. It is easy to see, in fact, that except for center-of-mass effects, the thresholds for the (γ,n) and $(n,2n)$ reactions are identical, and these reactions are therefore important for the same nuclei. However, unlike the $(n,2n)$ reaction, the (γ,n) reaction continues to produce neutrons even after a reactor is shut down, owing to the continuing decay of the fission products.

II.8 NUCLEAR FISSION

II.8.1 FISSION PHYSICS

The binding energy per nucleon in atomic nuclei reaches a maximum of 8.7 MeV for nuclei mass numbers of about 50 (Fig. II.12). Hence it is possible to produce more tightly bound nuclei and thereby release energy by either fusing together lighter nuclei (nuclear fusion) or inducing a heavy nucleus into fissioning into two nuclei of intermediate mass number (nuclear fission). The observed stability of heavy nuclei against spontaneous fission is due to the short-range nuclear forces within the nucleus giving rise to a potential energy barrier that must be overcome before the nucleus will fission. The size of this fission barrier is typically 6-9 MeV in most heavy nuclei of interest. Hence to induce nuclear fission, one must add a sufficient amount of energy to the heavy nucleus to overcome this fission barrier.

This can be done in a variety of ways. One could simply slam an energetic particle (with kinetic energy greater than the fission barrier) into the nucleus. An example of such a reaction would be photofission, in which a high-energy gamma strikes a heavy nucleus, thereby inducing fission. An alternative scheme would be to let the heavy nucleus capture a neutron. Then the binding energy of the added neutron itself might be sufficient to overcome the fission barrier and induce fission.

This latter process can in fact occur in certain heavy nuclei as ^{233}U, ^{235}U, ^{239}Pu, and ^{241}Pu. Such nuclides that can be induced to fission

with neutrons of essentially zero kinetic energy (or of more relevance to nuclear reactor applications, thermal neutrons having very small kinetic energies, at least compared to nuclear energies) are referred to as fissile nuclides. We will see later that such fissile nuclides represent the principal fuels used in fission chain-reacting systems.

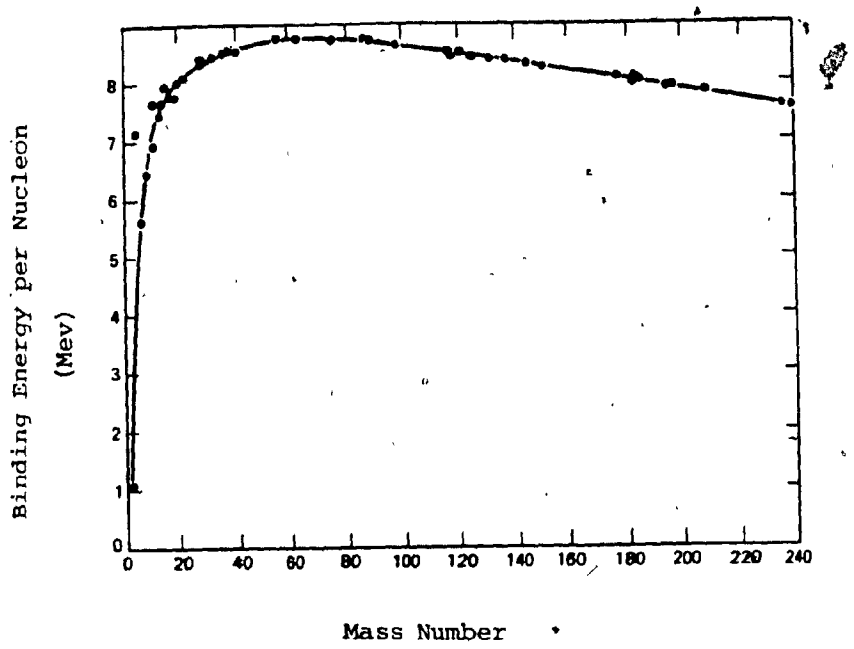


Fig. II.12 Binding Energy per Nucleon Versus Mass Number.

With most heavy nuclides, the additional binding energy provided by a captured neutron is not sufficient to push the heavy nucleus over the fission barrier. Frequently, however, one can add a dash of extra energy to the neutron, for instance by giving it a kinetic energy of an MeV or so, and this is sufficient to lift the nucleus the rest of the way over the barrier to cause fission. Nuclides that can be fissioned with such "fast" neutrons are referred to as fissionable. Examples are ^{232}Th, ^{238}U, and ^{240}Pu (as well as fissile nuclei such as

^{235}U). Although such fissionable nuclides do play an important role as nuclear fuels they are unable to sustain by themselves a stable fission chain reaction and hence must always be used in combination with a fissile nuclide such as ^{235}U or ^{239}Pu.

There is also a small possibility that certain heavy nuclei will fission spontaneously via the barrier penetration mechanism familiar from quantum mechanics. However the probability for such an event is quite low in most nuclides of interest as nuclear fuels. For example, the half-life for spontaneous fission in ^{238}U is some 6.5×10^{15} years. However even this very slow spontaneous fission rate can be of importance in nuclear systems, since even a few neutrons can be rapidly multiplied to appreciable numbers in a growing chain reaction.

II.8.2 FISSION CROSS SECTIONS

We noted earlier that nuclear fission is a process that proceeds via compound nucleus formation, such as does radiative capture. Hence it is not surprising that fission cross sections show considerable resonance structure. In Figs. II.13, II.14, II.15 we have shown the fission cross sections characterizing the principle fissile nuclides ^{233}U, ^{235}U, ^{239}Pu, taken from ENDF/B-IV.

The indicated cross section behavior is very similar to that of radiative capture cross sections. However this would be expected since we have seen that compound nucleus formation via neutron absorption is essentially independent of the mode of compound nucleus disintegration or decay, for example, via fission or gamma emission. It is particularly important to note that the fission cross section is over two orders of magnitude larger for

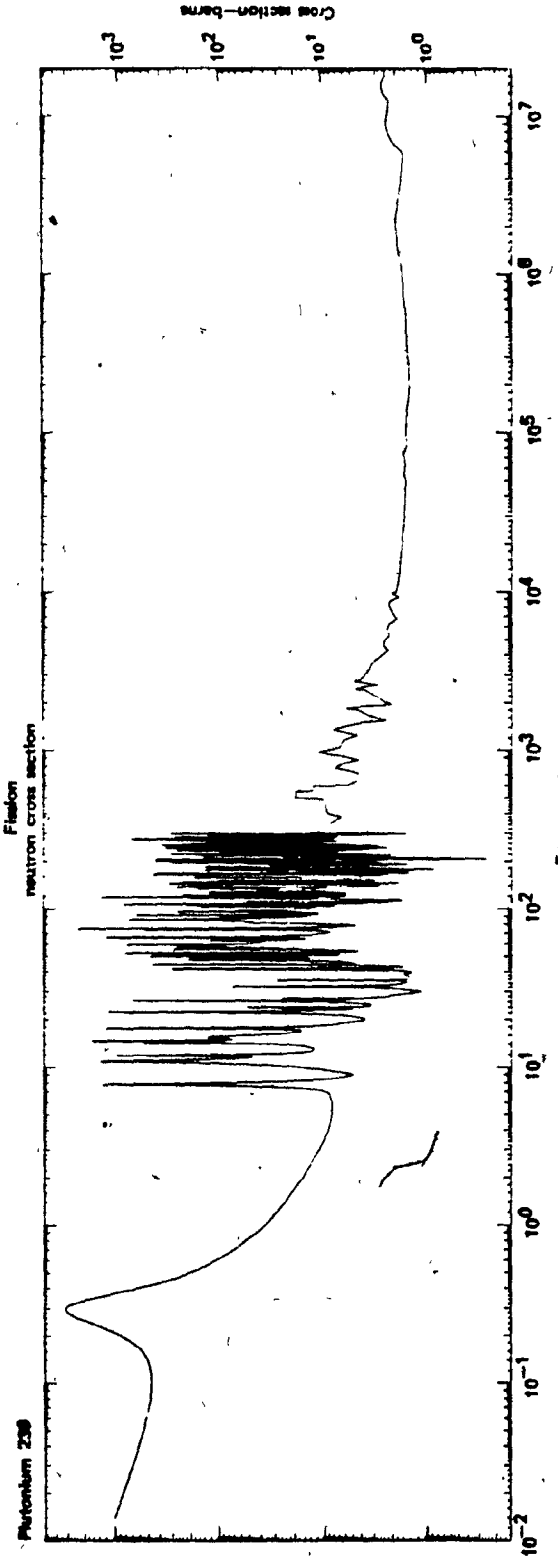


Fig. II.13 Fission Cross Section of Pu^{239}.

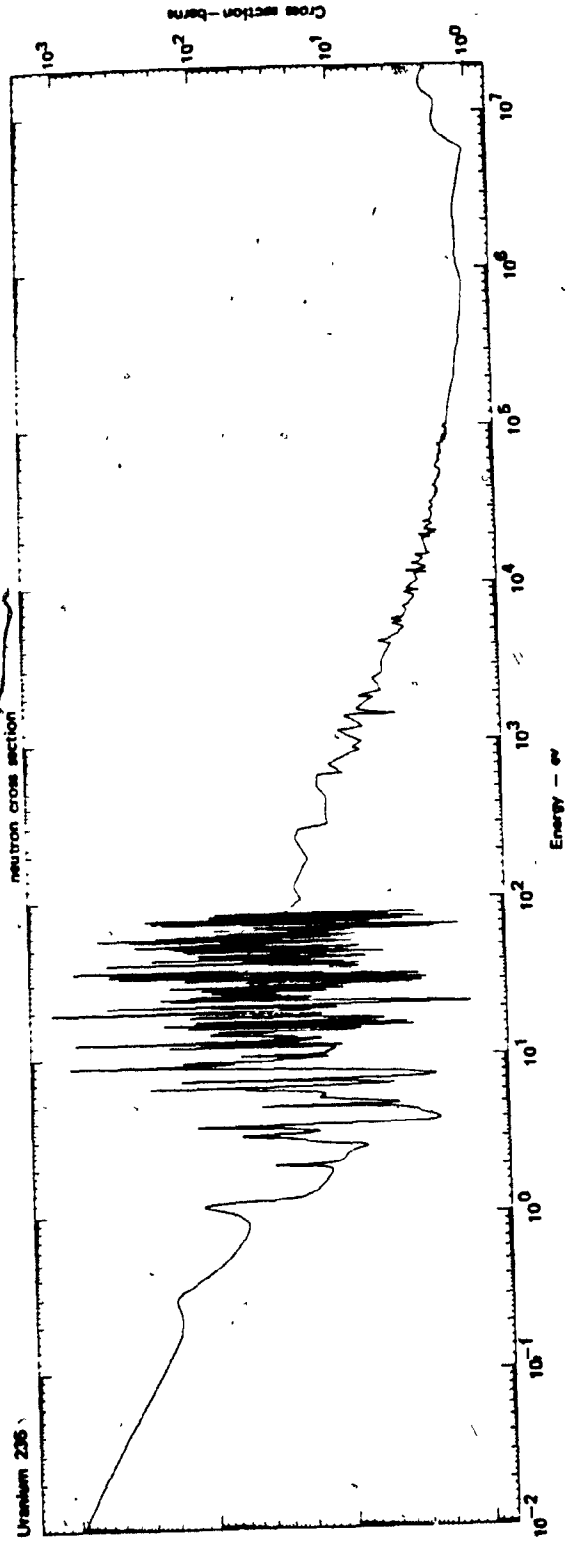


Fig. II.14 Fission Cross Section of U-235.

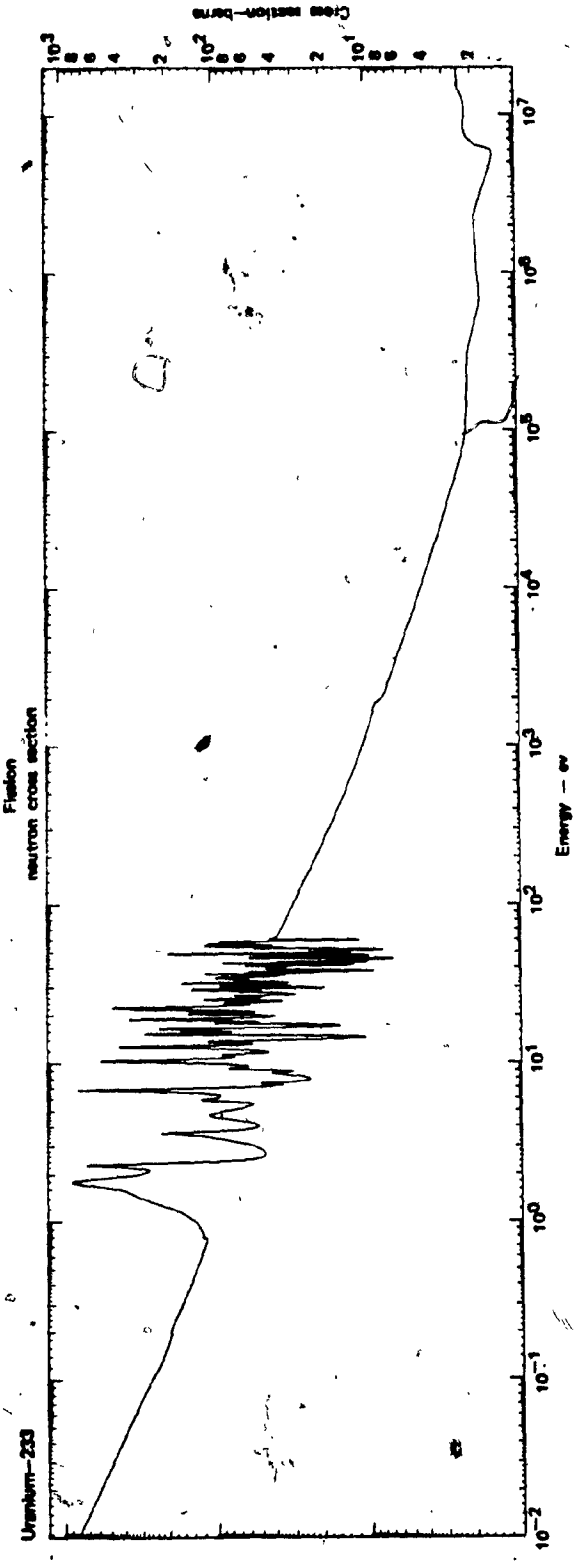


Fig. II.15 Fission Cross Section of U-233.

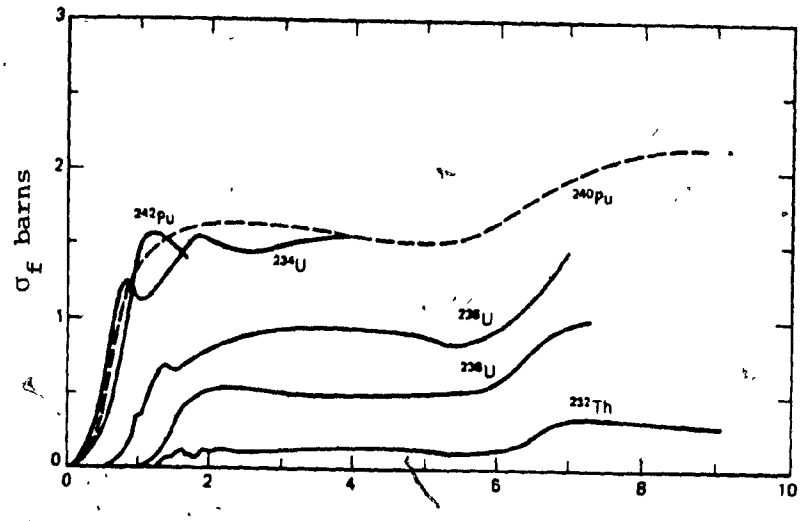


Fig. II.16 Fission Cross Sections of Principal Fissionable Isotopes.

low-energy or thermal neutrons than for high-energy fast neutrons (above 1 keV). The thermal neutron fission cross sections are indeed enormous for these fissile isotopes, ranging up to thousands of barns in magnitude. Such behavior will prove of very considerable importance in our later studies of nuclear reactors.

We have also indicated the fission cross sections characterizing the principal fissionable nuclides of interest, ^{232}Th, ^{238}U, and ^{240}Pu (see Figure II.16). This cross section behavior is somewhat different than that characterizing fissile nuclides since fissionable nuclides can only be fissioned by sufficiently high-energy neutrons. This implies that their fission cross sections will have a threshold energy, below which the cross section drops to zero. Even above this threshold energy (roughly 1 MeV), the fission cross sections are quite low, being less than two barns.

When a neutron is absorbed by a fissile isotope such as ^{235}U, it may induce that isotope to fission. Yet is is also possible that the compound nucleus formed by the neutron absorption, $^{236}U^*$, might simply decay to its ground state by gamma emission. The relative balance between the probability of fission and radiative capture is an extremely important factor in nuclear reactor applications. We characterize this balance by the capture-to-fission ratio, defined by

$$\alpha \equiv \frac{\sigma_\gamma}{\sigma_f} \qquad \text{(II.89)}$$

This ratio depends not only on the isotope of interest, but as well on the incident neutron energy E. It is plotted in Fig. II.17 for the three primary fissile nuclides. It can be seen that most neutron absroption in

such isotopes leads to fission events (with the exception of a small range of $\alpha > 1$ for ^{235}U). It should be noticed that α decreases quite appreciably above 0.1 MeV. This latter fact will prove to be of considerable importance when we discuss the concept of a fast breeder reactor.

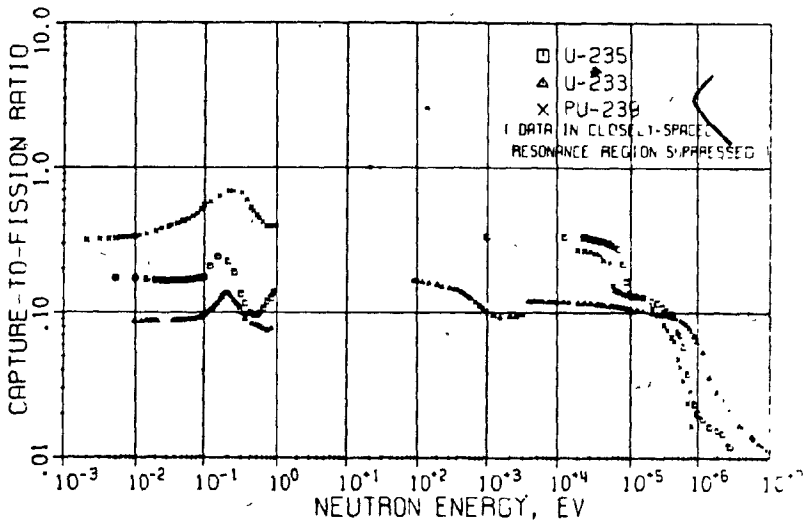


Fig. II.17 Variation of α with energy for U^{23}, U^{25}, Pu^{29}.

II.9 THE CONSEQUENCES OF FISSION: Particles Emitted

II.9.1 FISSION FRAGMENTS

With very rare exceptions a nucleus that fissions splits into two fragments.

Because of certain nuclear stability effects having to do with the number of particles in the nucleus, the two fragments are unlikely to be of equal

mass. In fact, curves of the probable distribution of masses of fission fragments (Fig. II.18) have a decided dip in them when the masses of the fragments are equal.

As can be seen only for the case of fission by high-energy neutrons is there a significant probability of an equal split.

The fission fragments, as they appear initially, are unstable with respect to the number of neutrons in their nuclei. Most commonly they become stable again through a series of β decays (emissions of electrons) which, in effect, transform the excess neutrons in the nucleus into protons. Thus, in the course of their radioactive decay to a stable end product, the fission fragments assume a sequence of chemical identities, each successive one having a nucleus containing an additional positive charge. Some of the members of these decay chains are nuclei that have a high probability for capturing neutrons. Such nuclei represent a "poison" in the system in that they compete for neutrons with the fissile material. Thus the details of the splitting curves shown in Figure II.8 are of some importance. It is unfortunate in this regard that Xe^{135}, which belongs to the mass-135 decay chain, is the isotope that has the highest probability for capturing low-energy neutrons. Examination of Figure II.18 shows that the mass-135 chain is one of the most likely products of a fission event.

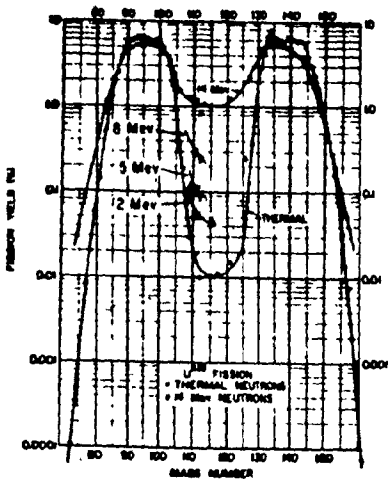


Fig. II.18 Mass-yield curve for fission of U^{235} by thermal, ~2-MeV (fission-spectrum), 5-MeV, 8-MeV, and 14-MeV neutrons.

II.9.2 NEUTRON EMISSION

Neutrons are also emitted in fission - on the average 2.5 of them, the particular number for any given fission event depending on the particular split that occurs. The average total number of neutrons emitted per fission is symbolized by ν. This quantity differs for the different fissionable materials and is slightly dependent on energy. Thus a more precise notation for the neutrons emitted per fission in, say U^{235} is $\nu^{235}(E')$, where E' is the kinetic energy of the neutron causing the fission.

The fraction of the ν neutrons emitted in fission that are emitted between the energies E and E + dE is symbolized by $X^j(E)$ being called the fission spectrum for isotope j. Strictly speaking $X^j(E)$ also depends on the energy of the neutron causing the fission. However this dependence is slight and can, if necessary, be accounted for by letting the $X^j(E)$ for different isotopes be almost indistinguishable. An empirical expression for the fission spectrum $X^j(E)$ that can be used in most

applications for all isotopes j, regardless of the energy of the neutron causing the fission is

$$X(E) = 0.453e^{-1.036E} \sinh\sqrt{(2.29E)} \qquad \text{(II.90)}$$

where E is in MeV. Since X(E) is a probability density, $\int_0^\infty X(E)dE = 1$. The curve is plotted as Fig. II.19. Note that the most probable energy of an emitted neutron is slightly below 1 MeV and that very few neutrons are emitted with an energy greater than 10 MeV.

The neutrons emitted in fission may be assumed to emerge with equal probability in all directions. In other words fission neutrons are produced isotropically. They preserve no "memory" of the direction of travel Ω of the neutron that caused the fission.

II.9.3 DELAYED NEUTRONS

Not all the ν neutrons emitted in an average fission event appear at once. A small fraction (0.65 percent for U^{235}) are emitted from certain of the fission fragments after a β decay makes these fragments unstable with respect to their neutron content. The β decay that precedes the emission of these delayed neutrons is a relatively slow process, the half-lives of the unstable fission fragments involved being between 0.2 sec and 54 sec. There are believed to be about twenty fission fragments that emit delayed neutrons. Such fission fragments are called delayed-neutron precursors. Each precursor has a unique half-life for β decay (after which the delayed neutron is emitted essentially instantaneously). The amount of precursor formed on the average as the result of a fission is, however, not

unique; it depends both on the isotope undergoing fission and on the energy of the neutron causing that fission. Thus the number C(t) of delayed-neutron procursors present at a time t after a single, average fission of fissionable isotope j is given by

$$C(t) = \sum_{i=1}^{20} \nu^j(E')\beta_i^j(E')e^{-\lambda ijt} \qquad \text{(II.91)}$$

where $\nu^j(E')$ is the total number of neutrons emitted when a neutron of energy E' causes fission in isotope j, $\beta_i^j(E')$ is the fraction of $\nu^j(E')$ that will be emitted from precursor i, and λ_i is the decay constant (λ_i = 0.693/half-life) of the fission fragment that constitutes the ith precursor

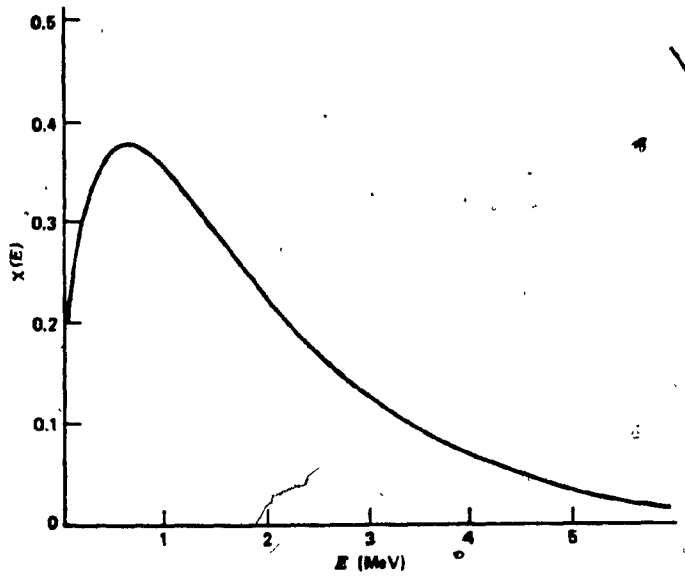


Fig. II.19 Fission Spectrum for Thermal Neutron Induced Fission in ^{235}U.

In practice it is found that an adequate representation of C(t) can be made if only six "effective" precursor "groups" are used (rather than the twenty physically real fragments). When this approximation is made, however, the six "equivalent" λ_j's are no longer physical quantities and

are, therefore, slightly dependent on j and E'.

Delayed neutrons (those coming from the precursors, as distinct from prompt neutrons which are emitted at essentially the instant the fission takes place) are extremely important in determining the time-dependent behavior of a reactor under non-steady-state conditions (when the number of neutrons created per second through fission differs from the number used up per second by absorption and leakage).

II.9.4 γ Rays, β Particles, and Neutrinos

In addition to producing fission fragments and neutrons, a fission event also results in the release of γ rays, β particles (often still referred to as "β rays"), and neutrinos.

The γ rays come from three sources: those released at the time of the fission (called prompt γ's), those released as the result of subsequent capture of some of the neutrons emitted in fission (capture γ's), and those following β decay to an excited state of a daughter nucleus.

It has already been pointed out that the β rays released as a result of the fission process come from the radioactive decay of the fission fragments and that neutrino release also accompanies this radioactive decay. In fact the neutrino, the existence of which was orginally postulated to account theoretically for the observed characteristics of β decay, was first observed experimentally with a nuclear reactor acting as the neutrino source.

II.10 THE CONSEQUENCES OF FISSION: Energy Released

The energy emitted as the result of a fission is transmitted to the local environment by the particles and photons released as a consequence of the fission. Thus, to see how this energy is converted into heat (the form desired for present power-generation methods), we shall follow the history of these emanations.

II.10.1 THE ENERGY FROM FISSION FRAGMENTS

The two fragments into which a fissioning nucleus splits quickly re-form into roughly spherical shapes, after which, since they are separated by a distance greater than the range of the nuclear force and are highly charged, they are pushed apart by a very great Coulomb repulsive force. Hence they pick up speed and go careening through the medium, repelling (by Coulomb interaction) other charged nuclei and thereby transmitting kinetic energy to them. These charged nuclei, in turn, push their neighbors. The net result is that the initial kinetic energy of the fission fragments ($\sim$ 168 MeV) is transmitted via Coulomb interaction in a cascade fashion to the nuclei in the vicinity (10^{-3} cm) of the site where the original fission took place. Thus the average kinetic energy of these neighboring nuclei (i.e., their temperature) increases in a time which, for purposes of reactor calculations, is instantaneous. About 82 percent of the energy released in fission is converted to a local increase in temperature in this manner.

II.10.2 DISPOSITION OF THE NEUTRON KINETIC ENERGY

The total kinetic energy of neutrons emitted in a fission is ~ 5 MeV. This energy, corresponding to a speed of $\sim 80 \times 10^6$ km/hr, is converted into heat as the neutrons collide with the nuclei in the reactor, thereby increasing the average kinetic energy of these nuclei. If they don't leak out of the reactor and are not absorbed, the neutrons thus slow down until their average energy approaches that of the atoms of the material making up the medium, ~ 0.025 eV, which corresponds to a neutron speed of ~ 8000 km/hr. Eventually (in 10^{-3} to 10^{-7} seconds, depending on the medium) the neutrons are absorbed.

II.10.3 β- AND γ-RAY ENERGY-DECAY HEAT

The prompt and delayed γ rays emitted at the time of fission and later on from the fission products each contribute about 7 MeV of energy. The amount of γ energy released as the result of neutron capture depends on the material in which the capture takes place but is usually in the range 3-12 MeV. Strictly speaking this capture-γ energy should not be counted as part of the energy released by fission since it really requires active contributions from other nuclear species. Nevertheless, since its release always accompanies the fission release and since it contributes to the overall reactor power level, we shall include it.

Since γ rays are not charged particles, they are not subject to Coulomb interaction. Instead they lose their energy by other kinds of electromagnetic interactions with charged particles. Specifically three mechanisms are involved: the photoelectric effect (most important for γ rays

of low energy, < 0.3 MeV), the Compton effect (most important at intermediate to high energies, 0.3-10 Mev), and pair production (most important at energies > 10 MeV). All these mechanisms result in an energy transfer to a charged particle, which then, because of Coulomb interactions, moves only a short distance before its energy is converted to heat by the cascade effect mentioned earlier. Thus, once a γ ray interacts with matter, the energy it transfers is deposited locally. However the γ ray itself may move a number of centimeters, or, occassionally, meters before losing all its energy. In this respect it is like the neutron, which also moves a number of centimeters, on the average, before being absorbed. Except in very small reactors neither γ-ray nor neutron leakage out of the reactor amounts to more than a few percent. However, even though most of the energy of these particles is absorbed within the reactor, the small leakage can be a severe biological hazard, and it is therefore necessary to provide a radiation shield for reactors.

The β rays released from radioactive decay of the fission fragments contribute an energy of about 8 MeV per fission, and, β rays being charged, this energy is immediately converted to heat locally.

Because radioactive decay can be a slow process, the energy of β and γ rays coming from fission fragments and from capture γ rays is transformed into heat relatively slowly. On the average the fragments from a single fission decay in time approximately as $t^{-1.2}$ ($1 \text{ sec} \leq t \leq 10^6$ sec). To be specific the average decay power (energy released per second) following a single fission is

$$P_d(t) = 2.66\, t^{-1.2} \text{ MeV/sec}, \quad t > 1 \text{ sec} \qquad \text{(II.92)}$$

where t is the time (in seconds) since the fission took place. If a reactor has been at a constant power corresponding to F fissions per second for a very long time and then at some time t_0 (prior to t) has been turned off, we must add the average decay power from all past fission to get the overall average decay power. To do this we note that the contribution to the total decay power at time t due to fissions which occurred in time dt' equals the number of fissions F dt' that occurred between t' and t' + dt' times $P_d(t + t')$. As a result the total decay power at t is

$$\int_{-\infty}^{t_0} (F\, dt')\, 2.66(t + t')^{-1.2} = \frac{2.66}{0.2} F(t + t_0)^{-0.2}, \quad (t + t_0) > 1 \text{ sec} \tag{II.93}$$

Thus the total decay power following sustained, constant power operation of a reactor falls off only as the 1/5th power of the time after shutdown. Only a few percent of the total reactor power is involved in this decay. However, for a reactor which operates at, say, 2400 megawatts, this is a substantial, long-lived decay power. Thus a power reactor must be both cooled and shielded after shutdown.

II.10.4 NEUTRINO ENERGY

About 12 MeV of the energy released in a fission is tied up in neutrinos. However, since these particles are not charged and have zero rest mass, the probability that they will interact with any of the atoms of material constituting the reactor is negligibly small. Hence the 12 MeV of neutrino energy out of the ≃ 200 MeV released by a fission is lost to the reactor. To some extent, however, this energy loss is made up by

the energy released by the capture γ's. Thus 200 MeV is a reasonably accurate approximation for the energy released in a reactor as the result of a fission.

II.10.5 THE OVERALL ENERGY BALANCE

We have not yet discussed the question of where the 200 MeV of energy released in fission "comes from". The answer is that it comes from the mass of the fissioned atom. Thus, if we measure the sum of the rest masses of all the products of a fission (fragments, neutrons and β's) and subtract that sum from the mass of the original fissionable atom, we get a mass difference which, when multiplied by the square of the velocity of light, gives (after a change of units) 200 MeV. Hence the immediate result of a fission is the conversion of mass energy into the kinetic energy of the fission fragments and prompt neutrons and the radiation energy of γ rays and neutrinos. The fission fragments subsequently decay, further converting mass energy into γ rays and neutrinos and into the kinetic energy of released β rays and delayed neutrons. The neutrons, by direct collisions with nuclei, and the γ rays, by electromagnetic interactions, transmit their energy to charged particles. Thus, except for the neutrino energy, all the energy released as the result of a fission is transmitted to charged particles. These, because of the long-range Coulomb interaction with the nuclei making up the material medium, transmit their energy to other charged nuclei in a cascade effect, thus raising the average kinetic energy, or temperature, of the material comprising the medium.

The amount of energy obtained from the fissioning of a small amount of

fissionable material is remarkable. A good figure to keep in mind in this regard is that a power level of one watt results from a fission rate of 3×10^{10} fissions per second. In an absolute sense this is a sizeable fission rate. But, if one remembers that there are 6×10^{23} atoms in 235 grams of U^{235}, it seems relatively small. In fact to sustain a power level of 4000 megawatts for a day requires the fissioning of only about 4 kilograms of fissionable material.

II.10.6 FISSION FUELS

Our previous discussion has indicated that there are a number of possibilities available for fueling a fission chain-reacting system. In particular, we have noted that the principal nuclides of concern in nuclear reactor applications are:

Fissile nuclides: ^{233}U, ^{235}U, ^{239}Pu, ^{241}Pu

while those susceptible to fast neutron fission are:

Fissionable nuclides: ^{232}Th, ^{238}U, ^{240}Pu, ^{242}Pu

Because of both the energy threshold that neutrons must exceed in order to induce fission in fissionable nuclides and the relatively large value of A characterizing such nuclides, only the first class of nuclides are capable of sustaining a fission chain reaction. Of the isotopes, only ^{235}U is found in nature - and then, only as 0.711% of natural uranium (which is composed primarily of ^{238}U).

Ways to obtain fissile isotopes, are illustrated in Figs. II.20, II.21. It is found that when certain nuclides absorb neutrons, they then undergo a sequence of radioactive disintegrations that eventually result in the formation of a fissile isotope. The two most important examples of such neutron transmutation reactions are:

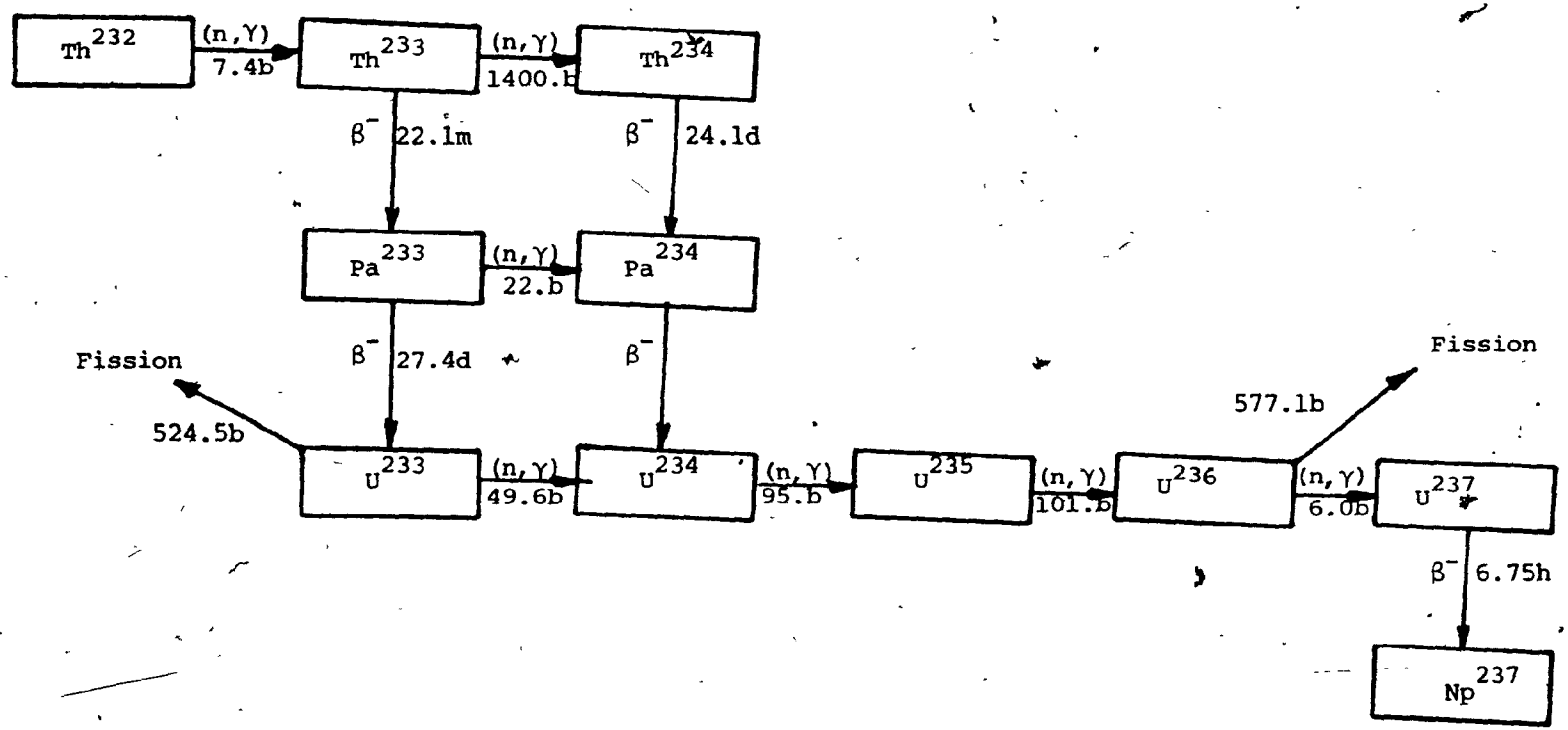
$$^{238}U(n,\gamma)\,^{239}U \xrightarrow[23\ \text{min}]{\beta-} {}^{239}Np \xrightarrow[2.3d]{\beta-} {}^{239}Pu \qquad (II.94)$$

$$^{232}Th(n,\gamma)\,^{233}Th \xrightarrow[22\ \text{min}]{\beta-} {}^{233}Pa \xrightarrow[27d]{\beta-} {}^{233}U$$

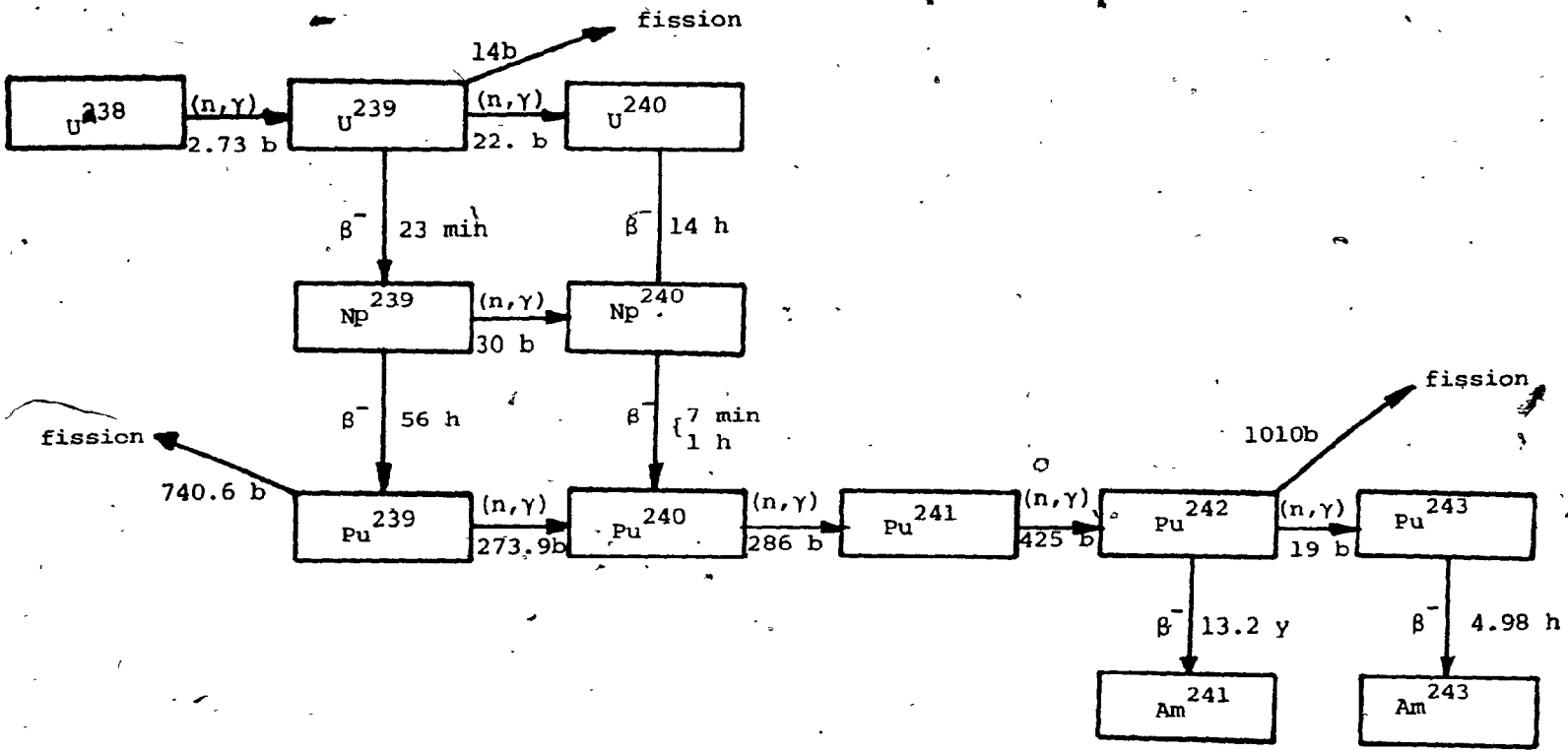
Isotopes that can be transmitted into fissile nuclides via neutron capture are referred to as fertile. The fertile isotopes of most interest are ^{238}U and ^{232}Th, which are in abundant supply throughout the world.

Yet where does one find the neutrons necessary for this process? In a nuclear reactor. Indeed since most present-day power reactors are fueled with natural or low-enrichment uranium that may contain as high as 98% ^{238}U, such transmutation processes will occur quite naturally as the fertile nuclei capture excess neutrons from the fission chain reaction. The key parameter in such processes is the number of neutrons produced in each fission reaction per neutron absorbed in the fuel nuclei. (Here we must remember that not all neutron absorptions in the fuel lead to fission - some result in radiative capture). We will define

$\eta \equiv$ average number of neutrons produced per neutron absorbed in fuel.



(Figure II.20) The chain of isotopes created by neutron irradiation of Th^{232}.



(Figure II.21) The chain of isotopes created by neutron irradiation of U^{238}

For a fuel composed of a single fissile isotope, we can write

$$\eta \equiv \nu\sigma_t/\sigma_a = \nu/(1 - \alpha) \qquad \text{(II.95)}$$

Most fuels, however, contain a mixture of isotopes. In this case, we would use the macroscopic fission and absorption cross sections characterizing each isotope to write

$$\eta = \frac{\sum_j \nu_j \Sigma_f^j}{\sum_j \Sigma_a^j} \qquad \text{(II.96)}$$

The dependence of this very important quantity on energy E is shown for the principal fissile isotopes in Fig. II.20. It should be noted that $\eta(E)$ is generally of the order of 2 for low-energy neutrons, but increases with energy above 0.1 MeV as the capture-to-fission ratio α falls off. If we are to attempt to utilize the neutrons "left over" from the chain reaction to convert fertile isotopes into fissile material, is is apparent that we require $\eta(E)$ to be at least greater than 1, since one neutron per fission is needed to sustain the chain reaction. Of course, a certain fraction of the fission neutrons will be absorbed in nonfuel materials, and others will leak out of the reactor and be lost to the chain reaction. Nevertheless it is apparent that $\eta(E)$ is sufficiently greater than unity to enable appreciable conversion using any of these isotopes.

Indeed it might even be possible to produce more fissile material than one depletes in maintaining the fission chain reaction. For this to occur, one would have to operate with fissile isotopes and neutron energies for which

$\eta(E)$ was greater than two, since one neutron would be needed to maintain the chain reaction, while one neutron would be used to produce a new fissile nucleus to replace the one destroyed in the fission reaction. Any excess over this (and over the number of neutrons lost to the chain reaction via nonproductive capture or leakage) could then be used to produce or breed new fissile material.

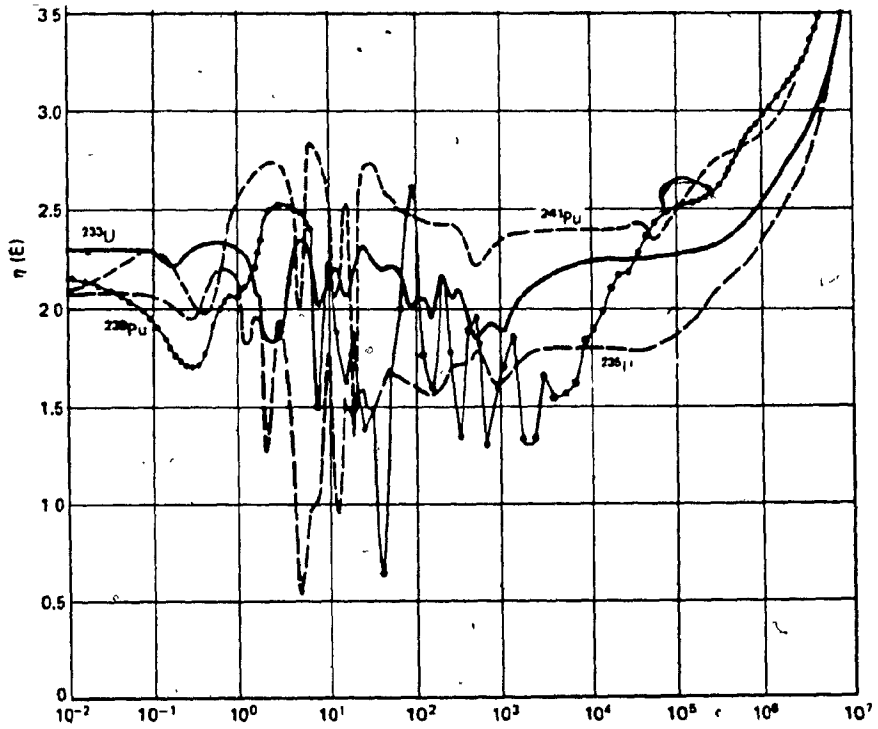


Fig. II.22 Variation of η with energy for ^{233}U, ^{235}U, ^{239}Pu and ^{241}Pu.

It is apparent from Fig. II.22 that the most favorable situation for accomplishing this would involve relatively fast neutrons in the 0.1 - 1 MeV range. The most suitable fuel would be ^{239}Pu. Such is the motivation behind the development of the fast breeder reactor which operates with a chain reaction in a $^{239}Pu/^{238}U$ fuel mixture maintained by fast neutrons in order to achieve this large value of η.

However if we recall the energy dependence of the fission cross section itself, it is apparent that it is more difficult to use fast neutrons to sustain the chain reaction, since the cross sections for fast fission are some two orders of magnitude smaller than those characterizing thermal neutrons. This suggests that it might be easier to achieve a sustained chain reaction using slow neutrons, since then the probability of fission is appreciably larger. Yet we must remember that the neutrons produced in the fission reaction are quite energetic with average energies in the MeV range. Hence in order to take advantage of the large fission cross sections for slow neutrons, one must slow down the fast neutrons to thermal energies (< 1 eV).

II.11 FISSION CYCLE: Criticality

In order to sustain a stable fission chain reaction and thereby achieve a constant production rate of fission energy, one must design a nuclear reactor in such a way that the rates of neutron absorption and leakage are balanced by the rate of fission neutron production.

It is evident from this that the principal elements in a nuclear chain reaction are the fuel and the chain carriers. The fuel of a nuclear reactor is fissionable material; the chain carriers are the neutrons. Common to all chain reactions is that the chain carriers react with fuel to liberate energy and produce new chain carriers. These new carriers react with more fuel and produce more energy and chain carriers, the process repeating itself from generation to generation.

The word "chain" thus refers to the fact that the carriers (i.e., the neutrons) are produced anew by the same energy-liberating process which they induce when they are absorbed in the fuel. Refer to Figure (II.21) for a pictorial fission chain reaction.

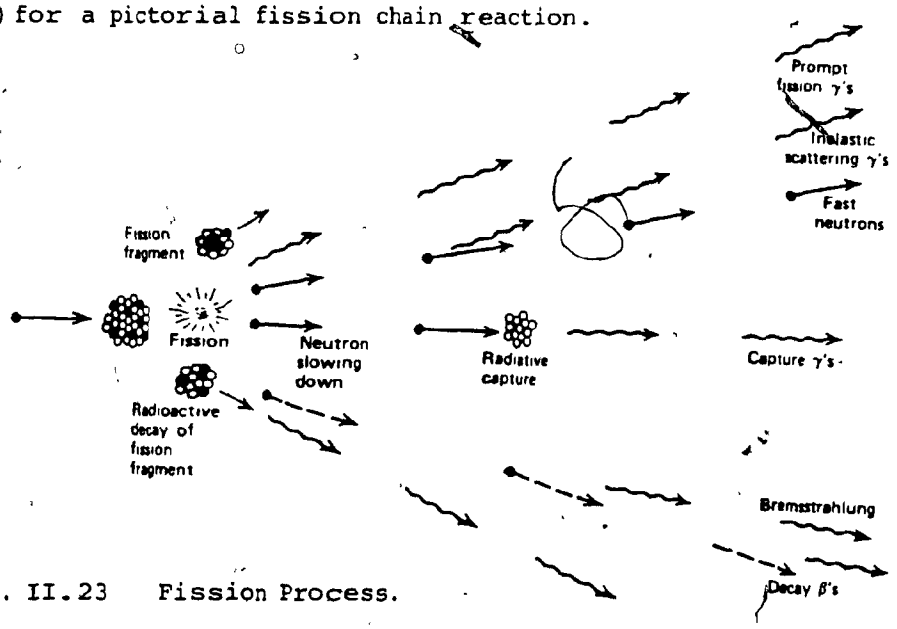


Fig. II.23 Fission Process.

We can express this process mathematically. Suppose that we could somehow measure the number of neutrons in the two successive fission generations, we would then define the ratio of these numbers as the multiplication factor k characterizing the chain reaction

$$k \equiv \text{Multiplication factor}$$

$$\equiv \frac{\text{Number of neutrons in one generation}}{\text{Number of neutrons in preceding generation}}$$

Notice that if $k = 1$, the number of neutrons in any of two consecutive fission generations will be the same, and hence the chain reaction will

be time independent. We refer to a system characterized by $k = 1$ as being critical.

By a similar argument we can conclude that if $k < 1$, the number of neutrons decreases from generation to generation, and hence the chain reaction dies out. We then refer to the system as being subcritical. Finally, if $k > 1$, then the chain reaction grows without bound as the number of neutrons in each successive generation is larger. Such a system is said to be supercritical.

In summary

$k < 1$	subcritical
$k = 1$	critical
$k > 1$	supercritical

The definition of the multiplication constant k in terms of successive fission neutron generations is sometimes known as the "life-cycle" point of view because of its similarity to biological population growth. This definition is a bit awkward, however, since it is usually rather difficult to determine the neutron generation time. For example, some neutrons may induce fission immediately after their birth in a fission reaction. Others may first slow down to thermal energies before inducing fission. Some neutrons may not induce fission reactions at all, but will instead be absorbed in nonproductive capture or leak out of the system.

A somewhat more practical definition of the multiplications factor k can

be given in terms of a neutron balance relation by defining ,

$$k \equiv \frac{\text{Rate of neutron production in reactor}}{\text{Rate of neutron loss (absorption plus capture in reaction)}} \equiv \frac{P(t)}{L(t)} \tag{II.97}$$

Here we have explicity noted that the production and loss rates may change with time (e.g., due to fuel consumption).

We will find the "neutron balance" definition of multiplication a somewhat more useful concept, since it is consistent with the approach that we will use to develop more elaborate models of nuclear reactor behavior in later chapters. In particular, we can then define the neutron lifetime, l, in an unambiguous fashion as

$$l \equiv \frac{N(t)}{L(t)} \tag{II.98}$$

where $N(t)$ is the total neutron population in the reactor at a time t. This latter approach is also particularly convenient for studying the time behavior of the neutron population in a reactor.

II.11.1 SIMPLE KINETICS OF CHAIN REACTIONS

Imagine that we could somehow count the number of neutrons $N(t)$ in a nuclear reactor at a time t. Then obviously the time rate of change of $N(t)$ is given by

$$\frac{dN}{dt} = \text{Production rate} - \text{Loss rate} = P(t) - L(t) \tag{II.99}$$

However if we use our definition of the multiplication factor k as given by the neutron balance relation, we can write

$$\frac{dN}{dt} = \left[\frac{P(t)}{L(t)} - 1 \right] L(t) = (k - 1)L(t) \qquad \text{(II.100)}$$

To proceed further we can use our definition of the neutron lifetime l to write

$$\frac{dN}{dt} = \frac{(k - 1)}{l} N(t) \qquad \text{(II.101)}$$

If we assume that both k and l are time-independent (of course they will not be in general), then we can solve this simple ordinary differential equation for the neutron population at any time t, assuming that there are initially N_0 neutrons in the reactor at time t = 0, to find

$$\boxed{N(t) = N_0 \exp\left(\frac{k - 1}{l} \right) t} \qquad \text{(II.102)}$$

In particular, note that this very simple model of nuclear kinetics agrees with our earlier definition of reactor criticality in terms of k (see Figure II.24). Yet this model also tells us that the growth or decay of the neutron population in a reactor obeys an exponential growth law. Such exponential growth is quite commonly found in the study of population dynamics. Indeed the study of the "neutron" population in a reactor core is mathematically rather similar to the study of biological populations, and hence the terminology of the latter field is frequently adopted in

reactor physics (e.g., generation, birth, life, virgin, daughter).

We will later find that the power level of a nuclear reactor is essentially proportional to its neutron population. Hence we can also regard the time behavior of the reactor power level as being exponential with a time constant or reactor period T given by

$$T = \frac{l}{k - 1} \qquad \text{(II.103)}$$

In particular is should be noted that as the multiplication factor k approaches unity, the reactor period T approaches infinity which corresponds to a time-independent neutron population or reactor power level.

However suppose that k is not equal to unity. Then how rapidly might we expect the power level of the reactor to change? Suppose, for the sake of illustration, we increased k to make the reactor ever so slightly supercritical by an amount of $k = 1.001$. Since the neutron lifetime in a typical power reactor is about 10^{-4} sec, we find this corresponds to a reactor period of $T = 0.1$ sec. Hence in one second the power level of the reactor will increase by a factor $e^{10} = 22{,}000$. Thus it appears that the reactor will respond very rapidly to changes in the multiplication factor. In fact, if a power reactor did indeed respond this rapidly, then it would be difficult to control the reactor power level, for a 0.1% change in the multiplication factor is rather common. Fortunately we have omitted something from this simple model which tends to greatly increase the neutron lifetime l and hence T, thereby slowing down the reactor time response. This is the effect of delayed neutrons on the chain reaction.

However this is a tale for another time, so we will leave our study of reactor kinetics with the promise of returning later to patch up this model in order to provide a more optimistic picture of nuclear reactor time behavior.

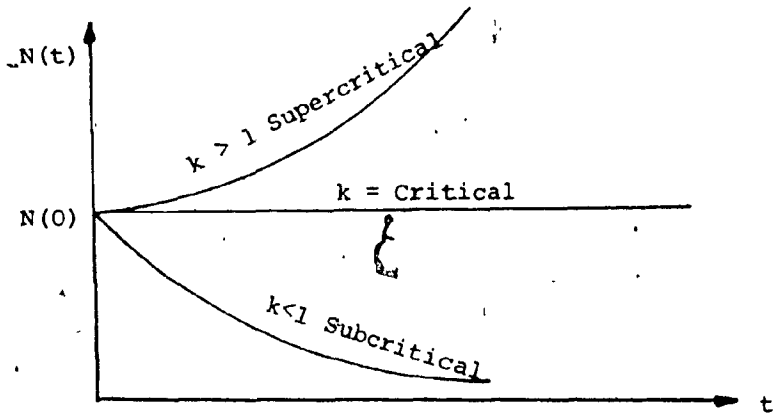


Fig. II.24 Time Behavior of the Number of Neutrons in a Reactor.

II.11.2 A FORMAL CALCULATION OF k: The Four-Factor Formula

Let us now turn our attention to the calculation of the multiplication factor for, say, a pile of uranium that one wishes to make into a nuclear reactor. We probably should add some coolant to remove fission heat and perhaps some structural material to hold the core together. However we will assume that we can treat these materials as intimately and homogeneously mixed so that the composition of the reactor is uniform.

Now to calculate k we must determine the possible fate of neutrons in a given fission generation. Fortunately this is rather easy to do since there are only two possible alternative destinies available to the neutron.

