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EXTENSIONS OF LÉVY-KHINTCHINE FORMULA AND BEURLING-DENY FORMULA IN SEMI-DIRICHLET FORMS SETTING

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Extensions of Lévy-Khintchine formula and Beurling-Deny formula in semi-Dirichlet forms setting

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Abstract

The Lévy-Khintchine formula or, more generally, Courrège's theorem characterizes the infinitesimal generator of a Lévy process or a Feller process on $\mathbf{R}^d$. For more general Markov processes, the formula that comes closest to such a characterization is the Beurling-Deny formula for symmetric Dirichlet forms. In this paper, we extend these celebrated structure results to include a general right process on a metrizable Lusin space, which is supposed to be associated with a semi-Dirichlet form. We start with decomposing a regular semi-Dirichlet form into the diffusion, jumping and killing parts. Then, we develop a local compactification and an integral representation for quasi-regular semi-Dirichlet forms. Finally, we extend the formulae of Lévy-Khintchine and Beurling-Deny in semi-Dirichlet forms setting through introducing a quasi-compatible metric.

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1. Introduction and setting

We consider a Lévy process $(X_t)_{t\geq 0}$ on some probability space $(\Omega, \mathcal{F}, P)$ taking values in the d-dimensional Euclidean space $\mathbf{R}^d$ with the characteristic exponent η, i.e. $E\{\exp(i\langle\lambda, X_t\rangle)\} = \exp(-t\eta(\lambda))$ for $\lambda \in \mathbf{R}^d$ and $t \geq 0$, where E denotes the expectation w.r.t. (with respect to) P. Hereafter, $\mathbf{R}^d$ is equipped with the standard product $\langle\cdot,\cdot\rangle$ and Euclidean norm $|\cdot|$. The celebrated

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Lévy-Khintchine formula (cf. e.g. [Be, p.3] or [Sa, p.37]) tells us that

$$\eta(\lambda) = i\langle b, \lambda\rangle + \frac{1}{2}Q(\lambda) + \int_{\mathbf{R}^d} \left(1 - e^{i\langle \lambda, x\rangle} + i\langle \lambda, x\rangle 1_{\{|x|\le 1\}}\right) \mu(dx),$$

where $b = (b_1, \ldots, b_d) \in \mathbf{R}^d, Q$ is a symmetric, nonnegative definite quadratic form on $\mathbf{R}^d$, and μ is a Lévy measure satisfying $\mu(\{0\}) = 0$ and $\int_{\mathbf{R}^d} |x|^2/(1 + |x|^2)\mu(dx) < \infty$. Or equivalently, the infinitesimal generator A of $(X_t)_{t\ge 0}$ is characterized by (cf. [Sa, Theorem 31.5])

$$\begin{aligned} Au(y) &= \sum_{i=1}^{d}(-b_i)\partial_i u(y) + \frac{1}{2}\sum_{i,j=1}^{d} Q_{ij}\partial_i\partial_j u(y) \\ &\quad + \int_{\mathbf{R}^d}\left(u(y+x) - u(y) - \sum_{i=1}^{d} x_i\partial_i u(y) 1_{\{|x|\le 1\}}(x)\right)\mu(dx) \end{aligned} \tag{1.1}$$

for $u \in C_0^\infty(\mathbf{R}^d)$. Hereafter, we use $C(\mathbf{R}^d)$ to denote the set of all continuous functions on $\mathbf{R}^d$ and use $C_0^\infty(\mathbf{R}^d)$ to denote the set of all infinitely differentiable functions on $\mathbf{R}^d$ with compact supports. If in addition μ satisfies $\int_{|x|\le 1} |x|\mu(dx) < \infty$, then (1.1) can be written as

$$Au(y) = \sum_{i=1}^{d}(-\bar{b}_i)\partial_i u(y) + \frac{1}{2}\sum_{i,j=1}^{d} Q_{ij}\partial_i\partial_j u(y) + \int_{\mathbf{R}^d}(u(y+x) - u(y))\mu(dx)$$

with $\bar{b}_i = b_i + \int_{|x|\le 1} x_i\mu(dx)$, $1 \le i \le d$.

In fact, decomposition (1.1) holds for more general Feller processes on $\mathbf{R}^d$. In [Co], Courrège proved that if A is a linear operator from $C_0^\infty(\mathbf{R}^d)$ to $C(\mathbf{R}^d)$ satisfying the positive maximum principle, i.e. $\sup_{x\in\mathbf{R}^d} u(x) = u(x_0) \ge 0$ implies $Au(x_0) \le 0$, then A is decomposed as

$$\begin{aligned} Au(y) &= -\gamma(y)u(y) + \langle l(y), \nabla u(y)\rangle + \frac{1}{2}\sum_{i,j=1}^{d} q_{ij}(y)\partial_i\partial_j u(y) \\ &\quad + \int_{\mathbf{R}^d}\left(u(y+x) - u(y) - \frac{\langle x, \nabla u(y)\rangle}{1+|x|^2}\right) N(y, dx), \end{aligned} \tag{1.2}$$

where $\gamma(y) \ge 0$, $l(y) \in \mathbf{R}^d$, $\bar{Q} = (q_{ij})_{1\le i,j\le d}$ is a symmetric, nonnegative definite quadratic form on $\mathbf{R}^d$, and $N(y, dx)$ is a kernel satisfying $\int_{\mathbf{R}^d} |x|^2/(1 + |x|^2)N(y, dx) < \infty$. We refer the readers to [J, §5.5] for more detailed discussion about the generators of Feller semigroups.

Set $\mathcal{E}(u, v) = \int_{\mathbf{R}^d} -(Au(y))v(y)dy$, $J(dx, dy) = (1/2)N(y, dx - y)dy$ and $K(dx) = \gamma(x)dx$. Then we may rewrite (1.2) for $u, v \in C_0^\infty(\mathbf{R}^d)$ and $\varepsilon > 0$ as

$$\mathcal{E}(u, v) = \mathcal{E}^{c,\varepsilon}(u, v) + \int_{|x-y|>\varepsilon} 2(u(y) - u(x))v(y)J(dx, dy) + \int_{\mathbf{R}^d} u(x)v(x)K(dx). \tag{1.3}$$

If $(u(y) - u(x))v(y)$ is symmetric principle value (abbreviated by S.P.V.) integrable w.r.t. the measure J, which means that $\lim_{\varepsilon\downarrow 0}\int_{|x-y|>\varepsilon} 2(u(y)-u(x))v(y)J(dx, dy)$ exists, then (1.3) becomes

$$\mathcal{E}(u, v) = \mathcal{E}^c(u, v) + S.P.V. \int_{\mathbf{R}^d\times\mathbf{R}^d\setminus d} 2(u(y) - u(x))v(y)J(dx, dy) + \int_{\mathbf{R}^d} u(x)v(x)K(dx), \tag{1.4}$$

where $\mathbf{R}^d \times \mathbf{R}^d \backslash d := \{(x,y) \in \mathbf{R}^d \times \mathbf{R}^d \,|\, x \neq y\}$ and $\mathcal{E}^c(u,v) := \lim_{\varepsilon \downarrow 0} \mathcal{E}^{c,\varepsilon}(u,v)$, which satisfies the left strong local property, in the sense that if u is constant on a neighborhood of the support of v then $\mathcal{E}^c(u,v) = 0$. If A is symmetric, then $(u(y)-u(x))v(y)$ is always S.P.V. integrable w.r.t. J and we can rewrite (1.4) in the following form

$$\mathcal{E}(u,v) = \mathcal{E}^c(u,v) + \int_{\mathbf{R}^d \times \mathbf{R}^d \backslash d} (u(y)-u(x))(v(y)-v(x))J(dx,dy) + \int_{\mathbf{R}^d} u(x)v(x)K(dx). \quad (1.5)$$

Note that (1.5) is nothing else but the classical Beurling-Deny formula in the theory of symmetric Dirichlet forms.

Suppose now that $(X_t)_{t\geq 0}$ is a general right (continuous strong Markov) process taking values in a metrizable Lusin space, i.e. a space topologically isomorphic to a Borel subset of a complete separable metric space. A structure result for the generator of $(X_t)_{t\geq 0}$ similar to (1.1) or (1.2) is not known (cf. [Sc]). The formula that comes closest to such a characterization is the Beurling-Deny formula for symmetric Dirichlet forms as in (1.5). Apart from other things, this formula provides us an analytic description of the sample path properties of $(X_t)_{t\geq 0}$. For this connection, the interested readers may refer to [FOT, Ch.5], [CFTYZ], [Mo], etc. In this paper, under the assumption that $(X_t)_{t\geq 0}$ is associated with a semi-Dirichlet form, we will establish some structure results for $(X_t)_{t\geq 0}$. In particular, we will extend the Beurling-Deny formula to semi-Dirichlet forms. For a nice representation of the Beurling-Deny formula for regular symmetric Dirichlet forms, we refer to [FOT]. For the extensions of the Beurling-Deny formula to quasi-regular symmetric Dirichlet forms see [AMR], [DMS] and [Ku]. Also, there have been some attempts of extending the Beurling-Deny formula to the non-symmetric case, see [Bl], [Ki], [CZ] and [Mat] (cf. Remarks 2.7 and 5.3). In [HMS], both the Beurling-Deny formula and LeJan's formula are extended to regular non-symmetric Dirichlet forms.

Now we establish our setting and notations. We refer the readers to [MOR] and [Fi] for more details. Let $(X_t)_{t\geq 0}$ be a right process taking values in a metrizable Lusin space E, $\mathcal{B}(E)$ the Borel σ-field of E, and m a σ-finite measure on $(E, \mathcal{B}(E))$. Suppose that $(X_t)_{t\geq 0}$ is associated with a semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$. We use $(\cdot,\cdot)$ to denote the inner product of $L^2(E;m)$. By [Fi], $(\mathcal{E}, D(\mathcal{E}))$ must be quasi-regular. Then, every element $u \in D(\mathcal{E})$ admits an $\mathcal{E}$-quasi-continuous m-version, which we denote by $\tilde{u}$. We use $\tilde{D}(\mathcal{E})$ to denote the set of all $\mathcal{E}$-quasi-continuous versions of elements in $D(\mathcal{E})$. Without loss of generality, we assume that every element $u \in \tilde{D}(\mathcal{E})$ is Borel measurable. Following [FOT], we say that a subset $A \subset E$ is *quasi-open* (respectively, *quasi-closed*) if there exists an $\mathcal{E}$-nest $\{F_k\}_{k\in\mathbf{N}}$ such that $F_k \cap A$ is relatively open (respectively, relatively closed) in F_k for each $k \in \mathbf{N}$. Let u be an m-a.e. defined function on E, then there exists a smallest (up to an $\mathcal{E}$-exceptional set) quasi-closed set F, which is called the *quasi-support* of u and is denoted by $\mathrm{supp}_q[u]$, such that $\int_{E\backslash F} |u(x)|m(dx) = 0$. We use the same notation for a function f (m-a.e. defined) on E and for the m-equivalence class of functions represented by f, if there is no risk of confusion.

The remainder of this paper is organized as follows. In Section 2, we present the decomposition of regular semi-Dirichlet forms. In Section 3, we develop a local compactification and an integral representation for quasi-regular semi-Dirichlet forms. In Sections 4 and 5, we give the decompositions of quasi-regular semi-Dirichlet forms and (non-symmetric) Dirichlet forms.

Part of the results of this paper have been announced in C. R. Math. Acad. Sci. Paris, see

[HM].

2. Decomposition of regular semi-Dirichlet form

Similar to a regular symmetric Dirichlet form (cf. [FOT, p.6]), we call a semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ regular if the following conditions hold:
(i) E is a locally compact separable metric space and m is a positive Radon measure on E with $\mathrm{supp}[m] = E$.
(ii) $C_0(E) \cap D(\mathcal{E})$ is dense in $D(\mathcal{E})$ w.r.t. the $\tilde{\mathcal{E}}_1^{1/2}$-norm.
(iii) $C_0(E) \cap D(\mathcal{E})$ is dense in $C_0(E)$ w.r.t. the uniform norm $\|\cdot\|_\infty$.

Hereafter, we use $\mathrm{supp}[\cdot]$ to denote the support of a measure or a function on E, use $\tilde{\mathcal{E}}$ to denote the symmetric part of $\mathcal{E}$, and use $C_0(E)$ to denote the set of all continuous functions on E with compact supports.

A subset $D \subset C_0(E) \cap D(\mathcal{E})$ is called a *core* if the following conditions hold:
(C.1) D is dense in $D(\mathcal{E})$ w.r.t. the $\tilde{\mathcal{E}}_1^{1/2}$-norm.
(C.2) D is dense in $C_0(E)$ w.r.t. the uniform norm $\|\cdot\|_\infty$.
(C.3) D is a linear lattice.
D is called a *special core* if in addition to (C.1)-(C.3), it holds that
(C.4) For any compact set K and relatively compact open set G with $K \subset G$, there exists a $u \in D$ such that $0 \le u \le 1, u|_K = 1$ and $u|_{E\setminus G} = 0$.

Throughout this section, we assume $(\mathcal{E}, D(\mathcal{E}))$ is a regular semi-Dirichlet form on $L^2(E;m)$. Denote the resolvent of $(\mathcal{E}, D(\mathcal{E}))$ by $(G_\alpha)_{\alpha>0}$ and define

$$\mathcal{E}^{(\beta)}(u,v) = \beta(u - \beta G_\beta u, v). \tag{2.1}$$

It is known that (cf., e.g. [MR, Theorem I.2.13(iii)])

$$\lim_{\beta\to\infty} \mathcal{E}^{(\beta)}(u,v) = \mathcal{E}(u,v) \ \text{ for all } u,v \in D(\mathcal{E}). \tag{2.2}$$

Lemma 2.1. *If S is a positive linear bounded operator on $L^2(E;m)$, then there is a unique positive Radon measure σ on the product space $E\times E$ satisfying that for $u,v \in L^2(E;m)$, $(Su,v) = \int_{E\times E} u(x)v(y)\sigma(dx,dy)$. If in addition S is sub-Markovian, then $\sigma(E\times A) \le m(A)$ for all $A \in \mathcal{B}(E)$.*

Proof. The proof is similar to [FOT, Lemma 1.4.1] and the only difference is that the measure σ given here is non-symmetric in general. □

Corollary 2.2. *There exists a unique positive Radon measure σ_β on $E\times E$ satisfying*

$$(\beta G_\beta u, v) = \int_{E\times E} u(x)v(y)\sigma_\beta(dx,dy) \quad \text{for } u,v \in L^2(E;m). \tag{2.3}$$

Moreover,

$$\sigma_\beta(E\times A) \le m(A) \quad \text{for all } A \in \mathcal{B}(E). \tag{2.4}$$

□

Lemma 2.3. *Let U be a relatively compact open subset of E. Then, for $u, v \in C_0(E) \cap D(\mathcal{E})$ with supports contained in U,*

$$\mathcal{E}^{(\beta)}(u,v) = \beta \int_{U\times U} (u(y)-u(x))v(y)\sigma_\beta(dx,dy) + \beta \int_U u(x)v(x)(1-\beta G_\beta I_U(x))m(dx). \quad (2.5)$$

Proof. Direct consequence of (2.1), (2.3) and (2.4). □

Lemma 2.4. *The following assertions hold:*
(i) For $u \in C_0(E)$, there exists a sequence $\{u_n\}_{n\in\mathbf{N}} \subset C_0(E)\cap D(\mathcal{E})$ such that $\mathrm{supp}[u_n] \subset \{x \in E | u(x) \neq 0\}$, $n \in \mathbf{N}$, and u_n converges to u uniformly as $n \to \infty$.
(ii) For any compact set F and relatively compact open set G with $F \subset G$, there exists $u \in C_0(E)\cap D(\mathcal{E})$ such that $0 \le u \le 1, u|_F = 1$ and $u|_{E\setminus G} = 0$.

