NOTICE

The quality of this microform is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us an inferior photocopy.

AVIS

La qualité de cette microforme dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de qualité inférieure.

Bivariate Lifetime Distributions

Hervé Benitah

A Project in The Department of Mathematics and Statistics

Presented in Partial Fulfillment of the requirements for the Degree of Master's of Science in Mathematics and Statistics at

Concordia University
Montreal, Quebec, Canada

September 1994

Typeset by $\mathcal{AMS}$-TEX

National Library of Canada
Bibliothèque nationale du Canada

Acquisitions and Bibliographic Services Branch
Direction des acquisitions et des services bibliographiques

395 Wellington Street
Ottawa, Ontario
K1A 0N4

395, rue Wellington
Ottawa (Ontario)
K1A 0N4

Your file Votre référence

Our file Notre référence

ISBN 0-612-01379-0

Table of Contents

Abstract

Bivariate Lifetime distributions

Hervé Benitah

A thorough investigation of a bivariate Gompertz hazard function for failure time (x, y) and its joint survival distribution is going to be considered. Interest is centered on whether a failure of the first type (i.e. person, machine) at $X = x$ increases the hazard in $t \geq x$ for failure of the second type. This result is then coupled with the Marshall-Olkin bivariate exponential law to create a general dependent bivariate Makeham law which allows for simultaneous death of the joint lives due to random accidents. Then a brief presentation of the general case, where no underlying distribution is assumed, will be shown.

Introduction

In practice, Actuaries assume independence between the two individual lives, since the dependence of these time-until-death random variable is very difficult to quantify. But in many situations involving joint life, there is a desire to obtain a joint life mortality table which exhibits a possible dependent bivariate structure.

For example the lifetimes for husband and wife may be correlated; flight of a twin-engine plane; if one engine goes dead, does it affect the life of the second engine?

The exponential distribution is considered as a useful statistical model since it is absolutely continuous and has constant failure rate everywhere in the univariate case. With this in mind, we will first develop a bivariate hazard function with the Gompertz marginal distribution, i.e. $\mu(x) = c_1 \exp(c_2 x)$, for $x > 0$, and then show that the most general bivariate Gompertz distribution possible has a hazard or force of mortality function of the form:

$$\mu(x,y) = a \exp(c_1 x + c_2 y + c_3 xy), \quad x, y > 0.$$

By piecing together a bivariate Gompertz law with bivariate exponential laws we are able to obtain a general dependent bivariate Makeham law. This allows for simultaneous failure of the joint lives due to a random accident where we will derive the general formula for the hazard function of an absolutely continuous bivariate Makeham distribution. We will then finalize this paper by deriving the joint bivariate survival function where no distribution is assumed.

3. The General Bivariate Gompertz Law

The problem of determining the most general bivariate Gompertz law (i.e. a bivariate law with Gompertz marginals) is of some interest since after about age 35, individual mortality is well approximated by Gompertz laws. Thus we are lead to the problem of finding a bivariate distribution $F(x,y)$ such that the univariate marginal distributions $F(x)$ and $F(y)$ are both Gompertz laws.

Since the Gompertz law has a hazard force of mortality of the form:

$$\mu(x) = c_1 \exp(c_2 x) \tag{1}$$

we wish to find $\mu(x,y)$ such that the marginal hazard functions are of the Gompertz type.

We are aided by the following result:

Lemma 1.

If l is a measurable function satisfying

$$l(y-x) = \sum_{i=1}^{n} a_i(x) b_i(y) \quad \textit{whenever} \quad x,y > 0 \tag{2}$$

Then l is necessarily an exponential polynomial with at most n terms, i.e.

$$l(x) = \sum_{i=1}^{n} a_i x^{j_i} \ \exp(c_i x) \tag{3}$$

for some choice $a_i, j_i, i = 1, 2, \ldots, n$ such that the j_i's are non-negative integers satisfying $\sum(j_i + 1) \leq n$ where the summation is over these $j_i > 0$ and the c_i's are complex constants.

Proof:

For a proof of this result consult Kemperman (1971) or Aczel (1966).