First it might leak out of the reactor and be lost to the chain reaction. If it does not leak out, then it must eventually be absorbed.* This absorption may correspond to a nonproductive capture event in either the fuel or other materials, or the absorption may induce a fission reaction, in which case a few fission neutron generation is produced. We can represent these destinies schematically, as shown in Figure II.25

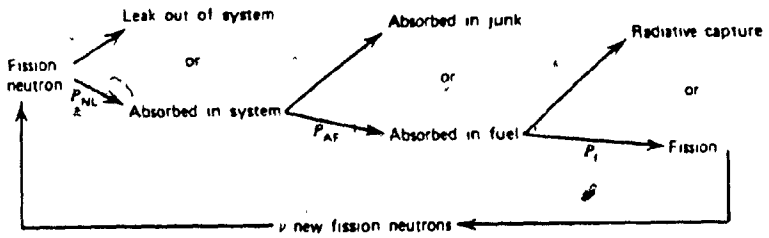


Fig. II.25. Fission Process Development.

To make this more formal, suppose we define the probabilities for each of these possible events as follows:

$P_{NL} \equiv$ Probability that neutron will not leak out of system before absorption

$P_{AF} \equiv$ Conditional probability that if neutron is absorbed, it will be absorbed in the fuel

$P_t \equiv$ Conditional probability that if neutron is absorbed in fuel, it will induce a fission reaction

* Of course, yet a thrid alternative would be a decay of the neutron into a proton, electron, and neutrino, but since the half-life for decay of a free neutron is 11.7 minutes, and the typical neutron lifetime I in the reactor is less than 10^{-3} sec, we can safely ignore this alternative.

These latter two conditional probabilities are easily calculated. The conditional probability for absorption in the fuel P_{AF} can be expressed simply as the ratio of the macroscopic absorption cross sections for the fuel Σ_a^F, and for the fuel plus the rest of the material in the core Σ_a. (We will usually indicate with a superscript the material to which we are referring. The absence of the superscript will imply that the macroscopic cross section is the total for all of the materials in the system.) Thus we can write

$$P_{AF} = \frac{\Sigma_a^F}{\Sigma_a} \tag{II.104}$$

It should be kept in mind that this expression has been introduced only for the situation in which the reactor has a uniform composition. Unfortunately for the reactor analyst all modern reactors have nonuniform compositions varying from point to point (e.g., due to fuel elements, coolant channels, support structure). In this more general case one can still use Eq.(II.104) if the macroscopic cross sections Σ_a are regarded as averages over the reactor. It should also be noted that we have not yet specified the neutron energy at which these cross sections are to be evaluated. Again, we will later find that the cross sections appearing in Eq. (II.104) must be appropriately averaged over energy, just as they are over space.

It is customary in reactor terminology to refer to this probability as the thermal utilization of the reactor and denote it by $P_{AF} \equiv f$. This term arose in the early analysis of thermal reactors in which essentially all fissions in the fuel were induced by thermal neutrons. In this

case the cross sections in f would be evaluated at thermal neutron energies and would represent the effectiveness of the fuel in competing with other materials in the reactor for the absorption of thermal neutrons, that is, the effectiveness with which the reactor utilized the thermal neutron in the fuel. The expression in Eq. II.104) actually applies to any type of reactor. However we will fall in line with convention and refer to it as the thermal utilization and denote it by "f".

The conditional probability for inducing a fission reaction in the fuel can also be expressed in terms of cross sections. In this case we simply take the ratio of the fission cross section to that of the absorption cross section (due to both fission and radiative capture) in the fuel material:

$$P_t = \frac{\Sigma_t^F}{\Sigma_a^F} = \frac{\sigma_t^F}{\sigma_a^F} \qquad \text{(II.105)}$$

We are now ready to utilize these probabilities to determine the multiplication factor k. The general scheme is to play a game of "follow the neutron". Suppose we start with N_1 neutrons present in the reactor in a given fission generation. Then with the help of the above probabilities and our diagram, we can compute the number of neutrons in the next generation as:

$$N_2 = \nu P_t P_{AF} P_{NL} N_1 \qquad \text{(II.106)}$$

$$N_2 = \eta f P_{NL} N_1 \qquad \text{(II.107)}$$

where we have recalled that $\eta = \nu(\sigma_t^F / \sigma_a^F)$ is the number of fission neutrons produced per absorption in the fuel. We can now use our definition of the multiplication factor k as being the ratio of the number of neutrons in two successive fission generations to write

$$k = \frac{N_2}{N_1} = \eta f P_{NL} \tag{II.108}$$

The nonleakage probability P_{NL} appearing in Eq. (II.108) and characterizing neutron leakage from the core is much more difficult to compute. It will require more elaborate mathematics (and in a realistic calculation, the use of a digital computer), and hence we will defer a discussion of it until later.

As a momentary detour, however, suppose our reactor were of infinite extent. Then since no neutrons could leak out, we immediately conclude that we must set the nonleakage probability $P_{NL} = 1$. The corresponding multiplication factor is then known as the infinite medium multiplication factor and denoted by

$$k_\infty = \eta f \tag{II.109}$$

Now of course no reactor is of infinite size. Nevertheless k_∞ is a useful parameter in reactor analysis since it essentially characterizes the multiplication properties of the material in the reactor as distinct from the geometry of the reactor core. Of course since $P_{NL} < 1$ more generally for a finite reactor from which some neutron leakage can occur, we must have $k_\infty > 1$ in order to have any chance of achieving a critical chain

reaction.

There are a couple of important modifications that must be introduced into this simple development in order to understand how the present generation of so-called "thermal" reactors works. We must account for the fact that the neutrons in a nuclear reactor have a distribution of energies. As we saw previously, the fission neutrons are born at very high energies in the MeV range. However the fission cross section is largest at very low energies - indeed, at those energies corresponding to neutrons in thermal equilibrium with the reactor core at a temperature T, e.g., for $T = 300^{\circ}C$, $E = kT = 0.05$ eV. Hence it is obviously to our advantage to try to slow down, or in the language of reactor physics, "moderate", the fast fission neutrons to take advantage of the fact that slow neutrons are more likely to induce fission reactions. This can be accomplished rather easily, simply by letting the fast neutrons collide with light nuclei, thereby losing some of their kinetic energy in elastic scattering collisions. The lighter the nucleus involved, the more kinetic energy per collision will be lost on the average by the neutron and hence the more effective the slowing down or moderation. It is well know that in CANDU reactors the moderator is D_2O (Heavy Water). Hence if we just let the fast neutrons rattle around in water for a bit, they will quickly slow down to the desired thermal energy. In this sense, we refer to water as a neutron moderator. Numerous other materials can be used as moderators in nuclear reactors, and we will discuss these in greater detail later.

The presence of such neutron moderation in a reactor suggests several modifications to our earlier calculation of the multiplication factor

k. Suppose we first modify our diagram of the various possible neutron destinies to take into account neutron energy as shown in Figure II.26 Now since most fissions will be induced by thermal neutrons, we will regard f and η as being evaluated at thermal neutron energies. For example, f would now refer to the ratio of thermal neutron absorptions in the fuel to total thermal neutron absorptions and thereby become more deserving of its designation as the "thermal" utilization. Similarly, η is now identified as the average number of fission neutrons produced per absorption of a thermal neutron in the fuel.

Then to account for processes that occur while a neutron is slowing down to thermal energies, we will introduce two new quantities. We first define a factor that takes account of the fact that, although most fissions will be induced in fissile material by thermal neutrons, some fissions will be induced in both fissile and fissionable material by fast neutrons. Hence we will scale up our earlier expression for k by a fast fission factor ε:

$$\varepsilon \equiv \frac{\text{Total number of fission neutrons (from both fast and thermal fission)}}{\text{Number of fission neutrons from thermal fissions}} \qquad \text{(II.110)}$$

The fast fission factor ε is usually quite close to unity in a thermal reactor with typical values ranging between $\varepsilon = 1.03$ and $\varepsilon = 1.15$.

The second factor we will introduce will characterize the possibility that the neutron might be absorbed while slowing down from fission to thermal energies. Since most absorptions occuring during the slowing down process correspond to resonance capture in heavy nuclei such as ^{238}U, we refer to

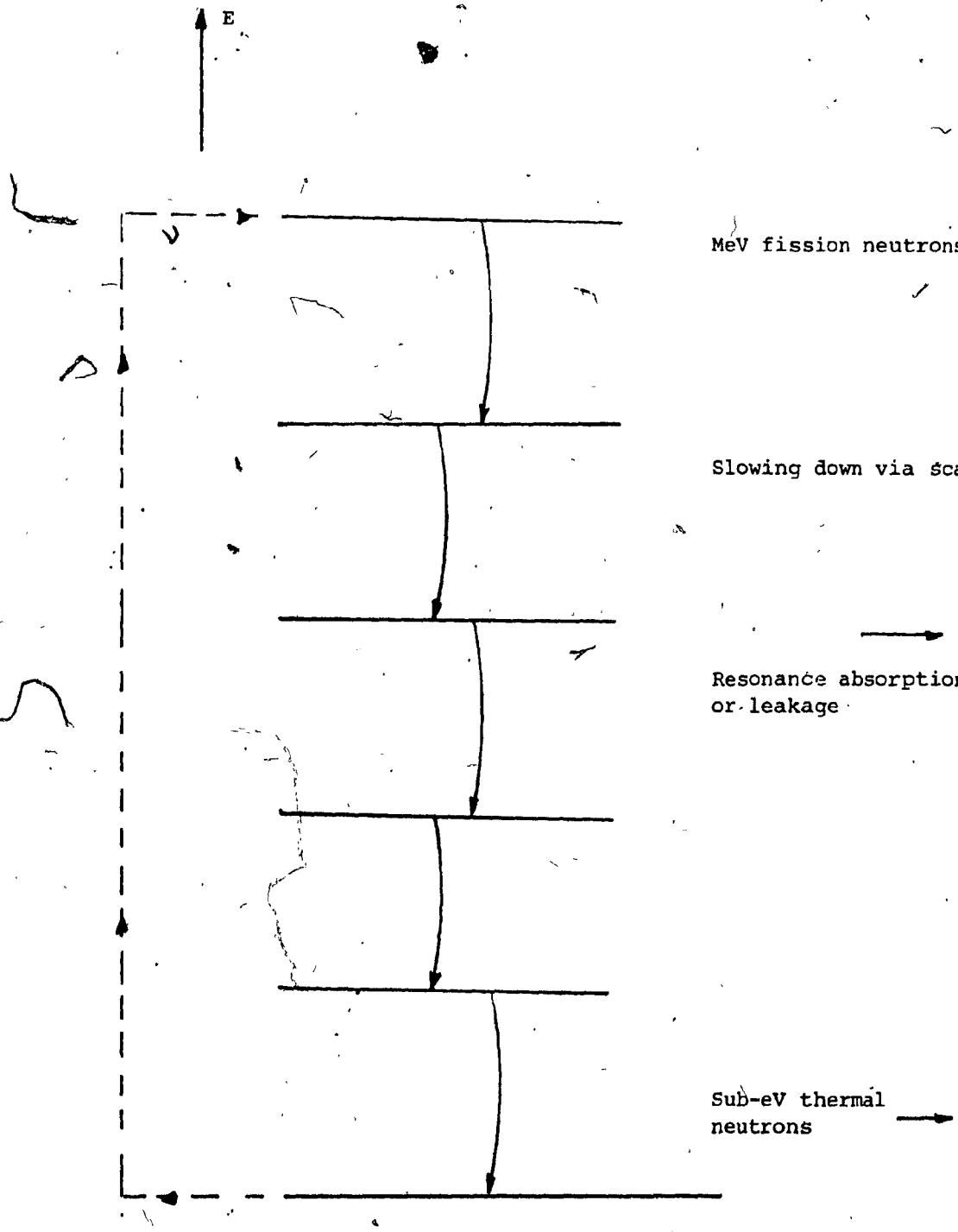


Fig. II.26 Process Characterizing a Neutron Generation in a Thermal Reactor.

this factor as the resonance escape probability p:

$$p \equiv \begin{array}{l}\text{Fraction of fission neutrons that manage to}\\ \text{slow down from fission to thermal energies}\\ \text{without being absorbed}\end{array} \qquad \text{(II.111)}$$

Finally it is useful to modify our definition of the nonleakage probability to take account of the fact that there will be two distinct phases of neutron leakage that will require two rather different types of analysis in our later work. First the neutron may leak out while slowing down. Indeed since the neutron mean free path is relatively large for high energies, such fast neutron leakage may be quite appreciable. A second leakage process may occur after the neutron has managed to slow down to thermal energies. After slowing down, the neutron may continue to scatter and eventually leak out before it has had an opportunity to be absorbed. To take account of these two processes, we will break up our earlier nonleakage probability as follows:

$$P_{NL} = P_{FNL}P_{TNL} \qquad \text{(II.112)}$$

where

$P_{FNL} \equiv$ Probability that fast neutron will not leak out (fast nonleakage)

$P_{TNL} \equiv$ Probability that thermal neutron will not leak out (thermal nonleakage)

If we now insert these new definitions into our earlier expressions (II.108) and (II.109), we find that the infinite medium multiplication factor becomes

$$k_\infty = \eta f p \varepsilon \qquad \text{(II.113)}$$

This is known as the four-factor formula. Moreover one now writes

$$k = \eta f p \varepsilon P_{FNL} P_{TNL} \qquad \text{(II.114)}$$

which is known, suprisingly enough, as the six-factor formula.

Hence provided we can calculate each of these factors, our critically condition $k = 1$ can then be easily checked (in the above example we fudged up the parameters a bit to yield a critical system). Of course, the calculation of these factors is quite difficult in general. Indeed one cannot really separate the various conditional probabilities as was done in these formulas. Instead alternative schemes based on iterative numerical methods must be used in practice to arrive at a critically condition.

Nevertheless the four-factor and six-factor formulas are quite useful because they provide insight into the various mechanisms involved in nuclear fission chain reactions and on rare occasions may actually be of use in making crude estimates in nuclear design. They are also useful in illustrating the trends of parameter variation in a core design.

For example, although η and ε are essentially fixed once the fuel has

been chosen, the thermal utilization f and resonance escape probability p can be varied considerably by changing the ratio of fuel density to moderator density. All of these parameters can be varied by using a heterogeneous lattice of fuel elements surrounded by moderator rather than a uniform, homogeneous mixture of fuel and moderator.

One can also vary the nonleakage probabilities by simply making the reactor core larger, or surrounding the reactor by a material with large scattering cross section so that some of the neutrons leaking out will be scattered back into the reactor. Actually when leakage is changed, there will be some change in the parameters in the four-factor formula as well since these are actually averages over the various neutron energies in the reactor, and this distribution of energies will vary with the amount of leakage. Such considerations have given rise to a somewhat different notation for the multiplication factor characterizing a finite system which is occasionally referred to as the effective multiplication factor and denoted by k_{eff}

$$k_{eff} = k_{\infty} P_{FNL} P_{TNL} \qquad \text{(II.115)}$$

There are other prescriptions for defining the multiplication factor. In particular we will introduce one of these schemes later when we consider the analytical treatment of the neutron energy dependence in more detail. However for now we will continue to regard the multiplication factor as the ratio between either the number of neutrons in two successive fission generations or the neutron production and loss rates in the reactor.

We can use the six-factor formula for the multiplication factor to gain a bit more insight into the goals of reactor design and operation. There are several ways to adjust k in the initial design of the reactor. One could first regard the size of the reactor as the design variable. Since the ratio of surface area to volume decreases as the reactor geometry is enlarged, one can control the relative importance of the leakage factors by adjusting the reactor size. For a given core composition (with k_∞ greater than 1, of course) there will be a certain size at which $k = 1$. An alternative way to achieve the same reduction in leakage is to surround the reactor with a scattering material that acts as a neutron reflector. Most thermal reactor cores are so large that leakage represents a rather small loss mechanism (typically about 3% of the neutrons leak out from the core in large thermal reactors).

Usually the core size and geometry for a power reactor are dictated by thermal considerations, for instance, the size of the core necessary to produce a given power output while being provided with sufficient cooling so that the temperature of the reactor materials will not become excessively high. The primary design variable at the disposal of the nuclear engineer is the core composition. In particular he can vary the composition (enrichment) and shape of the fuel, the ratio of fuel to moderator density, the type of moderator, coolant and structural materials used, or the manner in which reactor multiplication is controlled. One would refer to the amount of fuel required to achieve a critical chain reaction as the critical mass of fuel.

In reality, however, a nuclear reactor is always loaded with much more fuel than is required merely to achieve $k = 1$. For example the PHW

is typically loaded with sufficient fuel to achieve a multiplication of about $k = 1.2$. This extra multiplication is required for several reasons. First if the reactor is to operate at power for a period of time, one must provide enough excess fuel to compensate for those fuel nuclei destroyed in fission reactions during the power production. Since most contemporaty reactors are run roughly one year between refuelling a sizable amount of excess fuel is needed to compensate for fuel burnup. A second motivation arises from the fact that the multiplication of a reactor tends to decrease as the reactor power level and temperature increase from ambient levels to operating levels. Additional multiplication is needed to compensate for this effect. Finally one must include enough extra multiplication to allow for reactor power level changes. For example, we have seen that if we wish to increase the reactor power level, we must temporarily adjust k to a value slightly greater than 1 so that the reactor is supercritical. The reactor can then be returned to critical when the desired power level has been reached.

Of course when this excess multiplication is not being used, some mechanism has to be provided to cancel it out to achieve reactor criticality. This is the function of reactor control mechanisms. Such control is usually achieved by introducing into the reactor core materials characterized by large absorption cross sections. They will then tend to eat up the excess neutrons produced in the chain reaction. In terms of our six-factor formula such absorbing materials lower the value of the thermal utilisation f, since they compete with the fuel for neutron absorption. Sometimes the absorber may be dissolved in the reactor coolant. When such control absorbers are used to hold down the excess multiplication introduced to compensate for fuel burnup, one refers to them as shim control.

They may also be used to force the reactor subcritical in the case of emergency; then they are known as scram control. Finally they may just be used to regulate the power level of the reactor; then they are referred to as maneuvering control elements.

The case with which such control elements can control the fission chain reaction will depend on how rapidly the reactor responds to variations in multiplication. Since fuel burnup occurs over very long periods of time (typically weeks or months), a rapid response of shim control is not required - which is fortunate, bacause rather large amounts of multiplication must be manipulated (typically changes of 10 - 20% in k). The normal power variations in the reactor are due to much smaller changes in multiplication (< 0.1%) and are characterized by essentially the reactor period T which in turn is proportional to the neutron lifetime l. However we saw earlier that the lifetime of prompt fission neutrons was quite short, typically about 10^{-4} sec. The effective neutron lifetime is greatly increased by the presence of delayed neutrons, however. We recall that about 0.7% of the neutrons produced in fission are delayed anywhere from 0.6 to 80 sec since they arise from fission product radioactive decay. Hence the effective neutron lifetime is actually the average of the prompt neutron lifetime and the average decay time of these delayed neutrons, properly weighted, of course, by their relative yield fractions. When this is taken into account, one finds that the effective neutron lifetime is almost two orders of magnitude longer, $l_{eff} \approx 10^{-1}$ seconds. Hence a multiplication of 0.1% would now correspond to a reactor period of T = 10 seconds, well within the control capability of a reactor control system.

II.12 INTRODUCTORY CONCEPTS

II.12.1 NEUTRON DENSITY AND FLUX

Having introduced the concepts of cross sections, fission, multiplication factor we now turn our attention to determine the distribution of neutrons in a nuclear reactor core. This requires accounting for the neutron motion about the core and neutron interactions with nuclei in the core. We will begin by defining the neutron density $N(\vec{r},t)$ at any point $\vec{r}$ in the reactor core by (Fig. II.27)

$$\boxed{N(r,t)d^3r \equiv \text{expected number of neutrons in } d^3r \text{ about } \vec{r} \text{ at a time } t} \qquad \text{(II.116)}$$

For simplicity we now define $N(\vec{r},t) \equiv N(r,t)$; $\vec{r} \to r$

The word "expected" has been inserted into this definition to indicate that this will be a statistical theory in which only mean or average values are calculated. (The actual neutron density one would obtain from a series of measurements would fluctuate about this mean value, of course.) The neutron density $N(r,t)$ is of interest because it allows us to calculate the rate at which nuclear reactions are occurring at any point in the reactor. To understand this, let us suppose for convenience that all the neutrons in the reactor have the same speed v. Now recall that one can express the frequency with which a neutron will experience a given neutron-nuclear reaction in terms of the macroscopic cross section characterizing that reaction Σ and the neutron speed υ as

$$\upsilon\Sigma = \text{interaction frequency}$$

Hence we can define the reaction-rate density F(r,t) at any point in the system by merely multiplying the neutron density N(r,t) by the interaction frequency $v\Sigma$:

$$F(\vec{r},t)d^3r \equiv v\Sigma N(\vec{r},t)d^3r \equiv \begin{matrix}\text{expected rate at which}\\ \text{interaction are occurring}\\ \text{in } d^3r \text{ about } r \text{ at time } t.\end{matrix} \qquad \text{(II.117)}$$

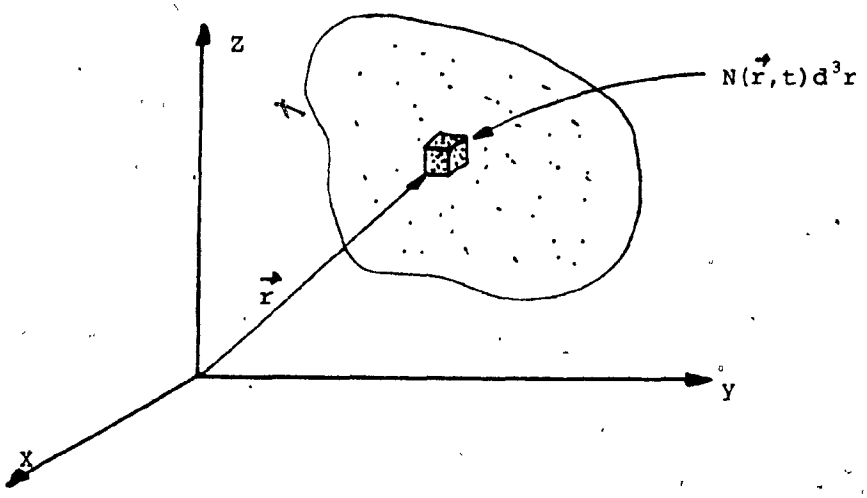


Fig. II.27 The Neutron Density N(r,t)

For example, if we consider a thermal neutron density of $N = 10^8$ cm^{-3} in a graphite medium, then using the total cross section tabulated in Appendix A of $\Sigma_t = 0.385$ cm^{-1} and a corresponding neutron speed of 2.2×10^5 cm/sec, we would find a reaction rate density of 8.47×10^{12} reactions/cm^3/sec. In this particular case, most of these reactions would consist of scattering collisions.

These concepts can easily be extended to the case in which the neutron density is different for various neutron energies E by defining

$$N(r,E,t)d^3rdE \equiv \begin{matrix}\text{expected number of neutrons in } d^3r \\ \text{about } r\text{, energies in } dE \text{ about } E\text{, at} \\ \text{time } t\end{matrix} \qquad \text{(II.118)}$$

Notice that this "density" is defined with respect to both space and energy. One can also generalize the concept of reaction rate density to include energy dependence as

$$F(r,E,t)d^3rdE = \upsilon\Sigma(E)N(r,E,t)d^3rdE \qquad \text{(II.119)}$$

The product $\upsilon N(r,t)$ arising in Eqs. (II.117 and (II.119) occurs very frequently in reactor theory, and therefore it is given a special name:

$$\boxed{\phi(r,t) \equiv \upsilon N(r,t) \equiv \text{neutron flux} \quad \text{cm}^{-2}.\text{sec}^{-1}} \qquad \text{(II.120)}$$

Although it will certainly prove convenient to work with $\phi(r,t)$ rather than $N(r,t)$ (since then one does not have to worry about including the neutron speed v in the reaction rate densities), the tradition in nuclear engineering of referring to this quantity as the neutron "flux" is very misleading. For $\phi(r,t)$ is not at all like the fluxes encountered in electromagnetic theory or heat conduction, since these latter fluxes are vector quantities, whereas $\phi(r,t)$ is a scalar quantity. Actually the "neutron current" $J(r,t)$, which we shall introduce momentarily, corresponds more closely to the conventional interpretation of a "flux". To avoid unnecessary confusion over this unfortunate convention, the student would probably do best at this point to think of the neutron flux as simply a convenient mathematical variable (speed x density) to use in computing reaction rates:

$$F(r,E,t) = \Sigma(E)\phi(r,E,t) \qquad \text{(II.121)}$$

A bit later we will introduce a physical interpretation of the neutron flux.

II.12.2 ANGULAR DENSITIES AND CURRENTS

The significance of the neutron density $N(r,t)$ or flux $\phi(r,t)$ in determining nuclear reaction rates leads us to search for an equation that describes these quantities. Unfortunately there is no exact equation that is satisfied by $N(r,t)$ or $\phi(r,t)$ - only approximate equations. To understand why, we must generalize the concept of the neutron density somewhat.

First let us determine just which variables characterize the state of an individual neutron. Certainly these include the neutron position r, energy E (or speed $v = (2E/m)^{1/2}$), and the time t at which the neutron is observed. Yet notice that to specify the state of the neutron we must also give its direction of motion characterized by the unit vector $\hat{\Omega} = v/|v|$. (Actually one could worry about specifying other variables such as the neutron spin; but for reactor calculations, the variables r, E, $\hat{\Omega}$, and t provide a sufficient description of the state of the neutron.)

Let us now generalize the concept of density by defining the angular neutron density that depends on all of these variables

$$n(r,E,\hat{\Omega},t)d^3rdEd\Omega \equiv \text{expected number of neutrons in } d^3r \text{ about } r, \text{ energy } dE \text{ about } E, \text{ moving in direction } \hat{\Omega} \text{ in solid angle } d\Omega \text{ at time } t \qquad \text{(II.122)}$$

The term "angular" arises from the fact that $n(r,E,\hat{\Omega},t)$ depends on the velocity spherical coordinate angles θ and ϕ specifying the neutron direction $\hat{\Omega}$ (Figure (II.28)). This is the most general neutron density function we need to define since it happens that one can derive an essentially exact equation, the neutron transport equation, for the angular neutron density $n(r,E,\hat{\Omega},t)$.

However before deriving this equation, it is useful to introduce several other definitions. We will first define the angular neutron flux in a manner similar to that in which we earlier defined the neutron flux, simply by multiplying the angular density by the neutron speed v:

$$\psi(r,E,\hat{\Omega},t) \equiv vn(r,E,\hat{\Omega},t) \qquad \text{(II.123)}$$

A related concept is the angular current density, defined by

$$\vec{j}(r,E,\hat{\Omega},t) \equiv v\hat{\Omega}n(r,E,\hat{\Omega},t) = \hat{\Omega}\psi(r,E,\hat{\Omega},t) \qquad \text{(II.124)}$$

Notice that since $\hat{\Omega}$ is a unit vector, the angular flux is actually nothing more than the magnitude of the angular current density

$$|\vec{j}| = |\hat{\Omega}|\psi = \psi \qquad \text{(II.125)}$$

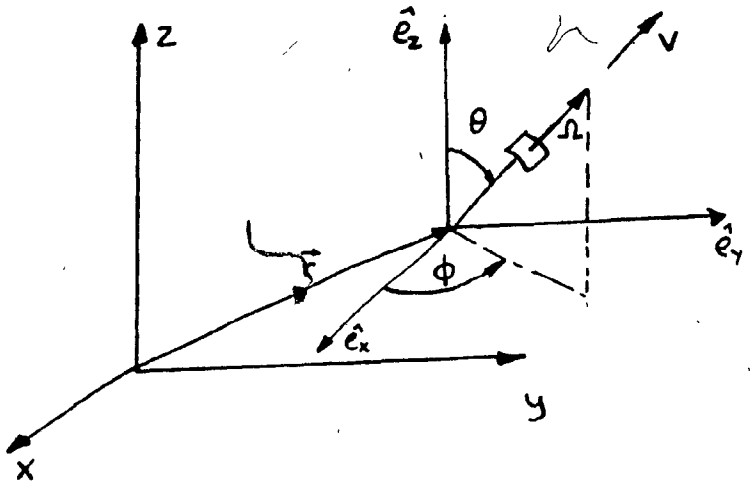


Fig. II.28 The neutron position and direction variables

The angular current density has a useful physical interpretation. Consider a small area dA at a point r. Here we will use the convention that $dA = \hat{e}_s dA$ where $\hat{e}_s$ is the unit vector normal to the surface. In particular note that if we consider a small area dA at a point r, then

$$j(r,E,\hat{\Omega},t).dAdEd\Omega \equiv \text{expected number of neutrons passing through an area dA per unit time with energy E in dE, direction } \hat{\Omega} \text{ in } d\Omega \text{ at time t} \qquad \text{(II.126)}$$

We can also define an angular interaction rate

$$f(r,E,\hat{\Omega},t) = v\Sigma(r,E)n(r,E,\hat{\Omega},t) = \Sigma(r,E)\psi(r,E,\hat{\Omega},t) \qquad \text{(II.127)}$$

All of these angle-dependent quantities can be related to our earlier definitions in Section (II.12.1) by simply integrating over the angular variables. For example:

$$N(r,E,t) = \int_{4\pi} d\Omega n(r,E,\hat{\Omega},t) \qquad \text{(II.128)}$$

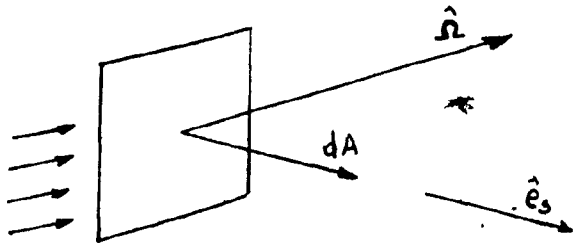


Fig. II.29 Neutrons incident on differential element of area dA

or $$N(r,t) = \int_0^\infty dE N(r,E,t) = \int_0^\infty dE \int_{4\pi} d\Omega n(r,E,\hat{\Omega},t) \qquad \text{(II.129)}$$

Sometimes quantities such as $N(r,t)$ and $\phi(r,t)$ which do not depend on $\hat{\Omega}$ are referred to as scalar or total densities and fluxes, to distinguish them from $n(r,E,\hat{\Omega},t)$ and $\psi(r,E,\hat{\Omega},t)$. We find this nomenclature cumbersome and will avoid it in our development.

Notice that if the angular density is independent of $\hat{\Omega}$ (i.e., it is isotropic) then we find that Eq. (II.128) demands the presence of a 4π normalization factor in the angular density

$$\boxed{n(r,E,\hat{\Omega},t) = \frac{1}{4\pi} N(r,E,t)} \qquad \text{(II.130)}$$

More generally, however $n(r,E,\hat{\Omega},t)$ will have a directional dependence - particularly if we are near a boundary or a source of neutrons, as a little geometrical reasoning applied to Figure II.30 should indicate.

In a similar fashion, we find

$$\phi(r,E,t) = \int_{4\pi} d\Omega \psi(r,E,\hat{\Omega},t) \qquad \text{(II.131)}$$

and

$$\boxed{\phi(r,t) = \int_0^\infty dE \phi(r,E,t) = \int_0^\infty dE \int_{4\pi} d\Omega \psi(r,E,\hat{\Omega},t)} \qquad \text{(II.132)}$$

Finally, we can define the neutron current density $J(r,E,t)$ in terms of the angular current density $j(r,E,\hat{\Omega},t)$ as

$$J(r,E,t) \equiv \int_{4\pi} d\Omega j(r,E,\hat{\Omega},t) \qquad \text{(II.133)}$$

and

$$\boxed{J(r,t) \equiv \int_0^\infty dE J(r,E,t) = \int_0^\infty dE \int_{4\pi} d\Omega j(r,E,\hat{\Omega},t)} \qquad \text{(II.134)}$$

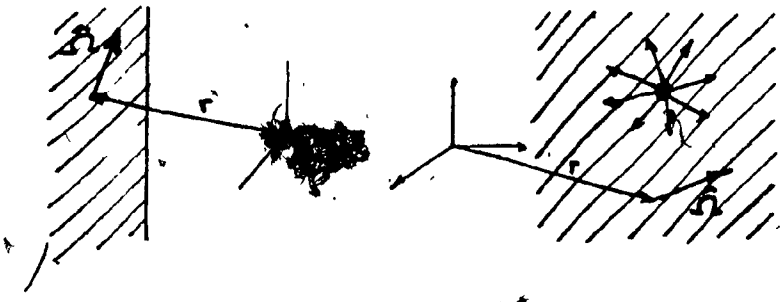


Fig. II.30 Anisotropies in the angular density

Notice that $J(r,t)$ is actually what would be referred to as the "flux" in other fields of physics, since if we have a small area dA at a position r, then

$$J(r,t)\cdot dA = \begin{array}{l}\text{net rate at which neutrons pass}\\ \text{through a surface area } dA\end{array} \qquad \text{(II.135)}$$

The units of both $J(r,t)$ and $\phi(r,t)$ are identical $\left(cm^{-2}.sec^{-1}\right)$. However J is a vector quantity that characterizes the net rate at which neutrons pass through a surface oriented in a given direction. Whereas ϕ simply characterizes the total rate at which neutrons pass through a unit area, regardless of orientation. Such an interpretation would suggest that J is a more convenient quantity for describing neutron leakage or flow (e.g., through the surface of the reactor core), while ϕ is more suitable for characterizing neutron reaction rates in which the total number of neutron interactions in a sample (e.g., a small foil) is of interest. Although the angular flux and current density are very simply related, we will find that there is no simple analogous relationship between J and ϕ. These concepts may appear a bit confusing at first, but they will become more familiar after we have illustrated their application in both our further theoretical development.

A closely related concept is that of the partial current densities, $J_{\pm}(r,t)$ which correspond to the total rates at which neutrons flow through a unit area from left to right (J_+) or right to left (J_-). If we recall our earlier definition of $j(r,E,\hat{\Omega},t)$ then it becomes apparent that

$$J_{\pm}(r,t) = \int_0^{\infty} dE \int_{2\pi\pm} d\Omega \hat{e}_s \cdot j(r,E,\hat{\Omega},t) \qquad \text{II.136}$$

where $2\pi^{\pm}$ is merely a convenient notation to indicate that the angular integration is performed only over directions with components along the surface normal ($2\pi^{+}$) or in the opposite direction ($2\pi^{-}$). For example, if we choose to define the polar coordinates that specify $\hat{\Omega}$ along the normal to the surface, then in the integration for J_{+}, ϕ would range from 0 to 2π, while θ would range only from 0 to $\pi/2$.

It is evident from this definition that

$$\hat{e}_s \cdot J(r,t) = \left[J_+(r,t) - J_-(r,t) \right] \qquad \text{(II.137)}$$

Hence J is sometimes referred to as the net current density, since it can be constructed as the sum of the partial current densities.

CHAPTER III

III NEUTRON TRANSPORT

III.I INTRODUCTION

The central problem of nuclear reactor theory, as pointed out previously, is the determination of the distribution of neutrons in the reactor. For it is the neutron distribution that determines the rate at which various nuclear reactions occur within the reactor. Furthermore by studying the behavior of the neutron population we will be able to infer the stability of the fission chain reaction. To determine the distribution of neutrons in the reactor we must investigate the process of neutron transport, that is, the motion of the neutrons as they stream about the reactor core, frequently scattering off of atomic nuclei and eventually either being absorbed or leaking out of the reactor. Most reactor studies treat the neutron motion as a diffusion process. In effect one assumes that neutrons tend to diffuse from regions of high neutron density to low neutron density, much as heat diffuses from regions of high to low temperature, or even more analogously, as one gas of molecules (corresponding to the neutrons) would diffuse through another (the nuclei) to reduce spatial variations in concentration.

Unfortunately, however, while the treatment of thermal conduction and gaseous diffusion as diffusion processes in usually found to be quite accurate, the treatment of neutron transport as a diffusion process has only limited validity. The reason for this failure is easily understood when it is noted that in most diffusion processes the diffusing particles are characterized by very frequent

collisions that give rise to very irregular, almost random, zigzag trajectories. However, we have seen that the cross section for neutron-nuclear collisions is quite small (about 10^{-24} cm^2). Hence neutrons tend to stream relatively large distances between interactions (recall that the mean free path characterizing fast neutrons is typically on the order of centimeters). Furthermore, the dimensions characterizing changes in reactor core composition are usually comparable to a neutron mfp (e.g., a reactor fuel pin is typically about 1 cm in diameter).

Hence we require a more accurate description of neutron transport that takes into account the relatively long neutron mfp and neutron streaming. Such a description has been borrowed from the kinetic theory of rarefied gases (which are also characterized by long mfp) - more precisely, the kinetic theory of gas mixtures. The fundamental equation describing dilute gases was first proposed more than one century ago by Boltzmann, and even today the Boltzmann equation remains the principal tool of the gas dynamicist. Its counterpart for the neutron "gas", the so-called neutron transport equation, is far younger (less than 40 years old), far simpler (e.g., it is a linear equation in contrast to the Boltzmann equation, which is nonlinear), but usually strikes far more terror in the hearts nuclear reactor physicists who are intimidated by its frightening reputation within the nuclear reactor community. Neutron transport theory has come to be associated with a hideous plethora of impenetrable mathematics, unwieldy formulas, and (eventually) the expenditure of enormous amounts of money on computer number-crunching.

The job of the reactor analyst is to develop suitable (i,e., calculationally feasible and accurate) approximations to it. Usually, however, only by comparing these various approximate theories to the transport equation from

which they originated can one really assess their range of validity.

There is another reason for including an introduction to the neutron transport equation in even an elementary discussion of nuclear reactor analysis. Although neutron diffusion theory is usually found adequate for reactor applications it owes its accuracy to various schemes that have been developed to "patch it up" using results from more accurate transport equation solutions. For example, we will find that the neutron diffusion equation is quite invalid near the boundary of a reactor, or near a highly absorbing material such as a fuel rod or a control element. Nevertheless we can continue to use diffusion theory to describe the reactor provided we fudge it a bit by inserting so-called "transport corrections" into the boundary conditions accompanying the diffusion equation.

III.2 THE NEUTRON TRANSPORT EQUATION

III.2.1 THE GENERAL BOLTZMANN EQUATION: TIME DEPENDENT

The exact equation for the angular neutron density in a system is derived by simply balancing the various mechanisms by which neutrons can be gained or lost from an arbitrary volume V within the system. That is, we will consider mechanisms that will change the number of neutrons in this volume that are characterized by a specific energy E and are traveling in a specific direction $\hat{\Omega}$. It is convenient to use a bit of vector calculus here, but hopefully this will not obscure the simple physics behind this equation (which is just the mathematical expression of a "count-the-neutrons" game).

To this end, consider any old arbrtrary volume V. The number of neutrons in V with energy E in dE and traveling in a direction $\hat{\Omega}$ in $d\Omega$ within this

volume is just

$$\left[\int_V n(r,E,\hat{\Omega},t)d^3r\right] dEd\Omega \qquad \text{(III.1)}$$

(Since $n(r,E,\hat{\Omega},t)$, is a "density" in E and $\hat{\Omega}$ space, we must multiply it by dE and $d\Omega$ in order to get a number.) The time rate of change of this number, then, is given by a balance relation

$$\frac{\partial}{\partial t}\left[\int_V n(r,E,\hat{\Omega},t)\ d^3r\right] dEd\Omega = \text{gain in V - loss from V.} \qquad \text{(III.2)}$$

If we assume that the arbitrary volume V is chosen not to depend on time, we can bring the time differentiation inside the spatial integration

$$\frac{\partial}{\partial t}\left[\int_V n,E,\hat{\Omega},t)\ d^3r\right] dEd\Omega = \left[\int_V \frac{\partial n}{\partial t}\ d^3r\right] dEd\Omega \qquad \text{(III.3)}$$

We will now classify the various ways that neutrons can appear or disappear from V, and then we will try to write mathematical expressions for each of these mechanisms in terms of the angular density $n(r,E,\hat{\Omega},t)$.

III.2.2 GAIN AND LOSS MECHANISMS

Gain mechanisms:

① Any neutron sources in V (e.g., fissions).

② Neutrons streaming into V through the surface S.

③ Neutrons of different E', $\hat{\Omega}'$ suffering a scattering collision in V that changes E', $\hat{\Omega}'$ into $E,\hat{\Omega}$ of interest.

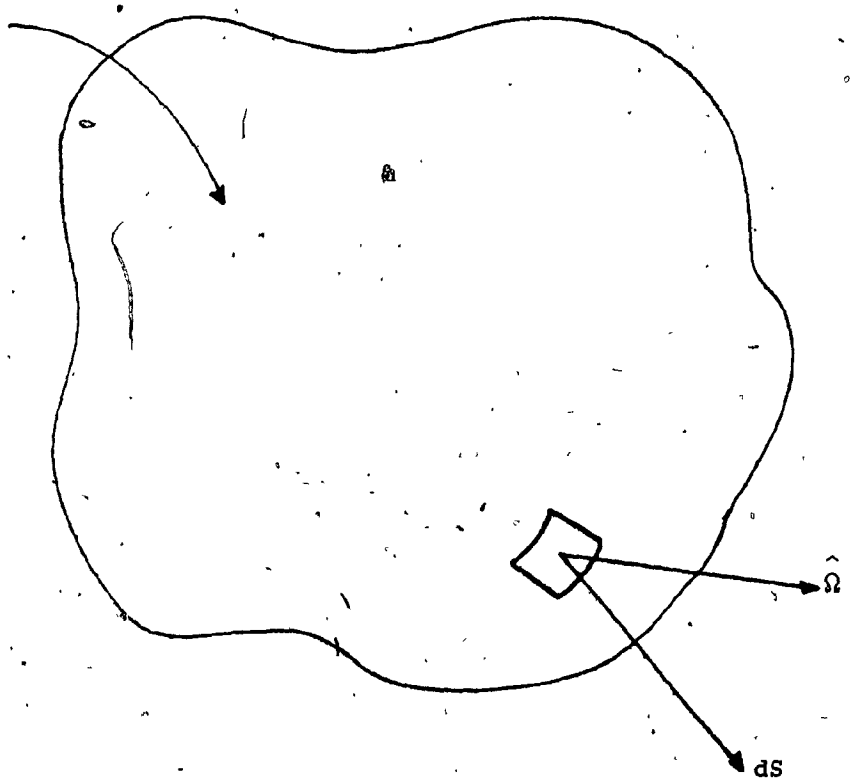


Figure III-1 An arbitrary Volume V with Surface Area S.

Loss mechanisms:

④ Neutrons leaking out through the surface S.

⑤ Neutrons in V suffering a collision. (It is obvious that an absorption interaction removes a neutron from V; and since by definition a scattering collision changes $E,\hat{\Omega}$ and since we are only keeping track of neutrons in V with this specific energy and direction, a scattering collision also amounts to a loss of neutrons). (See figure III.1 for physical interpretation.)

III.2.3 MATHEMATICAL EXPRESSIONS FOR GAIN AND LOSS MECHANISMS

We can now write a mathematical expression for each of these contributions.

① Source terms: If we define

$$s(r,E,\hat{\Omega},t)\ d^3r dE d\Omega \equiv \text{rate of source neutrons appearing in } d^3r \text{ about } r,\ dE \text{ about } E, \text{ and } d\Omega \text{ about } \hat{\Omega} \qquad \text{(III.4)}$$

then obviously the contribution to the point ① in Section (III.2.2).

$$① = \int_V s(r,E,\hat{\Omega},t)\ d^3r\ dE d\Omega \qquad \text{(III.5)}$$

This term was really easy-we only needed to define a source density, $s(r,E,\hat{\Omega},t)$.

⑤ Loss due to collisions in V: The rate at which neutrons suffer collisions at a point r is

$$f_t(r,E,\hat{\Omega},t) = v\Sigma_t(r,E)n(r,E,\hat{\Omega},t) \qquad \text{(III.6)}$$

Hence integrating this collision rate over the volume V, we find

$$⑤ = \left[\int_V v\Sigma_t(r,E)n(r,E,\hat{\Omega},t)d^3r\right] dEd\Omega \qquad \text{(III.7)}$$

3. Gain due to neutrons scattering into dE about E, $d\Omega$ about $\hat{\Omega}$ from other energies E′ and directions $\hat{\Omega}'$: If we recall from Chapter II that the probability of scattering from E′, $\hat{\Omega}'$, to E, $\hat{\Omega}$ is given in terms of the double-differential scattering cross section, then the rate at which neutrons scatter from E′, $\hat{\Omega}'$ to E, $\hat{\Omega}$ is

$$\left[\int_V v'\Sigma_s(E'\to E,\hat{\Omega}'\ \hat{\Omega})n(r,E',\hat{\Omega}',t)d^3r\right] dEd\Omega \qquad \text{(III.8)}$$

However we must consider contributions from any E′, $\hat{\Omega}'$. Hence

$$③ = \left[\int_V d^3r \int_{4\pi} d\Omega' \int_0^\infty dE' v'\Sigma_s(E'\to E,\hat{\Omega}'\to\hat{\Omega})n(r,E'\hat{\Omega},t)\right] dEd\Omega \qquad \text{(III.9)}$$

This is known as the inscattering term since it characterizes neutrons scattering from energies or directions into dEdΩ.

②-④ Leakage into or from the volume V: We will combine these terms together and calculate the net leakage through the surface S. If we use the concept of the angular current density $j(r,E,\hat{\Omega},t)$, we can write the rate at which neutrons of E, $\hat{\Omega}$ leak out of a piece of the surface, dS, as

$$j(r,E,\hat{\Omega},t).dS = v\hat{\Omega}n(r,E,\hat{\Omega},t).dS \qquad \text{(III.10)}$$

Hence the leakage contribution over the entire surface areas S is

$$④ - ② = \int_S dS.\upsilon\hat{\Omega}n(r,E,\Omega,t). \qquad \text{(III.11)}$$

We can rewrite this in terms of a volume integral if we use Gauss's theorem (reference 4)

$$\int_s dS.A(r) = \int_V d^3r\nabla.A(r), \qquad \text{(III.12)}$$

to find

$$\left[\int_s dS.\upsilon\hat{\Omega}n(r,E,\hat{\Omega},t)\right] dEd\Omega = \left[\int_V d^3r\nabla.\upsilon\hat{\Omega}n(r,E,\hat{\Omega},t)\right]dEd\Omega$$

$$= \left[\int_V d^3r\upsilon\Omega\cdot\nabla n(r,E,\hat{\Omega},t)\right]dEd\Omega \qquad \text{(III.13)}$$

Here we have noted that

$$\nabla.\upsilon\hat{\Omega} = \upsilon\Omega\cdot\nabla \qquad \text{(III.14)}$$

Since $\hat{\Omega}$ and υ do not depend on r.

If we now combine all of these terms such that

rate of change of number of neutrons in V = ① + ② + ③ - ④ - ⑤ ,

then we find

$$\int_V d^3r\left[\frac{\partial n}{\partial t} + \upsilon\hat{\Omega}.\nabla n + \upsilon\Sigma_t n(r,E,\hat{\Omega},t)\right.$$

(cont'd)

$$-\int_0^\infty dE' \int_{4\pi} d\Omega' v' \Sigma_s(E' \to E, \hat{\Omega}' \to \hat{\Omega}) n(r,E',\hat{\Omega},t) - s(r,E,\hat{\Omega},t) \Big] dE d\Omega = 0. \tag{III.15}$$

However we now apply the fact that the volume V was quite arbitrarily chosen. Hence the only way for the integral to vanish for any V is for its integrand to be identically zero - that is,

$$\int_{\text{any } V} d^3 r f(r) = 0 \qquad f(r) \equiv 0. \tag{III.16}$$

Hence we arrive at a balance relation

$$\boxed{\begin{aligned} &\frac{\partial n}{\partial t} + v\hat{\Omega}.\nabla n + v_t n(r,E,\hat{\Omega},t) \\ &= \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' v' \Sigma_s(E' \to E, \hat{\Omega}' \to \hat{\Omega}) n(r,E',\hat{\Omega},t) + s(r,E,\hat{\Omega},t). \end{aligned}} \tag{III.17}$$

This is known as the NEUTRON TRANSPORT EQUATION.

III.2.4 COMMENTS ON NEUTRON TRANSPORT EQUATION

Several general features of the equation (III.17) should be noted: First, it is a linear equation in the unknown dependent variable $n(r,E,\hat{\Omega},t)$ with seven independent variables ($r = x,y,z; E; \hat{\Omega}=\theta,\phi; t$). Since it contains both derivatives in space and time as well as integrals over angle and energy, it is known as an "integrodifferential" equation.

However the presence of the derivatives suggest that we must also specify appropriate initial and boundary conditions for the angular density. Since only a single time derivative appears in the equation, we can simply choose the initial condition to be the specification of the initial

value of the angular density for all positions, energies, and directions:

$$\text{Initial condition: } n(\mathbf{r},E,\hat{\Omega},0) = n_0(\mathbf{r},E,\hat{\Omega}), \text{ all } \mathbf{r}, E, \hat{\Omega}. \tag{III.18}$$

The boundary conditions will depend on the particular problem of interest. In the case of nonreentrant surface then appropriate boundary conditions would simply express the fact that there can be no neutrons entering the system from the outside. That is, we require the angular neutron density on the surface to vanish for all inward directions

$$n(\mathbf{r}_s,E,\hat{\Omega},t) = 0 \quad \text{if } \hat{\Omega}.\hat{\mathbf{e}}_s < 0, \text{ for all } \mathbf{r}_s \text{ on } S, \tag{III.19}$$

where $\mathbf{r}_s$ denotes a point on the surface S.

It is convenient to rewrite the neutron transport equation along with its initial and boundary conditions in terms of the angular flux

$$\frac{1}{v}\frac{\partial \psi}{\partial t} + \hat{\Omega}.\nabla\psi + \Sigma_t(\mathbf{r},E)\psi(\mathbf{r},E,\hat{\Omega},t)$$

$$= \int_{4\pi} d\Omega' \int_0^\infty dE' \Sigma_s(E'\to E,\hat{\Omega}'\to\hat{\Omega},)\psi(\mathbf{r},E',\hat{\Omega}',t) + s(\mathbf{r},E,\hat{\Omega},t), \tag{III.20}$$

$$\text{Initial condition:} \quad \psi(\mathbf{r},E,\hat{\Omega},0) = \psi_0(\mathbf{r},E,\hat{\Omega}), \tag{III.21}$$

$$\text{Boundary condition:} \quad \psi(\mathbf{r}_s,E,\hat{\Omega},t) = 0 \text{ if } \hat{\Omega}.\hat{\mathbf{e}}_s<0, \tag{III.22}$$

$$\text{all } \mathbf{r}_s \text{ on } S.$$

III.3 SIMPLIFIED CASES OF THE NEUTRON TRANSPORT EQUATION

III.3.1 ONE-DIMENSIONAL FORM

Suppose we try to make the Eq. (III.20) a little bit less abstract by applying it to the special case in which there is plane symmetry, that is, where the neutron flux depends only on a single spatial coordinate, say X (as shown in Fig. III.2). Then the directional derivative $\hat{\Omega}.\nabla$ reduces to

$$\boxed{\Omega.\nabla\psi(x) = \left[\Omega_x \frac{\partial}{\partial x} + \Omega_y \frac{\partial}{\partial y} + \Omega_z \frac{\partial}{\partial z}\right] \psi(x) = \Omega_x \frac{\partial\psi}{\partial x}} \qquad \text{(III.23)}$$

For convenience, we will choose our angular coordinate system with its polar coordinate axis in the x-direction. Then $\Omega_x = \cos\theta$. The assumption of plane symmetry also implies that there is no dependence on the azimuthal angle ϕ. Hence the one-dimensional form of the transport equation becomes

$$\boxed{\begin{aligned} &\frac{1}{v}\frac{\partial\psi}{\partial t} + \cos\theta \frac{\partial\psi}{\partial x} + \Sigma_t\psi(x,E,\theta,t) \\ &= \int_0^{\pi} d\theta' \sin\theta' \int_0^{\infty} dE' \Sigma_s(E'\to E,\theta'\to\theta)\psi(x,E',\theta') + s(x,E,\theta,t) \end{aligned}} \qquad \text{(III.24)}$$

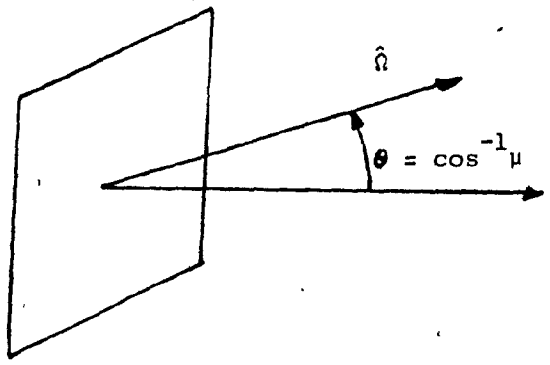


Fig. III.2 Coordinates Characterizing Plane Symmetry.

A final modification of the angular variable is useful. It is customary to rewrite this Eq. (III.24) terms of a new variable, $\mu \equiv \cos\theta$. Note that as θ ranges between 0 and π, μ ranges from 1 to -1. Hence the usual form of the one-dimensional transport equation is written as

$$\frac{1}{v}\frac{\partial \psi}{\partial t} + \mu \frac{\partial \psi}{\partial x} + \Sigma_t \psi(x,E,\mu,t)$$

$$= \int_{-1}^{+1} d\mu' \int_0^\infty dE' \Sigma_s(E' \to E, \mu' \to \mu)\psi(x,E',\psi',t) + s(xE,\psi,t). \tag{III.25}$$

We can easily generalize Eq. (III.20) to include nuclear fission by including a component in the source term to account for fission neutrons. The rate at which neutrons with energy E' and direction $\hat{\Omega}'$ induce fission events is just $\Sigma_f(E')\psi(r,E',\hat{\Omega}',t)$. If $\nu(E')$ is the average number of fission neutrons produced by a fission induced by a neutron of energy E', then the total rate at which fission neutrons are born at a position r is just

$$\int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \nu(E')\Sigma_f(E')\psi(r,E',\hat{\Omega}',t). \tag{III.26}$$

These fission neutrons will have an energy distribution given by the fission spectrum $\chi(E)$. If we assume that they are emitted isotropically, then the fission source term we should include in the transport equation is just

$$s_f(r,E,\hat{\Omega},t) = \frac{\chi(E)}{4\pi} \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \nu(E')\Sigma_f(E')\psi(r,E',\hat{\Omega}',t). \tag{III.27}$$

Actually we should qualify this argument a bit by admitting that we have assumed all of the fission neutrons to appear instantaneously at the time of fission. Hence s_f is actually the source term corresponding to prompt fission neutrons. Delayed fission neutrons will be included later.

The neutron transport equation provides an essentially exact description of the neutron distribution within the reactor (at least, provided one is supplied with appropriate cross section information). Its solution would yield the angular flux $\psi(r,E,\hat{\Omega},t)$ containing essentially all the information (actually considerably more) we require concerning the nuclear behavior of the reactor. All we have to do is solve this equation.

Yet notice that: (a) the neutron transport equation has seven independent variables x,y,z,θ,ϕ,E,t, (b) the dependence of the macroscopic cross sections on position r is extremely complicated because of the complex, nonuniform structure of most reactor cores, and (c) as we have seen in Chapter II, the cross section dependence on energy is also extremely complicated including resonance.