Proof. By the regularity of $(\mathcal{E}, D(\mathcal{E}))$ and [Ku, Lemma 2.1(ii)], this lemma can be proved similarly to the case of Dirichlet forms. □

Definition 2.5. Denote by d the diagonal of $E \times E$.
(i) A subset $A \subset E \times E\setminus d$ is said to be symmetric if its indicator function I_A is symmetric, i.e. $I_A(x,y) = I_A(y,x)$ for all $(x,y) \in E \times E\setminus d$.
(ii) Let J be a Radon measure on $E \times E\setminus d$. A measurable function f on $E \times E\setminus d$ is said to be integrable w.r.t. J in the sense of *symmetric principle value* (abbreviated by *S.P.V. integrable*), if f is integrable on each relatively compact symmetric subset $A \subset E \times E\setminus d$ and for any increasing sequence of relatively compact symmetric sets $\{A_n\}_{n\ge1}$ with $\cup_{n=1}^\infty A_n = E \times E\setminus d$, the limit

$$S.P.V. \int_{E\times E\setminus d} f(x,y)J(dx,dy) := \lim_{n\to\infty} \int_{A_n} f(x,y)J(dx,dy)$$

exists and is independent of the specific choice of the sequence $\{A_n\}_{n\ge1}$.

Theorem 2.6. *(i) There exist a unique positive Radon measure J on $E \times E\setminus d$ and a unique positive Radon measure K on E such that for $v \in C_0(E)\cap D(\mathcal{E})$ and $u \in I(v)$,*

$$\mathcal{E}(u,v) = \int_{E\times E\setminus d} 2(u(y)-u(x))v(y)J(dx,dy) + \int_E u(x)v(x)K(dx), \quad (2.6)$$

where $I(v) := \{u \in C_0(E)\cap D(\mathcal{E}) | u$ is constant on a neighbourhood of $\mathrm{supp}[v]\}$.
(ii) Denote $\mathcal{A}(v) := \{u \in C_0(E)\cap D(\mathcal{E}) | (u(y)-u(x))v(y)$ is S.P.V. integrable w.r.t. $J\}$. Then we have the following unique decomposition

$$\begin{aligned}\mathcal{E}(u,v) \quad &= \quad \mathcal{E}^c(u,v) + S.P.V. \int_{E\times E\setminus d} 2(u(y)-u(x))v(y)J(dx,dy) \\ &\quad + \int_E u(x)v(x)K(dx) \quad \text{for } v \in C_0(E)\cap D(\mathcal{E}) \text{ and } u \in \mathcal{A}(v), \quad (2.7)\end{aligned}$$

where $\mathcal{E}^c(u,v)$ satisfies the left strong local property in the sense that $I(v) \subset \mathcal{A}(v)$ and $\mathcal{E}^c(u,v) = 0$ whenever $v \in C_0(E)\cap D(\mathcal{E})$, $u \in I(v)$.

Proof. (i) The uniqueness of J and K satisfying (2.6) can be proved in the same way as in [FOT, Theorem 3.2.1] by virtue of Lemma 2.4(i). The existence of J can be proved similarly to [FOT, Theorem 3.2.1]. Moreover, $(\beta/2)\sigma_\beta \to J$ vaguely on $E \times E\backslash d$ as $\beta \to \infty$.

To show the existence of K, we fix a relatively compact open set U. For any compact subset F of U, by Lemma 2.4(ii), there exist $u, v \in C_0(E) \cap D(\mathcal{E})$ satisfying $\mathrm{supp}[u] \cup \mathrm{supp}[v] \subset U$, such that $v|_F \equiv 1, v \geq 0$, $u|_{\mathrm{supp}[v]} \equiv 1$ and $0 \leq u \leq 1$. Then, we get by (2.5) that

$$\begin{aligned} \int_F \beta(1-\beta G_\beta I_U(x))m(dx) &\leq \beta\int_U u(x)v(x)(1-\beta G_\beta I_U(x))m(dx) \\ &\leq \beta\int_U u(x)v(x)(1-\beta G_\beta I_U(x))m(dx) \\ &\quad +\beta\int_{U\times U}(u(y)-u(x))v(y)\sigma_\beta(dx,dy) \\ &= \mathcal{E}^{(\beta)}(u,v). \end{aligned} \tag{2.8}$$

Now it follows from (2.8) that the family of measures $\{\beta(1-\beta G_\beta I_U(x))m(dx)\}_{\beta>0}$ are uniformly bounded on any compact subset of U. Let $\bar\rho$ be a metric compatible with the topology of E, $\{U_l\}_{l\geq 1}$ an increasing sequence of relatively compact open sets satisfying $\cup_{l=1}^\infty U_l = E$, and $\{\delta_l\}_{l\geq 1}$ ($\delta_l \downarrow 0$) a decreasing sequence of positive numbers such that $U_l \times U_l\backslash\{(x,y)|\bar\rho(x,y) < \delta_l\}$ is a continuous set of J for each l. Note that such $\{U_l\}$ and $\{\delta_l\}$ always exist. Then, there exist an increasing sequence $\{\beta_n\}_{n\in\mathbf{N}}$ satisfying $\beta_n \to \infty$ as $n \to \infty$ and a positive Radon measure K_l on U_l such that for each $l \geq 1$,

$$\beta_n(1-\beta_n G_{\beta_n} I_{U_l})\cdot m \to K_l \text{ vaguely on } U_l \text{ as } n\to\infty. \tag{2.9}$$

Extend K_l to E by setting $K_l(A) := K_l(A\cap U_l)$ for any Borel subset A of E. By (2.9), for each compact subset F of E, there exists l_0 such that $\{K_l(F)\}_{l\geq l_0}$ is non-increasing. Consequently, there exists a Radon measure K on E such that

$$K_l \to K \text{ vaguely on } E \text{ as } l\to\infty. \tag{2.10}$$

Denote $\Gamma_l := U_l \times U_l\backslash\{(x,y)|\bar\rho(x,y) < \delta_l\}$. Let $v \in C_0(E)\cap D(\mathcal{E})$ and $u \in I(v)$. Suppose that $u(x) = \alpha$ on a neighborhood of $\mathrm{supp}[v]$ for some constant α. Then, we get by (2.2) and (2.5) that

$$\begin{aligned} \mathcal{E}(u,v) &= \lim_{n\to\infty}\frac{\beta_n}{2}\int_{U_l\times U_l,\bar\rho(x,y)<\delta_l} 2(u(y)-u(x))v(y)\sigma_{\beta_n}(dx,dy) \\ &\quad +\int_{\Gamma_l}2(u(y)-u(x))v(y)J(dx,dy)+\int_{U_l}u(x)v(x)K_l(dx) \end{aligned}$$

provided $l \geq l_1$ for some large enough l_1. Letting $l\to\infty$, we get

$$\mathcal{E}(u,v) = \int_{E\times E\backslash d}2(u(y)-u(x))v(y)J(dx,dy)+\int_E u(x)v(x)K(dx),$$

where the integrability of $(u(y)-u(x))v(y)$ follows from the fact that for any $y\in\mathrm{supp}[v]$,

$$(u(y)-u(x))v(y) = (\alpha-u(x))v(y) = (\alpha-u(x))^+v(y)-(\alpha-u(x))^-v(y),$$

and either $\text{supp}[(\alpha-u(x))^{+}v(y)]$ or $\text{supp}[(\alpha-u(x))^{-}v(y)]$ must be contained in Γ_{l_1} for some large l_1, since u has a compact support. Thus, the measure K constructed in (2.10) satisfies (2.6), which in turn implies that K is independent of the specific choice of $\{U_l\}_{l\geq 1}$ and $\{\delta_l\}_{l\geq 1}$ by the uniqueness of K.

(ii) For $v \in C_0(E)\cap D(\mathcal{E})$ and $u \in \mathcal{A}(v)$, define

$$\mathcal{E}^c(u,v) := \lim_{n\to\infty}\frac{\beta_n}{2}\int_{U_l\times U_l,\,\bar{\rho}(x,y)<\delta_l} 2(u(y)-u(x))v(y)\sigma_{\beta_n}(dx,dy). \tag{2.11}$$

Then, we obtain decomposition (2.7) by the proof of (i) above. The uniqueness is obvious by (i) and the left strong local property of $\mathcal{E}^c(u,v)$ follows from (2.11). The proof is complete. $\square$

Remark 2.7. (i) As in the setting of Dirichlet forms, J and K respectively represent the *jumping* and *killing* measures of the process $(X_t)_{t\geq 0}$. For any $\mathcal{E}$-exceptional set N, $J(E\times N\backslash d) = J(N\times E\backslash d) = 0$ and $K(N)=0$ (cf. [Hu1]).
(ii) Let D be a special core of $(\mathcal{E}, D(\mathcal{E}))$. If (2.6) holds for any $v\in D$ and $u \in D\cap I(v)$, then the measures J and K are unique.
(iii) Note that if $v\in C_0(E)\cap D(\mathcal{E})$ and $u\in I(v)$ then $\mathcal{E}^c(u,v)=0$, since $I(v)\subset \mathcal{A}(v)$. In this case, decomposition (2.7) has been obtained in [Ki, Lemma 2.14] in Dirichlet forms setting. Further, Chen and Zhao [CZ, (A.15)] extended the result to non-symmetric Dirichlet forms in the extended sense that only the sub-Markovian property of the dual semigroup of the α-subprocess is assumed for some $\alpha>0$, rather than that for the original process (that is $\alpha=0$).
(iv) Mataloni [Mat, Theorems 2.7 and 2.8] has obtained the decomposition like (2.7) in Dirichlet forms setting but without introducing the notion of S.P.V. integral and the constraint that $u\in\mathcal{A}(v)$. These conditions are essential and cannot be dropped. The interested readers may refer to [HMS] for a counterexample. We thank Kazuhiro Kuwae for drawing our attention to the paper [Mat].

We now extend Theorem 2.6 for later use. Let $v\in\tilde{D}(\mathcal{E})$. We define

$$I'(v) := \{u\in\tilde{D}(\mathcal{E})|u \text{ is constant } \mathcal{E}\text{-q.e. on a quasi-open set containing } \text{supp}[v]\}.$$

Lemma 2.8. *Let v be a bounded function in $\tilde{D}(\mathcal{E})$ such that* $\text{supp}[v]$ *is compact. If $u\in I(v)$, then*

$$\mathcal{E}(u,v) = \int_{E\times E\backslash d} 2(u(y)-u(x))v(y)J(dx,dy) + \int_E u(x)v(x)K(dx).$$

Proof. We assume $0\leq v\leq M$ for some constant $M>0$, and $u|_G=\alpha$ for some constant α and some open set $G\supset \text{supp}[v]$. Since E is a locally compact separable metric space, there exists a relatively compact open set G_1 such that $\text{supp}[v]\subset G_1\subset \bar{G}_1\subset G$. By Lemma 2.4(ii), there exists a $w\in C_0(E)\cap D(\mathcal{E})$ satisfying $0\leq w\leq M$, $w|_{\text{supp}[v]}=M$ and $w|_{E\backslash G_1}=0$. By the regularity of $(\mathcal{E},D(\mathcal{E}))$, there exists a sequence $\{v_n'\}_{n\in\mathbf{N}}\subset C_0(E)\cap D(\mathcal{E})$ such that v_n' is $\mathcal{E}_1$-convergent to v as $n\to\infty$. Set $v_n := (v_n'\vee 0)\wedge w$. Then by [MR, Lemma I.2.12], there exists a subsequence $\{v_{n_k}\}_{k\in\mathbf{N}}$ of $\{v_n\}_{n\in\mathbf{N}}$ such that the Cesàro sum $w_n := (1/n)\sum_{k=1}^n v_{n_k}$ is $\mathcal{E}_1$-convergent to $(v\vee 0)\wedge w = v$ as

$n \to \infty$. Obviously, $\text{supp}[w_n] \subset \bar{G}_1 \subset G$. By Theorem 2.6(i),

$$\mathcal{E}(u, w_n) = \int_{E\times E\backslash d} 2(u(y) - u(x))w_n(y)J(dx, dy) + \int_E u(x)w_n(x)K(dx). \tag{2.12}$$

There exists an $\mathcal{E}$-exceptional set N such that $w_n(x) \to v(x)$ for all $x \in E\backslash N$ by [MOR, Proposition 2.18(i)]. Note that $0 \le w_n \le M$, $n \in N$, $\text{supp}[uw_n] \subset \text{supp}[w_n] \subset \bar{G}_1$ and $\bar{G}_1$ is compact, $\lim_{n\to\infty} \int_E u(x)w_n(x)K(dx) = \int_E u(x)v(x)K(dx)$ by the dominated convergence theorem and Remark 2.7(i). Since $u = u \wedge \alpha - (u \wedge \alpha - u)$, we assume without loss of generality that $u \le \alpha$. By Theorem 2.6(i), $2(u(y) - u(x))w(y)$ is integrable w.r.t. J on $E \times E\backslash d$. Noting that $0 \le w_n \le w$, we obtain by the dominated convergence theorem, Remark 2.7(i) and (2.12) that

$$\begin{aligned}
\int_{E\times E\backslash d} 2(u(y) - u(x))v(y)J(dx, dy) &= \lim_{n\to\infty} \int_{E\times E\backslash d} 2(u(y) - u(x))w_n(y)J(dx, dy) \\
&= \lim_{n\to\infty} \left[\mathcal{E}(u, w_n) - \int_E u(x)w_n(x)K(dx)\right] \\
&= \mathcal{E}(u, v) - \int_E u(x)v(x)K(dx).
\end{aligned}$$

The proof is complete. □

Theorem 2.9. *Let v be a bounded function in $\tilde{D}(\mathcal{E})$ such that* $\text{supp}[v]$ *is compact. If $u \in I'(v)$, then*

$$\mathcal{E}(u, v) = \int_{E\times E\backslash d} 2(u(y) - u(x))v(y)J(dx, dy) + \int_E u(x)v(x)K(dx).$$

Proof. We assume without loss of generality that $v \ge 0$. Since $u \in I'(v)$, there exist a quasi-open set $G_1 \supset \text{supp}[v]$ and a constant α such that $u|_{G_1} = \alpha$ $\mathcal{E}$-q.e. Since X is a locally compact separable metric space, there exists a relatively compact open set G_2 such that $\text{supp}[v] \subset G_2$. By Lemma 2.4(ii), there exists an $s \in C_0(E) \cap D(\mathcal{E})$ such that $s|_{\overline{G}_2} \equiv \alpha$. Then, $G_1 \cap G_2$ is also a quasi-open set containing $\text{supp}[v]$ and $(u - s)|_{G_1\cap G_2} = 0$ $\mathcal{E}$-q.e. Consequently, we may assume without loss of generality that $\alpha = 0$ by Lemma 2.8. Moreover, since $u = u \wedge 0 - (u \wedge 0 - u)$, we may only consider the case that $u \le 0$.

Set $G := E\backslash\text{supp}[v]$. Then G is an open set and $u \in D(\mathcal{E}_G)$, where $D(\mathcal{E}_G) := \{u \in D(\mathcal{E})|u = 0\ m\text{-a.e. on } E\backslash G\}$. For $u, v \in D(\mathcal{E}_G)$, define $\mathcal{E}_G(u, v) := \mathcal{E}(u, v)$. Then, $(\mathcal{E}_G, D(\mathcal{E}_G))$ is a regular semi-Dirichlet form on $L^2(G; m)$ (cf. [Hu2]). Hence there exists a sequence $\{f_n\}_{n\in\mathbf{N}} \subset C_0(G) \cap D(\mathcal{E}_G)$ such that f_n is $\mathcal{E}_{G,1}$-convergent to u as $n \to \infty$. Since $u \le 0$, we may assume that $f_n \le 0$, $\forall n \in \mathbf{N}$. Otherwise, we may replace $\{f_n\}_{n\ge1}$ with the Cesàro sums of a subsequence of $\{f_n \wedge 0\}_{n\in\mathbf{N}}$.