Let us now consider a hazard function which corresponds to a bivariate Gompertz law (i.e. Gompertz marginals), then the following equation must hold:

$$\mu(x,y) = A(x) \exp\{B(x)y\} = C(y) \exp\{D(y)x\}, \quad x,y > 0, \tag{4}$$

we see from (4) that

$$\mu(u-v,0) = A(u-v)\exp\{B(u-v)\cdot 0\}$$
$$= C(0)\exp\{D(0)\cdot(u-v)\}$$

$$\Rightarrow A(u-v) = C(0)\exp\{D(0)\cdot(u-v)\}$$
$$= C(0)\exp\{u\cdot D(0) - vD(0)\}$$
$$= C(0)\exp\{D(0)\cdot u\}\cdot\exp\{-D(0)\cdot v\}$$

This is precisely the form (3), and thus, by the lemma, the function $A(x)$ is an exponential polynomial with one term

$$i.e.\quad A(x) = C(0)\exp\{D(0)x\}\quad \text{for}\quad x > 0$$

By the same way,

$$\mu(0,u-v) = A(0)\exp\{B(0)\cdot(u-v)\}$$
$$= C(u-v)\exp\{D(u-v)\cdot 0\}$$

$$\Rightarrow C(u-v) = A(0)\exp\{B(0)\cdot(u-v)\}$$
$$= A(0)\exp\{uB(0) - vB(0)\}$$
$$= A(0)\exp\{B(0)u\}\cdot\exp\{-B(0)v\}$$

And again this is precisely the form (3), and thus, by the lemma, the function $c(x)$ is an exponential polynomial with one term

$$i.e.\quad C(x) = A(0)\exp\{B(0)\cdot x\}\quad \text{for}\quad x > 0.$$

Substituting $A(x)$ and $c(x)$ into (4) we get for $x, y > 0$

$$\mu(x,y) = C(0)\exp\{D(0)x\}\cdot\exp\{B(x)y\}$$
$$= A(0)\exp\{B(0)y\}\cdot\exp\{D(y)x\}$$

$$\Rightarrow \mu(x,y) = C(0)\exp\{D(0)x + B(x)y\}$$
$$= A(0)\exp\{B(0)y + D(y)x\}$$

let $C(0) = \exp\{lnC(0)\}$ and $A(0) = \exp\{lnA(0)\}$

$$\begin{aligned}\mu(x,y) &= \exp\{lnA(0) + B(0)y + D(y)x\} \\ &= \exp\{lnC(0) + D(0)x + B(x)y\}\end{aligned}$$

Therefore

$$lnC(0) + D(0)x + B(x)y = lnA(0) + B(0)y + D(y)x$$

Assuming that the derivatives with respect to y on both sides exist

$$\Rightarrow B(x) = B(0) + D'(y)x$$

$\Rightarrow B(x)$ is a linear function of x i.e. $B(x) = \alpha + \beta x$. By the same reasoning, taking the derivative with respect to x on both sides, we get

$$\Rightarrow D(0) + B'(x)y = D(y)$$

$\Rightarrow D(y)$ is a linear function of y i.e. $D(y) = \alpha' + \beta' y$.

Knowing that $\mu(x,y) = C(0)\exp\{D(0)x + B(x)y\}$ and substituting the value of $D(0)$ and $B(x)$ we get

$$\begin{aligned}\mu(x,y) =& C(0)\exp\{\alpha' x + (\alpha + \beta x)y\} \\ =& C(0)\exp\{\alpha' x + \alpha y + \beta xy\}\end{aligned}$$

Finally

$$\mu(x,y) = a_1 \exp\{c_1 x + c_2 y + c_3 xy\} \tag{5}$$

It is the most general bivariate hazard function consistent with the univariate marginal Gompertz distribution.

The Gompertz bivariate survival function is then given by

$$S_G(t,s) = P(x > t, y > s) = \exp\{-\int_0^s \int_0^t \mu(x,y)dxdy\}, \quad s,t > 0 \tag{6}$$

Gompertz only assumed age deterioration. He therefore assumed that $P(x = y) = 0$ which implies that the possibility of a catastrophe (i.e. accident) resulting in simultaneous failure of both components is ruled out. But this is not realistic, since simultaneous deaths do occur.

By combining a bivariate Gompertz law with bivariate exponential laws, we are able to obtain a general dependent bivariate Makeham law allowing for simultaneous failure of the component lives. In this case, $Pr(x = y) \neq 0$.

