ON THE SQUARE INTEGRABILITY OF SOME REPRESENTATIONS OF $SU(1,1)$

NASSER SAAD

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ABSTRACT

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NASSER SAAD

We survey the theory of square integrable group representations for the non-compact, semisimple lie group $SU(1,1)$, which has three different series of representations: principal, complementary and discrete. As a result of our survey we conclude that the discrete series representations are the only ones which are square integrable in the classical sense. In addition, we study the square integrability of the principal series representations over the homogeneous spaces $SU(1,1)/A$ and $SU(1,1)/N$, where A and N are closed subgroups of $SU(1,1)$. This latter is a generalization of an earlier proposal by Perelomov.

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To Gamil and Ezzat Attia

CONTENTS

INTRODUCTION

The square integrability of group representations[1] plays a fundamental role in constructing a coherent states system[2]. The importance of the latter comes from its wide usage through various branches of mathematical physics [22]. Coherent state techniques can for example be used in the theory of frames [6] or geometric quantization [2] or wavelet analysis [17].

The remarkable correlation between square integrable group representations and associated coherent state systems has been investigated by many authors [4],[17], [21], [25].

In the standard theory [17], a system of coherent states can be constructed whenever one has a square integrable representation of a locally compact group G (see chapter II for the construction). A more general construction was proposed by Perelomov [25], where the coherent states, labelled by the points of the coset space G/K, K being a stationary subgroup, satisfy an alternative definition of square integrability [19]. However, neither of these methods are applicable to certain group representation appearing in physics such as, for example, the Galilei group [2] or the Poincaré groups $P_+^\uparrow(1,1)$ or $P_+^\uparrow(1,3)$ [4].

A further generalization of Perelomov's method was proposed by Ali et al [4], where the authors constructed a coherent states system on a homogeneous space G/H, provided there is a unitary irreducible representation U of the group G which is square integrable modulo the closed subgroup H and the Borel section $\sigma : G/H \to G$ such that the following integral converges weakly to a bounded, positive, invertible operator A_σ:

$$\int_{X=G/H} \lambda(\sigma(x),x)U(\sigma(x))|\eta >< \eta|U(\sigma(x))^* d\mu(x) = A_\sigma$$

---

[1] That is , $(T(g)\eta,\eta) \in L^2(G)$, where $T(g), g \in G$, is a unitary irreducible representation of G and η is some fixed vector in the representation space.

[2] We refer to Klauber and Skagerstam [22] for a general introduction to coherent states

here μ is a quasi-invariant measure on X and $\lambda : G \times X \to \mathbb{R}^+$ is the Radon-Nikodym derivative, $\lambda(g,x) = d\mu(g^{-1}.x)/d\mu(x)$.

Thus, the vectors

$$\eta_{\sigma(x)} = \sqrt{\lambda(\sigma(x),x)}U(\sigma(x))\eta, \quad x \in X$$

form a system of coherent states with all expected properties [4].

The purpose of this thesis is to study this generalization for the principal series representations of the non-compact semi-simple Lie group $SU(1,1)$, which are not square integrable in the classical sense. The organisation of this thesis is as follows: After a short introduction to the theory of group representation; we study:
in chapter (I), the isomorphism between the group $SU(1,1)$ and the group $SL(2,\mathbb{R})$, providing a detailed construction of the principal series representation induced from the minimal parabolic subgroup. We give independent proofs for almost all the results stated in this chapter using the notion of the cocycles [30].
In chapter (II), we survey the theory of the square integrable group representations in the classical sense [13], examining the affine group as an example of applying this theory [17]. We also study the theory of square integrable group representations over a homogeneous space as a generalization of Perelomov's construction by using the Poincaré group as an example.
In chapter (III), we try to apply the generalization given by Ali at el [4] to the principal series representation of the group $SU(1,1)$, using it's isomorphism with the group $SL(2,\mathbb{R})$. The corresponding subgroups of $SL(2,\mathbb{R})$ allow for easier construction of the Borel sections.

PRELIMINARIES

We collect together some basic facts from the theory of group representations which will be useful later on.

Let G denote a separable locally compact group (denoted s.l.c. group) with identity e, and $g, g', \ldots$ elements in G. It is well known [14, chapter (14)] that there exists in such a group a left invariant (written dg) and a right invariant (written $d_r g$) Haar measure. Therefore,

$$d(g'g) = dg, \text{that is} \quad \int_G f(g'g)dg = \int_G f(g)dg \tag{0.1}$$

$$d_r(gg') = d_r g, \text{that is} \quad \int_G f(gg')d_r g = \int_G f(g)d_r g \tag{0.2}$$

for $g, g' \in G$ and f belongs to the set of continuous functions on G with compact support (denoted $C_c(G)$). One has

$$dg = \Delta(g)d_r g \tag{0.3}$$

$$dgg' = \Delta(g')dg \tag{0.4}$$

where $\Delta : G \to \mathbb{R}^+$, called the modular function, satisfies

$$\left.\begin{aligned} \Delta(g) &> 1 \\ \Delta(gg') &= \Delta(g)\Delta(g') \\ \Delta(e) &= 1 \end{aligned}\right\} \tag{0.5}$$

and one has the following formula

$$\int_G f(g^{-1})dg = \int_G f(g)\Delta(g^{-1})dg = \int_G f(g)d_r g. \tag{0.6}$$

If $\Delta(g) \equiv 1$ [as the case of compact, abelain and semisimple Lie groups], the group G is said to be *unimodular* and we have in this case

$$\int_G f(g^{-1})dg = \int_G f(g)dg. \tag{0.7}$$

Let H be a closed subgroup of G. It is well known [14] that there exists a unique (up to equivalence) *quasi-invariant* measure μ on the homogeneous space $X = G/H$, that is for any Borel subset E of X and any $g \in G$, $\mu(E) = 0$ if and only if $\mu(g.E) = 0$. Therefore, there exists a strictly positive Borel function ρ on G satisfying for every $h \in H$,

$$\rho(gh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(g), \quad (g \in G) \tag{0.8}$$

where Δ_H and Δ_G are modular functions (see equation (0.5)) for H and G, respectively. The function ρ related to the *quasi-invariant* measure μ by the formula,

$$\int_G f(g)\rho(g)dg = \int_X d\mu(x) \int_H f(gh)dh, \tag{0.9}$$

for $f \in C_c(G)$.

Thus, for each $f \in C_c(G/H)$, one has the relation,

$$\int_X f(g.x)d\mu(x) = \int_X \sigma(g^{-1}, x)f(x)d\mu(x) \tag{0.10}$$

where $\sigma : G \times X \to \mathbb{R}^+$ can be obtained using the Radon-Nikodym theorem [14] as,

$$\sigma(g^{-1}, x) = \frac{\rho(g^{-1}.x)}{\rho(x)} = \frac{d\mu(g^{-1}.x)}{d\mu(x)}, \tag{0.11}$$

such that, for $g, g' \in G$,

$$\left.\begin{aligned} \sigma(gg', x) &= \sigma(g, g'x)\sigma(g', x) \\ \sigma(g^{-1}, x) &= \sigma(g, g^{-1}x)^{-1} \end{aligned}\right\}. \tag{0.12}$$

Note that, if G and H are unimodular, then the homogeneous space X admits an *invariant* measure.

The following theorem will be important for the sequel:

Mackey decomposition theorem [10; 23]. *Let G be a s.l.c. group and H be a closed subgroup of G. Then there exists a Borel set S of G (that is $G = S.H$) such that every element $g \in G$ can be uniquely represented in the form:*

$$g = s(x).h, \quad x \in X = G/H, h \in H \quad \bullet \tag{0.13}$$

The map $s : X \to G$ is called a *Borel section.* It satisfies the condition:

$$p(s(x)) = x, \text{where} \quad p(g) = gH \in X.$$

By *unitary representations* of a s.l.c. group G, we mean homomorphisms U of G into the group $\mathcal{U}(\mathfrak{H})$ of unitary operators on a separable Hilbert space $\mathfrak{H}$ that are (strongly) continuous:

$$||U(g_n)\varphi - U(g)\varphi|| \longrightarrow 0, \text{ as } \quad g_n \to g, \text{ for all } \quad \varphi \in \mathfrak{H}. \tag{0.14}$$

It's well known [14] that (strong)-continuity follows from the measurability of the maps

$$g \to (U(g)\varphi|\varphi'), \quad \varphi, \varphi' \in \mathfrak{H}. \tag{0.15}$$

All the representations, we deal with in this thesis, are continuous even if we do not mention that explicitly.

The Hilbert space $\mathfrak{H}$ is called the *representation space.* the dimension d of $\mathfrak{H}$ is called the *dimension* (or *degree*) of the representation U.

Let $d_r g$ be the right Haar measure on G. For every $g \in G$, let $U_r(g)$ be the operator in $L^2(G, d_r g)$ defined by

$$(U_r(g')\varphi)(g) = \varphi(gg'), \quad \varphi \in L^2(G, d_r g) \tag{0.16}$$

Then $U_r(g')$ is a continuous unitary representation of G in $L^2(G, d_r g)$, called the *right regular representation*. Similarly, the *left regular representation* U_l is defined by

$$(U_l(g')\varphi)(g) = \varphi({g'}^{-1}g), \quad \varphi \in L^2(G, dg). \tag{0.17}$$

We shall mainly be concerned with *irreducible* representations: those allowing no proper, closed, invariant subspaces under U in $\mathfrak{H}$. The irreducibility of U is equivalent to saying that for any $\varphi \in \mathfrak{H}$, $\varphi \neq 0$, the set $D = \{U(g)\varphi \mid g \in G\}$ is *dense* in $\mathfrak{H}$.

Schur's Lemma: If $T \in \mathfrak{L}(\mathfrak{H})$, the set of linear bounded operators on $\mathfrak{H}$, and if T commutes with $U(g)$ for all $g \in G$, i.e. $UT = TU$, then

$$T = \lambda \mathrm{I}, \quad \text{for some} \quad \lambda \in \mathbb{C} \quad \bullet \tag{0.18}$$

Let U_1, U_2 be two unitary representations of G on the Hilbert spaces $\mathfrak{H}_1$, $\mathfrak{H}_2$ respectively. An intertwining operator between U_1, and U_2 is a bounded linear operator: $T : \mathfrak{H}_1 \to \mathfrak{H}_2$, such that,

$$U_1(g)T = TU_2(g), \quad \text{for all} \quad g \in G. \tag{0.19}$$

The representations U_1, U_2 are unitarily equivalent, if there is an invertible isometry (an invertible linear mapping preserving the norm) of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$. For example, let $T : L^2(G, dg) \to L^2(G, d_r g)$, defined by

$$(T\varphi)(g) = \varphi(g^{-1}) \tag{0.20}$$

then T is a unitary map by (0.6) and

$$TU_l(g)T^{-1} = U_r(g), \quad \text{for all} \quad g \in G; \tag{0.21}$$

Indeed,

$$\begin{aligned}(TU_l(g)T^{-1}f)(g') &= (U_l(g)T^{-1})(g^{-1}) \\ &= (T^{-1}f)(g^{-1}g'^{-1}) \\ &= (T^{-1}f)((g'g)^{-1}) \\ &= f(g'g) \\ &= (U_r(g)f)(g')\end{aligned}$$

consequently, the left regular and the right regular representation are unitarily equivalent.

If U is a representation of G on $\mathfrak{H}$ and H is a closed subgroup of G, we may form the restricted representation $U|_H$ on $\mathfrak{H}$. If U is a representation of G on $\mathfrak{H}$ and $\mathcal{H} \subset \mathfrak{H}$ a closed, invariant under U, $U|_{\mathcal{H}}$ defines a subrepresentation of U given by the action of $U(G)$ on $\mathcal{H}$. Then $\mathcal{H}^{\perp}$ is also an invariant subspace under $U(G)$, and $\mathfrak{H} = \mathcal{H} \oplus \mathcal{H}^{\perp}$.

We now note several conventions that we employ throughout:

* All the groups, we deal with in this thesis, are Lie, therefore locally compact, groups.
* All the Hilbert spaces are separable, complex, even if we do not mention that explicitly.
* All the functions are complex valued and measurable.
* All the operators are linear on appropriate Hilbert spaces.
* The symbol • denotes the end of definition or of the proof of a theorem or a lemma.
* The notation (i.j.k) signifies equation k of chapter i in section j.

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CHAPTER I

THE ISOMORPHISM BETWEEN $SL(2,\mathbb{R})$ AND $SU(1,1)$

The group $SL(2,\mathbb{R}) \cong SU(1,1)$ has the three different series of irreducible unitary representations; the principal continuous series, the discrete series, and the complementary series, in addition to the trivial one. In the present chapter we summarize the properties of the groups $SL(2,\mathbb{R})$ and $SU(1,1)$ and their representations.

The construction of the unitary maps from the representation spaces for $SL(2,\mathbb{R})$ to the representation spaces for $SU(1,1)$ is discussed explicitly. Also, the construction of the principal continuous series representation of $SL(2,\mathbb{R})$ induced by the minimal parabolic subgroup defined by (I.1.14) is given.

I.1. The group $SL(2,\mathbb{R})$

Let G denote the 2×2 real unimodular[1] special linear group, i.e.

$$G = SL(2,\mathbb{R}) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad det(g) = 1, \quad a,b,c,d \in \mathbb{R}\}. \qquad \text{(I.1.1)}$$

The lie algebra of G, denoted by $\mathfrak{sl}(2,\mathbb{R})$, consists of 2×2 real traceless[2] matrices. A basis for $\mathfrak{sl}(2,\mathbb{R})$ is

$$X_0 = 1/2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_1 = 1/2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

satisfying the commutation relations

$$[X_0, X_1] = X_0 - X_2, \quad [X_0, X_2] = X_1, \quad [X_1, X_2] = X_2. \qquad \text{(I.1.2)}$$

The corresponding elements[3] in the group G are

$$k_\theta = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix} = exp(\theta X_0), \quad \theta \in \mathbb{R}, \qquad \text{(I.1.3)}$$

[1] In the sense that determinant $g = 1$

[2] Indeed, $det(g) = det(e^{tX}) = e^{t(trX)} = 1$, implies $tr(X) = 0$ for all $t \in \mathbb{R}$ and $X \in \mathfrak{sl}(2,\mathbb{R})$.

[3] Consider the power series representations of $\sin\theta/2, \cos\theta/2$ and $e^{t/2}$.

$$a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} = exp(tX_1), \quad t \in \mathbb{R}, \tag{I.1.4}$$

$$n_\xi = \begin{pmatrix} 1 & \xi \\ o & 1 \end{pmatrix} = exp(\xi X_2), \quad \xi \in \mathbb{R}. \tag{I.1.5}$$

Particular 1-parameter subgroups of G are defined by

$$K = \{k_\theta, 0 \leq \theta < 4\pi\}, \quad A = \{a_t, t \in \mathbb{R}\}, \quad N = \{n_\xi, \xi \in \mathbb{R}\} \tag{I.1.6}$$

and K is a maximal compact subgroup of G.

The noncompactness of G follows immediately from (I.1.6). Moreover, from (I.1.2) we can show that $\mathfrak{sl}(2,\mathbb{R})$ is a semisimple Lie group (we will provide the proof in detail for $SU(1,1)$; see discussion following (I.2.17)).

Lemma (I.1.1). *Any element $g \in G$ is uniquely decomposable in the form*

$$g = k_\theta a_t n_\xi \tag{I.1.7}$$

where $0 \leq \theta < 4\pi, t, \xi \in \mathbb{R}$. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then the triple (θ, t, ξ) is given by the relation

$$e^{\imath\theta/2} = \frac{a - \imath c}{\sqrt{a^2 + c^2}}, \quad e^t = a^2 + c^2, \quad \xi = \frac{ab + cd}{a^2 + c^2}. \tag{I.1.8}$$

Proof. Straightforward, by matrix multiplication•

The decomposition (I.1.7) is called the Iwasawa decomposition for G, which shows that the mapping $(k_\theta, a_t, n_\xi) \to k_\theta a_t n_\xi$ is a diffeomorphism of $K \times A \times N$ onto G. From the uniqueness of the Iwasawa decomposition we have for $g \in G$ and $\theta \in \mathbb{R}$,

$$gk_\theta = u_{g.\theta} a_{t(g,\theta)} n_{\xi(g,\theta)}. \tag{I.1.9}$$

and we can easily prove using (I.1.9) and (I.1.8),

$$e^{\imath g.\theta/2} = \frac{(a - \imath c)\cos\theta/2 + (-b + \imath d)\sin\theta/2}{|(a - \imath c)\cos\theta/2 + (-b + \imath d)\sin\theta/2|}, \tag{I.1.10}$$

$$e^{t(g,\theta)} = |(a - \imath c)\cos\theta/2 + (-b + \imath d)\sin\theta/2|^2, \tag{I.1.11}$$

$$\xi(g,\theta) = \frac{(ab+cd)\cos\theta + 1/2(a^2-b^2+c^2-d^2)\sin\theta}{|(a-\imath c)\cos\theta/2 + (-b+\imath d)\sin\theta/2|^2}. \tag{I.1.12}$$

Let $A^+ = \{a_t : t > 0\}$ and $A^- = \{a_t : t < 0\}$, we can define the Cartan decomposition for $SL(2,\mathbb{R})$ as $G = K.A^+.K$. That is each g can be decomposed into the form

$$g = k_\theta a_t k_\psi \tag{I.1.13}$$

where $\psi \in \mathbb{R}$.

Let M be the centralizer of A in K;

$$M = \{k_\theta \in K : \quad k_\theta X k_\theta^{-1} = X \quad \text{for each} \quad X \in \mathfrak{a}, \mathfrak{a} \quad \text{is the Lie algebra of A}\},$$

then

$$M = \{\pm I\}$$

where I is the identity matrix, and

$$P = MAN = \{\pm \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a \in \mathbb{R}\setminus 0, b \in \mathbb{R}\} \tag{I.1.14}$$

is called *the minimal parabolic subgroup of G*.