For $n \in N$, we define

$$u_n := \begin{cases} f_n & \text{on } G, \\ 0 & \text{on } E\backslash G. \end{cases}$$

Then $u_n \in C_0(E) \cap D(\mathcal{E}), u_n \leq 0$, $\mathrm{supp}[u_n] \subset \mathrm{supp}[f_n] \subset G$, $n \in N$, and u_n is $\mathcal{E}_1$-convergent to u as $n \to \infty$. Since $\mathrm{supp}[f_n]$ is compact, for each $n \in \mathbf{N}$, there exists an open set $V_n \supset \mathrm{supp}[v]$ such that $u_n|_{V_n} \equiv 0$. By Lemma 2.8,

$$\begin{aligned} \mathcal{E}(u_n, v) &= \int_{E\times E\backslash d} 2(u_n(y) - u_n(x))v(y)J(dx, dy) + \int_E u_n(x)v(x)K(dx) \\ &= -\int_{E\times E\backslash d} 2u_n(x)v(y)J(dx, dy). \end{aligned} \tag{2.13}$$

By [MOR, Proposition 2.18(i)], there exists an $\mathcal{E}$-exceptional set N such that $u_n(x) \to u(x)$ as $n \to \infty$ for all $x \in E\backslash N$. Then by Remark 2.7(i), Fatou's lemma and (2.13),

$$\begin{aligned} \int_{E\times E\backslash d} -2u(x)v(y)J(dx, dy) &\leq \liminf_{n\to\infty} \int_{E\times E\backslash d} -2u_n(x)v(y)J(dx, dy) \\ &= \liminf_{n\to\infty} \mathcal{E}(u_n, v) \\ &= \mathcal{E}(u, v). \end{aligned} \tag{2.14}$$

Noting that $v \geq 0, u \leq 0, u_n \leq 0, \forall n \in \mathbf{N}$, we obtain by Remark 2.7(i) and the dominated convergence theorem that

$$\begin{aligned} \int_{E\times E\backslash d} -2u(x)v(y)J(dx, dy) &\geq \int_{E\times E\backslash d} \lim_{n\to\infty} ((-2u_n(x)) \wedge (-2u(x)))v(y)J(dx, dy) \\ &= \lim_{n\to\infty} \int_{E\times E\backslash d} -2(u_n \vee u)(x)v(y)J(dx, dy). \end{aligned} \tag{2.15}$$

We claim that

$$\mathcal{E}(u_n \vee u, v) = \int_{E\times E\backslash d} -2(u_n \vee u)(x)v(y)J(dx, dy). \tag{2.16}$$

Since $u_n \vee u \in D(\mathcal{E}_G)$, by the regularity of $(\mathcal{E}_G, D(\mathcal{E}_G))$, there exists a sequence $\{g'_k\}_{k\in\mathbf{N}} \subset C_0(G) \cap D(\mathcal{E}_G)$ such that g'_k is $\mathcal{E}_{G,1}$-convergent to $u_n \vee u$ as $k \to \infty$. Since $u_n \in C_0(E) \cap D(\mathcal{E})$, there exists a constant $M > 0$ such that $-M \leq u_n \vee u \leq 0$. Obviously, $\mathrm{supp}[u_n \vee u] \subset \mathrm{supp}[u_n]$ is compact. By Lemma 2.4(ii), there exists a $w \in C_0(E) \cap D(\mathcal{E})$ such that $-M \leq w \leq 0$, $w|_{\mathrm{supp}[u_n \vee u]} = -M$ and $\mathrm{supp}[w] \subset G$. For $k \in \mathbf{N}$, define $g_k := (g'_k \wedge 0) \vee w$. Then by [MR, Lemma I.2.12], there exists a subsequence $\{g_{k_l}\}_{l\in\mathbf{N}}$ of $\{g_k\}_{k\in\mathbf{N}}$ such that the Cesàro sum $w_m := (1/m)\sum_{l=1}^m g_{k_l}$ is $\mathcal{E}_1$-convergent to $((u_n \vee u) \wedge 0) \vee w = u_n \vee u$ as $m \to \infty$. Similar to (2.13), we get

$$\begin{aligned} \mathcal{E}(w_m, v) &= \int_{E\times E\backslash d} 2(w_m(y) - w_m(x))v(y)J(dx, dy) + \int_E w_m(x)v(x)K(dx) \\ &= \int_{E\times E\backslash d} -2w_m(x)v(y)J(dx, dy). \end{aligned} \tag{2.17}$$

Note that $-w_n(x) \leq -w(x)$ and $-w(x)v(y) = (w(y) - w(x))v(y)$ is integrable w.r.t. J on $E \times E\backslash d$ by Lemma 2.8. By [MOR, Proposition 2.18(i)], there exists an $\mathcal{E}$-exceptional set N' such that

$w_m(x) \to (u_n \vee u)(x)$ as $m \to \infty$ for all $x \in E\backslash N'$. By the dominated convergence theorem, Remark 2.7(i) and (2.17), we get

$$\begin{aligned}\int_{E\times E\backslash d} -2(u_n \vee u)(x)v(y)J(dx,dy) &= \int_{E\times E\backslash d} \lim_{m\to\infty} -2w_m(x)v(y)J(dx,dy) \\ &= \lim_{m\to\infty}\int_{E\times E\backslash d} -2w_m(x)v(y)J(dx,dy) \\ &= \lim_{m\to\infty}\mathcal{E}(w_m,v) \\ &= \mathcal{E}(u_n\vee u,v).\end{aligned}$$

Thus (2.16) holds.

By (2.16) and the fact that u_n is $\mathcal{E}_1$-convergent to u as $n \to \infty$,

$$\lim_{n\to\infty}\int_{E\times E\backslash d} -2(u_n \vee u)(x)v(y)J(dx,dy) = \lim_{n\to\infty}\mathcal{E}(u_n\vee u,v) = \mathcal{E}(u,v). \tag{2.18}$$

Finally, by (2.14), (2.15), (2.18) and the fact that $u = 0$ $\mathcal{E}$-q.e. on supp$[v]$, we get

$$\begin{aligned}\mathcal{E}(u,v) &= \int_{E\times E\backslash d} -2u(x)v(y)J(dx,dy) \\ &= \int_{E\times E\backslash d} 2(u(y)-u(x))v(y)J(dx,dy) + \int_E u(x)v(x)K(dx),\end{aligned}$$

which completes the proof. □

3. Local compactification and integral representation of quasi-regular semi-Dirichlet form

First, we recall some basic results about quasi-regular semi-Dirichlet forms. We refer the readers to [MOR, Definition 3.5] for the definition of quasi-regular semi-Dirichlet form. Throughout this section, we let E be a metrizable Lusin space and m a σ-finite measure on $(E,\mathcal{B}(E))$.

Proposition 3.1. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form on $L^2(E;m)$. Then*
(i) $D(\mathcal{E})$ is separable w.r.t. the $\tilde{\mathcal{E}}_1^{1/2}$-norm.
(ii) Each element $u \in D(\mathcal{E})$ has an $\mathcal{E}$-quasi-continuous m-version, which we denote by $\tilde{u}$.
(iii) Let $\{F_k\}_{k\in\mathbf{N}}$ be an $\mathcal{E}$-nest and suppose that supp$[I_{F_k}\cdot m]$ *exists for each $k \in \mathbf{N}$. Set $F'_k :=$* supp$[I_{F_k}\cdot m]$*. Then $\{F'_k\}_{k\in\mathbf{N}}$ is also an $\mathcal{E}$-nest.*
(iv) If f is $\mathcal{E}$-quasi-continuous and $f \geq 0$ m-a.e. on an open subset U of E, then $f \geq 0$ $\mathcal{E}$-q.e. on U. In particular, $\tilde{u}$ is $\mathcal{E}$-q.e. unique for any $u \in D(\mathcal{E})$.
(v) If D is a dense subset of $D(\mathcal{E})$, then there exist an $\mathcal{E}$-exceptional set $N \subset E$ and $\mathcal{E}$-quasi-continuous m-versions $\tilde{u}$ such that $\{\tilde{u}|u \in D\}$ separates the points of $E\backslash N$.
(vi) Fix a $\varphi \in L^2(E;m)$ satisfying $0 < \varphi \leq 1$ m-a.e. Set $g := G_1\varphi$. Let h be a fixed $\mathcal{E}$-quasi-continuous m-version of g, and $\hat{h}$ a fixed $\mathcal{E}$-quasi-continuous m-version of the 1-reduced function of h w.r.t. the dual form $(\hat{\mathcal{E}}, D(\mathcal{E}))$. Hereafter we define $\hat{\mathcal{E}}(u,v) := \mathcal{E}(v,u)$, $\forall u,v \in D(\mathcal{E})$.

Then, there exists an $\mathcal{E}$-nest $\{F_k^h\}_{k\in\mathbf{N}}$ such that $h \in C(\{F_k^h\}), \hat{h} \in C(\{F_k^h\}), \hat{h}(x) \geq h(x)$ for all $x \in \cup_{k\geq 1} F_k$, and

$$\inf\{h(x)|x \in F_k^h\} > 0 \quad \text{for all } k \in \mathbf{N}.$$

Proof. We refer to [MOR, Proposition 3.6] for the proofs of (i), (ii), (iv) and (v).
(iii) It can be proved similarly to [MR, Proposition III.3.8].
(vi) Following the proof of [MR, Proposition III.3.6], we know that there exists an $\mathcal{E}$-nest $\{F_k^{(1)}\}_{k\in\mathbf{N}}$ such that $\inf\{h(x)|x \in F_k^{(1)}\} > 0$ for all $k \in \mathbf{N}$. Since $\hat{h}$ is a reduced function of h, $\hat{h} \geq h$ m-a.e. and thus $\hat{h} \geq h$ $\mathcal{E}$-q.e. Hence, there exists an $\mathcal{E}$-nest $\{F_k^{(2)}\}_{k\in\mathbf{N}}$ such that $\hat{h}(x) \geq h(x)$ for each $x \in \cup_{k\geq 1} F_k^{(2)}$. Let $\{F_k^{(3)}\}_{k\in\mathbf{N}}$ be an $\mathcal{E}$-nest such that $h \in C(\{F_k^h\})$ and $\hat{h} \in C(\{F_k^h\})$. We set $F_k^h := F_k^{(1)} \cap F_k^{(2)} \cap F_k^{(3)}$ for $k \in \mathbf{N}$. Then $\{F_k^h\}_{k\in\mathbf{N}}$ is a desired $\mathcal{E}$-nest. □

Lemma 3.2. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form on $L^2(E; m)$. Then, there exists a countable subset D_0^+ of $D(\mathcal{E})$ consisting of bounded 1-excessive functions such that $D_0^+ - D_0^+$ is dense in $D(\mathcal{E})$.*

Proof. By the quasi-regularity of $(\mathcal{E}, D(\mathcal{E}))$ and [Ku, Lemma 2.1], one can prove this lemma similarly to [MR, Proposition IV.3.4(ii)]. □

Lemma 3.3. *Denote $F := \{u \in \tilde{D}(\mathcal{E})|u = u_1 - u_2$ for two 1-excessive functions $u_1, u_2 \in D(\mathcal{E})$ and $|u| \leq ch$ for some constant $c > 0\}$, where h is specified by Proposition 3.1(vi). Then for any $u, v \in F$ and any $c_1, c_2 \in Q$, $u \wedge v, u \wedge 1, u \wedge (v + 1), c_1u + c_2v \in F$. Hereafter, Q denotes the set of all rational numbers.*

Proof. Let $u = u_1 - u_2, v = v_1 - v_2$ be as in the definition of F. Then

$$u \wedge v = (u_1 - u_2) \wedge (v_1 - v_2) = (u_1 + v_2) \wedge (v_1 + u_2) - (u_2 + v_2),$$

and $(u_1+u_2)\wedge(v_1+u_2)$, u_2+v_2 are 1-excessive functions in $D(\mathcal{E})$. Obviously, $|u\wedge v|$ is dominated by ch for some constant $c > 0$ and is $\mathcal{E}$-quasi-continuous. Hence $u \wedge v \in F$. Similarly, one can check that $u \wedge 1, u \wedge (v + 1), c_1u + c_2v \in F$. □

Proposition 3.4. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form on $L^2(E; m)$. Then, there exists a countable set D of $\mathcal{E}$-quasi-continuous functions such that the corresponding m-classes form a dense subset of $D(\mathcal{E})$ satisfying the following properties:*
(i) $u \wedge v, u \wedge 1, u \wedge (v + 1), c_1u + c_2v \in D$ for all $u, v \in D$ and $c_1, c_2 \in Q$.
(ii) $h \in D$, where h is specified by Proposition 3.1(vi).
(iii) Each u in D is bounded and $|u| \leq ch$ for some constant $c > 0$.
(iv) There exists an $\mathcal{E}$-nest $\{F_k\}_{k\in\mathbf{N}}$ consisting of compact metrizable sets such that $D \cup \{\hat{h}\} \subset C(\{F_k\})$, D separates the points of $Y := \cup_{k\geq 1} F_k$, and $F_k \subset F_k^h$ with F_k^h being specified by Proposition 3.1(vi). Moreover, $F_k = \mathrm{supp}[I_{F_k} \cdot m]$ for each k.

Proof. Let D_0^+, F and $\{F_k^h\}_{k\in\mathbf{N}}$ be specified by Lemma 3.2, Lemma 3.3 and Proposition 3.1(vi), respectively. For $u \in D_0^+$ and $k \in \mathbf{N}$, set $u_k = u - u_{(F_k^h)^c} \wedge u$. We fix an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}_k$ of u_k such that $\tilde{u}_k = 0$ on $E\backslash F_k^h$. Then, $\{\tilde{u}_k|u \in D_0^+, k \in \mathbf{N}\} \cup \{h\} \subset F$. By Lemma

3.3 and [FOT, Lemma 7.1.1], there exists a countable subset D of F such that
a) $\{\tilde{u}_k | u \in D_0^+, k \in \mathbf{N}\} \cup \{h\} \subset D$.
b) $u \wedge v, u \wedge 1, u \wedge (v+1) \in D$ for all $u, v \in D$.
c) $c_1 u + c_2 v \in D$ for all $u, v \in D$ and $c_1, c_2 \in Q$.

Now assertions (i), (ii) and (iii) are obvious. One can check that for $u \in D_0^+$, there exists a subsequence $\{u_{k_l}\}_{l \in \mathbf{N}}$ of $\{u_k\}_{k \in \mathbf{N}}$ such that the Cesàro sum $w_n := (1/n)\sum_{l=1}^n u_{k_l} \to u$ in $D(\mathcal{E})$ as $n \to \infty$. Hence, by a), c) and Lemma 3.2, we know that D is dense in $D(\mathcal{E})$. By Proposition 3.1(v), there exists an $\mathcal{E}$-exceptional set N such that D separates the points of $E \backslash N$. Let $\{F_{1k}\}_{k \in \mathbf{N}}$ be an $\mathcal{E}$-nest such that $N \subset \cap_{k \geq 1}(E \backslash F_{1k})$ and $\{F_{2k}\}_{k \in \mathbf{N}}$ an $\mathcal{E}$-nest such that $D \cup \{\hat{h}\} \subset C(\{F_{2k}\})$. By the quasi-regularity of $(\mathcal{E}, D(\mathcal{E}))$, there exists an $\mathcal{E}$-nest $\{F_{3k}\}_{k \in \mathbf{N}}$ consisting of compact metrizable sets. Set $F'_k := F_{1k} \cap F_{2k} \cap F_{3k} \cap F_k^h$ and $F_k := \mathrm{supp}[I_{F'_k} \cdot m]$. Then, $\{F_k\}_{k \in \mathbf{N}}$ is an $\mathcal{E}$-nest satisfying (iv). □

Let $(\mathcal{E}, D(\mathcal{E}))$ be a semi-Dirichlet form on $L^2(E; m)$ and $E^\sharp$ another Hausdorff topological space with Borel σ-field $\mathcal{B}(E^\sharp)$. Suppose that N is an $\mathcal{E}$-exceptional set. Set $Y = E \backslash N$. Suppose that j is a $\mathcal{B}(Y)/\mathcal{B}(E^\sharp)$-measurable map from Y into $E^\sharp$. Let $m \circ j^{-1}$ be the image measure of m on $(E^\sharp, \mathcal{B}(E^\sharp))$. If $u^\sharp$ is $m \circ j^{-1}$-a.e. defined on $E^\sharp$, then $u^\sharp \circ j$ is m-a.e. defined on E since $m(N) = 0$. Define $j^* u^\sharp = u^\sharp \circ j$ m-a.e. for $u^\sharp \in L^2(E^\sharp; m \circ j^{-1})$. Then, j^* is an isometric map from $L^2(E^\sharp, m \circ j^{-1})$ into $L^2(E; m)$.