2.1 Bivariate Gompertz law assuming independence for the individual components

In practice, whenever we express life table functions in the bivariate case, we use an assumption of independence for the individual lives. This simplifies the calculations. We begin with the assumption that mortality follows Gompertz's law.

i.e. $\mu(x) = c_1 \exp\,(c_2 x)$ and $\mu(y) = c_1{}' \exp\,(c_2{}' y)$.

Since both individual lives are independent

$$f(x,y) = f(x) \cdot f(y)$$

$$\begin{aligned} \Rightarrow S(x,y) =& Pr(X > x, Y > y) \\ =& Pr(X > x) \cdot Pr(Y > y) \\ =& S(x) \cdot S(y). \end{aligned} \tag{7}$$

$$\begin{aligned} S(x) &= \exp\,(-\int_0^x h(t)dt) \\ &= \exp\,(-\int_0^x c_1 \exp\,(c_2 t)dt) \\ &= \exp\,[-\frac{c_1}{c_2} \exp\,(c_2 t)\,|_0^x] \\ &= \exp\,[\frac{c_1}{c_2}(1 - \exp\,(c_2 x))] \end{aligned} \tag{8}$$

Therefore,

$$\begin{aligned} S(x,y) &= \exp\,[\frac{c_1}{c_2}(1 - \exp\,(c_2 x))] \cdot \exp\,[\frac{c_1'}{c_2'}(1 - \exp\,(c_2' y)] \\ &= \exp\,[\frac{c_1}{c_2} + \frac{c_1'}{c_2'} - \frac{c_1}{c_2} \exp\,(c_2 x) - \frac{c_1'}{c_2'} \exp\,(c_2' y)] \\ &= \exp\,[b - m \exp\,(c_2 x) - m' \exp\,(c_2' y)] \end{aligned}$$

where $c_2 \neq 0$, $c_2' \neq 0$, $x > 0$ and $y > 0$.

Since the Gompertz law has an absolutely continuous bivariate survival function $S(x,y)$ with density function $f(x,y)$, the bivariate failure rate at (x, y) is given by

$$\begin{aligned}
\mu(x,y) =& \frac{f(x,y)}{S(x,y)} \\
=& \frac{f(x,y)}{Pr(X > x, Y > y)} \\
=& \frac{f(x) \cdot f(y)}{Pr(X > x) \cdot Pr(Y > y)} \\
=& \frac{f(x) \cdot f(y)}{S(x) \cdot S(y)} \\
=& \frac{f(x)}{S(x)} \cdot \frac{f(y)}{S(y)} \\
=& \mu(x) \cdot \mu(y) \\
=& c_1 \exp\,(c_2 x) \cdot c_1' \exp\,(c_2' y) \\
=& c_1 c_1' \exp\,(c_2 x + c_2' y) \\
=& a_1 \exp\,(c_2 x + c_2' y) \qquad (10)
\end{aligned}$$

Comparing the failure rate in both cases, we notice that in the dependent case there is an extra term that is multiplied (i.e. $\exp\,(c_3 xy)$), where c_3 can represent the degree of dependence between x and y.

As mentioned previously, the construction of life tables for a bivariate lifetime distribution is very difficult. So we seek to substitute a single-life status (w) for a bivariate life status (x, y). To do this, we want the two failure rates to be the same.

$$\mu(x,y) = \mu(w).$$

As an example, we will assume that the two lifes are independent and each component follows a Gompertz law (i.e. $\mu(x) = Be^{cx}$).

$$\begin{aligned}
\mu(x,y) =& \mu(w) \\
\mu(x) \cdot \mu(y) =& \mu(w) \\
B \exp\,(cx) \cdot B \exp\,(cy) =& B \exp\,(cw) \\
B \exp\,(cx + cy) =& \exp\,(cw) \\
\exp\,(log B) \exp\,(cx + cy) =& \exp\,(cw) \\
\exp\,(B' + cx + cy) =& \exp\,(cw) \\
B' + cx + cy =& cw
\end{aligned}$$

$$\Rightarrow \quad w = x + y + B''$$

where, to be realistic, $B'' < 0$.

This defines the desired w. Therefore the need for a two-dimensional array has been replaced by the need for a one-dimensional array, but typically w and B'' will not be integer and therefore the determination of its values will require interpolation in the single array.