As pointed out in [18, chapter (10)], any semi-simple Lie group G is unimodular. The following lemma provides us with a method to construct an invariant Haar measure on G.

Lemma (I.1.2). *Let G be a unimodular locally compact group, having the decomposition $G = K.P$, with Haar measure dg, and let $d_l k$ (resp. $d_r p$) be the left (resp. right) Haar measure for K (resp. P). Then if $\alpha(p)$ is the modular function of P, we have*

$$dg = d_l k d_r p = \alpha(p) d_l k d_l p, \quad (g = kp) \tag{I.1.15}$$

Moreover, suppose $P = A.N$ where A and N are closed subgroups, both unimodular, with N normal in P. Then there is an analytic homomorphism δ of A into $\mathbb{R}^+$ such that, for each $a \in A$,

$$d(ana^{-1}) = \delta(a)^2 dn, \tag{I.1.16}$$

and

$$\alpha(p) = \alpha(an) = \delta(a)^2, \tag{I.1.17}$$

so that

$$d_l p = dadn, \quad d_r p = \delta(a)^2 dadn, \quad (b = an). \tag{I.1.18}$$

Proof. See [31, Chapter (4), Lemma (1) and (2)]•

A straightforward calculation shows that for $SL(2,\mathbb{R})$ the groups K, A, N and G have bi-invariant measures which we normalize as follows :

$$\left.\begin{aligned} dk_\theta &= \frac{d\theta}{4\pi} \\ da_t &= dt \\ dn_\xi &= d\xi \end{aligned}\right\} \tag{I.1.19}$$

and

$$\begin{aligned} dg &= e^t dk_\theta da_t dn_\xi \\ &= \frac{1}{4\pi} e^t d\theta dt d\xi. \end{aligned} \tag{I.1.20}$$

To prove the last statement, it is sufficient to compute the invariant Haar measure of P by using equation (I.1.18) and then combining it with equation (I.1.15). This is done in the appendix.

From the Cartan decomposition, there exists [28, chapter (5), proposition (5.2)] an alternative measure to (I.1.20) of $SL(2,\mathbb{R})$ defined as

$$dg = 2\pi \sinh t d\theta dt d\xi \tag{I.1.21}$$

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I.2. The unitary irreducible representations (UIR)

Bargmann [9, 1948] has determined all the UIR's of the group G. They fall into four classes:

(1) *The two principal continuous series:*

$$g \to U_g^{j,s}, \quad s \in \mathbb{C}, \quad Re(s) = 1/2, \quad j \in \{0, 1/2\}$$

excluding the case $\{j, s\} = \{1/2, 1/2\}$.

(2) *The complementary series:*

$$g \to U^{0,\sigma}(g), \quad 1/2 < \sigma < 1, \quad \sigma \in \mathbb{R}.$$

(3) *The two discrete series:*

$$g \to V_n^{\pm}(g)$$

where

$$n = 1, 3/2, 2, 5/2, \ldots \quad \text{for} \quad V_n^+(g)$$

and

$$n = -1, -3/2, -2, -5/2, \ldots \quad \text{for} \quad V_n^-(g).$$

(4) *Others: There is the trivial representation (which is the only finite dimensional UIR), and there are "the two limits of the discrete series".*

The explicit forms of these representations and the Hilbert spaces on which they act will be given below.

Actually, Bargmann constructs representations of the pseudo-unitary group $G_0 = SU(1,1)$ which can be defined as the group of all complex unimodular linear transformations leaving the form $|z_0|^2 - |z_1|^2, \quad z_0, z_1 \in \mathbb{C}$, invariant.

A general element g_0 of G_0 corresponds to a matrix

$$g_0 \longrightarrow \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \tag{I.2.1}$$

where

$$|\alpha|^2 - |\beta|^2 = 1,$$

with α, β being complex numbers and the bars denoting complex conjugation.

It is known that G and G_0 are conjugate subgroups of

$$GL(2,\mathbb{C}) = \{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ a_{11}a_{22} - a_{21}a_{12} \neq 0,\ a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{C}\}. \tag{I.2.2}$$

More specifically, if

$$c = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \in GL(2,\mathbb{C})$$

then

$$G_0 = c.G.c^{-1} \qquad (c^{-1} = c^*,\ * = \text{transpose, complex conjugate} \tag{I.2.3}$$

and we have for

$$g_0 = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}, \text{and} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\alpha = 1/2[(a+d) + (b-c)i] \tag{I.2.4}$$

$$\beta = 1/2[(a-d) - (b+c)i]. \tag{I.2.5}$$

One also has the corresponding subgroups of G_0 (see (I.2.18)):

$$K_0 = c.K.c^{-1} = \{k'_\theta, 0 \leq \theta \leq 2\pi\}$$

$$A_0 = c.A.c^{-1} = \{A'_t, -\infty \leq t \leq \infty\}$$

$$N_0 = c.N.c^{-1} = \{N'_\xi, -\infty \leq \xi \leq \infty\}.$$

Moreover, one can get that

$$g = \begin{pmatrix} \alpha_1 + \beta_1 & \alpha_2 - \beta_2 \\ -\alpha_2 - \beta_2 & \alpha_1 - \beta_1 \end{pmatrix} \quad (G = c^{-1}.G_0.c) \tag{I.2.6}$$

where

$$\alpha = \alpha_1 + i\alpha_2 \qquad \beta = \beta_1 + i\beta_2.$$

Thus, we have established an isomorphism between the two groups G_0 and G.

By virtue of lemma (1), any element in G_0 can be written uniquely, in the sense of the Iwasawa decomposition as

$$g_0 = k'_\theta a'_t n'_\xi, \tag{I.2.7}$$

where the equation (I.1.8) now reads

$$e^{i\theta/2} = \frac{\alpha+\beta}{|\alpha+\beta|}, \quad e^t = |\alpha+\beta|^2, \quad \xi = \frac{Im(\alpha\beta)}{|\alpha+\beta|^2}. \tag{I.2.8}$$

Moreover, for $g_0 \in G_0$

$$g_0 k'_\theta = k'_{g_0.\theta} a'_{t(g_0,\theta)} n'_{\xi(g_0,\theta)}, \tag{I.2.9}$$

where

$$e^{i(g_0.\theta)} = \frac{\alpha e^{i\theta/2} + \beta e^{-i\theta/2}}{|\alpha e^{i\theta/2} + \beta e^{-i\theta/2}}, \tag{I.2.10}$$

$$e^{t(g_0,\theta)} = |\alpha\xi + \beta|^2, \tag{I.2.11}$$

$$e^{\xi(g_0,\theta)} = \frac{Im(\alpha\beta e^{i\theta})}{|\alpha\xi+\beta|^2}. \tag{I.2.12}$$

We have from (I.2.12) and the complex conjugate of equation (I.2.11),

$$e^{i(\theta/2 - g_0.\theta/2)} = \frac{\overline{\beta}\xi + \overline{\alpha}}{|\overline{\beta}\xi + \overline{\alpha}|}, \quad \xi = e^{i\theta}. \tag{I.2.13}$$

We can extend this equation to an element in $\mathbb{R}$. By taking the complex conjugate of (I.1.12) and from (I.1.11), we have for $\xi = e^{i\theta}, g_0 \in G_0,$ and $\quad g \in G$:

$$e^{i(\theta/2 - g_0.\theta/2)} = e^{i(\theta/2 - g.\theta/2)} \tag{I.2.14}$$

which can be proved as follows:

$$\begin{aligned} e^{\frac{1}{2}(\theta - g_0.\theta)} &\overset{(I.2.13)}{=} \frac{\overline{\beta}e^{i\theta} + \overline{\alpha}}{|\overline{\beta}e^{i\theta} + \overline{\alpha}} \\ &\overset{(I.2.4-5)}{=} e^{i\theta/2}\frac{(a+ci)\cos\theta/2 - (b+di)\sin\theta/2}{|(a+ci)\cos\theta/2 - (b+di)\sin\theta/2|} \\ &\overset{(I.1.10)}{=} e^{i(\theta - g.\theta)/2} \end{aligned}$$

Note that, we used the fact

$$\cos\theta/2 = 1/2(e^{i\theta/2} + e^{-i\theta/2}), \quad \sin\theta/2 = 1/2(e^{i\theta/2} - e^{-i\theta/2}).$$

We claim that

$$e^{i(\theta/2 - g.\theta/2)} = \frac{cx+d}{|cx+d|}. \tag{1.2.15}$$

(We need some further discussion to prove this equation; However, note that $x \in \mathbb{R}$ can be written, using equation (1.2.23) below , in the form $x = c^{-1}.e^{i\theta}$)

For completeness, it might be useful to mention that the pseudo-orthogonal group $SO(2,1)$ of Lorentz transformations on $2+1$ dimensional space–time is locally isomorphic to G and also to G_0. The homomorphism between $SO(2,1)$ and the group G_0, for example, is easily displayed through the action of the latter on hermitian matrices U associated to the triples (u_1, u_2, u_3).

Indeed, let

$$U = \begin{pmatrix} u_3 & u_1 + iu_2 \\ u_1 - iu_2 & u_3 \end{pmatrix}, u_1, u_2, u_3 \in \mathbb{R},$$

be a Hermitian matrix with determinant

$$det(U) = \sum_{\mu,\nu=1}^{3} g_{\mu\nu} u_\mu u_\nu = u_3^2 - u_1^2 - u_2^2.$$

Then, if $g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$ and $g^* = \begin{pmatrix} \overline{\alpha} & \beta \\ \overline{\beta} & \alpha \end{pmatrix}$ its Hermitian adjoint, we have

$$U' = gUg^* = \begin{pmatrix} u_3' & u_1' + iu_2' \\ u_1' - iu_2' & u_3' \end{pmatrix}.$$

again a hermitian matrix of the same type, and with

$$u_1' = Re(\alpha^2 + \beta^2)u_1 - Im(\alpha^2 - \beta^2)u_2 + 2Re(\alpha\beta)u_3$$

$$u_2' = Im(\alpha^2 + \beta^2)u_1 + Re(\alpha^2 - \beta^2)u_2 + 2Im(\alpha\beta)u_3$$

$$u_3' = 2Re(\alpha\overline{\beta})u_1 - 2Im(\alpha\overline{\beta})u_2 + (|\alpha|^2 + |\beta|^2)u_3$$

one easily verifies that the 3×3 matrix

$$\begin{pmatrix} Re(\alpha^2 + \beta^2) & -Im(\alpha^2 - \beta^2) & 2Re(\alpha\beta) \\ Im(\alpha^2 + \beta^2) & Re(\alpha^2 - \beta^2) & 2Im(\alpha\beta) \\ 2Re(\alpha\overline{\beta}) & -2Im(\alpha\overline{\beta}) & (|\alpha|^2 + |\beta|^2) \end{pmatrix}$$

is an element of $SO(2,1)$.

This homomorphism is two-to-one in the sense that the two elements $\pm g \in G_0$ are mapped to $U' \in SO(2,1)$ with kernel $\pm I$.

Bargmann actually wrote G_0 differently, as the subgroup of

$$SL(2,\mathbb{C}) = \{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \gamma\beta = 1, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C} \}$$

which satisfies the following relation[4]:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

that so $\gamma = \overline{\beta}$ and $\delta = \overline{\alpha}$.

Therefore, the Lie algebra $\mathfrak{su}(1,1)$ of the group G_0 consists of all matrices satisfying the following conditions:

$$\left. \begin{aligned} tr(X) &= 0 \\ X^* \sigma_3 + \sigma_3 X &= 0 \end{aligned} \right\} . \tag{I.2.16}$$

Hereafter "tr" stands for the ordinary matrix trace $tr(g) = \sum g_{ii}$. A basis for $\mathfrak{su}(1,1)$ is

$$X_0 = \frac{i}{2}\sigma_3 = 1/2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$$X_1 = \frac{1}{2}\sigma_1 = 1/2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$X_2 = \frac{1}{2}\sigma_2 = 1/2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

---

[4] In other words; the elements of G_0 satisfy $g_0^* \sigma_3 g_0 = \sigma_3$ where σ_k; k=1,2,3 are the Pauli matrices

satisfying the commutation relations:

$$[X_0, X_1] = -X_2, \quad [X_0, X_2] = X_1, \quad [X_1, X_2] = X_0. \qquad (1.2.17)$$

The corresponding elements in G_0 are

$$k'_\theta = \begin{pmatrix} e^{\frac{\theta}{2}\imath} & 0 \\ 0 & e^{-\frac{\theta}{2}\imath} \end{pmatrix} = \exp(\theta X_0)$$

$$a'_t = \begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix} = \exp(tX_1)$$

$$n'_\xi = \begin{pmatrix} \cosh\frac{\xi}{2} & -\imath\sinh\frac{\xi}{2} \\ \imath\sinh\frac{\xi}{2} & \cosh\frac{\xi}{2} \end{pmatrix} = \exp(\xi X_2). \qquad (1.2.18)$$

The noncompactness of G_0 follows immediately from (I.2.18). From (I.2.17), we can prove that the Killing form B ([18],[10]) is nondegenerate, i.e.

$$det(B(X_\imath, X_\jmath)) \neq 0 \quad 1 \leq i, j \leq 3$$

where

$$B(X_\imath, X_\jmath) = tr(\text{ad } X_\imath \text{ ad } X_\jmath),$$

and the homomorphism $X \longrightarrow \text{ad}X$ is defined by:

$$\text{ad } X_\imath(X_\jmath) = [X_\imath, X_\jmath].$$

Indeed, from (I.2.17), we have

$$\text{ad } X_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{ad } X_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{ad } X_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and a straightforward computation shows that

$$det(B(X_\imath, X_\jmath)) = -2\delta_{\imath\jmath} = \begin{cases} -2 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus, the Lie algebra $sl(1,1)$ is semisimple[5]. From this, it follows that the group $SU(1,1)$ is a non-compact, semi-simple Lie group.

It is well known that the group $GL(2,\mathbb{C})$ acts on the complex plane $\mathbb{C}$ as a group of linear fractional transformations.

For $\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right) \in GL(2,\mathbb{C})$, $z \in \mathbb{C}$, this action is given by

$$g.z = \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}}. \tag{I.2.19}$$

Set

$$\sigma(g,z) = \frac{d(g.z)}{dz} = \frac{det(g)}{(a_{21}z + a_{22})^2}; \tag{I.2.20}$$

a straightforward calculation shows that

$$(g_1 g_2.z) = g_1.(g_2.z)$$

and

$$\left.\begin{aligned} \sigma(g_1 g_2, z) &= \sigma(g_1, g_2.z)\sigma(g_2, z) \\ \sigma(e,z) &= 1, \end{aligned}\right\} \tag{I.2.21}$$

where e is the identity matrix.

The linear fractional transformation corresponding to c is given by

$$\xi = c.z = \frac{z-i}{z+i} \tag{I.2.22}$$

and its inverse is

$$z = c^{-1}.\xi = i\frac{1+\xi}{1-\xi}. \tag{I.2.23}$$

The transformation $\xi = c.z$ takes the upper half plane $C^+ = \{z \in \mathbb{C} | Im(z) > 0\}$ onto the open unit disk $D = \{\xi \in \mathbb{C} || \xi | < 1\}$, and the real axis $\mathbb{R}$ onto the unit circle $U = \{\xi \in \mathbb{C} || \xi | = 1\}$. This is the transformation that we use to construct the intertwining operators connecting the representations of G_0 and G.

★ ★ ★

[5] Cartan's Criterion: A finite-dimensional Lie algebra is semisimple if and only if its Killing form is nondegenerate.

I.3. The two continuous principal series

For G_0, the representation space of the principal series is the Hilbert space $L^2(U, d\xi), (d\xi = \frac{d\theta}{2\pi}, \xi = e^{i\theta} \in U)$, of functions defined on the unit circle

$$U = \{\xi \in \mathbb{C} | |\xi| = 1\}, \tag{I.3.1}$$

satisfying

$$\int_U |f(\xi)|^2 d\xi < \infty,$$

with an inner product

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta. \tag{I.3.2}$$

The action of G_0 on U is given by

$$\xi \to g_0.\xi = \frac{\alpha\xi + \beta}{\bar{\beta}.\xi + \bar{\alpha}} \tag{I.3.3}$$

where $g_0 = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in G_0$ and $\xi \in U$.

One can prove that G_0 acts on U as a group of transformations, that is

$$(g_0 g_0').\xi = g_0.(g_0'.\xi) \tag{I.3.4}$$

for $g_0, g' \in G_0$. Moreover, using (I.3.3) and (I.3.4), we can prove that

$$\left.\begin{aligned} \sigma(g_0 g_0', \xi) &= \sigma(g_0, g_0'.\xi)\sigma(g_0', \xi), \\ \sigma(e, \xi) &= 1. \end{aligned}\right\} \tag{I.3.5}$$

Where (see equation (I.2.20))

$$\sigma(g_0, \xi) = \frac{d(g_0.\xi)}{d\xi} = (\bar{\beta}\xi + \bar{\alpha})^{-2}. \tag{I.3.6}$$

It is clear that, for any complex number $s \in \mathbb{C}$ and $g_0,\ g_0' \in G_0$

$$|\sigma(g_0 g_0', \xi)|^s = |\sigma(g_0, g_0'.\xi)|^s |\sigma(g_0', \xi)|^s. \tag{I.3.7}$$

Let the map $v : G_0 \times U \to U$ be defined by

$$v(g_0, \xi) = \frac{\overline{\beta}\xi + \overline{\alpha}}{|\overline{\beta}\xi + \overline{\alpha}|}, \tag{I.3.8}$$

for

$$g_0 = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in G_0, \quad \text{and} \quad \xi \in U.$$

It is easy to show that v is a continuous function and satisfies the multiplier equation

$$v(g_0 g_0', \xi) = v(g_0, g_0'.\xi)v(g_0', \xi). \tag{I.3.9}$$

Definition (I.3.1) *Let $j \in \{0, 1/2\}$, and $s \in \mathbb{C}, Re(s) = 1/2$; the UIR's of the principal series for G_0 are defined by:*

$$\boxed{U^{j,s}_{g_0} f(\xi) = |\sigma(g_0^{-1}, \xi)|^s (v(g_0^{-1}, \xi))^{2j} f(g_0^{-1}.\xi)} \tag{I.3.10}$$

for $f \in L^2(U, d\xi)$ and $g_0^{-1} = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$ excluding the case $\{j, s\} = \{1/2, 1/2\}$•

It can be checked without difficulty [28, chapter (5), proposition (5.2)] that $U^{j,s}$ is actually a representation and if $Re(s) = 1/2$ then $U^{j,s}$ is unitary. Moreover, the unitary representation is irreducible for $j \in 0, 1/2$ and $s \in \mathbb{C}$ except when $\{j, s\} = \{1/2, 1/2\}$.