We define

$$\begin{cases} D(\mathcal{E}^j) = \{u^\sharp \in L^2(E^\sharp; m \circ j^{-1}) \,|\, j^* u^\sharp \in D(\mathcal{E})\}, \\ \mathcal{E}^j(u^\sharp, v^\sharp) = \mathcal{E}(j^* u^\sharp, j^* v^\sharp), \quad \forall u^\sharp, v^\sharp \in D(\mathcal{E}^j). \end{cases}$$

Then $(\mathcal{E}^j, D(\mathcal{E}^j))$ is called the image of $(\mathcal{E}, D(\mathcal{E}))$ under j. If j^* is onto then one can check that $(\mathcal{E}^j, D(\mathcal{E}^j))$ is a semi-Dirichlet form by [Ku, Proposition 2.2].

Theorem 3.5. (**local compactification**) *Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form on $L^2(E; m)$. Then, there exist an $\mathcal{E}$-nest $\{F_k\}_{k \in \mathbf{N}}$ consisting of compact metrizable subsets of E and a locally compact separable metric space $Y^\sharp$ such that*
(i) $Y^\sharp$ is a local compactification of $Y := \cup_{k \geq 1} F_k$ in the sense that $Y^\sharp$ is a locally compact space containing Y as a dense subset and $\mathcal{B}(Y) = \{A \in \mathcal{B}(Y^\sharp) | A \subset Y\}$.
(ii) The trace topologies on F_k induced by E and $Y^\sharp$ coincide for each $k \in \mathbf{N}$.
(iii) The image $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$ of $(\mathcal{E}, D(\mathcal{E}))$ under the inclusion map: $i : Y \subset Y^\sharp$ is a regular semi-Dirichlet form on $L^2(Y^\sharp; m^\sharp)$, where $m^\sharp := m \circ i^{-1}$ is the image measure of m on $(Y^\sharp, \mathcal{B}(Y^\sharp))$.

Proof. Let D be a countable dense subset of $\tilde{D}(\mathcal{E})$ specified by Proposition 3.4, say $D := \{u_n | n \in \mathbf{N}\}$ with $u_1 = h$, where h is specified by Proposition 3.1(vi). Let $\{F_k\}_{k \in \mathbf{N}}$ be an $\mathcal{E}$-nest specified by Proposition 3.4(iv) and $Y := \cup_{k \geq 1} F_k$. Then, by Proposition 3.1(vi) and Proposition 3.4,
(D.1) $u_1 > 0$ on Y.
(D.2) For any $u \in D$, there exists $c > 0$ such that $|u| \leq c u_1$ on Y.
(D.3) $D \subset C(\{F_k\})$ and D separates the points of Y.
(D.4) $u \wedge v, u \wedge 1, u \wedge (v+1), c_1 u + c_2 v \in D$ for all $u, v \in D$ and $c_1, c_2 \in Q$.

Set $g_n := (2/\pi)\arctan u_n, n \in \mathbf{N}$, and define a metric ρ on Y by

$$\rho(x,y) := \sum_{n=1}^{\infty} 2^{-n}|g_n(x) - g_n(y)|, \ x, y \in Y.$$

Since D separates the points of Y, (Y, ρ) is isometric to a subset of $[-1,1]^{\mathbf{N}}$ and thus the completion $(\bar{Y}, \rho)$ is a compact metric space. All g_n, u_n have unique continuous extensions $\bar{g}_n, \bar{u}_n$ to $\bar{Y}$ and, clearly, $\{\bar{g}_n | n \in \mathbf{N}\}$ separates the points of $\bar{Y}$ and so does $\{\bar{u}_n | n \in \mathbf{N}\}$. Set $Y^\sharp := \{x \in \bar{Y} | \bar{u}_1(x) > 0\}$. Then $(Y^\sharp, \rho)$ is a locally compact separable metric space. By (D.1), $Y \subset Y^\sharp$. For each $n \in \mathbf{N}$, we denote by $u_n^\sharp$ the restriction of $\bar{u}_n$ to $Y^\sharp$. Set $D^\sharp := \{u_n^\sharp | n \in \mathbf{N}\}$. We claim that

$$D^\sharp \text{ is dense in } C_\infty(Y^\sharp) \text{ w.r.t. the uniform norm } \|\cdot\|_\infty, \tag{3.1}$$

where $C_\infty(Y^\sharp) := \{f \in C(Y^\sharp) | \{f \geq \varepsilon\} \text{ is compact for any } \varepsilon > 0\}$.

For $u, v \in D$ and $c_1, c_2 \in Q$, by the uniqueness of continuous extensions, $u^\sharp \wedge v^\sharp = (u \wedge v)^\sharp, u^\sharp \wedge 1 = (u \wedge 1)^\sharp, u^\sharp \wedge (v^\sharp + 1) = (u \wedge (v+1))^\sharp$, and $c_1 u^\sharp + c_2 v^\sharp = (c_1 u + c_2 v)^\sharp$. Hence $D^\sharp$ is a Q-linear lattice satisfying

$$u^\sharp \wedge v^\sharp, u^\sharp \wedge 1, u^\sharp \wedge (v^\sharp + 1) \in D^\sharp, \ \forall u^\sharp, v^\sharp \in D^\sharp. \tag{3.2}$$

Set $\tilde{D}^\sharp := \{u^\sharp + r | u^\sharp \in D^\sharp, r \in Q\}$. Then, one can check that $\tilde{D}^\sharp$ is a Q-linear lattice by (3.2). Since $u_1^\sharp \in D^\sharp$ is strictly positive on $Y^\sharp$ and $D^\sharp$ separates the points of $Y^\sharp$, (3.1) holds by the Stone-Weierstrass theorem. Now assertions (i), (ii) and (iii) can be proved in the same way as in [MR, Theorem VI.l.2]. □

Let $\phi \in L^2(E;m)$ be such that $0 < \phi \leq 1$ m-a.e. and $\phi^\sharp$ the corresponding element of ϕ in $L^2(Y^\sharp; m^\sharp)$. Following [MOR, Definition 2.11], we introduce the capacity Cap_ϕ (respectively, $\mathrm{Cap}^\sharp_{\phi^\sharp}$) w.r.t. $(\mathcal{E}, D(\mathcal{E}))$ (respectively, $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$).

Corollary 3.6. *(i) If $\{E_k\}_{k\in\mathbf{N}}$ is an $\mathcal{E}^\sharp$-nest, then $\{F_k \cap E_k\}_{k\in\mathbf{N}}$ is an $\mathcal{E}$-nest and vice versa.*
(ii) $N^\sharp \subset Y^\sharp$ is $\mathcal{E}^\sharp$-exceptional if and only if $N^\sharp \cap Y$ is $\mathcal{E}$-exceptional. In particular, $\mathrm{cap}^\sharp_{\phi^\sharp}(Y^\sharp \backslash Y) = 0$.
(iii) A function $u^\sharp : Y^\sharp \to \mathbf{R}$ is $\mathcal{E}^\sharp$-quasi-continuous if and only if $u^\sharp \circ i$ is $\mathcal{E}$-quasi-continuous.
(iv) $\mathrm{cap}^\sharp_{\phi^\sharp}(A^\sharp) = \mathrm{cap}_\phi(A^\sharp \cap Y)$, $\forall A^\sharp \subset Y^\sharp$.

Proof. The proof is similar to the case of Dirichlet forms (cf. [MR, Corollary VI.1.4]). □

Now let $m^\sharp$ be a σ-finite Borel measure on $E^\sharp$, $(\mathcal{E}, D(\mathcal{E}))$ and $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$ two semi-Dirichlet forms on $L^2(E;m)$ and $L^2(E^\sharp; m^\sharp)$, respectively. All the notations w.r.t. $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$ will be marked by "$\sharp$".

Definition 3.7. $(\mathcal{E}, D(\mathcal{E}))$ is said to be *quasi-homeomorphic* to $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$, if there exists a map $j : \cup_{k\geq 1} F_k \to \cup_{k \geq 1} F_k^\sharp$, where $\{F_k\}_{k\in\mathbf{N}}$ is an $\mathcal{E}$-nest in E and $\{F_k^\sharp\}_{k\in\mathbf{N}}$ an $\mathcal{E}^\sharp$-nest in $E^\sharp$, such that
(i) j is a topological homeomorphism from F_k onto $F_k^\sharp$ for each $k \in \mathbf{N}$.
(ii) $m^\sharp = m \circ j^{-1}$.

(iii) $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp)) = (\mathcal{E}^j, D(\mathcal{E}^j))$, where $(\mathcal{E}^j, D(\mathcal{E}^j))$ is the image of $(\mathcal{E}, D(\mathcal{E}))$ under j.
The map j is called a *quasi-homeomorphism* from $(\mathcal{E}, D(\mathcal{E}))$ to $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$.

Theorem 3.8. *A semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ is quasi-regular if and only if it is quasi-homeomorphic to a regular semi-Dirichlet form $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$ on $L^2(E^\sharp;m^\sharp)$.*

Proof. (i) "if"-part: Similar to the setting of Dirichlet forms (cf. [CMR, Theorem 3.7]).
(ii) "only if"-part: Direct consequence of Theorem 3.5. □

Theorem 3.9. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form on $L^2(E;m)$. Suppose that $u \in \tilde{D}(\mathcal{E})$ and u is constant $\mathcal{E}$-q.e. on a quasi-open set U of E. Set $\mathcal{L}_U := \{v \in \tilde{D}(\mathcal{E}) | \operatorname{supp}_q[v] \subset U\}$. Then, there exists a unique σ-finite signed Borel measure J_u on U such that*

$$\mathcal{E}(u,v) = \int_U v(y) J_u(dy) \text{ for all } v \in \mathcal{L}_U \tag{3.3}$$

and J_u charges no $\mathcal{E}$-exceptional sets.

Proof. Suppose that $u|_U = \alpha$ $\mathcal{E}$-q.e. for some constant α. We first prove the theorem under the additional assumption that $u \leq \alpha$ $\mathcal{E}$-q.e. The basic idea of the proof is from [DM, Theorem 1]. For $v \in \mathcal{L}_U$, define $Lv = \mathcal{E}(u,v)$. Then L is a linear functional on $\mathcal{L}_U$ satisfying
(i) If $v \in \mathcal{L}_U$ and $v \geq 0$ $\mathcal{E}$-q.e., then $Lv \geq 0$.
(ii) If $\{v_n\}_{n\in\mathbf{N}} \subset \mathcal{L}_U$ and $\mathcal{E}_1(v_n, v_n) \to 0$ as $n \to \infty$, then $Lv_n \to 0$ as $n \to \infty$.
Assertion (ii) is obvious. Assertion (i) is true since $Lv = \lim_{\beta\to\infty} \beta(u - \beta G_\beta u, v) = \lim_{\beta\to\infty} \beta(\alpha - \beta G_\beta u, v) \geq 0$.

Suppose that $\{v_n\}_{n\in\mathbf{N}} \subset \mathcal{L}_U$ is a decreasing sequence such that $v_n(x) \downarrow 0$ for all $x \in E$. We will show that $Lv_n \downarrow 0$. To this end, set $\mathcal{L} := \{f \in \tilde{D}(\mathcal{E}) | f \geq v_1 \ m\text{-a.e.}\}$. By [MOR, Proposition 2.8] (replacing U with E), there exists a unique $v \in \mathcal{L}$ such that $\mathcal{E}_1(v,v) \leq \mathcal{E}_1(v,f)$, $\forall f \in \mathcal{L}$; $\mathcal{E}_1(v,w) \geq 0$, $\forall w \in D(\mathcal{E})$ satisfying $w \geq 0\, m$-a.e. Hence v is 1-excessive (cf. [MOR, Theorem 2.4]). By the quasi-regularity of $(\mathcal{E}, D(\mathcal{E}))$, there exists an $\mathcal{E}$-nest $\{F_k\}_{k\in\mathbf{N}}$ consisting of compact sets such that $v_n \in C(\{F_k\})$ for each $n \in \mathbf{N}$. Let $F_k^c := E \backslash F_k$ and $v_{F_k^c}$ be the 1-reduced function of v on F_k^c (cf. [MOR, Proposition 2.8]). By [MOR, Proposition 2.8] and [MR, Lemma I.2.12], one can check that $v_{F_k^c}$ converges weakly to 0 in $(D(\mathcal{E}), \tilde{\mathcal{E}}_1)$ as $k \to \infty$. Since $v_{F_k^c}$ is decreasing (cf. [MOR, Proposition 2.8 (iv)]) and 1-excessive,

$$\mathcal{E}_1(v_{F_k^c}, v_{F_k^c}) \leq \mathcal{E}_1(v_{F_k^c}, v_{F_1^c}) \to 0.$$

Set $u_k := v_1 \wedge \tilde{v}_{F_k^c}$. It is easy to see that $\sup_{k\in\mathbf{N}} \mathcal{E}(u_k, u_k) < \infty$ and $\lim_{k\to\infty} \|u_k\|_{L^2(E;m)} = 0$. Then, by [MR, Lemma I.2.12], there exists a subsequence $\{u_{k_l}\}_{l\in\mathbf{N}}$ of $\{u_k\}_{k\in\mathbf{N}}$ such that the Cesàro sum $w_k := (1/k)\sum_{l=1}^k u_{k_l}$ converges to 0 in $D(\mathcal{E})$, i.e. $\mathcal{E}_1(w_k, w_k) \to 0$, as $k \to \infty$. By the definition of $\mathcal{L}_U$, we know that $w_k, v_1 \wedge \tilde{v} \in \mathcal{L}_U$. By [Ku, Lemma 2.1(ii)], $\mathcal{E}_1((v_1 \wedge \tilde{v}) \wedge (1/j), (v_1 \wedge \tilde{v}) \wedge (1/j)) \to 0$ as $j \to \infty$. By assertion (ii), for arbitrary $\delta > 0$, there exist k_0, j_0 such that $L(w_k) \leq \delta$, $\forall k \geq k_0$, and $L((v_1 \wedge \tilde{v}) \wedge (1/j)) \leq \delta$, $\forall j \geq j_0$. Since $v_n \downarrow 0$ and v_n is continuous on the compact set F_{k_0}, there exists $n_0 \in N$ such that $v_n \leq (1/j_0)$ on F_{k_0} for any $n \geq n_0$ and thus $v_n \leq (v_1 \wedge \tilde{v}) \wedge (1/j_0) + w_{k_0}$ $\mathcal{E}$-q.e. Hence $Lv_n \leq L((v_1 \wedge \tilde{v}) \wedge (1/j_0) + w_{k_0}) \leq 2\delta$, $\forall n \geq n_0$, i.e. $Lv_n \downarrow 0$ as $n \to \infty$.

Since $\mathcal{L}_U$ is a linear lattice, L is a Daniell integral on $\mathcal{L}_U$. Then, there exists a Borel measure J_u on $\sigma\{v : v \in \mathcal{L}_U\}$ satisfying (3.3) by Daniell's theorem. Let N be an arbitrary $\mathcal{E}$-exceptional set. Since $I_N = 0$ $\mathcal{E}$-q.e., $I_N \in \mathcal{L}_U$ and $\int_E I_N(x)J_U(dx) = LI_N = 0$ by assertion (i). Thus $J_u(N) = 0$.