III.3.2 THE ONE-SPEED APPROXIMATION

It is frequently convenient to suppress the neutron energy dependence by assuming that one can characterize the neutrons by a single energy or speed. We will find later that if one chooses the appropriate effective cross sections, such a representation will in fact frequently yield a reasonable description of the reactor. However for now we will intro-

duce the one-speed approximation in a rather artificial manner by simply assuming that the neutron energy does not change in a scattering collision. This can be inserted into the transport equation (Eq. III.20) in a rather convenient manner by simply assuming a differential scattering cross section of the form

$$\Sigma_s(E' \to E, \hat{\Omega}' \to \hat{\Omega}) = \Sigma_s(E, \hat{\Omega}' \to \hat{\Omega})\,\delta(E' - E), \tag{III.28}$$

where $\delta(E' - E)$ is the Dirac δ-function defined by the property

$$\int dx' f(x')\,\delta(x - x') = f(x) \tag{III.29}$$

for any sufficiently well-behaved function $f(x)$. (Consult Ref. 5 for a more detailed discussion of the δ-function.) Using this definition, the inscattering term in Eq. (III.20) becomes

$$\int_{4\pi} d\Omega' \int_0^{\infty} dE' \Sigma_s(E' \to E, \hat{\Omega}' \to \hat{\Omega})\,\psi(r, E', \hat{\Omega}', t) = \int_{4\pi} d\Omega' \Sigma_s(E, \hat{\Omega}' \to \hat{\Omega})\,\psi(r, E, \hat{\Omega}', t). \tag{III.30}$$

Since all of the terms in the transport equation are now evaluated at the same energy, we may as well eliminate the explicit dependence on energy to write the one-speed neutron transport equation as

$$\boxed{\frac{1}{v}\frac{\partial \psi}{\partial t} + \hat{\Omega}\cdot\nabla\psi + \Sigma_t(r)\psi(r, \hat{\Omega}, t) = \int_{4\pi} d\hat{\Omega}' \Sigma_s(\hat{\Omega}' \to \hat{\Omega})\,\psi(r, \hat{\Omega}', t) + s(r, \hat{\Omega}, t)} \tag{III.31}$$

III.3.3 ISOTROPIC SOURCES AND SCATTERING

One major simplification that can be introduced into the transport equation arises when one assumes both isotropic neutron sources

$$s(r,\hat{\Omega},t) = \frac{1}{4\pi} S(r,t) \tag{III.32}$$

and isotropic scattering (in the LAB system)

$$\Sigma_s(\hat{\Omega}'\rightarrow\hat{\Omega}) = \frac{1}{4\pi} \Sigma_s. \tag{III.33}$$

The assumption of isotropic neutron sources is usually not too restrictive since most sources such as fission are indeed essentially isotropic. Unfortunately although neutron scattering is usually isotropic in the CM system, it is far from isotropic in the LAB system, particularly for low mass number scatterers such as hydrogen. Undeterred by such physical considerations, we will assume for the moment that isotropic scattering is present. Then the one-speed transport equation simplifies still further to

$$\boxed{\frac{1}{v}\frac{\partial\psi}{\partial t} + \hat{\Omega}.\nabla\psi+\Sigma_t\psi(r,\hat{\Omega},t) = \frac{\Sigma_s}{4\pi}\int_{4\pi} d\Omega'\psi(r,\hat{\Omega},t) + \frac{S(r,t)}{4\pi}} \tag{III.34}$$

III.3.4 STEADY-STATE

Thus far we have mutilated the energy and angular dependence of the transport equation in the interest of mathematical expediency - and still have not arrived at anything we can hope to solve (at least analytically). So in frustration we now turn our attention to the remaining time and spatial variables. First we will completely eliminate the time variable by agreeing to consider only steady-state transport problems. Then Eq. (III.34) simplifies to

$$\hat{\Omega}\cdot\nabla\psi+\Sigma_t\psi(r,\hat{\Omega}) = \frac{\Sigma_s}{\pi}\int_{4\pi} d\hat{\Omega}'\psi(r,\hat{\Omega}') + \frac{S(r)}{4\pi} \qquad \text{(III.35)}$$

Next, we will assume that the system under study has uniform composition such that the cross sections do not depend on position. Finally we will simplify the system geometry, for example, by considering only planar or spherical symmetry. In case of planar symmetry we arrive at a rather simple-looking equation

$$\mu\frac{\partial\psi}{\partial x} + \Sigma_t\psi(x,\mu) = \frac{\Sigma_s}{2}\int_{-1}^{+1} d\mu'\psi(x,\mu') + \frac{S(x)}{2} \qquad \text{(III.36)}$$

This equation can actually be solved analytically but only with rather sophisticated mathematical techniques. So even after a number of rather questionable approximations, one arrives at an equation that can still only be solved with great difficulty.

III.4 THE DIFFUSION APPROXIMATION

III.4.1 THE NEUTRON CONTINUITY EQUATION

For most reactor calculations, the details of the angular dependence of the flux are not necessary, and we really only need to calculate the angle-integrated flux:

$$\phi(r,E,t) = \int_{4\pi} d\hat{\Omega}\psi(r,E,\hat{\Omega},t) \qquad \text{(III.37)}$$

in order to calculate nuclear reaction rates and hence study the chain reaction (e.g., by calculating the multiplication factor k).

Surely we can formulate an equation for $\phi(r,E,t)$ by simply integrating the transport equation over angle. Let's try and see what happens. That is, we will integrate each term of the transport equation (III.20) over the direction variable $\hat{\Omega}$:

$$\int_{4\pi} d\Omega \frac{1}{v}\frac{\partial\psi}{\partial t} + \int_{4\pi} d\Omega\hat{\Omega}\cdot\nabla\psi + \int_{4\pi} d\Omega\Sigma_t\psi \qquad \text{(III.38)}$$

① ② ③

$$= \int_{4\pi} d\Omega \int_{4\pi} d\Omega' \int_0^\infty dE'\Sigma_s(E'\to E,\hat{\Omega}'\to\hat{\Omega})\,\psi(r,E',\hat{\Omega}',t) + \int_{4\pi} d\hat{\Omega}\, s(r,E,\hat{\Omega},t)$$

④ ⑤

We can simplify each of these terms somewhat by the straightforward manipulations indicated below:

$$① = \int_{4\pi} d\Omega \frac{1}{v} \frac{\partial \psi}{\partial t} = \frac{1}{v} \frac{\partial}{\partial t} \int_{4\pi} d\Omega \psi = \frac{1}{v} \frac{\partial \phi}{\partial t} \tag{III.39}$$

$$② = \int_{4\pi} d\Omega \Sigma_t \psi = \Sigma_t \int_{4\pi} d\Omega \psi = \Sigma_t \phi \tag{III.40}$$

$$③ = \int_{4\pi} d\Omega s(r,E,\hat{\Omega},t) \equiv S(r,E,t) \tag{III.41}$$

Here we have used our earlier expression Eq.(II.131) for the neutron flux ϕ in terms of the angular flux ψ and also simply defined a source term $S(r,E,t)$. To evaluate the inscattering term ④ we first recal that $\Sigma_s(E' \to E, \hat{\Omega}' \to \hat{\Omega})$ usually depends only on the scattering angle cosine $\mu_0 = \hat{\Omega}' . \hat{\Omega}$. This implies that

$$\int_{4\pi} d\Omega \Sigma_s(E' \to E, \hat{\Omega}' \to \hat{\Omega}) = 2\pi \int_{-1}^{+1} d\mu_0 \Sigma_s(E' \to E, \mu_0) = \Sigma_s(E' \to E) \tag{III.42}$$

where $\Sigma_s(E \to E)$ is just the "single" differential scattering cross section defined in Eq.(III.37). Hence we can interchange the order of integrations over $\hat{\Omega}$ and $\hat{\Omega}'$ to write:

$$④ = \int_{4\pi} d\Omega' \int_0^\infty dE' \int_{4\pi} d\Omega \Sigma_s(E' \to E, \hat{\Omega}' \to \hat{\Omega}) \psi(r,E',\hat{\Omega}',t) \tag{III.43}$$

$$= \int_0^\infty dE' \Sigma_s(E' \to E) \int_{4\pi} d\Omega' \psi(r,E',\hat{\Omega}',t)$$

$$= \int_0^\infty dE' \Sigma_s(E' \to E) \phi(r,E',t)$$

So far everything is straightforward; but unfortunately the last term ② cannot be evaluated in terms of $\phi(r,E,t)$. In fact we find that ② must

be evaluated in terms of the neutron current J by using Eq. (II.132):

$$② = \int_{4\pi} d\Omega\, \hat{\Omega}.\nabla\psi = \nabla.\int_{4\pi} d\Omega\hat{\Omega}\psi = \nabla.J(r,E,t) \qquad \text{(III.44)}$$

If we now rewrite Eq. (III.38) it takes the form

$$\boxed{\frac{1}{\upsilon}\frac{\partial\phi}{\partial t} + \nabla.J(r,E,t) + \Sigma_t(r,E)\ \phi\ (r,E,t) = \int_0^\infty d\hat{E}'\Sigma_s(E' \to E)\ \phi\ (r,E',t) + S(r,E,t)} \qquad \text{(III.45)}$$

This is known as the neutron continuity equation, since it is just the mathematical statement of neutron balance.

It is important to note that this equation contains two unknowns, $\phi(r,E,t)$ and $J(r,E,t)$, unlike the neutron transport equation, which only contained one unknown, the angular flux $\psi(r,E,\hat{\Omega},t)$.

It is impossible to express $J(r,E,t)$ in terms of $\phi(r,E,t)$ in a general and exact manner. This is more apparent if we recall the definitions Eqs. (II.130) and (II.132):

$$\phi(r,E,t) \equiv \int_{4\pi} d\Omega\psi(r,E,\hat{\Omega},t) \qquad \text{(III.46)}$$

$$J(r,E,t) \equiv \int_{4\pi} d\Omega\hat{\Omega}\psi(r,E,\hat{\Omega},t) \qquad \text{(III.47)}$$

It is obvious that these two quantities are entirely different functions, although they can both be expressed in terms of an angular integral of the angular flux $\psi(r,E,\hat{\Omega},t)$. Hence there is no reason why one would expect these functions to be simply related.

Undaunted by our failure to find a simple equation for $\phi(r,E,t)$, suppose we shift our attention instead to developing an equation for the current density $J(r,E,t)$. By comparing the definitions in Eqs.(II.130) and (II.132) above, we are tempted to try multiplying the transport equation by $\hat{\Omega}$ and then integrating once again over angle. Actually since the direction variable $\hat{\Omega}$ is a vector,

$$\hat{\Omega} = \hat{e}_x \sin\theta\cos\phi + \hat{e}_y \sin\theta\sin\phi + \hat{e}_z \cos\theta \qquad \text{(III.48)}$$

$$\Omega_x \qquad \Omega_y \qquad \Omega_z$$

we would multiply the transport equation by each component separately and integrate. For example, the Ω_x component would yield

$$\int_{4\pi} d\Omega\Omega_x \frac{1}{v}\frac{\partial\psi}{\partial t} + \int_{4\pi} d\Omega\Omega_x \hat{\Omega}\cdot\psi + \int_{4\pi} d\Omega\Omega_x \Sigma_t \psi \qquad \text{(III.49)}$$

$$\textcircled{1} \qquad \textcircled{2} \qquad \textcircled{3}$$

$$= \int_{4\pi} d\Omega\Omega_x \int_{4\pi} d\Omega' \int_0^\infty dE' \Sigma_s(E' \to E, \hat{\Omega}' \to \hat{\Omega})\, \psi(r,E',\hat{\Omega}',t) + \int_{4\pi} d\Omega\Omega_x s(r,E,\hat{\Omega},t)$$

$$\textcircled{4} \qquad \textcircled{5}$$

Each of these terms can be simplified in a manner similar to that used in deriving the neutron continuity equation Eq. (III.45) :

$$\textcircled{1} = \frac{1}{v}\frac{\partial}{\partial t}\int_{4\pi} d\Omega\Omega_x \psi = \frac{1}{v}\frac{\partial J_x}{\partial t} \qquad \text{(III.50)}$$

$$\textcircled{3} = \Sigma_t \int_{4\pi} d\Omega\Omega_x \psi = \Sigma_t J_x \qquad \text{(III.51)}$$

$$\textcircled{5} = \int_{4\pi} d\Omega\Omega_x s(r,E,\hat{\Omega},t) \equiv S_{1x}(r,E,t) \qquad \text{(III.52)}$$

Now to handle the inscattering term ④ , write

$$④ = \int_{4\pi} d\Omega' \int_0^\infty dE' \left[\int_{4\pi} d\Omega\Omega_x \Sigma_s(E' \to E, \hat{\Omega}' \to \hat{\Omega}) \right] \psi(r,E',\hat{\Omega},t) \qquad \text{(III.53)}$$

Next we do something a bit sneaky. Since $\hat{\Omega}$ is a unit vector, we can write $\hat{\Omega}'.\hat{\Omega}' = 1$. We will insert this into Eq. (III.53) so that we can rewrite it as:

$$④ = \int_0^\infty dE' \int_{4\pi} d\Omega' \left[\int_{4\pi} d\Omega\Omega_x \Omega'_x \Sigma_s(E' \to E, \hat{\Omega}' \to \hat{\Omega}) \right] \cdot \hat{\Omega}' \psi(r,E',\hat{\Omega}',t) \qquad \text{(III.54)}$$

Now we recal again that $\Sigma_s(E' \to E, \hat{\Omega}' \to \hat{\Omega})$ depends only on the cosine of the scattering angle $\mu_0 = \hat{\Omega}'.\hat{\Omega}$. Thus we can write

$$④ = 3 \int_0^\infty dE' \int_{4\pi} d\Omega' \left[\int_{4\pi} d\Omega\Omega_x \Omega'_x \Sigma_s(E' \to E, \hat{\Omega}'.\hat{\Omega}) \right] \Omega'_x \psi(r,E',\hat{\Omega},t) \qquad \text{(III.55)}$$

We will define

$$\int_{4\pi} d\Omega\Omega_x \Omega'_x \Sigma_s(E' \to E, \hat{\Omega}'.\hat{\Omega}) = \frac{1}{3} \int_{4\pi} d\Omega\hat{\Omega}.\hat{\Omega}' \Sigma_s(E' \to E, \hat{\Omega}'.\hat{\Omega})$$

$$= \frac{2\pi}{3} \int_{-1}^{+1} d\mu_0 \mu_0 \Sigma_s(E' \to E, \mu_0) \equiv \frac{1}{3} \Sigma_{s1}(E' \to E) \qquad \text{(III.56)}$$

so that ④ finally becomes

$$④ \quad \int_0^\infty dE' \Sigma_{s1}(E' \to E) \int_{4\pi} d\Omega' \Omega'_x \psi(r,E',\hat{\Omega}',t) = \int_0^\infty dE' \Sigma_{s1}(E' \to E) \, J_x(r,E',t) \qquad \text{(III.57)}$$

Thus far each of our terms has been expressed in terms of the current

density J_x, except for the source term S_{1x} which is a known term. But we still haven't considered the streaming term:

$$\textcircled{2} = \int_{4\pi} d\Omega \Omega_x \hat{\Omega}.\nabla\psi = \nabla. \int_{4\pi} d\Omega \Omega_x \hat{\Omega}\psi(r,E,\hat{\Omega},t) \qquad \text{(III.58)}$$

A quick glance at the integral term confirms our fears; once again the streaming term has kicked out yet another new unknown. To see this more clearly, we can combine these results along with similar results for Ω_y and Ω_z to write Eq. (III.49) as an equation for the current density J:

$$\frac{1}{v}\frac{\partial J}{\partial t} + \nabla. \int_{4\pi} d\Omega\, \hat{\Omega}\hat{\Omega}\psi(r,E,\hat{\Omega},t) + \Sigma_t J(r,E,t)$$

$$= \int_0^\infty dE' \Sigma_{s1}(E' \to E) J(r,E',t) + S_1(r,E,t) \qquad \text{(III.59)}$$

Here we have taken the luxury of using a symbolic notation of writing two vectors together, $\hat{\Omega}\hat{\Omega}$. We interpret this as a convenient notation for taking each of the various combinations of components $\Omega_x\Omega_x, \Omega_x\Omega_y, \ldots, \Omega_z\Omega_z$ separately to construct a quantity with nine components, $\Pi_{xx}, \Pi_{xy}, \ldots, \Pi_{zz}$. Such quantities are referred to as tensors (or, in this case, dyadics), get so formal here. It should be evident that we can get a new equation for $\Pi(r,E,t)$ by multiplying the transport equation by $\hat{\Omega}\hat{\Omega}$ and integrating, but this new equation will contain yet another unknown,

$$\int_{4\pi} d\Omega\, \hat{\Omega}\hat{\Omega}\hat{\Omega}\hat{\Omega}\psi(r,E,\hat{\Omega},t) \qquad \text{(III.60)}$$

Hence all we are doing by multiplying by $\hat{\Omega}^n$ and integrating is generating

an infinite set of coupled equations (which we can't solve). Incidently, it should be apparent that the culprit is the "streaming" or "leakage" term $\hat{\Omega}.\nabla\psi$ which contains a factor of $\hat{\Omega}$ and hence generates the new unknowns in each equation.

The only way to cut off this chain of equations is to introduce an approximation. We shall do this by assuming that the angular flux is only weakly dependent on angle, and in so doing, we will generate the neutron diffusion equation.

III.4.2 THE ONE-SPEED DIFFUSION EQUATION

We now turn our attention toward the development of an approximate description of neutron transport more amenable to calculation than the neutron transport equation itself. To make life simple, we will first work within the one-speed approximation represented by Eq.(III.31). Let us first note the explicit forms taken by the neutron conservation equation and the corresponding equation for the current density J in the one-speed case:

$$\frac{1}{v}\frac{\partial \phi}{\partial t} + \nabla.J + \Sigma_t \phi(r,t) = \Sigma_s \phi(r,t) + S(r,t) \qquad \text{(III.61)}$$

$$\frac{1}{v}\frac{\partial J}{\partial t} + \nabla. \int_{4\pi} d\Omega\, \hat{\Omega}\hat{\Omega}\psi(r,\hat{\Omega},t) + \Sigma_t J(r,t) = \bar{\mu}_0 \Sigma_s J(r,t) + S_1(r,t) \qquad \text{(III.62)}$$

Here we have noted explicitly the simplifications that occur in the in-scattering term when the one-speed approximation is introduced. More specifically,

$$\int_0^\infty dE' \Sigma_s(E' \to E)\ \phi\ (r,E',t) + \Sigma_s \phi\,(r,t) \qquad \text{(III.63)}$$

and

$$\int_0^\infty dE' \Sigma_{s1}(E' \to E)\ J\ (r,E,t) + \Sigma_{s1} J(r,t) \qquad \text{(III.64)}$$

and

$$\Sigma_{s1} = 2\pi \int_{-1}^{+1} d\mu_0 \mu_0 \Sigma_s(\hat{\Omega}'.\hat{\Omega}) = \bar{\mu}_0 \Sigma_s \qquad \text{(III.65)}$$

where we have defined the average scattering angle cosine $\bar{\mu}_0$ as

$$\bar{\mu}_0 \equiv \hat{\Omega}.\hat{\Omega}' = \frac{2\pi}{\Sigma_s} \int_{-1}^{+1} d\mu_0 \mu_0 \Sigma_s(\mu_0) \qquad \text{(III.66)}$$

$$= \frac{1}{4\pi\Sigma_s} \int_{4\pi} d\Omega \int_{4\pi} d\Omega' \hat{\Omega}.\hat{\Omega}' \Sigma_s(\hat{\Omega}.\hat{\Omega}')$$

As an aside, it should be noted that one can easily calculate $\bar{\mu}_0$ for the case of elastic scattering from stationary nuclei when s-wave scattering is present. For then we know that in the CM system, $\sigma_{CM}(\theta_C) = \sigma_s/4\pi$.

Hence if we use $\sigma_{CM}(\theta_C)d\Omega_C = \sigma_L(\theta_L)d\Omega_L$, we find (III.67)

$$\bar{\mu}_0 = \frac{2\pi}{\Sigma s}\int_0^\pi \sin\theta_C d\theta_C \cos\theta_L \Sigma_{CM}(\theta_C)$$

$$= \frac{1}{2}\int_0^\pi \sin\theta_C \cos\theta_L d\theta_C$$

But recall

$$\cos\theta_L = \frac{1 + A\cos\theta_C}{\sqrt{A^2 + 2A\cos\theta_C + 1}} \qquad \text{(III.68)}$$

If we substitute this into Eq.(III.67) and perform the integration, we find the very simple result

$$\bar{\mu}_0 = \frac{2}{3A} \qquad \text{(III.69)}$$

Now that we have justified the forms of Eqs.(III.61) and (III.62) let us consider how we might eliminate the annoying appearance of the third unknown, $\int d\Omega \hat{\Omega}\hat{\Omega}\psi(r,\hat{\Omega},t)$. We will accomplish this by assuming that the angular flux is only weakly dependent on angle. To be more specific, we will expand the angular flux in angle as

$$\psi(r,\hat{\Omega},t) = \psi_0(r,t) + \psi_{1x}(r,t)\Omega_x + \psi_{1y}(r,t)\Omega_y + \psi_{1z}(r,t)\Omega_z - \ldots \qquad \text{(III.70)}$$

and neglect all terms of higher than linear order in $\hat{\Omega}$. Actually a slightly different notation for the unknown functions ψ_0, ψ_{1x}, ψ_{1y}, ψ_{1z} is useful.

Write Eq. (III.70) as

$$\psi(r,\hat{\Omega},t) \approx \frac{1}{4\pi}\phi + \frac{3}{4\pi}\left[J_x\Omega_x + J_y\Omega_y + J_z\Omega_z\right]$$

$$\approx \frac{1}{4\pi}\phi(r,t) + \frac{3}{4\pi}J(r,t)\cdot\hat{\Omega} \qquad \text{(III.71)}$$

Notice that we have labeled the unknown expansion coefficients as the flux and current. That this notation is perfectly consistent can be seen by noting from Eq. (III.71) that

$$\int_{4\pi} d\Omega\psi(r,\hat{\Omega},t) = \phi(r,t)\,\frac{1}{4\pi}\int_{4\pi} d\Omega + \frac{3}{4\pi}J(r,t)\cdot\int_{4\pi} d\Omega\,\Omega = \phi(r,t) \qquad \text{(III.72)}$$

and,

$$\int_{4\pi} d\Omega\hat{\Omega}\psi(r,\hat{\Omega},t) = \phi(r,t)\,\frac{1}{4\pi}\int_{4\pi} d\Omega\hat{\Omega} + \frac{3}{4\pi}\,J_x(r,t)\int_{4\pi} d\Omega\hat{\Omega}_x\hat{\Omega}$$

$$+ J_y(r,t)\int_{4\pi} d\Omega\hat{\Omega}_y\hat{\Omega} + J_z(r,t)\int_{4\pi} d\Omega\hat{\Omega}_z\hat{\Omega} \qquad \text{(III.73)}$$

However it can easily be demonstrated that the integral of the product of any two components of $\hat{\Omega}$ gives

$$\int_{4\pi} d\Omega\hat{\Omega}_i\hat{\Omega}_j = \begin{cases} \frac{4\pi}{3} & i = j \\ 0 & i \neq j \end{cases} \quad i,j = x,y,z \qquad \text{(III.74)}$$

Hence

$$\int_{4\pi} d\Omega\,\hat{\Omega}\psi(r,\hat{\Omega},t) = J(r,t) \qquad \text{(III.75)}$$

Of course Eqs. (III.72) and (III.75) are identical to our original definitions of the flux and current earlier in Eqs. (II.130) and (II.132).

We will now use the approximate form of the angular flux in Eq. (III.71) evaluate the second term in Eq. (III.62):

$$\nabla . \int_{4\pi} d\Omega \; \hat{\Omega}\hat{\Omega}\psi(r,\hat{\Omega},t) \;=\; \nabla . \int_{4\pi} d\Omega \; \hat{\Omega}\hat{\Omega} \left[\; \frac{1}{4\pi} \phi + \frac{3}{4\pi} J . \hat{\Omega} \; \right] \tag{III.76}$$

Next note that the integral of the product of any odd number of components of $\hat{\Omega}$ vanishes by symmetry:

$$\int_{4\pi} d\Omega \; \hat{\Omega}_x^l \hat{\Omega}_y^m \hat{\Omega}_z^n \;=\; 0 \quad \text{if } l,m \text{ or } n \text{ is odd} \tag{III.77}$$

If we use both Eqs. (III.74) and (III.77) we can evaluate

$$\nabla . \int_{4\pi} d\Omega \; \hat{\Omega}\hat{\Omega}\psi \;=\; \frac{1}{3} \nabla\phi(r,t) \tag{III.78}$$

Hence by assuming that the angular flux depends only weakly on angle - more specially, that the angular flux is only <u>linearly anisotropic</u> - we have managed to express the third unknown appearing in Eq. (III.62) in terms the neutron flux $\phi(r,t)$. We have now achieved our goal of obtaining a closed set of two equations for two unknowns, $\phi(r,t)$ and $J(r,t)$:

$$\frac{1}{v} \frac{\partial \phi}{\partial t} + \nabla . J + \Sigma_a(r) \; \phi \quad (r,t) \;=\; S(r,t) \tag{III.79}$$

$$\frac{1}{v} \frac{\partial J}{\partial t} + \frac{1}{3} \nabla\phi + \Sigma_{tr}(r) \; J \; (r,t) \;=\; S_1(r,t) \tag{III.80}$$

dependent diffusion coefficient as defined by Eq. (III.101). If we now substitute this into Eq. (III.93), we arrive at the energy-dependent diffusion equation

$$\frac{1}{v}\frac{\partial \phi}{\partial t} - \nabla \cdot D(r,E)\nabla\phi + \Sigma_t(r,E)\phi(r,E,t) = \int_0^\infty dE' \Sigma_s(E' \to E)\phi(r,E',t) + S(r,E,t) \tag{III.102}$$

This equation plays a very important role in nuclear reactor analysis since it is frequently taken as the starting point for the derivation of the multigroup diffusion equations. These latter equations represent the fundamental tool used in modern nuclear reactor analysis.

III.4.4 DIFFUSION THEORY BOUNDARY CONDITIONS

Since the neutron diffusion equation has derivatives in both space and time, it is apparent that one must assign suitable boundary and initial conditions to complete the specification of any particular problem. Since the diffusion equation itself is only an approximation to the more exact transport equation, we might suspect that we can use the transport theory boundary conditions as a guide in our development of appropriate diffusion boundary conditions. It will suffice to consider this development within the one-speed approximation.

III.4.4.1 MATHEMATICAL BOUNDARIES ON FLUX

Although strictly speaking they are not boundary conditions, we should first mention those mathematical properties that the function $\phi(r,t)$ must exhibit in order to represent a physically realizable neutron flux. For example, $\phi(r,t)$ must be a real function. Furthermore since both the neutron speed v and density N cannot be negative, we must require that $\phi(r,t)$ be greater than or equal to zero. In most cases we can also require that $\phi(r,t)$ be bounded. However we should add here that one occasionally encounters pathological models of physical neutron sources that cause $\phi(r,t)$ to diverge.

III.4.4.2 BOUNDARIES AT INTERFACES

Consider next an interface between two regions of differing cross

sections. Now clearly the correct transport boundary condition is that

$$\psi_1(r_s,\hat{\Omega},t) = \psi_2(r_s,\hat{\Omega},t) \text{ for all } \hat{\Omega}, \qquad \text{(III.103)}$$

where ψ_1 is the angular flux in region 1, while ψ_2 is the angular flux in region 2. This condition ensures conservation of neutrons across the boundary and can easily be derived directly from the transport equation.

Unfortunately we cannot satisfy this boundary condition exactly using diffusion theory. At best, we can only ensure that angular moments of Eq. (III.103) are satisfied. And since diffusion theory yields only the first two moments of the angular flux, $\phi(r,t)$, and $J(r,t)$, at best we can demand

$$\int_{4\pi} d\hat{\Omega}\psi_1(r_s,\hat{\Omega},t) = \int_{4\pi} d\hat{\Omega}\psi_2(r_s,\hat{\Omega},t) \rightarrow \phi_1(r_s,t) = \phi_2(r_s,t), \qquad \text{(III.104)}$$

and

$$\int_{4\pi} d\hat{\Omega}\hat{\Omega}\psi_1(r_s,\hat{\Omega},t) = \int_{4\pi} d\hat{\Omega}\hat{\Omega}\psi_2(r_s,\hat{\Omega},t) \rightarrow J_1(r_s,t) = J_2(r_s,t) \qquad \text{(III.105)}$$

Hence the interface diffusion theory boundary conditions are simply those corresponding to continuity of flux and current density across the interface:

$$\begin{aligned} \phi_1(r_s,t) &= \phi_2(r_s,t) \\ -D_1\nabla\phi_1(r_s,t) &= -D_2\nabla\phi_2(r_s,t) \end{aligned} \qquad \text{(III.106)}$$

We will occasionally find it mathematically expedient to imagine an infinitestimally thin source of neutrons S at an interface boundary. Then the interface boundary conditions are modified to read:

$$\begin{aligned} \phi_1(r_s,t) &= \phi_2(r_s,t) \\ \hat{e}_s \cdot J_2(r_s,t) - \hat{e}_s \cdot J_1(r_s,t) &= S(t), \end{aligned} \qquad \text{(III.107)}$$

where $\hat{e}_s$ is the unit normal to the surface.

III.4.4.3 VACUUM BOUNDARIES

Recall that our transport theory boundary condition was merely a mathematically statement that there could be no incoming neutrons at a free or vacuum boundary

$$(r_s,\hat{\Omega},t) = 0 \quad \text{for } \hat{\Omega}\cdot dS<0, \text{ all } r_s \text{ on } S. \qquad \text{(III.108)}$$

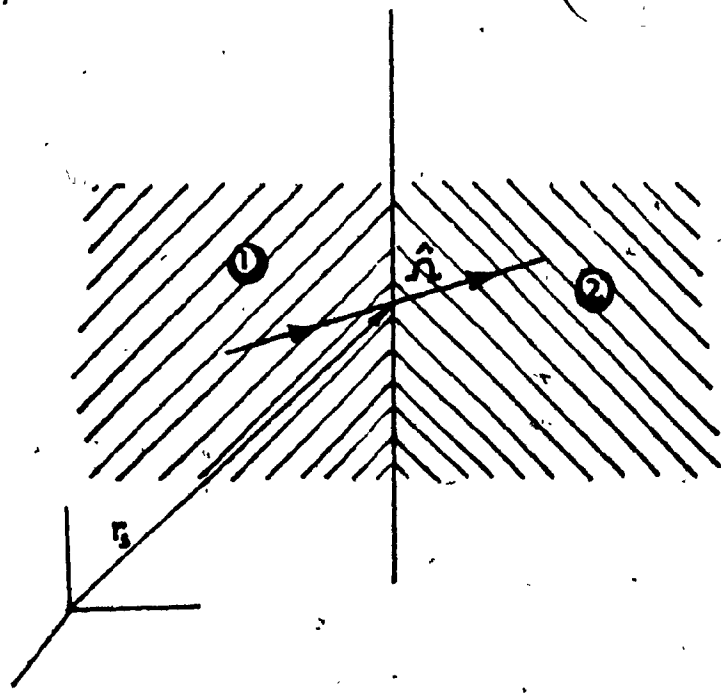


Fig. III.4 An Interface Boundary.

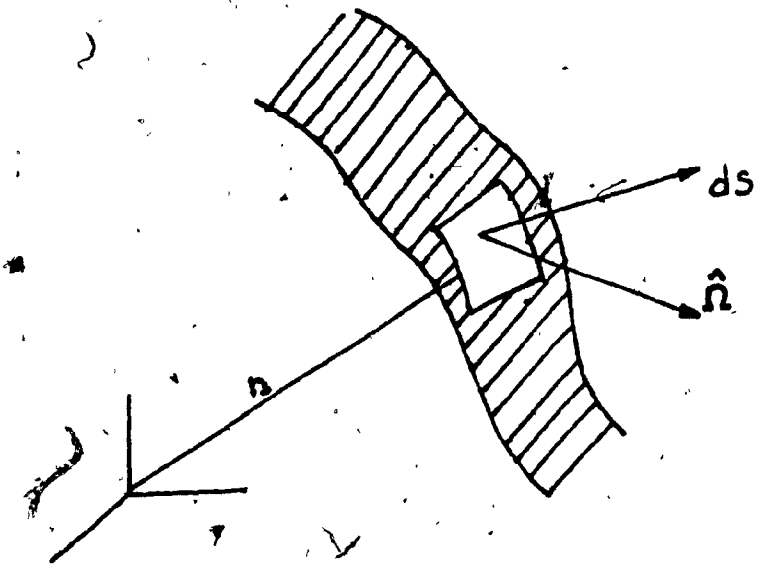


Fig. III.5 A/vacuum Boundary.

Once again diffusion theory will only be able to approximate this boundary condition. Notice in particular that the boundary condtion is given only over half of the range of solid angle (corresponding to incoming neutrons). Hence suppose we again seek to satisfy the transport boundary condition in an "integral" sense by demanding

$$\int_{2\pi^-} d\hat{\Omega}\hat{e}_s\cdot\hat{\Omega}\psi(r_s,\hat{\Omega},t) = \int_{2\pi^-} d\hat{\Omega}\hat{e}_s\cdot j(r_s,\hat{\Omega},t) \equiv J_-(r_s,t) = 0 \quad \text{(III.109)}$$

where we have recognized that this integral condition is equivalent to demanding that the inwardly directed partial current J_- vanish on the boundary.

Unfortunately diffusion theory is capable of only approximating even this integral condition, since it cannot yield the exact form for $J_\pm$. Indeed if we use the P_1 approximation (III.71) for the angular flux, we find that the partial current densities $J_\pm$ are approximated in diffusion theory by

$$J_\pm(r,t) = \int_{2\pi^\pm}\psi d\hat{\Omega}\hat{e}_s\cdot\hat{\Omega}\ (r,\hat{\Omega},t) \cong \frac{1}{4}\phi\ (r,t) \pm \frac{D}{2}\hat{e}_s\cdot\nabla\phi(r,t). \quad \text{(III.110)}$$

Hence our diffusion theory approximation to the transport boundary condition Eq. (III.108) is just

$$J_-(r_s,t) = \frac{1}{4}\phi(r_s,t) + \frac{D}{2}\hat{e}_s\cdot\nabla\phi(r_s,t) = 0 \quad \text{(III.111)}$$

For convenience consider this boundary condition applied to a one-dimensional geometry with the boundary at $\chi = \chi_s$

$$J_-(\chi_s) = \frac{1}{4}\phi(\chi_s) + \frac{D}{2}\left.\frac{d\phi}{d\chi}\right|_{\chi_s} = 0$$

or

$$\frac{1}{\phi(\chi_s)}\left.\frac{d\phi}{d\chi}\right|_{\chi_s} = -\frac{1}{2D} \qquad \text{(III.112)}$$

Notice that this relation implies that if we "extrapolated" the flux linearly beyond the boundary, it would vanish at a point

$$\tilde{\chi} \equiv \chi_s + 2D = \chi_s + \frac{2}{3}\lambda_{tr} \qquad \text{(III.113)}$$

For this reason, one frequently replaces the vacuum boundary condition

$$J_-(\chi_s) = 0 \qquad \text{(III.114)}$$

by the slightly simpler condition

$$\phi(\tilde{\chi}_s) = 0 \qquad \text{(III.115)}$$

where $\tilde{\chi}_s$ is referred to as the "extrapolated" boundary. More advanced transport theory calculations of the extrapolated boundary indicate that one should choose

$$\tilde{\chi} = \chi_s + z_0 \qquad \text{(III.116)}$$

where the "extrapolation length" z_0 for plane geometries is given by

$$z_0 = 0.7104\lambda_{tr} \qquad \text{(III.117)}$$

These boundary conditions complete our description of neutron transport within the diffusion approximation. The neutron diffusion equation will play a very fundamental role in our development of nuclear reactor analysis methods. We will begin our study of nuclear reactor behavior using one-speed diffusion theory, since while this description has only limited quantitative validity, it does allow us to illustrate rather easily the principal concepts of nuclear reactor theory, as well as to develop the mathematical techniques used in more sophisticated models.

III.4.4.4 SUMMARY OF THE ONE-SPEED DIFFUSION MODEL

To summarize then, the model we will initially use to describe the neutron population in a nuclear reactor consists of the neutron diffusion equation:

$$\frac{1}{v}\frac{\partial \phi}{\partial t} - \nabla \cdot D(r)\nabla\phi + \Sigma_a(r)\phi(r,t) = S(r,t) \qquad \text{(III.118)}$$

along with suitable initial conditions:

$$\phi(r,0) = \phi_0(r), \quad \text{all } r \qquad \text{(III.119)}$$

and boundary conditions:

a) Free surface: $\phi(\tilde{r}_s,t) = 0$ {or $J_-(r_s,t) = 0$}

(III.120)

b) Interface: ϕ and normal component of J continuous across

interface

c) $0 < \phi(r,t) < \infty$ (except in the neighbourhood of localized sources).

Here the diffusion coefficient $D = \lambda_{tr}/3 = \{3(\Sigma_t - \bar{\mu}_0\Sigma_s)\}^{-1}$ while the extrapolation length characterizing a free surface boundary condition is $z_0 = 0.7104\lambda_{tr}$.

The equation has been thoroughly studied in mathematics and physics alike for years, since it also describes processes such as heat condition; gas diffusion, and even a wave function (notice, if we stick an "i" in front of the time derivative, we have essentially just the Schrödinger equation familiar from quantum mechanics).

In many cases we will deal with situations for which the medium in which the neutrons are diffusing is uniform or homogeneous such that D and Σ_a do not depend on position. Then the one-speed diffusion equation simplifies to

$$\frac{1}{v}\frac{\partial\phi}{\partial t} - D\nabla^2\phi + \Sigma_a\phi(r,t) = S(r,t) \qquad \text{(III.121)}$$

The explicit form taken by this equation will depend on the specific coordinate system in which we choose to express the spatial variable r. ∇^2 is called the Laplacian operator which can take different form for different geometires (Ref. 5).

III.4.4.5 THE STEADY STATE DIFFUSION EQUATION

We will frequently consider situations in which the flux is not a

function of time. Then Eq. (III.121) becomes

$$-D\nabla^2 \phi(r) + \Sigma_a \phi(r) = S(r) \tag{III.122}$$

This equation is known as the Helmholtz equation and is also a very familiar in mathematical physics. It is useful to divide by -D to rewrite Eq. (III.122) as

$$\nabla^2 \phi(r) - \frac{1}{L^2} \phi(r) = - \frac{S(r)}{D} \tag{III.123}$$

where we have defined the neutron diffusion length L

$$L \equiv \sqrt{D/\Sigma_a} \tag{III.124}$$

L is essentially a measure of how far the neutrons will diffuse from a source before they are absorbed.

We now turn our attention to the application of this model to some important problems in nuclear reactor theory. We will first study neutron diffusion in "nonmultiplying" media - that is, media containing no flux in fissile material and begin our investigation of nuclear reactor core physics.

III.5 NEUTRON DIFFUSION IN NONMULTIPLYING MEDIA

The neutron diffusion theory may apply to multiplying media as well as to non multiplying media.

We will first apply Eq. (III-122) to study the diffusion of neutrons from a steady-state source in a nonmultiplying medium. All of the mathematical techniques we will use are standard methods which arise in the solution of ordinary or partial differential equation boundary value problems.

III.5.1 Elementary Solutions of the Diffusion Equation

III.5.1.1 PLANE SOURCE IN AN INFINITE MEDIUM

The simplest problem in neutron diffusion theory is that of an infinitely wide plane source located at the origin of an infinite, homogeneous medium. The source is assumed to be emitting neutrons isotropically at a rate of S_0 neutrons/cm^2·sec. Since both the source plane and the medium are of infinite extent, the neutron flux $\Phi(r) \rightarrow \phi(x)$ can only be a function of the distance x from the source plane. Hence the diffusion equation {Eq. III-123} reduces to the one-dimensional form

$$\boxed{\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S(x)}{D} = -\frac{S_0}{D}\delta(x)} \qquad \text{(III-125)}$$

where we have mathematically modeled the source by a Dirac δ-function.

Hence we just have an infomogeneous ordinary differential equation to solve with a slightly weird source. Note that if we restrict $x \neq 0$, the source term disappears from Eq. (III-125)

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0, \quad x \neq 0. \qquad \text{III-126}$$

Our approach will be to solve this homogeneous equation for $x \neq 0$, and then use a boundary condition at $x=0$ to "fix up" these solutions. We would obtain this boundary condition directly by integrating Eq. (III-125) from $x=0-\epsilon$ to $x=0+\epsilon$ across the source plane and then taking the limit as $\epsilon \to 0$ to find

$$-D\frac{d\phi}{dx}\bigg|_{+\epsilon} + D\frac{d\phi}{dx}\bigg|_{-\epsilon} = J_x(0^+) - J_x(0^-) = S_0. \qquad \text{III-127}$$

If we use the symmetry of the geometry to assert that $J_x(0^+) = -J_x(0^-) \equiv J(0)$, then our boundary condition at the source plane becomes just

$$\lim_{x \to 0^+} J(x) = \frac{S_0}{2}. \qquad \text{III-128}$$

This source boundary condition makes sense physically, since it merely says that the net neutron current at the origin on either side must be just half of the total source strength.

We are not through with boundary conditions yet. Since we have a second-order derivative, d^2/dx^2, we need another boundary condition.

We will use the boundary condition of finite flux as $x \to \infty$.

Hence the mathematical problem to be solved is

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = 0, \qquad x>0,$$

with boundary conditions:

$$\text{(a)} \quad \lim_{x\to 0+} -D\frac{d\phi}{dx} = \frac{S_0}{2} \qquad \text{III-129}$$

$$\text{(b)} \quad \lim_{x\to\infty} \phi(x) < \infty.$$

We will then use symmetry to infer the solution for $x<0$.

To solve this equation, we note the general solution

$$\phi(x) = A \exp(-\frac{x}{L}) + B \exp(\frac{x}{L}). \qquad \text{III-130}$$

Applying the boundary conditions, we find

$$\text{(b)} \Rightarrow B = 0$$

$$\text{(a)} \Rightarrow \lim_{x\to 0} -D(-\frac{A}{L}\exp(-\frac{x}{L})) = \frac{AD}{L} = \frac{S_0}{2}$$

or

$$A = \frac{S_0 L}{2D}.$$

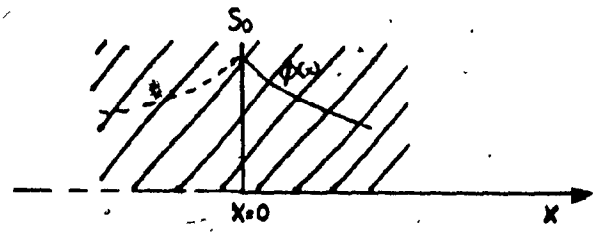


Figure III-6 A plane source of neutrons in an infinite medium

Hence our solution is

$$\phi(x) = \frac{S_0 L}{2D} \exp\left(-\frac{x}{L}\right), \quad x > 0 \qquad \text{III-131}$$

and by symmetry we can infer

$$\phi(x) = \frac{S_0 L}{2D} \exp\left(\frac{x}{L}\right), \quad x < 0. \qquad \text{III-132}$$

Hence the neutron flux falls off exponentially as one moves away from the source plane with a characteristic decay length of L. As one might expect, the larger L (i,e., the smaller Σ_a), the less the neutron flux is attenuated as we move into the medium. Notice also that the magnitude of the flux is proportional to the source. That is, doubling the source strength will double the neutron flux $\phi(x)$ at any position x, but this should have been anticipated since the neutron diffusion equation is linear and hence the principle of superposition holds.

III.5.1.2 POINT SOURCE IN AN INFINITE MEDIUM

As a variation on this therme, let us repeat this calculation for the case of an isotropic point source emitting S_0 neutrons/sec at the origin of an infinite medium. Since the source is isotropic, there can be no dependence of the neutron flux on angle. Hence the diffusion equation in spherical coordinates reduces to

$$\boxed{\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\phi}{dr} - \frac{1}{L^2}\phi(r) = 0, \qquad r > 0.} \qquad \text{III-133}$$

We will use our previous problem as a guide, and seek solutions such that the boundary conditions are

$$\text{(a)} \quad \lim_{r \to 0} 4\pi r^2 J(r) = S_0$$

$$\text{(b)} \quad \lim_{r \to \infty} \phi(r) < \infty$$

One can readily verify that the fundamental solutions to Eq. (III-133) are of the form $r^{-1}\exp(\pm r/L)$: hence we are led to seek

$$\phi(r) = A \frac{\exp(-r/L)}{r} + B \frac{\exp(r/L)}{r} \qquad \text{(III-134)}$$

Applying the boundary conditions, we find that (b) implies that we choose $B = 0$, while (a) implies that $A = S_0/4\pi D$. Hence the solution is

$$\phi(r) = \frac{S_0 \exp(-r/L)}{4\pi r D} \qquad \text{(III-135)}$$

An interesting application of this result is to calculate the mean-square distance to absorption in a nonmultiplying medium. Note the number of neutrons absorbed between r and r + dr is just

$$\frac{S_0 \exp(-r/L)}{4\pi r D} \; (4\pi r^2 dr) \; (\Sigma_a) \qquad \text{(III-136)}$$

and thus the probability that the neutron is absorbed in dr is just

$$p(r)\,dr = \frac{r}{L^2}\exp\left(-\frac{r}{L}\right)dr. \qquad \text{III-137}$$

We can then calculate

$$\langle r^2\rangle = \int_0^\infty dr\, r^2 p(r) = 6L^2. \qquad \text{III-138}$$

Hence the neutron-diffision length L has the interesting physical interpretation as being $1/\sqrt{6}$ of the root mean square (rms) distance to absorption

$$L^2 = \frac{1}{6}\langle r^2\rangle. \qquad \text{III-139}$$

That is, L measures the distance to which the neutron will diffuse (on the average) away from the source before it is absorbed. It should be stressed that we have calculated the rms distance from the source to the point of absorption, not the total path length traveled by the neutron. This path length will be very much longer since the neutron suffers a great many scattering collisions before it is finally absorbed. For example, in graphite the thermal diffusion length is 59 cm. Hence the rms distance to absorption from a point source is $(\langle r^2\rangle)^{\frac{1}{2}} = \sqrt{6}\ L = 144$ cm. If we recall that the mfp characterizing thermal neutrons in graphite is 2.5 cm and also recall that the average number of scattering collisions suffered by the neutron before absorption is 1500, then it is apparent

that the average path length or track length traveled by the neutron is about 3700 cm, considerably larger than $\sqrt{6}L$.

One can actually use Eq. (III-139) to define the neutron diffision length in situations in which diffusion theory would not apply (e,g., strongly anisotropic scattering or large absorption). Then one would first determine the flux $\phi(r)$ resulting from an isotropic point source using transport theory (or whatever description is relevant) and then calculate L by using

$$L^2 = \frac{1}{6}\langle r^2\rangle = \frac{\int d^3r\, r^2\phi(r)}{\int d^3r\phi(r)} \qquad \text{III-140}$$

where the integral is taken over all space.

The cylindrical geometry problem of a line source at the origin of an infinite medium can be worked out in a very similar way.

III-5.1.3 FINITE SLAB GEOMETRIES

Let us now modify our isotropic plane source by assuming that it is imbedded at a center of a slab of nonmultiplying material of width a surrounded on both sides by a vacuum (see Figure III-6). We will

set up this problem in a manner very similar to that for infinite plane geometry, except that we will add vacuum boundary conditions on either end of the slab.

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\,\phi(x) = 0, \quad x \neq 0,$$

with boundary conditions:

$$\text{(a)} \quad \lim_{x \to 0\pm} -D\frac{d\phi}{dx} = \frac{S_0}{2} \qquad \text{III-141}$$

$$\text{(b)} \quad \phi(\pm \frac{\tilde{a}}{2}) = 0.$$

Here we have replaced the boundary condition at infinity by the vacuum boundary condition - in this case, using an extrapolated boundary $\tilde{a}/2 = a/2 + z_0$. If we again seek a general solution of the form of Eq. (III-130), then applying the boundary condition (b) implies

$$\phi(\frac{\tilde{a}}{2}) = 0 = A \exp(-\frac{\tilde{a}}{2L}) + B \exp(\frac{\tilde{a}}{2L}) \rightarrow B = -A \exp(-\frac{\tilde{a}}{L}).$$

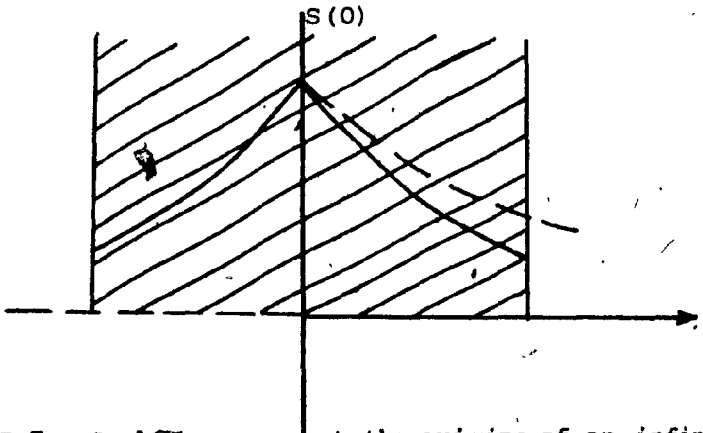


Figure III-7 A plane source at the origine of an infinite slab.

Then boundary condition (a) implies

$$A = \frac{SL}{2D}\left[(1 + \exp(-\frac{\tilde{a}}{L}))\right]^{-1}.$$

Our final solution is therefore

$$\phi(x) = \frac{SL}{2D}\,\frac{\exp(-\frac{x}{L}) - \exp -(\frac{\tilde{a}-x}{L})}{1 + \exp(-\frac{\tilde{a}}{L})} = \frac{SL}{2D}\left[\frac{\sinh\,(\tilde{a}-2|x|)}{2L}\right] \Big/ \cosh\left(\frac{\tilde{a}}{2L}\right) \qquad \text{III-142}$$

This solution is sketched in Figure III-7. It looks somewhat similar to the infinite medium result (Eq. III-131), except for dropping off more rapidly near the boundaries due to neutron leakage. We should mention once again that the solution is not valid within several mfp of the vacuum boundary (just as it is not valid near the source plane at the origin).

III.5.1.4 TWO REGIONS SLAB GEOMETRY

A variant on the above problem involves replacing the vacuum by a material of different composition than the slab itself (as sketched in Figure III-8). The general procedure for attacking such multiregion problems is to seek solutions of the diffusion equation characterizing each region, and then match these solutions using interface boundary conditions. Once again we can use geometrical symmetry to allow us to restrict our attention to the range $0<x<\infty$.

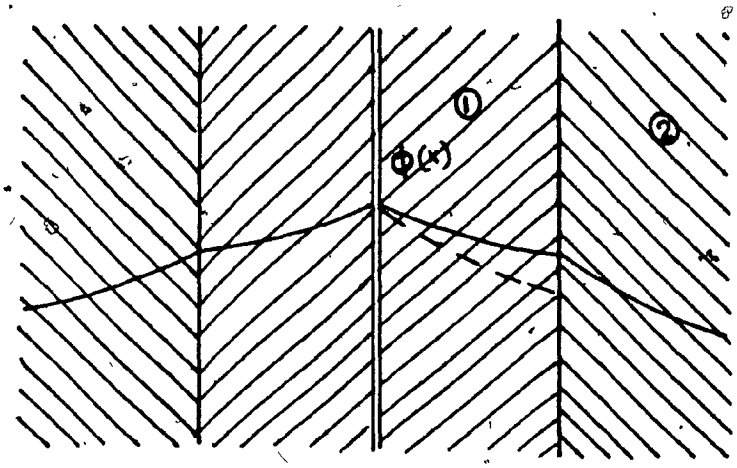


Figure III-8 Multiregion or reflected slab

In region (1) we will seek a solution of

$$\frac{d^2\phi}{dx^2} - \frac{1}{L_1^2}\,\phi_1(x) = 0, \qquad 0 < x < \frac{a}{2} \qquad \text{III-143}$$

where $L_1 = \sqrt{D_1/\Sigma a_1}$ is the diffusion length characterizing region (1). Similarly in region (2)

$$\frac{d^2\phi 2}{dx^2} - \frac{1}{L_2^2}\,\phi_2(x) = 0, \qquad \frac{a}{2} < x < \infty.$$

Now we will need several boundary conditions. We can use our earlier conditions (Eq, III-129).

$$\text{(a)} \quad \lim_{x \to 0+} J_1(x) = \frac{S_0}{2}$$

$$\text{(b)} \quad \phi_2(x) < \infty \quad \text{as} \quad x \to \infty.$$

In addition we will use the interface conditions

$$\text{(c)} \quad \phi_1\left(\frac{a}{2}\right) = \phi_2\left(\frac{a}{2}\right)$$

$$\text{(d)} \quad J_1\left(\frac{a}{2}\right) = J_2\left(\frac{a}{2}\right) \quad \text{or} \quad D_1 \left.\frac{d\phi_1}{dx}\right|_{\frac{a}{2}} = -D_2 \left.\frac{d\phi_2}{dx}\right|_{\frac{a}{2}}$$

Using our earlier work as a guide, we can seek general solutions

$$\phi_1(x) = A_1 \cosh \frac{x}{L_1} + B_1 \sinh \frac{x}{L} \qquad \text{in region (1)}$$

$$\phi_2(x) = A_x \exp\left(\frac{-x}{L_2}\right) + B_2 \exp\left(\frac{x}{L_2}\right) \qquad \text{in region (2)}$$

and apply the boundary condition. We find

$$B_2 = 0, \quad B_1 = -\frac{SL_1}{SD_1}$$

$$A_1 = \frac{SL_1}{2D_1} \cdot \frac{D_1L_2\cosh\left(\frac{a}{2L_1}\right) + D_2L_1\sinh\left(\frac{a}{2L_1}\right)}{D_2L_1\cosh\left(\frac{a}{2L_2}\right) + D_1L_2\sinh\left(\frac{a}{2L_1}\right)}, \qquad \text{III-144}$$

$$A_2 = \frac{SL_1L_2}{2} \cdot \frac{\exp\left(\frac{a}{2L_2}\right)}{D_2L_1\cosh\left(\frac{a}{2L_2}\right) + D_1L_2\sinh\left(\frac{a}{2L_1}\right)}$$

We have sketched the form of this solution in Figure (III-8). Several features of this solution are of some interest. Note that while the neutron flux is continuous across the interface, the derivative of the flux is not. This latter discontinuity is, of course, a consequence of the fact that the diffusion coefficients in the two regions differ; hence to obtain continuity of current J, we must allow a jump in $d\phi/dx$ across the interface.

We have compared the solution for this problem with that obtained earlier for the slab surrounded by a vacuum. It should be noted that the flux in the central region falls off somewhat more slowly when the vacuum is replaced by a diffusing material. This can be readily understood by noting that the material surrounding the slab will tend to scatter neutrons back into the slab that would have otherwise been lost to the vacuum. Such materials used to reduce neutron leakage are known as reflectors. Any material with a large scattering cross section and low absorption cross section would make a suitable neutron reflector. For example, the heavy water surrounding PHWR cores act as reflectors. In the HTGR,

graphite blocks are added to the top and bottom of the reactor core to serve as neutron reflectors.

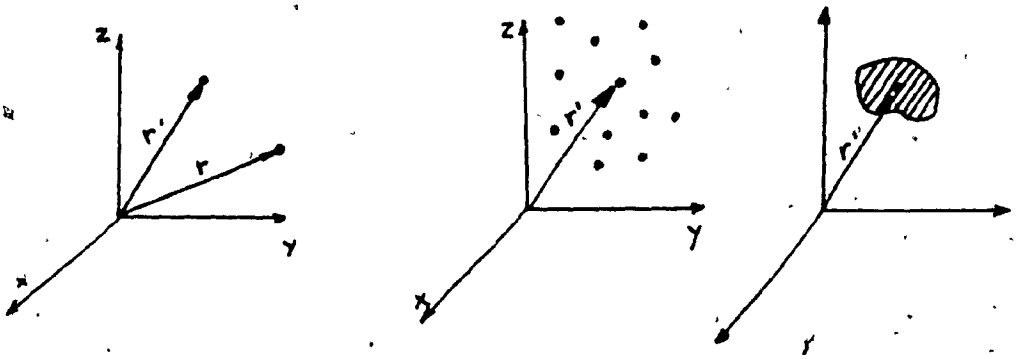
III-5.1.5 GENERAL DIFFUSION PROBLEMS

Recall that the neutron flux resulting from an isotropic point source of strength S_0 located at the origin of an infinite medium was found to be

$$\phi(r) = \frac{S_0 \exp(-\frac{r}{L})}{4\pi r D} \qquad \text{III-145}$$

Suppose this source was located at the point r' instead. Then the flux could be found by a simple coordinate translation as

$$\phi(r) = S_0 \frac{\exp(-\frac{|r-r'|}{L})}{4\pi D|r-r'|} \qquad \text{III-146}$$



Simple point source Several Point Sources Distributed Source

Figure III-9 Superposition of several point sources.

Next suppose we have several point sources at positions r_i', each of strength S_i. Then we can use the fact that the diffusion equation is linear to invoke the principle of superposition and write

$$\phi(r) = \sum_i \frac{S_i \exp\left(-\frac{|r - r_i'|}{L}\right)}{4\pi D|r-r_i'|} \qquad \text{III-147}$$

Finally suppose we have an arbitrary distribution of sources characterized by a source density $S(r)$. Then the flux resulting from this distributed source is just

$$\phi(r) = \int d^3r' \frac{\exp\left(-\frac{|r-r'|}{L}\right)}{4\pi D|r-r'|} S(r'). \qquad \text{III-148}$$

This is frequently rewritten as

$$\phi(r) = \int d^3r' G_{pt}(r,r')S(r'), \qquad \text{III-149}$$

where

$$G_{pt}(r,r') \equiv \frac{\exp\left(-\frac{|r-r'|}{L}\right)}{4\pi D|r-r'|} \qquad \text{III-150}$$

is known as the point diffusion kernel for an infinite medium. (The expression kernel is a mathematical term used to denote a function of several variables (including the variables of integration) in an integral of the form

$$\int dx' K(x,x') f(x').$$

Here $K(x,x')$ would be the known kernel, while $f(x)$ might either be known (as the source density $S(r)$) or unknown as the flux ψ in the inscattering term of the transport equation,

$$\int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(E' \to E, \hat{\Omega}' \to \hat{\Omega}) \psi'(r,E',\hat{\Omega},t) \qquad \text{III-152}$$

where $\Sigma_s(E' \to E, \hat{\Omega}' \to \hat{\Omega})$ is known as the scattering kernel).