In Bargmann's notation, the series $U^{0,s}, Im(s) \geq 0$ and $U^{1/2,s}, Im(s) > 0$ corresponds to the series C^0_q, with $q = s(1-s)$, and $C^{1/2,s}_q, Im(s) > 0$, with $q = s(1-s)$, respectively.

For $x \in \mathbb{R}$, (I.2.22) gives

$$c.x = \frac{x - i}{x + i} \in U$$

and so (I.2.20) gives

$$\sigma(c, x) = \frac{2i}{(x + i)^2} \,. \tag{I.3.11}$$

Considering the action (I.3.11), one can define for $j = \{0, 1/2\}, s \in \mathbb{C}$, a map $L_{j,s} : \varphi \to f$ which maps $L^2(U, d\xi)$ into $L^2(\mathbb{R}, dx)$ by:

$$f(x) = (L_{j,s}\varphi)(x) = \frac{1}{\sqrt{2\pi}} |\sigma(c, x)|^s v(c, x)^{2j} \varphi(c.x) \tag{I.3.12}$$

where $v(c,x) = \frac{x+i}{|x+i|}$ satisfying the equation (I.2.21) and the term $\frac{1}{\sqrt{2\pi}}$ is taken for future convenience.

On the other hand, for $\xi \in U$, (I.2.23) gives

$$c^{-1}.\xi = i\frac{1+\xi}{1-\xi} = \cot\frac{\theta}{2} \in \mathbb{R}$$

so that

$$\sigma(c^{-1}.\xi) = \frac{-2i}{(1-\xi)^2}\ . \tag{I.3.13}$$

If f is defined on $\mathbb{R}$, $j = 0, 1/2, s \in \mathbb{C}$, and $v(c^{-1},\xi) = \frac{1-\xi}{|1-\xi|}$, then we can define a map

$$\varphi(\xi) = (L_{j,s}^{-1}f)(\xi) = \sqrt{2\pi}|\sigma(c^{-1},\xi)|^s v(c^{-1},\xi)^{2j} f(c^{-1}.\xi) \tag{I.3.14}$$

which maps $L^2(\mathbb{R}, dx)$ onto $L^2(U, d\xi)$.

Lemma (I.3.2). *For $j = 0, 1/2$, $s \in \mathbb{C}$*

(1) $L_{j,s}^{-1}L_{j,s} = I_U$, *the identity operator defined on* $L^2(U, d\xi)$
$L_{j,s}L_{j,s}^{-1} = I_{\mathbb{R}}$, *the identity operator defined on* $L^2(\mathbb{R}, dx)$

(2) $L_{j,s}$ *is an isometry from* $L^2(U, d\xi)$ *onto* $L^2(\mathbb{R}, dx)$;

(3) $L_{j,s}^{-1}$ *is an isometry from* $L^2(\mathbb{R}, dx)$ *onto* $L^2(U, d\xi)$.

(4) *Let* $j = \{0, 1/2\}, s \in \mathbb{C}$ *with* $Re(s) = 1/2$, *put*

$$V_g^{j,s} = L_{j,s}U_{g_0}^{j,s}L_{j,s}^{-1} \qquad (g = cg_0c^{-1})$$

then $V_g^{j,s}$ *are UIR's of* G *given by*

$$\boxed{(V_g^{j,s}f)(x) = |\sigma(g^{-1}_{\cdot}, x)|^s [sign(cx+d)]^{2j} f(g^{-1}.c)} \tag{I.3.15}$$

Proof. Let φ be defined on U

$$
\begin{aligned}
(L_{j,s}^{-1}L_{j,s}\varphi)(\xi) &= \sqrt{2\pi}|\sigma(c^{-1},\xi)|^s\left(\frac{1-\xi}{|1-\xi|}\right)^{2j}(L_{j,s}\varphi)\left(i\frac{1+\xi}{1-\xi}\right)\\
&= \sqrt{2\pi}|\sigma(c^{-1},\xi)|^s\left(\frac{1-\xi}{|1-\xi|}\right)^{2j}\\
&\qquad \frac{1}{\sqrt{2\pi}}|\sigma(c,c^{-1}.\xi)|^s\left(\frac{i\left(\frac{1+\xi}{1-\xi}\right)+i}{|i\left(\frac{1+\xi}{1-\xi}\right)+i|}\right)^{2j}\varphi\left(\frac{i\left(\frac{1+\xi}{1-\xi}\right)-i}{i\left(\frac{1+\xi}{1-\xi}\right)+i}\right)\\
&= |\sigma(c^{-1},\xi)|^s|\sigma(c,c^{-1}.\xi)|^s\left(\frac{1-\xi}{|1-\xi|}\right)^{2j}\left(\frac{|1-\xi|}{1-\xi}\right)^{2j}\varphi(\xi)\\
&= \varphi(\xi)
\end{aligned}
$$

which proves that $L_{j,s}^{-1}L_{j,s} = I_U$; similarly, $L_{j,s}L_{j,s}^{-1} = I_{\mathbb{R}}$. To prove (2), let $\varphi,\psi \in L^2(U,d\xi)$

$$
\begin{aligned}
(L_{j,s}\varphi, L_{j,s}\psi) &= \frac{1}{2\pi}\int_{\mathbb{R}}|\sigma(c,x)|^s\left(\frac{x+i}{|x+i|}\right)^{2j}\varphi\left(\frac{x-i}{x+i}\right)\\
&\qquad \overline{|\sigma(c,x)|^{\overline{s}}\left(\frac{x+i}{|x+i|}\right)^{2j}\psi\left(\frac{x-i}{x+i}\right)}dx\\
&= \frac{1}{2\pi}\int_R|\sigma(c,x)|\varphi(c.x)\overline{\psi(c.x)}dx\\
&= \frac{1}{2\pi}\int_0^{2\pi}\varphi(e^{i\theta})\overline{\psi(e^{i\theta})}d\theta\\
&= \int_U\varphi(\xi)\overline{\psi(\xi)}d\xi\\
&= (\varphi,\psi).
\end{aligned}
$$

In the same fashion we can prove (3). To prove (4), using (I.3.12), (I.3.10) and (I.3.14) we have,

$$
\begin{aligned}
(V_g^{j,s}f)(x) &= (L_{j,s}U_{g_0}^{j,s}L_{j,s}^{-1}f)(x)\\
&= \frac{1}{\sqrt{2\pi}}|\sigma(c,x)|^s(v(c,x))^{2j}(U_{g_0}^{j,s}L_{j,s}^{-1}f)(c.x)
\end{aligned}
$$

$$= \frac{1}{\sqrt{2\pi}} |\sigma(c.x)|^s (v(c,x))^{2j} |\sigma(g_0, c.x)|^s (v(g_0, c.x))^{2j} L_{j,s}^{-1} f(g_0.c.x)$$

$$= |\sigma(c,x)|^s (v(c,x))^{2j} |\sigma(g_0, c.x)|^s (v(g_0, c.x))^{2j}$$

$$|\sigma(c^{-1}, g_0.c.x)|^s (v(c^{-1}, g_0.c.x))^{2j} f(c^{-1}.g_0.c.x)$$

$$= |\sigma(c,x)|^s |\sigma(c^{-1}.g_0, c.x)|^s (v(c,x))^{2j} (v(c^{-1}.g_0, c.x))^{2j} f(c^{-1}.g_0.c.x)$$

$$= |\sigma(c^{-1}.g_0.c, x)|^s (v(c^{-1}.g_0.c, x))^{2j} f(c^{-1}.g_0.c.x)$$

$$= |\sigma(g,x)|^s (v(g,x))^{2j} f(g.x)$$

$$= |\sigma(g,x)|^s [sign(cx+d)]^{2j} f(g.x) \bullet$$

⋆ ⋆ ⋆

§ INDUCED REPRESENTATION

Given any UIR V of G, we can obtain in general a reducible representation of a closed subgroup H of G simply by restricting the domain of V to H. It is quite natural to ask, if given a UIR of the subgroup H whether we can construct a UIR on G . The answer is affirmative and has been studied by Weil for compact groups [32], by Mackey for arbitrary locally compact group [23], and by Bruhat for Lie groups [12].

Here we will give a concrete description of the inducing construction in the general setting, followed by a detailed construction of the induced representations of the group $G = SL(2,\mathbb{R})$, using a given unitary representation of the closed subgroup P defined above (see (I.1.16)).

THE CONSTRUCTION

Let H be a closed subgroup of a separable locally compact group G and $h \to L_h$ be a unitary representation of H in a separable Hilbert space $\mathfrak{H}$.

Denote by $\mathfrak{H}^L$ the set of all functions $\psi : G \to \mathfrak{H}$ such that:

(1) $(\psi(g), \varphi)$ is a Borel function of g for each $\varphi \in \mathfrak{H}$;

(2) $\psi(gh) = L^{-1}(h)\psi(g) \quad$ for all $\quad g \in G, h \in H$;

(3) $\int_{X=G/H} |\psi(gh)|^2_{\mathfrak{H}} d\mu(x) = \int_X |\psi(g)|^2_{\mathfrak{H}} d\mu(x) < \infty$

It is known [10, page 474–5 or Wa, page 366-7] that $\mathfrak{H}^L$ is a separable Hilbert space with inner product

$$(\psi, \varphi)_{\mathfrak{H}^L} = \int_X (\psi, \varphi)_{\mathfrak{H}} d\mu(x).$$

The map $g \to U_g^L$ given by

$$\boxed{(U_g^L \psi)(g') = \sigma(g, g^{-1}g')^{1/2} \psi(g^{-1}g')}, \tag{I.3.16}$$

where

$$\sigma(g^{-1}, g') = \frac{d\mu(g^{-1}\hat{g})}{d\mu(\hat{g})} \qquad (\hat{g} = g'h \in X) \tag{I.3.17}$$

is the Radon–Nikodym derivative of the quasi–invariant measure μ in X and satisfies

$$\sigma(gg', *) = \sigma(g, g'.*)\sigma(g', *) \qquad (g, g' \in G),$$

defines a unitary representation of G in $\mathfrak{H}^L$ called the **representation of G induced by L**, or simply, the **induced representation**.

Suppose that for each class $x \in X = G/H$, we choose a "*representative*" $s(x) \in G$ such that

$$x = s(x).h, \quad h \in H, \quad x \in X.$$

For any $g \in G, g.x$ is the class of the elements $gs(x)$ mod H, and therefore we may write

$$gs(x) = s(g.x)h(g, x)$$

where

$$h(g, x) = s(g.x)^{-1} g s(x) \tag{I.3.18}$$

belongs to H and,

$$\begin{aligned} h(gg', x) &= s(gg'.x)^{-1} gg' s(x) \\ &= s(g.(g'.x))^{-1} g s(g'.x) h(g', x) \qquad (I.3.19) \\ &= h(g, g'.x) h(g', x) \end{aligned}$$

for $g, g' \in G$ and

$$h(e, x) = e.$$

When $x = x_0 = H$, we take $s(x_0) = e$ and since $x = s(x).x_0$, we have

$$\begin{aligned} s(x) = s(x)e = s(x)s(x_0) \\ &= s(s(x).x_0)h(s(x), x_0) \qquad \text{from (I.3.18)} \\ &= s(x)h(s(x), x_0) \end{aligned}$$

which implies $h(s(x), x_0) = e$, hence

$$\begin{aligned} h(g, x) &= h(g, s(x).x_0) \\ &= h(gs(x), x_0) h(s(x), x_0)^{-1} \qquad \text{from (I.3.19)} \\ &= h(gs(x), x_0). \end{aligned}$$

Also note that for $u \in H$, $u.x_0 = x_0$, and therefore

$$\begin{aligned} h(u, x_0) &= s(u.x_0)^{-1} u s(x_0) \\ &= s(x_0)^{-1} u s(x_0) \\ &= u \qquad (I.3.20) \end{aligned}$$

and

$$\begin{aligned} h(gu, x_0) &= h(g, u.x_0) h(u, x_0) \\ &= h(g, x_0) u. \qquad (I.3.21) \end{aligned}$$

Given a unitary representation L of H in $\mathfrak{H}$, let $\mathcal{U}(\mathfrak{H})$ be the group of unitary operators on $\mathfrak{H}$, then an L-cocycle is a map

$$\gamma : G \times G/H \to \mathcal{U}(\mathfrak{H})$$

that is

$$\gamma(g,x) = L(h(g,x))$$

(Indeed, we can prove that γ is a unitary representation on $\mathfrak{H}$ as follows:

$$\begin{aligned}\gamma(gu,x_0) &= L(h(gu,x_0))\\ &= L(h(g,u.x_0))L(h(u,x_0))\\ &= L(h(g,x_0))L(h(u,x_0))\\ &= \gamma(g,x_0)\gamma(u,x_0)\end{aligned}$$

that is

$$\gamma(gu) = \gamma(g)\gamma(u)$$

moreover,

$$\begin{aligned}\gamma(g,x)\gamma(g,x)^{-1} &= L(h(g,x))L(h(g^{-1},g.x))\\ &= L(h(g^{-1}g,x))\\ &= 1)\end{aligned}$$

such that

(1) $\gamma(gg',x) = \gamma(g,g'.x)\gamma(g',x)$ for $g,g' \in G, x \in X.$

Indeed we can easily prove that as follows:

$$\begin{aligned}\gamma(gg',x) &= L(h(gg',x))\\ &= L(h(g,g'.x)h(g',x))\\ &= L(h(g,g'.x))L(h(g',x))\\ &= \gamma(g,g'.x)\gamma(g',x);\end{aligned}$$

(2) $\gamma(u, x_0) = L(u)$ for $u \in H, x_0 = H$

note that

$$\gamma(u, x_0) = L(h(u, x_0)) \overset{(I.3.20)}{=} L(u);$$

(3) $(\gamma(g, x)\psi, \varphi)_{\mathfrak{H}}$ is a Borel function for all $\psi, \varphi \in \mathfrak{H}$.

Let μ be a quasi–invariant measure in X and γ an L–cocycle. Then one can define a unitary representation (which is an alternative form of the induced representation) V_g on $L^2(X)$, the space of square integrable measurable functions on the homogeneous space $X = G/H$, by

$$\boxed{(V_g\psi)(x) = \sigma(g, g^{-1}.x)^{1/2}\gamma(g, g^{-1}.x)\psi(g^{-1}.x)} \tag{I.3.22}$$

for $g \in G$, $x \in X$.

Next, we construct [V, chapter (3)] an isometry between the Hilbert spaces $\mathfrak{H}^L$ and $L^2(X)$ (which proves the equivalence of (I.3.16) and (I.3.22)).

Define $\gamma_s(g, x) = L(h(g, x))$ which satisfies

$$\begin{aligned} \gamma_s(gg', x) &= L(h(gg', x)) \\ &= L(h(g, g'.x)h(g', x)) \\ &= L(h(g, g'.x))L(h(g', x)) \\ &= \gamma_s(g, g'.x)\gamma_s(g', x) \end{aligned}$$

and

$$\gamma_s(e, x) = id_U \quad \text{the identity operator in} \quad \mathcal{U}(\mathfrak{H})$$

i.e. γ_s is a L-cocycle. Then for $u \in H$,

$$\gamma_s(u, x_0) = L(h(u, x_0)) = L(u). \tag{I.3.23}$$

If γ is any L-cocycle, then

$$\begin{aligned}\gamma(g,x) &= \gamma(g,s(x)x_0)\\ &= \gamma(gs(x),x_0)\gamma(s(x),x_0)^{-1}\\ &= \gamma(s(g.x)h(g,x),x_0)\gamma(s(x),x_0)^{-1}\\ &= \gamma(s(g.x),h(g,x)x_0)\gamma(h(g,x),x_0)\gamma(s(x),x_0)^{-1}\\ &= \gamma(s(g.x),x_0)L(h(g,x))\gamma(s(x),x_0)^{-1}\\ &= \gamma(s(g.x),x_0)\gamma_s(g,x)\gamma(s(x),x_0)^{-1} \qquad \text{(I.3.24)}\end{aligned}$$

So if we put

$$\beta(x) = \gamma(s(x),x_0)$$

then

$$\gamma(g,x) = \beta(g.x)\gamma_s(g,x)\beta(x)^{-1} \qquad \text{(I.3.25)}$$

One can show now [BR, Lemma 1, page 473] that the map

$$f \longrightarrow \beta f$$

is an isometry in $L^2(X)$ which takes the representation defined by γ to one defined by γ_s.