Through the "local-compactification" of quasi-regular semi-Dirichlet forms (cf. Theorem 3.5), we can find two $\mathcal{E}$-nests $\{F_k^{(1)}\}_{k\in\mathbf{N}}$ and $\{F_k^{(2)}\}_{k\in\mathbf{N}}$ satisfying that for any $k, m \in \mathbf{N}$ and any compact set $F \subset F_k^{(1)} \cap F_m^{(2)} \cap U$, there exists a sequence $\{s_n\}_{n\in\mathbf{N}}$ of $\mathcal{E}$-quasi-continuous elements in $D(\mathcal{E})$ such that $s_n|_F \equiv 1, s_n \downarrow I_F$, and $\mathrm{supp}_q[s_n] \subset U$ (cf. the existence part of Theorem 4.1 below for a detailed proof). Hence $F \in \sigma(v : v \in \mathcal{L}_U)$ and

$$J_u(F) = \lim_{n\to\infty} \int_U s_n(y)J_u(dy) = \lim_{n\to\infty} \mathcal{E}(u, s_n). \tag{3.4}$$

Since k, m and F are arbitrary, $\mathcal{B}(\cup_{m\geq 1} \cup_{k\geq 1} F_k^{(1)} \cap F_m^{(2)} \cap U) \subset \sigma(v : v \in \mathcal{L}_U)$. Note that $N_1 := U\backslash(\cup_{k\geq 1} \cup_{m\geq 1} F_k^{(1)} \cap F_m^{(2)})$ is an $\mathcal{E}$-exceptional set. We define the Borel measure J_u on U by setting $J_u(N_1) = 0$. By (3.4), J_u is σ-finite and unique.

Now we consider the general case. Note that

$$\mathcal{E}(u, v) = \mathcal{E}(u - u \wedge \alpha, v) + \mathcal{E}(u \wedge \alpha, v) = -\mathcal{E}(u \wedge \alpha - u, v) + \mathcal{E}(u \wedge \alpha, v). \tag{3.5}$$

We respectively apply the above proof to $(u \wedge \alpha - u)$ and $u \wedge \alpha$, and obtain the corresponding Borel measures $J_{u\wedge\alpha-u}$ and $J_{u\wedge\alpha}$. Set $J_u = J_{u\wedge\alpha} - J_{u\wedge\alpha-u}$. Then, J_u is the desired signed Borel measure. The proof is complete. □

In the next section, we will employ the signed Borel measure J_u given in Theorem 3.9 and the local compactification method developed in Theorem 3.5 to obtain the jumping measure J and the killing measure K of a quasi-regular semi-Dirichlet form, see Theorem 4.1 below and its proof.

4. Decomposition of quasi-regular semi-Dirichlet form

Throughout this section, we let E be a metrizable Lusin space, m a σ-finite measure on $(E, \mathcal{B}(E))$ and $(\mathcal{E}, D(\mathcal{E}))$ a quasi-regular semi-Dirichlet form on $L^2(E; m)$. A metric ρ on E is called a *quasi-compatible metric* if the Borel σ-field induced by ρ coincides with $\mathcal{B}(E)$ and there exists an $\mathcal{E}$-nest $\{F_k\}_{k\in\mathbf{N}}$ such that ρ is compatible with the trace topology on F_k for each $k \in \mathbf{N}$.

Let J be a σ-finite positive Borel measure on $E \times E\backslash d$. A measurable function f on $E \times E\backslash d$ is said to be integrable w.r.t. J in the sense of *symmetric principle value* (abbreviated by *S.P.V. integrable*), if there exists an increasing sequence $\{A_n\}_{n\geq 1}$ of subsets of $E \times E\backslash d$ satisfying $J(\,(E \times E\backslash d) \setminus (\cup_n A_n)\,) = 0$, $I_{A_n}(x, y) = I_{A_n}(y, x)$ for all $x, y \in E$, $n \geq 1$, and f is integrable on each A_n, and for any sequence $\{A_n\}_{n\geq 1}$ with the above properties, the limit

$$S.P.V. \int_{E\times E\backslash d} f(x, y)J(dx, dy) := \lim_{n\to\infty} \int_{A_n} f(x, y)J(dx, dy)$$

exists and is independent of the specific choice of the sequence $\{A_n\}_{n\geq 1}$.

Theorem 4.1. *(i) There exist a unique σ-finite positive Borel measure J on $E \times E\backslash d$ and a unique σ-finite positive Borel measure K on E satisfying the following properties:*
(a) $J(N \times E\backslash d) = J(E \times N\backslash d) = 0$ and $K(N) = 0$ for any $\mathcal{E}$-exceptional set N.
(b) For $v \in \tilde{D}(\mathcal{E})$ and $u \in I_q(v)$,

$$\mathcal{E}(u,v) = \int_{E\times E\backslash d} 2(u(y)-u(x))v(y)J(dx,dy) + \int_E u(y)v(y)K(dy), \tag{4.1}$$

where $I_q[v] := \{u \in \tilde{D}(\mathcal{E}) | u$ is constant $\mathcal{E}$-q.e. on a quasi-open set containing $\mathrm{supp}_q[v]\}$.

(ii) Define

$$\tilde{\mathcal{A}}(v) := \{u \in \tilde{D}(\mathcal{E}) |\, (u(y)-u(x))v(y) \text{ is S.P.V. integrable w.r.t. } J \text{ and } u(x)v(x) \text{ is integrable w.r.t. } K\}. \tag{4.2}$$

Then we have the following unique decomposition

$$\begin{aligned}\mathcal{E}(u,v) &= \mathcal{E}^c(u,v) + S.P.V. \int_{E\times E\backslash d} 2(u(y)-u(x))v(y)J(dx,dy) \\ &\quad + \int_E u(x)v(x)K(dx) \quad \text{for } v \in \tilde{D}(\mathcal{E}) \text{ and } u \in \tilde{\mathcal{A}}(v),\end{aligned} \tag{4.3}$$

where $\mathcal{E}^c$ satisfies the left strong local property in the sense that $I_q[v] \subset \tilde{\mathcal{A}}(v)$ and $\mathcal{E}^c(u,v) = 0$ whenever $v \in \tilde{D}(\mathcal{E})$ and $u \in I_q(v)$.

Proof. (i) *Existence:* For $v \in \tilde{D}(\mathcal{E})$ and $u \in I_q(v)$, there exist a quasi-open set $U \supset \mathrm{supp}_q[v]$ and a constant α such that $u = \alpha$ $\mathcal{E}$-q.e. on U. To prove (4.1), we assume without loss of generality that $\alpha \geq 0$. Further, by (3.5), we can assume that $u \leq \alpha$ $\mathcal{E}$-q.e. By Theorem 3.9, there exists a unique σ-finite signed Borel measure J_u on U such that

$$\mathcal{E}(u,w) = \int_U w(y)J_u(dy) \tag{4.4}$$

for any $w \in \mathcal{L}_U = \{f \in \tilde{D}(\mathcal{E}) \,|\, \mathrm{supp}_q[f] \subset U\}$.

Let $\{F_k\}_{k\in\mathbf{N}}, Y := \cup_{k\geq 1} F_k$ and $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$ be specified by Theorem 3.5, where $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$ is a regular semi-Dirichlet form on $L^2(Y^\sharp; m^\sharp)$. Then, by Theorem 2.6, there exist a unique positive Radon measure $J^\sharp$ on $Y^\sharp \times Y^\sharp\backslash d$ and a unique positive Radon measure $K^\sharp$ on $Y^\sharp$ such that for $v^\sharp \in C_0(Y^\sharp) \cap D(\mathcal{E}^\sharp)$ and $u^\sharp \in I^\sharp(v^\sharp)$,

$$\mathcal{E}(u^\sharp, v^\sharp) = \int_{Y^\sharp\times Y^\sharp\backslash d} 2\left(u^\sharp(y) - u^\sharp(x)\right) v^\sharp(y)J^\sharp(dx,dy) + \int_{Y^\sharp} u^\sharp(y)v^\sharp(y)K^\sharp(dy),$$

where $I^\sharp(v^\sharp)$ is defined similarly to $I(v)$ as in Theorem 2.6.

Extend $J^\sharp|_{Y\times Y\backslash d}$ to a measure J on $E \times E\backslash d$ by setting $J(A) := J^\sharp(A \cap (Y \times Y\backslash d)), \forall A \in \mathcal{B}(E \times E\backslash d)$, and extend $K^\sharp|_Y$ to a measure K on E by setting $K(B) := K^\sharp(B \cap Y), \forall B \in \mathcal{B}(E)$. We will show that on the quasi-open set U,

$$J_u(dy) = \int_E \{2(u(y)-u(x))J(dx,dy) + u(y)K(dy)\}. \tag{4.5}$$

Note that the measures $\int_E 2(u(y)-u(x))J(dx,dy)$ and $u(y)K(dy)$ are nonnegative on U by the assumptions that $u|_U=\alpha$, $u\le\alpha$, $\mathcal{E}$-q.e., and $\alpha\ge 0$. Then, (4.1) follows from (4.4) and (4.5). In the following, we show that (4.5) holds.

Since U is quasi-open, there exists an $\mathcal{E}$-nest $\{F_k^U\}_{k\in\mathbf{N}}$ such that $F_k^U\cap U$ is open relative to F_k^U for each $k\in\mathbf{N}$. Set $F_k^{(1)}:=F_k^U\cap F_k$. Then $\{F_k^{(1)}\}_{k\in\mathbf{N}}$ is an $\mathcal{E}$-nest and $F_k^{(1)}\cap U$ is open relative to $F_k^{(1)}$. Let h be specified by Proposition 3.1(vi). Set $g_l:=h-h_{(F_l^{(1)})^c}\wedge h$, where $(F_l^{(1)})^c:=E\backslash F_l^{(1)}$. We fix an $\mathcal{E}$-quasi-continuous version $\tilde{g}_l$ of g_l such that $\tilde{g}_l|_{(F_l^{(1)})^c}=0$. Since $\tilde{g}_l$ is $\mathcal{E}_1$-convergent to h as $l\to\infty$ (cf. [MOR, Proposition 2.18(i)]), there exist a subsequence of $\{\tilde{g}_l\}_{l\in\mathbf{N}}$, which we still denote by $\{\tilde{g}_l\}_{l\in\mathbf{N}}$, and an $\mathcal{E}$-nest $\{F_k^{(2)}\}_{k\in\mathbf{N}}$ such that $F_k^{(2)}\subset F_k$ and $\tilde{g}_l$ converges to h uniformly on each $F_k^{(2)}$ as $l\to\infty$.

Since the trace topologies on F_k induced by E and $Y^\sharp$ are the same, $Y^\sharp$ is a locally compact separable metric space and J_u charges no $\mathcal{E}$-exceptional sets, it is sufficient to show that for any $k,m\in\mathbf{N}$ and any compact set $F\subset F_m^{(2)}\cap F_k^{(1)}\cap U$,

$$J_u(F)=\int_F\left(\int_E\{2(u(y)-u(x))v(y)J(dx,dy)+u(y)K(dy)\}\right). \tag{4.6}$$

Since $\inf\{h(x)|x\in F_m\}>0$ (cf. Proposition 3.1(vi)), $F_m^{(2)}\subset F_m$, and $\tilde{g}_l$ converges to h uniformly on each $F_m^{(2)}$, there exist $l>k$ and a constant $\delta_l>0$ such that $\tilde{g}_l\ge\delta_l$ on $F_m^{(2)}$. Set $g_F:=((1/\delta_l)\tilde{g}_l)\wedge 1$. Then, $g_F|_{F_m^{(2)}}\equiv 1$ and $g_F|_{(F_l^{(1)})^c}\equiv 0$.

Since F is compact and $F_l^{(1)}\cap U$ is open in $F_l^{(1)}$, there exists an open set G_l (relative to $F_l^{(1)}$) such that $F\subset G_l\subset\bar{G}_l^{F_l^{(1)}}\subset F_l^{(1)}\cap U$, where $\bar{G}_l^{F_l^{(1)}}$ is the closure of G_l in $F_l^{(1)}$. Since F is also compact in $Y^\sharp$ and $G_l\cup(Y^\sharp\backslash F_l^{(1)})$ is open in $Y^\sharp$, by the regularity of $(\mathcal{E}^\sharp,D(\mathcal{E}^\sharp))$, there exists a sequence $\{f_n^\sharp\}_{n\in\mathbf{N}}\subset C_0(Y^\sharp)\cap D(\mathcal{E}^\sharp)$ such that $f_n^\sharp\ge 0$, $f_n^\sharp\downarrow I_F$, and $\mathrm{supp}[f_n^\sharp]\subset G_l\cup(Y^\sharp\backslash F_l^{(1)})$. Define f_n to be $f_n^\sharp$ on Y and zero on $E\backslash Y$ ($Y=\cup_{k\ge1}F_k$). Then $f_n\in D(\mathcal{E})$ (cf. Corollary 3.6(iii)). Set $s_n:=f_n\wedge g_F$. Then $s_n|_F\equiv 1, s_n\downarrow I_F$, and $\{x\in E|s_n(x)\neq 0\}\subset G_l\subset\bar{G}_l^{F_l^{(1)}}\subset F_l^{(1)}\cap U\subset U$. Since $F_l^{(1)}\subset F_l$ and F_l is compact, $\bar{G}_l^{F_l^{(1)}}$ is a compact set. Consequently, $\mathrm{supp}_q[s_n]\subset$ q.e. $\mathrm{supp}[s_n]\subset\bar{G}_l^{F_l^{(1)}}\subset U$, where "$\subset$ q.e." means "$\subset$" except for an $\mathcal{E}$-exceptional set. Thus $s_n\in\mathcal{L}_U$ and

$$J_u(F)=\lim_{n\to\infty}\int_E s_n(y)J_u(dy)=\lim_{n\to\infty}\mathcal{E}(u,s_n). \tag{4.7}$$

Define $u^\sharp$ to be u on Y and zero on $Y^\sharp\backslash Y$. Similarly, define $s_n^\sharp$ to be s_n on Y and zero on $Y^\sharp\backslash Y$. Then, $u^\sharp,s_n^\sharp\in D(\mathcal{E}^\sharp)$. Since for each $k\in\mathbf{N}$, the trace topologies on F_k induced by E and $Y^\sharp$ are the same, $\mathrm{supp}[s_n^\sharp]\subset\bar{G}_l^{F_l^{(1)}}\subset U\cap Y$. It is easy to see that $U\cap Y$ is a quasi-open set w.r.t. $(\mathcal{E}^\sharp,D(\mathcal{E}^\sharp))$. Since $u^\sharp|_{U\cap Y}=u|_{U\cap Y}$, by Corollary 3.6, $u^\sharp=\alpha$ $\mathcal{E}^\sharp$-q.e. on $U\cap Y$. By the definition of $s_n^\sharp$, we know that $s_n^\sharp$ is bounded and $\{x\in Y^\sharp|s_n^\sharp\neq 0\}\subset\mathrm{supp}[s_n]\subset\bar{G}_l^{F_l^{(1)}}\subset F_l^{(1)}\subset F_l$. Now by Theorem 2.9 and Remark 2.7(i) we get

$$\mathcal{E}^\sharp(u^\sharp,s_n^\sharp)=\int_{Y^\sharp\times Y^\sharp\backslash d}2(u^\sharp(y)-u^\sharp(x))s_n^\sharp(y)J^\sharp(dx,dy)+\int_{Y^\sharp}u^\sharp(y)s_n^\sharp(y)K^\sharp(dy)$$

$$= \int_{Y\times Y\backslash d} 2(u^\sharp(y)-u^\sharp(x))s_n^\sharp(y)J^\sharp(dx,dy)+\int_Y u^\sharp(y)s_n^\sharp(y)K^\sharp(dy). \quad (4.8)$$

By the definitions of J and K and Theorem 3.5, we obtain from (4.8) that

$$\begin{aligned}\mathcal{E}(u,s_n) &= \mathcal{E}^\sharp(u^\sharp,s_n^\sharp)\\ &= \int_{E\times E\backslash d} 2(u(y)-u(x))s_n(y)J(dx,dy)+\int_E u(y)s_n(y)K(dy). \quad (4.9)\end{aligned}$$

By (4.7), (4.9) and the dominated convergence theorem, we obtain (4.6).