As second example of such kernels, consider the flux resulting from a plane source at the origin

$$\phi(x) = \frac{SL}{2D} \exp\left(-\frac{|x|}{L}\right). \qquad \text{III-153}$$

If this had been located at x', then the flux at a position x would be

$$\phi(x) = \frac{SL}{2D} \exp\left(-\frac{|x - x'|}{L}\right). \qquad \text{III-154}$$

Hence in general, for $S \to S(x')\, dx'$, we find

$$\phi(x) = \int_{-\infty}^{\infty} dx' \frac{L}{2D} \exp\left(-\frac{|x-x'|}{L}\right) S(x') \qquad \text{III-155}$$

$$= \int_{-\infty}^{\infty} dx' G_{pl}(x,x') S(x'),$$

where we have identified the plane source diffusion kernel for an infinite medium.

$$G_{pl}(x,x') = \frac{L}{2D}\exp\left(-\left|\frac{x-x'}{L}\right|\right) \qquad \text{III-156}$$

III.5.2 EIGENFUNCTION EXPANSION METHODS

One of the most powerful methods available for solving boundary value problems is to seek the solution as an expansion in the set of normal modes or eigenfunctions characterizing the geometry of interest. As an example we will attempt to determine the neutron flux resulting from an arbitary distributed source in a finite slab of width a. That is, we wish to solve

$$\frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi(x) = -\frac{S(x)}{D}, \quad -\frac{a}{2} \leq x \leq \frac{a}{2}, \qquad \text{III-157}$$

subject to the vacuum boundary conditions:

(a) $\phi\left(\frac{\tilde{a}}{2}\right) = 0$

(b) $\phi\left(-\frac{\tilde{a}}{2}\right) = 0$

Since we have taken the source S(x) to be arbitrary, we cannot assume symmetry to restrict our attention to the range $0 \leq x \leq \tilde{a}/2$.

We begin by considering a homogeneous problem very similar to Eq. (III-157)

$$\frac{d^2\psi}{dx^2} + B^2\psi(x) = 0 \rightarrow \frac{d^2\psi}{dx^2} = -B^2\psi(x)$$

with boundary conditions: III-158

$$\psi\left(\frac{\tilde{a}}{2}\right) = 0 = \psi\left(\frac{-\tilde{a}}{2}\right).$$

Here B^2 is just an arbitrary parameter - at least for the moment. Let us now solve this associated homogeneous problem by noting the general solution

$$\psi(x) = A_1 \cos Bx + A_2 \sin Bx \qquad \text{III-159}$$

Our boundary conditions require

$$\psi\left(\pm\frac{\tilde{a}}{2}\right) = A_1\cos\left(\frac{B\tilde{a}}{2}\right) \pm A_2 \sin\left(\frac{B\tilde{a}}{2}\right). \qquad \text{III-160}$$

Adding and subtracting these equations, we find that we must simultaneously require

$$A_1\cos\left(\frac{B\tilde{a}}{2}\right) = 0$$

and

$$A_2\sin\left(\frac{B\tilde{a}}{2}\right) = 0 \qquad \text{III-161}$$

Certainly we cannot set A_1 and A_2 equal to zero since then we would have the "trivial" solution $\psi(x) \equiv 0$. Instead we must choose the parameter B such that these conditions are satisfied. Of course there are many values of B for which this will occur. For example, if we choose $A_2=0$, then any

$$B=B_n \equiv \frac{n\pi}{a} \qquad \text{III-162}$$

will give rise to a solution

$$\psi_n(x) = A_n \cos \frac{n\pi x}{a}, \quad n = 1,3,5,... \qquad \text{III-163}$$

which obviously satisfies both the differential equation (eq. III-158) and the boundary conditions. Alternatively, we could have chosen $A_1 = 0$, in which case

$$B = B_n \equiv \frac{n\pi}{a}, \quad n = 2,4,6,... \qquad \text{III-164}$$

yields solutions

$$\psi_n(x) = A_n \sin \frac{n\pi x}{a}, \quad n = 2,4,6,... \qquad \text{III-165}$$

Hence our homegeneous problem can be solved only for certain values of the parameter B. One refers to the values of B^2 for which nontrivial solutions exist to the homogenous problem as eigenvalues:

$$\text{Eigenvalues: } B_n^2 = \left(\frac{n\pi}{a}\right)^2; \quad n = 1,2,... \qquad \text{III-166}$$

The corresponding solutions are referred to as the eigenfunctions of the problem:

Eigenfunctions:

$$\psi_n(x) = \begin{cases} A_n \cos\left(\frac{n\pi x}{\tilde{a}}\right), & n=1,3,5,\ldots \\ A_n \sin\left(\frac{n\pi x}{\tilde{a}}\right), & n=2,4,6,\ldots \end{cases} \qquad \text{III-167}$$

We may have already encountered eigenvalue problems in a somewhat different form:

$$H\psi_n = \lambda_n \psi_n \qquad \text{III-168}$$

where ψ_n is the eigenfunction corresponding to the eigenvalue λ_n. However by comparing this form with Eq.(III-158), one can easily identify $H = d^2 / dx^2$, $\lambda \rightarrow -B^2$, and $\psi_n \rightarrow \psi_n(x)$.

Notice that in the example Eq. (III-158) we actually find two types of eigenfunctions: the cosine functions corresponding to odd n and symmetric about the origin, and the sine functions corresponding to even n and antisymmetric about the origin. Had we restricted ourselves to symmetric sources $S(x) = S(-x)$, we could have eliminated the antisymmetric solutions (Eq. III-165) from further consideration. We have sketched the first few eigenfunctions for the slab geometry in Figure III-10

In acoustics these eigenfunctions would be identified as the normal modes or natural harmonics of the system, and this terminology is frequently carried over to reactor analysis. Notice that the A_n are still undetermined and are, in fact, arbitrary. These can be chosen in a number of ways, but for now we will just set $A_n = 1$ for convenience.

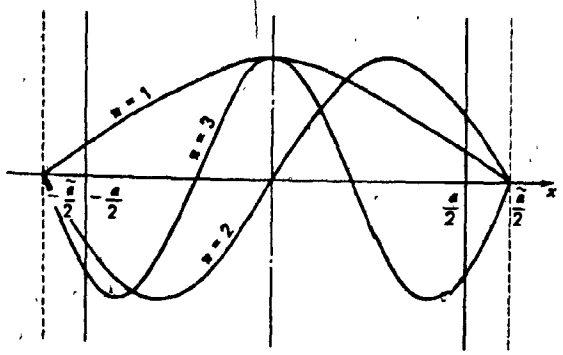


Figure III-10 The eigenfunctions for a slab geometry

So now this auxiliary problem has given us an infinite set of solutions $\psi_n(x)$. Which have a couple of very useful properties. First notice that the product of any two of these functions will vanish when integrated over the slab unless the functions are identical:

$$\int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx\psi_m(x)\psi_n(x) = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\tilde{a}}{2}, & \text{if } m = n \end{cases} \qquad \text{III-169}$$

This property is known as orthogonality and proves to be of very considerable usefulness, as we will see in a moment.

The second property of the eigenfunctions $\psi_n(x)$ is that they form a complete set in the mathematical sense that any reasonable "well-behaved" function f(x) can be represented as a linear combination of the $\psi_n(x)$:

$$f(x) = \sum_{n=1}^{\infty} c_n\psi_n(x) \qquad \text{III-170}$$

Of course, such a representation is only of formal interest unless some scheme is available for determining the expansion coefficients C_n, but this is where orthogonality comes in handy. Multiply Eq.(III-170) by $\psi_m(x)$ and integrate over x to find

$$\int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx\psi_m(x)f(x) = \sum_{n=1}^{\infty} C_n \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx\psi_m(x)\psi_n(x) = C_m \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx\psi_m^2(x)$$

or

$$\text{III-171}$$

$$C_m = \frac{2}{\tilde{a}} \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dxf(x)\psi_m(x).$$

Hence given any function $f(x)$, we can evaluate the appropriate expansion coefficients C_m by a simple integration.

With this background, we are finally ready to return to solve our original boundary value problem. As we mentioned, the essential idea is to seek the solution as an expansion in the eigenfunctions $\psi_n(x)$:

$$\phi(x) = \sum_{n=1}^{\infty} C_n \psi_n(x). \qquad \text{III-172}$$

We will also expand the source term in a similar fashion

$$S(x) = \sum_{n=1}^{\infty} S_n \psi_n(x). \qquad \text{III-173}$$

Notice that since $S(x)$ is known, we can use orthogonality to determine the source expansion coefficients (as in Eq. III-173) as

$$S_n = \frac{2}{\tilde{a}} \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx S(x) \psi_n(x) \qquad \text{III-174}$$

Of course since $\phi(x)$ is unknown, we cannot determine the C_n in a similar fashion, but that is just what we can use the original equation (Eq. III-157) to accomplish. If we substitute Eqs. (III-172) and (III-73) into Eq. (III-158), we find

$$\sum_{n=1}^{\infty} C_n \left[\frac{d^2\psi_n}{dx^2} - \frac{1}{L^2}\psi_n \right] = \frac{1}{D} \sum_{n=1}^{\infty} S_n \psi_n. \qquad \text{III-175}$$

However using Eq. (III-158), we can eliminate $d^2\psi_n / dx^2$ to find

$$\sum_{n=1}^{\infty} C_n \left(B_n^2 + \frac{1}{L^2} \right) \psi_n = \frac{1}{D} \sum_{n=1}^{\infty} S_n \psi_n . \qquad \text{III-176}$$

Thus we are left with one equation for an infinite number of unknowns, the C_n. Fortunately orthogonality once again comes to our rescrue. Multiply by $\psi_m(x)$, integrate over x, and use orthogonality to find

$$C_n = \frac{S_n / D}{B_n^2 + \frac{1}{L^2}} = \frac{S_n \Sigma_a}{1 + L^2 B_n^2} \qquad \text{III-177}$$

Thus we find the flux for any source distribution as

$$\phi(x) = \frac{1}{\Sigma_a} \sum_{n=1}^{\infty} \frac{S_n}{1 + L^2 B_n^2} \psi_n(x) \qquad \text{III-178}$$

where $\psi_n(x) = \cos(n\pi x/\tilde{a})$, n odd and $\psi_n(x) = \sin(n\pi x / \tilde{a})$, n even.

We can rewrite this in a bit more familiar form if we substitute Eq. III-174 into Eq. III-178 and rearrange things a bit:

$$\phi(x) = \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx' \left[\frac{2}{\tilde{a}\Sigma_a} \sum_{n=1}^{\infty} \frac{\psi_n(x)\psi_n(x')}{1 + L^2 B_n^2} \right] S(x') \qquad \text{III-179}$$

$$= \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx' G_{pl}(x,x') S(x') .$$

Where G_{pl} is given by:

$$G_{pl}(x,x') = \frac{2}{a\,\Sigma_a} \sum_{n=1}^{\infty} \frac{\psi_n(x)\psi_n(x')}{1 + L^2 B_n^2} \qquad \text{III-180}$$

Note in particular that the Green's function for a finite geometry is no longer a displacement kernel, that is, a function only of $x - x'$, as it was for infinite geometries.

III.6 THE ONE SPEED DIFFUSION THEORY IN MULTIPLYING MEDIA

III.6.1 GENERAL

The diffusion of neutrons in a nuclear reactor which contains fissile materials dictates the use of a fission term in the diffusion equation. To this end, let us first recall the sequence of events involved in a fission chain reaction. Specifically we consider processes occurring in thermal reactors. (see Fig. III.11). Fission neutrons are born at high energies in the MeV range. It is possible that such fast neutrons induce fission in either fissile (^{235}U or ^{239}Pu) or fissionable isotopes (^{238}U). It is far more likely that the fast fission neutrons will be moderated to lower energies by elastic scattering collisions with light moderator nuclei (e.g., $^{2}_{1}D$ or $^{12}_{6}C$). As the fission neutrons are slowed down, they pass through energies comparable to the absorption resonances in heavy nuclei such as ^{238}U and hence experience an appreciable probability of being absorbed. They may also leak out of the reactor core during this slowing down process. In a thermal reactor, however, over 85-90% of the neutrons will manage to slow down to thermal energies. They will then diffuse about the reactor core until they either leak from the core or are absorbed. If they are absorbed in the fuel, then they may induce a new fission, thereby repeating the cycle.

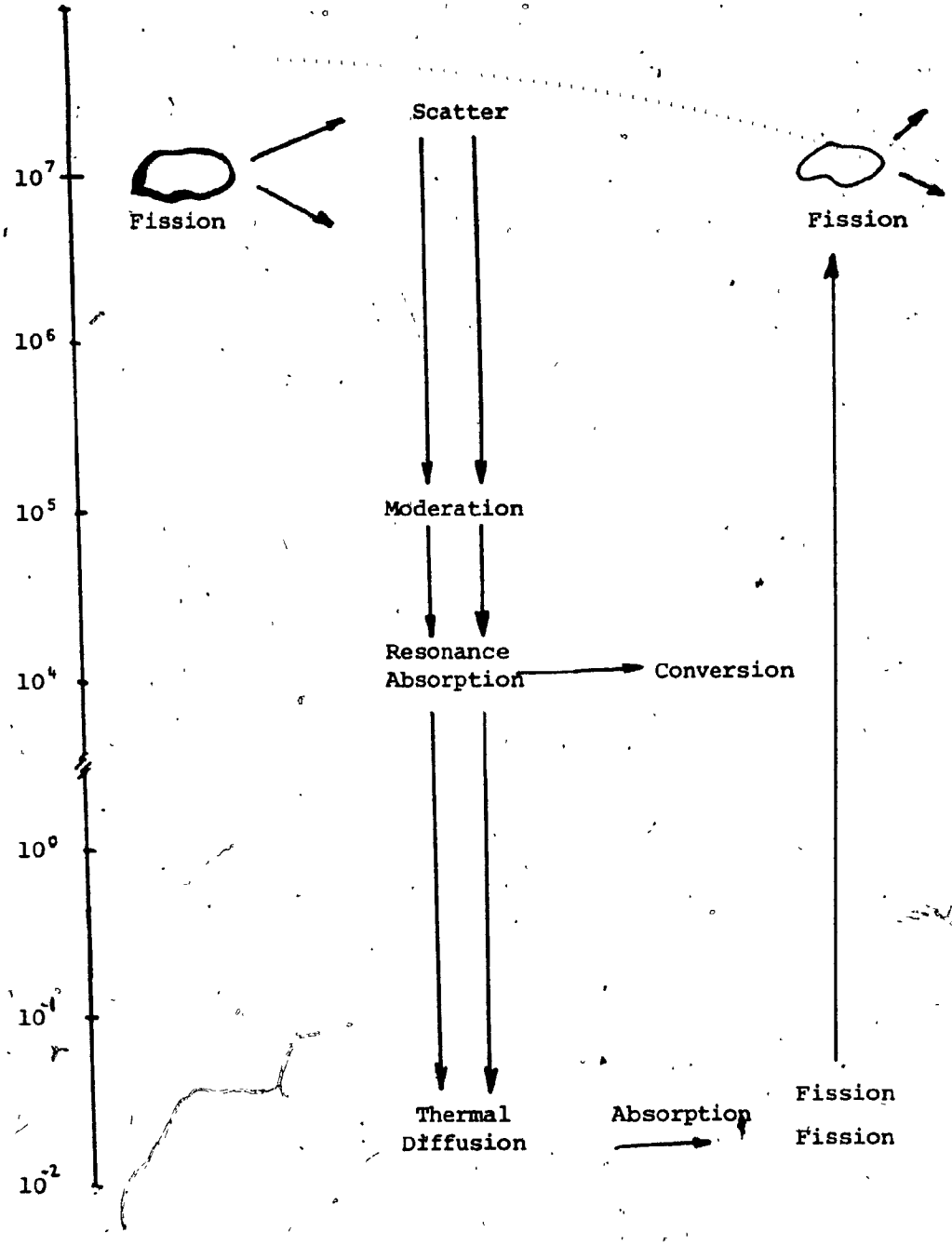


Fig. III.11 Process Occurring in Thermal Reactors.

III.6.2 THE FISSION SOURCE TERM

It is evident from previous sections that the slowing down process of neutrons born from fission is energy dependent. However, we will assume that diffusion, absorption, and fission all occur at the same energy. Then a term to represent fissions can easily be derived by noting that if $\Sigma_f\phi(r,t)$ is the fission reaction rate density, then the rate at which fission neutrons appear in the reactor - that is, the "fission source" - is given by

$$S_f(r,t) = \nu\Sigma_f\phi(r,t) \tag{III.181}$$

If this is the only source of neutrons in the reactor, then the appropriate diffusion equation becomes

$$\frac{1}{v}\frac{\partial\phi}{\partial t} - \nabla\cdot D\nabla\phi + \Sigma_a\phi(r,t) = \nu\Sigma_f\phi(r,t) \tag{III.182}$$

Note here that we can identify the various components of the macroscopic absorption cross section which appear in Eq. (III.182) as:

and

$$\Sigma_a = \Sigma_a^{\text{moderator}} + \Sigma_a^{\text{structure}} + \Sigma_a^{\text{coolant}} + \Sigma_a^{\text{fuel}}$$

$$\Sigma_a^{\text{fuel}} = \Sigma_\gamma^{\text{fuel}} + \Sigma_f^{\text{fuel}} \tag{III.183}$$

III.6.3 THE TIME-DEPENDENT "SLAB" REACTOR (ONE-SPEED)

III.6.3.1 GENERAL SOLUTION

Considering a uniform slab of fissile material characterized by cross

sections Σ_a, Σ_{tr}, and Σ_f. This unrealistic appearing "slab reactor" is chosen to introduce many of the concepts of nuclear reactor analysis, since its one dimensional geometry greatly facilitates the detailed solution of the one speed diffusion equation. The appropriate mathematical description of the

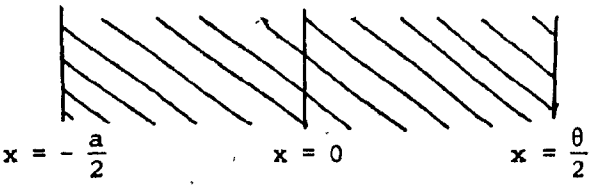


Fig. III.12 The Slab Reactor.

neutron flux in such a reactor is

$$\boxed{\frac{1}{v}\frac{\partial\phi}{\partial t} - D\frac{\partial^2\phi}{\partial x^2} + \Sigma_a\phi(x,t) = \nu\Sigma_f\phi(x,t)} \qquad \text{(III.184)}$$

with initial condition: $\phi(x,0) = \phi_0(-x)$ (symmetric)

and boundary conditions: $\phi\left(\frac{\tilde{a}}{2},t\right) = \phi\left(-\frac{\tilde{a}}{2},t\right) = 0$ (III.185)

The initial flux is assumed to be symmetric.

Unlike our earlier studies of time-independent neutron diffusion, we are now faced with a partial differential equation to solve. The simplest way to solve the equation is to use separation of variables by seeking a solution of the form

$$\boxed{\phi(x,t) = \psi(x)T(t).} \qquad \text{(III.186)}$$

If we substitute this form into Eq.(III.184) and divide by $\psi(x)T(t)$, we find

$$\frac{1}{T}\frac{dT}{dt} = \frac{\upsilon}{\psi}\left[D\frac{d^2\psi}{dx^2} + (\nu\Sigma_f - \Sigma_a)\psi(x)\right] = \text{constant} \equiv -\lambda \qquad \text{(III.187)}$$

Here we have noted that since we have a function only of x set equal to a function only of t, both terms must in fact be equal to a constant. We name this constant separation constant and indicate it by $-\lambda$. However, λ is as yet unknown.

Hence the separation of variables given by Eq.(III.186) has reduced the original partial differential equation in two variables to two ordinary differential equations:

$$\boxed{\begin{aligned} &\frac{dT}{dt} = -\lambda T(t) \\ &D\frac{d^2\psi}{dx^2} + (\nu\Sigma_f - \Sigma_a)\psi(x) = \frac{\lambda}{\upsilon}\psi(x) \end{aligned}} \qquad \text{(III.188)}$$

We can easily solve the time-dependent equation

$$T(t) = T(0)e^{-\lambda t} \qquad \text{(III.189)}$$

where T(0) is an initial value which must be determined later. To solve the space-dependent equation, we must tack on the boundary conditions:

$$D\frac{d^2\psi}{dx^2} + \left(\frac{\lambda}{\upsilon} + \nu\Sigma_f - \Sigma_a\right)\psi(x) = 0, \qquad \text{(III.190)}$$

$$\text{Boundary condition: } \psi\left(\frac{a}{2}\right) = \psi\left(-\frac{a}{2}\right) = 0 \qquad \text{(III.191)}$$

Here λ is still to be determined. However we recall the eigenvalue problem

$$\frac{d^2\psi_n}{dx^2} + B_n^2 \psi_n(x) = 0$$

$$\psi_n(\tilde{\frac{a}{2}}) = \psi_n(-\tilde{\frac{a}{2}}) = 0 \qquad \text{(III.192)}$$

has symmetric solutions (we are only interested in symmetric solutions since $\phi_0(x)$ is symmetric):

$$\text{eigenfunctions:} \quad \psi_n(x) = \cos B_n x$$

$$\text{eigenvalues:} \quad B_n^2 = (\frac{n\pi}{\tilde{a}})^2 \qquad n = 1,3,5,... \qquad \text{(III.193)}$$

If we identify Eq.(III.190) as the same problem, it is apparent that we must choose

$$\lambda = \upsilon\Sigma_a + \upsilon D B_n^2 - \upsilon\nu\Sigma_f \equiv \lambda_n \qquad n = 1,3,5,... \qquad \text{(III.194)}$$

These values of λ_n are known as the time eigenvalues of the equation, since they characterize the time decay in Eq.(III.189). The general solution to Eq.(III.184) must therefore be of the form

$$\phi(x,t) = \sum_{\substack{n \\ \text{odd}}} A_n \exp(-\lambda_n t) \cos\frac{n\pi x}{\tilde{a}} \qquad \text{(III.195)}$$

This solution automatically satisfies the boundary conditions. To

termine the A_n, we use the initial condition to write:

$$\text{Initial condition: } \phi(x,0) = \phi_0(x) = \sum_{\substack{n \\ \text{odd}}} A_n \cos \frac{n\pi x}{\tilde{a}} \qquad \text{(III.196)}$$

Using orthogonality, we find

$$A_n = \frac{2}{\tilde{a}} \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx \phi_0(x) \cos \frac{n\pi x}{\tilde{a}} \qquad \text{(III.197)}$$

Thus we have found that the flux (for any symmetric initial distribution) can be represented as a superposition of modes, each mode weighted by an exponential factor:

$$\boxed{\phi(x,t) = \sum_{\substack{n \\ \text{odd}}} \left[\frac{2}{\tilde{a}} \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx' \phi_0(x') \cos B_n x' \right] \exp(-\lambda_n t) \cos B_n x \,,} \qquad \text{(III.198)}$$

where the time eigenvalues λ_n are given by

$$\lambda_n = \upsilon\Sigma_a + \upsilon D B_n^2 - \upsilon\nu\Sigma_f, \qquad B_n = \frac{n\pi}{\tilde{a}} \qquad \text{(III.199)}$$

III.6.3.1.1 LONG TIME BEHAVIOR

Notice that one can order $B_1^2 < B_3^3 < \ldots B_n^2 = (n\pi/\tilde{a})^2$ . . . Hence the time eigenvalues must similarly be ordered such that $\lambda_1 < \lambda_3 < \lambda_5 < \lambda_7 < \ldots$ This means that the modes corresponding to larger n decay out more rapidly in time. If we wait long enough, then only the fundamental mode remains:

$$\phi(x,t) \sim A_1 \exp(-\lambda_1 t) \cos B_1 x \quad \text{as} \quad t \to \infty \qquad \text{(III-200)}$$

This implies that regardless of the initial shape of $\phi_0(x)$ the flux will decay into the fundamental mode shape. Of course, we have implicitly assumed that A_1 will not be zero. The coefficient of the fundamental mode is just

$$A_1 = \frac{2}{\tilde{a}} \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx' \phi_0(x') \cos \frac{\pi x'}{\tilde{a}}. \qquad \text{(III.201)}$$

Since $\phi_0(x)$ must be nonnegative in the slab to represent a physically realizable flux, then it is apparent that $A_1 > 0$.

Actually for sufficiently large Σ_f, $-\lambda_n$ may be positive corresponding to an exponentially growing flux. However the same argument will hold since $-\lambda_1 > -\lambda_3 > \ldots$. Hence regardless of whether the flux grows or decays, it will eventually approach a "persistent" or fundamental cosine distribution.

It is customary to refer to the value of B_n^2 characterizing this mode as

$$B_1^2 = (\frac{\pi}{\tilde{a}})^2 \equiv B_g^2 \equiv \text{geometric buckling.} \qquad \text{(III.202)}$$

This nomenclature is used since B_n^2 is a measure of the curvature of the mode shape

$$B_n^2 = -\frac{1}{\psi_n} \frac{d^2\psi_n}{dx^2}. \qquad \text{(III.203)}$$

Since there will be a larger current density J and hence leakage induced by a mode with larger curvature of buckling, we might expect that the mode with least curvature will persist in time the longest.

III.6.3.1.2 THE CRITICALITY CONDITION

The criticality condition in a nuclear reactor can be interpreted as that situation for which the neutron flux is time-independent. That is, to make the fission chain reaction steady-state. We will define this situation to be that of reactor criticality:

> Criticality: ≡ When a time-independent neutron flux can be sustained in the reactor (in the absence of sources other than fission).

If we write out the general solution for the flux

$$\phi(x,t) = A_1 \exp(-\lambda_1 t) \cos B_1 x + \sum_{\substack{n=3 \\ n:\text{odd}}}^{\infty} A_n \exp(-\lambda_n t) \cos B_n x \qquad \text{(III.204)}$$

it is evident that the requirements for a time-independent flux is just that the fundamental time eigenvalue vanish

$$\lambda_1 = 0 = \upsilon(\Sigma_a - \nu\Sigma_f) + \upsilon D B_1^2 \ , \qquad \text{(III.205)}$$

since the higher modes will have negative λ_n and decay out in time, leaving just

$$\phi(x,t) = A_1 \cos B_1 x \not\equiv \text{function of time}. \tag{III.206}$$

If we rewrite this "criticality condition" using the notation $B_1^2 = B_g^2$, then we find we must require

$$\boxed{\frac{\nu\Sigma_f - \Sigma_a}{D} = B_g^2} \tag{III.207}$$

It has become customary to refer to

$$\boxed{\frac{\nu\Sigma_f - \Sigma_a}{D} \equiv B_m^2 \equiv \text{material buckling}} \tag{III.208}$$

Since it depends only on the material composition of the reactor core (whereas B_g^2 depends only on core geometry). Hence our criticality condition can be written very concisely as:

$$\text{(material composition) } B_m^2 = B_g^2 \text{ (core geometry)}. \tag{III.209}$$

Thus to achieve a critical reactor, we must either adjust the size (B_g^2) or the core composition (B_m^2) such that $B_m^2 = B_g^2$. We also note:

$$\begin{aligned} B_m^2 > B_g^2 &\rightarrow \lambda_1 < 0 \rightarrow \text{supercritical} \\ B_m^2 = B_g^2 &\rightarrow \lambda_1 = 0 = \text{critical} \\ B_m^2 < B_g^2 &\rightarrow \lambda_1 > 0 \rightarrow \text{subcritical} \end{aligned} \tag{III.210}$$

In particular notice that by increasing the core size we decrease B_g^2, while by increasing the concentration of fissile material we increase Σ_f and hence B_m^2. Both of these modifications would, therefore, tend to enhance core multiplication.

We can write the time eigenvalue as

$$\lambda_1 = \upsilon\Sigma_a(1 + L^2B_g^2)\left(1 - \frac{\nu\Sigma_f/\Sigma_a}{1 + L^2B_g^2}\right) \qquad \text{(III.211)}$$

where

$$L^2 \equiv \left(\sqrt{\frac{D}{\Sigma_a}}\right)^2 = \frac{D}{\Sigma_a} \qquad \text{(III.212)}$$

Now recalling that $(\upsilon\Sigma)^{-1}$ is the mean lifetime for a given neutron-nuclear reaction to occur, $(\upsilon\Sigma_a)^{-1}$ is just the mean lifetime of a neutron to absorption. Furthermore we can identify (refer to earlier Eg. no.)

$$\frac{\nu\Sigma_f}{\Sigma_a} = \frac{\nu\Sigma_f^{\text{Fuel}}}{\Sigma_a^{\text{Fuel}}} \cdot \frac{\Sigma_a^{\text{Fuel}}}{\Sigma_a} = \eta f = k_\infty \qquad \text{(III.213)}$$

Now the only remaining task is to identify $(1 + L^2B_g^2)^{-1}$.

We recall that the rate of neutron leakage is given by:

$$\text{Leakage rate} = \int_S dS \cdot J = \int_V d^3r\, \nabla\cdot J = -\int_V d^3r\, D\nabla^2\phi \qquad \text{(III.214)}$$

Where we have used both the Gauss theorem(B-6) and diffusion approximation (III.86). Hence we can write

$$\frac{\text{Rate of neutron absorption}}{\text{Rate of neutron absorption \& leakage}} = \frac{\int_V d^3r \Sigma_a \phi}{\int_V d^3r \Sigma_a \phi - \int_V d^3r D\nabla^2\phi}$$

$$v = \frac{\Sigma_a \int_V d^3r\phi}{\Sigma_a \int_V d^3r\phi + B_g^2 D \int_V d^3r\phi} = \frac{1}{1 + L^2B_g^2} \qquad \text{(III.215)}$$

However, we can identify this ratio as just

$$\text{Non Leakage probability} \equiv P_{NL} \equiv \frac{1}{1 + L^2B_g^2} \qquad \text{(III.216)}$$

Therefore we can interpret

$$\left(\frac{1}{v\Sigma_a}\right) \frac{1}{1 + L^2B_g^2} = P_{NL} \left(\frac{1}{v\Sigma_a}\right) \equiv 1 \equiv \text{Neutron lifetime in a finite reactor.} \qquad \text{(III.217)}$$

Since we have just reduced the lifetime to absorption in an infinite medium to take account of neutron leakage. If we now combine Eqs.(III.213) and (III.217) we find that the multiplication factor k for this model becomes just:

$$k = \eta f P_{NL} = \frac{\nu\Sigma_f}{\Sigma_a} \cdot \frac{1}{1 + L^2B_g^2} \qquad \text{(III.218)}$$

Thus we can identify our fundamental time eigenvalue as just the inverse of the reactor period

$$-\lambda_1 = \frac{k-1}{L} = \frac{1}{T} \qquad \text{(III.219)}$$

If we also recall from equation (III-199) that

$$\lambda_1 = \nu D(B_g^2 - B_m^2) \tag{III.220}$$

then it is apparent that the various forms of the criticality condition are indeed equivalent:

$$\lambda_1 = 0 \leftrightarrow B_g^2 = B_m^2 \leftrightarrow k = 1 \tag{III.221}$$

III.6.4 THE CRITICALITY CONDITION FOR MORE GENERAL BARE GEOMETRIES

III.6.4.1 A RIGHT CIRCULAR CYLINDRICAL CORE (ONE REGION)

The most common reactor core shape is that of a right circular cylinder of height H and radius R. (Actually a sphere would be the more optimum geometry from the aspect of minimizing neutron leakage, but spheres are very inconvenient geometries to pass coolants through). The appropriate form of the Helmholtz equation is then

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} + B^2 \phi(r,z) = 0,} \tag{III.222}$$

subject to boundary conditions

$$\phi(\tilde{R},z) = 0 = \phi\left(r, \pm\frac{\tilde{H}}{2}\right). \tag{III.223}$$

Since this is a homogeneous partial differential equation, we can seek its solution using separation of variables

$$\phi(r,z) = R(r)Z(z) \tag{III.224}$$

Then if we substitute this form into Eq.(III.222) we arrive at two ordinary

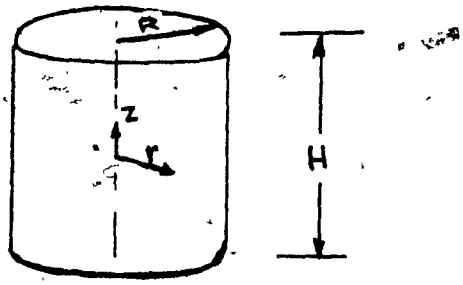


Figure III.12 Finite Cylindrical Reactor Core.

differential equations

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \alpha^2 R(r) = 0 \qquad R(\tilde{R}) = 0 \tag{III.225a}$$

$$\frac{d^2Z}{dz^2} + \lambda^2 Z(z) = 0, \qquad Z\left(\pm\frac{\tilde{H}}{2}\right) = 0 \tag{III.225b}$$

where the separation constants α^2 and λ^2 are constrained by the relationship $B^2 = \alpha^2 + \lambda^2$. Each of these equations represents a separate eigenvalue problem that one can use to determine α and λ (and hence B^2). The eigenfunctions and eigenvalues of the axial equation are well known to us

$$Z_n(z) = \cos\left(\frac{n\pi z}{\tilde{H}}\right), \qquad \lambda_n^2 = \left(\frac{n\pi}{\tilde{H}}\right)^2, \qquad n = 1,3,\ldots \tag{III.226}$$

To construct the eigenfunctions of the radial Eq. (III.225a) we first identify its general solution in terms of zeroth order Bessel functions

$$\boxed{R(r) = AJ_0(\alpha r) + CY_0(\alpha r)} \qquad \text{(III.227)}$$

Since $Y_0(\alpha r) \to \infty$ as $r \to 0$, we must set $C \equiv 0$. Applying our boundary condition at $r = \tilde{R}$, we find

$$R(\tilde{R}) = AJ_0(\alpha\tilde{R}) = 0 \to \alpha\tilde{R} = \nu_n, \qquad \text{(III.228)}$$

where ν_n are the zeros of J_0. In particular, the smallest such zero is $\nu_0 = 2.405 \ldots$ (kind of like π to a Bessel function). Hence we find the eigenfunctions and eigenvalues generated by the radial equation (III.225b) are just

$$R_n(r) = J_0\left(\frac{\nu_n r}{\tilde{R}}\right), \qquad \alpha_n^2 = (\nu_n/\tilde{R})^2, \quad n = 0,1,\ldots \qquad \text{(III.229)}$$

Therefore, consistent with our prescription of seeking the smallest value of B^2 as our geometric buckling, we find

$$B_g^2 = \left(\frac{\nu_0}{\tilde{R}}\right)^2 + \left(\frac{\pi}{\tilde{H}}\right)^2, \qquad \text{(III.230)}$$

corresponding to a spatial flux shape

$$\phi(r,z) = AJ_0\left(\frac{\nu_0 r}{\tilde{R}}\right) \cos\left(\frac{\pi z}{\tilde{H}}\right) \qquad \text{(III.231)}$$

Since this is the geometry most frequently encountered in reactor design, it is useful to calculate the normalization factor A in terms of the core power level P by noting

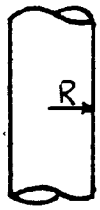
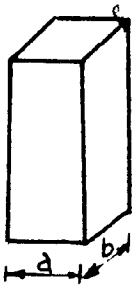
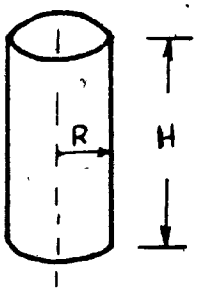
$$P = \int_V d^3r w_f \Sigma_f \phi(r) = w_f \Sigma_f 2\pi A \int_0^{\tilde{R}} dr\, r J_0\left(\frac{\nu_0 r}{\tilde{R}}\right) \int_{-\frac{\tilde{H}}{2}}^{\frac{\tilde{H}}{2}} dz \cos\left(\frac{\pi z}{\tilde{H}}\right) = \frac{w_f \Sigma_f A 4 V J_1(\nu_0)}{\pi \nu_0}. \quad \text{(III.232)}$$

Thus we find

$$A = \frac{3.63P}{w_f \Sigma_f V}, \quad V = \pi \tilde{R}^2 \tilde{H} \quad \text{(III.233)}$$

where w_f is the energy released per fission event.

For convenience we have tabulated the geometric buckling and critical flux profile for most common core geometries.

	Geometric Buckling B_g^2	Flux Profile
	$\left(\frac{\pi}{\tilde{a}}\right)^2$	$\cos\frac{\pi x}{\tilde{a}}$
	$\left(\frac{\nu_0}{\tilde{R}}\right)^2$	$J_0\left(\frac{\nu_0 r}{\tilde{R}}\right)$
	$\left(\frac{\pi}{\tilde{R}}\right)^2$	$r^{-1}\sin\left(\frac{\pi r}{\tilde{R}}\right)$
	$\left(\frac{\pi}{\tilde{a}}\right)^2+\left(\frac{\pi}{\tilde{b}}\right)^2+\left(\frac{\pi}{\tilde{c}}\right)^2$	$\cos\left(\frac{\pi x}{\tilde{a}}\right)\cos\left(\frac{\pi y}{\tilde{b}}\right)\cos\left(\frac{\pi z}{\tilde{c}}\right)$
	$\left(\frac{\nu_0}{\tilde{R}}\right)^2+\left(\frac{\pi}{\tilde{H}}\right)^2$	$J_0\left(\frac{\nu_0 r}{\tilde{R}}\right)\cos\left(\frac{\pi z}{\tilde{H}}\right)$

III.7 MULTIGROUP DIFFUSION THEORY

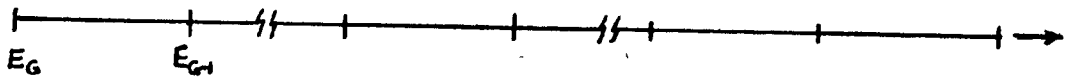
The one speed diffusion theory which has been derived in previous sections, certainly suffices to introduce most of the important concepts of reactor analysis as well as many of the computational methods used in modern reactor design. It can even be used on occasion to provide useful qualitative information such as in preliminary survey design studies. However for most of the problems encountered in practical nuclear reactor design the one-speed diffusion model is simply not adequate.

Two very significant assumptions were made in deriving the one-speed diffusion model. We first assumed that the angular flux was only weakly dependent on angle (linearly anisotropic, in fact) so that the diffusion approximation was valid. Usually this assumption is reasonably well satisfied in large power reactors provided we take care to modify the analysis a bit in the vicinity of strong absorbers, interfaces, and boundaries to account for transport effects.

The principal deficiency of the model is the assumption that all of the neutrons can be characterized by only a single speed or energy. As we have seen, the neutrons in a reactor have energies spanning the range from 10MeV down to less than 0.01 eV - some nine orders of magnitude. Furthermore, we have noted that neutron-nuclear cross sections depend rather sensitively on the incident neutron energy. Hence it is not surprising that practical reactor calculations will require a more realistic treatment of the neutron energy dependence. (Indeed it is

surprising that the one-speed diffusion equation works at all. Its success depends on a very judicious choice of the one-speed cross sections that appear in the equation).

We will now allow the neutron flux to depend on energy, but rather than treat the neutron energy variable E as a continuous variable, we will immediately discretize it into energy intervals or groups. That is, we will break the neutron energy range into G energy groups, as shown schematically below:



We use a backward indexing scheme, because it corresponds physically to the fact that the neutron usually loses energy during its lifetime (and mathematically to the fact that one always solves the discretized equations starting at high energies and working successively to lower energies).

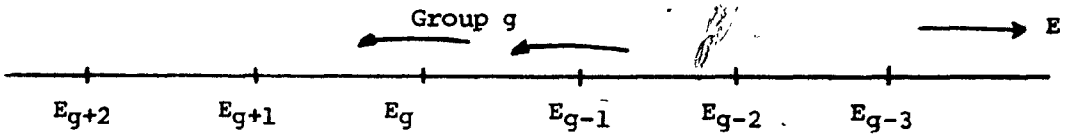
As in our earlier discrete ordinates approach, it would be possible to discretize $\phi(r,E,t)$ by considering it only to be defined at each energy mesh point E_g. However it is more convenient to define the discretized fluxes instead to be the integrals of $\phi(r,E,t)$ over the energies of each group, such that the multigroup fluxes $\phi_g(r,t)$, represent the total flux of all neutrons with energies E in the group $E_g<E<E_{g-1}$. Then our task is to determine equations for $\phi_g(r,t)$. We

will find that these equations take the form of a set of diffusion equations describing the neutrons in each energy group. The equations are coupled to one another since neutrons may experience changes in energy and hence pass from group to group. For example fission neutrons will usually be born in the highest energy groups and then cascade downward in energy from group to group as they are moderated by scattering collisions.

Recalling the rather detailed dependence of neutron cross sections on neutron energy E, one might expect that a great many such energy groups would be necessary to adequately describe a nuclear reactor. Surprisingly enough, however, most nuclear reactor calculations achieve sufficient accuracy using only few-group diffusion descriptions. The ability to describe a reactor adequately with a relatively small number of energy groups is not simply fortuitious, but rather is a consequence of a carefulchoice of the energy-averaged cross sections that characterize the neutrons in each group.

III.7.1 DERIVATION OF THE MULTIGROUP DIFFUSION EQUATIONS

Perhaps the most straightforward manner in which to arrive at the form of the multigroup diffusion equations is to apply the concept of neutron balance to a given energy group by balancing the ways in which neutrons can enter or leave this group. Consider then a typical energy group g:



After a bit of reflection, it should be apparent that such a balance would read as follows:

$$\begin{bmatrix}\text{Time rate of change of neutrons in group g}\end{bmatrix} = -\begin{bmatrix}\text{change due to leakage}\end{bmatrix} - \begin{bmatrix}\text{absorption in group g}\end{bmatrix} + \begin{bmatrix}\text{source neutrons appearing in group g}\end{bmatrix} - \begin{bmatrix}\text{neutrons scattering out of group g}\end{bmatrix} + \begin{bmatrix}\text{neutrons scattering into group g}\end{bmatrix} \qquad \text{(III.234)}$$

It should be noted that we have taken explicit account of the fact that a scattering collision can change the neutron energy and hence either remove it from the group g, or if it is initially in another group g′, scatter it to an energy in the group g. We will characterize the probability for scattering a neutron from a group g′ to the group g by something akin to the differential scattering cross section $\Sigma_s(E_{g'} \to E_g)$ (a so-called group-transfer cross section), $\Sigma_{sg'g}$. Note that the cross section characterizing the probability that a neutron will scatter out of the group g is then given by

$$\Sigma_{sg} = \sum_{g'=1}^{G} \Sigma_{sgg'}. \qquad \text{(III.235)}$$

We will similarly define an absorption cross section characterizing the

group g, Σ_{ag}, and a source term S_g giving the rate at which source neutrons appear in group g. Finally we will define a diffusion coefficient D_g so that the leakage from group g can be written within the diffusion approximation as $\nabla \cdot D_g \nabla \phi_g$. If we combine all of these terms, we find a mathematical representation of the balance relations (Eq. (III.234).

$$\frac{1}{v_g}\frac{\partial \phi_g}{\partial t} = \nabla \cdot D \nabla \phi - \Sigma_{ag}\phi_g + S_g - \Sigma_{sg}\phi_g + \sum_{g'=1}^{G} \Sigma_{sg'}\phi_{g'}, \quad g=1,2,\ldots G. \tag{III.236}$$

If we separate out that component of the source due to fissions, then we can write

$$S_g = \chi_g \sum_{g'=1}^{G} \nu_{g'}\Sigma_{fg'}\phi_{g'} + S_g^{ext} \tag{III.237}$$

where χ_g is the probability that a fission neutron will be born with an energy in group g while $\Sigma_{fg'}$ is the fission cross section characterizing a group g' and $\nu_{g'}$ is the average number of fission neutrons released in a fission reaction induced by a neutron in group g'.

Hence we now have a set of G coupled diffusion equations for the G unknown group fluxes $\phi_g(r,t)$. It shouldn't take much imagination to see that many of the same techniques that we used for the one-speed (or one-group) model will also hold for the G-group system.

The more serious problem concerns just how one determines the group

constants that appear in these equations:

$$\nu_g,\ D_g,\ \Sigma_{ag},\Sigma_{sg},\Sigma_{sg'g},\chi_g,\Sigma_{fg},\nu_g \qquad \text{(III.238)}$$

III.7.2 DERIVATION OF THE MULTIGROUP EQUATIONS FROM ENERGY-DEPENDENT DIFFUSION THEORY

Perhaps the most satisfying manner in which to derive the multigroup diffusion equations characterizing the average behavior of neutrons in each energy group is to integrate (i.e., average) the equation for the energy-dependent neutron flux, $\phi(r,E,t)$, over a given group, $E_g<E<E_{g-1}$. We will assume that this flux can be adequately described by the energy-dependent diffusion equation:

$$\frac{1}{v}\frac{\partial\phi}{\partial t} - \nabla\cdot D\nabla\phi - \Sigma_t\phi(r,E,t) = \int_0^\infty dE'\Sigma_s(E'\to E)\phi(r,E',t) + \chi(E)\int_0^\infty dE' v(E')\Sigma_f(E')\phi(r,E',t) + S_{ext}(r,E,t) \qquad \text{(III.239)}$$

Notice that we have inserted the explicit form for the fission source developed earlier.

We will begin by eliminating the energy variable in the energy-dependent diffusion equation by integrating Eq. (III.239) over the gth energy group characterized by energies $E_g<E<E_{g-1}$:

$$\frac{\partial}{\partial t}\int_{Eg}^{Eg-1} dE\,\frac{1}{v}\phi - \nabla\cdot\int_{Eg}^{Eg-1} dED\nabla\phi + \int_{Eg}^{Eg-1} dE\Sigma_t\phi =$$

$$-\int_{E_g}^{E_{g-1}} dE \int_0^\infty dE' \Sigma_s(E' \to E)\phi(r,E',t) - \int_{E_g}^{E_{g-1}} dE S. \qquad \text{(III.240)}$$

We will proceed further by making some formal definitions. First define the neutron flux in group g as

$$\phi_g(r,t) \equiv \int_{E_g}^{E_{g-1}} dE \phi(r,E,t) \qquad \text{(III.241)}$$

Next define the total cross section for group g as

$$\Sigma_{tg} \equiv \frac{1}{\phi_g} \int_{E_g}^{E_{g-1}} dE \Sigma_t(E)\phi(r,E,t) \qquad \text{(III.242)}$$

the diffusion coefficient for group g as

$$D_g \equiv \frac{\int_{E_g}^{E_{g-1}} dE\ D\ (E)\nabla_j\phi(r,E,t)}{\int_{E_g}^{E_{g-1}} dE\ \nabla_j\phi(r,E,t)} \qquad \text{(III.243)}$$

and the neutron speed characterizing group g as

$$\frac{1}{v_g} \equiv \frac{1}{\phi_g} \int_{E_g}^{E_{g-1}} dE\ \frac{1}{v}\ \phi(r.E.t) \qquad \text{(III.244)}$$

The scattering term requires a bit more work. If we break up the integral over E' to write

$$\int_{E_g}^{E_{g-1}} dE \int_0^\infty dE' \Sigma_s(E' \to E)\phi(r,E',t)$$

$$= \sum_{g'=1}^{G} \int_{E_g}^{E_{g-1}} dE \int_{E_{g'}}^{E_{g'-1}} dE' \Sigma_s(E' \to E)\ \phi\ (r,E',t) \qquad \text{(III.245)}$$

then it becomes evident that we want to define the group-transfer cross section as

$$\Sigma_{sg'g} \equiv \frac{1}{\phi_{g'}} \int_{E_g}^{E_{g-1}} dE \int_{E_{g'}}^{E_{g'-1}} dE' \Sigma_s(E' \to E)\phi(r,E',t) \qquad \text{(III.246)}$$

A very similar procedure is followed for the fission term by writing

$$\int_{E_g}^{E_{g-1}} dE S_f(r,E,t) = \int_{E_g}^{E_{g-1}} dE \chi(E) \left[\sum_{g'=1}^{G} \int_{E_{g'}}^{E_{g'-1}} dE' \nu(E')\Sigma_f(E')\phi(r,E',t) \right] \qquad \text{(III.247)}$$

and then defining the fission cross section for group g' as

$$\nu_{g'}\Sigma_{fg'} \equiv \frac{1}{\phi_{g'}} \int_{E_{g'}}^{E_{g'-1}} dE' \nu(E')\Sigma_f(E')\phi(r,E',t) \qquad \text{(III.248)}$$

while defining

$$\chi_g = \int_{E_g}^{E_{g-1}} dE \chi(E) \qquad \text{(III.249)}$$

If we now use these purely formal definitions to rewrite Eq.(III.240) we arrive directly at the multigroup diffusion equations:

$$\frac{1}{v_g}\frac{\partial \phi}{\partial t} - \nabla \cdot D_g \nabla \phi + \Sigma_{tg}\phi_g(r,t) = \sum_{g'=1}^{G} \Sigma_{sg'g}\phi_{g'} + \chi_g \sum_{g'=1}^{G} \nu_{g'}\Sigma_{fg'}\phi_{g'} + S_g$$

$$g = 1,2,\ldots,G \qquad \text{(III.250)}$$

Several comments concerning these equations are necessary. The multigroup diffusion equations (III.250) are still quite exact (within the diffusion approximation, that is), but they are also quite formal in the sense that the group constants are as yet undetermined.

The decision of how many groups we need depends on the problem one is considering. For example, in very crude survey calculations of thermal reactors, two groups (one to characterize fast neutrons and the other to characterize thermal neutrons) may be sufficient. Most LWR calculations are performed using a four-group diffusion model (three fast groups, one thermal group) while gas-cooled reactor analysis typically uses seven to nine groups. For fine detail, one may have to go as high as 20 groups (this is particularly true in fast-reactor calculations), and in spectrum calculations, the number of groups can range as high as 1500 (so-called microgroup structure), which is almost as detailed as the tabulated cross section data itself.

Let us look now in a bit more detail at the structure of the multigroup diffusion equation. In particular, consider the scattering term in the equation. Recall that in our study of the kinematics of neutron scattering collisions in Chapter 2, we noted that if the incident neutron energy E was subsequently greater than the thermal energy of the target nuclei (typically less that 0.1 eV), the neutron could never gain energy in a scattering collision. Such "fast" neutrons will only slow down in a scattering collision. Hence in these fast groups, we can set

$$\Sigma_{sg' g} = 0, \text{ for } g' > g$$

Since most few-group diffusion calculations utilize only one thermal group to describe the neutrons with E<1eV (assuming that neutrons cannot scatter up out of the thermal group), we can generally simplify

the scattering term to write

$$\sum_{g'=1}^{G} \Sigma_{sg'g}\phi_{g'} = \sum_{g'=1}^{g-1} \Sigma_{sg'g}\phi_{g'} + \Sigma_{sgg}\phi_g$$

Here we have taken care to separate out the in-group scattering term Σ_{sgg} which characterizes the probability that a neutron can suffer a scattering collision and lose sufficiently little energy that it will still remain within the group. It is customary to transfer this term to the left-hand side of the multigroup equation (III.236) and to define a removal cross section,

$$\Sigma_{Rg} \equiv \Sigma_{tg} - \Sigma_{sgg}$$

which characterizes the probability that a neutron will be removed from the group g by a collision. Note that the removal cross section is sometimes defined such that it does not contain absorption Σ_{ag}. We will use the above definition in our development, however. We will see later that the neglect of upscattering (that is, the assumption that the neutron can never gain or scatter up in energy in a collision) greatly simplifies the solution of the multigroup diffusion equations.

One frequently achieves an additional simplification of the multigroup equations by choosing the group spacing such that neutrons will only scatter to the next lowest group - that is, such that

$$\sum_{g'=1}^{G} \Sigma_{sg'g}\phi_g = \Sigma_{sg-1,g}\phi_{g-1} + \Sigma_{sgg}\phi_g$$

In this case, one refers to the multigroup equations as being directly coupled. (Fig. III.13) If we recall from Chapter II that a neutron of energy E cannot scatter to an energy below αE in a single elastic scattering collision, then it is apparent that to achieve direct coupling we should choose our group spacing such that $E_{g-1}/E_g > 1/\alpha$. For heavier moderators (e.g. ^{12}C), this is easy to do. Unfortunately in hydrogeneous moderators $\alpha_H = 0$, and hence direct coupling cannot be strictly achieved. However, if one chooses $E_{g-1}/E_g > 150$, then the probability of the neutron "skipping" the next lowest group in a scattering collision with hydrogen is less than 1%, and hence direct coupling is effectively achieved.

We will most frequently be concerned with situations in which both the time dependence and the presence of an external source can be ignored (e.g., criticality calculations).

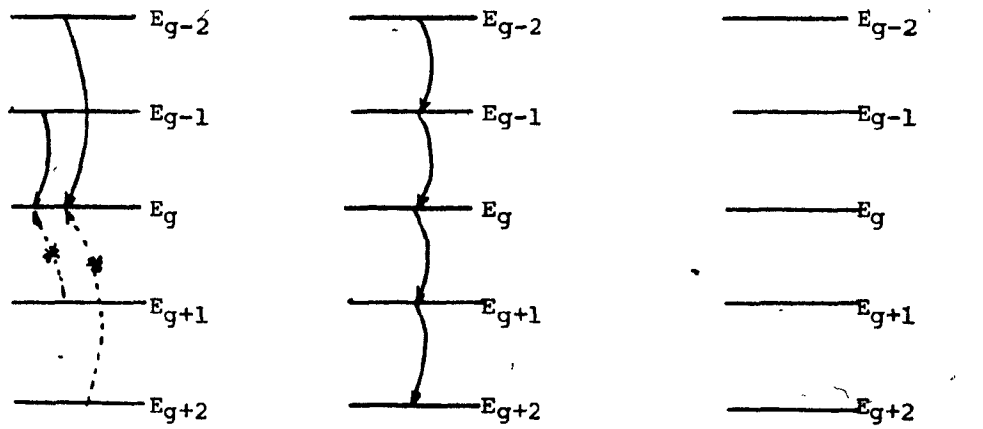


Figure III.13 Multigroup Coupling.

In this case the multigroup equations can be written as

$$-\nabla\cdot D_g \nabla\phi_g + \Sigma_{Rg}\phi_g = \sum_{g'=1}^{g-1} \Sigma_{sg'g}\phi_{g'} + \frac{1}{k}\chi_g \sum_{g'=1}^{G} \nu_{g'}\Sigma_{fg'}\phi_{g'}. \qquad \text{(III.254)}$$

The structure of these equations can be seen more clearly in matrix form.

$$\underbrace{\begin{bmatrix} -\nabla\cdot D_1\nabla+\Sigma_{R_1} & 0 & 0 & \cdots \\ -\Sigma_{s_{12}} & -\nabla\cdot D_2\nabla+\Sigma_{R_2} & 0 & \cdots \\ -\Sigma_{s_{13}} & -\Sigma_{s_{23}} & -\nabla\cdot D_3\nabla+\Sigma_{R_e} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}}_{\underline{\underline{M}}} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \end{bmatrix}}_{\underline{\phi}}$$

$$\underbrace{\begin{bmatrix} \nu_1\chi_1\Sigma_{f_1} & \nu_2\chi_1\Sigma_{f_2} & \cdots \\ \nu_1\chi_2\Sigma_{f_1} & \nu_2\chi_2\Sigma_{f_2} & \cdots \\ \nu_1\chi_3\Sigma_{f_1} & \nu_2\chi_3\Sigma_{f_2} & \cdots \\ \vdots & \vdots & \end{bmatrix}}_{\underline{\underline{F}}} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \end{bmatrix}}_{\underline{\phi}}$$

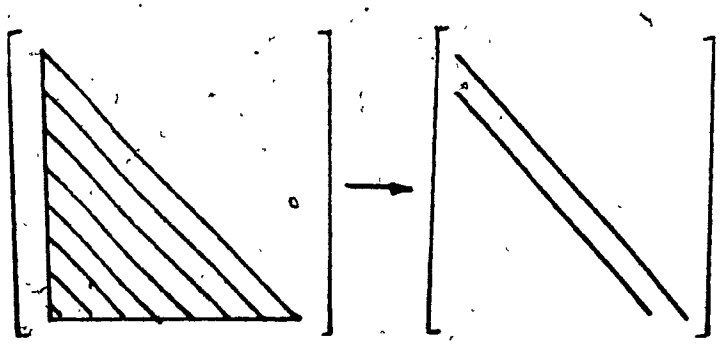
or shortly

$$\underline{\underline{M}}\,\underline{\phi} = \frac{1}{k}\underline{\underline{F}}\,\underline{\phi}$$

where we have inserted the usual criticality eigenvalue k. Notice in particular that the neglect of upscattering has led to a lower triangular form for the "diffusion" matrix M. The fission matrix F is full, however,

since fission neutrons induced by a neutron absorption in a lower group will appear distributed among the higher energy groups.

In the case of directly coupled groups, M becomes a simple bidagonal matrix of the form

$\underline{M} =$ 

III.7.3 SIMPLE APPLICATIONS OF THE MULTIGROUP DIFFUSION MODEL

III.7.3.1 ONE-GROUP DIFFUSION THEORY

First suppose we set up the "one-group" diffusion equation by defining $E_0 = \infty$ and $E_1 = 0$. Then if we note that

$$\int_0^\infty dE\, \chi(E) = 1,$$

and

$$\int_0^\infty dE' \Sigma_s(E \to E') = \Sigma_s(E)$$

1 Group { $E_0 = \infty$; $E_1 = 0$ } (III.256)

we find that the multigroup equations yield an old friend, the one-speed diffusion equation

$$\frac{1}{v}\frac{\partial \phi}{\partial t} - \nabla \cdot D \nabla \phi + \Sigma_a \phi(r,t) = \nu \Sigma_f \phi. \qquad \text{(III.257)}$$

Of course it should be stressed that this equation is still of only formal significance until we provide some prescription for calculating the group constants (that is, the intragroup flux - which, in this case, is $\phi(r,E,t)$). Nevertheless it is comforting to know that if we chose the group constants properly, even one-speed diffusion theory could give an accurate description of nuclear reactor behavior.