The induced representations of $SL(2,\mathbb{R})$

We are interested in constructing the representations of the group $G = SL(2,\mathbb{R})$ induced by the finite dimensional irreducible representations (i.e. characters) of the minimal parabolic subgroup

$$P = MAN = A'N = \{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}, a \in \mathbb{R}^*, b \in \mathbb{R}\}.$$

First of all, we can easily show that P is a semi-direct product of A' and N where N is an invariant subgroup of P. Because the character is constant on N, $\chi(n) = \chi(ana^{-1})$, then [10, chapter (6)] every finite dimensional representation of the group P is trivial on N and hence uniquely determined by a representation of $A' = P/N$. Indeed we have the following lemma [10, chapter (19)],

Lemma (I.3.3). *A finite continuous irreducible representation L of P in a space $\mathfrak{H}$ has the form*

$$L_{man} = \chi(a)L_m, \qquad m \in M, a \in A, n \in N$$

where χ is the irreducible representation of A and $m \to L_m$ is a continuous irreducible representation of M in $\mathfrak{H}$ •

Since, $M = \{\pm I\}$, it has only two irreducible nonequivalent representations given by

$$L^{2j} = [sign(a)]^{2j}, \quad j = 0, 1/2$$

consequently, by Lemma (I.3.3), the irreducible finite-dimensional unitary representations of P are one-dimensional [10, chapter(6)] and we have

$$man = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \to \chi(a)L^{2j} \tag{I.3.26}$$

where

$$\chi(a) = |a|^{i\nu}, \quad \nu \in \mathbb{R}.$$

Using Mackey's decomposition (see Preliminaries), every element in G which satisfies the condition $a \neq 0$ can be represented in the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

The remaining elements in G of the form $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$, $c = -b^{-1}$ (see (I.1.1)) can be represented as

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}.$$

where we will disregard them in the following considerations because they are of measure zero in $X = G/P$. Therefore, any element $g \in G$ can be represented as

$$g = s(x)h \tag{I.3.27}$$

where

$$s(x) = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}, \quad x \in \mathbb{R}.$$

For our convenience, let us put

$$g^{-1} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

then,

$$\begin{aligned} s(x)^{-1}g^{-1} &= \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{cx+d} & c \\ 0 & cx+d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{ax+b}{cx+d} & 1 \end{pmatrix}. \\ &= h(g^{-1},x)s(g^{-1},x) \end{aligned} \tag{I.3.28}$$

And the action of g on x is

$$g^{-1}.x = \frac{ax+b}{cx+d}. \tag{I.3.29}$$

Put $\sigma(g^{-1},x) = \frac{d(g^{-1}.x)}{dx}$, then

$$\sigma(g^{-1},x) = |cx+d|^{-2}$$

satisfies (I.2.21) and the induced representation U^L on the Hilbert space $\mathfrak{H}^L$ [see (I.3.16)] is now defined by setting for $f \in \mathfrak{H}^L$

$$\boxed{(U_g^L f)(g') = |cx+d|^{-1} f(g^{-1}g').} \tag{I.3.30}$$

On the other hand, from (I.3.28) we have immediately

$$h(g^{-1},x) = \begin{pmatrix} (cx+d)^{-1} & c \\ 0 & (cx+d). \end{pmatrix} \tag{I.3.31}$$

thus by virtue of equations (I.3.26) and (I.3.31)

$$\gamma(g^{-1},x) = |cx+d|^{-i\nu}[sign(cx+d)]^{2j}, \qquad \nu \in \mathbb{R} \tag{I.3.32}$$

and

$$\boxed{(V_g\psi)(x) = |cx+d|^{-1-i\nu}[sign(cx+d)]^{2j}\psi(g^{-1}.x)}. \qquad (I.3.33)$$

To prove that all these representations are irreducible on $L^2(G/H, dx)$ except the one corresponding to $\nu = 0, \quad j = 1/2$ see [28, chapter 5] and [10, chapter (19)].

It is remarkable to realize in that case, the equation (I.3.33) is the same as $V_g^{s,j}$ defined by equation (I.3.15).

The remaining part of this construction is to prove the equavilence between the induced representation defined by (I.3.30) and the one defined by equation (I.3.33).

Let $L^{j,\nu}$ be a unitary irreducible representation on P. In particular,

$$L^{j,\nu}(a_t n_\xi) = |cx+d|^{-i\nu} \quad \text{see (I.3.26) and (I.3.32).}$$

We can extend $L^{j,\nu}$ to a function on G, by setting

$$L^{j,\nu}(k_\theta a_t n_\xi) = L^{j,\nu}(k_\theta)L^{j,\nu}(a_t n_\xi). \qquad (I.3.34)$$

Indeed, since G/P is locally isomorphic to the subgroup $K = \{k_\theta, \theta \in \mathbb{R}\}$, then $s(x) \cong k_\theta$. However, since K is a commutative group isomorphic to $\mathbb{R}$, then any irreducible unitary representations is one dimensional and equivalent to

$$L^{j,\nu}(k_\theta) = \chi_j(k_\theta) = e^{ij\theta}. \qquad (I.3.35)$$

Therefore,

$$L^{j,\nu}(k_\theta a_t n_\xi) = \chi_j(k_\theta)L^{j,\nu}(a_t n_\xi) \qquad (I.3.36)$$

satisfies

$$L^{j,\nu}(gh) = L^{j,\nu}(g)L^{j,\nu}(h) \quad \text{for all} \quad g \in G, h \in P. \qquad (I.3.37)$$

Let $f \in \mathfrak{H}^L$, and define

$$f_1(g) = L^{j,\nu}(g)f(g). \qquad (I.3.38)$$

Then f_1 is actually a function on G/P, indeed

$$\begin{aligned} f_1(gh) &= L^{j,\nu}(gh)f(gh) \\ &= L^{j,\nu}(g)L^{j,\nu}(h)L^{j,\nu}(h)^-1f(g) \\ &= L^{j,\nu}(g)f(g) \end{aligned} \tag{I.3.39}$$

and therefore,

$$\varphi(x) = f_1(gh) = L^{j,v}(g)f(g) \quad \text{for} \quad x = gP \in G/P. \tag{I.3.40}$$

It's straightforward to show that $A : f \to \varphi$ is indeed an isometry from $\mathfrak{H}^L$ onto $L^2(\mathbb{R}, dx)$.

Put $g = k_\theta$, then

$$\varphi(x) = \chi_j(k_\theta)f(k_\theta), \quad x = k_\theta P \tag{I.3.41}$$

and if we have

$$\varphi_1(x) = L^{j,\nu}(k_\theta)(U_g^L f)(k_\theta)$$

then from equations (I.3.30), (I.1.4), (I.3.41), (I.3.40) and (I.2.15), we have

$$\begin{aligned} \varphi_1(x) &= L^{j,\nu}(k_\theta)|cx+d|^{-1}f(g^{-1}.k_\theta) \\ &= \chi_j(k_\theta)|cx+d|^{-1}f(k_{g^{-1}.\theta)}a_{t(g^{-1},\theta)}n_{\xi(g^{-1},\theta)}) \\ &= \chi_j(k_\theta)|cx+d|^{-1}L(a_{t(g^{-1},\theta)}n_{\xi(g^{-1},\theta)})^{-1}f(k_{g^{-1}.\theta}) \\ &= \chi_j(k_\theta)|cx+d|^{-1}|cx+d|^{-i\nu}f(k_{g^{-1}.\theta}) \\ &= \chi_j(k_\theta)|cx+d|^{-1-i\nu}\chi_j(k_{g_1.\theta})^{-1}\varphi(g^{-1}.x) \\ &= \left(e^{(i\theta/2-ig^{-1}.\theta/2)}\right)^{2j}|cx+d|^{-1-i\nu}\varphi(g^{-1}.x) \\ &= [sign(cx+d)]^{2j}|cx+d|^{-1-i\nu}\varphi(g^{-1}.x) \end{aligned}$$

which is the same as the R.H.S. of (I.3.15), and we have proved that:

$$A.U_g^L.A^{-1} = V_g$$

i.e. U^L is a unitary representation of G equivalent to $V_g^{j,\nu}$.

* * *

I.4. The complementary series

From definition (I.3.10), we know that the operator $U_{g_0}^{0,s}$ is unitary if and only if $s = 1/2 + \imath\nu, \nu \in \mathbb{R}$. However, if $\nu = 0$ and $\sigma = Re(s) \neq 1/2, U_{g_0}^{0,\sigma}$ is no longer unitary.

It is possible to define [28] a Hilbert space $\mathfrak{H}_\sigma$ so that the operator $U_{g_0}^{\nu,\sigma}$ is unitary for G_0 and $1/2 < \sigma < 1$. Indeed, the Hilbert space $\mathfrak{H}_\sigma$ is the completion of the set of holomorphic functions on U having finite norm with respect to the inner product

$$< \varphi, \psi >= C_\sigma \int_U \int_U \frac{\varphi(\xi)\overline{\psi(\eta)}}{|1 - Re(\xi\overline{\eta})|^{1-\sigma}} d\xi d\eta \tag{I.4.1}$$

where

$$d\xi = \frac{d\theta}{2\pi}, \text{and} \quad d\eta = \frac{d\phi}{2\pi}$$

for

$$\xi = e^{\imath\theta}, \text{and} \quad \eta = e^{\imath\phi}$$

and

$$C_\sigma = \frac{\Gamma(1/2)\Gamma(\sigma)}{2^{\sigma-1}\Gamma(\sigma - 1/2)}, \quad (\Gamma(1/2) = \sqrt{\pi}). \tag{I.4.2}$$

The construction leads us to defining a unitary irreducible representation $U_{g_0}^{0,\sigma}$ called the complementary series representation for G_0 where

$$\boxed{(U_{g_0}^{0,\sigma}\varphi)(\xi) = |\sigma(g_0^{-1}, \xi)|^\sigma \varphi\left(\frac{\alpha\xi + \beta}{\overline{\beta}\xi + \overline{\alpha}}\right).} \tag{I.4.3}$$

and $g_0^{-1} = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$, $\sigma(g_0^{-1}, \xi) = \frac{1}{|\overline{\beta}\xi + \overline{\alpha}|^2}$.

The corresponding complementary series representation $V_g^{0,\sigma}$ of G is realized on the Hilbert space $\mathfrak{H}'_\sigma$, which is the completion of the set of holomorphic functions on the real line with the norm

$$||f||^2 = C_\sigma \int_{\mathbb{R}} \int_{\mathbb{R}} |x - y|^{-2(1-\sigma)} f(x)\overline{f(y)} dx dy < \infty \tag{I.4.4.}$$

By putting $j=0, s=\sigma$ in (I.3.12), we can define a map $L_{0,\sigma} : \varphi \to f$, i.e.

$$f(x) = (L_{0,\sigma}\varphi)(x) = \frac{1}{\sqrt{2}\pi}|J(c,x)|^{\sigma}\varphi(c.x) \tag{I.4.5}$$

where

$$|J(c,x)| = \frac{2}{|x+\imath|^2} \tag{I.4.6}$$

satisfies the multiplier equation (I.2.21), and $c.x = \frac{x-\imath}{x+\imath}$. ,maps $\mathfrak{H}_\sigma$ into $\mathfrak{H}'_\sigma$.

Moreover, from (I.3.14) we can define a map $L^{-1}_{0,\sigma} : f \to \varphi$, i.e.

$$\varphi(\xi) = (L^{-1}_{0,\sigma}f)(\xi) = \sqrt{2}\pi|J(c^{-1},\xi)|^{\sigma}f(c^{-1}.\xi) \tag{I.4.7}$$

where

$$|J(c^{-1},\xi)| = \frac{2}{|1-\xi|^2}, \text{and} \quad c^{-1}.\xi = \imath\frac{1+\xi}{1-\xi} \tag{I.4.8}$$

maps $\mathfrak{H}'_\sigma$ into $\mathfrak{H}_\sigma$ as will be shown in the following lemma.

Lemma (I.4.1).

(1) $L_{0,\sigma}L^{-1}_{0,\sigma} = I_{\mathbb{R}}$, *the identity operator in* $\mathfrak{H}'_\sigma$.

$L^{-1}_{0,\sigma}L_{0,\sigma} = I_U$, *the identity operator in* $\mathfrak{H}_\sigma$.

(2) $L_{0,\sigma}$ *is an isometry from* $\mathfrak{H}_\sigma$ *onto* $\mathfrak{H}'_\sigma$.

$L^{-1}_{0,\sigma}$ *is an isometry from* $\mathfrak{H}'_\sigma$ *onto* $\mathfrak{H}_\sigma$.

(3) $V^{0,\sigma}_g = L_{0,\sigma}.U^{0,\sigma}_{g_0}.L^{-1}_{0,\sigma}$ i.e. $V^{0,\sigma}_g$ $(g = c^{-1}.g_0.c)$ *is a unitary representation of the group* G *on the Hilbert space* $\mathfrak{H}'_\sigma$. *The representation operator is given by*

$$\boxed{(V^{0,\sigma}_g f)(x) = |J(g^{-1},x)|^{\sigma}f(g^{-1}.x)} \tag{I.4.9}$$

where $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\quad J(g^{-1},x) = \frac{1}{(cx+d)^2}$.

Proof. To prove (1), let $f \in \mathfrak{H}'_\sigma$ then

$$\begin{aligned}(L_{0,\sigma}L^{-1}_{0,\sigma}f)(x) &= \frac{1}{\sqrt{2}\pi}|J(c,x)|^{\sigma}(L^{-1}_{0,\sigma}f)(c.x)\\ &= \frac{1}{\sqrt{2}\pi}|J(c,x)|^{\sigma}\sqrt{2}\pi|J(c^{-1},c.x)|^{\sigma}f(c^{-1}.c.x)\\ &= |J(c^{-1}c,x)|^{\sigma}f(x)\\ &= f(x)\end{aligned}$$

Therefore, $L_{0,\sigma}L_{0,\sigma}^{-1} = I_{\mathfrak{H}}$, and similarly we can prove that

$$L_{0,\sigma}^{-1}L_{0,\sigma} = I_U.$$

To prove (2), let

$$\xi = e^{\imath\theta} = \frac{x-\imath}{x+\imath} \in U, \text{and} \quad \eta = e^{\imath\phi} = \frac{y-\imath}{y+\imath} \in U$$

then

$$d\xi = \frac{d\theta}{2\pi} = \frac{dx}{\pi|x+i|^2}, \text{and} \quad d\eta = \frac{d\phi}{2\pi} = \frac{dy}{\pi|y+\imath|^2} \tag{I.4.10}$$

and by a direct calculation

$$1 - Re(\xi\overline{\eta}) = \frac{2(x-y)^2}{(x^2+1)(y^2+1)} = \frac{2|x-y|^2}{|x+\imath|^2|y+\imath|^2}. \tag{I.4.11}$$

For $\varphi \in \mathfrak{H}_\sigma$, we have

$$\begin{aligned}
(L_{0,\sigma}\varphi, L_{0,\sigma}\varphi)_{\mathfrak{H}'_\sigma} &= C_\sigma \frac{2^{\sigma-1}}{\pi^2} \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \frac{|x+\imath|^{-2\sigma}|y+\imath|^{-2\sigma}}{|x-y|^{2(1-\sigma)}} \varphi\left(\frac{x-\imath}{x+\imath}\right)\overline{\varphi\left(\frac{y-\imath}{y+\imath}\right)} dxdy \\
&= C_\sigma \frac{2^{\sigma-1}}{\pi^2} \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \left(\frac{|x-y|^2}{|x+\imath|^2|y+\imath|^2}\right)^\sigma \frac{1}{|x-y|^2} \\
&\qquad \varphi\left(\frac{x-\imath}{x+\imath}\right)\overline{\varphi\left(\frac{y-\imath}{y+\imath}\right)} dxdy \\
&= C_\sigma 2^\sigma \int\limits_U\int\limits_U \left(\frac{|1-Re(\xi\overline{\eta})|}{2}\right)^\sigma \frac{1}{|1-Re(\xi\overline{\eta})|} \\
&\qquad \varphi(\xi)\overline{\varphi(\eta)} \frac{dx}{\pi|x+\imath|^2}\frac{dy}{\pi|y+\imath|^2} \\
&= C_\sigma \int\limits_U\int\limits_U \frac{\varphi(\xi)\overline{\varphi(\eta)}}{|1-Re(\xi\overline{\eta})|^{1-\sigma}} d\xi d\eta \\
&= (\varphi,\varphi)_{\mathfrak{H}_\sigma}
\end{aligned}$$

and we have proved (2). To prove (3),

$$\begin{aligned}
(V_g^{0,\sigma} f)(x) &= (L_{0,\sigma} U_{g_0}^{0,\sigma} L_{0,\sigma}^{-1} f)(x) \\
&= \frac{1}{\sqrt{2}\pi} |\sigma(c,x)|^{\sigma} (U_{g_0}^{0,\sigma} L_{0,\sigma}^{-1} f)(c.x) \\
&= \frac{1}{\sqrt{2}\pi} |\sigma(c,x)|^{\sigma} |\sigma(g_0, c.x)|^{\sigma} L_{0,\sigma}^{-1} f(g_0.c.x) \\
&= |\sigma(c,x)|^{\sigma} |\sigma(g_0, c.x)|^{\sigma} |\sigma(c^{-1}, g_0.c.x)|^{\sigma} f(c^{-1}.g_0.c.x) \\
&= |\sigma(c^{-1}.g_0.c, x)|^{\sigma} f(c^{-1}.g_0.c.x) \\
&= |\sigma(g^{-1}, x)|^{\sigma} f(g^{-1}.x) \bullet
\end{aligned}$$

$\star \quad \star \quad \star$

I.5. The two discrete series

For G_0, the representation space [28, chapter (5)] of the discrete series is the Hilbert space

$$\mathfrak{H}^n = L^2(D, \frac{2n-1}{\pi}(1-|z|^2)^{2n-2} d^2 z) \tag{I.5.1}$$

of holomorphic functions defined on the unit disk

$$D = \{z = x + \imath y \in \mathbb{C} | \quad |z| < 1\} \tag{I.5.2}$$

with the inner product

$$(\varphi|\psi)_n = \frac{2n-1}{\pi} \int_D \varphi(z)\overline{\psi(z)}(1-|z|^2)^{2n-2} d^2 z \quad (dz = dxdy) \tag{I.5.3}$$

which can be written in a polar form ($z = re^{\imath\theta}$) as,

$$(\varphi|\psi) = \frac{2n-1}{\pi} \int_0^{2\pi} \int_0^1 \varphi(re^{\imath\theta})\overline{\psi(re^{\imath\theta})}(1-r^2)^{2n-2} r dr d\theta. \tag{I.5.4}$$

We can easily prove that the constant vector $\mathbb{I}$, $\mathbb{I}(z) = 1$ for all $z \in D$, is in $\mathfrak{H}^n$ with a norm $\| \mathbb{I} \| = 1$, indeed

$$\begin{aligned} \| \mathbb{I} \|^2 &= \frac{2n-1}{\pi} \int_0^{2\pi} \int_0^1 (1-r^2)^{2n-2} r dr d\theta \\ &= \frac{2n-1}{2\pi} \int_0^{2\pi} d\theta \int_0^1 (1-t)^{2n-2} dt \qquad (r^2 = t) \\ &= 1. \end{aligned} \tag{I.5.5}$$

The UIR's $U^{\pm}$ are defined by

$$\boxed{(U^{\pm}_{g_0}\varphi)(z) = (\overline{\beta}\, z + \overline{\alpha})^{-2n} \varphi\left(\frac{\alpha\, z + \beta}{\overline{\beta}\, z + \overline{\alpha}}\right)} \tag{I.5.6}$$

where $n = 1, 2/3, 2, 5/2, \ldots$ for U^+ and $n = -1, -2/3, -2, -5/2, \ldots$ for U^-.