3. Bivariate Makeham law allowing simultaneous failure

We now consider the problem of determining a bivariate Makeham law. It is a model for which $Pr(X = Y) \neq 0$. The hazard function can be split into two parts, the "deterioration" component which according to Gompertz measures the inability to oppose destruction or bodily aging, and the "accident" component which is constant across ages and gives rise to the exponential factor of survival. It was with this addition of the "accident" component to the Gompertz hazard function that Makeham made his contribution. He postulated that the survival of an individual depended upon surviving the statistically independent causes of "deterioration" as measured by Gompertz, and "accidents" as measured by the exponential law with constant hazard.

We shall couple the bivariate Gompertz hazard function with the appropriate generalization of the exponential law given by Marshall and Olkin (1967) to develop the "shock" model.

Random shocks or accidents are assumed to be non-anticipatory and occur according to Poisson processes.

The intensity parameter of the process $N_i(t)$ of shocks which are fatal only to person i is d_i where $i = 1, 2$.

Due to the simultaneous exposure of the two persons, there is an intensity d_3 for random accidents (i.e. automobile crashes) which could be fatal to both individuals simultaneously: $N_3(t)$ is the third process. N_1 is a Poisson random variable with parameter d_1x. N_2 is a Poisson random variable with parameter d_2y. N_3 is a Poisson random variable with parameter d_3 max (x, y). We use max (x, y) in N_3 since in practice, two individuals are deemed to have died simultaneously if the second death occurs within 30 to 60 days of the first. Thus, a single accident can be fatal to both and death be deemed simultaneous even if one outlives the other by a few hours, days, or weeks.

By simply looking at the "accident" component we see that the bivariate survival function is

$$\begin{aligned} S(x,y) &= Pr(X > x, Y > y) \\ &= Pr(N_1 = 0, N_2 = 0, N_3 = 0) \end{aligned}$$

Since N_1, N_2, N_3 are independent processes

$$\begin{aligned} S(x,y) =& Pr(N_1 = 0) \cdot Pr(N_2 = 0) \cdot Pr(N_3 = 0) \\ &= \exp\ (-d_1 x) \cdot\ \exp\ (-d_2 y) \cdot\ \exp\ (-d_3 \max\ (x,y)) \\ S(x,y) &= \exp\ [-d_1 x - d_2 y - d_3 \max\ (x,y)] \end{aligned} \tag{11}$$

for $x, y > 0$ where $d_1, d_2 > 0$ and $d_3 \geq 0$.

Now adding the deteriorating component: Let W_i denote the time of death of person i due to "deterioration" alone. Then (w_1, w_2) is a bivariate Gompertz random variable with parameter c_1, c_2, c_3. It follows that (w_1, w_2) has a joint hazard function as follows:

$$\mu(w_1, w_2) = a_1\ \exp\ [c_1 w_1 + c_2 w_2 + c_3 w_1 w_2] \tag{12}$$

Since all the random variables $(w_1, w_2), N_1, N_2$ and N_3 are assumed to be independent, the general bivariate Makeham model, with the deterioration component, has the following survival function:

$$\begin{aligned} Sm(x,y) =& Pr(X > x, Y > y) \\ =& Pr[(w_1, w_2) > (x,y), N_1(x) = 0, N_2(y) = 0, N_3(\max\ (x,y)) = 0] \\ =& Pr[(w_1, w_2) > (x,y)] \cdot Pr[N_1(x) = 0] \\ & \cdot Pr[N_2(y) = 0] \cdot Pr[N_3(\max\ (x,y)) = 0] \\ Sm(x,y) =& S_G(x,y) \cdot\ \exp\ (-d_1 x) \cdot\ \exp\ (-d_2 y) \cdot\ \exp\ (-d_3\ \max\ (x,y)) \\ Sm(x,y) =& S_G(x,y) \cdot \exp\ [-d_1 x - d_2 y - d_3\ \max\ (x,y)] \end{aligned} \tag{13}$$

where $S_G(x,y)$ is the bivariate Gompertz survival function defined in (6). Equation (13) is the general Makeham bivariate distribution which has a Makeham marginal distribution and whose accident component has a lack of memory property (because of the exponential distribution), i.e. $S(x+t, y+t) = S(x,y) \cdot S(t,t)$. We will now look at the general case, where we do not assume a distribution and it is absolutely continuous everywhere.