III-7.3.2 TWO-GROUP THEORY

The one group theory fails at boundaries of two regions (1) and (2) of the reactor, the core region and the reflector for example.

At an interface between core and reflector material, the net leakage of high-energy neutrons, which are originally created by fission in the core, is from the core to the reflector, whereas the net leakage of low-energy neutrons, created in abundance by the superior moderating power of the reflector material, is in the opposite direction. A one-group model cannot describe this process.

Two-group theory represents an attempt to improve the accuracy with which the flux can be described near such interfaces. The basic idea is to split the energy spectrum into two rauges:
The thermal spectrum and the fast spectrum. If for thermal spectrum usually it is choosen the energy range 0-lev while for fast spectrum the rauge lev-∞, then we can identify

$$\phi_1(r,t) = \int_{E_1}^{E_0} dE\phi(r,E,t) \equiv \text{fast flux}, \qquad \text{III-258}$$

$$\phi_2(r,t) = \int_{E_2}^{E_1} dE\phi(r,E,t) \equiv \text{thermal flux}. \qquad \text{III-259}$$

We can simplify the group constants for this model somewhat. Consider first the fission spectrum. Since essentially all fission neutrons are born in the fast group

$$\boxed{\chi_1 = \int_{E_1}^{E_0} dE\chi(E) = 1, \quad \chi_2 = \int_{E_2}^{E_1} dE\chi(E) = 0.} \qquad \text{III-260}$$

Hence the fission source will only appear in the fast group equition:

$$\boxed{S_{f1} = \nu_1\Sigma_{f1}\phi_1 + \nu_2\Sigma_{t2}\phi_2} \qquad \text{III-261}$$

$$\boxed{S_{f2} = 0} \qquad \text{III-262}$$

We can proceed to calculate the scattering and removal cross section.

First since there is no slowing down out of the thermal group,

$$\int_{E_2 = 0}^{E_1 \sim 1eV} dE\Sigma_s(E'\to E) = \Sigma_s(E'), \quad E_2 < E < E_1 \qquad \text{III-263}$$

Hence we find

$$\Sigma_{s_{22}} = \frac{1}{\phi_2}\int_{E_2}^{E_1} dE \int_{E_2}^{E_1} dE'\Sigma_s(E'\to E)\phi(r,E') = \frac{1}{\phi_2}\int_{E_2}^{E_1} dE'\Sigma_s(e')\phi(r,E') = \Sigma_{s_2} \qquad \text{III-264}$$

Thus the removal cross section for the thermal group is just

$$\Sigma_{R_2} = \Sigma_{t_2} - \Sigma_{s_{22}} = \Sigma_{t_2} - \Sigma_{s_2} = \Sigma_{a_2}, \qquad \text{III-265}$$

as we might have expected. The remainder of the group constants are defined as before in the previous section. In practice they would be calculated by first performing a fine spectrum calculation for the group of interest, and then averaging the appropriate cross section data over this spectrum to obtain the group constants.

We will consider the application of two-group diffusion theory to a reactor criticality calculation. Then we can set both the time derivatives and the external source terms equal to zero to write the two-group diffusion equation as

$$\boxed{\begin{aligned} -\nabla . D_1 \nabla \phi_1 + \Sigma_{R_1} \phi_1 &= \frac{1}{k} \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2, \\ -\nabla . D_2 \nabla \phi_2 + \Sigma_{a_2} \phi_2 &= \Sigma_{s_{t2}} \phi_1 \end{aligned}} \qquad \text{III-266}$$

Notice that we have inserted a multiplication factor (1/k) in front of the fission source term since we are eventually going to be performing a criticality search. Also notice that while the source terms in the fast group correspond to fission neutrons, the source term in the thermal group is due only to slowing down from the fast group.

As a specific illustration, we will apply the two-group diffusion equations, (Eq. III-266), to analyze the criticality of a bare, uniform reactor assuming that both fast and thermal fluxes can be characterized by the same spatial shape $\tilde{\psi}(r)$:

$$\nabla^2 \psi + B^2 \psi(r) = 0, \; \psi(r_s) = 0. \qquad \text{III-267}$$

We have omitted the subscript g from the geometric buckling $B_g^2 = B^2$ so as not to confuse it with the group index g. Then if we substitute

$$\phi_1(r) = \phi_1\psi(r), \qquad \phi_2(r) = \phi_2\psi(r) \qquad \text{III-268}$$

into (Eq. III-266), we find the algebraic equations

$$(D_1B^2 + \Sigma_{R_1} - K^{-1}\nu_1\Sigma_{f_1})\,\phi_1 - k^{-1}\nu_2\,\Sigma_{f_2}\phi_2 = 0, \qquad \text{III-269}$$

$$-\Sigma_{12}\phi_1 + (D_2B^2 + \Sigma_{a_2})\,\phi_2 = 0.$$

However this algebraic system has a solution if and only if

$$(D_1B^2 + \Sigma_{R_1} - \frac{\nu_1\Sigma_{f_1}}{K})\,(D_2B^2 + \Sigma_{a_2}) - \frac{\nu_2\Sigma_{f_2}\Sigma_{s_{12}}}{K} = 0 \qquad \text{III-270}$$

We can now solve for the value of the multiplication factor k, which will yield a nontrivial solution of the two-group equations:

$$k = \frac{\nu_1\Sigma_{f_1}}{\Sigma_{R_1} + D_1B^2} + \frac{\Sigma_{s_{12}}}{(\Sigma_{R_1} + D_1B_2)}\,\frac{\nu_2\Sigma_{f_2}}{\Sigma_{a_2} + D_2B^2} \qquad \text{III-271}$$

It is of interest for us to see if we can relate this expression to our earlier expressions for k-notably the six-factor formula. First notice that the first term in (Eq. III-271) represents neutron multiplication due to fissions occurring in the fast group, whereas the

second term represents multiplication due to thermal fission. Since we expect the thermal fission contribution to be dominant in those situations in which such a two-group analysis makes sence, let us first examine

$$k_2 = \frac{\Sigma_{s_{12}}}{(\Sigma_{R_1} + D_1 B^2)} \frac{\nu_2 \Sigma_{f_2}}{(\Sigma_{a_2} + D_2 B^2)}$$

$$= \frac{\Sigma_{s_{12}}/\Sigma_{R_1}}{(1 + L_1^2 B^2)} \frac{\nu_2 (\Sigma_{f_2}/\Sigma_{a_2})}{(1 + L_2^2 B^2)} . \qquad \text{III-272}$$

From previous discussions, it is evident that

$$P_{NL_1} = (1 + L_1^2 B^2)^{-1}, \quad P_{NL_2} = (1 + L_2^2 B^2)^{-1} \qquad \text{III-273}$$

are just the fast and thermal nonleakage probabilities. Notice that the diffusion length L_1 characterizing the fast group is defined somewhat differently as

$$L_1^2 \equiv \frac{D_1}{\Sigma_{R_1}} = \frac{D_1}{\Sigma_{a_1} + \Sigma_{s_{12}}} \qquad \text{III-274}$$

but this is consistent with our earlier definition of the diffusion length, since both Σ_a and $\Sigma_{s_{12}}$ act to remove neutrons from the fast group. The only unidentified term is the ratio $\Sigma_{s_{12}}/\Sigma_{R_1}$. However for a homogeneous reactor we know that this ratio is just:

$$\frac{\text{Rate at which neutrons slow down to thermal group}}{\text{Rate at which neutrons are removed from fast group}} = \frac{\int d^3r \Sigma_{s_{12}} \phi_1(r)}{\int d^3r \Sigma_{R_1} \phi_1(r)} = \frac{\Sigma_{s_{12}}}{\Sigma_{R_1}} = p, \qquad \text{III-275}$$

which we can identify as the resonance escape probability p characterizing slowing down from group 1 to group 2. Hence

$$k_2 = \eta_2 f_2 p P_{NL_1} P_{NL_2}. \qquad \text{III-276}$$

In a very similar manner we can identify that fast multiplication factor as

$$k_1 = \frac{\nu_1 \Sigma_{f_1} / \Sigma_{R_1}}{(1 + L_1^2 B^2} = \eta_1 f_1 P_{NL_1}, \qquad \text{III-277}$$

where $\eta_1 = \nu_1 \Sigma^F_{f_1} / \Sigma^F_{a_1}$ and we have defined a "fast utilization factor" $f_1 = \Sigma^F_{a_1} / \Sigma^F_{R_1}$ in analogy to the thermal utilization f_2.

To complete our identification with the usual six-factor formula, we evidently must identify the fast fission factor ϵ as just

$$\epsilon = (1 + \frac{k_1}{k_2}) = \left[1 + \left(\frac{\nu_1 \Sigma_{f_1}}{\nu_2 \Sigma_{f_2}}\right)\left(\frac{\Sigma_{a_2} + D_2 B^2}{\Sigma_{s_{12}}}\right)\right] \qquad \text{III-278}$$

Then we find

$$k = k_1 + k_2 = \epsilon k_2 = \eta_2 f_2 p \epsilon P_{NL_1} P_{NL_2} =$$

$$= \eta_{th} f_{th} p \epsilon P_{FNL} P_{TNL}, \qquad \text{III-279}$$

The two-group diffusion model can be used to demonstrate a number of the various applications of the multigroup formalism. For example, one frequently wishes to generate the group constants for a few-group calculation using the neutron spectrum generated by a many-group calculation. Such a procedure is known as group collapsing, since it expresses few-group constants in terms of many-group constants.

To illustrate this we can derive expressions for the one-group constants in terms of two-group constants. For example,

$$\Sigma_a = \frac{\int_{E_2}^{E_o} dE\Sigma_a(E)\phi(E)}{\int_{E_2}^{E_o} d\Sigma\phi(E)} = \frac{\int_{E_1}^{E_o} dE\Sigma_a(E)\phi(E) + \int_{E_2}^{E_1} dE\Sigma_a(E)\phi(E)}{\int_{E_1}^{E_o} dE\phi(E) + \int_{E_2}^{E_1} dE\,\phi(E)} \qquad \text{III-280}$$

$$= \frac{\Sigma_{R_1}\phi_1 + \Sigma_{a_2}\phi_2 - \Sigma_{s_{12}}\phi_1}{\phi_1 + \phi_2},$$

or using (Eq. III-269) to eliminate ϕ_2 in terms of ϕ_1:

$$\Sigma_a = \frac{(\Sigma_{R_1} - \Sigma_{s_{12}})(D_2B^2 + \Sigma_{a_2}) + \Sigma_{a_2}\Sigma_{s_{12}}}{D_2B^2 + \Sigma_{a_2} + \Sigma_{s_{12}}} \qquad \text{III-281}$$

The remaining one-group constants can be given as

$$= \frac{D_1\phi_1 + D_2\phi_2}{\phi_1 + \phi_2} \qquad \frac{(D_2B^2 + \Sigma_{a_2})D_1 + \Sigma_{s_{12}}D_2}{D_2B^2 + \Sigma_{a_2} + \Sigma_{s_{12}}}, \qquad \text{III-282}$$

$$\nu\Sigma_f = \frac{\nu_1\Sigma_{f_1}\phi_1 + \nu_2\Sigma_{f_2}\phi_2}{\phi_1 \quad \phi_2} = \frac{(D_2B^2 + \Sigma_{a_2})\nu_1\Sigma_{f_1} + \nu_2\Sigma_{s_{12}}\Sigma_{f_2}}{D_2B^2 + \Sigma_{a_2} + \Sigma_{s_{12}}} \qquad \text{III-283}$$

III.8 KINETIC EQUATION

Reactor dynamics is concerned with time behavior of neutrons in an arbitrary medium whose nuclear and geometric properties may change with time. In obtaining the kinetic equations, we shall present the description of various physical phenomena influencing the temporal behavior of neutrons in detail to provide an adequate understanding of the physics of reactor dynamics, and to point out the relationships between various phenomena in a sufficiently precise manner.

The basic equation of deterministic reactor kinetics may be obtained by simply extending the basic flux equation to the time dependent case.

However, since we shall be concerned exclusively with the diffusion theory approximation we shall immediately apply Fick's Law, and base the analysis on the continuous energy diffusion equation (III.45). We must then incorporate into this equation the fact that the neutron density may be changing with time and that some neutrons produced as the result of fission events may be delayed. Delayed neutrons come from certain fission fragments "precursors" which, after a β-decay, happen to be unstable with respect to the number of neutrons in their nuclei. Once the β-decay occurs, a neutron appears essentially instantaneously. To determine the rate of neutron emission from this source it is necessary to keep account of the concentration of all the "delayed neutron precursors" which, when they emit a β particle, become neutron-unstable. If the concentration of the i-th precursor at location r in the reactor is

$C_i(r,t)$ (precursor nuclei/cc)

and its time constant for β-decay is λ_i (sec^{-1}), we have:

Total number of delayed neutrons emitted per cc per sec at point $\vec{r}$ and time t

$$= \sum_{i=1}^{I} \lambda_i C_i(r,t) \qquad \text{(III.284)}$$

Where the sum is over all the different fission fragments that are delayed-neutron emitters. There are about twenty of these, but most of them have such a small yield that it is possible to fit the data by assuming I=6. The delayed emitters are created by fission. Thus, if β_i^J is the fraction of the neutrons from a fission of isotope J (U^{235}, P_u^{249}, U^{238}, etc.) that eventually appear from the decay of precursor i, we have:

$$\sum_J \beta_i^J \int_0^\infty \nu^J \Sigma_f^J(r,E,t)\, \Phi(r,E,t)\, dE \qquad \text{(III.285)}$$

= Number of delayed precursors of the i-th kind created per cc per sec at r and t.

It follows that the rate of change of Ci (r,t) for fuel that is stationary in space is given by:

$$\boxed{\frac{\partial Ci(r,t)}{\partial t} = \sum_J \beta_i^J \int_o^\infty \nu^J \Sigma_f^J (r,E,t)\phi(r,E,t)\, dE - \lambda_i\, Ci(r,t)} \qquad \text{(III.286)}$$

The migration of the precursor before the emission of a delayed neutron can be neglected in solid-fuel reactors, because they lose their kinetic energy very rapidly as a consequence of their large electric charge. They are stopped within a short distance from the point of their formation by fission. It is therefore a good approximation to assume that in a solid-fuel reactor, which we shall mainly be concerned with (CANDU type for example), the delayed neutrons as well as the prompt neutrons are produced at the same point in the space where the fission event takes place.

Delayed neutrons are emitted with energies that are, on the average, lower than those of prompt neutrons. Designating the spectrum of emission energies for the i-th precursor by $\chi_i(E)$ we have from (III.284).

Rate at which neutrons appear in dVdE because of the decay of delayed precursors

$$= \sum_i \chi_i(E)\ \lambda_i\ C_i(r,E)\ dVdE \qquad \text{(III.287)}$$

Designating the spectrum of emission energies for the prompt neutrons from isotope J by $\chi_p^J(E)$ we also have:

Rate at which prompt neutrons appear in dVdE =

$$\{\sum_J \chi_p^J (E)\ (1-\beta^J) \int_0^\infty \nu \Sigma_f^{J'J} (\vec{r},E,t)\phi(r,E,)\} \times dVdE \quad \text{(III.288)}$$

where β^J is the total yield of delayed precursors from isotope J;

$$\beta^J \equiv \sum_{i=1}^{I} \lambda_i^J \beta_i^J$$

If, for the sake of completeness, we also assume that there is an additional neutron source (usually referred to as the "external" source and taken to include, for example, neutrons from the interaction of radium γ rays with beryllium or of deuterons with deuterium in an accelerator) which isotropically emits $q(r,E,t)$ dVdE neutrons per sec. into dEdV; then we can extend the continuous-energy diffusion equation (III.45) to the time dependent case by expressing mathematically the fact that

> "The net rate of increase of neutrons in dVdE is the difference between their rate of appearance and their rate of disappearance."

$$\frac{d}{dt}\left[\frac{1}{v}\,\phi(r,E,t)\,dV dE\right] =$$

$$\left\{\sum_J \chi_p^J(E)\,(1-\beta^J)\int_0^\infty \nu^J \Sigma_f^J(r,E',t)\phi(r,E',t)\,dE'\right\} dV dE$$

$$+ \sum_i \chi_{d_i}(E)\lambda_i\, Ci\,(r,t)\,dV\,dE - q(r,E,t)\,dV dE$$

$$+ \nabla\cdot D\,(r,E,t)\nabla\phi(r,E,t)\,dV\,dE - \Sigma_t(r,E,t)\phi(r,E,t)\,dV dE$$

$$- \left\{\int_0^\infty \Sigma_s(r,E'\to E,t)\phi(r,E',t)\,dE'\right\} dE dV$$

(III.289)

If we introduce two integral operators A and F^j defined through their operation on any function f(r,E,t) by:

$$A_f = \Sigma_t(r,E,t)\, f(r,E,t) - \int_0^\infty \Sigma_s(r,E'\to E,t) f(r,E',t)\, dE' \qquad \text{(III.290)}$$

$$F^j f = \int_0^\infty \nu^j \Sigma_f^j(r,E',t)\, f(r,E',t)\, dE' \qquad \text{(III.291)}$$

then the time-dependent continuous-energy diffusion equation and the rate of change of Ci(r,t) for a reactor in which the fuel is stationary become:

1. $$\nabla\cdot D(r,E,t)\nabla\phi(r,E,t) - Ad(r,E,t) + 9(r,E,t) + \sum_J \chi_p^j(E)\,(1-\beta^j)\, F^j\,\phi(r,E',t) + \sum_{i=1}^{I} \chi_i(E)\lambda_i\, Ci\,(r,t) = \frac{1}{v}\,\frac{\partial\phi(r,E,t)}{\partial t} \qquad \text{(III.292)}$$

2. $$\sum_J \beta_i^j\, F^j\phi(r,E',t) - \lambda_i\, Ci(r,t) = \frac{\partial Ci(r,t)}{\partial t} \qquad \text{(III.293)}$$

which constitute the basis for the development of reactor kinetics.

III.8.1 THE POINT KINETIC EQUATIONS

Since mean free paths are fairly long and since the lifetimes of neutrons in a nuclear reactor are quite short, the effects of local perturbations on $\Phi(r,E,t)$ will quickly spread throughout a reactor. The immediate consequence of perturbing a reactor locally (for example by changing a control-rod position slightly) is thus a readjustment in the shape of the flux. In many cases this readjustment is slight and is completed in a few milliseconds; after that the readjusted shape rises or falls as a whole depending on whether the initial perturbation increased or decreased k_{eff}. For reactors in which transients proceed in this manner, merely being able to predict the change in the level (i.e. average value) of the flux is sufficient to permit a very accurate prediction of the consequences of perturbation. Thus, instead of having to face the very difficult problem of solving the time-dependent continuous energy diffusion equation in full detail, we shall find a simple set of equations that specify how the overall magnitude of the flux changes with time. A measure of the level of the flux is the integral of $\Phi(r,E,t)$ over all energy and over the reactor volume. However to conform with convention, and because it is conceptually simpler, we shall deal with the integral of the number density $\frac{1}{v(E)}\Phi(r,E,t)$. Weighting this number density with a function $W(r,E)$ defined over the same spatial and energy domain as flux $\Phi(r,E,t)$ and independent of time, we introduce a quantity $T(t)$, sometimes called amplitude function defined as:

$$T(t) = \int_V dv \int_0^\infty de\ W(r,E)\ \frac{\eta}{\nu(E)}\ \phi(r,E,t) \qquad \text{(III.294)}$$

Clearly T(t) is a weighted integral of the total number of neutrons present in the reactor at any time. (If W(r,E) = 1, T(t) is exactly the total number of neutrons present).

To derive equations for the amplitude function we simply multiply the first equation of the time-dependent c-e-d-eq'n by W(r,t) and the second by $\chi_i(E)$ W(r,E) and integrate them both over energy and volume. It is convenient, however, to make a clearer distinction between the shape of the flux and its amplitude. Accordingly we define a "shape function" S(r,E,t) by:

$$S(r,E,t) \equiv \frac{\phi(r,E,t)}{T(t)} \qquad \text{(III.295)}$$

and substitute the product S(r,E,t) T(t) rather than $\phi(r,E,t)$ into the time dependent c-e-d equation. Also to conform to convention, we add and substract the term,

$$\sum_J \sum_{i=1}^{I} \chi_i(E)\beta_i^j F^j \phi(r,E,t) \qquad \text{(III.296)}$$

The result of this substitution and of the weighting and integrating procedure is then:

$$\int_{\text{reactor}} dv \int dE\, W(r,E)\left[\nabla\cdot D\,(r,E,t)\cdot\nabla S(r,E,t) - AS(r,E,t)\right.$$

$$\left. + \sum_{J}\left\{\chi_p^J(E)\,(1-\beta^J) + \sum_{i=1}^{I}\chi_i(E)\,\beta_i^J\right\} F^J\cdot S(r,E,t)\right] T(t)$$

$$= \left\{\int_{\text{reactor}} dv\int dE\, W(r,E)\sum_{i,J}\chi_i^J(E)\,\beta_i^J F^J S(r,E,t)\right\} T(t)$$

$$+ \sum_{i=1}^{I}\lambda_i\left\{\int_{\text{reactor}} dv\int dE\, W(r,E)\chi_i(E)\,Ci_i(r,E)\right\} +$$

$$+\int_{\text{reactor}} dv\int dE\, W(r,E)\, q\,(r,E,t) = \frac{\partial}{\partial t}\left\{\int_V dv\int dE\, W(r,E)\,\frac{1}{v}\,S(r,E,t)\,T(t)\right. \qquad \text{(III.297)}$$

$$\{\int_V dv \int_0^\infty dE\ W(r,E)\ \chi_i(E) \sum_J \beta_i^j F^j\ S(r,E,t)\}\ T(t)$$

$$-\lambda_i \{\int_{reactor} dv \int_0^\infty dE\ W(r,E)\ \chi_i(E)\ Ci(r,E)\}$$

$$= \frac{\partial}{\partial t} \{\int_{reactor} dv \int_0^\infty dE\ W(\vec{r},E)\ \chi_i(E)\ Ci(r,E)\} \qquad \text{(III.298)}$$

Now because of

$$T(t) \equiv \int_{v \text{ reactor}} dv \int_0^\infty dE\ W(r,E) \frac{1}{\nu(E)}\ \phi(r,E,t) \qquad \text{(III.299)}$$

and

$$S(r,E,t) \equiv \frac{\phi(r,E,t)}{T(t)} \qquad \text{(III.300)}$$

$$T(t) = \int_v dv \int_0^\infty dE\ W(r,E) \frac{1}{\nu(E)}\ \phi\ (r,E,t)$$

$$= \left\{ \int_v dv \int_0^\infty dE\ W(r,E,t) \frac{1}{\nu(E)}\ S(r,E,t) \right\} T(t) \qquad \text{(III.301)}$$

$$\int_v dv \int_0^\infty dE\ W(r,t) \frac{1}{\nu(E)}\ S(r,E,t) = 1 \qquad \text{(III.302)}$$

for all t

That is, the normalization of the (time-dependent) shape function is such that the integral (III.302) is a constant, independent of time.

It follows that we may rearrange the RHS of time-dependent c-e-d-eq'n as:

$$\frac{\partial}{\partial t}\left\{\int dv \int dE\ W(r,E)\frac{1}{\nu(E)}\ S(r,E,t)\ T(t)\right.$$

$$= \left\{\int dv \int dE\ W(r,E)\frac{1}{\nu(E)}\ S(r,E,t)\right\}\ \frac{d\ T(t)}{dt} \qquad \text{(III.303)}$$

$$\frac{\partial}{\partial t}\left\{\int dv \int dE\ W(r,E)\ \chi_i\ (E)\ Ci(r,t)\right\} =$$

$$= \left\{\int dv \int dE\ W(r,E)\frac{1}{\nu(E)}\ S(r,E,t)\right\}\ x$$

$$x\ \frac{d}{dt}\left\{\frac{\int dv \int dE\ W(r,E)\chi_i\ (E)\ Ci(r,E)}{\int dv \int dE\ W(r,E)\frac{1}{\nu(E)}\ S(r,E,t)}\right. \qquad \text{(III.304)}$$

Then if we divide the time dependent c-e-d-eq'n by

$$\int dv \int dE\ W(r,E)\ \sum_j \left\{\chi_p^j\ (E)\ (1-\beta^j) + \sum_{i=1}^{I} \chi_i(E)\ \beta_i^j\right\} F^j\ S\ (r,E,t) \qquad \text{(III.305)}$$

and define:

$$e(t) \equiv \frac{\int dv \int dE\ W\{\nabla\cdot D\ \nabla S - AS + \sum_j\ (\chi_p^j\ \{1-\beta^j) + \Sigma_i \chi_i \beta_i^j\}\ F^j S\}}{\int dv \int dE\ W\ \sum_j \{\chi_p^j\ (1-\beta^j) + \Sigma_i \chi_i \beta_i^j\}\ F^j S} \qquad \text{(III.306)}$$

$$\beta_i(t) \equiv \frac{\int dv \int dE\ W\ \Sigma_j\ \chi_i\ \beta_i^j\ F^j S}{\int dv \int dE\ W\ \sum_j \{\chi_p^j\ (1-\beta^j) + \sum_i\ \chi_i\ \beta_i^j\}\ F^j S} \qquad \text{(III.307)}$$

$$\beta(t) \equiv \sum_{i=1}^{I} \beta_i(t) \qquad \text{(III.308)}$$

$$\Lambda(t) \equiv \frac{\int dv \int dE\ W \frac{1}{v} S}{\int dv \int dE\ W \sum_J \{\chi_p^j (1-\beta^j) + \sum_i \chi_i \beta_i^j\} F^j S} \tag{III.309}$$

$$Q(t) \equiv \frac{\int dv \int dE\ W_9}{\int dv \int dE\ W \frac{1}{v} S} \tag{III.310}$$

and

$$Ci(t) \equiv \frac{\int dv \int dE\ W \chi i\ Ci(r,t)}{\int dv \int dE\ W \frac{1}{v} S} \tag{III.311}$$

Where all the space, energy and time dependency have been suppressed, we obtain

$$(p - \beta)\ T(t) + \Lambda \sum_{i=1}^{I} \lambda i\ Ci\ (t) + \Lambda Q(t) = \Lambda \frac{dT(t)}{dt} \tag{III.312}$$

$$\beta i\ T(t) - \lambda i \Lambda Ci(t) = \Lambda \frac{dCi(t)}{dt} \tag{III.313}$$

or

$$\boxed{\frac{dT(t)}{dt} = \frac{\rho - \beta}{\Lambda}\ T(t) + \sum_{i=1}^{I} \lambda i\ Ci(t) + Q(t)} \tag{III.314}$$

$$\boxed{\frac{dCi(t)}{dt} = \frac{\beta i}{\Lambda}\ T(t) - \lambda i\ Ci(t)} \tag{III.315}$$

$$(i = 1,2,3,\ldots I)$$

The above equations are called "Point Kinetics Equations". They form the basis of almost all transient design analysis performed for reactors in operation today.

They are derived without making any approximations. Therefore, if we use the values of the kinetics parameters $\rho(t)$, $\beta i(t)$, $Q(t)$, $\Lambda(t)$, $Ci(t)$ as defined, the solution of the point kinetics equations for $T(t)$ will be exactly the same as if it had solved the time-dependent continuous energy diffusion equation for $\Phi(r,E,t)$ and applied the definition

$$T(t) \equiv \int_V dv \int_0^\infty dE\ W(r,E)\ \frac{1}{V(E)}\ \Phi(r,E,t)$$

However, since we must know $\Phi(r,E,t)$ to get the kinetic parameters, the pratical utility of the kinetics equations depends on our ability to obtain a reasonable accurate value of $S(r,E,t)$ without actually solving the time-dependent c-e-d-eq'n.

CHAPTER IV

IV.0 FUEL DEPLETION ANALYSIS

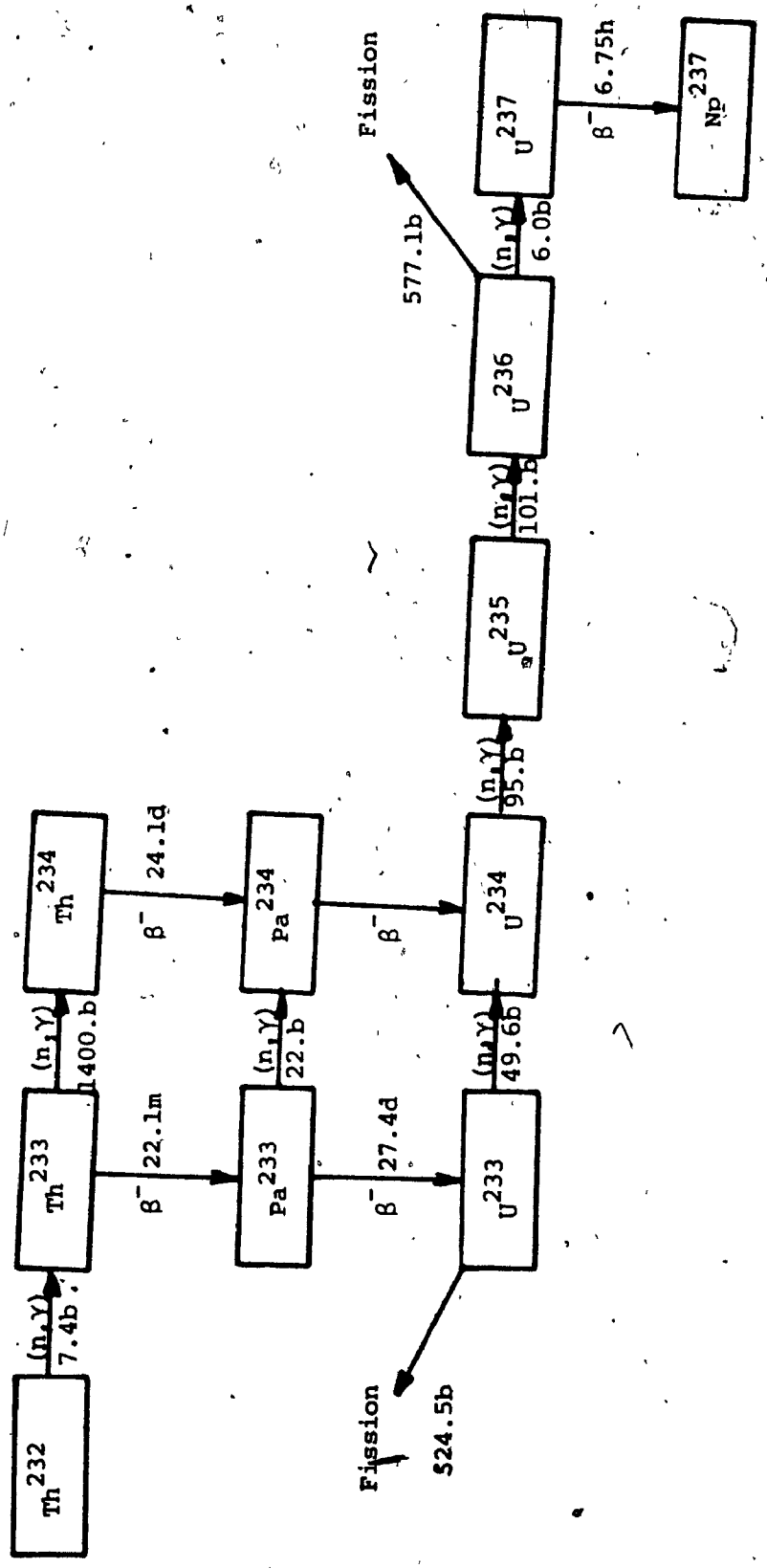
IV.1 INTRODUCTION

To sustain a power level of one watt in a reactor requires that there be approximately 3×10^{10} fission per second. Since only about 80 percent of the neutrons absorbed in fuel actually lead to fission, we conclude that the destruction rate of fissionable material in a reactor operating at a power level of one watt is approximately 3.75×10^{10} nuclei per second. Some simple arithmetic leads to the conclusion that a thousand-megawatt (electric) power plant operating at a thermal efficiency of 30 percent burns fissionable material at a rate of approximately 0.8 gm/min. Compared with, say, the fuel burned to get a jet transport into the air, this figure is astonishingly low. Nevertheless it implies that, in a year of such power operation, a nuclear reactor will burn about 1.5 metric tonnes of fissionable material.

Fuel depletion analysis is concerned with predicting the long-term changes in reactor fuel composition caused by exposure to neutron flux during reactor operation. Such changes have an important bearing on the operating life of a reactor, as well as on its stability and control. One must first ensure that the shift in the core power distribution that accompanies fuel burnup does not result in the exceeding of core thermal limitations. Sufficient excess reactivity must be provided in the fresh core loading to achieve the desired fuel exposure, and, of course, a detailed analysis of core composition is necessary in order to optimize fuel exposure to achieve minimum power costs as well as to determine the value of discharged fuel.

A variety of nuclear processes must be monitored during a depletion study. These include, of coures, the consumption of fissile nuclides (fuel burnup). However, one must also account for the conversion of fertile isotopes into fissile isotopes and the production of numerous fission products (Chapter V). The fuel depletion analysis is normally performed in three steps.

First we shall consider how the concentrations of nuclear materials are changed in the presence of a neutron flux. Next we shall examine how to translate these changes into the computation of energy-group parameters. Finally we shall discuss problems of reactor design and control arising from depletion and its consequences.



(Figure IV-2) The chain of isotopes created by neutron irradiation of Th^{232}.

atomic number of the nucleus and the second being the last digit of the mass number, the concentrations of the isotopes of interest change according to the equations

$$\frac{\partial n^{24}}{\partial t} = -\sigma_a^{24}\phi_1 n^{24}, \qquad \text{IV-7}$$

$$\frac{\partial n^{25}}{\partial t} = \sigma_\gamma^{24}\phi_1 n^{24} - \sigma_a^{25}\phi_1 n^{25}, \qquad \text{IV-8}$$

$$\frac{\partial n^{26}}{\partial t} = \sigma_\gamma^{25}\phi_1 n^{25} - \sigma_a^{26}\phi_1 n^{26}, \qquad \text{IV-9}$$

$$\frac{\partial n^{28}}{\partial t} = \sigma_a^{28}\phi_1 n^{28}, \qquad \text{IV-10}$$

$$\frac{\partial n^{29}}{\partial t} = \sigma_\gamma^{28}\phi_1 n^{28} - (\sigma_a^{29}\phi_1 + \lambda^{29})n^{29}, \qquad \text{IV-11}$$

$$\frac{\partial n^{39}}{\partial t} = \lambda^{29} n^{29} - (\sigma_a^{39}\phi_1 + \lambda^{39})n^{39}, \qquad \text{IV-12}$$

$$\frac{\partial n^{49}}{\partial t} = \lambda^{39} n^{39} - \sigma_a^{49}\phi_1 n^{49} \qquad \text{IV-13}$$

It is important in these equations to distinguish the σ_a^j, defined in (IV-2), which include fission, from the σ_γ^j which are cross sections for the (n,γ) reactions only. (For example σ_a^{25} is about seven times greater than σ_γ^{25}). It should also be kept in mind that the σ_a^j in (IV-6, -13), are one-group cross sections defined by (IV-6); they are only approximately equal to the 0.025-eV values displayed on the chain diagrams. In fact, for isotopes that absorb strongly in the resonance range, they may be several times larger than the numbers shown on the chain diagrams.

Nevertheless we can get a rough idea of the magnitudes of the one-group numbers σ_a^j by using the 0.025-eV values of the corresponding energy-dependent cross sections.

With good approximation we can say that for highly enriched fuel, at most the first three equations of (IV-6 to -13) need to be considered; and for many cases it is legitimate to neglect the effect of U^{24} entirely.

For slightly enriched reactors or reactors initially fuelled with plutonium, the last four equations of (IV-6, to -13) become quite important and, in fact, must be augmented by equations describing the concentrations of the higher isotopes of plutonium appearing in Figure IV.1 . Thus the equations required in addition to (IV-6 to -13) to describe the fission chains in a slightly enriched reactor or one initially fueled with plutonium are:

$$\frac{\partial n^{40}}{\partial t} = \sigma_\gamma^{49}\phi_1 n^{49} - \sigma_a^{40}\phi_1 n^{40},$$

$$\frac{\partial n^{41}}{\partial t} = \sigma_\gamma^{40}\phi_1 n^{40} - (\sigma_a^{41}\phi_1 - \lambda^{41})n^{41}, \qquad \text{IV-14}$$

$$\frac{\partial n^{42}}{\partial t} = \sigma_\gamma^{41}\phi_1 n^{41} - \sigma_a^{42}\phi_1 n^{42}.$$

Something can be said about the solution of the fuel depletion equations and mostly to the solution of equations involving Pu^{40} which introduces non-linearity in the set of equations.

In fact the Pu^{40} absorption is almost entirely due to its 1.06 eV

resonance; approximately

$$\sigma_a^{40} = (\sigma_o^{40} + \frac{\alpha}{\sqrt{1 - \beta n^{40}}}) \qquad \text{IV-15}$$

where $\sigma_o^{40}, \alpha, \beta$, are contants and n^{40} is the Pu^{40} concentration when solving the equation

$$\frac{\partial n^{40}}{\partial t} = \sigma_\gamma^{49}\Phi_1 n^{49} - \sigma_a^{40}\Phi_1 n^{40} \qquad \text{IV-16}$$

the dependence of σ_a^{40} on n^{40} cannot, in CANDU's and LWR's be ignored.

The equation is therefore non-linear and is best intepreted numerically with timesteps chosen so that the result does not change (within a given tolerance) the calculation is repeated for halved timesteps.

CHAPTER V

V. FISSION PRODUCTS

V.1 Introduction

It was mentioned in chapter II that one of the consequences of fission process is the release of fission fragments. The nuclear reactor core is the area of a nuclear power station where fuel is destroyed by the presence of the neutron flux and as a result a variety of fission products are generated. Most (if not all) of the fission products are radioactive giving rise to chains of isotopes as they decay to stable nuclei.

The members of these chains have absorption cross sections for 0.025 eV neutrons which vary from a few tenths of a barn to over two million barns (xenon or samarium for example). Moreover, when neutrons are absorbed in these nuclei, other neutrons - absorbing isotopes are created.

Thus a detailed description of how $\Sigma_a(r,t)$ changes with time because of the creation of fission products requires that the concentrations of hundreds of nuclei be determined as functions of time. And these creation rates depend on both the magnitude and the energy dependence of $\Phi(r,E,t)$ as well as on the time.

The behavior of the various isotopes toward neutrons is very complicated.

The complexity is mostly due to the fact that neutron absorption cross section is energy dependent. In case of neutrons born from fission in a nuclear reactor we know that the mean fission neutron energy ranges up to 2×10^6 eV. In a thermal reactor, these neutrons are slowed down by collision with the moderator atoms until they are in thermal equilibrium with the moderator and have an average energy of .025 eV. Therefore, a spectrum of neutron energies exist in a reactor. When neutrons reach thermal equilibrium with the moderator, their energies are determined by the thermal energies of the moderator atoms, and the neutron energy spectrum becomes a Maxwellian distribution at the temperature (T in ^{o}K) of the moderator material.

Setting the kinetic energy of neutron motion equal to the thermal energy of the moderator, we can obtain the most probable velocity of the neutron in thermal equilibrium. For a moderator of room temperature, the neutron velocity is 2200 m/sec. To obtain a reaction rate, one would use the neutron density of neutrons, this most probable velocity of 2200 m/sec, and the neutron cross section of the energy corresponding to a 2200 m/sec velocity.

V. 2 FISSION YIELDS

The binary fission process can occur in many different modes. A very

large number of fission products are known, ranging in the thermal neutron fission of U-235 from zinc (atomic number Z = 30) to dysprosium (Z = 66); and from mass number A = 72 to A = 161. Fission into two equal fragments is by no means the most probable mode in thermal neutron fission. Asymmetric modes are much more favored, the maximum fission product yields occurring at A = 95 and A = 138. (See Fig. V-1). The asymmetry appears to become less pronounced with increasing bombarding energy of the neutron.

Since the neutron to proton ratio is about 1.6 for ^{235}U and about 1.3 for the stable elements in the fission product region, the primary products of fission are always on the neutron excess side of stability. Each such product decays by successive β-decay to a stable isobar. Chains with as many as six β-decay have been established, and undoubtedly some fission products further removed from stability have escaped detection because of their very short half-lives. No neutron-deficient nuclides have been found among the products of thermal neutron fission; however, a few so-called shielded nuclides occur among the fission products. A shielded nuclide is one that has a stable isobar one unit lower in Z so that it is not formed as a daughter product in a β-decay chain. Examples are Rb-86 and Cs-136. The fission yield of such a nuclide is presumably due entirely to its direct formation as a primary product.

If we add a third dimension (E) to the chart of nuclides then the line of stability lies at the bottom of a valley and the sides of the valley are populated by fission products which, being unstable will decay (fall) to the bottom of the valley becoming stable nuclides. One side of the

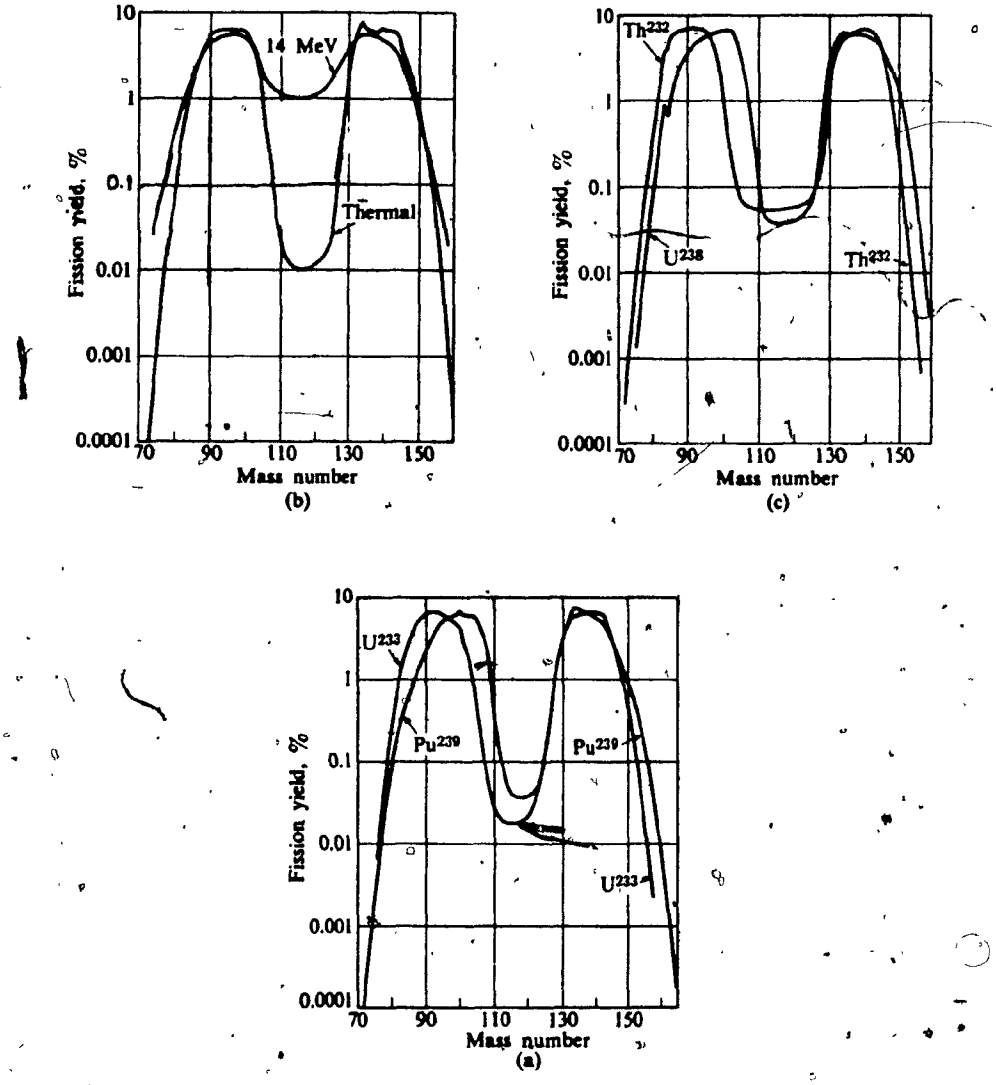


Fig. V-1. Fission Yields: Thermal and Fast Neutrons.

valley would contain isotopes with an excess of neutrons, while the other side of the valley would contain isotopes having an excess of protons.

The total cumulative yield is the percentage of the total fissions that lead directly or indirectly to that nuclide. Since two fission products are emitted for every binary fission, the total cumulative yields of all the isobaric chains should add up to 200%. Direct or independent fission yields have been measured for a relatively few nuclides in some isobaric chains.

The independent yield of a particular fission product is the direct or instantaneous yield of that nuclide without any contribution from decay of preceding members of the isobaric chain. The measured total cumulative yield of a particular fission product represents the sum of its independent yield and of the independent yields of all its precursors.

In addition to binary fission, ternary fission, or break-up into three products also occurs, but less frequently than binary fission. This process is a source of light-charged particles such as: H^3, He^3, He^4, Be^7 etc... The long-lived, or stable, light nuclides can build up in the nuclear fuel of an operating reactor. The CANDU REACTORS OPERATING with U-238-U-235. The ternary thermal neutron fission has tritium H^3 production yield in the range of .0068 to .014%.

V.3 THE FISSION PRODUCT TIME BEHAVIOR

The splitting of ^{233}U, ^{235}U, ^{238}U, ^{239}Pu into lighter nuclei cannot be predicted precisely because of its statistical behavior, but we can analyse the time behavior of two complementary fission products, k and j of the decay chains A and B {Ref. to figure (V-2)}.

It was pointed out in chapter II that the consequences of fission process are:

- Release of energy
- Release of fission fragments

To summarize the release of fission fragments we only need to say that as a result of fission we have

- neutrons
- γ-rays
- neutrinos
- fission products

Assuming that the neutron, γ-rays, and neutrino emission are well known and well established we concentrate on time behavior of fission products starting by illustrating the basic mechanism how fission products decay to stable nuclei (Figures V-3 and V-4).

The reason of presenting figures V-3 and V-4 is that most of fission processes are binary events (e.g. two main fission products are released

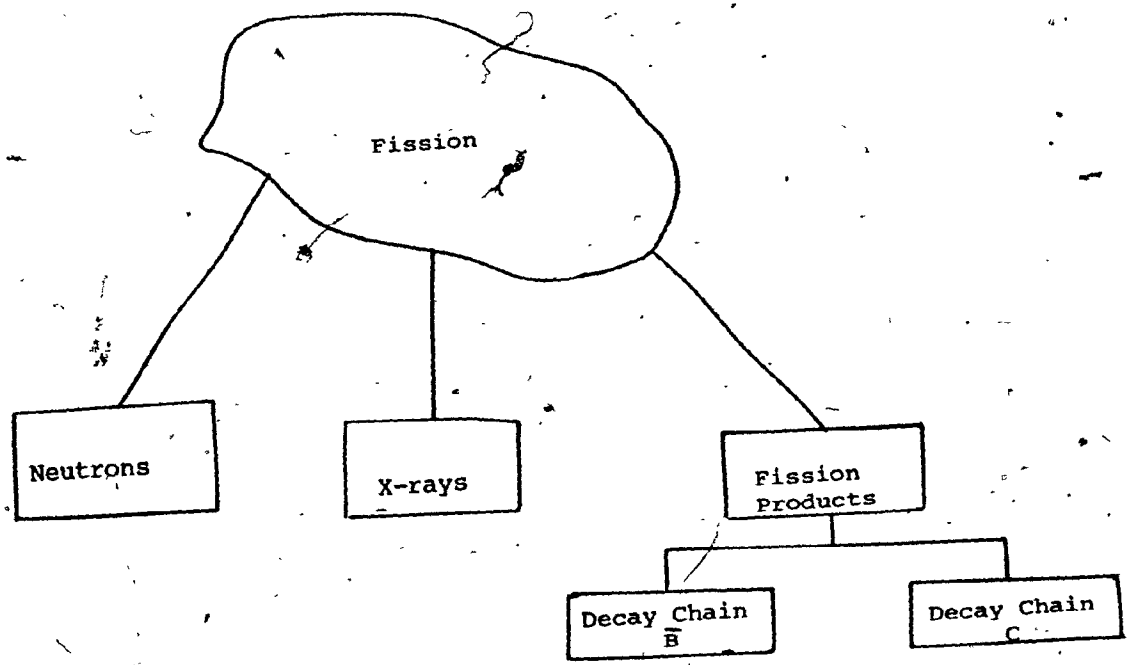


Figure V-2 Basic Fission Process Events

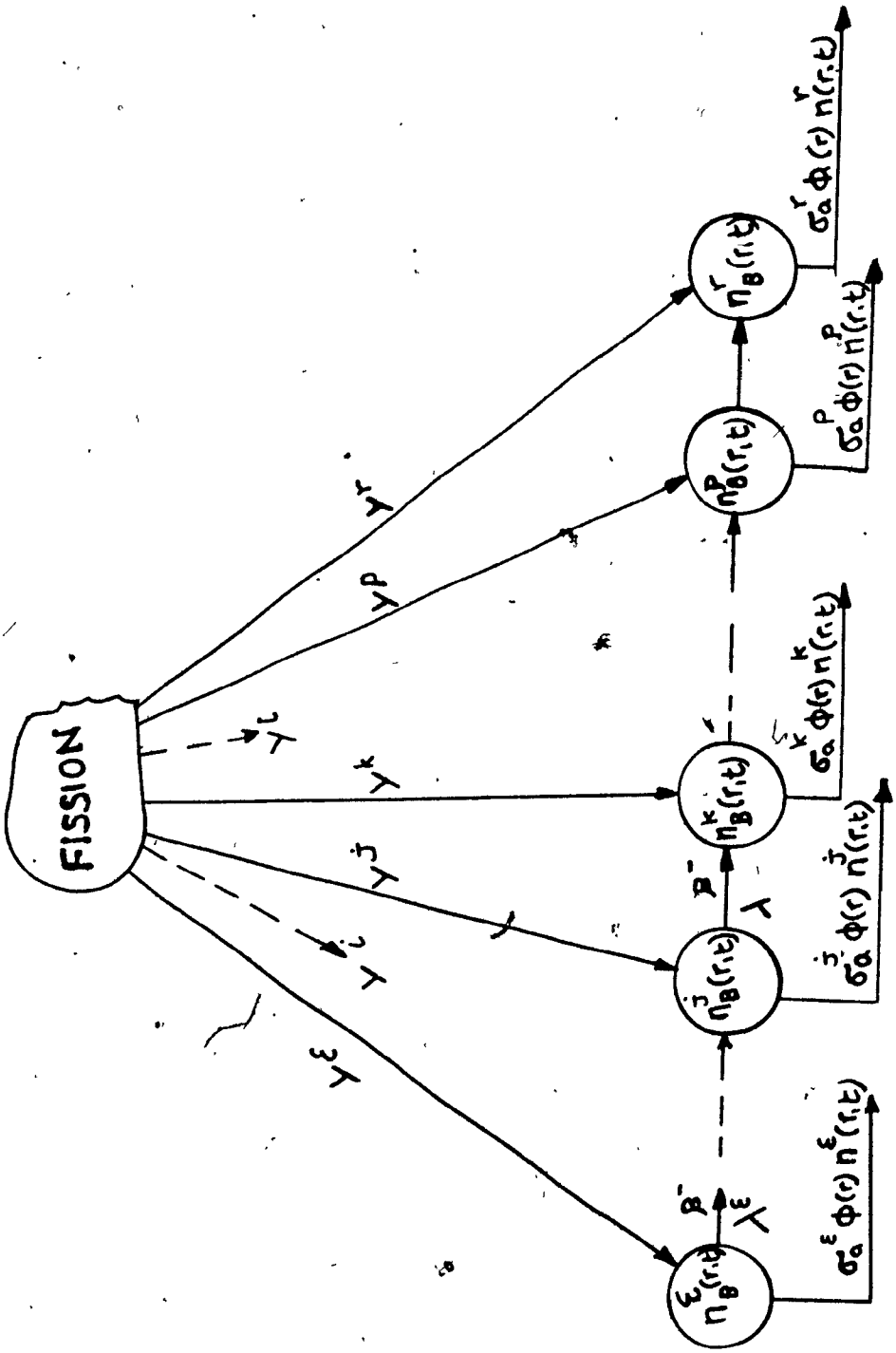


Figure V-3 Fission decay scheme for chain B

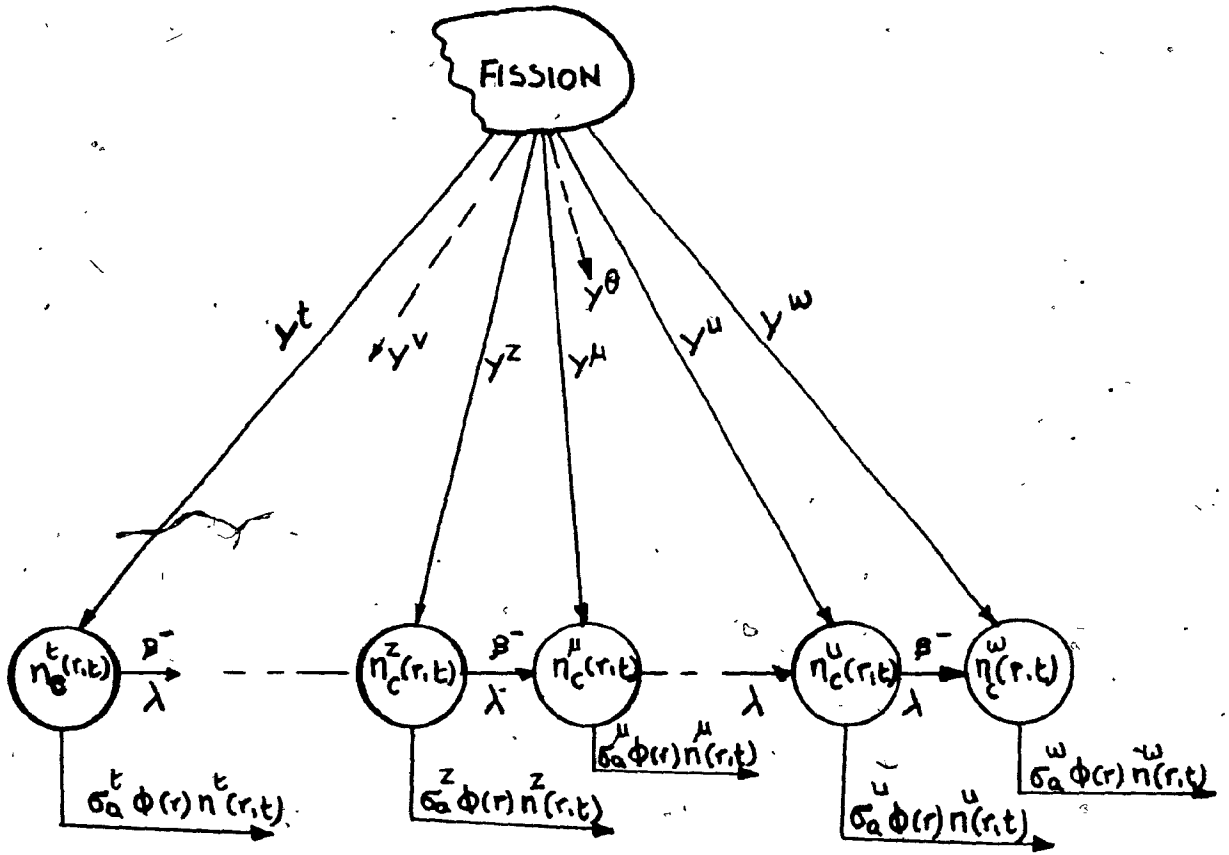


Figure V-4 Fission decay scheme for chain C

from the splitting of ^{235}U, ^{230}Pu, ^{233}U, ^{238}U).

In figure V-3 we give the production and decay possibilities for isotopes having mass number B, while in figure V-4 we give production and decay possibilities for isotopes having mass number C. It is obvious that

$$B + C = A \quad \text{e.g.}$$

should add to ^{235}U, ^{233}U, ^{238}U or ^{239}Pu depending which isotope constitutes the fuel.

V-4 Fission Product Concentration and Definitions

Nuclear interactions in which neutrons are absorbed by fission products and β-particles are emitted, change the identity of the interacting nuclei. Thus if we wish to compute the isotopic concentration of isotopes k and j for decay chain C or the isotopic concentration of isotopes u and w for decay chain D it is convenient to establish a list of definitions characterizing the constants and variables used in the analysis.

From figures V-3 and V-4 we can define:

$n_C^k(r,t) \equiv$ concentration of isotope k at location r, and time t, in nuclei per unit volume.