For G, the representation space is the Hilbert space $\mathfrak{H}_n$ of all holomorphic functions f on the upper half plane

$$\mathbb{C}^+ = \{z \in \mathbb{C} | Im(z) > 0\} \tag{I.5.7}$$

with finite norm

$$||f||^2 = (2n-1) \iint\limits_{\mathbb{C}^+} |f(z)|^2 y^{2n-2} dx dy \tag{I.5.8}$$

where $z = x + iy$.

Lemma (I.5.1).

(1) *Let L_n be the mapping from $\mathfrak{H}^n$ into $\mathfrak{H}_n$ given by*

$$f(z) = (L_n\varphi)(z) = \frac{2^{2n-1}}{\sqrt{\pi}}(z+i)^{-2n}\varphi(\frac{z-i}{z+i})$$

for $f \in \mathfrak{H}_n$ and $\varphi \in \mathfrak{H}^n$. Then L_n is an isometry from $\mathfrak{H}^n$ onto $\mathfrak{H}_n$.

(2) *Let L_n^{-1} be the mapping from $\mathfrak{H}_n$ into $\mathfrak{H}^n$ given by*

$$\varphi(\xi) = (L_n^{-1}f)(\xi) = \frac{\sqrt{\pi}}{2^{-2n+1}}(1-\xi)^{-2n} f(i\frac{1+\xi}{1-\xi})$$

for $\varphi \in \mathfrak{H}^n$ and $f \in \mathfrak{H}_n$. Then L_n^{-1} is an isometry from $\mathfrak{H}_n$ onto $\mathfrak{H}^n$.

(3) *Put*

$$V_g^{\pm} = L_n U_{g_0}^{\pm} L_n^{-1}$$

then V^+ is an UIR of G on the the Hilbert space $\mathfrak{H}_n$ which is unitary equivalent to U_{g_0}. the representation operator V^+ is given by

$$(V_g^{\pm} f)(z) = (cz+d)^{-2n} f\left(\frac{az+b}{cz+d}\right)$$

for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

Proof. Let

$$\xi = c.z = \frac{z-i}{z+i} = u + iv \qquad (z=x+iy,)$$

then, we can prove that:

$$dxdy = \frac{|z+i|^4}{4} dudv$$

and

$$1 - |\xi|^2 = \frac{4y}{|z+i|^2}.$$

Use these equations, and following a procedure similar to the proof of Lemma (6) we can prove the present lemma•

* * *

CHAPTER II

SQUARE INTEGRABILITY OF GROUP REPRESENTATIONS AND ASSOCIATED COHERENT STATES

The history of coherent states goes back to the early days of quantum mechanics. In 1926 Schrödinger [29] specified the existence of a certain class of states that displayed the classical behavior of the oscillator. This set of states have come to be known in the literature [20] as the coherent states of the harmonic oscillator or Schrödinger coherent states. They have a number of interesting properties:

(1) They minimize the Heisenberg uncertainty relation $\Delta x \Delta p \geq \hbar/2$ (for these states, $\Delta x \Delta p = \hbar/2$).

(2) They are eigenstates in a Hilbert space of the annihilation (destruction) operator:

$$a|z>= z|z>,$$

where z is a complex eigenvalue; a is the conjugate to the creation operator

$$[a, a^*] = 1, \quad [a, 1] = [a^*, 1] = 0.$$

(3) They are created from the ground state $|0>$ by a unitary displacement operator:

$$\{exp[za^* - z^*a]\}|0>= |z> .$$

These properties are all equivalent. In fact, usually one of them is adopted as the definition of the harmonic oscillator coherent states.

In the 1960's the concept of coherent states (introduced by Glauber) came into wide usage through the new field of quantum optics [15], and many authors popularized their use [22].

Due to the very interesting properties of coherent states, various attempts have been made to generalize the concept. One such successful generalization to arbitrary Lie groups is due to Perelomov [25] , who constructed what is referred to in the literature as "generalized coherent states". This in turn was a generalization of Barut and Giradello [11] who presented the generalized coherent states as the eigenstates of the ladder operators

$$L^{\pm} = \frac{\imath}{\sqrt{2}}(L_{13} \pm \imath L_{23})$$

where in the case of $SU(1,1)$, for example, $L_{12} = \frac{1}{2}\sigma_3, \quad L_{13} = \frac{\imath}{2}\sigma_1, \quad L_{23} = \frac{\imath}{2}\sigma_2$ (see chapter (I), equation (I.2.17)) of non-compact groups

According to Perelomov's generalization, the construction of a set of generalized coherent states is based on the relation

$$|\eta_g >= T(g)|\eta_0 > \qquad (g \in G)$$

where $T(g)$ is a unitary irreducible representation (UIR) of Lie group G , and $|\eta_0 >$ is a fixed vector in the Hilbert space $\mathfrak{H}$ of the representation $T(g)$. Then $\{|\eta_g >\}$ is the set of generalized coherent states (GCS). It turn out in this case that the square integrability condition of the UIR is not critical in the construction. However, the assiocated coherent states lack many of the nice properties of the square integrable ones [3]. One of these interesting properties is the "resolution of identity";

$$\frac{1}{d}\int_G |\eta_g >< \eta_g| dg = \mathbb{I}$$

where $d > 0$ is a constant number depending on the normalization of the invariant Haar measure dg on G. That is, for $\eta, \varphi \in \mathfrak{H}$;

$$\frac{1}{d}\int_G | < T(g)\eta|\varphi > |^2 dg = \| \varphi \|^2 \ \| \eta \|^2$$

which means that the coefficient $g \rightarrow < T(g)\eta|\varphi >$ is "square integrable" on G (cf. section (II.1) below).

However, for certain group representations, Perelomov's generalization does not always work (cf section (II.2)). One of the earlier attempts to recuperate this lack was due to Prugovečki [27], for the Poincaré group $P_+^{\uparrow}(1,1)$.

Recently, Ali et al were able to generalize Perelomov's construction and give a theory applicable in many cases where the Perelomov theory fails.

The objective of the present chapter is twofold: first to analyse the theory of square integrable group representations for a locally compact group and built what is known in the literature as "standard coherent states". As an example we examine the affine group which leads to the affine coherent states.

The second objective is to study the theory of square integrable group representations over a coset space G/H, where we will study that in two situations:

(1) When H is a closed stability subgroup for a vector $\eta \in \mathfrak{H}$, under the action of a unitary irreducible representation.

(2) When H is an arbitrary closed subgroup of G. Here, we study, as an example, the Poincaré group in 1-space and 1-time dimensions.

⋆ ⋆ ⋆

II.1. The standard theory of square integrable group representations:

The standard theory may be found in Dixmier [13]. However, we will not restrict ourselves to the case of unimodular[1] locally compact groups [17].

We shall begin by formulating, in a general setting, certain results concerning the square integrability of the representations of a locally compact group and the orthogonality relations as well. Details of the proofs will be provided in appropriate places.

Let G be a locally compact group with left invariant Haar measure dg and right invariant Haar measure $d_r g$. Let $U : g \to U(g), \quad (g \in G)$, be a strongly continuous unitary irreducible representation in a Hilbert space[2] $\mathfrak{H}$.

Definition (II.1.1) *The UIR U is said to be square integrable if the there exists a non-zero vector $\eta \in \mathfrak{H}$ such that:*

$$\boxed{\int_G |(U(g)\eta|\varphi)|^2 dg < +\infty, \quad \varphi \in \mathfrak{H}.} \tag{II.1.1}$$

Equation (II.1.1) is called the admissibility condition and the vector η is said to be admissible •

Note that:

(1) The admissibility condition holds even if we replace the left invariant Haar measure dg by the right invariant measure $d_r g$. Indeed, from the unitarity

[1] In the sense that the left invariant measure is also right invariant.

[2] We will use throughout this chapter the following notation for the inner product; say for $L^2(\mathbb{R}, dt)$

$$(\varphi|\eta) = \int_{-\infty}^{\infty} \overline{\varphi(t)}\eta(t)dt$$

of U and from (0.6), we have

$$\int_G |(U(g)\eta|\varphi)|^2 dg = \int_G |(\eta|U(g^{-1})\varphi)|^2 dg$$
$$= \int_G |(U(g^{-1})\eta|\varphi)|^2 dg$$
$$= \int_G |(U(g)\eta|\varphi)|^2 d_r g. \qquad \text{(II.1.2)}$$

Therefore, in the following computations only the left invariant Haar measure will be used since analogous results hold for the right invariant measure as well, in virtue of equation (II.1.2).

(2) If U is square integrable then the set $\mathfrak{D}$ of admissible vectors is dense in $\mathfrak{H}$.

(3) In the case of a unimodular group, $\mathfrak{D} = \mathfrak{H}$.

Lemma (II.1.2). *If $U_i : G \to \mathfrak{L}(\mathfrak{H}_i)$; $i = 1, 2$ are unitarily equivalent and if U_1 is square-integrable then so is U_2.*

Proof. Let $T : \mathfrak{H}_2 \to \mathfrak{H}_1$ be a unitary map and $U_1 T = T U_2$. Then, for $\eta, \varphi \in \mathfrak{H}_2$,

$$\int_G |(U_2(g)\eta|\varphi)|^2 dg = \int_G |(T^{-1}U_1(g)T\eta|\varphi)|^2 dg$$
$$= \int_G |(U_1(g)T\eta|T\varphi)|^2 dg < +\infty.$$

Consequently, any representation unitarily equivalent to a square integrable representation is also square integrable •

Lemma (II.1.3). *If $U_i : G \to \mathcal{U}(\mathfrak{H}_i)$ $(i = 1, 2)$ are inequivalent square integrable representations of the unimodular group G. Then for $\eta, \varphi \in \mathfrak{H}_1$, and $\eta', \varphi' \in \mathfrak{H}_2$, we have*

$$\int_G \overline{(U_1(g)\eta|\varphi)}(U_2(g)\eta'|\varphi')dg = 0 \qquad \text{(II.1.3)}$$

On the other hand, if the two representations are equivalent under a unitary mapping T of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$, and if G is unimodular we have

$$\int_G \overline{(U_1(g)\eta|\varphi)}(U_2(g)\eta'|\varphi')dg = \frac{1}{d}\overline{(T\eta|\eta')}(T\varphi|\varphi') \tag{II.1.4}$$

where d is the formal degree of the representation $U_1 \cong U_2$.

Proof. See [19; theorem (1)]•

The formal degree d appearing in the above lemma is analogous to the dimension of the representation in the case of compact group. Indeed, if G is compact group then every UIR of G is finite dimensional and we have the following fact [14; Chapter (VII), proposition (24)]:

Theorem (I.1.4). *If G is a compact group and $U : G \to \mathcal{U}(\mathfrak{H})$ is irreducible (therefore, finite dimension). Then*

$$\int_G \overline{(U(g)\eta|\varphi)}(U(g)\eta'|\varphi')dg = \frac{1}{d}\overline{(\eta|\eta')}(\varphi|\varphi') \quad \eta\eta'\varphi\varphi' \in \mathfrak{H} \tag{II.1.5}$$

and d is the dimension of U in the usual sense. Consequently, any UIR of a compact group is square integrable •

The square-integrability of the group representations of a compact group is also consequence of the well known fact of the weak measurability of the maps $g \mapsto (U(g)\eta|\eta')$ [see (0.1)] and the following lemma [13, Chapter (V), Proposition (2.1)]:

Lemma (II.1.5). *A locally compact group G has finite measure if and only if it is compact.*

Lemma (II.1.6). *Let U be a square integrable representation of a locally compact group G [not necessarily unimodular]. Then there is a unique, positive, invertible operator C on $\mathfrak{H}$, such that the following holds:*

(1) *The set of admissible vectors coincides with the domain of C.*

(2) *Let η and η' be any two admissible vectors and let φ and φ' be any two vectors in $\mathfrak{H}$. Then*

$$\int_G \overline{(U(g)\eta|\varphi)}(U(g)\eta'|\varphi')dg = \overline{(C\eta|C\eta')}(\varphi|\varphi') \tag{II.1.6}$$

(3) *If the group G is unimodular, then C is a multiple of the identity ,*

$$C = \lambda I, \qquad (\lambda \in \mathbb{R}^+) \tag{II.1.7}$$

and we have the standard orthogonality relations:

$$\int_G \overline{(U(g)\eta|\varphi)}(U(g)\eta'|\varphi')dg = \lambda\overline{(\eta|\eta')}(\varphi|\varphi') \tag{II.1.8}$$

where $d = \frac{1}{\lambda}$, the formal degree, depends only on the square integrable representation U.

Proof. See [17, theorem 3.1]•

Remarks:

(1) If $\eta = \eta' = \varphi = \varphi'$ in (II.1.2), then for $\eta \in \mathfrak{D}$, one has

$$(C\eta|C\eta) = \frac{1}{\| \eta \|^2}\int_G |(U(g)\eta|\eta)|^2 dg \tag{II.1.9}$$

and one can associate to η, the positive number:

$$c_\eta = \frac{1}{\| \eta \|^2}\int_G |(U(g)\eta|\eta)|^2 dg. \tag{II.1.10}$$

(2) If $\eta = \eta'$, then one has from (II.1.2) and (II.1.4)

$$\int_G \overline{(U(g)\eta|\varphi)}(U(g)\eta|\varphi')dg = \frac{\int_G |(U(g)\eta|\eta)|^2 dg}{||\eta||^2}(\varphi|\varphi'). \tag{II.1.11}$$

Theorem (II.1.7). *Let U be a square integrable representation of G acting on $\mathfrak{H}$ and let η be a nonzero admissible vector,*

(1) *The mapping*

$$L_\eta : \mathfrak{H} \longrightarrow L^2(G, dg)$$

defined by

$$(L_\eta \varphi)(g) = \frac{1}{\sqrt{c_\eta}}(U(g)\eta|\varphi), \quad \text{for} \quad \varphi \in \mathfrak{H}, g \in G \tag{II.1.12}$$

is a linear isometry onto a proper[3] closed subspace $\mathfrak{H}_\eta$ of $L^2(G, dg)$, That is, for every $\varphi, \varphi' \in \mathfrak{H}$, we have

$$(L_\eta\varphi|L_\eta\varphi') = \int_G \overline{(L_\eta\varphi)(g)}(L_\eta\varphi')(g)dg = (\varphi|\varphi'). \tag{II.1.13}$$

(2) *The subspace $\mathfrak{H}_\eta \subset L^2(G, dg)$ is a reproducing kernel Hilbert space (cf. appendix A of [6]), so the corresponding projection operator*

$$\mathbb{P}_\eta = L_\eta L_\eta^*, \qquad (\mathfrak{H}_\eta = \mathbb{P}_\eta L^2(G, dg)) \tag{II.1.14}$$

has the reproducing kernel $K_\eta : G \times G \rightarrow \mathbb{C}$,

$$(\mathbb{P}_\eta\varphi)(g) = \int_G K_\eta(g, g')\varphi(g')dg', \quad \varphi \in L^2(G, dg) \tag{II.1.15}$$

$$K_\eta(g, g') = \frac{1}{c_\eta}(U(g'^{-1}g)\eta|\eta) \tag{II.1.16}$$

defined by the function

$$k_\eta(g) = \frac{1}{c_\eta}(U(g)\eta|\eta). \tag{II.1.17}$$

[3] $\mathfrak{H}_\eta \neq \{0\}$ or $L^2(G, dg)$.

Proof. See [17 section (4) and (5)]•

Remarks.

(1) From (0.11), and the definition of the L_g, we have

$$\begin{aligned}(U_l(g')L_g\varphi)(g) &= (L_g\varphi)(g'^{-1}g)\\ &= \frac{1}{\sqrt{c_\eta}}(U(g'^{-1}g\eta|\varphi)\\ &= \frac{1}{\sqrt{c_\eta}}(U(g)\eta|U(g')\varphi)\\ &= (L_\eta U(g')\varphi)(g) \qquad \text{(II.1.18)}\end{aligned}$$

So L_η is an intertwining operator between the square integrable representation U and the restriction of the left regular representation U_l. Consequently, every square integrable group representation is unitarily equivalent to a subrepresentation of the left regular representation.