Since J_u charges no $\mathcal{E}$-exceptional sets, it is easy to show that property (a) holds (this can also be deduced by the definitions of J and K and Remark 2.7(i)), which completes the proof of the existence.

Uniqueness: Let $J^\sharp$ and $K^\sharp$ be as in the existence part. Suppose that there exists another pair of measures J' and K' satisfying properties (a) and (b). Extend $J'|_{Y\times Y\backslash d}$ to a measure J^* on $Y^\sharp\times Y^\sharp\backslash d$ by setting $J^*(A):=J'(A\cap(Y\times Y\backslash d))$ for any $A\in\mathcal{B}(Y^\sharp\times Y^\sharp\backslash d)$. Similarly, extend K' to a measure K^* on $Y^\sharp$. For $v^\sharp\in C_0(Y^\sharp)\cap D(\mathcal{E}^\sharp), u^\sharp\in I^\sharp(v^\sharp)$, define v to be $v^\sharp$ on Y and zero on $E\backslash Y$. Similarly, we define u. By Corollary 3.6, one can easily check that $u,v\in\tilde{D}(\mathcal{E})$ and $u\in I_q(v)$. By Theorem 2.6, Theorem 3.5 and Remark 2.7(i),

$$\begin{aligned}&\int_{Y^\sharp\times Y^\sharp\backslash d} 2(u^\sharp(y)-u^\sharp(x))v^\sharp(y)J^\sharp(dx,dy)+\int_{Y^\sharp} u^\sharp(y)v^\sharp(y)K^\sharp(dy)\\ &= \mathcal{E}^\sharp(u^\sharp,v^\sharp)\\ &= \mathcal{E}(u,v)\\ &= \int_{E\times E\backslash d} 2(u(y)-u(x))v(y)J'(dx,dy)+\int_E u(y)v(y)K'(dy)\\ &= \int_{Y^\sharp\times Y^\sharp\backslash d} 2(u^\sharp(y)-u^\sharp(x))v^\sharp(y)J^*(dx,dy)+\int_{Y^\sharp} u^\sharp(y)v^\sharp(y)K^*(dy).\end{aligned}$$

It follows that $J^\sharp=J^*$ on $Y^\sharp\times Y^\sharp\backslash d$ and $K^\sharp=K^*$ on $Y^\sharp$. Then $J=J'$ on $Y\times Y\backslash d$ and $K=K'$ on Y. Since $E\backslash Y$ is an $\mathcal{E}$-exceptional set, $J=J'$ and $K=K'$ by property (a), which completes the proof.

(ii) Let J and K be the measures specified by (i). For $v\in\tilde{D}(\mathcal{E})$, we define $\tilde{\mathcal{A}}(v)$ by (4.2). Then, for $v\in\tilde{D}(\mathcal{E})$ and $u\in\tilde{\mathcal{A}}(v)$, we obtain decomposition (4.3) by simply setting

$$\mathcal{E}^c(u,v):=\mathcal{E}(u,v)-S.P.V.\int_{E\times E\backslash d} 2(u(y)-u(x))v(y)J(dx,dy)-\int_E u(x)v(x)K(dx).$$

By the proof of (i), one finds that for any $v\in\tilde{D}(\mathcal{E})$ and $u\in I_q[v]$, $(u(y)-u(x))v(y)$ is integrable w.r.t. J (and thus S.P.V. integrable w.r.t. J) and $u(x)v(x)$ is integrable w.r.t. K. Then $I_q[v]\subset\tilde{\mathcal{A}}(v)$. Further, by (4.1) and (4.3), we know that $\mathcal{E}^c(u,v)=0$ whenever $v\in\tilde{D}(\mathcal{E})$ and $u\in I_q[v]$. Hence $\mathcal{E}^c$ satisfies the left strong local property.

Now we show the uniqueness of decomposition (4.3). For $v\in\tilde{D}(\mathcal{E})$ and $u\in I_q[v]$, we have

$$\mathcal{E}(u,v)=S.P.V.\int_{E\times E\backslash d} 2(u(y)-u(x))v(y)J(dx,dy)+\int_E u(x)v(x)K(dx). \quad (4.10)$$

By the definition of $I_q[v]$, there exist a quasi-open set $U \supset \mathrm{supp}_q[v]$ and a constant α such that $u|_U = \alpha$ $\mathcal{E}$-q.e. As in the existence part of (i), without loss of generality, we can assume that $v \geq 0$, $\alpha \geq 0$ and $u \leq \alpha$. Let $\{A_n\}_{n\geq 1}$ be an increasing sequence of subsets of $E \times E \backslash d$ as in the definition of "S.P.V. integrable" such that

$$S.P.V. \int_{E\times E\backslash d} 2(u(y)-u(x))v(y)J(dx,dy) = \lim_{n\to\infty} \int_{A_n} 2(u(y)-u(x))v(y)J(dx,dy). \quad (4.11)$$

Noting that $(u(y)-u(x))v(y) \geq 0$ $\mathcal{E}$-q.e., we obtain from property (a) of (i), Fatou's Lemma and (4.11) that

$$\begin{aligned} &\int_{E\times E\backslash d} 2(u(y)-u(x))v(y)J(dx,dy) \\ &\quad = \int_{E\times E\backslash d} \lim_{n\to\infty} 2(u(y)-u(x))v(y)I_{A_n}(x,y)J(dx,dy) \\ &\quad \leq \lim_{n\to\infty} \int_{A_n} 2(u(y)-u(x))v(y)J(dx,dy) \\ &\quad < \infty. \end{aligned}$$

Then $2(u(y)-u(x))v(y)$ is integrable w.r.t. J on $E \times E\backslash d$. Thus the uniqueness of J and K follows from (4.10) and (i) and therefore decomposition (4.3) is unique. □

Theorem 4.1 is an extension of the classical Beurling-Deny formula (cf. (1.5)), noting that if $(\mathcal{E}, D(\mathcal{E}))$ is a regular symmetric Dirichlet form then $\tilde{\mathcal{A}}(v) = \tilde{D}(\mathcal{E})$ for any $v \in \tilde{D}(\mathcal{E})$ and

$$\begin{aligned} &S.P.V. \int_{E\times E\backslash d} 2(u(y)-u(x))v(y)J(dx,dy) \\ &\qquad = \int_{E\times E\backslash d} (u(y)-u(x))(v(y)-v(x))J(dx,dy). \end{aligned}$$

As in the case of Lévy processes (cf. [HMS, Example 4.1]), we can find some sufficient conditions to ensure that decomposition (4.3) holds for all u, v in a special quasi-core (cf. Theorem 4.8 below), which is defined as follows.

Definition 4.2. A subset $\tilde{D}$ of $\tilde{D}(\mathcal{E})$ is called a *quasi-core* of $(\mathcal{E}, D(\mathcal{E}))$ if the following conditions hold:
(QC.1) $\tilde{D}$ is dense in $D(\mathcal{E})$ w.r.t. the $\tilde{\mathcal{E}}_1^{1/2}$-norm;
(QC.2) $\tilde{D}$ is a linear lattice and $u, v \in \tilde{D}$ implies $u \wedge 1, u \wedge (v+1) \in \tilde{D}$;
(QC.3) There exist a countable family $\{u_n\}_{n\in\mathbf{N}} \subset \tilde{D}$ and an $\mathcal{E}$-exceptional set N such that $\{u_n\}_{n\in\mathbf{N}}$ separates the points of $E \setminus N$.
$\tilde{D}$ is said to be a *special quasi-core* if in addition to (QC.1)-(QC.3), it holds that
(QC.4) For any $v \in \tilde{D}$, there exists $u \in \tilde{D}$ such that $u = 1$ $\mathcal{E}$-q.e. on a quasi-open set containing $\mathrm{supp}_q[v]$.

Note that by (QC.2), if $\tilde{D}$ is a quasi-core, then it satisfies
(QC.2′) $u \in \tilde{D}$ implies $u^+ \wedge 1 \in \tilde{D}$, hereafter $u^+ := u \vee 0$.

Let h, $\hat{h}$ and $\{F_k^h\}_{k\in\mathbf{N}}$ be specified by Proposition 3.1(vi). By the quasi-regularity of $(\mathcal{E}, D(\mathcal{E}))$, we can assume that F_k^h is compact for each $k \in \mathbf{N}$. For $k \in \mathbf{N}$, set $h_k := h - h_{(F_k^h)^c} \wedge h$. We fix an $\mathcal{E}$-quasi-continuous m-version $\tilde{h}_k$ of h_k such that $\tilde{h}_k|_{(F_k^h)^c} = 0$. Since $\tilde{h}_k$ converges to h in $D(\mathcal{E})$ as $k \to \infty$, by [MOR, Proposition 2.18(i)], there exist a subsequence of $\{\tilde{h}_k\}$, which we denote again by $\{\tilde{h}_k\}$, and an $\mathcal{E}$-nest $\{F_l^{(1)}\}_{l\in\mathbf{N}}$ such that $\tilde{h}_k$ converges to h uniformly on each $F_l^{(1)}$ as $k \to \infty$. Without loss of generality we may assume that $F_k^{(1)} \subset F_k^h$ for each k. Then for each $F_k^{(1)}$ we can find an $\tilde{h}_j$, for some large enough j, such that $\inf\{\tilde{h}_j(x)|x \in F_k^{(1)}\} > 0$. Let D_0^+ be specified by Lemma 3.2. For $u \in D_0^+$ and $k \in \mathbf{N}$, set $u_k := u - u_{(F_k^{(1)})^c} \wedge u$. We fix an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}_k$ of u_k such that $\tilde{u}_k|_{(F_k^{(1)})^c} = 0$. Define

$$D_2' := \{\tilde{u}_k | u \in D_0^+, k \in \mathbf{N}\} \cup \{\tilde{h}_k | k \in \mathbf{N}\} \cup \{0\} \tag{4.12}$$

and

$$D_2 := \{u - u \wedge \varepsilon \mid u \in D_2', \varepsilon \in Q_+\}, \tag{4.13}$$

where 0 is the constant function 0, Q_+ is the set of all positive rational numbers. Note that $(D_2 - D_2)$ is a countable set and is dense in $D(\mathcal{E})$. Hence there exists an $\mathcal{E}$-nest $\{F_k^{(2)}\}_{k\in\mathbf{N}}$ such that $(D_2 - D_2)$ separates the points of $\cup_{k\geq 1} F_k^{(2)}$. We now slightly modify the proof of Theorem 3.5 by adding $D_2' \cup (D_2 - D_2) \cup \{\hat{h}\}$ to D and modifying $\{F_k\}_{k\in\mathbf{N}}$ so that $F_k \subset F_k^{(1)} \cap F_k^{(2)}$ for each k and $D_2' \cup (D_2 - D_2) \cup \{\hat{h}\} \subset C(\{F_k\})$. We can check that with the above modification the proof of Theorem 3.5 is still valid provided that we set $u_1 = \hat{h}$.

Let J be specified by Theorem 4.1. Let $Y = \cup_{k=1}^{\infty} F_k, Y^\sharp, m^\sharp$ and $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$ be as in Theorem 3.5 with the above enlarged D and modified $\{F_k\}_{k\in\mathbf{N}}$. Define

$$\begin{aligned} D_1 := \{u \in \tilde{D}_b(\mathcal{E}) \,|\, u = u^\sharp \text{ on } Y \text{ for some } u^\sharp \in D(\mathcal{E}^\sharp) \\ \text{such that } \mathrm{supp}[u^\sharp] \text{ is compact in } Y^\sharp\}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} D_1' := \{u \in \cup_{k\geq 1} D(\mathcal{E})_{F_k^h} \,|\, u = u_1 - u_2 \text{ for two bounded} \\ \text{1-excessive functions } u_1, u_2 \in \tilde{D}(\mathcal{E})\} \end{aligned} \tag{4.15}$$

and

$$D_1'' := \left\{ u \in \tilde{D}_b(\mathcal{E}) \,\middle|\, \int_{E\times E\backslash d} (u(y) - u(x))^2 \hat{h}(y) J(dx, dy) < \infty \right\}, \tag{4.16}$$

where $\tilde{D}_b(\mathcal{E})$ denotes all the bounded elements in $\tilde{D}(\mathcal{E})$.

Lemma 4.3. $(D_2 - D_2) \subset D_1 \cap D_1' \cap D_1''$.

Proof. By the construction of D_2 above and the definitions of D_1 and D_1', we have that $(D_2 - D_2) \subset D_1 \cap D_1'$. In the following, we will show that $(D_2 - D_2) \subset D_1''$. Let u be an arbitrary function of $D_2 - D_2$. Then there exist two bounded 1-excessive functions $u_1, u_2 \in D(\mathcal{E})$ and some $k \in \mathbf{N}$ such that $u = u_1 - u_2$ and $u \in D(\mathcal{E})_{F_k^h}$. We claim that

$$\begin{aligned} &\int_{E\times E\backslash d}(u(y)-u(x))^2\hat{h}(y)J(dx,dy)\\ &\qquad \le \ \|\hat{h}I_{F_k^h}\|_\infty\left[\mathcal{E}_1(u_1+u_2,u_1+u_2)+(\|u_1\|_{L^2(E;m)}+\|u_2\|_{L^2(E;m)})^2+\frac{1}{2}\|u\|_{L^2(E;m)}\right]. \end{aligned} \tag{4.17}$$

The notations w.r.t. $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$ are marked by "$\sharp$". Since $u \in D(\mathcal{E})_{F_k^h}$, by Theorem 3.5 and Corollary 3.6,

$$\begin{aligned} \beta(u-\beta G_\beta u, u\hat{h}) &= \beta(u^\sharp, u^\sharp\hat{h}^\sharp) - \beta(\beta G_\beta^\sharp u^\sharp, u^\sharp\hat{h}^\sharp)\\ &= \beta\int_{Y^\sharp}(u^\sharp(x))^2\hat{h}^\sharp(x)m^\sharp(dx) - \beta\int_{Y^\sharp\times Y^\sharp}u^\sharp(x)u^\sharp(y)\hat{h}^\sharp(y)\sigma_\beta^\sharp(dx,dy)\\ &= \beta\int_{F_k^h\cap Y}(u^\sharp(x))^2\hat{h}^\sharp(x)m^\sharp(dx) - \beta\int_{(F_k^h\cap Y)\times(F_k^h\cap Y)}u^\sharp(x)u^\sharp(y)\hat{h}^\sharp(y)\sigma_\beta^\sharp(dx,dy)\\ &= \frac{\beta}{2}\int_{(F_k^h\cap Y)\times(F_k^h\cap Y)}(u^\sharp(x)-u^\sharp(y))^2\hat{h}^\sharp(y)\sigma_\beta^\sharp(dx,dy)\\ &\quad +\beta\int_{F_k^h\cap Y}(u^\sharp(x))^2\left[\hat{h}^\sharp(x)-\frac{\hat{h}^\sharp(x)}{2}\beta G_\beta^\sharp I_{F_k^h\cap Y}(x)\right.\\ &\qquad\qquad \left.-\frac{1}{2}\beta\hat{G}_\beta^\sharp(\hat{h}^\sharp\cdot I_{F_k^h\cap Y})(x)\right]m^\sharp(dx), \end{aligned} \tag{4.18}$$

where $\sigma_\beta^\sharp$ is the positive Radon measure on $Y^\sharp$ such that for $u^\sharp, v^\sharp \in L^2(Y^\sharp, m^\sharp)$ (cf. Corollary 2.2),

$$(\beta G_\beta^\sharp u^\sharp, v^\sharp) = \int_{Y^\sharp}u^\sharp(x)v^\sharp(y)\sigma_\beta^\sharp(dx,dy).$$