$n_C^j(r,t) \equiv$ concentration of isotope j at location r and time t, in nuclei per unit volume.

$\lambda^k \equiv$ β-decay constant for isotope k in sec^{-1}

$\lambda^j \equiv$ β-decay constant for isotope j in sec^{-1}

$y^k \equiv$ effective fraction of fission product k (yield)

$y^j \equiv$ effective fraction of fission product j (yield)

$\sigma_a^k(E) \equiv$ probability of absorption of one neutron with energy E, by isotope k per unit path length and per unit number density.

$\sigma_a^j(E) \equiv$ probality of absorption of one neutron with energy E, by isotope j, per unit path length and per unit number density .

$n_D^w(r,t) \equiv$ concentration of isotope w at location r, and time t, in nuclei per unit volume.

$n_D^\varepsilon(r,t) \equiv$ concentration of isotope ε at location r, and at time t, in nuclei per unit volume.

$\lambda^w \equiv$ β-decay constant for isotope w in sec^{-1}

$\lambda^\varepsilon \equiv$ β-decay constant for isotope ε in sec^{-1}

$y^w \equiv$ effective fraction of fission product w (yield)

$y^\varepsilon \equiv$ effective fraction of fission product ε (yield)

$\sigma_a^w(E) \equiv$ probability that isotope w will absorb one neutron with energy E per unit path length, per unit number density.

$\sigma_a^\varepsilon(E) \equiv$ probality that isotope ε will absorb one neutron with energy E per unit path length, per unit number density.

$\Sigma_t(r,E,t) \equiv$ Macroscopic cross section characterizing the probability that a neutron will undergo any type of reaction per unit path length.

$\Sigma_f(r,E,t) \equiv$ time-dependent macroscopic fission cross section defined as:

$$\Sigma_f(r,E,t) = N(r,t)\sigma_f(r,E).$$

$\Sigma_a(r,E,t) \equiv$ time-dependent macroscopic absorption cross section defined as:

$$\Sigma_a(r,E,t) = N(r,t)\sigma_a(r,E).$$

$\Sigma_s(r,E,t) \equiv$ time-dependent macroscopic scattering cross section defined as:

$$\Sigma_s(r,E,t) = N(n,t)\sigma_s(r,E).$$

$\Phi(r,E,t) \equiv$ angle integrated flux:

$$\Phi(r,E,t) = \int_{4\pi} d\Omega\psi(r,E,\hat{\Omega},t)$$

$$= \psi(r,E,\hat{\Omega},t) = \nu n(r,E,\hat{\Omega},t)d^3r dE,d\Omega$$

$n(r,E,\Omega,t)d^3r d\Omega dE \equiv$ expected number of neutrons in d^3r about r, energy dE about E, moving in direction Ω in solid angle $d\Omega$, at time t.

Generally microscopic and macroscopic cross sections are dependent on space, energy and time, however, whereas the "macro", Σ, may vary heavily with time, because they are proportional to number densities, the "micros", σ, are usually time-dependent only through the time-dependence of the neutron spectrum (energy distribution).

Neutron spectrum variations are very slow and can be computed by successive steady-state "snap shots", so that the σ's do not require an ad hoc dynamic calculation.

Among the definitions and in figures V-3, V-4, y^k, y^j, y^w, y^ε..... are average fission yield of isotopes k, j, w, ε..... To be precise, y^k for example, is the average number of nuclei of isotope k created per fission in the material present at location r at time t. It is an average not only over a large number of fissions, but also over the particular nuclei fissioning near point r and over the energies of the neutron causing fission. (We could break the term y^k down into a sum over fissionable isotopes and energy groups each having its own yield, but for notational simplicity we do not). Note that, since almost all fissions yield two fragments, the sum of the y^k over all fragments is very close to 2.0.

V-5 Mathematical Description of Fission Products: Creation and Description

The steps in writing the rate of change of the various isotope concentrations due to fission are very similiar to those used for fuel depletion analysis in Chapter IV. However, fission product time behavior is much more complex due to the various processes by which fission products change their identity. Except for the first and last isotopes appearing in figures V-3, V-4, any isotope in the middle of the chain decay has a very similar creation rate, providing the masses span the relatively flat-topped peak of the mass yield curve (V-1). The rate of change of isotope k along the chain decay C or the rate of change of isotope w along the chain decay D is dictated by the same mathematical model.

Because of this similarity only one isotope in one of the chain decays is analyzed: isotope k for example. From figure V-3 we can see that the concentration of isotope k, defined as $n^k(r,t)$, in nuclei / cc changes with time because:

① isotope k can be created directly from fission.

② isotope k can be created by decay of isotope j.

③ isotope k can be destroyed be absorbing one neutron with energy E.

④ isotope k can be destroyed because of its β-decay into isotope l.

The net rate of change of isotope k is then given symbolically by:

$$\frac{\partial n^k(r,t)}{\partial t} = ① + ② - ③ - ④ \qquad (V\text{-}1)$$

The contribution to Equation V-1 from term ① is mathematically expressed by:

$$① = y^k \Sigma_f(r,E,t)\phi(r,E,t).$$

The contribution to Equation V-1 from term ② is expressed by:

$$② = \lambda^j n^j(r,t).$$

The contribution from ③ and ④ is:

$$③ = \sigma_a^k(E)\ \phi(r,E,t)\ n^k(r,t).$$

$$④ = \lambda^k n^k(r,t).$$

Hence equation V-1 takes the form:

$$\frac{\partial n^k(r,t)}{\partial t} = y^k \int_0^\infty \Sigma_f(r,E,t)\phi(r,E,t)dE + \lambda^j n^j(r,t) - \lambda^k n^k(r,t) - \int_0^\infty \sigma_a^k(E)\phi(r,E,t)\, n^k(r,t)dE \qquad \text{V-2}$$

The same procedure applies for isotope j in the decay chain C with the only difference that k superscript is now j and j changes to i. With this change Equation V-2 applied to isotope j takes the form:

$$\frac{\partial n^j(r,t)}{\partial t} = y^j \int_0^\infty \Sigma_f(r,E,t)\phi(r,E,t)^{dE} + \lambda^i n^i(r,t) - \lambda^j n^j(r,t) - \int_0^\infty \sigma_a^j(E)\,(r,E,t)n^j(r,t)dE \qquad \text{V-3}$$

In a similar fashion the rate of change of isotopes u and w in the decay chain D are given by:

$$\frac{\partial n^u(r,t)}{\partial t} = y^u \int_0^\infty dE\Sigma_f(r,E,t)\phi(r,E,t) + \lambda^\theta n^\theta(r,t) - \lambda^u n^u(r,t) - \int_0^\infty dE\sigma^u{}_a(E)\phi(r,E,t)n^u(r,t) \qquad \text{V-4}$$

and

$$\frac{\partial n^w(r,t)}{\partial t} = y^w \int_0^\infty \Sigma_f(r,E,t)\phi(r,E,t)dE + \lambda^u n^u(r,t) - \int_0^\infty dE\sigma_a^w(E)\phi(r,E,t)n^w(r,t) \qquad \text{V-5}$$

V-6 FICTITIOUS FISSION PRODUCTS

Equations V-2, V-3, V-4, V-5 can be used to describe the behavior of any two close isotopes in the decay chains shown in figures V-3 and V-4. In practice we are only interested in those fission-fragment chains which have large y and large σ_a. The effect of the rest of the chains on neutron absorption can be represented quite accurately by defining a few fictitious isotopes having artificial neutron absorption cross sections and radioactive-decay constants.

These fictitious isotopes are defined as obeying V-2 through V-5 (generally with decay term set equal to zero). The fictitious parameter analysis reduces the assumption that there is only one artificial isotope, that it does not decay, and that, when it absorbs a neutron in accord with its subsequent absorption in the chain - also has the fictitious cross section σ_a^f.

This last assumption implies that, in effect, when a neutron is absorbed in the fictitious isotope, the isotope is not destroyed. Thus equations V-2 through V-5 for fictitious isotopes become:

$$\frac{\partial n^{ff}(r,t)}{\partial t} = Y^{ff} \int_0^\infty \Sigma_f(r,E,t)\phi(r,E,t)dE \qquad (V\text{-}6)$$

where the superscript ff stands for "fictitious fission fragments".

$y^{ff}\sigma_a^{ff}$ is the one-group microscopic absorption cross section for the

fictitious fission fragment, the corresponding microscopic cross section at a time t in the life of the fuel is:

$$\Sigma^{ff}(r,t) \equiv n^{ff}(r,t)\,\sigma_a^{ff} =$$

$$= y^{ff}\sigma_a^{ff}\int_o^t dt' \int_o^\infty \Sigma_f(r,E,t')\phi(r,E,t')\,dE \qquad V\text{-}7$$

Since $\int_o^t dt' \int_o^\infty dE\ \Sigma_f(r,E,t')\phi(r,E,t')$ is the number of fission per unit volume that have occurred at r in time t, and since we are assuming, in effect, that a fictitious nucleus of this type, once created, is never destroyed, the quantity $y^{ff}\sigma_a^{ff}$ is the approximate number of "barn per fission" introduced into the reactor due to fission fragments for which yield, and microscopic absorption cross sections are not unduly large. Valves of $y^{ff}\,\sigma_a^{ff}$ for thermal reactors generally lie in the range 40 - 50 barns per fission. Thus, if the average fission density in the fuel is 100 watts/cc, the value of $\Sigma^{ff}(r,t)$ at the end of four months (10^7 sec), is in the range of .0012 - .0015 cm^{-1}. A typical value for the one-group microscopic fission cross section in a thermal reactor is .1cm^{-1}. Thus the fraction of neutrons absorbed in the less important fission products after four months of continuous full-power operation is still small compared to the fraction absorbed in the fuel. Of course, as reactor depletion goes on, $\Sigma^{ff}(r,t)$ continue to increase and eventually accounts for a significant fraction of the neutrons absorbed in the reactor. Unless these fission fragments are removed from the fuel, they will eventually make it impossible to maintain criticality.

V-7 FISSION PRODUCT POISONING IN CANDU REACTORS

The use of heavy water as moderator and as primary heat transport system in CANDU reactors makes it possible to use natural uranium as fuel. The reasons of this possibility have been outlined in chapter II. However, the splitting of U^{235} and some small fraction of U^{238} originates a large number of fission products. Certain fission products possess extremely large thermal neutron absorption cross sections. Of particular concern are ^{135}Xe and ^{149}Sm whose absorption cross sections are shown in figure V-4. The significance of these extremely strongly absorbing fission products is compounded by their relatively large fission yields.

The buildup of such fission product poisons can appreciably affect core multiplication and hence reactor operation. It should be noted here (Figure II-9) that since these absorption cross sections fall off quite rapidly for neutron energies above 1eV, fission product poisoning is therefore primarily of concern in CANDU reactors and other thermal reactors. In order to illustrate the importance of this phenonenon, let us make a very simple estimate of the reactivity change induced in a reactor core by the addition of a fission product poison. To first order, the effects of such poisons enter through the thermal utilization factor f. If we imagine introducing a poison characterized by a macroscopic cross section Σ_a^P uniformly throughout a homogeneous reactor core, then we can write the thermal utilization characterizing the "poisoned" core as:

$$f = \frac{\Sigma_a^F}{\Sigma_a^F + \Sigma_a^M + \Sigma_a^P} \qquad \text{V-8}$$

Hence we can compute the reactivity change from a critical reactor in which $\Sigma_a^P = 0$

$$\Delta\rho \equiv \rho(\Sigma_a^P) - \rho(\Sigma_a^{P=0}) = \rho' - \rho \qquad \text{(V-9)}$$

Now if we recall our earlier definition of reactivity as given by the six - factor formula.

$$\rho = \frac{k-1}{k}, \quad k = \eta\, \rho\varepsilon f\, P_{FNL}\, P_{TNL} \qquad \text{(V-10)}$$

and note that

$$P_{TNL} = \frac{1}{1 + L^2 B_g^2}, \quad L^2 = \frac{D}{\Sigma_a} = \frac{1}{3\Sigma_{tr}\Sigma_a} \qquad \text{(V-11)}$$

will also change as Σ_a^P changes, then we can compute

$$\Delta\rho = \frac{f' - f}{f'} + \frac{P'_{TNL} - P_{TNL}}{P'_{TNL}} \qquad \text{(V-12)}$$

If we use (V-8) and V-11 we find

$$\frac{f' - f}{f'} = -\frac{\Sigma_a^P}{\Sigma_a^f + \Sigma_a^m} \qquad \text{(V-13)}$$

and

$$\frac{P'_{TNL} - P_{TNL}}{P'_{TNL}} = \frac{L^2 B_g^2}{1 + L^2 B_g^2}\left[\frac{\Sigma_a^P}{\Sigma_a} + \frac{\Sigma_{tr}^P}{\Sigma_{tr}}\right] \qquad \text{(V-14)}$$

Hence

$$\Delta\rho = \frac{\Sigma_a^P}{\Sigma_a} \frac{1}{1 + L^2 B_g^2} \left[1 - L^2 B_g^2 \frac{\Sigma_{tr}^P / \Sigma_a^P}{\Sigma_{tr} / \Sigma_a} \right]$$

$$\simeq \frac{\Sigma_a^P / \Sigma_a}{1 + L^2 B_g^2} \qquad \text{(V-15)}$$

Here we have noted that the second term in brackets is usually quite small ($\leq 10^3$) and can be ignored. In fact for many large power reactor cores, leakage is sufficiently small that $L^2 B_g^2 \ll 1$ and to first order, the reactivity change due to such a poison is just the fraction of the total macroscopic absorption cross section due to the poison.

Now recall that the macroscopic cross section for the poison is given by $\Sigma_a^P = N^P \sigma_a^P$, where N^P is the number density of the poison while σa^P is its thermal absorption cross section. Hence in order to estimate the reactivity change due to fission product poisoning, we must calculate the number density of the poisoning isotope N^P at any time t.

To determine $N^P(t)$, we must solve the rate equations V-2 and V-3 or V-4 and V-5 describing the various productions and decay processes that can affect the poisoning concentration. We will illustrate the procedure by taking into consideration the two most important fission products:

$$Xe^{135} \text{ and } Pm^{149}$$

V-8 THE XENON Xe^{135} AS FISSION PRODUCT POISONING

Xe^{135} is the most significant fission product because of its enormous thermal neutron absorption cross section and its relatively large fission yield.

The fact that most of the Xe^{135} is produced by decay with considerable delay after the originating fission entails a very delicate prompt-positive, delayed-negative flux-to-reactivity feedback; this feedback is the cause of the so-called "Xenon oscillations", one of the most challenging control problems in thermal reactors. (This topic will be fully expanded in part II of the Thesis (e.g. Ph.D.).

Xe^{135} can be produced not only directly as a fission product but may also result from the β-decay of Te^{135} to I^{135} and further decay to Xe^{135}. The figure (V-5) gives all possible ways of Xe^{135} production.

To start the analysis of $^{135}X_e$ behavior in the reactor we define a series of quantities. From figure V-5 and assuming the analysis for one-group energy (thermal) we define:

- $Sb(r,t)$ = Atomic number density of isotope Sb^{135} (atoms/cm^3)

- $Te(r,t)$ = Atomic number density of isotope Te^{135} (atoms/cm^3)

- $I(r,t)$ = Atomic number density of isotope I^{135} (atoms/cm^3)

- $Xe(r,t)$ = Atomic number density of isotope Xe^{135} (atoms/cm^3)

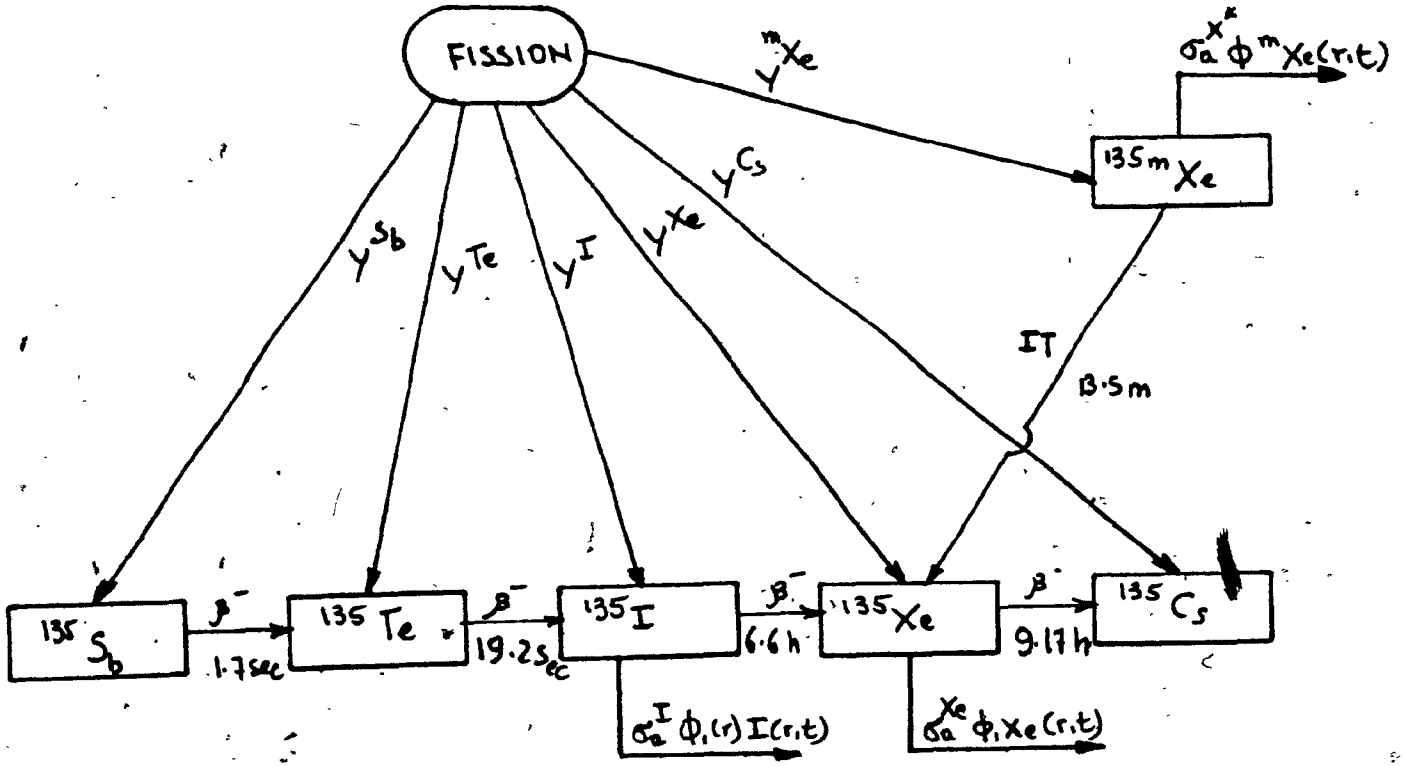


Figure V-5 The production and decay scheme for Xe^{135}.

$-Cs(r,t)$ = Atomic number density of isotope CS^{135}

y^{Sb} = Effective fraction of fission product Sb^{135}

y^{Te} = Effective fraction of fission product Te^{135}

y^{I} = Effective fraction of fission product I^{135}

y^{xe} = Effective fraction of fission product Xe^{135}

σ_a^{Sb} = Thermal neutron absorption cross section for isotope Sb^{135} (cm^2)

σ_a^{Te} = Thermal neutron absorption cross section for isotope TE^{135} (cm^2)

σ_a^{I} = Thermal neutron absorption cross section for isotope I^{135} (cm^2)

σ_a^{Xe} = Thermal neutron absorption cross section for isotope Xe^{135} (cm^2)

$\Sigma_f(r,t)$ = Microscopic fission cross section in cm^{-1}

$\phi(r,t)$ = Thermal neutron flux in neutrons/cm^2 sec.

λ^{Sb} = β-decay constant for fission product Sb^{135}

λ^{Te} = β-decay constant for fission product Te^{135}

λ^{I} = β-decay constant for fission product I^{135}

λ^{Xe} = β-decay constant for fission product Xe^{135}

Based on rate equations (V-2) and (V-3) or (V-4) and (V-5) there would be five rate equations to cover the decay chain for A=135 starting from Sb^{135} down to Cs^{135}. The five equations would be all coupled and very hard to solve.

To by-pass this complexity and also to stay very close to practical approach the following approximations are made:

a) From figure (V-5) we can see that the decay from Sb^{135} to Te^{135} and from Te^{135} to I^{135} can be taken to be instantaneous (i.e. we can say that I^{135} is produced directly from fission. Furthermore we will ignore the short-lived isomeric state Xe^{135m}, and assume that all I^{135} nuclei will decay to the ground state Xe^{135}. Because Xe^{135} is considered to be the most important isotope in the decay chain for A = 135, we can stop the chain at Xe^{135}. Also being the thermal neutron absorption cross sections of the isotopes in the chain, except for that of Xe^{135} which is very large, of no significance we will neglect them in the analysis. Hence the effective decay scheme for A = 135 simplifies to that shown in figure V-6

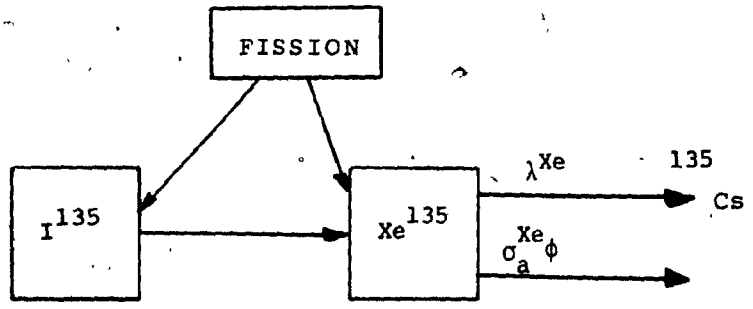


Fig. V.6. Simplified Decay Scheme for A = 135

Under the assumptions we have just made equations (V-2) and (V-3) applied to the actual decay scheme because:

$$\frac{\partial I(r,t)}{\partial t} = \gamma^I \int_o^\infty \Sigma_f(r,E't)\phi(r,E,t)\, dE - \lambda^I I(r,t) \qquad \text{(V-16)}$$

$$\frac{\partial Xe(r,t)}{\partial t} = \gamma^{Xe} \int_o^\infty dE\, \Sigma_f(r,E,t)\phi(r,E,t) + \lambda^I I(r,t) - \int_o^\infty dE \sigma_a^{Xe}(E)\phi(r,E,t)Xe(r,t)$$

$$- \lambda^{Xe} Xe(r,t) \qquad \text{(V-17)}$$

Equations (V-16) and (V-17) are the key to study all phenomenons concerning X^{135} and I^{135} behavior in the thermal reactors at startup, shutdown and various power changes.

If we assume $\phi_1(r)\Sigma_{f1}(r)$, $\sigma_{a1}^{Xe}(r)$, (the appropriate spectrum-averaged, one-group parameters) to be constant in time, and further we assume the flux to be one-group type: then equations (V-16) and (V-17) reduce to:

$$\frac{\partial I(r,t)}{\partial t} = \gamma^I \Sigma_{f1}(r)\phi_1(r,t) - \lambda^I I(\vec{r},t) \qquad \text{(V-18)}$$

$$\frac{\partial Xe(r,t)}{\partial t} = \gamma^{Xe}\Sigma_{f1}(r)\phi_1(r,t) + \lambda^I I(r,t) - \lambda^{Xe} Xe(r,t) - \sigma_{a1}^{Xe}(r)\phi(r,t)\, X(r,t)$$

(V-19)

Notice that in the equation (V-19) we have included a capture loss term for the fact that capture will deplete the Xe^{135} concentration (as well

as neutrons of course).

Equations (V-18) and (V-19) can only be solved if we assume the flux behavior to be known as a function of space and time. If this is the case integration of equation (V-18) will give: [App. A]

$$I(r,t) = \{I(r,0) + y^I \int_0^t dt' \; \Sigma_{f1}(r)\phi_1(r,t) \exp(\lambda^I t')\} \times \exp(-\lambda^I t) \qquad \text{(V-20)}$$

Substitution of the solution (V-20) into the Xenon equation and integration over the time interval (0,t) will give the Xe^{135} concentration as a function of time at a particular position r:

$$Xe(r,t) = \{X(r,0) + \int_0^t dt' \left[(\lambda^I I(r,t') + y^{Xe}\Sigma_{f1}(r)\phi_1(r,t')\right] \cdot$$

$$\exp\left\{\int_0^{t'} dt''(\lambda^{Xe} + \sigma_{a_1}^{Xe}(r)\phi_1(r,t''))\right\} \exp\left\{-\int_0^t dt'' \; (\lambda^{Xe} + \sigma_{a1}^{Xe}(r)\phi_1(r,t'')\right\} \qquad \text{(V-21)}$$

of course for any complicated flux behavior, the solutions (V-20) and (V-21) will require numerical integrations, and until we specify the form of $\phi_1(r,t)$ this formal solution is of little use in understanding the behavior of the xenon concentration in the reactor.

V-9 THE SAMARIUM Sm^{149} AS FISSION PRODUCT CHAIN

The second most important reactor fuel poison is Sm^{149}.

The isotope Sm^{149} appears in a nuclear reactor as the result of the chain events shown in figure (V-7).

From the figure (V-7) we can say that for practical purposes the 1.0 sec. 2.3 m, and 1.73 h half-lives of Ce^{148}, Pr^{149}, and Nd^{149} are so much faster than the 53.1 h half-life

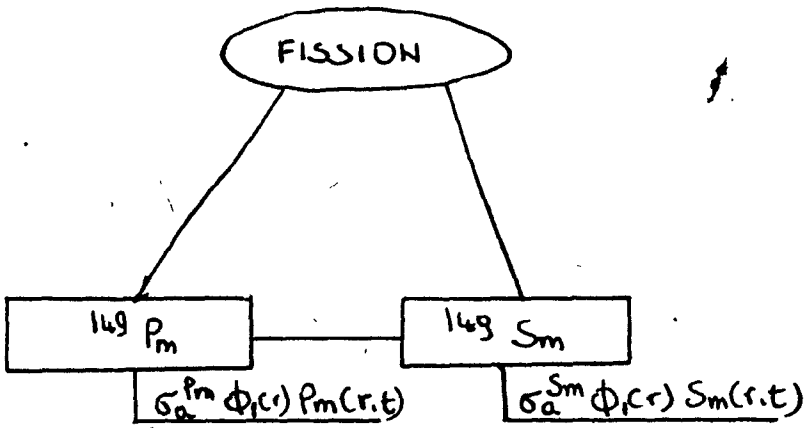


Figure V.8 Simplified Decay Scheme for A = 149

of the promethium = 149 that we may assume the later isotope appears directly from fission. Under this assumption the figure (V-7) reduces to (V-8).

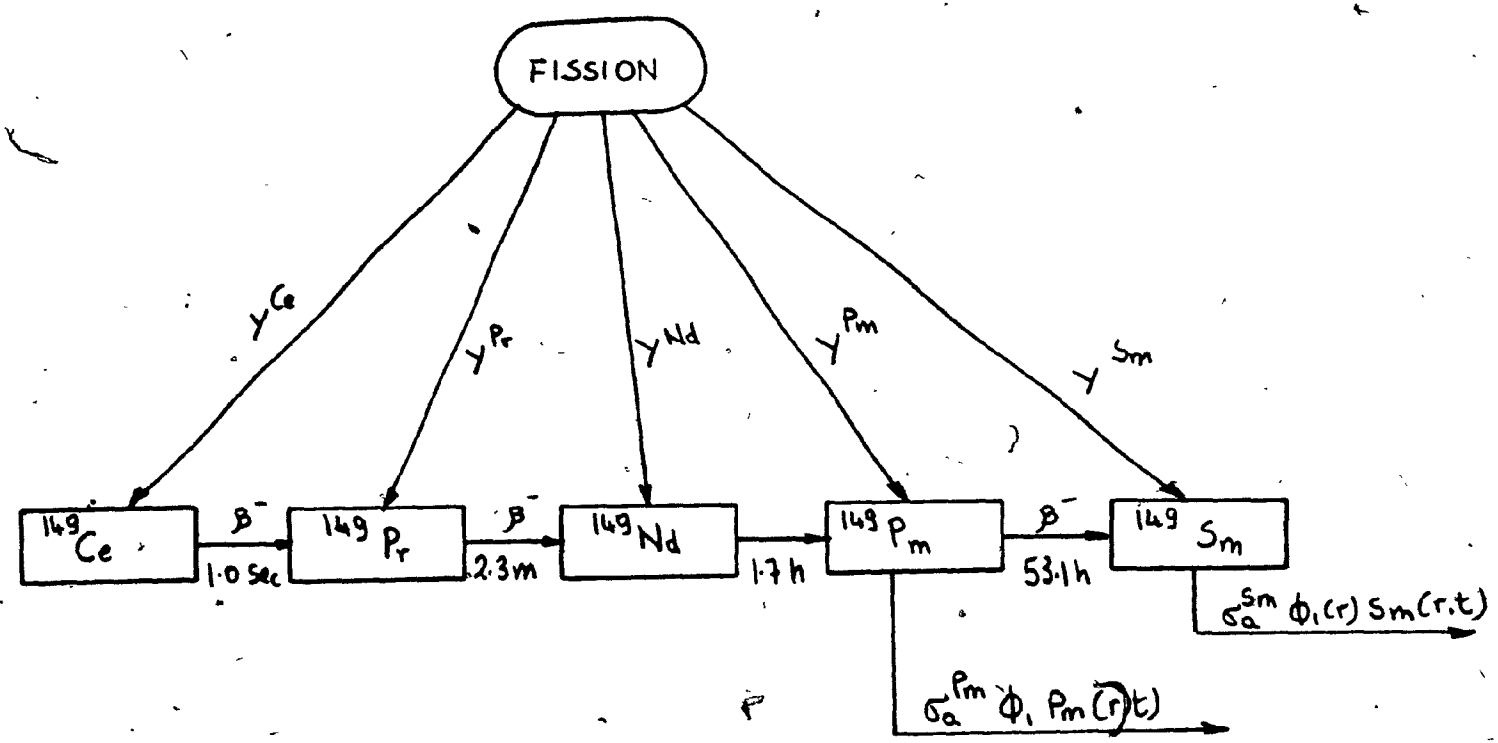


Figure V-7. The production and decay scheme for Sm^{149}

Thus if we use $Pm(r,t)$ and $Sm(r,t)$ to denote the concentrations of Pm^{149} and Sm^{149} nuclei at point r and time t and under the assumptions we have just made the rate Equations (V-2) and (V-3) applied to the actual scheme become:

$$\frac{\partial Pm(r,t)}{\partial t} = y^{Pm}\int_0^\infty dE\ \Sigma_f(r,E,t)\phi(r,E,t) - \lambda^{Pm}\ Pm(r,t) - \int_0^\infty dE\ \sigma_{a_1}^{Pm}(E)\phi(r,E,t)Pm(r,t) \qquad (V\text{-}22)$$

$$\frac{\partial Sm(r,t)}{\partial t} = y^{Sm}\int_0^\infty dE\ \Sigma_f(r,E,t)\phi(r,E,t) + \lambda^{Pm}\ Pm(r,t) - \int_0^\infty dE\,\sigma_a^{Sm}(E)\phi(r,E,t)Sm(r,t) \qquad (V\text{-}23)$$

Equations (V-22) and (V-23) are the key to study all phenomena concerning time dependence of isotopes Pm^{149} and Sm^{149} in thermal power reactors following startup, shutdown and power changes.

If we assume $\Sigma_{f1}(r)$, σ_{a1}^{Sm}, the appropriately spectrum averaged one-group parameters to be constant in time, and if for practical application we assume $\sigma_{a_1}^{Pm}(E)$ to be zero for flux levels encountered in modern power reactor, and further we assume the flux to be a one-group type; then equations (V- 2) and (V-23) reduce to:

$$\frac{\partial Pm(r,t)}{\partial t} = y^{Pm}\ \Sigma_{f1}(r)\phi_1(r) - \lambda^{Pm}Pm(r,t) \qquad (V\text{-}24)$$

$$\frac{\partial Sm(r,t)}{\partial t} = y^{Sm}\ \Sigma_{f1}(r)\phi_1(r) + \lambda^{Pm}Pm(r,t) - \sigma_{a_1}^{Sm}\ \phi_1(r)Sm(r,t) \qquad (V\text{-}25)$$

Equations (V-24) and (V-25) can only be solved if we assume the flux behavior to be known as a funcion of space and time. If this is the case integration of equation (V-24) will give:

$$Pm(r,t) = \{Pm(r,0) - \int_o^t dt' \, \Sigma_{f1}(r)\phi_1(r,t)\exp(\lambda^{Pm}t')\} \times \exp(-\lambda^{Pm}t) \qquad (V\text{-}26)$$

Substitution of the solution (V-26) into the samarium equation and integration over time interval (0,t) will give the Sm^{149} concentration as a function of time at a particular point r:

$$Sm(r,t) = \{Sm(r,0) + \int_o^t dt' \left[(\lambda^{Pm}Pm(r,t) + y^{Sm}\Sigma_{f1}(r)\phi_1(r,t')\right] \exp(\int_o^{t'} dt'' \; \sigma_{a_1}^{Sm}\phi_1(r,t''))\} \exp\left(-\int_o^t dt'' \; \sigma_{a_1}^{Sm}\phi_1(r,t''\right)$$

of course the same comments made for the solutions of I(r,t) and Xe(r,t) apply to the solutions of Pm(r,t) and Sm(r,t). Hence it is a good exercise to examine the time behavior of I^{135}, Xe^{135}, Pm^{149} and Sm^{149} concentrations for several particularly simple examples of flux behavior.

V-10 AT BEGINNING OF LIFE

STARTUP OF A CLEAN CORE REACTOR

Suppose we suddenly bring the reactor to a steady state flux level ϕ_o at time t = 0. Assuming that prior to this time, the reactor core has been in a shutdown state with zero fission product poisoning concentration (i.e. a "clean core"). We need only: Substitute a time-independent flux $\phi(r,t) = \phi_{01}(r)$ into the general solutions (V-20) and (V-21) and use the identical conditions:

$$I(r,0) = Xe(r,0) = 0 \tag{V-28}$$

to find the time behavior of the fission product concentrations following startup:

$$I(r,t) = I(t) = 0 + \frac{y^I \Sigma_{f1} \phi_{01}}{\lambda^I} \left[\exp(\lambda^I t')\right]_o^t \exp(-\lambda^I t) =$$

$$= \frac{y^I \Sigma_{f1} \phi_{o1}}{\lambda^I} \left[\exp(\lambda^I t) - 1\right] \exp(-\lambda^I t)$$

$$= \frac{y^I \Sigma_{f1} \phi_{o1}}{\lambda^I} \left[1 - \exp(-\lambda^I t)\right] \tag{V-29}$$

substitution of Eq. (V-29) into Eq. (V-21) will yield the following result:

$$X(r,t) \equiv X(t) = \{0 + \int_o^t dt' \frac{\lambda^I y^I \Sigma_{f1}\phi_{01}}{\lambda^I}\left[1-\exp(-\lambda^I t')\right] + y^{Xe}\Sigma_{f1}\phi_{01}$$

$$\exp \int_o^{t'} dt'' \left[\lambda^{Xe} + \sigma_{a1}^{Xe}\phi_{01}\right] \} \exp\left[-\int_o^t dt''$$

$$\left(\lambda^{Xe} + \sigma_{\sigma 1}^{Xe}\phi_{01}\right)\Big] \qquad (V\text{-}30)$$

Integration of equation (V-30) will yield the solution for xenon time behavior following reactor startup

$$X(t) = \frac{y^I \Sigma_{f1}\phi_{01}}{\lambda^{Xe} + \sigma_{a1}^{Xe}\phi_{01}} - \frac{y^I\Sigma_{f1}\phi_{01}}{\lambda^{Xe} + \sigma_{a1}^{Xe}\phi_{01}}\left[\exp -(\lambda^{Xe}+\sigma_{a1}^{Xe}\phi_{01})t\right]$$

$$- \frac{y^I \Sigma_{f1}\phi_{01}}{\lambda^{Xe} - \lambda^I + \sigma_{a1}^{Xe}\phi_{01}} \exp\left(-\lambda^I t\right) + \frac{y^I\Sigma_{f1}\phi_{01}}{\lambda^{Xe} - \lambda^I + \sigma_{a1}^{Xe}\phi_{01}}\left[\exp -(\lambda^{Xe} + \sigma_{a1}^{Xe}\phi_{01})t\right]$$

$$- \frac{y^{Xe}\Sigma_{f1}\phi_{01}}{\lambda^{Xe} + \sigma_{a1}^{Xe}\phi_{01}} \exp\left[-(\lambda^{Xe}+\sigma_{a1}^{Xe}\phi_{01})t\right] + \frac{y^{Xe}\Sigma_{f1}\phi_{01}}{\lambda^{Xe}+\sigma_{a1}^{Xe}\phi_{01}} \qquad (V\text{-}31)$$

$$\boxed{X(t) = \frac{(y^{Xe} + y^I)\,\Sigma_{f1}\,\phi_{01}}{\lambda^{Xe} + \sigma_{a1}^{Xe}\,\phi_{01}}\left[1 - \exp -(\lambda^{Xe} + \sigma_{a1}^{Xe}\,\phi_{01})t\right]$$

$$+ \frac{y^I\Sigma_{f1}\phi_{01}}{\lambda^{Xe} - \lambda^I + \sigma_{a1}^{Xe}\,\phi_{01}}\left[\exp -(\lambda^{Xe}+\sigma_{a1}^{Xe}\,\phi_{01}) - \exp(-\lambda^I t)\right]}$$

V-11 Sm^{149} AT BEGINNING OF LIFE: STARTUP OF CLEAN REACTOR

Applying the same conditions as the previous section, in analogy with Equation (V-28) we have

$$Pm(r,0) = Sm(r,0) = 0. \qquad \text{(V-33)}$$

With these assumptions, equations (V-26) and (V-27) to find the time behavior of fission product concentrations following reactor startup become

$$\boxed{Pm(r,t) = \frac{y^{Pm}\Sigma_{f1}(r)\phi_1(r)}{\lambda^{Pm}}\left[1 - \exp - (\lambda^{Pm}t)\right]} \qquad \text{(V-34)}$$

Substitution of the solution (V-34) into (V-27) and performing the desired integration applying the initial conditions (V-33) we get:

$$\boxed{\begin{aligned} Sm(r,t) \equiv Sm(t) = {} & \int_0^t dt' \; y^{Pm}\Sigma_{f1}\,\phi_{01}\left[1 - \exp - (\lambda^{Pm}t)\right] + \\ & y^{Sm}\Sigma_{f1}\exp\left[\int_0^t dt''\; \sigma_{a1}^{Sm}\phi_{01}\right]\exp\left[-\int_0^t dt''\,(\sigma_{a1}^{Sm}\phi_{01}\right] \end{aligned}}$$

(V-35)

Integration of equation (V-35) will yield the solution for Sm^{149} time behavior following reactor startup

$$S_m(t) = \frac{(y^{Sm} + y^{Pm})\,\Sigma_{f1}\,\phi_{01}}{\sigma_{a1}^{Sm}\,\phi_{01}} \left[1 - \exp\left(-\sigma_{a1}^{Sm}\,\phi_{01}\,t\right)\right]$$

$$+ \frac{y^{Pm}\Sigma_{f1}\,\phi_{01}}{\sigma_{a1}^{Sm}\,\phi_{01} - \lambda^{Pm}} \left[\exp\left\{-\sigma_{a1}^{Sm}\,\phi_{01}\,t\right\} - \exp\left\{-\lambda^{Pm}t\right\}\right] \qquad (V\text{-}36)$$

In summary the time behavior of the most important fission products following reactor startup is given by the following table:

TABLE V-1

ISOTOPE	SOLUTION
I^{135}	$I(t) = \frac{y^{I}\Sigma_{f1}\,\phi_{01}}{\lambda^{I}} \left[1 - \exp(-\lambda^{I}t)\right]$
Xe^{135}	$Xe(t) = \frac{(y^{Xe} + y^{I})\,\Sigma_{f1}\,\phi_{01}}{\lambda^{Xe} + \sigma_{a1}^{Xe}\,\phi_1} \left[1 - \exp\left\{-(\lambda^{Xe} + \sigma_{a1}^{Xe}\,\phi_1(r)\,t\right\}\right]$ $+ \frac{y^{I}\Sigma_{f1}\,\phi_{01}}{\lambda^{Xe} - \lambda^{I} + \sigma_{a1}^{Xe}\,\phi_{01}} \left\{\exp\left[-(\lambda^{Xe} + \sigma_{a1}^{Xe}\,\phi_{01}\right] - \exp(-\lambda^{I}t)\right\}$
Pm^{149}	$Pm(t) = \frac{y^{Pm}\Sigma_{f1}\phi_{01}}{\lambda^{Pm}}\left[1 - \exp(-\lambda^{Pm}t)\right]$

149 Sm	$Sm(t) = \frac{(y^{Sm} + y^{Pm})\,\Sigma_{f1}\,\phi_{01}}{\sigma_{a1}^{Sm}\,\phi_{01}}\left[1 - \exp\left(-\sigma_{a1}^{Sm}\,\phi_{01}\,t\right)\right] + \frac{y^{Pm}\Sigma_{f1}\phi_{01}}{\sigma_{a1}^{Sm}\,\phi_{01}\,\lambda^{Pm}}\left[\exp\left(-\sigma_{a1}^{Sm}\,\phi_{01}\,t\right) - \exp\left(-\lambda^{Pm}t\right)\right]$

The time behavior of the concentrations in table V-1 are shown in Figs.(V-12) to (V-27) pages 337 -352. In particular it should be noted that these concentrations eventually level off at equilibrium levels for long times following startup.

$$I(t) \to \infty \quad t \to \infty \qquad = \frac{y^{I}\Sigma_{f1}\phi_{01}}{\lambda^{I}} \tag{V-37}$$

$$Xe(t) \to Xe^{\infty} \quad t \to \infty \qquad = \frac{(y^{I} + y^{Xe})\,\Sigma_{f1}\,\phi_{01}}{\lambda^{Xe} + \sigma_{a1}^{Xe}\,\phi_{01}} \tag{V-38}$$

$$Pm(t) \to Pm^{\infty} \quad t \to \infty \qquad = \frac{y^{Pm}\Sigma_{f1}\,\phi_{01}}{\lambda^{Pm}} \tag{V-39}$$

$$Sm(t) \to Sm^{\infty} \quad t \to \infty \qquad = \frac{(y^{Sm} + y^{Pm})\,\Sigma_{f1}}{\sigma_{a1}^{Sm}} \tag{V-40}$$

That is, the concentrations of these fission product poisons in a reactor operating at constant flux will eventually saturate at those equilibrium values for which the production of poisons from fission is just balanced by the decay and neutron capture losses of the poisons. These equilibrium concentrations could also have been determined directly from the rate equations V-18, V-19, V-22 and V-23 themselves by merely setting

$$\frac{\partial I(r,t)}{\partial t} = \frac{\partial\ Xe(r,t)}{\partial t} = \frac{\partial Pm(r,t)}{\partial t} = \frac{\partial Sm(r,t)}{\partial t} = 0 \qquad \text{(V-41)}$$

Note also the $Sm^{(\infty)}$ value is independent of flux.

V-12 REACTOR SHUTDOWN

We have seen that when a nuclear reactor power is raised from zero to some steady-state value, time-dependent fission product concentrations $Sm(r,t)$ and $Xe(r,t)$ rise. We also saw that the two poisons level off at particular values [equation (V-30) and equation (V-31)] for Sm^{149} and for Xe^{135} respectively.

Now let us see what happens to Sm and Xe upon shutting the reactor down.

V-12-1 XENON TRANSIENTS FOLLOWING REACTOR SHUTDOWN

For the sake of simplicity, we shall consider a reactor in which nuclear properties are and stay uniform (even if time-dependent); we shall also ignore flux shape distortions due to non-uniform xenon effects.

Given the rate equations describing the Iodine-135 and Xenon-135 concentration time behavior, it is of particular interest to look at time behavior of these nuclides following reactor shutdown.

We suppose that after operating a reactor for a long period of time at a constant flux level ϕ_o, we suddenly shut the reactor down. An examination of the rate equations (V-16) and (V-17) reveal several consequences of setting the flux ϕ suddenly equal to zero.

First the removal of Xe^{135} due to neutron capture ceases, leaving the Xe^{135} decay as the sole removal mechanism. However the production of Xe^{135} via decay of I^{135} will continue for sometime. Since the half-life of I^{135} is shorter than that of the Xe^{135}, the Xe^{135} concentration may initially build-up before decaying out. In this case, the increasing Xe load injects a negative reactivity for a number of hours. To study this more explicitly, we solve the rate equations (V-18) and (V-19) characterizing the shutdown reactor:

Putting $\phi_1(r,t) = 0$. Equations (V-18) and (V-19) became:

$$\frac{\partial I(r,t)}{\partial t} = -\lambda^I I(r,t) \qquad \text{(V-42)}$$

$$\frac{\partial Xe(r,t)}{\partial t} = \lambda^I I(r,t) - \lambda^{Xe} Xe(r,t). \qquad \text{(V-43)}$$

subject to the initial conditions that at the time of shutdown ($t = t_s = 0$), the iodine and xenon concentrations have alterned their equilibrium values given by equations (V-37) and (V-38) respectively. The iodine equation (V-42) is easy to solve:

$$\boxed{I(r,t) = I_{\infty}(r) \exp(-\lambda^{I} t).} \qquad \text{(V-44)}$$

We can now insert (V-44) into (V-43) and get:

$$\frac{\partial Xe(r,t)}{\partial t} = \lambda^{I} I_{\infty}(r) \exp(-\lambda^{I} t) - \lambda^{Xe} Xe(r,t) \qquad \text{(V-45)}$$

The complete solution of (V-45) is

$$X(r,t) = \{X(r,0) + \int_0^t dt' \lambda^{I} I(r,t') \exp\left(\int_0^{t'} dt'' \lambda^{Xe}\right)\} \cdot \exp\left(-\int_0^t dt'' \lambda^{Xe}\right) \qquad \text{(V-46)}$$

Integrating of equation (V-46) term by term gives:

$$\int_0^t dt'' \lambda^{Xe} = \lambda^{Xe} t \rightarrow \exp\left(-\int_0^t dt'' \lambda^{Xe}\right) = \exp(-\lambda^{Xe} t)$$

$$\int_0^{t'} dt'' \lambda^{Xe} = \lambda^{Xe} t' \rightarrow \exp\left(\int_0^{t'} dt'' \lambda^{Xe}\right) = \exp(\lambda^{Xe} t')$$

$$\int_0^t dt' \lambda^{I} I(r,t') \exp(\lambda^{Xe} t') =$$

$$= \int_0^t dt' \lambda^{I} I_{\infty}(r) \exp(-\lambda^{I} t') \exp \lambda^{Xe} t' =$$

$$\lambda^I I_\infty (\vec{r}) \int_o^t dt' \exp\left[-(\lambda^I - \lambda^{Xe})t'\right] =$$

$$= \frac{\lambda^I I_\infty(r)}{\lambda^{Xe} - \lambda^I}\left[\exp\left(\lambda^{Xe} - \lambda^I\right)t - 1\right]$$

and finally:

$$\left\{\int_o^t dt' \lambda^I I(r,t') \exp \int_o^{t'} dt'' \lambda^{Xe}\right\} \exp -\int_o^t dt'' \lambda^{Xe} =$$

$$= \frac{\lambda^I I_\infty(\vec{r})}{\lambda^I - \lambda^{Xe}}\left[1 - \exp\left(\lambda^{Xe} - \lambda^I\right)t\right] \exp -\left(\lambda^{Xe}t\right) =$$

$$= \frac{\lambda^I I_\infty(r)}{\lambda^I - \lambda^{Xe}}\left[\exp(-\lambda^{Xe}t) - \exp(-\lambda^I t)\right]$$

Grouping of terms and substitution of $Xe(r,t_s)$ in (V-46) gives the general solution for $Xe(r,t)$ after shutdown:

$$Xe(r,t) = \frac{(y^I + y^{Xe})\,\Sigma_{f1}\,\phi_{o1}}{\lambda^{Xe} + \sigma_{a1}^{Xe}\,\phi_{01}(r)} \exp(-\lambda^{Xe}t)$$

$$+ \frac{\lambda^I I_\infty}{\lambda^I - \lambda^{Xe}}\left[\exp(-\lambda^{Xe}t) - \exp(-\lambda^I t)\right] \qquad \text{V-47}$$

V-12-2 Sm^{149} TRANSIENTS FOLLOWING REACTOR SHUTDOWN

If equilibirum conditions have been attained and the reactor has then been shutdown, the time behavior of Pm^{149} and Sm^{149} is given by equation (V-24) and (V-25) with ϕ (r) = 0.

The result is:

$$\frac{\partial Pm(r,t)}{\partial t} = -\lambda^{Pm} Pm(r,t) \qquad \text{(V-48)}$$

$$\frac{\partial Sm(r,t)}{\partial t} = \lambda^{Pm} Pm(r,t) \qquad \text{(V-49)}$$

Solution of equations V-48 and V-49 gives:

$$Pm(r,t) \equiv P(t) = Pm_{\infty} \exp(-\lambda^{Pm} t) \qquad \text{(V-50)}$$

One can now insert this into the samarium equation (V-49) to find:

$$\frac{\partial Sm(r,t)}{\partial t} = \lambda^{Pm} Pm_{\infty} \exp(-\lambda^{Pm} t) \qquad \text{(V-51)}$$

Integration of equation (V-51) gives:

$$Sm(r,t) \equiv Sm(t) = \lambda^{Pm}\, Pm_{\infty} \int_0^t \exp\,(-\lambda^{Pm} t')\, dt'$$

$$= -\frac{\lambda^{Pm}\, Pm_{\infty}}{\lambda^{Pm}} \left[\exp(-\lambda^{Pm} t) - 1 \right] \qquad \text{(V-52)}$$

$$= Pm_{\infty} \left[1 - \exp\,(-\lambda^{Pm} t) \right]$$

Substitution of Pm_{∞} given by (V-39) into (V-51) and (V-52) gives:

$$\boxed{Pm(t)_{shutdown} = \frac{y^{Pm} \Sigma_{f1} \phi_{01}}{\lambda^{Pm}} \exp\,(-\lambda^{Pm} t)} \qquad \text{(V-53)}$$

and

$$\boxed{Sm(t)_{shutdown} = \frac{y^{Pm} \Sigma_{f1} \phi_{01}}{\lambda^{Pm}} \left[1 - \exp\,(-\lambda^{Pm} t) \right]} \qquad \text{(V-54)}$$

In summary, the time behavior of the most important fission products (^{135}Xe and ^{149}Sm) and their precursors following reactor shutdown is given by Table V-2:

TABLE V-2

ISOTOPE	SOLUTION AT SHUTDOWN
$^{135}_{I}$	$I(t) = \dfrac{\gamma^{I}\Sigma_{f1}\phi_1}{\lambda^{I}} \exp(-\lambda^{I}t)$
$^{135}_{Xe}$	$Xe(t) = \dfrac{(\gamma^{I} + \gamma^{Xe}\Sigma_{f1}\phi_1(r)}{\lambda^{Xe} + \sigma_{a1}^{Xe}\phi_{01}} \left[\exp(-\lambda^{Xe}t) + \dfrac{\gamma^{I}\Sigma_{f1}\phi_1}{\lambda^{I} - \lambda^{Xe}}\right]$ $\left[\exp(-\lambda^{Xe}t) - \exp(-\lambda^{I}t)\right]$
$^{149}_{Pm}$	$P_m(t) = \dfrac{\gamma^{Pm}\Sigma_{f1}\phi_{01}}{\lambda^{Pm}} \exp(-\lambda^{Pm}t)$
$^{149}_{Sm}$	$S_m(t) = \dfrac{\gamma^{Pm}\Sigma_{f1}\phi_{01}}{\lambda^{Pm}} \left[1 - \exp(-\lambda^{Pm}t)\right]$

V-13 EFFECT OF Xe POISONING ON REACTIVITY

The equilibrium concentration Xe^{∞} V-13 is of interest because we can substitute it into (Eq.V-15) to determine the negative reactivity which the xenon will contribute under steady-state power conditions and following reactor shutdown. From section V-7 we know that

$$\Delta\rho\tilde{=} = -\frac{\Sigma_a^P}{\Sigma_a}\left(\frac{1}{1 + L^2B_g^2}\right)$$

If we neglect the term $L^2B_g^2$ ($<10^{-3}$) which is quite legitimate in large power reactor cores, the reactivity change is just the fraction of total macroscopic absorption cross section due to the poison then we have

$$\Delta\rho\tilde{=} = -\frac{\Sigma_a^P}{\Sigma_a} \qquad \text{(V-55)}$$

and for the xenon we have

$$\Delta\rho\tilde{=} = -\frac{\Sigma_a^{Xe}}{\Sigma_a} \simeq -\frac{n^{Xe}\sigma_{al}^{Xe}}{\Sigma_a}$$

$$\simeq -\frac{\sigma_{al}^{Xe}(y^I + y^{Xe})\Sigma_{fl}\,\phi_{01}}{\Sigma_a\,(\lambda^{Xe} + \sigma_{al}^{Xe}\,\phi_{01})} \qquad \text{(V-56)}$$

If we consider the fact that in a large, critical thermal reactor, leakage can be ignored (approximate)

$$k = (\nu \Sigma_f / \Sigma_a) p\varepsilon = 1 \text{ implies}$$

$$\frac{\Sigma_f}{\Sigma a} = \frac{1}{\nu p \varepsilon} \simeq \frac{1}{\nu} \tag{V-57}$$

and therefore

$$\Delta\rho \cong - \frac{\sigma_{al}^{Xe}(y^I + y^{Xe})\phi_{01}}{\nu(\lambda^{Xe} + \sigma_{al}^{Xe}\phi_{01})} \tag{V-58}$$

Notice this reactivity will depend on the flux level ϕ_{01}. However for large flux levels, characteristic of large power reactors, (Ref. 1, 2, 3)

$$\phi_{01} \gg \frac{\lambda^x}{\sigma_{al}^{Xe}} = .756 \times 10^{13} \text{ cm}^{-2} \cdot \text{sec}^{-1} \tag{V-59}$$

and the xenon reactivity will approach a maximum valve of

$$\Delta\rho\text{+} = - \frac{(y^I + y^{Xe})}{\nu} \tag{V-60}$$

This amount to $\Delta\rho \simeq .026$ for both U^{233} and U^{235} - fueled reactors, a rather sizable reactivity. This of course, must be accounted for in determining excess reactivity and control system requirements.

At shutdown we have:

$$\Delta\rho(t)/\text{shutdown} \simeq Xe^{\infty}\exp(-\lambda^{Xe}t) + \frac{\lambda^{I}I^{\infty}}{\lambda^{I}-\lambda^{Xe}}\left[\exp(-\lambda^{Xe}t) - \exp(-\lambda^{I}t)\,\sigma_{a1}^{Xe}\right]$$

$$\Sigma_a$$

$$\simeq\{(y^{I}+y^{Xe})\Sigma_{f1}\,\phi_{01}\exp(-\lambda^{Xe}t) + \frac{y^{I}\Sigma_{f1}\,\phi_{01}}{\lambda^{I}-\lambda^{Xe}}\left[\exp(-\lambda^{Xe}t - \exp(-\lambda^{I}t)\right]\}$$

$$\lambda^{Xe} + \sigma_{a1}^{Xe}\phi_{01}$$

$$\frac{\sigma_{a1}^{Xe}}{\Sigma_a} \qquad \text{(V-61)}$$

$$\simeq\left\{\frac{(y^{I}+y^{Xe})\phi_{01}\exp(-\lambda^{Xe}t)\sigma_{a1}^{Xe}}{\lambda^{Xe}+\sigma_{a1}^{Xe}\phi_{01}} + \frac{y^{I}\phi_{01}\sigma_{a1}^{Xe}}{\lambda^{I}-\lambda^{Xe}}\left[\exp(-\lambda^{Xe}t)-\exp(-\lambda^{I}t)\right]\right\}\frac{\Sigma_{f1}}{\Sigma_a}$$

$$\simeq -\frac{1}{\upsilon\varepsilon p}\left\{\frac{(y^{I}+y^{Xe})\phi_{01}}{\frac{\lambda^{Xe}}{\sigma_{a1}^{Xe}}+\phi_{01}}\exp(-\lambda^{Xe}t) + \frac{y^{I}\phi_{01}\sigma_{a1}^{Xe}}{\lambda^{I}-\lambda^{Xe}}\left[\exp(-\lambda^{Xe}t)-\exp(-\lambda^{I}t\right]\right\}$$

typical values for $|\Delta\rho(t)|$ are plotted at the end of the chapter. The maximum value of negative reactivity depends quite sensitively on the flux level prior to shutdown. In fact unless:

$$\phi_{01} > \frac{y^{Xe}\lambda^{Xe}}{y^{I}\sigma_{a1}^{Xe}} \qquad \text{(V-62)}$$

no buildup of xenon following shutdown will occur. However since the ratio in (V-62) is quite small (4×10^{11} and 3×10^{12} cm^{-2} sec^{-1} in U-235 and U-233 fueled reactors, respectively), a xenon transient buildup will occur following shutdown in most power reactors.