(2) From (II.1.11), we may deduce, for arbitrary $\varphi, \varphi' \in \mathfrak{H}$,

$$\frac{1}{C_\eta}\int\limits_G (\varphi|U(g)\eta)(U(g)\eta|\varphi')dg = (\varphi|\varphi') \qquad \text{(II.1.19)}$$

So, the resolution of identity (in the Dirac notation):

$$\frac{1}{C_\eta}\int\limits_G |U(g)\eta><U(g)\eta|dg = \mathbb{I} \quad \text{(identity operator in } \mathfrak{H}) \qquad \text{(II.1.20)}$$

holds on $\mathfrak{H}$.

(3) From (II.1.12) and (II.1.13)

$$(\mathbb{P}_\eta\varphi)(g) = \begin{cases} \varphi(g), & \text{if } \varphi(g) \in \mathfrak{H}_\eta \\ 0, & \text{if } \varphi(g) \in \mathfrak{H}_\eta^\perp \end{cases} \qquad \text{(II.1.21)}$$

and therefore, the adjoint L_η^* coincides with the inverse in the range of an isometric operator,

$$L_\eta^{-1}\Phi = \frac{1}{c_\eta}\int\limits_G \Phi(g')(U(g)\eta)(g')dg' \quad (\Phi \in \mathfrak{H}_\eta). \qquad \text{(II.1.22)}$$

Note that, an entirely analogous result of the theorem (II.1.7) holds for the right regular representation U_r and $L^2(G, d_r g)$. Thus, there exists an isometry

$$R_\eta : \mathfrak{H} \to L^2(G, d_r g)$$

defined by

$$(R_\eta \varphi)(g) = \frac{1}{\sqrt{c_\eta}}(U(g^{-1})\eta|\varphi), \text{for} \quad \varphi \in \mathcal{H}, g \in G, \tag{II.1.23}$$

which intertwines the square integrable representation and the right regular representation. The reproducing kernel is

$$K_\eta^r(g, g') = \frac{1}{c_\eta}(U(gg')^{-1}\eta|\eta). \tag{II.1.24}$$

Consequently, every square integrable representation is unitarily equivalent to a subrepresentation of the regular representation. In other words, square integrable representations belong to the discrete series of the representations of G. This result can be written as: Let U be a unitary irreducible representation of G on a Hilbert space $\mathfrak{H}$, then the following conditions are equivalent:

(1) U is square integrable;

(2) U is unitarily equivalent to a subrepresentation of the left (or right) regular representation.

Definition (II.1.8) *Let $U(g)$ be a strongly continuous, irreducible, unitary representation of G into a Hilbert space $\mathfrak{H}$. Let η be chosen arbitrary but fixed non zero vector in $\mathfrak{H}$ such that the admissibility condition (II.1.1) holds. Then the subset of $\mathfrak{H}$ generated by operating on η with $U(g); g \in G$, i.e*

$$\mathbb{G} = \{|\eta_g >= U(g)|\eta > | \quad g \in G\} \tag{II.1.25}$$

is an overcomplete family of vectors, called "standard" coherent states associated to the representation U•

The affine group

The affine group "ax+b" (denoted G') is the set of linear affine transformations on the real line:

$$x \to ax + b, \quad x \in \mathbb{R}, \quad a > 0, \quad b \in \mathbb{R} \tag{II.1.26}$$

with the group law:

$$(a,b)(a',b') = (aa', ab' + b). \tag{II.1.27}$$

It is a s.l.c. non-unimodular group with left Haar measure

$$d\mu_{(a,b)} = \frac{dadb}{a^2} \tag{II.1.28}$$

and right Haar measure

$$d_r\mu_{(a,b)} = \frac{dadb}{a}. \tag{II.1.29}$$

(the proof is given in the appendix).

§ *The representations:*

It is well known [8],[16] that there exist only two unitary inequivalent, irreducible representations $U^{\pm}$ on invariant subspaces $L^2(\mathbb{R}^+, dt)$ and $L^2(\mathbb{R}^-, dt)$ of $L^2(\mathbb{R}, dt)$ where

$$L^2(\mathbb{R}, dt) = L^2(\mathbb{R}^+, dt) \oplus L^2(\mathbb{R}^-, dt) \tag{II.1.30}$$

and

$$\left.\begin{aligned} L^2(\mathbb{R}^+, dt) &= \{\varphi \in L^2(\mathbb{R}, dt) | \varphi(t) = 0 \quad \text{if} \quad t < 0\} \\ L^2(\mathbb{R}^-, dt) &= \{\varphi \in L^2(\mathbb{R}, dt) | \varphi(t) = 0 \quad \text{if} \quad t > 0\} \end{aligned}\right\} \tag{II.1.31}$$

The representations $U^{\pm}$ on $L^2(\mathbb{R}^+, dt)$

$$(U^{\pm}_{(a,b)}\varphi)(t) = a^{1/2} e^{\pm ibt} \varphi(at) \tag{II.1.32}$$

for $\varphi \in L^2(\mathbb{R}^+, dt)$.

§ *The admissibility condition.*

Our goal is to prove the square integrability of the above representations. So, we will prove the admissibility condition (II.1.1) for $U^-_{(a,b)}$ with respect to the left Haar measure (resp. right Haar measure) and the proof for $U^+_{(a,b)}$ is almost identical. Let $\eta, \varphi \in L^2(\mathbb{R}^+, dt)$ then from (II.1.1) and (II.1.32) we have:

$$\iint_{\mathbb{R}} (\varphi | U^-_{(a,b)}\eta)(U^-_{(a,b)}\eta|\varphi) \frac{dadb}{a^2} = \iiiint_{\mathbb{R}} a e^{ib(t-t')} \overline{\varphi(t)}\varphi(t')\eta(at)\overline{\eta(at')}dtdt' \frac{dadb}{a^2} \tag{II.1.33}$$

using the Fubini theorem and the definition of delta measure[4]

$$\int_{\mathbb{R}} e^{ib(t-t')}db = 2\pi\delta(t-t'), \tag{II.1.34}$$

The right hand side of (II.1.33) becomes,

$$2\pi \| \varphi \|^2 \int_{\mathbb{R}} |\eta(at)|^2 \frac{da}{a}.$$

Indeed, $\int_{\mathbb{R}} |\eta(w)|^2 dw/w, \quad (w = at)$ is not finite for all $\eta \in L^2(\mathbb{R}^+, dt)$, for example, if $\eta(w) = e^{-w}$, the integral will go to infinity, so let $\mathfrak{D}$ be the subset of $L^2(\mathbb{R}^+, dt)$ for which the above integral is finite[5], i.e. the set of admissible vectors (or admissible analyzing wavelets [17]) of $L^2(\mathbb{R}^+, dt)$;

$$\mathfrak{D} = \{\eta \in L^2(\mathbb{R}^+, dt) | \int_{\mathbb{R}} |\eta(w)|^2 \frac{dw}{w} < +\infty\}. \tag{II.1.35}$$

A similar calculation can be done to prove the admissibility condition with respect to the right Haar measure. From theorem (II.1.6-2) we have,

$$(C\eta|C\eta) = 2\pi \int_{\mathbb{R}} |\eta(w)|^2 \frac{dw}{w} \tag{II.1.36}$$

[4] If f is infinity differentiable, then $\int f(t)\delta(t-t')dt = f(t')$

[5] For example, $\eta(w) = w^{1/2}e^{-w}$

where we can define a positive unbounded operator

$$C\eta(w) = \frac{1}{\sqrt{w}}\eta(w) \qquad \text{(II.1.37)}$$

For $\eta, \eta' \in \mathfrak{D}$ and arbitrary φ, φ' in $L^2(\mathbb{R}^+, dt)$, we have

$$\int_{\mathbb{R}} (\varphi|U^-_{(a,b)}\eta)(U^-_{(a,b)}\eta'|\varphi')\frac{dadb}{a^2} = (C\eta'|C\eta)(\varphi|\varphi'). \qquad \text{(II.1.38)}$$

Now, if we take one vector and fix it in $\mathfrak{D}$ for example

$$\eta_0(w) = \sqrt{w}e^{-w}, \qquad \text{(II.1.39)}$$

then, by virtue of (II.1.10), we can associate to η_0 a positive number

$$c_{\eta_o} = 2\pi \int_{\mathbb{R}} e^{-2w}dw = \pi. \qquad \text{(II.1.40)}$$

The resolution of identity follows now from (II.1.38). Indeed, from (II.1.20)

$$\frac{1}{\pi}\int_{\mathbb{R}} |U^-_{(a,b)}\eta_0 >< U^-_{(a,b)}\eta_0|\frac{dadb}{a^2} = \mathbb{I}.$$

§ The isometry :

According to theorem (II.1.7-1), we can now define a transform (or integral wavelet transform) between $L^2(\mathbb{R}^+, dt)$ and $L^2(G', dadb/a^2)$ as:

$$\begin{aligned}(L_{\eta_0}\varphi)(a,b) &= \frac{1}{\sqrt{\pi}}(U^-_{(a,b)}\eta_0|\varphi)\frac{dadb}{a^2} \\ &= \frac{a}{\sqrt{\pi}}\int_{\mathbb{R}} \sqrt{t}e^{ibt}e^{-at}\varphi(t)dt \qquad \text{(II.1.41)}\end{aligned}$$

and we can prove its isometry as follows:

$$\begin{aligned}(L_{\eta_0}\varphi|L_{\eta_0}\varphi) &= \iint_{\mathbb{R}} \overline{(L_{\eta_0}\varphi(t))}(L_{\eta_0}\varphi)(t')\frac{dadb}{a^2} \\ &= \frac{1}{\pi}\iiiint_{\mathbb{R}} \sqrt{tt'}e^{ib(t'-t)}e^{-a(t+t')}\overline{\varphi(t)}\varphi(t')dtdt'dadb \\ &= 2\iint_{\mathbb{R}} te^{-2at}|\varphi(t)|^2dtda \qquad \text{(from (II.1.34))} \\ &= \int_{\mathbb{R}} |\varphi(t)|^2dt \qquad \left(\text{since } \int_{\mathbb{R}} te^{-2at}da = 1/2\right) \\ &= (\varphi|\varphi)_{L^2(\mathbb{R}^+,dt)}\end{aligned}$$

for $\varphi \in L^2(\mathbb{R}, dt)$.

The next step in the analysis is to characterize the range of the transform $L_{\eta_0}\varphi$. It was suggested in [8] and [26] to consider $C^{\pm} \subset L^2(G', \frac{da\,db}{a^2})$ (or $\mathfrak{H}_{\pm 1/2}$ in the terminology of [26]), the space of the analytic functions defined on the $\substack{upper \\ lower}$ half plane, square integrable with respect to the measure

$$Im(z)^{\pm 1/2} dRe(z) dIm(z) = a^{\pm 1/2} db\,da$$

where $z = b + \imath a$. For details of such spaces and related problems, we refer to the references mentioned above.

However, the reproducing kernel of the range of the transform L_{η_0} is given now by the function (II.1.17) as:

$$K_{\eta_0}(a, b) = \frac{a}{\pi} \int_{\mathbb{R}} e^{-\imath z t} e^{-t} t\,dt \qquad \text{(II.1.44)}$$

for $z = b + \imath a$.

Applying U^- to η_0 gives a coherent state system [17].

⋆ ⋆ ⋆

II.2. Square integrability with respect to a homogeneous space :

As we have seen in definition (II.1.1), square integrability depends on the finiteness of an integral over the entire group which, in many cases, is too strong. However, it often happens that one has an analogous situation over a homogeneous space $X = G/H$.

Perelomov's construction:

Let U be the UIR of G acting in the Hilbert space $\mathfrak{H}$; η_0 some fixed vector in $\mathfrak{H}$; as we mentioned in the introduction the construction of a set of generalized coherent states is based on the relation

$$|\eta_g> = T(g)|\eta_0> . \tag{II.2.1}$$

Let H be a closed subgroup of G that leaves η_0 invariant up to a phase (H stationary subgroup) :

$$T(h)|\eta_0> = e^{i\alpha_h}|\eta_0> . \tag{II.2.2}$$

Then from (II.2.1),

$$\begin{aligned} |\eta_{gh}> &= T(gh)|\eta_0> \\ &= T(g)T(h)|\eta_0> \\ &= e^{i\alpha_h}T(g)|\eta_0> . \end{aligned}$$

This relation shows that the coherent state $|\eta_g>$ is uniquely parameterized by the point $x = gh$ of the coset space $X = G/H$ corresponding to the element g:

$$|\eta_g> = e^{i\alpha_g}|\eta_x> . \tag{II.2.3}$$

It can be seen that the set of coherent states is defined whenever the integral

$$\int_X |<\eta_0|\eta_x>|^2 d\mu(x) = d$$

where $d\mu(x)$ denotes a "measure" on X, converges, that is, d is a finite constant. In such a case

$$\frac{1}{d}\int_X |\eta_x><\eta_x|d\mu(x) = \mathbb{I} \qquad \text{(II.2.4)}$$

and one can expand any arbitrary state $|\eta_y>$ in the coherent states,

$$|\eta_y>= \frac{1}{d}\int_X <\eta_x|\eta_y>|\eta_x> d\mu(x) \qquad \text{(II.2.5)}$$

where $K(x,y) = \frac{1>}{d} \; <\eta_x|\eta_y>$ is not arbitrary but satisfies the integral equation

$$K(x,z) = \int_X K(x,y)K(y,z)d\mu(y) \qquad \text{(II.2.6)}$$

which means that K is reproducing kernel.

Perelomov's monograph contains many examples. One of these examples is the discrete series representation of the group $SU(1,1)$ which we will study in chapter (III).

However, Perelomov's method does not always work and cannot be applied for certain group representations, for example Poincaré group. Ali et al [4] introduced a generalization for Perelomov's construction where Grossmann et al's construction (cf. section II.1) appears as a particular case. The following is adapted from [4] (for the details see [4,I]), and as an example we look at the case of the Poincaré group.

Consider a homogeneous space $X = G/H$, where H is an arbitrary closed subgroup of G, as we saw in the preliminaries that there exists up to equivalence a quasi-invariant $\mu(x)$ on X (see (0.8–0.12)). Let

$$\sigma : X \to G$$

be a measurable section of G satisfying (I.3.18), and $F : X \to \mathfrak{L}(\mathfrak{H})^1$ be a μ-measureable function into the bounded positive operators on $\mathfrak{H}$. Consider the rank-1 operator

$$F = |\eta)(\eta| \tag{II.2.7}$$

and a section σ. Define a positive operator valued function $F_\sigma : X \to \mathfrak{L}(\mathfrak{H})^+$:

$$\left.\begin{aligned} F_\sigma(x) &= U(\sigma(x))FU(\sigma(x))^* \\ &= |U(\sigma(x))\eta)(U(\sigma(x))\eta| \end{aligned}\right\} \tag{II.2.8}$$

Definition (II.2.1) [4,II] *The representation U is said to be square integrable mod (H,σ), if together with F, there exists a positive, bounded, invertible operator A_σ on $\mathfrak{H}$, such that:*

$$\boxed{\int_X F_\sigma(x)d\mu(x) = A_\sigma} \tag{II.2.9}$$

the integral converging weakly, that is, for $\varphi, \psi \in \mathfrak{H}$:

$$\boxed{\int_X (\varphi|U(x)\eta)(U(x)\eta|\psi)d\mu(x) = (\varphi|A_\sigma\psi)\bullet} \tag{II.2.10}$$

Notice:

i) If U is square integrable mod (H,σ), it is also square integrable mod (H,σ'), where σ' is any other section for which

$$\begin{aligned} A_{\sigma'} &= \int_X F_{\sigma'}(x)d\mu(x) \\ &= \int_X U(\sigma'(x))FU(\sigma'(x))^* d\mu(x) \end{aligned} \tag{II.2.11}$$

exists as a bounded positive operator with positive, self-adjoint inverse $A_{\sigma'}^{-1}$.

ii) To get coherent states [4,II], if U is square integrable in the sense of definition (II.2.1), define the vectors:

$$\eta_{\sigma(x)} = U(\sigma(x))\eta. \qquad \eta \in \mathfrak{H} \tag{II.2.12}$$

Then the family of coherent states:

$$\mathbb{G}_\sigma = \{\eta_{\sigma(x)}|x \in X\} \tag{II.2.13}$$

is an overcomplete set of vectors with all the desired properties [4,I].