4. General case: Bivariate model for ordered pairs

Let S represent the age at failure of the first member of an ordered pair of individuals, and let T represent the age at failure of the second member. Here $f(s,t)$ represents the joint age at failure distribution. It is convenient to define the following functions derived from $f(s,t)$.

(1) The bivariate survival function:

$$\begin{aligned} S(s,t) &= Pr(S > s, T > t) \\ &= \int_s^\infty \int_t^\infty f(u,v)dudv \end{aligned} \tag{14}$$

(2) The hazard function for T given that S survives to s is

$$\begin{aligned} h_T(t/S \geq s) &= \frac{\partial}{\partial t}\{-\log S(s,t)\} \\ &= \frac{-\frac{\partial}{\partial t} S(s,t)}{S(s,t)} \end{aligned} \tag{15}$$

(3) The hazard function for S given that T survives to t is:

$$\begin{aligned} h_s(s/T \geq t) &= \frac{\partial}{\partial s}\{-\log S(s,t)\} \\ &= \frac{-\frac{\partial}{\partial s} S(s,t)}{S(s,t)} \end{aligned} \tag{16}$$

(4) The bivariate failure rate.

$$h_{(S,T)}(s,t) = \frac{f(s,t)}{S(s,t)} = \frac{f(s,t)}{1 - Pr(S \leq s \text{ or } T \leq t)} \tag{17}$$

getting another expression of $h_{(S,T)}(s,t)$

$$\begin{aligned} h_{(S,T)}(s,t) &= \frac{f(s,t)}{1 - Pr(S \leq s) - Pr(T \leq t) + Pr(S \leq s, T \leq t)} \\ &= \frac{f(s,t)}{1 + F_{(S,T)}(s,t) - F_s(s) - F_T(t)} \\ &= \frac{f(s,t)}{1 + F_{(S,T)}(s,t) - F(s,\infty) - F(\infty,t)} \end{aligned} \tag{18}$$

Note: $F(s,\infty) = \int_0^s \int_0^\infty f(u,v)dudv$

Notice that in the case of independence we have that

$$\begin{aligned} h_{(S,T)}(s,t) &= \frac{f(s,t)}{S(s,t)} \\ &= \frac{f_S(s)f_T(t)}{S_S(s)S_T(t)} \\ &= \frac{f_S(s)}{S_S(s)} \cdot \frac{f_T(t)}{S_T(t)} \\ &= h_S(s) \cdot h_T(t) \end{aligned} \tag{19}$$

where $h_S(s)$ and $h_T(t)$ are the corresponding univariate failure rates.

Proposition 1:

$$h_S(s/T=t) = -\frac{\partial}{\partial s} \log \{-\frac{\partial}{\partial t} S(s,t)\} \tag{20}$$

Proof:

For any continuous bivariate survivor function $S(s,t)$, the marginal survival functions are:

$$S(s) = S(s,0) = \int_s^\infty \int_0^\infty f(u,v)dudv$$
$$S(t) = S(0,t) = \int_0^\infty \int_t^\infty f(u,v)dudv$$

The conditional survivor function of S given $T \geq t$ is

$$\begin{aligned} S(s/T \geq t) &= \frac{Pr(S \geq s, T \geq t)}{f(T \geq t)} \\ &= \frac{S(s,t)}{S(t)} \end{aligned} \tag{21}$$

and the conditional survivor function of S given $T = t$ is

$$S(s/T=t) = \frac{Pr(S > s, T = t)}{Pr(T = t)} \tag{22}$$

But

$$\begin{aligned} \frac{\partial}{\partial t} S(t) &= \frac{\partial}{\partial t} Pr(T > t) \\ &= \frac{\partial}{\partial t} \int_t^\infty f(u)du \\ &= -f(T = t) \end{aligned}$$
$$\Rightarrow \quad f(T=t) = -\frac{\partial}{\partial t} S(t).$$

By the same way

$$
\begin{aligned}
\frac{\partial}{\partial t}S(s,t) =& \frac{\partial}{\partial t}Pr(S > s, T > t) \\
=& \frac{\partial}{\partial t}\int_s^\infty \int_t^\infty f(u,v)dvdu \\
=& -\int_s^\infty f(u,t)du \\
=& -Pr(S > s, T = t) \\
\Rightarrow \quad Pr(S > s, T = t) =& -\frac{\partial}{\partial t}S(s,t).
\end{aligned}
$$