Since $\hat{h}$ is 1-coexcessive w.r.t. $(\mathcal{E}, D(\mathcal{E}))$ (cf. Proposition 3.1(vi)), $\hat{h}^\sharp$ is 1-coexcessive w.r.t. $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$. Hence, for $\beta > 0$, $\beta\hat{G}_{\beta+1}^\sharp\hat{h}^\sharp \le \hat{h}^\sharp$ $m^\sharp$-a.e. Then, one obtains from (4.18) that

$$\begin{aligned} &\lim_{\beta\to\infty}\frac{\beta}{2}\int_{(F_k^h\cap Y)\times(F_k^h\cap Y)}(u^\sharp(x)-u^\sharp(y))^2\hat{h}^\sharp(y)\sigma_\beta^\sharp(dx,dy)\\ &\qquad\qquad \le \ \lim_{\beta\to\infty}\beta(u-\beta G_\beta u, u\hat{h}) + \frac{1}{2}\int_{F_k^h\cap Y}(u^\sharp(x))^2\hat{h}^\sharp(x)m^\sharp(dx)\\ &\qquad\qquad \le \ \lim_{\beta\to\infty}\beta(u-\beta G_\beta u, u\hat{h}) + \frac{1}{2}\int_E u^2(x)\hat{h}(x)m(dx). \end{aligned} \tag{4.19}$$

Note that

$$\begin{aligned} \beta(u-\beta G_\beta u, u\hat{h}) &= \beta(\,(u_1-\beta G_\beta u_1)-(u_2-\beta G_\beta u_2),(u_1-u_2)\hat{h}I_{F_k^h})\\ &= \beta(u_1-\beta G_\beta u_1, u_1\hat{h}I_{F_k^h}) - \beta(u_1-\beta G_\beta u_1, u_2\hat{h}I_{F_k^h})\\ &\quad -\beta(u_2-\beta G_\beta u_2, u_1\hat{h}I_{F_k^h}) + \beta(u_2-\beta G_\beta u_2, u_2\hat{h}I_{F_k^h})\\ &:= I_1 - I_2 - I_3 + I_4. \end{aligned}$$

One finds that

$$\begin{aligned}\lim_{\beta\to\infty} I_1 &= \lim_{\beta\to\infty}\beta(u_1-(\beta-1)G_{(\beta-1)+1}u_1, u_1\hat{h}I_{F_k^h})-(\beta G_\beta u_1, u_1\hat{h}I_{F_k^h})\\ &\le \|\hat{h}I_{F_k^h}\|_\infty[\mathcal{E}_1(u_1,u_1)+\|u_1\|^2_{L^2(E;m)}].\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{\beta\to\infty} I_2 &\le \|\hat{h}I_{F_k^h}\|_\infty[\mathcal{E}_1(u_1,u_2)+\|u_1\|_{L^2(E;m)}\|u_2\|_{L^2(E;m)}],\\ \lim_{\beta\to\infty} I_3 &\le \|\hat{h}I_{F_k^h}\|_\infty[\mathcal{E}_1(u_2,u_1)+\|u_2\|_{L^2(E;m)}\|u_1\|_{L^2(E;m)}],\\ \lim_{\beta\to\infty} I_4 &\le \|\hat{h}I_{F_k^h}\|_\infty[\mathcal{E}_1(u_2,u_2)+\|u_2\|^2_{L^2(E;m)}].\end{aligned}$$

Hence, we get

$$\lim_{\beta\to\infty}\beta(u-\beta G_\beta u, u\hat{h}) \le \|\hat{h}I_{F_k^h}\|_\infty[\mathcal{E}_1(u_1+u_2,u_1+u_2)+(\|u_1\|_{L^2(E;m)}+\|u_2\|_{L^2(E;m)})^2]. \quad (4.20)$$

Let $\rho^\sharp$ be a metric compatible with the topology of $Y^\sharp$, $\{G_l^\sharp\}_{l\in\mathbf{N}}$ an increasing sequence of relatively compact open sets satisfying $\cup_{l\ge1}G_l^\sharp=Y^\sharp$, and $\{\delta_l^\sharp\}_{l\in\mathbf{N}}$ ($\delta_l^\sharp\downarrow 0$) a decreasing sequence of numbers such that $\{(x,y)\in G_l^\sharp\times G_l^\sharp|\rho^\sharp(x,y)\ge\delta_l^\sharp\}$ is a continuous set w.r.t. $J^\sharp$ for each l. Note that u and $\hat{h}$ are in the enlarged D. Hence $u^\sharp$ and $\hat{h}^\sharp$ are continuous on $Y^\sharp$. Following the proof of Theorem 2.6, there exists a subsequence $\{\beta_n\}_{n\in\mathbf{N}}$ such that

$$\begin{aligned}&\int_{Y^\sharp\times Y^\sharp\setminus d}(u^\sharp(x)-u^\sharp(y))^2\hat{h}^\sharp(y)J^\sharp(dx,dy)\\ &\quad= \lim_{l\to\infty}\lim_{\beta_n\to\infty}\frac{\beta_n}{2}\int_{G_l^\sharp\times G_l^\sharp,\rho^\sharp(x,y)\ge\delta_l^\sharp}(u^\sharp(x)-u^\sharp(y))^2\hat{h}^\sharp(y)\sigma^\sharp_{\beta_n}(dx,dy)\\ &\quad\le \lim_{l\to\infty}\lim_{\beta_n\to\infty}\frac{\beta_n}{2}\int_{G_l^\sharp\times G_l^\sharp}(u^\sharp(x)-u^\sharp(y))^2\hat{h}^\sharp(y)\sigma^\sharp_{\beta_n}(dx,dy). \quad (4.21)\end{aligned}$$

Since for any $u\in(D_2-D_2)$, the support supp$[u^\sharp]$ of $u^\sharp$ is compact, we have that supp$[u^\sharp]\subset G_l^\sharp$ for some l. Then, without loss of generality, we can replace $F_k^h\cap Y$ with $G_l^\sharp$ in (4.18) and (4.19). Consequently, we obtain (4.17) from (4.19)-(4.21). Thus $u\in D_1''$ and $(D_2-D_2)\subset D_1''$ since $u\in(D_2-D_2)$ is arbitrary. Therefore $(D_2-D_2)\subset D_1\cap D_1'\cap D_1''$ and the proof is complete. □

Proposition 4.4. *Let J and K be specified by Theorem 4.1. Denote by D^* all the elements $u\in\tilde{D}(\mathcal{E})$ such that*

$$\int_{E\times\{u\neq0\}\setminus d}(u(y)-u(x))^2J(dx,dy)+\int_E u^2(x)K(dx)<\infty.$$

Then, D^* is dense in $D(\mathcal{E})$. Moreover, D^* contains a special quasi-core $\tilde{D}$.

Proof. With the same notations as in Lemma 4.3, for any $u\in D_1$, let $u^\sharp$ be as in the definition of D_1 (cf. (4.14)) and let $Y^\sharp, K^\sharp$ be as in the proof of Theorem 4.1. Then by Theorem 3.5 and

Theorem 4.1, we have that $\int_E u^2(x)K(dx) = \int_{Y^\sharp}(u^\sharp(x))^2 K^\sharp(dx) \leq \|u\|_\infty^2 K^\sharp(\mathrm{supp}[u^\sharp]) < \infty$. Now by (4.15), (4.16) and the fact $\inf\{\hat{h}(x)|x \in F_k^h\} > 0$ for all $k \in \mathbf{N}$ (cf. Proposition 3.1(vi) and Proposition 3.4(iv)), we find that $\int_{E\times\{u\neq 0\}\backslash d}(u(y)-u(x))^2 J(dx,dy) < \infty$ for any $u \in D_1' \cap D_1''$. Consequently $(D_1 \cap D_1' \cap D_1'') \subset D^*$. Since $D_0^+ - D_0^+$ is dense in $D(\mathcal{E})$ (cf. Lemma 3.2), hence $D_2 - D_2$ is dense in $D(\mathcal{E})$. Thus, by Lemma 4.3, $(D_1 \cap D_1' \cap D_1'')$ is dense in $D(\mathcal{E})$ and therefore D^* is dense in $D(\mathcal{E})$.

To show that D^* contains a special quasi-core, we let $\tilde{D}$ be the smallest linear lattice containing $D_2 - D_2$ and being closed under the operations $u \wedge 1, u \wedge (v+1)$ for $u, v \in \tilde{D}$. Noticing that $D_2 - D_2$ is dense in $D(\mathcal{E})$ and $D_2 - D_2$ separates the points of $\cup_{k\geq 1} F_k^{(2)}$, by the above construction $\tilde{D}$ satisfies (QC.1)- (QC.3) of Definition 4.2. Moreover, by Lemma 4.3 we can check that $\tilde{D} \subset (D_1 \cap D_1' \cap D_1'')$ and hence $\tilde{D} \subset D^*$. Thus to prove that D^* contains a special quasi-core, we need only to check that $\tilde{D}$ satisfies (QC.4) of Definition 4.2. To this end, we write $D_2' := \{u_n | n \in \mathbf{N}\}$. Set $g_n := (2/\pi)\arctan u_n, n \in \mathbf{N}$, and define a new metric ρ_0 on $Y := \cup_{k\geq 1} F_k$ by

$$\rho_0(x,y) := \sum_{n=1}^{\infty} 2^{-n}|g_n(x) - g_n(y)|,\ x, y \in Y.$$

Let $\bar{Y}$ be the completion of Y w.r.t. the metric ρ_0 and set

$$\tilde{Y} = \bigcup_{k\geq 1}\left\{x \in \bar{Y} \,\middle|\, \tilde{h}_k^\sharp(x) > 0\right\}, \tag{4.22}$$

where $\tilde{h}_k^\sharp$ is the continuous extension of $\tilde{h}_k|_Y$ to $\bar{Y}$. Then $Y \subset \tilde{Y}$ since $F_k \subset F_k^{(1)}$. Each $u \in \tilde{D}$ is continuous w.r.t the metric ρ_0. Let $\tilde{D}^\sharp$ be the collection of all the continuous extensions to $\tilde{Y}$ of the elements of $\tilde{D}$. For $u \in \tilde{D}$, there exist a constant $c > 0$ and $m \in \mathbf{N}$ such that $|u| \leq c\sum_{j=1}^m \tilde{h}_j$, which together with (4.22) and the fact that $\tilde{D}$ separates the points of $\tilde{Y}$ imply that $\tilde{D}^\sharp \subset C_\infty(\tilde{Y})$ and $\tilde{D}^\sharp$ is dense in $C_\infty(\tilde{Y})$ w.r.t. the uniform norm $\|\cdot\|_\infty$. Furthermore, by virtue of (4.13) we can check that $\tilde{D}^\sharp$ is indeed contained in $C_0(\tilde{Y})$ and hence is uniformly dense in $C_0(\tilde{Y})$. In particular, for any $v^\sharp \in \tilde{D}^\sharp$, there exists $u^\sharp \in \tilde{D}^\sharp$ such that $u^\sharp = 1$ on a open set of $\tilde{Y}$ containing $\mathrm{supp}[v^\sharp]$. Thus $\tilde{D}$ fulfills (QC.4) since the trace topologies on F_k induced by E and $\tilde{Y}$ are the same, which completes the proof. □

In the sequel, we denote by D_b^* all the bounded elements in D^*.

Theorem 4.5. *Let J and K be specified by Theorem 4.1.*
(i) There exist a quasi-compatible metric ρ on E and a special quasi-core $\tilde{D} \subset D_b^*$ satisfying the following properties:
(ρ.1) $\int_{E\times\{u\neq 0\}\backslash d}\rho^2(x,y)J(dx,dy) < \infty$ for all $u \in \tilde{D}$.
(ρ.2) Any $u \in \tilde{D}$ is $\mathcal{E}$-q.e. ρ-Lipschitz in the sense that

$$|u(y) - u(x)| \leq C\rho(x,y),\ \forall x, y \in E\backslash N$$

for some constant $C > 0$ and some $\mathcal{E}$-exceptional set N.

(ii) Let ρ and $\tilde{D}$ be specified by (i). Then for any $\varepsilon > 0$ and any $u, v \in \tilde{D}$, we have the following

decomposition

$$\begin{aligned}\mathcal{E}(u,v) &= \mathcal{E}^{\rho,\varepsilon}(u,v)+\int_{\rho(x,y)>\varepsilon}2(u(y)-u(x))v(y)J(dx,dy)\\ &\quad+\int_E u(x)v(x)K(dx),\end{aligned}\tag{4.23}$$

where $\mathcal{E}^{\rho,\varepsilon}$ *is a bilinear form with domain* $\tilde{D}$ *and satisfies*

$$\mathcal{E}^{\rho,\varepsilon}(u,v)=\int_{\rho(x,y)<\varepsilon}2(u(y)-u(x))v(y)J(dx,dy)\ \ \text{for } v\in\tilde{D}\text{ and } u\in\tilde{D}\cap I_q(v).\tag{4.24}$$

Moreover, if $(u(y)-u(x))v(y)$ *is S.P.V. integrable w.r.t.* J *then* $\lim_{\varepsilon\downarrow 0}\mathcal{E}^{\rho,\varepsilon}(u,v)=\mathcal{E}^c(u,v)$*, where* $\mathcal{E}^c(u,v)$ *is specified by (4.3).*

Proof. (i) A metric ρ and a special quasi-core $\tilde{D}$ satisfying the theorem are not unique. Below we provide an existence result using Proposition 4.4. Let (D_2-D_2) and $Y=\cup_{k\geq 1}F_k$ be as in the proof of Proposition 4.4. Then (D_2-D_2) is a countable subset of $\tilde{D}(\mathcal{E})$ separating the points of Y. Write $(D_2-D_2)=\{u_n|n\in\mathbf{N}\}$. Since $(D_2-D_2)\subset(D_1\cap D_1'\cap D_1'')$ (cf. Lemma 4.3), by (4.16), for each $u_n\in(D_2-D_2)$ there exists a constant M_n such that

$$\int_{E\times E\backslash d}(u_n(y)-u_n(x))^2\hat{h}(y)J(dx,dy)\leq M_n.\tag{4.25}$$

Let $\bar{d}$ be a metric on E compatible with its topology. We define a metric ρ on E by

$$\rho(x,y)=\begin{cases}\bar{d}(x,y), & x,y\in E\backslash Y,\\ \infty, & x\in Y,y\in E\backslash Y\text{ or } y\in Y,x\in E\backslash Y,\\ \left(\sum_{n=1}^{\infty}2^{-n}\frac{(u_n(x)-u_n(y))^2}{1+\|u_n\|_\infty+M_n}\right)^{1/2}, & x,y\in Y.\end{cases}\tag{4.26}$$

Since (D_2-D_2) separates the points of Y, ρ is a metric on E. Since F_k is compact and $u_n\in(D_2-D_2)$ is continuous on F_k for each k, it is easy to check that ρ is a quasi-compatible metric on E.

Let $\tilde{D}$ be the special quasi-core constructed in the proof of Proposition 4.4. By the construction, one finds that $\tilde{D}\subset D_b^*$. By (4.12) and (4.13), for $u\in\tilde{D}$, there exists $k\in\mathbf{N}$ such that $u\in D(\mathcal{E})_{F_k^h}$. Since $\inf\{\hat{h}(x)|x\in F_k^h\}>0$, there exists a constant $\delta>0$ such that $\hat{h}|_{F_k^h}\geq\delta$. Since $\{F_k\}_{k\in\mathbf{N}}$ is an $\mathcal{E}$-nest, hence $E\backslash Y$ is an $\mathcal{E}$-exceptional set. Consequently, by property (a) of Theorem 4.1(i), (4.25) and (4.26), it holds that

$$\begin{aligned}\int_{E\times\{u\neq 0\}\backslash d}\rho^2(x,y)J(dx,dy) &= \sum_{n=1}^{\infty}2^{-n}\int_{E\times\{u\neq 0\}\backslash d}\frac{(u_n(y)-u_n(x))^2}{1+\|u_n\|_\infty+M_n}J(dx,dy)\\ &\leq \frac{1}{\delta}\sum_{n=1}^{\infty}2^{-n}\int_{E\times\{u\neq 0\}\backslash d}\frac{(u_n(y)-u_n(x))^2}{1+\|u_n\|_\infty+M_n}\hat{h}(y)J(dx,dy)\\ &\leq \frac{1}{\delta}.\end{aligned}$$

Thus (ρ.1) holds. Further, by our construction, (ρ.2) holds for any $u \in (D_2 - D_2)$ and hence for any $u \in \tilde{D}$.