Poincaré group in 2-dimensions

The Poincaré group in two dimensions, denoted $P_+^\uparrow(1,1)$, is the semi-direct product of the group of two dimensional space-time translations T^2 and the group of pure Lorentz transformations $L_+^\uparrow(1,1)$. The general element g consists of a space-time translation a and a Lorentz boost Λ,

$$g = (a, \Lambda)$$

with

$$\left.\begin{aligned} a &= (a_0, \mathbf{a}) \in \mathbb{R}^2 \\ \Lambda = \Lambda_p &= \begin{pmatrix} p_0/m & \mathbf{p}/m \\ \mathbf{p}/m & p_0/m \end{pmatrix}, \quad m > 0, p = (p_0, \mathbf{p}) \in \mathcal{V}_m^+ \end{aligned}\right\} \tag{II.2.16}$$

where

$$\mathcal{V}_m^+ = \{p = (p_0, \mathbf{p}) \in \mathbb{R}^2 | p_0 > 0, \quad p_0^2 - \mathbf{p}^2 = m^2\} \tag{II.2.17}$$

is the forward mass hyperbola. the group multiplication law is:

$$\begin{aligned} gg' &= ((a_0, \mathbf{a}), \Lambda_p)((a'_0, \mathbf{a}'), \Lambda_{p'}) \\ &= ((a_0, \mathbf{a}) + \Lambda_p.(a'_0, \mathbf{a}'), \Lambda\Lambda') \end{aligned} \tag{II.2.18}$$

This group is s.l.c. unimodular, with invariant Haar measure

$$da_0 d\mathbf{a} d\mathbf{p}/p_0. \tag{II.2.19}$$

The elements Λ_p of the Lorentz group act transitively on $\mathcal{V}_m^+$ according to the action:

$$k \to k' = \Lambda_p k, \quad k = (k_0, \mathbf{k}) \in \mathcal{V}_m^+ \tag{II.2.20}$$

and the corresponding invariant measure on $\mathcal{V}_m^+$ is $d\mathbf{k}/k_0$ which can be seen as follows:

From (II.2.12), we can rewrite $k_0^2 - \mathbf{k}^2 = m^2$ using hyperbolic functions, with $k_0 = m\cosh\theta$, $\mathbf{k} = m\sinh\theta$. the invariant measure can be seen to be

$$d\theta = \frac{d\mathbf{k}}{m\sqrt{1+\sinh^2\theta}} = \frac{d\mathbf{k}}{k_0}. \tag{II.1.21}$$

§ The representations:

Consider the well known unitary irreducible Wigner representation U_w acting on φ in the Hilbert space

$$\mathfrak{H}_w = L^2(\mathcal{V}_m^+, d\mathbf{k}/k_0) = \{\varphi(k), k \in \mathcal{V}_m^+ | \int_{\mathcal{V}_m^+} |\varphi(k)|^2 \frac{d\mathbf{k}}{k_0} < \infty\}, \tag{II.1.22}$$

by:

$$(U_w(a,\Lambda)\varphi)(k) = e^{ik.a}\varphi(\Lambda_p^{-1}k), \quad \varphi \in \mathfrak{H}_w \tag{II.2.23}$$

where

$$k.a = k_0 a_0 - \mathbf{k}.\mathbf{a}.$$

First, we can see that this representation is not square integrable in the usual sense (definition (II.1.1)). Indeed, from (II.1.1) we have for $\eta, \varphi \in \mathfrak{H}_w$,

$$\begin{aligned}\int_{P_+^\uparrow(1,1)} (\varphi|U_w(g)\eta)_{\mathfrak{H}_w}(U(g)\eta|\varphi)_{\mathfrak{H}_w} da_0 d\mathbf{a}d\mathbf{p}/p_0 &= \iiint_{\mathbb{R}} \iint_{\mathcal{V}_m^+} \overline{\varphi(k)}e^{ik.a}\eta(\Lambda_p^{-1}k) \\ &\quad e^{-ik'.a}\overline{\eta(\Lambda_p^{-1}k')}\frac{d\mathbf{k}}{k_0}\frac{d\mathbf{k}'}{k_0'}dad\mathbf{a}\frac{d\mathbf{p}}{p_0} \\ &= \int_{\mathbb{R}} \frac{d\mathbf{p}}{p_0}\int \ldots \\ &= \infty\end{aligned}$$

Second, consider the time translation subgroup T,

$$T = \{g \in P_+^\uparrow(1,1) | g = ((a_0, 0), I_2), \quad a_0 \in \mathbb{R}\} \tag{II.2.24}$$

where I_2 is the 2×2 identity matrix.

It's clear that T is not a stationary subgroup under U_w of any vector $\eta \in \mathfrak{H}_w$. Thus, Perelomov's construction fails to give a system of coherent states.

Consider the left coset space $\Gamma_l = P_+^\uparrow(1,1)/T$ (resp. the right coset space Γ_r $T \setminus P_+^\uparrow(1,1)$). A direct computation shows that any element $g \in P_+^\uparrow(1,1)/T$ (resp. $g \in T \setminus P_+^\uparrow(1,1)$) can be factorized in the following form:

$$((q_0, \mathbf{q}), \Lambda_p) = ((0, \mathbf{q}'), \Lambda_p)((q_0', 0), I_2) \tag{II.2.25}$$

where

$$\left.\begin{aligned} \mathbf{q}' &= \mathbf{q} - \frac{q_0}{p_0}\mathbf{p} \\ q_0' &= mq_0/p_0 \end{aligned}\right\}$$

(resp. $((q_0, \mathbf{q}), \Lambda_p) = ((q_0', 0), I_2)((0, \mathbf{q}'), \Lambda_p)$ where $q_0' = q$, $\quad \mathbf{q}' = \mathbf{q}$).

Therefore the points in $\Gamma_{l,r}$ can be parameterized by $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^2$, and the map $\sigma(\mathbf{q}, \mathbf{p}) : \Gamma_{l,r} \to P_+^\uparrow(1,1)$ defined by

$$\sigma_0(\mathbf{q}, \mathbf{p}) = ((0, \mathbf{q}), \Lambda_p), \quad p = (p_0 = \sqrt{\mathbf{p}^2 + m^2}, \mathbf{p}) \tag{II.2.26}$$

is a Borel section (zero section in the terminology of [4]). According to the general theory [4], an arbitrary measurable section may be written as:

$$\begin{aligned} \sigma(\mathbf{q}, \mathbf{p}) &= \sigma_0(\mathbf{q}, \mathbf{p})((f(\mathbf{q}, \mathbf{p}), 0), I_2) \\ &= (\hat{q}, \Lambda_{\hat{p}}) \end{aligned} \tag{II.2.27}$$

with

$$\hat{p} = p, \quad \hat{q}_0 = \frac{p_0}{m} f(\mathbf{q}, \mathbf{p}), \quad \hat{\mathbf{q}} = \mathbf{q} + \frac{\mathbf{p}}{m} f(\mathbf{q}, \mathbf{p}) \tag{II.1.28}$$

where f is a measurable real-valued function. In order to ensure finiteness of the integral (II.2.9), one works with affine sections [4, II]

$$f(\mathbf{q}, \mathbf{p}) = \varphi(\mathbf{p}) + \mathbf{q}.\theta(\mathbf{p}) \tag{II.2.29}$$

where θ is a function of $\mathbf{p}$ alone and satisfies

$$-\frac{p_0 - \mathbf{p}}{m} < \theta(\mathbf{p}) < \frac{p_0 + \mathbf{p}}{m},$$

while φ may be actually set equal to zero. (see [4], in particular section 3.A.; Caution: φ or θ are not vectors in $\mathfrak{H}_w$)

Then

$$\hat{q}_0 = \frac{p_0 \hat{\mathbf{q}}.\theta(\mathbf{p})}{m + \mathbf{p}.\theta(\mathbf{p})}. \tag{II.2.30}$$

A number of special sections are discussed in [4]. For σ_0, $\theta(\mathbf{p}) = 0$ and for σ_s, $\theta(\mathbf{p}) = \frac{\mathbf{p}}{p_0+m}$; here we consider $\theta(\mathbf{p}) = \frac{\mathbf{p}}{m}$ which leads to σ_{DB} :

$$f(\mathbf{q},\mathbf{p}) = \frac{\mathbf{q}.\mathbf{p}}{m}, \qquad (\varphi(\mathbf{p}) = 0) \tag{II.2.31}$$

therefore, from (II.2.26), we have

$$\sigma_{DB}(\mathbf{q},\mathbf{p}) = ((\frac{p_0}{m^2}(\mathbf{q}.\mathbf{p}), \frac{p_0^2}{m^2}\mathbf{q}), \Lambda_p). \tag{II.2.32}$$

An arbitrary element $((q_0,\mathbf{q}),\Lambda_p) \in P_+^\uparrow(1,1)$ can be written according to Γ_l as (see appendix (B) for detailed computations):

$$\begin{aligned}((q_0,\mathbf{q}),\Lambda_p) = ((\frac{p_0}{m}(\mathbf{q}.\mathbf{p}) - \frac{q_0}{m^2}\mathbf{p}^2, \frac{p_0}{m^2}(p_0\mathbf{q} - q_0\mathbf{p})), \Lambda_p)\times \\ \times ((\frac{1}{m}(q_0 p_0 - \mathbf{q}.\mathbf{p}), 0), I_2)\end{aligned} \tag{II.2.33}$$

The left action of $P_+^\uparrow(1,1)$ on Γ_l is then the following:

$$(\mathbf{q},\mathbf{p}) \to (\mathbf{p}',\mathbf{q}') = (a,\Lambda_k)(\mathbf{q},\mathbf{p}) \tag{II.2.34}$$

for arbitrary $(a,\Lambda_k) \in P_+^\uparrow(1,1)$, where:

$$\mathbf{q}' = \frac{p_0(k_0\mathbf{a} - a_0\mathbf{k}) - \mathbf{p}.(a_0k_0 - \mathbf{a}.\mathbf{k}) + mp_0\mathbf{q}}{(k_0p_0 + \mathbf{k}.\mathbf{p})} \tag{II.2.35}$$

$$\mathbf{p}' = \frac{1}{m}(k_0\mathbf{p} + p_0\mathbf{k}) \tag{II.2.36}$$

A straightforward computation then shows that the measure

$$d\mu_l(\mathbf{q},\mathbf{p}) = d\mathbf{q}d\mathbf{p} \tag{II.2.37}$$

is invariant on Γ_l.

The proof of the square integrability of the representation U_w $mod(T,\sigma_{DB})$ and the construction of associated coherent states follow directly from the general theory in [4].

* * *

CHAPTER III

SQUARE INTEGRABILITY AND THE PRINCIPAL SERIES REPRESENTATIONS OF $SU(1,1)$

In this chapter, we will study the square integrability of some representations of $SU(1,1)$.

From chapter (I), we know that the isomorphism of $SU(1,1)$ and $SL(2,\mathbb{R})$ allows us to work with them without distinction.

From chapter (II), we conclude (theorem (II.1.7)) that representations which are square integrable in the usual sense belong to the discrete series representation of the group. In particular, the discrete series representation U_n^+ of the group $SU(1,1) \cong SL(2,\mathbb{R})$ (see chapter (I) section (5)) are square integrable.

To prove that precisely, let us first introduce the following result:

> Let U be a UIR of a semisimple Lie group[1] G acting on a Hilbert space $\mathfrak{H}$, then from the Cartan decomposition $G = K.A^+.K$, where K is compact subgroup of G, with a suitable invariant measure: $dg = \delta(a)dkdadk'$, where $\delta(a)$ has to be determined [18]. (For $SU(1,1)$; $dg_0 = 2\pi \sinh tdkdtdk'$, chapter I equationI.1.21)) From theorem (II.1.4), we can say in general that U is square integrable if and only if there exists a non zero vector $\eta \in \mathfrak{H}$ such that
>
> $$\boxed{\int_{A+} |(U(a)\eta|\eta)|^2 \delta(a)da < \infty.} \qquad \text{(III.0.1)}$$

We showed in chapter (I) section (5) that the constant vector $\mathbb{I}$ belongs to $\mathfrak{H}^n$, the representation space of the discrete seriesrepresentations U_n^+ defined by (I.5.1). Thus, for

$$(U_{g_0}^+\varphi)(z) = (\overline{\beta}\ z + \overline{\alpha})^{-2n}\varphi(\frac{\alpha\ z + \beta}{\overline{\beta}\ z + \overline{\alpha}})$$

[1] For example, $SU(1,1)$ and $SL(2,\mathbb{R})$, see (I.1.4)

where $g_0^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ and $n = 1, 3/2, 2, 5/2, \ldots$, we have from (III.0.1), where $a^{-1} = \begin{pmatrix} \cosh t/2 & -\sinh t/2 \\ -\sinh t/2 & \cosh t/2 \end{pmatrix}$, and $dz = r dr d\theta$

$$
\begin{aligned}
\int\limits_{A^+} |(U^+(a)\mathbb{1}|\mathbb{1})|^2 \delta(a) da &= \frac{2(2n-1)^2}{\pi} \int\limits_0^\infty \sinh(t) dt \\
&\qquad \left| \int\limits_{|z|<1} (-\sinh(t/2) z + \cosh(t/2))^{-2n} (1-|z|^2)^{2n-2} dz \right|^2 \\
&= \frac{2(2n-1)^2}{\pi} \int\limits_0^\infty \sinh(t) (\cosh(t/2))^{-4n} dt \\
&\qquad \left| \int\limits_{|z|<1} \frac{(1-|z|^2)^{2n-2}}{(1-\tanh(t/2)\, z)^{2n}} dz \right|^2 \\
&= \frac{2(2n-1)^2}{\pi} \int\limits_0^\infty \sinh(t) (\cosh(t/2))^{-4n} dt \\
&\qquad \left| \sum_{k=0}^\infty \binom{2n}{k} \tanh^k t/2 \int\limits_{z<1} z^k (1-|z|^2)^{2n-2} dz \right|^2 \\
&= \frac{2(2n-1)^2}{\pi} \int\limits_0^\infty \sinh(t) (\cosh(t/2))^{-4n} dt \\
&\qquad \left| 2\pi \int\limits_0^1 (1-r^2)^{2n-2} r dr \right|^2 \\
&= 2\pi(2n-1)^2 \int\limits_0^\infty \sinh(t) (\cosh(t/2))^{-4n} dt \\
&= 4\pi(2n-1)^2 \int\limits_0^\infty \sinh(t/2) (\cosh(t/2))^{-4n+1} dt \\
&= 8\pi \int\limits_1^\infty u^{-4n+1} du = \frac{4\pi}{2n-1}
\end{aligned}
$$

which is finite. The formal degree (see theorem (II.1.3)) of U^+ is $\frac{2n-1}{4\pi}$.

To show how Perelomov type of coherent states are derived from the square integrability of the representation U^+, let us refer to $\mathbb{I}$ as $|\eta_0> = |0>$ (the lowest weight function in Perelomov's terminology); acting on it by $U^+_{g_0}$, $g_0 \in G_0 = SU(1,1)$, we get the system of states:

$$|\eta_{g_0} >= T(g_0)|0 >= (\bar{\beta}\ z + \bar{\alpha})^{-2n}. \qquad \text{(III.0.3)}$$

From Mackey decomposition, any element $g_0 \in G_0$ can be decomposed uniquely as

$$g_0 = s(\xi).k$$

where

$$s(\xi) = \begin{pmatrix} (1-|\xi|^2)^{-1/2} & \xi(1-|\xi|^2)^{-1/2} \\ \bar{\xi}(1-|\xi|^2)^{-1/2} & (1-|\xi|^2)^{-1/2} \end{pmatrix} \qquad (\xi = \frac{\beta}{\bar{\alpha}})$$

and

$$k = \begin{pmatrix} \frac{\alpha}{|\alpha|} & 0 \\ 0 & \frac{\bar{\alpha}}{|\alpha|} \end{pmatrix} \in K. \qquad \text{(see (I.1.18))}$$

Then equation (III.0.3) can be written as:

$$|\eta_{g_0} >= |\xi >= (1 - \xi\ z)^{-2n}(1 - |\xi|^2)^n. \qquad \text{(III.0.4)}$$

The set $\{|\xi >\}$ is just the system of "coherent states" with all expected properties [25].

On the other hand, we may show that for the principal series representation (I.3.10), and consequently the complementary series representation (I.4.3), the integrals:

$$\int_{A^+} \overline{(U_a^{0,s}\eta|\varphi)_{L^2(U,d\xi)}}(U_a^{0,s}\eta|\varphi)_{L^2(U,d\xi)} \sinh(t)dt \qquad \text{(III.0.5)}$$

for $\eta, \varphi \in L^2(U, d\xi)$ with the usual inner product (I.3.2), and

$$\int_{A^+} \overline{(U_a^{0,\sigma}\eta|\varphi)_{\mathfrak{H}_\sigma}}(U_a^{0,\sigma}\eta|\varphi)_{\mathfrak{H}_\sigma} \sinh(t)dt \qquad \text{(III.0.6)}$$

where $\eta, \varphi \in \mathfrak{H}_\sigma$ with inner product (I.4.1) are **divergent**.

The success of the Ali et al technique prompts us to inquire into the possibility of applying this technique to the principal series representation of the group $SU(1,1) \cong SL(2,\mathbb{R})$. We will attempt to investigate this in the rest of the present chapter. Our approach will be to study the generalization of [4] for the group $SU(1,1) \cong SL(2,\mathbb{R})$ mod certain subgroups A and N (see chapter (I), section 1).

III.1. The homogeneous space G/A and $A \setminus G$:

Consider the closed subgroup

$$A = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad a \neq 0 \in \mathbb{R}\}$$

of $G = SL(2,\mathbb{R})$(see equation (I.1.4)). It's clear that A is not a stationary subgroup for any vector in the Hilbert space $L^2(U, d\xi)$ (see section (I.3)).

By using Mackey's decomposition (see preliminaries), any element of G that satisfies the condition $a \neq 0$ can be uniquely written in the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ba \\ ca^{-1} & da \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \qquad \text{(III.1.1)}$$

with respect to the left coset space $X_l = G/A$ or

$$g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & ba^{-1} \\ ca & ad \end{pmatrix} \qquad \text{(III.1.3)}$$

with respect to the right coset space $X_r = A \setminus G$.

Therefore, the Borel set has the form

$$S = \{s(\underline{x}) = \begin{pmatrix} 1 & x_1 \\ x_2 & 1 + x_1 x_2 \end{pmatrix}, \quad x_1 = ba, x_2 = ca^{-1} \in \mathbf{R}\} \qquad \text{(III.1.2)}$$

and the map $s(\underline{x}) : X_{l,r} \to G = SL(2,\mathbb{R})$ defines a Borel section [23].

Note that, the Borel set corresponding to the remaining elements in G, $c = -b^{-1}$, will be of measure zero in X_l and X_r therefore we will disregard it in our consideration.

It's clear that the homogeneous space $X_{l,r} \cong \mathbb{R}^2$ and the points in both X_l and X_r can be parameterized by $(x_1, x_2) \in \mathbb{R}^2$.