Therefore the conditional survival probability $S(s/T = t)$ is

$$
\begin{aligned}
S(s/T = t) =& \frac{Pr(S > s, T = t)}{Pr(T = t)} \\
=& \frac{-\frac{\partial}{\partial t}S(s,t)}{-\frac{\partial}{\partial t}S(t)}
\end{aligned}
$$

Thus we have that

$$
\begin{aligned}
h_S(s/T = t) =& -\frac{\partial}{\partial s}\log S(s/T = t) \\
=& -\frac{\partial}{\partial s}\log \left[\frac{-\frac{\partial}{\partial t}S(s,t)}{-\frac{\partial}{\partial t}S(t)}\right] \\
=& -\frac{\partial}{\partial s}[\log [-\frac{\partial}{\partial t}S(s,t)] + \frac{\partial}{\partial t}S(t)] \\
=& -\frac{\partial}{\partial s}\log [-\frac{\partial}{\partial t}S(s,t)] + 0 \\
=& -\frac{\partial}{\partial s}\log [-\frac{\partial}{\partial t}S(s,t)] \qquad \text{Q.E.D.}
\end{aligned}
$$

Clayton postulated in his 1978 paper a relation of the form

$$
h_S(s/T = t) = (1 + \phi)h_S(s/T \geq t) \tag{23}
$$

where the parameter ϕ measures the degree of association between s and t, independence being implied by $\phi = 0$. Here ϕ represents a positive association (i.e. $\phi > 0$) between the hazard function of the conditional distribution of S, given $T = t$ and S given $T \geq t$.

Theorem 1:

From Clayton's postulation (23) the bivariate survival function has the form:

$$S(s,t) = [(\frac{1}{S_S(s)})^{\phi} + (\frac{1}{S_T(t)})^{\phi} - 1]^{-\frac{1}{\phi}} \tag{24}$$

Proof:

Substituting (16) and (20) into Clayton's relation (23) we get:

$$-\frac{\partial}{\partial s} \log\,[-\frac{\partial}{\partial t}S(s,t)] = (1+\phi)[-\frac{\partial}{\partial s} \log\, S(s,t)]$$

Integrating over $(0,s)$ both sides:

$$\begin{aligned}
\int_0^s \frac{\partial}{\partial s} \log\,[-\frac{\partial}{\partial t}S(s,t)]ds &= (1+\phi)[\int_0^s \frac{\partial}{\partial s} \log\, S(s,t)ds] \\
\frac{\partial}{\partial s}\int_0^s \log\,[-\frac{\partial}{\partial t}S(s,t)]ds &= (1+\phi)[\frac{\partial}{\partial s}\int_0^s \log\, S(s,t)ds] \\
\Rightarrow \log[-\frac{\partial}{\partial t}S(s,t)] - \log\,[-\frac{\partial}{\partial t}\frac{S(0,t)}{S_T(t)}] &= (1+\phi)[\log\, S(s,t) - \log\,\frac{S(0,t)}{S_T(t)}] \\
\Rightarrow \log\,[\frac{\frac{\partial}{\partial t}S(s,t)}{\frac{\partial}{\partial t}S_T(t)}] &= \log\,[(\frac{S(s,t)}{S_T(t)})^{1+\phi}]
\end{aligned}$$

taking the exponential on both sides

$$\frac{\frac{\partial}{\partial t}S(s,t)}{\frac{\partial}{\partial t}S_T(t)} = [\frac{S(s,t)}{S_T(t)}]^{1+\phi}$$

$$\Rightarrow \frac{\frac{\partial}{\partial t}S(s,t)}{[S(s,t)]^{1+\phi}} = \frac{\frac{\partial}{\partial t}S_T(t)}{[S_T(t)]^{1+\phi}}$$