We can calculate the time at which the maximum negative reactivity occurs as: (differentiating and equating to zero)

$$t_{max} = \frac{1}{\lambda^I - \lambda^{Xe}} \ln \left[\frac{\lambda_I/\lambda_{Xe}}{1 + \frac{\lambda^{Xe}}{\lambda^I}\left(\frac{\lambda^I}{\lambda^{Xe}} - 1\right)\frac{Xe^\infty}{I^\infty}} \right]$$

$$= \frac{1}{\lambda^I - \lambda^{Xe}} \ln \left[\frac{\frac{\lambda^I}{\lambda^{Xe}}}{1 + \frac{\lambda^{Xe}}{\lambda^I}\left(\frac{\lambda^{Xe}}{\lambda^I} - 1\right) \quad \frac{y^I + y^{Xe}\phi_{01}}{\lambda^{Xe} + \sigma_{al}^{Xe}\phi_{01}} \Big/ \frac{y^I \phi_{01}}{\lambda^I}} \right] \qquad \text{(V-63)}$$

and for $\phi_{01} \gg \frac{\lambda^{Xe}}{\sigma_{al}^{Xe}}$ (V-64)

$$t_{max} = \frac{1}{\lambda^I - \lambda^{Xe}} \ln \left(\frac{\lambda^I}{\lambda^{Xe}}\right) \approx 11.6 \text{ h.} \qquad \text{(V-65)}$$

(Note in (Fig. V-11) how curves' maxima begin at t = 0 for small fluxes, but approach this value (11.6 h.) at higher fluxes.)

V-14 SPATIAL EFFECTS IN FISSION PRODUCT POISONING

In our earlier estimate of the reactivity change caused by fission product poisoning, we merely computed the changes in the thermal utilization as the poison concentration built up. However such an estimate is actually only valid for a uniform concentration of poison in an infinite core. More generally, we know that the poison concentration will depend on the spatial variation of the flux in a finite core, and hence our estimate of reactivity worth of the poison is only of limited validity. We can improve this estimate by using perturbation theory.

Let us compute the reactivity due to the equilibrium xenon concentration in a core operating with a steady-state flux $\phi_0(r)$. Then we know

$$\Sigma_a^{Xe}(r) = \sigma_a^{Xe} N^{Xe}(r) = \frac{(y_I - y_X)\Sigma_f(r)\phi_0(r)}{(\lambda^{Xe}/\sigma_a^{Xe} + \phi_0(r)} \tag{V-66}$$

If we treat this as a perturbation in the absorption cross section, $\delta\Sigma_a \to \Sigma_a^{Xe}(r)$, then we can use the first-order perturbation result to find the reactivity change due to the xenon as (Ref. 4)

$$\Delta\rho = -\frac{\int d^3r \phi_0^2(r)\Sigma_a^{Xe}(r)}{\nu\int d^3r \phi_0^2(r)\Sigma_f(r)} = -\frac{(y^I + y^{Xe})}{\nu} \frac{\int d^3r \frac{\Sigma_f(r)\phi_0^3(r)}{\lambda^{Xe}/\sigma_a^{Xe} + \phi_0(r)}}{\int d^3r \phi_0^2(r)\Sigma_f(r)} \tag{V-67}$$

This expression can be evaluated numerically to compute the equilibrium reactivity worth of the xenon poisoning in a core. Note that for large power levels, $\phi_0 >> \lambda^{Xe}/\sigma_a^{Xe}$, this reduces to just our earlier approximate

result for an infinite medium [Eq V.60]

$$\Delta\rho \underset{\phi_0 >> \lambda^{Xe}/\sigma_a^{Xe}}{\rightarrow} -\frac{(y_I + y_X)}{\nu} \equiv \Delta\rho_\infty \qquad \text{(V-68)}$$

For low flux levels, $\phi_0 << \lambda^{Xe}/\sigma_a^{Xe}$, in an uniform core (e.g., a low-power research reactor)

$$\Delta\rho \underset{\phi_0 << \lambda^{Xe}/\sigma_a^{Xe}}{\rightarrow} -\frac{(y_I + y_X)}{\nu} \frac{\int d^3r \phi_0^3(r)}{\int d^3r \phi_0^2(r)} \qquad \text{(V-69)}$$

For example, in a slab, one finds that in this limit

$$\Delta\rho \rightarrow {}^4/_3 \, \Delta\rho_\infty \qquad \text{(V-70)}$$

where $\Delta\rho_\infty$ is the reactivity change characterizing an infinite medium model. That is, the reactivity worth is some 33% larger when the spacial variation of the flux is taken into account. In a low-power reactor core, this spatial weighting factor is more typically between 2.5 and 3.0.

V-14.1 XENON-INDUCED POWER OSCILLATIONS IN LARGE POWER REACTORS: CANDU

Thus far we have ignored the effect of the reactivity introduced by the fission product poisons on the neutronic behavior of the core. The interaction between xenon fission product buildup and the changes in the

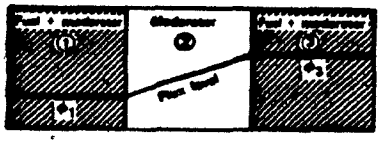
neutron flux distribution that accompanies local changes in reactivity can lead to spatial oscillations in the power distribution in a large thermal reactor core.

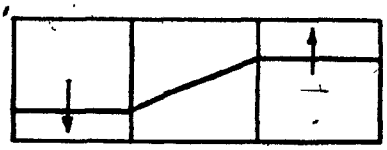
To explain this phenomenon, let us first consider a point-reactor model in which the reactor has been operating at a steady-state flux level for a period of time. Then our earlier study of xenon fission product poisoning has indicated that there will be a saturation in the Xe^{135} concentration resulting from the balance of Xe^{135} production (via direct fission and I^{135} decay) and loss due to decay to Cs^{135} and transmutation (via neutron absorption) to Xe^{136}. Now suppose a small perturbation increase in the flux occurs. Then Xe^{135} will transmute more rapidly to Xe^{136} (instantaneously), depleting the Xe^{135} concentration, hence decreasing the absorption and increasing the reactivity and the flux. However the increased flux transmutes even more Xe^{135}, and hence the initial flux perturbation grows with time (unstable). Such an instability can only exist for power levels higher than a certain threshold value. For U^{235}-fueled reactors, this threshold is 3×10^{11} neutrons/cm^2.sec. Below this threshold the stabilizing effect of the direct xenon yield from fission is more important than the destabilizing effect of the xenon decaying from I^{135}.

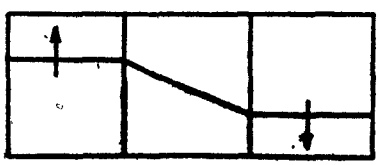
Actually this type of instability is relatively unimportant in practical reactor operations, since it is easily controlled by normal control rod movement. A much more serious spatial xenon instability can arise, however, which requires a more complex control rod program. To understand this, consider the very simple model illustrated in Figure (V-9) consisting of two coupled xenon-unstable point reactors, separated by a region of nonmultiplying material. Suppose further that there is a control system

keeping the total power of all three core regions constant (although the flux or power in an individual region is not constant).

A slight increase in the power level on one side of the core will give rise to the unstable xenon process described for a point reactor. Since the control system keeps the total power constant, the flux on the other side decreases. This process continues with an increasingly steeper tilt in the flux resulting. Two effects will limit this tilt: (a) burnup of most of the xenon on the high flux side and (b) the steep flux tilt which creates a flux gradient and hence a current that carries all the excess neutrons being produced to the other side. The flux will remain tilted for several hours. Eventually the high flux side will have created an I^{135} concentration much greater than that originally present. Since the decay half-life of I^{135} is 6.7 hours, more xenon will be created after this delay period. Similarly on the low side less xenon will be created. This reverses the flux tilt eventually and produces a side-to-side oscillation with a period of 15-30 hours. Thus the xenon process tends to be self-limiting and produces the effect of a moving "hot spot" to the reactor operator.







A simple model of xenon-induced power oscillations.

Figure V-9

The oscillation of the power distribution in a reactor can cause changes in the power-peaking factor F_q^N. (Ref. 2) However this can be easily accounted for in design to ensure that thermal performance constraints are not exceeded. Such xenon-induced power oscillations are more of an operational than a safety problem.

Although such oscillations can be controlled by control rod motion, there is strong motivation to design the reactor core in such a way that xenon oscillations are minimized, since the load-following requirements of a power reactor operating on a utility power grid imply that such control adjustments may incur considerable economic penalty. In general, negative feedback mechanisms such as moderator void formation or the Doppler effect will tend to suppress xenon-induced power oscillations.

The stability of the spatial power distribution with respect to xenon-induced oscillations will decrease with increasing core size or decreasing neutron migration length. As a rule of thumb, xenon oscillations will be a problem if the reactor core is over 30 migration lengths in size (Ref. 5). Since this is the case with most large power reactor cores, xenon oscillations can represent a serious design and control consideration that will be the topic for the extension of this work.

V.15 PLOTS OF FISSION PRODUCT POISON CONCENTRATIONS

FIGURE	TITLE
V.10 and V.11	Reactivity Change at Shutdown
V.12 and V.13	X^{135} at Startup
V.14 and V.15	Xe^{135} at Shutdown
V.16 and V.17	I^{135} at Startup
V.18 and V.19	I^{135} at Shutdown
V.20 and V.21	Sm^{149} at Startup
V.22 and V.23	Sm^{149} at Shutdown
V.24 and V.25	Pm^{149} at Startup
V.26 and V.27	Pm^{149} at Shutdown

Data for these plots are taken from Appendix C. [Ref. 5]