§ The measure:

Consider the left action of G on X_l, i.e.

$$(x_1, x_2) \longrightarrow (x_1', x_2') = g.(x_1, x_2) \tag{III.1.4}$$

then we have:

$$\begin{aligned} g.s(\underline{x}) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ x_2 & 1 + x_1 x_2 \end{pmatrix} \\ &= \begin{pmatrix} a + bx_2 & (a + bx_2)x_1 + b \\ c + dx_2 & (c + dx_2)x_1 + d \end{pmatrix} \\ &= \begin{pmatrix} 1 & ((a + bx_2)x_1 + b)(a + bx_2) \\ \frac{c+dx_2}{a+bx_2} & ((c + dx_2)x_1 + d)(a + bx_2) \end{pmatrix} \begin{pmatrix} a + bx_2 & 0 \\ 0 & (a + bx_2)^{-1} \end{pmatrix} \\ &= s(g.\underline{x}).h(g, \underline{x}) \end{aligned}$$

where

$$h(g, \underline{x}) = \begin{pmatrix} a + bx_2 & 0 \\ 0 & (a + bx_2)^{-1} \end{pmatrix} \tag{III.1.5}$$

satisfying the multiplier equation:

$$h(gg', \underline{x}) = h(g, g'.\underline{x})h(g', \underline{x})$$

moreover

$$h(g, \underline{x}) = s(g.\underline{x})^{-1} g.s(\underline{x})$$

Under this action

$$\left.\begin{aligned} x_1' &= (a + bx_2)^2 x_1 + b(a + bx_2) \\ x_2' &= \frac{c + dx_2}{a + bx_2} \end{aligned}\right\} \tag{III.1.6}$$

It's not diffecult to see that

$$1 + x_1' x_2' = ((c + dx_2)x_1 + d)(a + bx_2)$$

Similarly, on X_r, we have for $g \in G$,

$$(x_1, x_2) \longrightarrow (x_1', x_2') = (x_1, x_2).g \qquad \text{(III.1.7)}$$

where now

$$\left. \begin{aligned} x_1' &= \frac{(b + dx_1)}{(a + cx_1)} \\ x_2' &= (a + cx_1)^2 x_2 + c(a + cx_1) \end{aligned} \right\} \qquad \text{(III.1.8)}$$

A straightforward computation shows that the measure

$$d\mu_{(x_1, x_2)} = dx_1 dx_2 \qquad \text{(III.1.9)}$$

is invariant on both X_l and X_r.

Indeed, from the definition of the quasi-invariant measure [see Preliminaries] and since both G and N are unimodular, both $X_l \cong \mathbb{R}^2$ and $X_r \cong \mathbb{R}^2$ admit the same invariant measure $dx_1 dx_2$.

§ The representations:

Consider the unitary irreducible representation of the principal series for $SL(2, \mathbb{R})$ (see equation (I.3.15))

$$(U_g^{j,s}\eta)(k) = |ck + d|^{-2s}[sign(ck + d)]^{2j}\eta(\frac{ak + b}{ck + d})$$

where,

$$g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad j = 0, 1/2, \text{and} \quad s = 1/2 + iv, v \in \mathbb{R}$$

with Hilbert space $L^2(\mathbb{R}, dx)$.

For the section $s(x_1, x_2)$, let us write,

$$\begin{aligned} (U^{j,s}_{s(x_1,x_2)}\eta)(k) = |x_2 k + (1 + x_1 x_2)|^{-2s}[sign(x_2 k + (1 + x_1 x_2))]^{2j} \\ \eta\left(\frac{k + x_1}{x_2 k + (1 + x_1 x_2)}\right) \end{aligned} \qquad \text{(III.1.10)}$$

Put $j = 0$, and define

$$\eta_{(x_1, x_2)}(k) = U^{0,s}_{s(x_1,x_2)}\eta(k) \qquad \text{(III.1.11)}$$

Now we attempt to find at least one non-zero vector, *if it is exists*, in $L^2(\mathbb{R}, dk)$ such that the function $f_{\varphi,\eta} : X_l \to \mathbb{C}$ defined by

$$f_{\varphi,\eta} = < \varphi|\eta_{x_1,x_2}) > \tag{III.1.12}$$

where,

$$< \varphi|\eta_{(x_1,x_2)} > = \int_{\mathbb{R}} \varphi(k)|x_2k + (1 + x_1x_2)|^{-2\bar{s}}\overline{\eta(\frac{k + x_1}{x_2k + (1 + x_1x_2)})}dx \tag{III.1.13}$$

is square integrable.

In other words, we search for a vector $\eta \in L^2(\mathbb{R}, dk)$ such that:

$$A_{s(x_1,x_2)} = \iint\limits_{\mathbb{R}} |\eta_{(x_1,x_2)} >< \eta_{(x_1,x_2)}|dx_1dx_2. \tag{III 1.14}$$

is a positive invertible operator in the weak sense.

Therefore, if $\varphi, \psi \in L^2(\mathbb{R}, dk)$, then

$$< \varphi|A\psi > = \iint\limits_{\mathbb{R}} < \varphi|\eta_{(x_1,x_2)} >< \eta_{(x_1,x_2)}|\psi > dx_1dx_2 \tag{III.1.15}$$

and we have the following integration,

$$< \varphi|A\psi > = \iiiint\limits_{\mathbb{R}} |x_2k + (1 + x_1x_2)|^{-2s}\eta(\frac{k + x_1}{x_2k + (1 + x_1x_2)})\varphi(k)$$
$$|x_2k' + (1 + x_1x_2)|^{-2\bar{s}}\overline{\eta(\frac{k' + x_1}{x_2k' + (1 + x_1x_2)})}\psi(k')dkdk'dx_1dx_2 \tag{III.1.16}$$

⋆ ⋆ ⋆

III.2. The homogeneous space G/N and $G\setminus N$:

Consider the closed subgroup

$$N = \{\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}, \xi \in \mathbb{R}\}$$

Again using Mackey's decomposition, every element $g \in SL(2,\mathbf{R})$ satisfying the condition $a \neq 0$ can be uniquely written in the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & ba^{-1} \\ 0 & 1 \end{pmatrix} \tag{III.2.1}$$

with respect to the left coset space G/N.

So the Borel set has the form

$$s(\underline{x}) = \{\begin{pmatrix} x_1 & 0 \\ x_2 & x_1^{-1} \end{pmatrix}, \quad x_1 = a \neq 0 \in \mathbf{R}, \quad x_2 = c \in \mathbb{R}\} \tag{III.2.2}$$

and the map $s(\underline{x}) : G/A \to G$ defines a Borel section.

§ The invariant measure:

Consider the action of the group on $s(\underline{x})$, i.e. if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R})$, then

$$g.s(\underline{x}) = \begin{pmatrix} ax_1 + bx_2 & bx_1^{-1} \\ cx_1 + dx_2 & dx_1^{-1} \end{pmatrix}$$

implies

$$g.s(\underline{x}) = s(g.\underline{x}).h$$

where

$$s(g.\underline{x}) = \begin{pmatrix} ax_1 + bx_2 & 0 \\ cx_1 + dx_2 & (ax_1 + bx_2)^{-1} \end{pmatrix}$$

and

$$h = h(g,\underline{x}) = \begin{pmatrix} 1 & b(ax_1 + bx_2)^{-1}/x_1 \\ 0 & 1 \end{pmatrix}$$

where again $h = h(g,\underline{x})$ satisfies the multiplier equation

$$h(gg',\underline{x}) = h(g,g'.\underline{x})h(g',\underline{x}).$$

Under the action of g, we get

$$(x_1, x_2) \longrightarrow (x_1', x_2') = (ax_1 + bx_2, cx_1 + dx_2) \tag{III.2.3}$$

and a straightforward computation shows that the measure

$$d\mu_{(x_1,x_2)} = dx_1 dx_2 \tag{III.2.4}$$

is invariant on the left coset space G/N.

If we use the principal series representation of $SL(2,\mathbb{R})$ for the section (III.2.2), we get:

$$(U^{j,s}_{s(\underline{x})}\varphi)(x) = |x_2 x + x_1^{-1}|^{-2s}[sign(x_2 x + x_1^{-1})]^{2j}\varphi(\frac{x_1 x}{x_2 x + x_1^{-1}}) \tag{III.2.5}$$

Put $j = 0$ and define

$$\eta_{(x_1,x_2)} = V^{0,s}_{s(\underline{x})}\eta(x) \tag{III.2.6}$$

then

$$< \varphi|\eta_{(x_1,x_2)} >= \int_{\mathbb{R}} \varphi(x)\overline{\eta_{(x_1,x_2)}(x)}dx \tag{III.2.7}$$

and

$$< \eta_{(x_1,x_2)}|\psi >= \int_{\mathbb{R}} \eta_{(x_1,x_2)}(x)\overline{\psi(x)}dx \tag{III.2.8}$$

Again, as in section (III.1), we get

$$< \varphi|A\psi > = \iint\limits_{\mathbb{R}} < \varphi|\eta_{(x_1,x_2)} >< \eta_{(x_1,x_2)}|\psi > dx_1 dx_2 \tag{III.2.9}$$

Using (III.2.6), then we have the following integration:

$$\begin{aligned} < \varphi|A\psi > &= \iiiint\limits_{\mathbb{R}} \varphi(x)|x_2 x + x_1^{-1}|^{-2\bar{s}}\overline{\eta(\frac{x_1 x}{x_2 x + x_1^{-1}})} \\ &\qquad |x_2 x' + x_1^{-1}|^{-2s}\eta(\frac{x_1 x'}{x_2 x + x_1^{-1}})\overline{\psi(x')}dx dx' dx_1 dx_2 \\ &= \iiiint\limits_{\mathbb{R}} |x_1|^{2\bar{s}}|x_1|^{2s}|x_1 x_2 x + 1|^{-2\bar{s}}|x_1 x_2 x + 1|^{-2s}\overline{\eta(\frac{x_1^2 x}{x_1 x_2 x + 1})} \\ &\qquad \eta(\frac{x_1^2 x'}{x_1 x_2 x' + 1})\varphi(x)\overline{\psi(x')}dx dx' dx_1 dx_2 \end{aligned} \tag{III.2.10}$$

Unfortunately both expression (III.1.16) and (III.2.10) are difficult to compute explicitly. Although we do not have good estimates at present, we guess that for certain sections or under some restrictions there is a possibility to prove the convergence of both integrations.

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* * *

APPENDIX (A)

In this appendix, we prove the formulas (I.1.22), (II.1.28,29):

To prove (I.1.22): First, note that the closed subgroups K, A and N are unimodular (that is, the left and right Haar measure conicide)); they are abelain groups. Moreover, the subgroup K is isomorphic to the torus $T = \mathbb{R}/4\pi$ [28] and we have proved equation (I.1.21).

Secondly, the group P is a semi-direct product of A and N, N is a normal subgroup of P, with the product law:

$$(a,n)(a',n') = (a.a', a(r').n), \quad a \in A, \quad n \in N$$

where $a(n)$ is the action of A on N. Indeed,

$$\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^t\xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

Then, from lemma (2) equation (I.1.18), we have

$$d(ana^{-1}) = a^2 dn, \quad (a = e^{t/2}, n = \xi),$$

and the modular function of P is e^t (see equation (I.1.19)) which satisfies (0.5).

Combine this with the measure of K (I.1.21) in (I.1.17) to get the desired formula:

$$\boxed{dg = \frac{1}{4\pi} e^t d\theta dt d\xi.}$$

To prove the invariance of this measure, that is $d(g_0' g_0) = dg_0$ for $g_0, g_0' \in G_0$ see [28, chapter 5 proposition (5.1)]

⋆ ⋆ ⋆

To prove (II.1.28), consider the left action $g \longrightarrow gg' \quad (g' \in G')$, i.e.

$$(a,b)(a',b') = (a'',b'')$$

where

$$a'' = aa'$$

$$b'' = ab' + b$$

A straightforward computation shows that the left invariant measure on G' is

$$dadb/a^2 .$$

On the other hand, if we consider the right action $g \to g'g \quad (g' \subset G)$ then

$$a'' = a'a$$

$$b'' = a'b + b'$$

and the right invariant measure is

$$dadb/|a| .$$

and we have proved equation (II.1.29).

⋆ ⋆ ⋆

APPENDIX (B)

In this appendix, we give the detailed computations of equations (II.2.33-36):

To prove (II.2.33) using Mackey decomposition, we can factorize any arbitrary element $((q_0, \mathbf{q}), \Lambda_p)$ with respect to $L_+^\uparrow(1,1)/T$ as:

$$((q_0,\mathbf{q}),\Lambda_p) = ((\frac{p_0}{m^2}(\mathbf{q}'.\mathbf{p}), \frac{p_0^2}{m^2}\mathbf{q}'), \Lambda_p).((f(\mathbf{q},\mathbf{p}),0), \mathbb{I}_2)$$

where, using (II.2.18) and comparing the coefficients,

$$\left.\begin{aligned} \mathbf{q}' &= \frac{p_0\mathbf{q} - q_0\mathbf{p}}{p_0} \\ f(\mathbf{q},\mathbf{p}) &= \frac{1}{m}(q_0 p_0 - \mathbf{q}.\mathbf{p}) \end{aligned}\right\}$$

From the action (II.2.34), we have:

$$((a_0,\mathbf{a}),\Lambda_k)((\frac{p_0}{m^2}(\mathbf{q}.\mathbf{p}), \frac{p_0^2}{m^2}\mathbf{q}), \Lambda_p) = ((a_0', \mathbf{a}'), \Lambda_{p_k})$$

where

$$a_0' = a_0 + \frac{k_0 p_0}{m^3}(\mathbf{q}.\mathbf{p}) + \frac{\mathbf{k}.\mathbf{q}}{m^3}p_0^2$$

$$\mathbf{a}' = \mathbf{a} + \frac{p_0}{m^3}(\mathbf{q}.\mathbf{p})\mathbf{k} + \frac{k_0 p_0^2}{m^3}\mathbf{q}$$

$$\Lambda_{p_k} = m^{-1}\begin{pmatrix} m^{-1}(k_0p_0 + \mathbf{k}.\mathbf{p}) & m^{-1}(k_0\mathbf{p} + p_0\mathbf{k}) \\ m^{-1}(k_0\mathbf{p} + p_0\mathbf{k}) & m^{-1}(k_0p_0 + \mathbf{k}.\mathbf{p}) \end{pmatrix}$$

We can rewrite this equation using (I.3.18) as:

$$((a_0',\mathbf{a}'),\Lambda_{p_k}) = (m^{-3}(k_0p_0 + \mathbf{k}.\mathbf{p})(\mathbf{q}'.m^{-1}(p_0\mathbf{k} + k_0\mathbf{p})), m^{-4}(k_0p_0 + \mathbf{k}.\mathbf{p})^2\mathbf{q}'), \Lambda_{p_k})$$
$$((f(\mathbf{q}', m^{-1}(k_0\mathbf{p} + p_0\mathbf{k})), 0), \mathbf{I}_2)$$

where in this case,

$$\mathbf{q}' = \frac{p_0(k_0\mathbf{a} - a_0\mathbf{k}) - \mathbf{p}.(a_0k_0 - \mathbf{a}.\mathbf{k}) + mp_0\mathbf{q}}{(k_0p_0 + \mathbf{k}.\mathbf{p})}$$

$$f(\mathbf{q}', m^{-1}(k_0\mathbf{p} + p_0\mathbf{k})) = m^2(k_op_0 + \mathbf{k}.\mathbf{p})^{-1}(a_0 + m^{-3}k_0p_0(\mathbf{q}.\mathbf{p}) + m^{-3}p_0^2(\mathbf{k}.\mathbf{q})$$
$$- m^{-4}\mathbf{q}'(k_0p_0 + \mathbf{k}.\mathbf{p})(p_0\mathbf{k} + k_0\mathbf{p}))$$

which proves (II.2.35).

List Of Important Symbols:

(The number at right indicates the page of the first appearance)

$\mathfrak{a}$:	Lie algebra of the subgroup A	(10)
adX :	The adjoint mapping of X in the Lie algebra	(17)
$B(X,Y)$:	Killing form	(17)
C :	Positive, invertible operator on Hilbert space	(45)
$\mathbb{C}$:	The field of complex numbers	(10)
$\mathfrak{D}$:	The set of the admissible vectors	(44)
$exp(X)$:	Exponential of X, an element of the group	(8)
G :	The group $SL(2,\mathbb{R})$	(8)
$\mathbb{G}$:	Coherent state system	(49)
G' :	Affine group "ax+b"	(50)
G_0 :	The group $SU(1,1)$	(12)
$GL(2,\mathbb{C})$:	General linear group, $g = \{g_{ij}\}, g_{ij} \in \mathbb{C}, \lvert det(g)\rvert \neq 0$	(13)
$\mathfrak{H}_\sigma$:	Representation Space of $U_{g_0}^{0,s}$	(33)
$\mathfrak{H}'$:	Representation space of $V_g^{0,s}$	(33)
$\mathfrak{H}^n$:	Representation space of $U_{g_0}^{\pm}$	(37)
$\mathfrak{H}_n$:	Representation space of $V_g^{\pm}$	(38)
$L^2(U,d\xi)$:	Representation space of $U_{g_0}^{j,s}$	(19)
$\mathfrak{L}(\mathfrak{H})$:	The set of Linear bounded operator of $\mathfrak{H}$	(39)
M :	centralizer of A in K	(10)
$\mathbb{R}$:	The set of real numbers	(8)
P :	The minimal parabolic subgroup of $SL(2,\mathbb{R})$	(10)
$P_+^\uparrow(1,1)$:	Poincaré group in 1 space and 1 time dimensions	(57)

$s(.)$:	Borel section (representative)	(24)
$SO(2,1)$:	Pseudo-orthogonal group on $2+1$ dimension	(15)
$\mathfrak{sl}(2,\mathbb{R})$:	Lie algebra of $SL(2,\mathbb{R})$	(8)
$\mathfrak{su}(1,1)$:	Lie algebra of $SU(1,1)$	(16)
$\mathcal{U}(\mathfrak{H})$:	The group of unitary operator on $\mathfrak{H}$	(39)
$U_g^{j,s}, V_{g_0}^{j,s}$:	Principal series representations of G, G_0	(19)
$U_g^{0,s}, V_{g_0}^{j,s}$:	Complementary series representations of G , G_0	(33)
$U_n^{\pm}, V_n^{\pm}$:	Discrete series representations of G, G_0	(37)