Now integrating both sides over $(0,t)$

$$\int_0^t \frac{\frac{\partial}{\partial t}S(s,t)}{[S(s,t)]^{1+\phi}}dt = \int_0^t \frac{\frac{\partial}{\partial t}S_T(t)}{[S_T(t)]^{1+\phi}}dt$$

$$u = S(s,t) \qquad\qquad u = S_T(t)$$

$$du = \frac{\partial}{\partial t}S(s,t)dt \qquad\qquad du = \frac{\partial}{\partial t}S_T(t)dt$$

$$\Rightarrow \int \frac{1}{u^{1+\phi}} du = \int \frac{1}{u^{1+\phi}} du$$

$$\Rightarrow [\frac{1}{S(s,t)}]^{\phi} |_0^t = [\frac{1}{S_T(t)}]^{\phi} |_0^t$$

$$\Rightarrow [\frac{1}{S(s,t)}]^{\phi} - [\frac{1}{S(s,0)}]^{\phi} = [\frac{1}{S_T(t)}]^{\phi} - [\frac{1}{S_T(0)}]^{\phi}$$

$$\Rightarrow [\frac{1}{S(s,t)}]^{\phi} = [\frac{1}{S_S(s)}]^{\phi} + [\frac{1}{S_T(t)}]^{\phi} - 1$$

So that the bivariate survival function takes the form of:

$$S(s,t) = [(\frac{1}{S_S(s)})^{\phi} + (\frac{1}{S_T(t)})^{\phi} - 1]^{-\frac{1}{\phi}}$$

Q.E.D.

Therefore, if $G(s) = Pr(S > s)$ and $H(t) = Pr(T > t)$ are continuous univariate survival functions with $G(0) = H(0) = 1$ and $\phi > 0$ (i.e. positive association between them), the bivariate survival function is given by

$$S(s,t) = [(\frac{1}{G(s)})^{\phi} + (\frac{1}{H(t)})^{\phi} - 1]^{-\frac{1}{\phi}} \tag{25}$$

This is a bivariate survival function with G and H as Marginals.

Corollary:

As $\phi \to 0$ then $S(s,t) = G(s) \cdot H(t)$, which corresponds to the independence between S and T.

Proof:

$$\log S(s,t) = -\frac{1}{\phi} \log [\exp(-\phi \log G) + \exp(-\phi \log H) - 1].$$

Using l'Hopital's rule:

$$\lim_{\phi \to 0} \log S(s,t) = \lim_{\phi \to 0} [\frac{e^{-\phi \log G}(-\log G) + e^{-\phi \log H}(-\log H)}{e^{-\phi \log G} + e^{-\phi \log H} - 1} \cdot \frac{1}{-1}]$$

$$= -1[-\log G - \log H]$$

$$= \log G + \log H$$

$$= \log GH$$

therefore

$$\lim_{\phi \to 0} S(s,t) = \exp(\log GH)$$

$$= G(s) \cdot H(t)$$

S and T are independent

Q.E.D.

Conclusion

We developed, with the help of Gompertz Marginal distribution, the most general bivariate Gompertz survival function possible. We showed that the force of Mortality function has the form:

$$\mu(x, y) = a \exp\ (c_1 x + c_2 y + c_3 xy)$$

We then obtained the general dependent bivariate Makeham survival function which allows for simultaneous failures by combining the Gompertz survival function with an "accident" component:

$$S_M(x, y) = S_G(x, y) \exp\ [-d_1 x - d_2 y - d_3 \max\ (x, y)]$$

We then finalized the paper by displaying the survival function in the general case where no underlying distribution is assumed.

More research can be done on this subject since in the Gompertz case, the constants c_1, c_2 and c_3 can be estimated and then we will be able to construct a two dimensional life table for this distribution.

We can also estimate ϕ by finding the maximum likelihood estimator when the marginal survivor functions are parameterized, say as exponential distributions with parameters p_1 and p_2.

References

[1] Aczel, J., *Lectures on functional equations and their applications*, Academic Press; New York, 1966.

[2] Brockett, Patrick L., *"General Bivariate Makeham Laws,"*, Scand. Actuarial Journal, (pp. 150-156), 1984.

[3] Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. and Nesbitt, C.J., *Actuarial Mathematics*, The Society of Actuaries; Illinois, 1986.

[4] Clayton, D.G., *"A model for association in bivariate life tables,"*, Biometrika, V65 (pp. 141-151), 1978.

[5] Cox, D.R. and Oakes, D., *Analysis of Survival Data*, Chapman and Hall; New York, 1984.

[6] Marshall, A.W. and Olkin, I., *"A multivariate exponential distribution,"*, Journal of Am. Statist. Assoc. V62 (pp. 30-44), 1967.

[7] Oakes, D., *"A model for association in bivariate survival data," Journal Royal Statistical Society, V44 (pp. 414-422), 1982.*