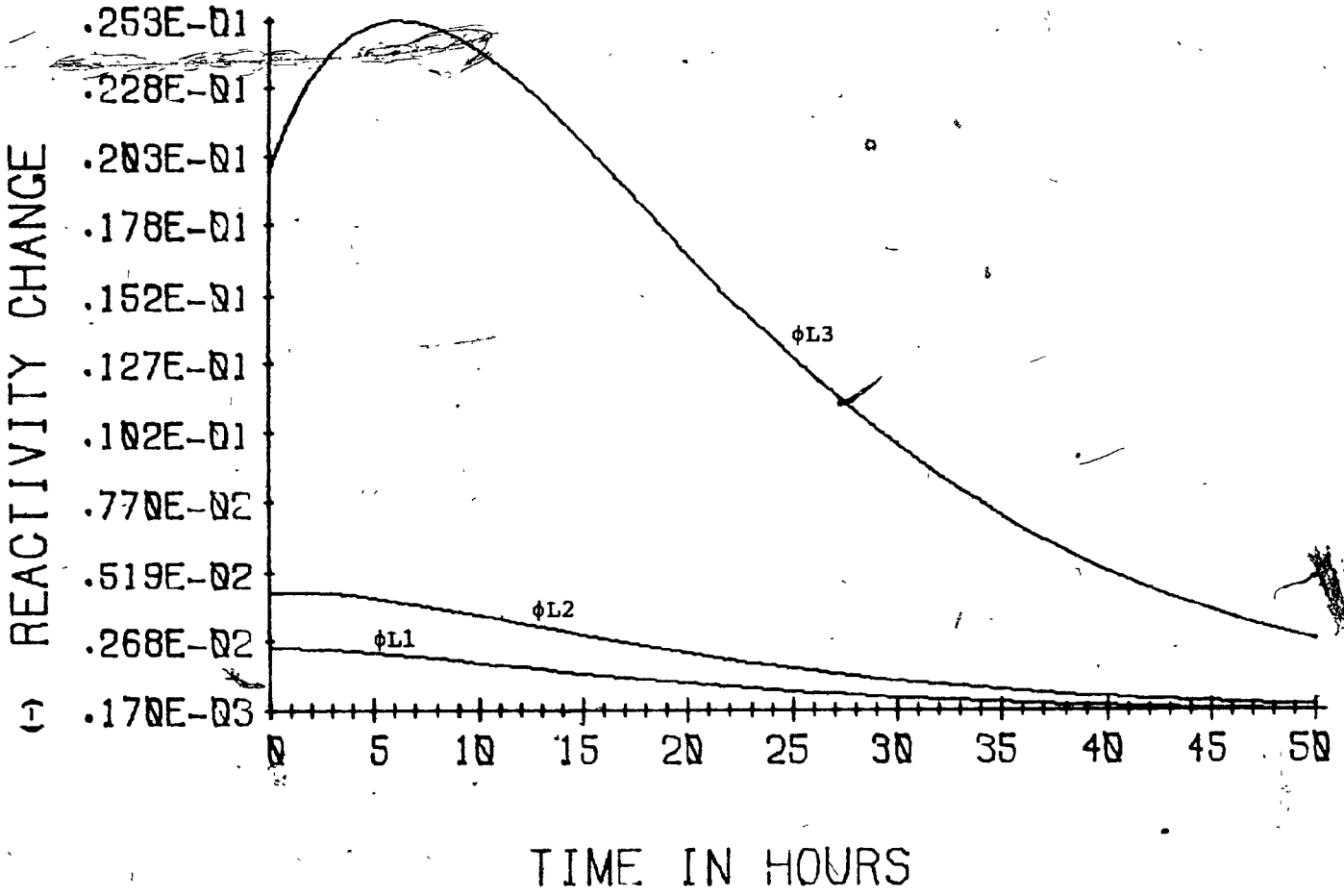


Figure V.10 Reactivity Time Behavior Following Reactor Shutdown

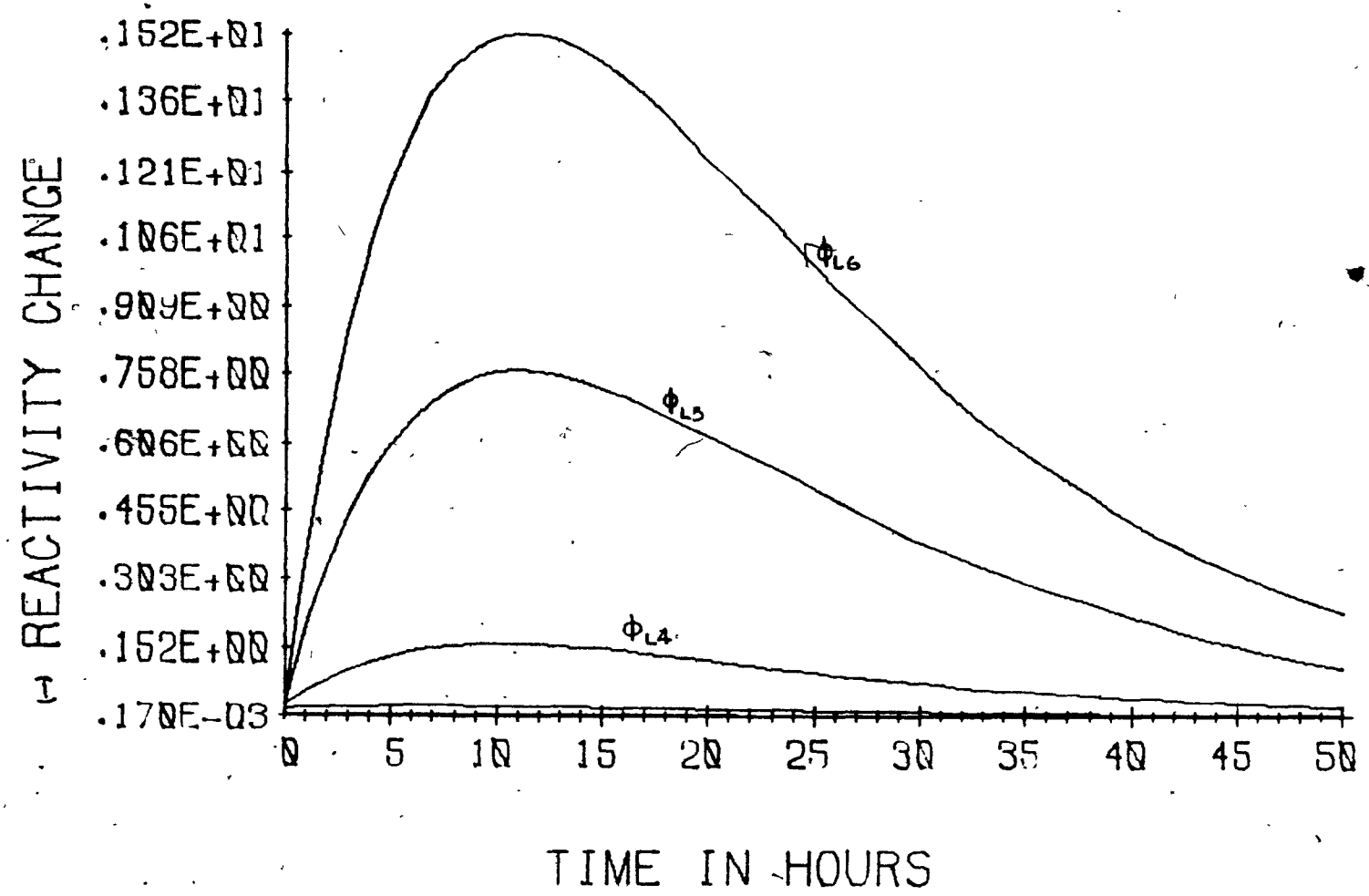


Figure V.11 Reactivity Time Behavior Following Reactor Shutdown

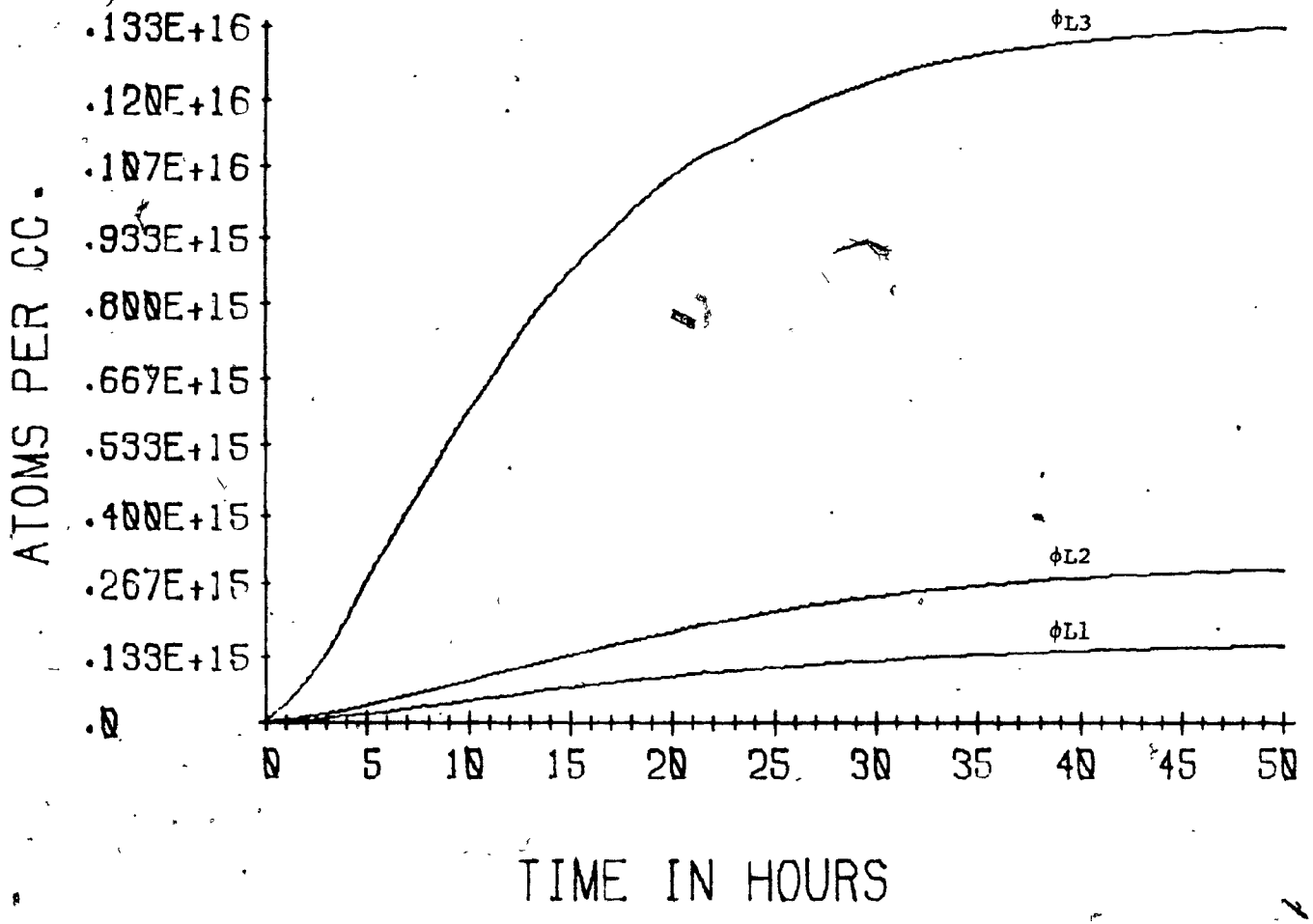


Figure V.12 Xenon Concentration Following Reactor Startup

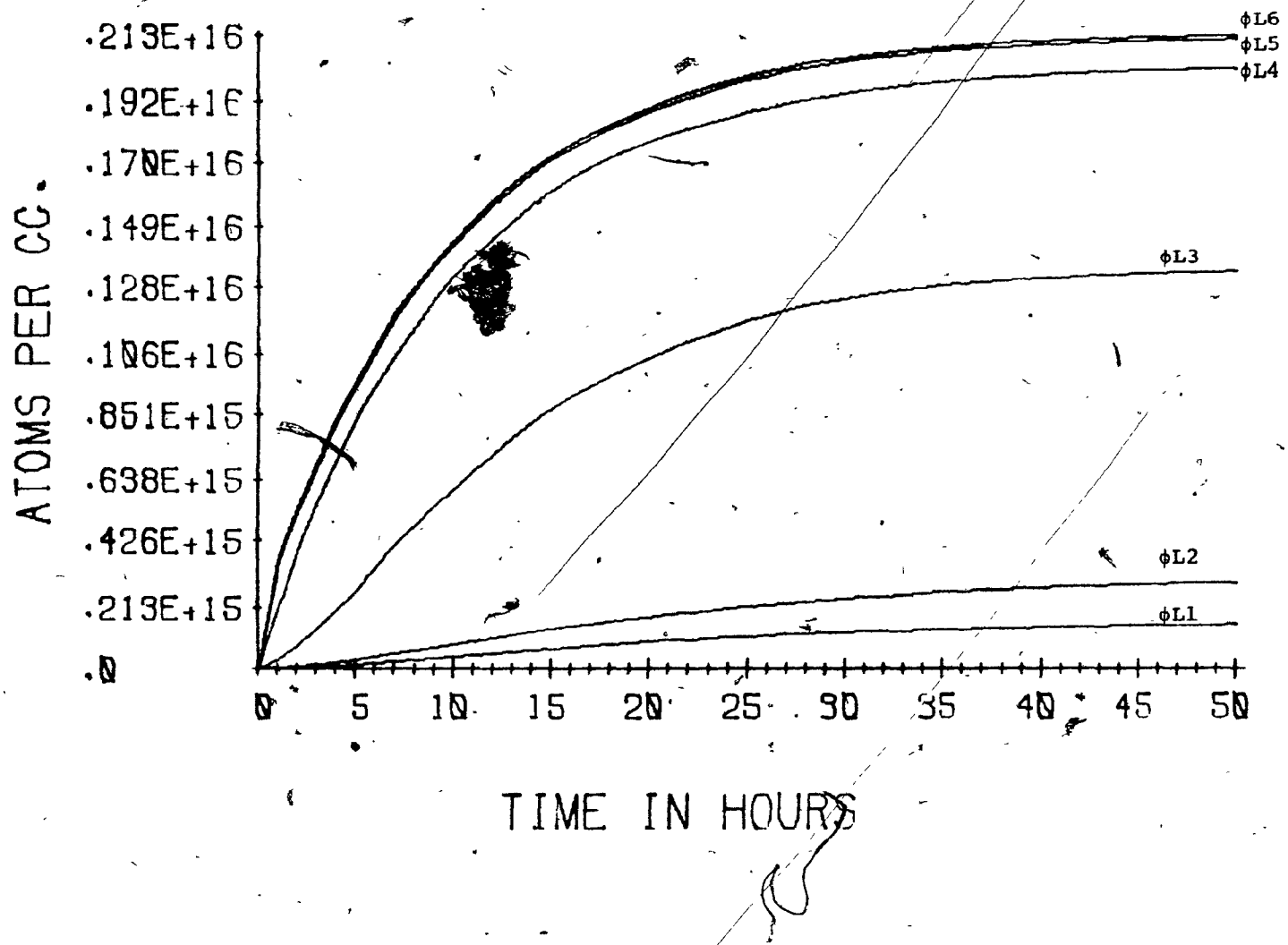


Figure V.13 Xenon Concentration Following Reactor Startup

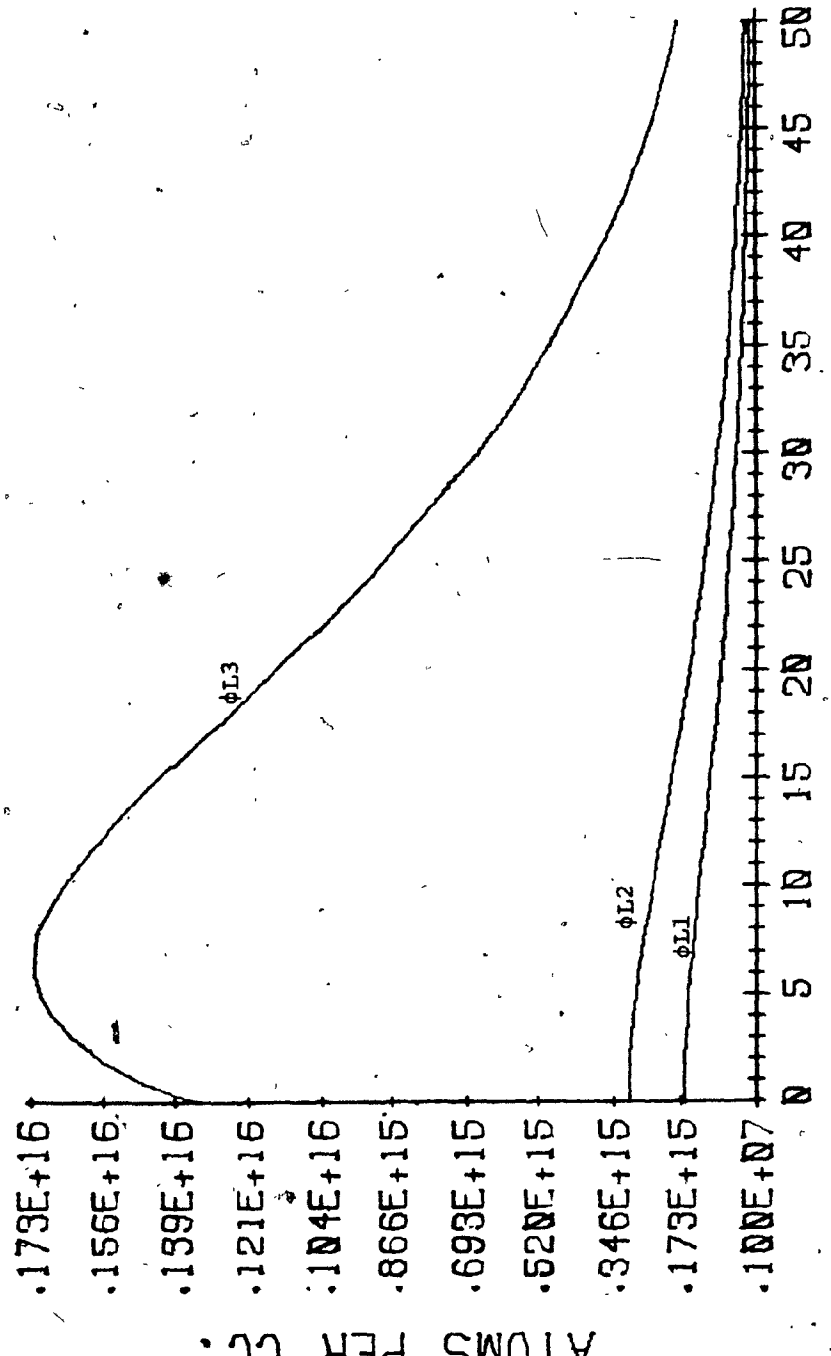


Figure V.14 Xenon Concentration Following Reactor Shutdown

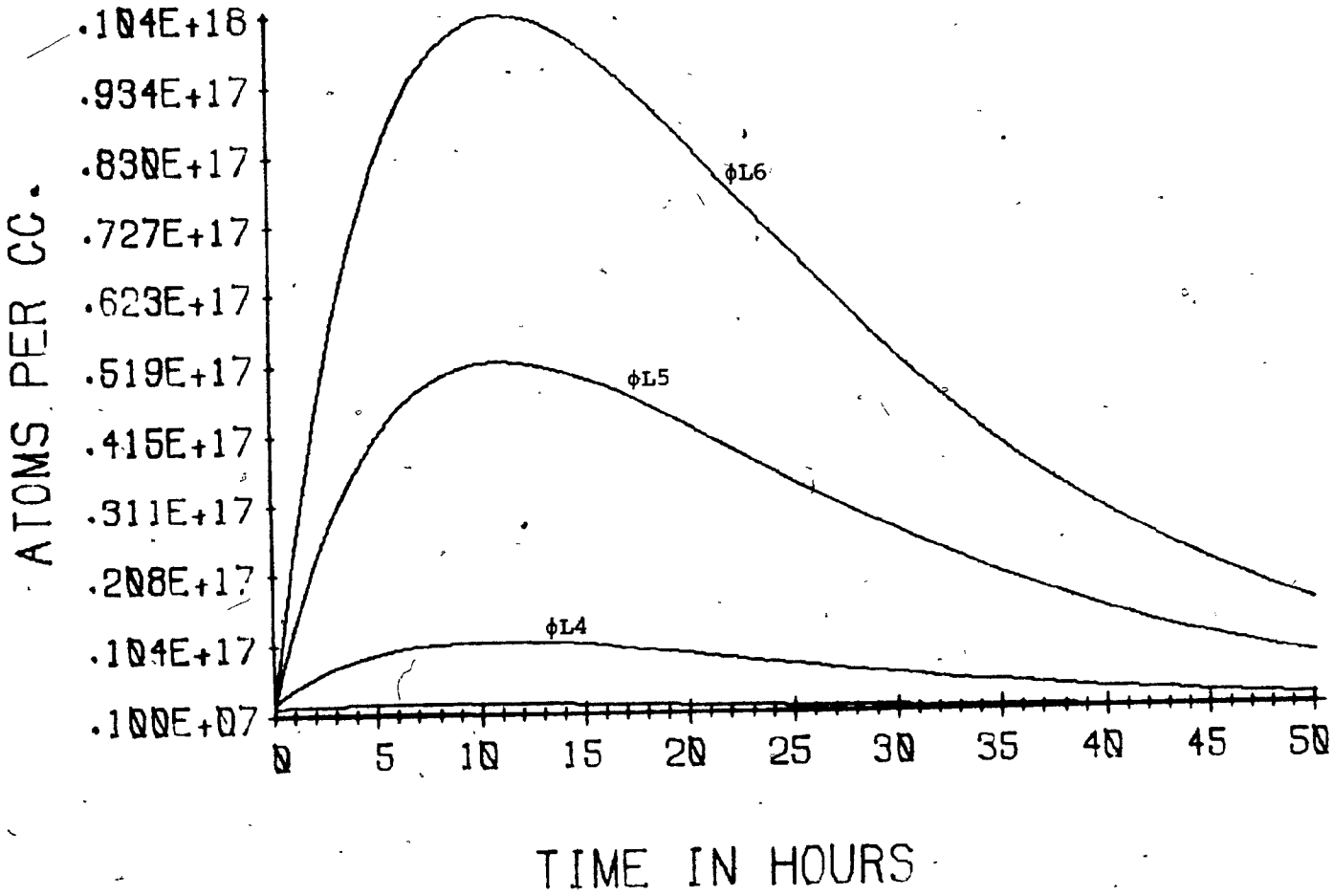


Figure V.15 Xenon Concentration Following Reactor Shutdown

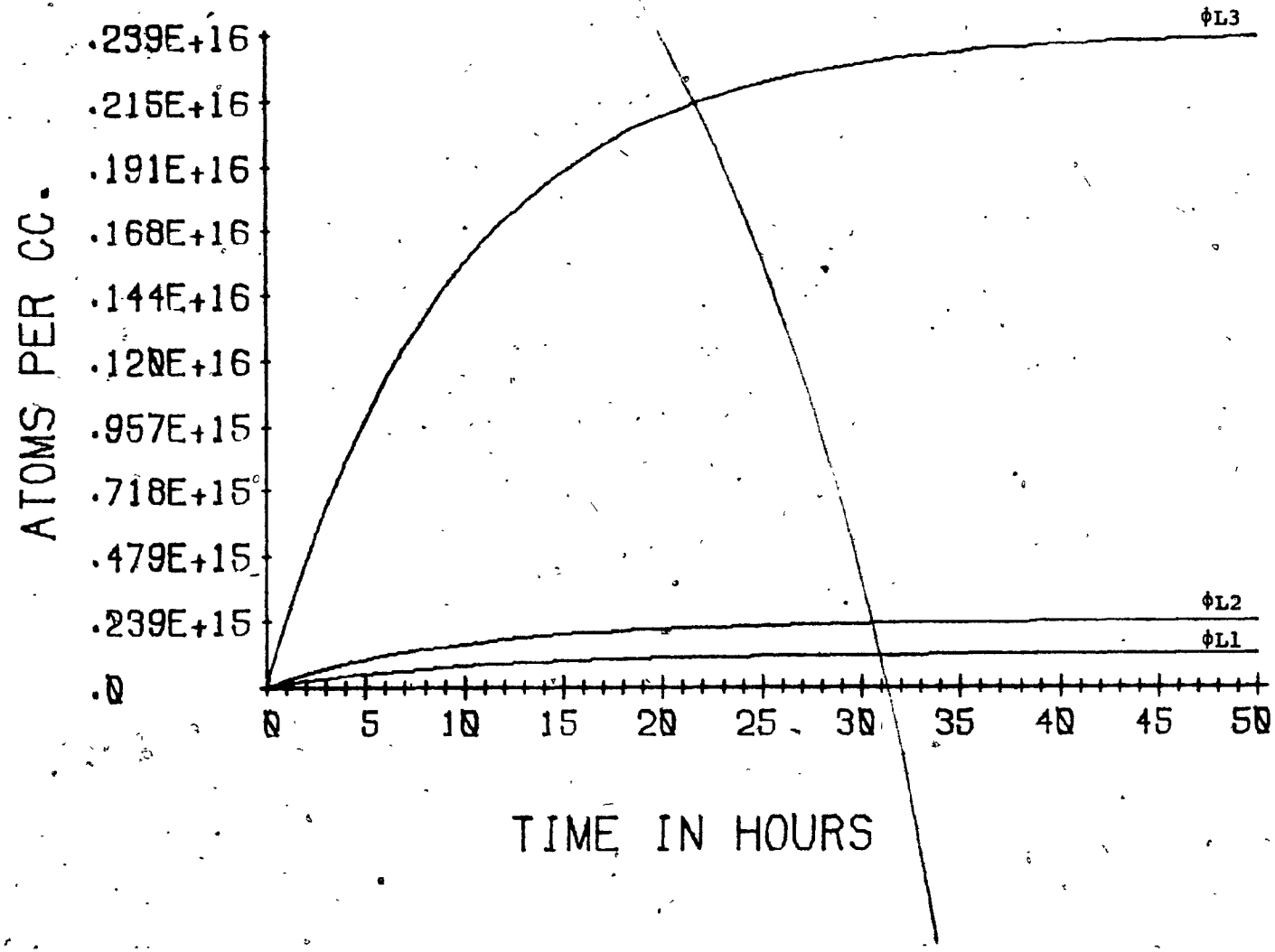


Figure V.16 I^{135} Concentration Following Reactor Shutdown

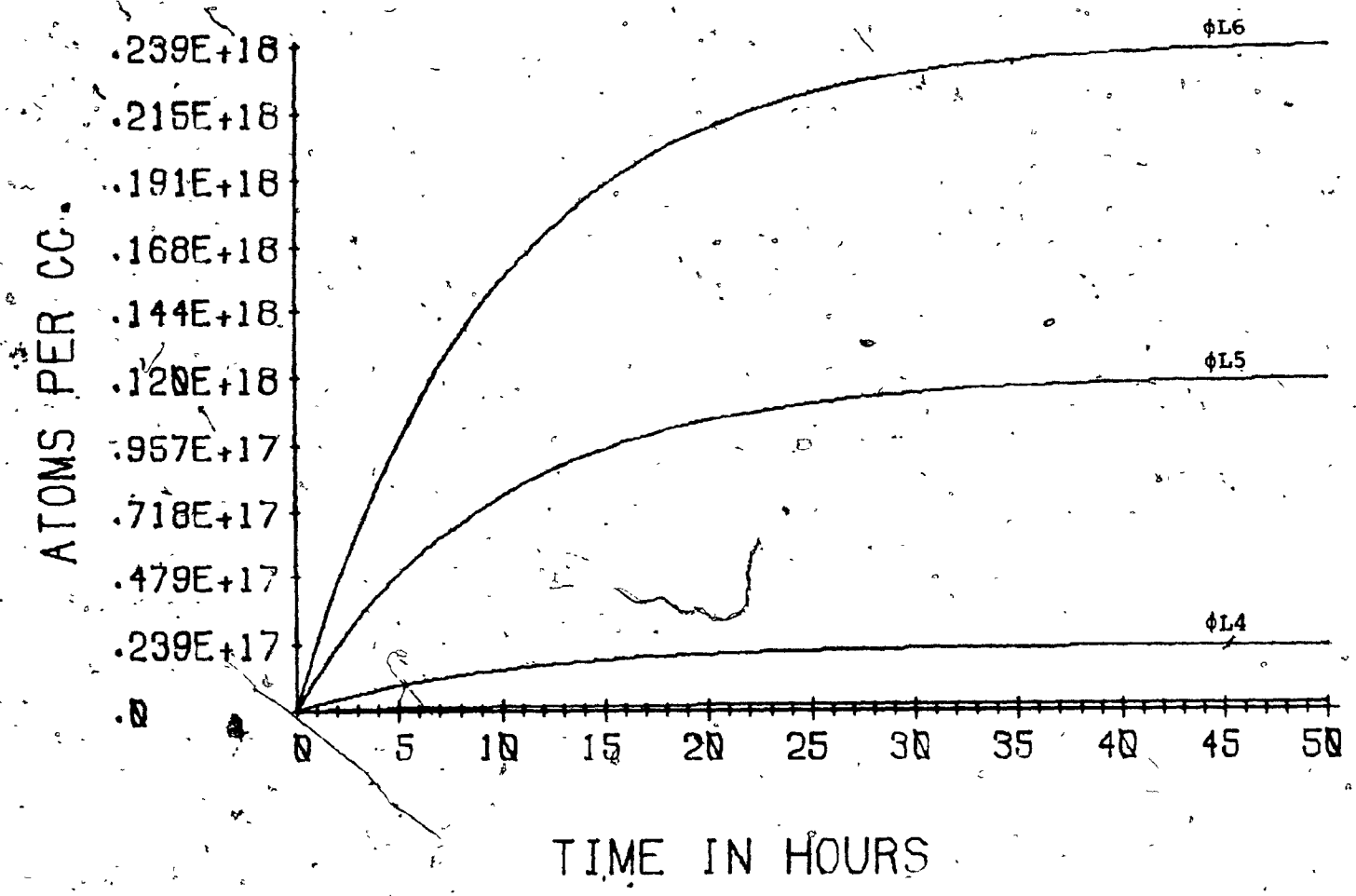


Figure V.17 I^{135} Concentration Following Reactor Startup

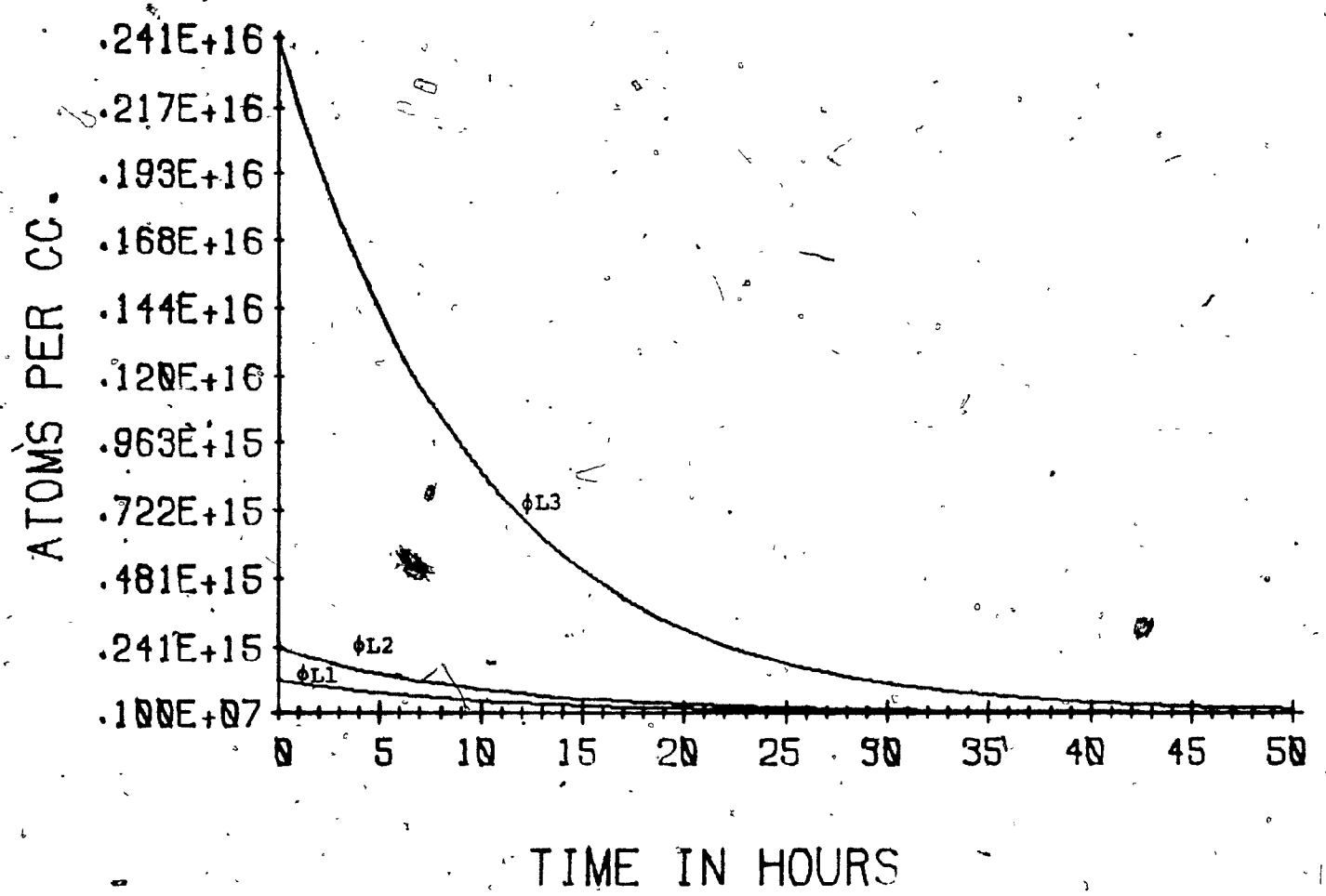


Figure V.18 I^{135} Concentration Following Reactor Shutdown

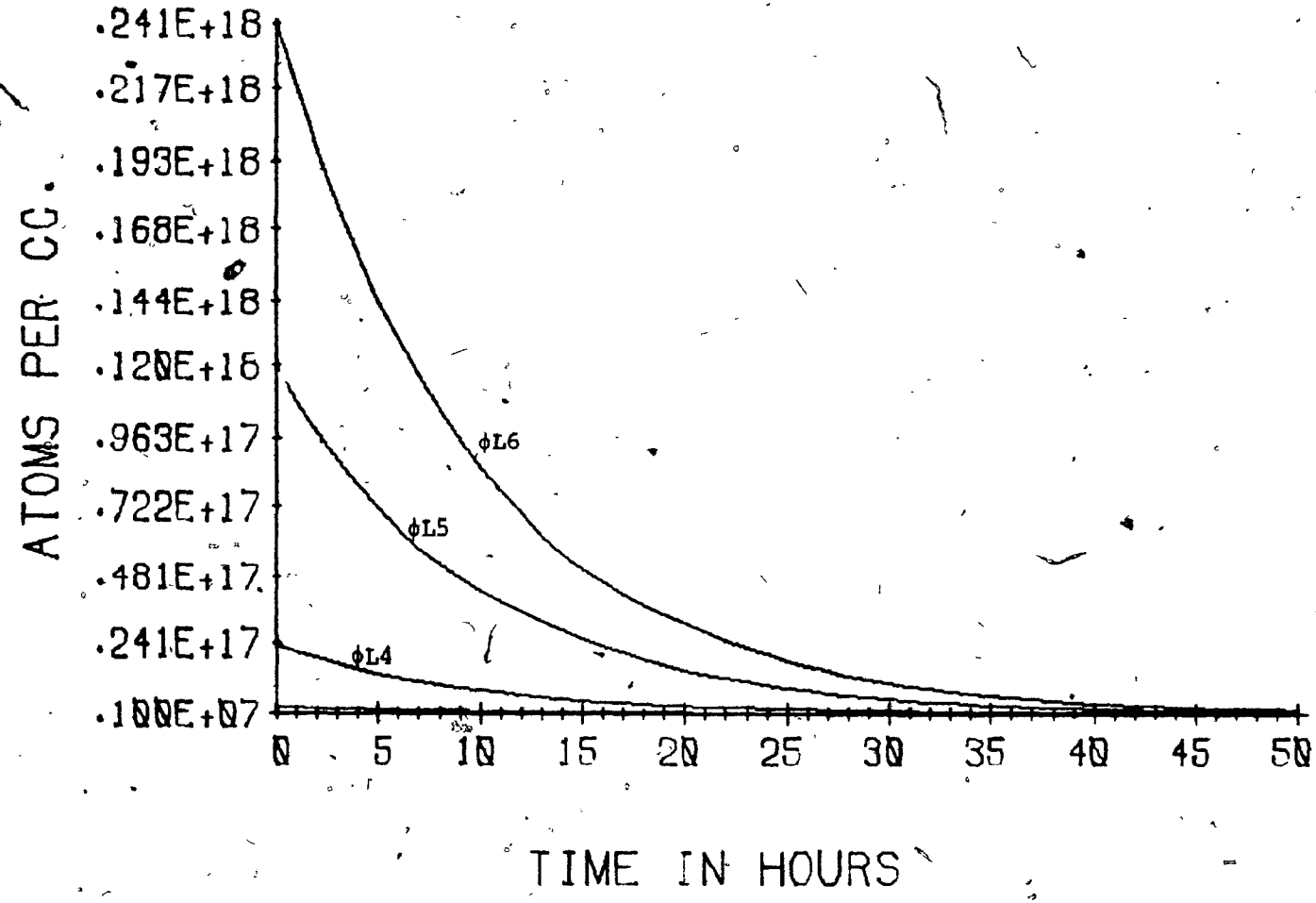


Figure V.19 I^{135} Concentration Following Reactor Shutdown

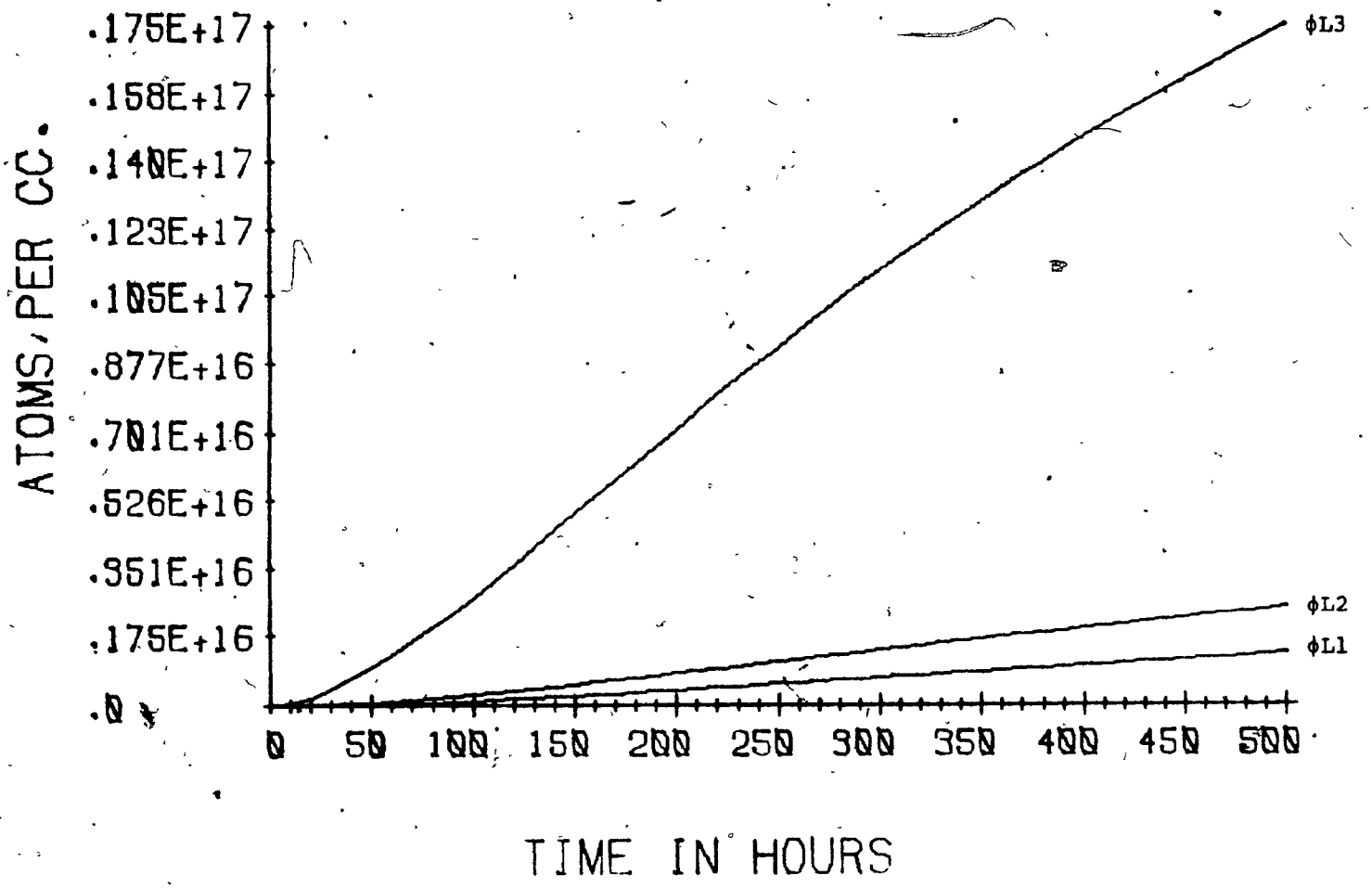


Figure V.20 Samarium Concentration Following Reactor Startup

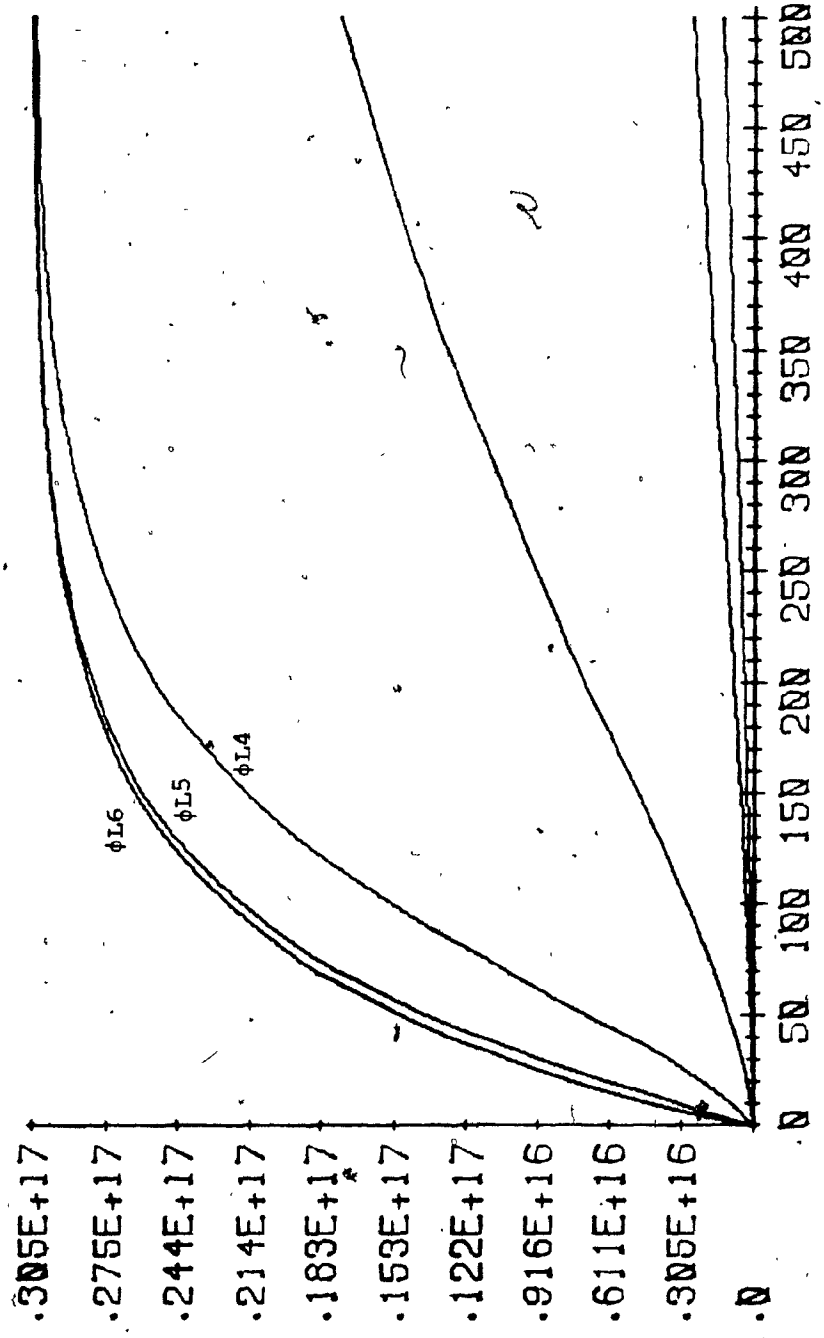


Figure V.21 Samarium Concentration Following Reactor Startup

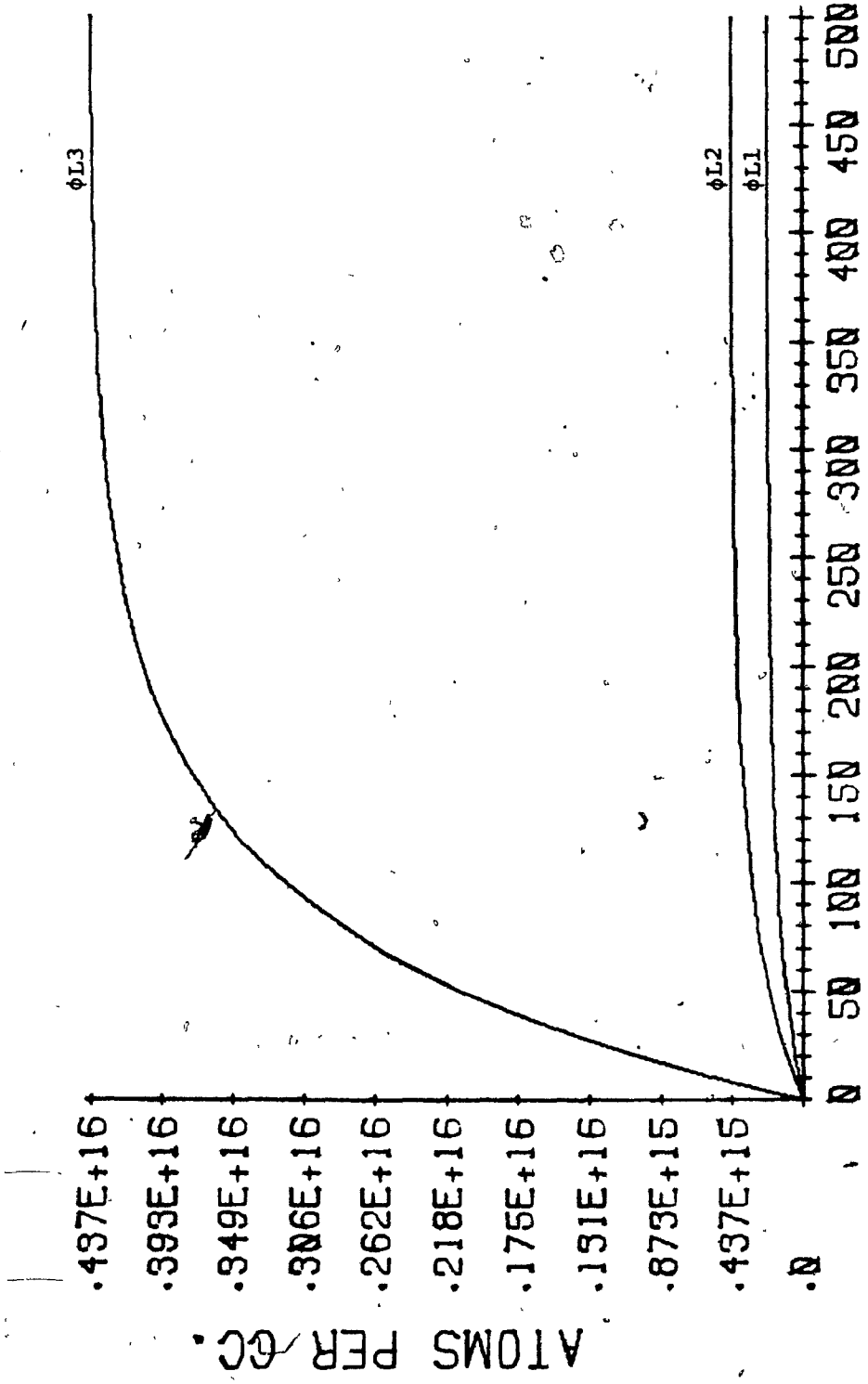


Figure V.22 Samarium Concentration Following Reactor Shutdown

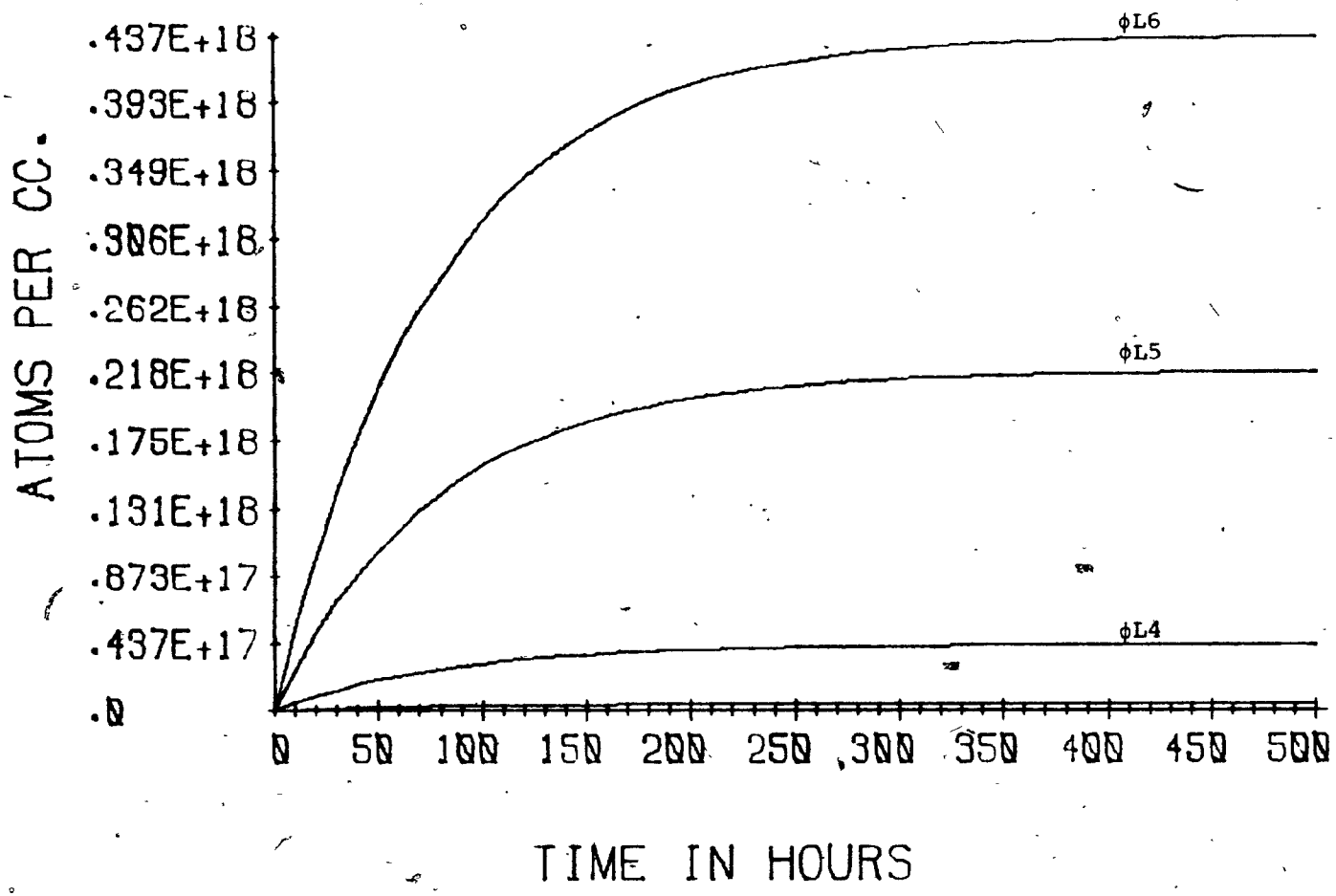


Figure V.23 Samarium Concentration Following Reactor Shutdown

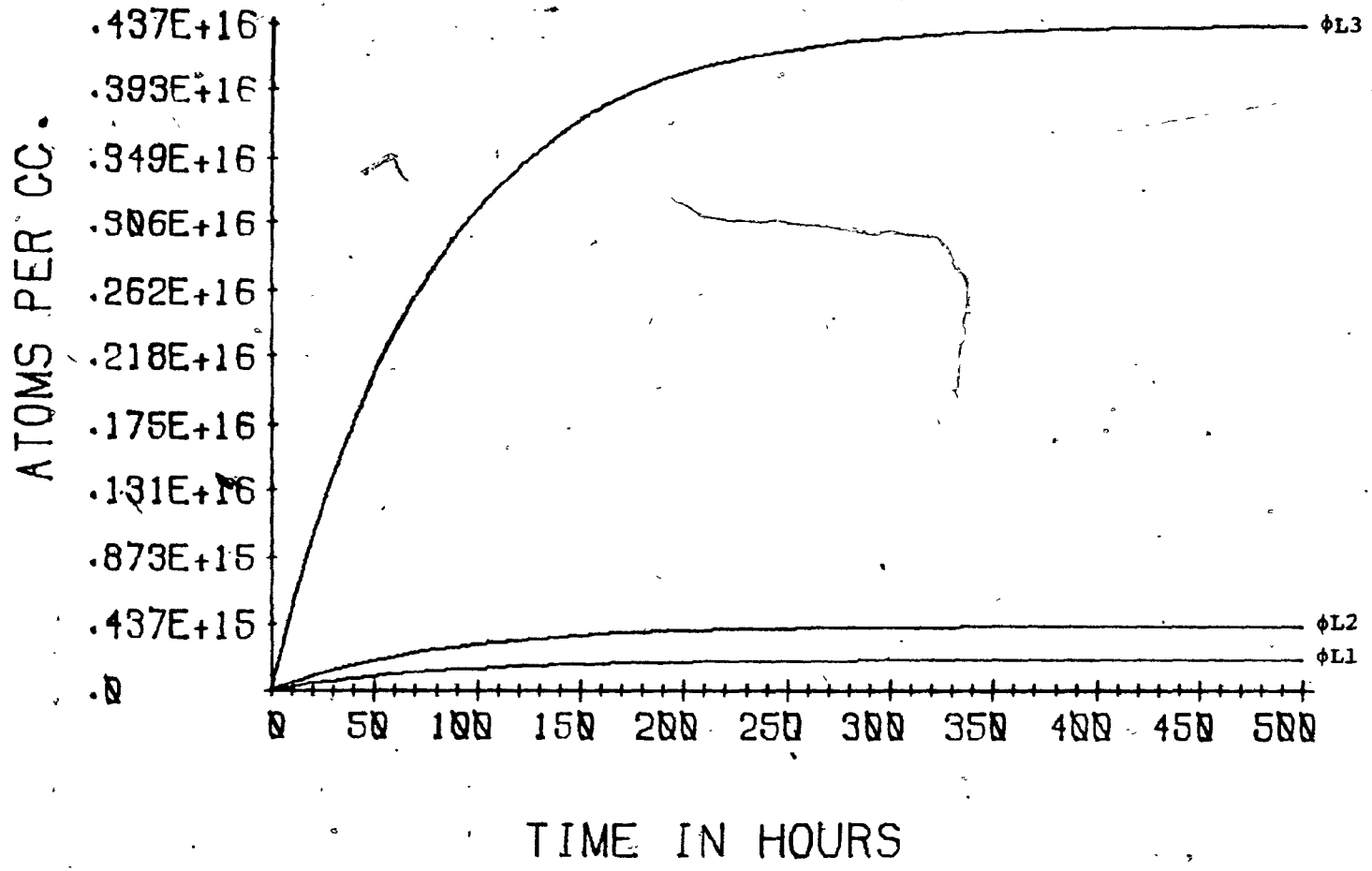


Figure V.24 Prometium Concentration at Startup

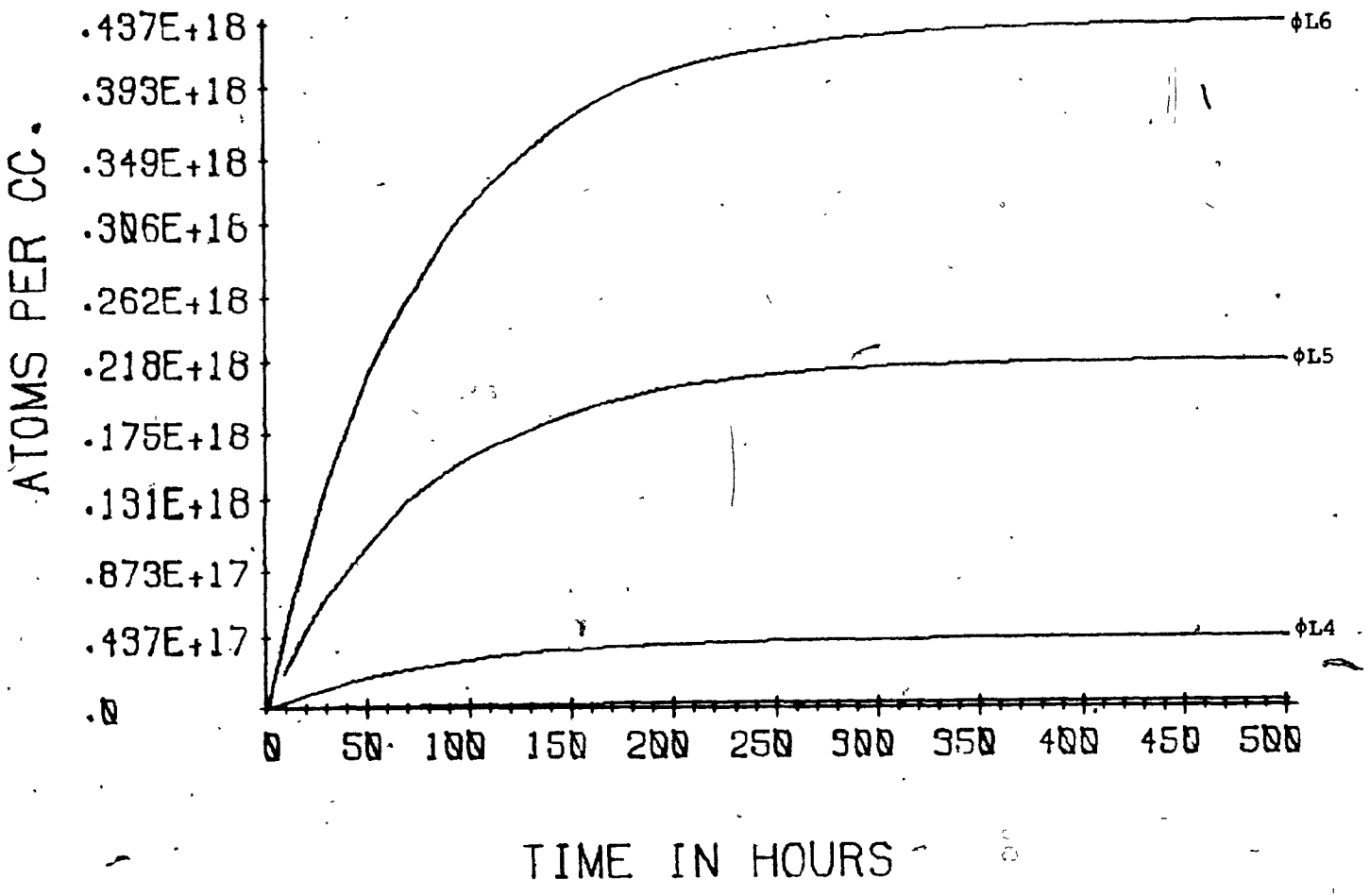


Figure V.25 Prometium Concentration at Startup

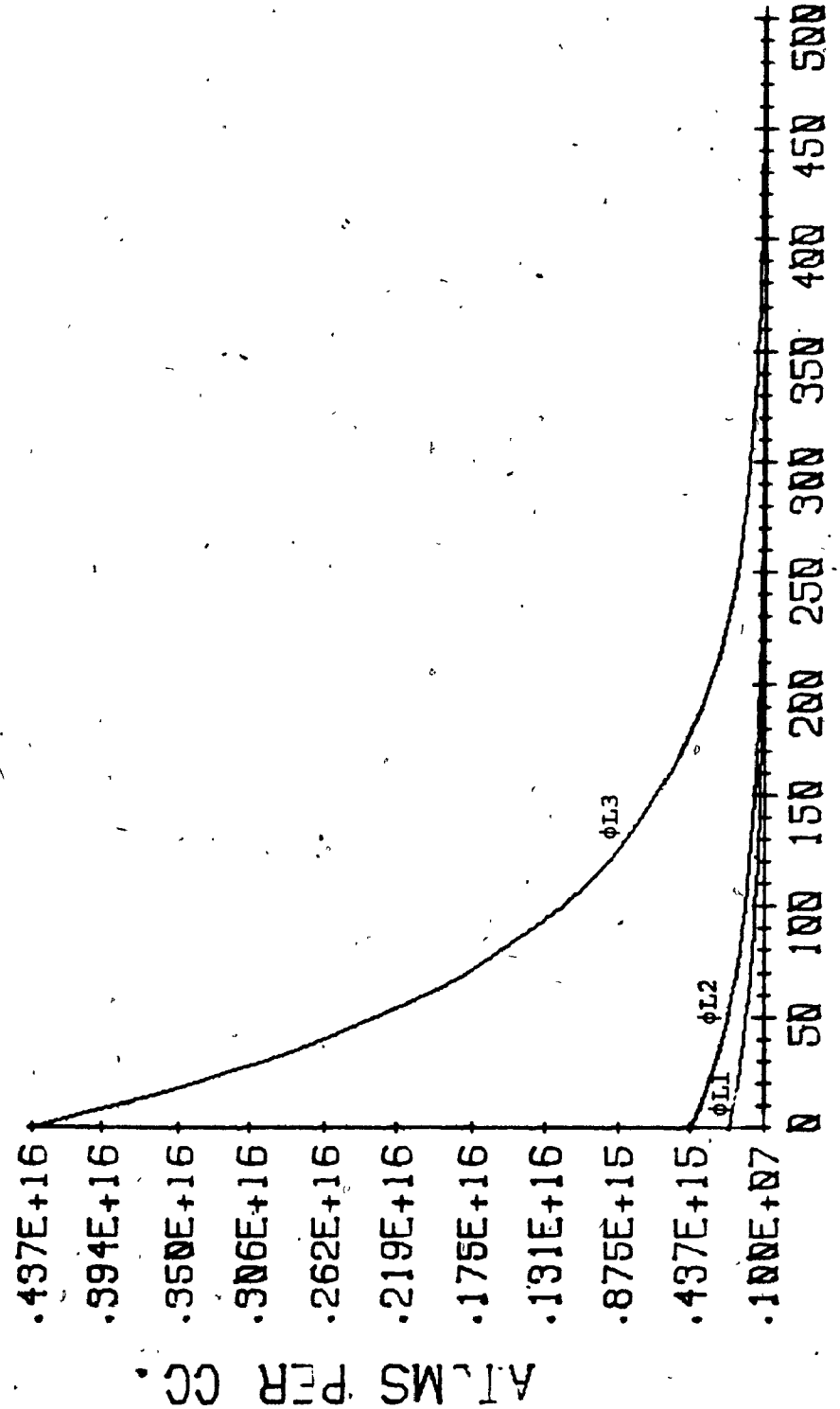


Figure V.26 Prometium Concentration at Shutdown

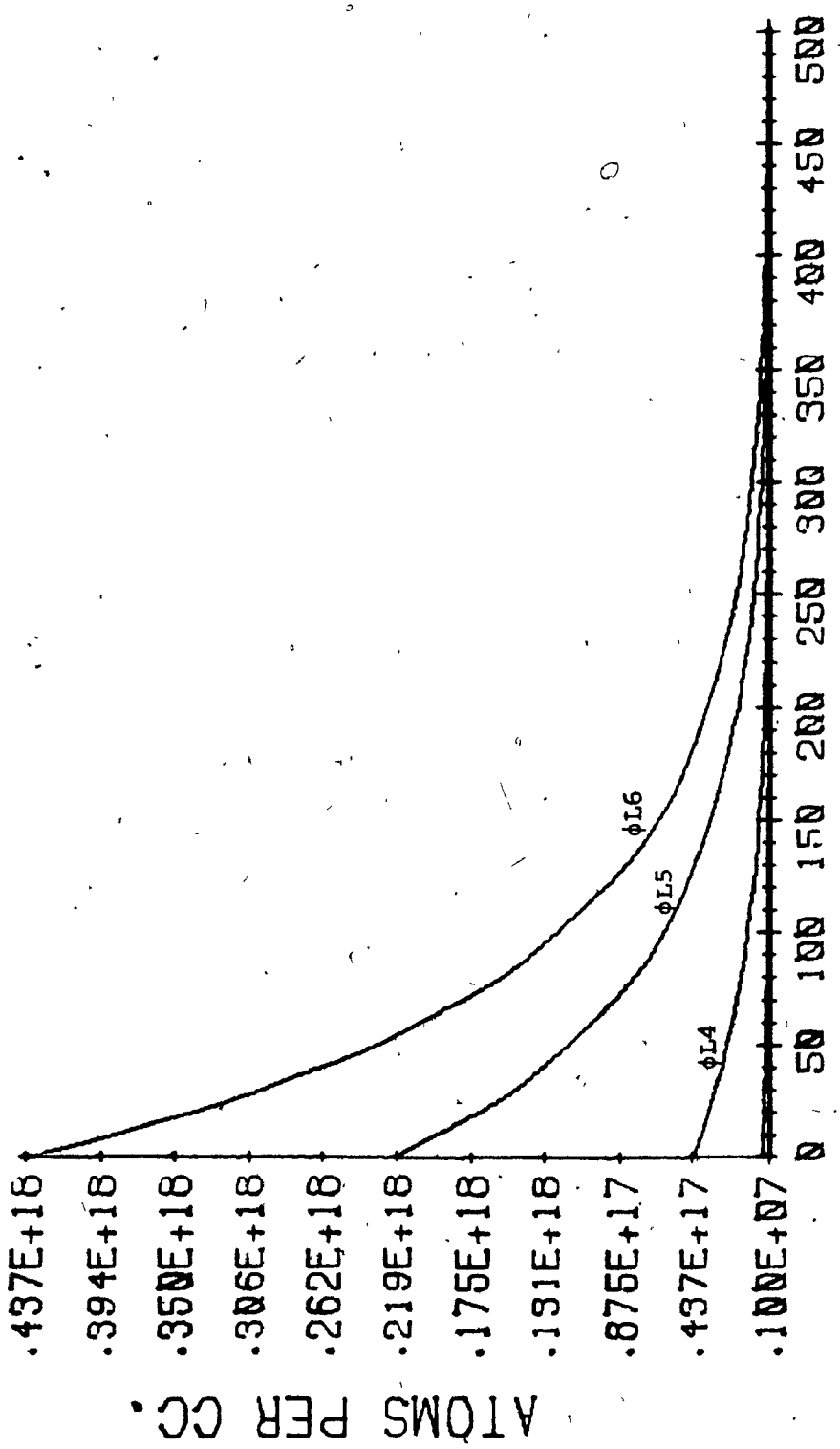


Figure V.27 Prometium Concentration at Shutdown

CHAPTER VI

VI.0 CONCLUSION

As pointed out in the introduction, this thesis is considered to be two-fold. First, I have given the basic elements of nuclear reactors in Chapter I, the detailed approach to reactor physics in Chapter II, the challenging derivation of the Boltzmann transport equation with its simplifications and the derivation of the one-group neutron diffusion equation in Chapter III.

Second, I have made extensive use of the one-group neutron diffusion equation to study the flux behavior in various reactor core geometries giving particular attention to the cylindrical reactor core geometry since the latter is one of the most economical and practical used in power reactors.

As an application of all the material included in Chapters I through IV, a detailed study of fission product poisoning in thermal reactors is given in Chapter V. Programs have been written to calculate and plot the density of the most important fission products in a low flux level reactor (Slowpoke reactor), CANDU power reactors (Darlington), and general high flux level thermal reactors. Listing of programs are found in Appendix D.

With the accomplishment of this thesis, I feel very confident to start the most advanced analysis of fission product power-induced oscillations mentioned at the end of Chapter V.

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APPENDIX A

Solution of Linear first order differential equation.

given

$$y' - f(x)y = r(x) \qquad (A-1)$$

If $r(x) = 0$ (A-1) is said to be homogeneous, otherwise is said to be non-homogeneous.

For homogeneous case we have

$$y' + f(x)y = 0$$

or separating variables we get

$$\frac{dy}{y} = -f(x)dx$$

and thus

$$\ln|y| = -\int f(x)dx + c$$

or

$$y(x) = c \exp(-\int f(x)dx)$$

with

$$c = \pm \exp(c^*) \text{ for } y \lesseqgtr 0$$

For nonhomogeneous cases we have:

$$y' + f(x)y = r(x) \qquad \text{(A-2)}$$

Multiply (A-2) by e^{h} and get (where $h(x) = \int f(x)dx$)

$$e^{h}(y' + f(x)y) = e^{h}r(x) \qquad \text{(A-3)}$$

Since

$$h'(x) = f(x)$$

we can write (A-3) as:

$$\frac{d}{dx}\left[y\, e^{h(x)}\right] = e^{h(x)}r(x) \qquad \text{(A-4)}$$

Now, integrating both sides of (A-4) we will get:

$$ye^{h(x)} = \int e^{h(x)}\, r(x)\, dx + c \qquad \text{(A-5)}$$

By dividing both sides of (A-5) by $e^{h(x)}$, we will get the desired formula:

$$y(x) = e^{-h(x)}\left[\int e^{h(x)}\, r(x)\, dx + c\right] \qquad \text{(A-6)}$$

with

$$h(x) = \int f(x)\, dx \qquad \text{(A-7)}$$

which represent the general solution of (A-2) in the form of an integral.

APPENDIX B

Some Useful Mathematical Formulas

(1) Solution of First-Order Linear Differential Equations:

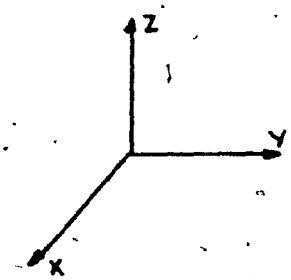
$$\frac{df}{dx} + a(x)f(x) = g(x) \tag{B-1}$$

$$f(x) = e^{-A(x)}\left[\int^x dx' e^{A(x')} g(x') + C\right], \quad A(x) \equiv \int^x dx'\, a(x'). \tag{B-2}$$

(2) Representation of Laplacian ∇^2 in Various Coordinate Systems:

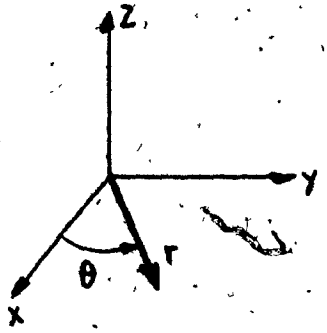
(a) Cartesian:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{B-3}$$



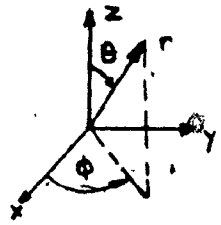
(b) Cylindrical:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{B-4}$$



(c) Spherical:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2} \qquad \text{(B-5)}$$



(4) Gauss' Divergence Theorem:

$$\int_v d^3r \nabla \cdot A = \int_S dS \hat{e}_s \cdot A \qquad \text{(B-6)}$$

where $\hat{e}_s$ is the unit vector normal to the surface element dS.

APPENDIX C

List of Flux Levels, thermal absorbing cross sections, fission product decay constants, and macroscopic fission cross sections, used in computing fission product poisoning concentrations.

$\phi_{L1} = 5 \times 10^{11}$ neutrons/cm^2 sec.

$\phi_{L2} = 10^{12}$ neutrons/cm^2 sec.

$\phi_{L3} = 10^{13}$ neutrons/cm^2 sec.

$\phi_{L4} = 9.7694 \times 10^{13}$ neutrons/cm^2 sec.

$\phi_{L5} = 5 \times 10^{14}$ neutrons/cu^2 sec.

$\phi_{L6} = 10^{15}$ neutrons/cm^2 sec.

$\sigma_a^{Xe} = 3.5 \times 10^{-18}$ cm^2

$\sigma_a^{Sm} = 5.6 \times 10^{-20}$ cm^2

$\Sigma_{f1} = .1112$ cm^{-1}

$\lambda^{Xe} = 2.09 \times 10^{-5}$ sec^{-1}

$\lambda^{I} = 2.87 \times 10^{-5}$ sec^{-1}

$\lambda^{Pm} = 3.56 \times 10^{-6}$ sec^{-1}

```
00010* THIS PROGRAM RUNS UNDER THE CFORT COMPILER ON A CDC
00020* CYBER 172 SYSTEM.  IT GENERATES A FILE TO BE ACCESSED
00030* BY A PLOTTING PROGRAM.
00040*
00050 PROGRAM REACT(INPUT,OUTPUT,TAPE1,TAPE2)
00060 REAL DRHR,NU,FLUX(7),QUOT1,QUOT2
00070 100 FORMAT(6X,E12.4)
00080 200 FORMAT(1H1,30X,*TIME BEHAVIOUR OF REACTIVITY WITH FLUX=*,
00090+E12.4,1X,11HN/CM**2*SEC)
00100 300 FORMAT(F4.1,E13.4)
00110 CALL GET(5HTAPE1,7HREACTIN,0,0)
00120* INPUT DATA IS ON FILE CALLED REACTIN.
00130 READ(1,100)NU
00140 READ(1,100)EPS
00150 READ(1,100)P
00160 READ(1,100)YIELDX
00170 READ(1,100)YIELDI
00180 READ(1,100)DECAYX
00190 READ(1,100)DECAYI
00200 READ(1,100)SIGMA
00210 PRINT 101,NU,EPS,P,YIELDI,YIELDX,DECAYX,DECAYI,SIGMA
00220 101 FORMAT(2X,8E12.4)
00230 DO 10 I=1,6
00240 READ(1,100)FLUX(I)
00250 10 CONTINUE
00260 PRINT102,(FLUX(I),I=1,6)
00270 102 FORMAT(2X,6E12.4)
00280 FACT1=1./(NU*EPS*P)
00290* 50 HOUR PLOT FOR 6 FLUXES.
00300 DO 20 I=1,6
00310 DO 30 K=1,51
00320 T=(K-1)*3600
00330  EXP1=DECAYX*T
00340 EXP1=EXP(-EXP1)
00350 EXP2=DECAYI*T
00360 EXP2=EXP(-EXP2)
00370 EXP3=EXP1-EXP2
00380*QUOT1=((YIELDI+YIELDX)*FLUX(I))/(DECAYX/SIGMA+FLUX(I))
00390 QUOT2=(YIELDI*FLUX(I)*SIGMA)/(DECAYI-DECAYX)
00400 DRHR=-FACT1*(QUOT1*EXP1+QUOT2*EXP3)
00410 T1=T/3600.
00420 WRITE(2,300)T1,DRHR
00430 30 CONTINUE
00440 20 CONTINUE
00450 ENDFILE 2
00460 REWIND 2
00470 CALL REPLACE(5HTAPE2,7HREACOUT,0,0)
00480* OUTPUT DATA FOR PLOTTING PROGRAM PLACED IN REACOUT.
00490 END
READY.
```

```
00010 PROGRAM XENON(INPUT,OUTPUT,TAPE1,TAPE2)
00020*
00030* STARTUP.  50 HOUR PLOT, 1 HOUR INCREMENTS, 6 FLUXES.
00040*
00050  DIMENSION FLUX(500)
00060  REAL NUM1,NUM2
00070 100 FORMAT(6X,E12.4)
00080 200 FORMAT(1H1,35X,*XENON TIME BEHAVIOUR AT REACTOR STARTUP*)
00090 300 FORMAT(1H ,5X,*TIME(HOURS)*,10X,*XENON*)
00100 400 FORMAT(F4.0,E13.4)
00110* INPUT DATA IN FILE XENONIN.
00120  CALL GET(5HTAPE1,7HXENONIN,0,0)
00130  READ(1,100)YIELDX
00140  READ(1,100)YIELDI
00150  READ(1,100)DECAYX
00160  READ(1,100)DECAYI
00170  READ(1,100)SIGMAF
00180  READ(1,100)SIGMA
00190  READ(1,100)(FLUX(I),I=1,6)
00200  DO 10 J=1,6
00210  NUM1=(YIELDX+YIELDI)*SIGMAF*FLUX(J)
00220  DENOM1=DECAYX+SIGMA*FLUX(J)
00230  FACT1=NUM1/DENOM1
00240  NUM2=YIELDI*SIGMAF*FLUX(J)
00250  DENOM2=DECAYX-DECAYI+SIGMA*FLUX(J)
00260  FACT2=NUM2/DENOM2
00270  DO 10 I=1,51
00280  T=(I-1)*3600.
00290  T1=(I-1)
00300  EXP1=DENOM1*T
00310  EXP1=EXP(-EXP1)
00320  EXP2=DECAYI*T
00330  EXP2=EXP(-EXP2)
00340  XEN=FACT1*(1.-EXP1)+FACT2*(EXP1-EXP2)
00350  WRITE(2,400)T1,XEN
00360 10 CONTINUE
00370  ENDFILE 2
00380  REWIND 2
00390* OUTPUT ON FILE XENOUT.
00400  CALL REPLACE(5HTAPE2,6HXENOUT,0,0)
00410  STOP
00420  END
READY.
```

```
00010 PROGRAM XENOND(INPUT,OUTPUT,TAPE1,TAPE2)
00020*
00030*SHUTDOWN.  50 HOUR PLOT, 1 HOUR INCREMENTS, 6 FLUXES.
00040*
00050  DIMENSION FLUX(500)
00060  REAL NUM1,NUM2
00070 100 FORMAT(6X,E12.4)
00080 200 FORMAT(1H1,35X,*XENON TIME BEHAVIOUR AT REACTOR SHUTDOWN*)
00090 300 FORMAT(1H ,5X,*TIME(HOURS)*,10X,*XENON*)
00100 400 FORMAT(F4.0,E13.4)
00110  CALL GET(5HTAPE1,7HXENONIN,0,0)
00120  READ(1,100)YIELDX
00130  READ(1,100)YIELDI
00140  READ(1,100)DECAYX
00150  READ(1,100)DECAYI
00160  READ(1,100)SIGMAF
00170  READ(1,100)SIGMA
00180  READ(1,100)(FLUX(I),I=1,6)
00190  DO 10 J=1,6
00200  NUM1=(YIELDX+YIELDI)*SIGMAF*FLUX(J)
00210  DENOM1=DECAYX+SIGMA*FLUX(J)
00220  FACT1=NUM1/DENOM1
00230  NUM2=YIELDI*SIGMAF*FLUX(J)
00240  DENOM2=DECAYI-DECAYX
00250  FACT2=NUM2/DENOM2
00260  DO 10 I=1,51
00270  T=(I-1)*3600.
00280  T1=(I-1)
00290  EXP1=DECAYX*T
00300  EXP1=EXP(-EXP1)
00310  EXP2=DECAYI*T
00320  EXP2=EXP(-EXP2)
00330  XEN=FACT1*(EXP1)+FACT2*(EXP1-EXP2)
00340  WRITE(2,400)T1,XEN
00350 10 CONTINUE
00360  ENDFILE 2
00370  REWIND 2
00380  CALL REPLACE(5HTAPE2,7HXENOUT,0,0)
00390  STOP
00400  END
READY.
```

```
00010 PROGRAM SAMAR(INPUT,OUTPUT,TAPE1,TAPE2)
00020*
00030* STARTUP.  500 HOUR PLOT, 10 HOUR INCREMENTS, 6 FLUXES.
00040*
00050 DIMENSION FLUX(500)
00060 REAL NUM1,NUM2
00070 100 FORMAT(6X,E12.4)
00080 200 FORMAT(1H1,35X,*SAMARIUM TIME BEHAVIOUR AT REACTOR STARTUP*)
00090 300 FORMAT(1H ,5X,*TIME(HOURS)*,10X,*SAMARIUM*)
00100 400 FORMAT(F4.0,E13.4)
00110 CALL GET(5HTAPE1,7HSAMARIN,0,0)
00120 READ(1,100)YIELDS
00130 READ(1,100)YIELDP
00140 READ(1,100)DECAY
00150 READ(1,100)SIGMAF
00160 READ(1,100)SIGMA
00170 READ(1,100)(FLUX(I),I=1,6)
00180 DO 10 J=1,6
00190 NUM1=(YIELDS+YIELDP)*SIGMAF*FLUX(J)
00200 DENOM1=SIGMA*FLUX(J)
00210 FACT1=NUM1/DENOM1
00220 NUM2=YIELDP*SIGMAF*FLUX(J)
00230 DENOM2=-DECAY+SIGMA*FLUX(J)
00240 FACT2=NUM2/DENOM2
00250 DO 10 I=1,51
00260 T=(I-1)*36000.
00265 T1=10.*(I-1)
00270 EXP1=DENOM1*T
00280 EXP1=EXP(-EXP1)
00290 EXP2=DECAY*T
00300 EXP2=EXP(-EXP2)
00310 SAM=FACT1*(1.-EXP1)+FACT2*(EXP1-EXP2)
00320 WRITE(2,400)T1,SAM
00330 10 CONTINUE
00340 ENDFILE 2
00350 REWIND 2
00360 CALL REPLACE(5HTAPE2,7HSAMROUT,0,0)
00370 STOP
00380 END
READY.
```

```
00010 PROGRAM SAMARD(INPUT,OUTPUT,TAPE1,TAPE2)
00020*
00030* SHUTDOWN.  500 HOUR PLOT, 10 HOUR INCREMENTS, 6 FLUXES.
00040*
00050 DIMENSION FLUX(500)
00070 100 FORMAT(6X,E12.4)
00080 200 FORMAT(1H1,35X,*SAMARIUM TIME BEHAVIOUR AT REACTOR SHUTDOWN*)
00090 300 FORMAT(1H ,5X,*TIME(HOURS)*,10X,*SAMARIUM*)
00100 400 FORMAT(F4.0,E13.4)
00110  CALL GET(5HTAPE1,7HSAMARIN,0,0)
00120 READ(1,100)YIELDS
00130 READ(1,100)YIELDP
00140 READ(1,100)DECAY
00150 READ(1,100)SIGMAF
00160 READ(1,100)SIGMA
00170 READ(1,100)(FLUX(I),I=1,6)
00180 DO 10 J=1,6
00190 FACT=YIELDP*SIGMAF*FLUX(J)/DECAY
00250 DO 10 I=1,51
00260 T=(I-1)*36000.
00265 T1=10.*(I-1)
00270 EXP1=DECAY*T
00280 EXP1=EXP(-EXP1)
00310 SAM=FACT*(1.-EXP1)
00320 WRITE(2,400)T1,SAM
00330 10 CONTINUE
00340 ENDFILE 2
00350 REWIND 2
00360 CALL REPLACE(5HTAPE2,7HSAMDOUT,0,0)
00370 STOP
00380 END
READY.
```

READY.


```
00010 PROGRAM PROMET(INPUT,OUTPUT,TAPE1,TAPE2)
00020*
00030* STARTUP.  500 HOUR PLOT, 10 HOUR INCREMENTS, 6 FLUXES.
00040*
00050 DIMENSION FLUX(500)
00060 REAL NUM1,NUM2
00070 100 FORMAT(6X,E12.4)
00080 200 FORMAT(1H1,35X,*PROMETHIUM TIME BEHAVIOUR AT REACTOR STARTUP*)
00090 300 FORMAT(1H ,5X,*TIME(HOURS)*,10X,*PROMETHIUM*)
00100 400 FORMAT(F4.0,E13.4)
00110 CALL GET(5HTAPE1,6HPROMIN,0,0)
00120 READ(1,100)YIELD
00130 READ(1,100)DECAY
00140 READ(1,100)SIGMAF
00150 READ(1,100)(FLUX(I),I=1,6)
00160 DO 10 J=1,6
00170 FACT=YIELD*SIGMAF*FLUX(J)/DECAY
00180 DO 10 I=1,51
00190 T=(I-1)*36000.
00195 T1=10.*(I-1)
00200 EXP1=DECAY*T
00210 EXP1=EXP(-EXP1)
00220 PROM=FACT*(1.-EXP1)
00230 WRITE(2,400)T1,PROM
00240 10 CONTINUE
00250 ENDFILE 2
00260 REWIND 2
00270 CALL REPLACE(5HTAPE2,7HPROMOUT,0,0)
00280 STOP
00290 END
READY.
```

```
      12 CALL PLOT(XJ,YJ,1)
   40 CONTINUE
      PRINT 99
      NI=NI+NF
      N2=N2+NF
      NG=NG-1
      IF(NG)1,1,17
    2 RES=YC+17.
      CALL RESFT(RES)
      STOP
      END
      SUBROUTINE DRAW(X,DX,DY,IP)
      DIMENSION X(3)
C   THIS ROUTINE LINKS SIMBOL TO PLOT.
C   IT COULD BE USED FOR ERROR CHECKING, E.G. NEGATIVE COORDINATES.
      TX=X(1)+DX
      TY=X(2)+DY
      IF(TX.LT.-0.1.OR.TX.GT.10.0)GOTO 1
      IF(TY.LT.-0.1.OR.TY.GT.2000.)GOTO 3
      CALL PLOT(TX,TY,IP)
      GOTO 2
    1 PRINT 4,TX
    4 FORMAT(*TX=*,F7.3)
      GOTO 2
    3 PRINT 5,TY
    5 FORMAT(*TY=*,F7.3)
    2 RETURN
      END
-EOF-
```

FIX 2.0 - BEGIN EDITING.

```
C PLOTMB2.  PLOTMR1 MODIFIED FOR 500 HR. PLOT.
C GET,TAPE3=FNAME OF INPUT FILE (SAMROUT,PROMOUT,ETC.)
C SEE FORMATS #100,157 FOR DATA INPUTS
C
      PROGRAM PLOTMR(INPUT,TAPE2,TAPE3,OUTPUT)
      DIMENSION Y(4096),KARAY(40),MARAY(11),NARAY(11)
      REAL LY,ILY
   98 FORMAT(*DAMN*)
   99 FORMAT(*OK*)
  100 FORMAT(//*ENTER NI,NF,NG,INC (FORMAT(4I4)).*
     C/*  12341234123412341234*)
  101 FORMAT(////*TAPE3  LONGER THAN 1024 CHARACTERS...*////)
  120 FORMAT(* YMAX=*E12.4/* YMIN=*E12.4/*    YM=*E12.4)
  121 FORMAT(* NXT=*I5)
  130 FORMAT(40H      ATOMS PER CC.  TIME IN HOURS        )
  140 FORMAT(I2)
  141 FORMAT(E13.6)
  142 FORMAT(/*COUNTS/CHANNEL LABELS*)
  150 FORMAT(4X,E13.4)
  156 FORMAT(I4)
  157 FORMAT(4I4)
  158 FORMAT(* NI   NF   NG   NU INC*/*12341234123412341234*/5I4)
  159 FORMAT(* EXPANSION FACTOR =*,F8.1)
  190 FORMAT(E8.3)
      CALL RESET(-1)
      YC=-7.5
      ILY=-3.5
    1 PRINT 100
      READ 157,NI,NF,NG,INC
      IF(NF)2,2,3
    3 EX=1.0
      LG=2HNO
      OVR=2HNO
      NU=NF*NG
      N2=NF
      DO 4 I=1,NU
      READ(3,150)Y(I)
      IF(EOF(3))13,4
    4 Y(I)=ABS(Y(I))
   13 REWIND 3
      IF(NU.GT.1024)PRINT 101
      NO=NF-NI
      NR=NO+1
      IF(LG.NE.3HLOG)GOTO 5
      DO 50 I=1,NU
      IF(Y(I).EQ.0.)Y(I)=1.0
      Y(I)=1000.*ALOG10(Y(I))
   50 CONTINUE
    5 PRINT 158,NI,NF,NG,NU,INC
      YMAX=0.
      YMIN=10.**6
      DO 10 I=1,NU
```

```
      IF(YMAX.GE.Y(I))GOTO 8
      IF(YMAX.GE.Y(I))GOTO 8
      YMAX=Y(I)
    8 IF(YMIN.LT.Y(I))GOTO 10
      YMIN=Y(I)
   10 CONTINUE
      YM=YMAX-YMIN
      XT=1.0
C  IF ND>65 INCREASE XT ACCORDINGLY
      LX=8
      LY=5
      ILY=ILY+8.5
      XL=1.30*NR
      YL=1.25*YM
      XF=LX/XL
      YF=LY/YL
      NRF=XL*XF
      YMF=YL*YF
      XC=1.5
      YC=YC+8.5
      NXT=XT
      XT=XT*XF
      PRINT 120,YMAX,YMIN,YM
      PRINT 121,NXT
      YT=0.1*YM
      NYT=YT
      YT=YT*YF
      XLY=0.20
      YLY=YT+YC
      XLX=3.0
      YLX=YC-1.0
      XNX=XC-0.4
      YNX=YC-0.3
      XMY=XC-1.1.
C  X AXIS
      CALL PLOT(XC,YC,-1)
      XX=XC
      L=6
      DO 60 K=NI,NF,NXT
      L=L-1
      IF(L.EQ.0)GOTO 21
      GOTO 22
   21 CALL SIMBOL(XX,YC,0.25,2,0.0,-2)
      L=5
      GOTO 59
   22 CALL SIMBOL(XX,YC,0.15,2,0.0,-2)
   59 XX=XX+XT
      IF(XX.GT.LX)GOTO 15
   60 CONTINUE
C  Y AXIS
   15 CALL PLOT(XC,YC,-1)
      IF(LG.NE.3HLOG)GOTO 6
      CALL PLOT(XC,ILY,1)
      GOTO 7
```

```
      DISPLAY OFF.
        6 YY=YC
          DO 70 K=1,10
          YY=YY+YT
          CALL SIMBOL(XC,YY,0.1,2,0.0,-2)
       70 CONTINUE
C *LABELY*
        7 ENCODE(40,130,KARAY(1))
          CALL PLOT(XLY,YLY,-1)
          CALL SIMBOL(XLY,YLY,0.2,KARAY(1),90.0,20)
C *LABELX*
          CALL PLOT(XLX,YLX,-1)
          CALL SIMBOL(XLX,YLX,0.2,KARAY(3),0.0,13)
C X LABELS...
          L=0
          NXT=5
          XT=5.*XT
          INC=NI-1
          DO 20 K=NI,NF,NXT
          L=L+1
          IF(XNX.GT.LX)GOTO 20
          ENCODE(4,156,NARAY(L))INC
          CALL SIMBOL(XNX,YNX,0.15,NARAY(L),0.0,4)
          PRINT 156,INC
          INC=INC+50
          XNX=XNX+XT
       20 CONTINUE
C Y LABELS...
          IF(LG.EQ.3HLOG)GO TO 14
          DO 30 K=1,11
          YMY=YM*(K-1)/10.
          YMYF=YMY*YF+YC-0.075
          IF(OVR.EQ.3HYES)GO TO 18
          YMY=YMY/EX+YMIN
          GO TO 19
       18 YMY=YMY+YMIN
       19 PRINT 190,YMY
          ENCODE(8,190,MARAY(K))YMY
          CALL PLOT(XMY,YMYF,-1)
          CALL SIMBOL(XMY,YMYF,0.15,MARAY(K),0.0,8)
       30 CONTINUE
          GO TO 17
       14 IM=1
          IN=2
          IF(YMAX.GT.2000.)IN=3
          IF(YMAX.GT.3000.)IN=4
          IF(YMAX.GT.4000.)IN=5
          IF(YMAX.GT.5000.)IN=6
          IF(YMAX.GT.6000.)IN=7
          IF(YMAX.GT.7000.)IN=8
          IF(YMIN.GT.2000.)IM=2
          IF(YMIN.GT.3000.)IM=3
          IF(YMIN.GT.4000.)IM=4
```

```
      IF(YMIN.GT.5000.)IM=5
      IF(YMIN.GT.6000.)IM=6
      ITEN=10
      DO 90 K=IM,IN
      YMY=1000*ALOG10(10.**K)
      YMYF=(YMY-YMIN)*YF+YC
      PRINT 405,YMYF,YC
  405 FORMAT(2F12.4)
      IF(YMYF.LT.YC)GOTO 9
      PRINT 141,YMY
      XMY=XC-0.2
      CALL PLOT(XMY,YMYF,-1)
      CALL PLOT(XC,YMYF,1)
      XMY=XC-0.7
      YMYF=YMYF-0.5*YT
      CALL PLOT(XMY,YMYF,-1)
      ENCODE(2,140,MARAY(K))ITEN
      CALL SIMBOL(XMY,YMYF,0.15,MARAY(K),0.0,2)
      XMY=XMY+0.2
      YMYF=YMYF+0.22
      ENCODE(2,140,MARAY(K))K
      CALL PLOT(XMY,YMYF,-1)
      CALL SIMBOL(XMY,YMYF,0.1,MARAY(K),0.0,2)
    9 XMY=XC-0.1
      DO 90 J=2,8,2
      YJ=J
      YMYF=(1000.*ALOG10(YJ*10.**K)-YMIN)*YF+YC
      IF(YMYF.GT.ILY)GOTO 17
      IF(YMYF.LT.YC)GOTO 90
      CALL PLOT(XMY,YMYF,-1)
      CALL PLOT(XC,YMYF,1)
   90 CONTINUE
C RAW DATA. EX=EXPANSION FACTOR
   17 PRINT 159,EX
      IF(Y(NI).GT.YMAX)Y(NI)=Y(NI)/EX
      YFF=(Y(NI)-YMIN)*YF+YC
      CALL PLOT(XC,YFF,-1)
      DO 40 J=NI,N2
      XJ=(J-NI)*XF+XC
      YJ=(Y(J)-YMIN)*EX*YF+YC
      IF(EX.EQ.1.0)GO TO 12
      IF(YJ.LE.ILY)GO TO 11
      IF(OVR.EQ.2HNO)GO TO 16
      YJ=(Y(J)-YMIN)*YF+YC
      GO TO 11
   16 YJ=YC
      CALL PLOT(XJ,YJ,-1)
      GO TO 40
   11 CALL PLOT(XJ,YJ,-1)
   12 CALL PLOT(XJ,YJ,1)
   40 CONTINUE
      PRINT 99
```

```
      NI=NI+NF
      N2=N2+NF
      NG=NG-1
      IF(NG)1,1,17
    2 RES=YC+17.
      CALL RESET(RES)
      STOP
      END
      SUBROUTINE DRAW(X,DX,DY,IP)
      DIMENSION X(3)
C  THIS ROUTINE LINKS SIMBOL TO PLOT.
C  IT COULD BE USED FOR ERROR CHECKING, E.G. NEGATIVE COORDINATES.
      TX=X(1)+DX
      TY=X(2)+DY
      IF(TX.LT.-0.1.OR.TX.GT.10.0)GOTO 1
      IF(TY.LT.-0.1.OR.TY.GT.2000.)GOTO 3
      CALL PLOT(TX,TY,IP)
      GOTO 2
    1 PRINT 4,TX
    4 FORMAT(*TX=*,F7.3)
      GOTO 2
    3 PRINT 5,TY
    5 FORMAT(*TY=*,F7.3)
    2 RETURN
      END
-EOF-
:
  NO FILE UPDATE.
CP  0.456 SECS.
READY.

READY.

FIX 2.0 - BEGIN EDITING.
```

```
C PLOTMB1.  PLOTMB WITH LABELY CHANGED AS IN FORMAT 130.
C GET,TAPE3=FNAME OF INPUT FILE (XENOUT, IODIOUT, ETC.) *
C SEE FORMATS #100,157 FOR DATA INPUT.
C
      PROGRAM PLOTMB1(INPUT,TAPE2,TAPE3,OUTPUT)
      DIMENSION Y(4096),KARAY(40),MARAY(11),NARAY(11)
      REAL LY,ILY
   98 FORMAT(*DAMN*)
   99 FORMAT(*OK*)
  100 FORMAT(//*ENTER NI,NF,NG,INC (FORMAT(4I4)).*
     C/*  1234123412341234*)
  101 FORMAT(////*TAPE3  LONGER THAN 1024 CHARACTERS...*////)
  120 FORMAT(* YMAX=*E12.4/* YMIN=*E12.4/*   YM=*E12.4)
  121 FORMAT(* NXT=*I5)
  130 FORMAT(40H   ATOMS PER CC.     TIME IN HOURS       )
  140 FORMAT(I2)
  141 FORMAT(E13.6)
  142 FORMAT(/*COUNTS/CHANNEL LABELS*)
  150 FORMAT(4X,E13.4)
  156 FORMAT(I4)
  157 FORMAT(4I4)
  158 FORMAT(* NI  NF  NG  NU INC*/*1234123412341234*/5I4)
  159 FORMAT(* EXPANSION FACTOR =*,F8.1)
  190 FORMAT(E8.3)
      CALL RESET(-1)
      YC=-7.5
      ILY=-3.5
    1 PRINT 100
      READ 157,NI,NF,NG,INC
      IF(NF)2,2,3
    3 EX=1.0
      LG=2HNO
      OVR=2HNO
      NU=NF*NG
      N2=NF
      DO 4 I=1,NU
      READ(3,150)Y(I)
      IF(EOF(3))13,4
    4 Y(I)=ABS(Y(I))
   13 REWIND 3
      IF(NU.GT.1024)PRINT 101
      ND=NF-NI
      NR=ND+1
      IF(LG.NE.3HLOG)GOTO 5
      DO 50 I=1,NU
      IF(Y(I).EQ.0.)Y(I)=1.0
      Y(I)=1000.*ALOG10(Y(I))
   50 CONTINUE
    5 PRINT 158,NI,NF,NG,NU,INC
      YMAX=0.
      YMIN=10.**6
      DO 10 I=1,NU
DISPLAY OFF.
```

```
      IF(YMAX.GE.Y(I))GOTO 8
      YMAX=Y(I)
    8 IF(YMIN.LT.Y(I))GOTO 10
      YMIN=Y(I)
   10 CONTINUE
      YM=YMAX-YMIN
      XT=1.0
C     IF ND>65 INCREASE XT ACCORDINGLY
      LX=8
      LY=5
      ILY=ILY+8.5
      XL=1.30*NR
      YL=1.25*YM
      XF=LX/XL
      YF=LY/YL
      NRF=XL*XF
      YMF=YL*YF
      XC=1.5
      YC=YC+8.5
      NXT=XT
      XT=XT*XF
      PRINT 120,YMAX,YMIN,YM
      PRINT 121,NXT
      YT=0.1*YM
      NYT=YT
      YT=YT*YF
      XLY=0.20
      YLY=YT+YC
      XLX=3.0
      YLX=YC-1.0
      XNX=XC-0.4
      YNX=YC-0.3
      XMY=XC-1.1
C     X AXIS
      CALL PLOT(XC,YC,-1)
      XX=XC
      L=6
      DO 60 K=NI,NF,NXT
      L=L-1
      IF(L.EQ.0)GOTO 21
      GOTO 22
   21 CALL SIMBOL(XX,YC,0.25,2,0.0,-2)
      L=5
      GOTO 59
   22 CALL SIMBOL(XX,YC,0.15,2,0.0,-2)
   59 XX=XX+XT
      IF(XX.GT.LX)GOTO 15
   60 CONTINUE
C     Y AXIS
   15 CALL PLOT(XC,YC,-1)
      IF(LG.NE.3HLOG)GOTO 6
      CALL PLOT(XC,ILY,1)
```

```
      GOTO 7
    6 YY=YC
      DO 70 K=1,10
      YY=YY+YT
      CALL SIMBOL(XC,YY,0.1,2,0.0,-2)
   70 CONTINUE
C *LABELY*
    7 ENCODE(40,130,KARAY(1))
      CALL PLOT(XLY,YLY,-1)
      CALL SIMBOL(XLY,YLY,0.2,KARAY(1),90.0,20)
C *LABELX*
      CALL PLOT(XLX,YLX,-1)
      CALL SIMBOL(XLX,YLX,0.2,KARAY(3),0.0,13)
C X LABELS...
      L=0
      NXT=INC
      XT=INC*XT
      INC=NI-1
      DO 20 K=NI,NF,NXT
      L=L+1
      IF(XNX.GT.LX)GOTO 20
      ENCODE(4,156,NARAY(L))INC
      CALL SIMBOL(XNX,YNX,0.15,NARAY(L),0.0,4)
      PRINT 156,INC
      INC=INC+NXT
      XNX=XNX+XT
   20 CONTINUE
C Y LABELS...
      IF(LG.EQ.3HLOG)GO TO 14
      DO 30 K=1,11
      YMY=YM*(K-1)/10.
      YMYF=YMY*YF+YC-0.075
      IF(OVR.EQ.3HYES)GO TO 18
      YMY=YMY/EX+YMIN
      GO TO 19
   18 YMY=YMY+YMIN
   19 PRINT 190,YMY
      ENCODE(8,190,MARAY(K))YMY
      CALL PLOT(XMY,YMYF,-1)
      CALL SIMBOL(XMY,YMYF,0.15,MARAY(K),0.0,8)
   30 CONTINUE
      GO TO 17
   14 IM=1
      IN=2
      IF(YMAX.GT.2000.)IN=3
      IF(YMAX.GT.3000.)IN=4
      IF(YMAX.GT.4000.)IN=5
      IF(YMAX.GT.5000.)IN=6
      IF(YMAX.GT.6000.)IN=7
      IF(YMAX.GT.7000.)IN=8
      IF(YMIN.GT.2000.)IM=2
      IF(YMIN.GT.3000.)IM=3
```

```
      IF(YMIN.GT.4000.)IM=4
      IF(YMIN.GT.5000.)IM=5
      IF(YMIN.GT.6000.)IM=6
      ITEN=10
      DO 90 K=IM,IN
      YMY=1000*ALOG10(10.**K)
      YMYF=(YMY-YMIN)*YF+YC
      PRINT 405,YMYF,YC
  405 FORMAT(2F12.4)
      IF(YMYF.LT.YC)GOTO 9
      PRINT 141,YMY
      XMY=XC-0.2
      CALL PLOT(XMY,YMYF,-1)
      CALL PLOT(XC,YMYF,1)
      XMY=XC-0.7
      YMYF=YMYF-0.5*YT
      CALL PLOT(XMY,YMYF,-1)
      ENCODE(2,140,MARAY(K))ITEN
      CALL SIMBOL(XMY,YMYF,0.15,MARAY(K),0.0,2)
      XMY=XMY+0.2
      YMYF=YMYF+0.22
      ENCODE(2,140,MARAY(K))K
      CALL PLOT(XMY,YMYF,-1)
      CALL SIMBOL(XMY,YMYF,0.1,MARAY(K),0.0,2)
    9 XMY=XC-0.1
      DO 90 J=2,8,2
      YJ=J
      YMYF=(1000.*ALOG10(YJ*10.**K)-YMIN)*YF+YC
      IF(YMYF.GT.ILY)GOTO 17
      IF(YMYF.LT.YC)GOTO 90
      CALL PLOT(XMY,YMYF,-1)
      CALL PLOT(XC,YMYF,1)
   90 CONTINUE
C RAW DATA. EX=EXPANSION FACTOR
   17 PRINT 159,EX
      IF(Y(NI).GT.YMAX)Y(NI)=Y(NI)/EX
      YFF=(Y(NI)-YMIN)*YF+YC
      CALL PLOT(XC,YFF,-1)
      DO 40 J=NI,N2
      XJ=(J-NI)*XF+XC
      YJ=(Y(J)-YMIN)*EX*YF+YC
      IF(EX.EQ.1.0)GO TO 12
      IF(YJ.LE.ILY)GO TO 11
      IF(OVR.EQ.2HNO)GO TO 16
      YJ=(Y(J)-YMIN)*YF+YC
      GO TO 11
   16 YJ=YC
      CALL PLOT(XJ,YJ,-1)
      GO TO 40
   11 CALL PLOT(XJ,YJ,-1)
   12 CALL PLOT(XJ,YJ,1)
   40 CONTINUE
```