(ii) If $u, v \in \tilde{D}$ ($\subset D_b^*$), then $u(x)v(x)$ is integrable w.r.t. K on E by the definition of D^*. We claim that $(u(y) - u(x))v(y)$ is integrable w.r.t. J on $\{(x, y) \in E \times E\backslash d | \rho(x, y) > \varepsilon\}$. In fact, for $u, v \in \tilde{D}$, we find that

$$\begin{aligned}
&\int_{\rho(x,y)>\varepsilon} |(u(y) - u(x))v(y)|J(dx, dy) \\
&\qquad = \int_{\{\rho(x,y)>\varepsilon,\, v(y)\neq 0\}} |(u(y) - u(x))v(y)|J(dx, dy) \\
&\qquad \leq \frac{2\|u\|_\infty \|v\|_\infty}{\varepsilon^2} \int_{E\times\{v\neq 0\}\backslash d} \rho^2(x, y)J(dx, dy). \qquad (4.27)
\end{aligned}$$

By (4.27) and (ρ.1), we have

$$\int_{\rho(x,y)>\varepsilon} |(u(y) - u(x))v(y)|J(dx, dy) < \infty.$$

Then, we obtain (4.23) by simply setting

$$\mathcal{E}^{\rho,\varepsilon}(u, v) \ := \ \mathcal{E}(u, v) - \left\{ \int_{\rho(x,y)>\varepsilon} 2(u(y) - u(x))v(y)J(dx, dy) + \int_E u(x)v(x)K(dx) \right\}. \quad (4.28)$$

(4.24) follows from (4.1) and (4.28). The last assertion follows from the definition of S.P.V. integral. □

Employing the concept of special quasi-core, we can show that the decomposition stated in Theorem 4.5 (ii) is unique in the sense of Theorem 4.7 below. We prepare first a lemma.

Lemma 4.6. *Suppose that $\bar{J}$ is a σ-finite positive Borel measure on $E \times E\backslash d$ satisfying $\bar{J}(N \times E\backslash d) = \bar{J}(E \times N\backslash d) = 0$ for any $\mathcal{E}$-exceptional set N, $\bar{K}$ is a σ-finite positive Borel measure on E charging no $\mathcal{E}$-exceptional sets, and $D \subset \tilde{D}(\mathcal{E})$ is a special quasi-core of $(\mathcal{E}, D(\mathcal{E}))$ consisting of bounded elements. If for any $v \in D$ and $u \in D \cap I_q[v]$, it holds that*

$$\mathcal{E}(u, v) = \int_{E\times E\backslash d} 2(u(y) - u(x))v(y)\bar{J}(dx, dy) + \int_E u(x)v(x)\bar{K}(dx), \qquad (4.29)$$

then $\bar{J} = J$ and $\bar{K} = K$, where J and K are specified by Theorem 4.1.

Proof. Since D is a special quasi-core, by (QC.1), (QC.3) and Proposition 3.1(i), there exist a countable family $\{v_n\}_{n\in\mathbf{N}} \subset D$ and an $\mathcal{E}$-exceptional set N_1 such that $\{v_n\}_{n\in\mathbf{N}}$ is dense in $D(\mathcal{E})$ and $\{v_n\}_{n\in\mathbf{N}}$ separates the points of $E\backslash N_1$. By (QC.4) and (QC.2$'$), for any $v_k \in \{v_n | \, n \in \mathbf{N}\}$ there exists an element $h_k \in D$ such that $h_k = 1$ $\mathcal{E}$-q.e. on a quasi-open set containing $\mathrm{supp}_q[v_k]$ and $0 \leq h_k \leq 1$. Then there exists an $\mathcal{E}$-exceptional set N_2 such that for any $x \in E\backslash N_2$ and any $k \in \mathbf{N}$, $v_k(x) \leq \|v_k\|_\infty h_k(x)$ and $\sup_{k\geq 1} h_k(x) > 0$. Let $\{F_{1k}\}_{k\in\mathbf{N}}$ be an $\mathcal{E}$-nest such that $(N_1 \cup N_2) \subset \cap_{k\geq 1}(E\backslash F_{1k})$. Let $\bar{D}$ be the smallest Q-linear lattice containing $\{v_k, h_k | \ k \in \mathbf{N}\}$

and being closed under the operations $u \wedge 1, u \wedge (v+1)$ for $u, v \in \bar{D}$. Then by [FOT, Lemma 7.1.1], $\bar{D}$ is a countable set. Let $\{F_{2k}\}_{k\in\mathbf{N}}$ be an $\mathcal{E}$-nest such that $\bar{D} \subset C(\{F_{2k}\})$. By the quasi-regularity of $(\mathcal{E}, D(\mathcal{E}))$, there exists an $\mathcal{E}$-nest $\{F_{3k}\}_{k\in\mathbf{N}}$ consisting of compact metrizable sets. Set $E'_k := F_{1k} \cap F_{2k} \cap F_{3k}$ and $E_k := \mathrm{supp}[I_{E'_k} \cdot m]$ for each k. Let $Y := \cup_{k=1}^{\infty} E_k$. Similar to the proof of Theorem 3.5 we can define a metric on Y with the functions of $\bar{D}$ and make a completion $\bar{Y}$ of Y. Set

$$Y^* := \bigcup_{k\geq 1} \{x \in \bar{Y} |\ h_k^*(x) > 0\},$$

where h_k^* is the continuous extension of $h_k|_Y$ to $\bar{Y}$. Then Y^* is a locally compact separable metric space and as in Theorem 3.5 we obtain a regular semi-Dirichlet form $(\mathcal{E}^*, D(\mathcal{E}^*))$. For $u \in \bar{D}$, we denote by u^* the continuous extension of $u|_Y$ to Y^*. Set $\bar{D}^* := \{u^* |\ u \in \bar{D}\}$ and

$$\bar{D}_0^* := \{u^* - (u^* \vee (-\varepsilon)) \wedge \varepsilon |\ u^* \in \bar{D}^*, \varepsilon \in R_+\},$$

where R_+ is the set of all positive real numbers. Let D^* be the smallest linear lattice containing $\bar{D}_0^*$ and being closed under the operation $u^* \to (u^*)^+ \wedge 1$. Further set

$$\tilde{D} := \{\tilde{u} \in \tilde{D}(\mathcal{E}) |\ \tilde{u} = u^* \text{ on } Y \text{ for some } u^* \in D^*\}.$$

Since $\bar{D} \subset D$ and D is a special quasi-core, we have that $\tilde{D} \subset D$.

In addition, we claim that D^* is a special core (cf. Section 2) of the regular semi-Dirichlet form $(\mathcal{E}^*, D(\mathcal{E}^*))$. By the definition, D^* is a linear lattice, i.e. (C.3) holds. Since $\{v_k\}_{k\geq 1} \subset \bar{D}$ is dense in $D(\mathcal{E})$, one finds that D^* is dense in $D(\mathcal{E}^*)$, i.e. (C.1) holds. By the constructions of $\bar{D}$ and Y^*, following the proof of Theorem 3.5, we get that $\bar{D}^* \subset C_\infty(Y^*)$ and is dense in $C_\infty(Y^*)$ w.r.t. the uniform norm. Then $\bar{D}_0^* \subset C_0(Y^*)$ and is dense in $C_0(Y^*)$ w.r.t. the uniform norm. Hence D^* is dense in $C_0(Y^*)$ w.r.t. the uniform norm, i.e. (C.2) holds. Since D^* is closed under the operation $u^* \to (u^*)^+ \wedge 1$, by (C.2) and the fact that Y^* is a locally compact separable metric space, one finds that (C.4) holds. Therefore D^* is a special core.

Extend $\bar{J}|_{Y\times Y\backslash d}$ to a measure $\bar{J}^*$ on $Y^* \times Y^* \backslash d$ by setting $\bar{J}^*(A) = \bar{J}(A \cap (Y \times Y\backslash d))$ for any $A \in \mathcal{B}(Y^* \times Y^*\backslash d)$. Extend $\bar{K}|_Y$ to a measure $\bar{K}^*$ on Y^* similarly. For any $v^* \in D^*$ and $u^* \in D^* \cap I^*(v^*)$, where $I^*(v^*)$ is defined similarly to $I(v)$ as in Theorem 2.6. Define v to be v^* on Y and zero on $E\backslash Y$. Similarly, we define u from u^*. Then $v \in D$ and $u \in D \cap I_q[v]$. By (4.29) we have

$$\begin{aligned} \mathcal{E}^*(u^*, v^*) &= \mathcal{E}(u, v) \\ &= \int_{E\times E\backslash d} 2(u(y) - u(x))v(y)\bar{J}(dx, dy) + \int_E u(x)v(x)\bar{K}(dx) \\ &= \int_{Y\times Y\backslash d} 2(u(y) - u(x))v(y)\bar{J}(dx, dy) + \int_Y u(x)v(x)\bar{K}(dx) \\ &= \int_{Y^*\times Y^*\backslash d} 2(u^*(y) - u^*(x))v^*(y)\bar{J}^*(dx, dy) + \int_{Y^*} u^*(x)v^*(x)\bar{K}^*(dx). \quad (4.30) \end{aligned}$$

By (4.30) and Remark 2.7(ii), we get that $\bar{J}^* = J^*$ and $\bar{K}^* = K^*$, here J^* and K^* are respectively the jumping and killing measures of $(\mathcal{E}^*, D(\mathcal{E}^*))$. Following the proof of Theorem 4.1(i), one finds that $J|_{Y\times Y\backslash d} = J^*|_{Y\times Y\backslash d}$, $K|_Y = K^*|_Y$. Therefore $\bar{J} = J$ and $\bar{K} = K$ since $E\backslash Y$ is an $\mathcal{E}$-exceptional set. □

Theorem 4.7. *Suppose that $\bar{J}$ is a σ-finite positive Borel measure on $E\times E\backslash d$ satisfying $\bar{J}(N\times E\backslash d) = \bar{J}(E\times N\backslash d) = 0$ for any $\mathcal{E}$-exceptional set N, $\bar{K}$ is a σ-finite positive Borel measure on E charging no $\mathcal{E}$-exceptional sets, ρ_1 is a quasi-compatible metric on E, $\tilde{D}_1 \subset \tilde{D}(\mathcal{E})_b$ is a special quasi-core, and for any $\varepsilon > 0$ and any $u, v \in \tilde{D}_1$, (4.23) and (4.24) hold with J, K, ρ and $\tilde{D}$ replaced by $\bar{J}, \bar{K}, \rho_1$ and $\tilde{D}_1$ respectively. Then we have that $\bar{J} = J$ and $\bar{K} = K$.*

Proof. By the assumption, for any $v \in \tilde{D}_1$ and $u \in \tilde{D}_1 \cap I_q[v]$ it holds that

$$\mathcal{E}(u,v) = \int_{E\times E\backslash d} 2(u(y)-u(x))v(y)\bar{J}(dx,dy) + \int_E u(x)v(x)\bar{K}(dx). \tag{4.31}$$

By (4.31) and Lemma 4.6, we get that $\bar{J} = J$ and $\bar{K} = K$. □

In what follows, we fix a quasi-compatible metric ρ satisfying Theorem 4.5(i). Write $\hat{J}(dx,dy) := J(dy,dx)$. We say that J is *symmetric* if $J = \hat{J}$. In general, J is not symmetric and $J - \hat{J}$ is a generalized signed measure, which is well defined and finite on each A_n for some countable partition $\{A_n\}_{n\in\mathbf{N}}$ of $E\times E\backslash d$. Denote by $J_1 := (J-\hat{J})^+$ the positive part of the Jordan decomposition of $(J-\hat{J})$. Set $J_0 := J - J_1$. One can check that J_0 is the largest symmetric σ-finite positive measure dominated by J. In particular, if J itself is symmetric then $J = J_0$.

Theorem 4.8. *Let J and D^* be as in Theorem 4.1. Write $J = J_0 + J_1$ as above.*

(i) If $J_1(E\times E\backslash d) < \infty$, then $(u(y)-u(x))v(y)$ is S.P.V. integrable w.r.t. J and thus (4.3) holds for all $u, v \in D^*_b$, where D^*_b is all the bounded elements of D^*. In particular, if J is symmetric, then (4.3) holds for all $u, v \in D^*_b$.
(ii) If we can find a quasi-compatible metric ρ satisfying (ρ.1) and (ρ.2) of Theorem 4.5(i) and satisfying further
(ρ.3) $\int_{E\times\{v\neq 0\}\backslash d}(\rho(x,y)\wedge 1)J_1(dx,dy) < \infty$ for all $v \in \tilde{D}$,
then $(u(y)-u(x))v(y)$ is S.P.V. integrable w.r.t. J and thus (4.3) holds for all $u, v \in \tilde{D}$, where $\tilde{D}$ is specified by Theorem 4.5(i).

Proof. (i) By the assumption $(u(y)-u(x))v(y)$ is integrable w.r.t. J_1 for any bounded u and v. Since $J = J_0 + J_1$, it is sufficient to show that $(u(y)-u(x))v(y)$ is S.P.V. integrable w.r.t. J_0 for any $u, v \in D^*_b$. Let $A \subset E\times E\backslash d$ be a symmetric set such that $(u(y)-u(x))v(y)$ is integrable on A, since J_0 is symmetric, we have

$$2\int_A (u(y)-u(x))v(y)J_0(dx,dy) = \int_A (u(y)-u(x))(v(y)-v(x))J_0(dx,dy),$$

therefore we need only to show that $(u(y)-u(x))^2$ is integrable w.r.t. J_0 for any $u \in D^*_b$. In deed, for $u \in D^*$, we have

$$\int_{E\times E\backslash d}(u(y)-u(x))^2 J_0(dx,dy) = \int_{E\times\{u\neq 0\}\backslash d}(u(y)-u(x))^2 J_0(dx,dy)$$

$$
\begin{aligned}
&+\int_{E\times\{u=0\}\backslash d}(u(y)-u(x))^2J_0(dx,dy)\\
:=\;& I_1+I_2,
\end{aligned}
$$

$$
\begin{aligned}
I_1 &\le \int_{E\times\{u\neq 0\}\backslash d}(u(y)-u(x))^2J(dx,dy)<\infty,\\
I_2 &= \int_{\{u\neq 0\}\times\{u=0\}\backslash d}(u(y)-u(x))^2J_0(dx,dy)\\
&\le \int_{E\times\{u\neq 0\}\backslash d}(u(x)-u(y))^2J_0(dx,dy)\\
&< \infty.
\end{aligned}
$$

(ii) We know from the proof of (i) above that for $u,v\in D^*$, $(u(y)-u(x))v(y)$ is S.P.V. integrable w.r.t. J_0. Hence to prove (ii), it is sufficient to show that for $u,v\in\tilde{D}$, $(u(y)-u(x))v(y)$ is S.P.V. integrable w.r.t. J_1. For $u,v\in\tilde{D}$, let C be an $\mathcal{E}$-q.e. Lipschitz constant of u. Then, by property (a) of Theorem 4.1(i),

$$
\begin{aligned}
&\int_{E\times E\backslash d}|(u(y)-u(x))v(y)|J_1(dx,dy)\\
&\quad\le \int_{E\times E\backslash d}C\rho(x,y)\;|v(y)|\;J_1(dx,dy)\\
&\quad= C\int_{\rho(x,y)\le 1}\rho(x,y)\;|v(y)|\;J_1(dx,dy)+C\int_{\rho(x,y)>1}\rho(x,y)\;|v(y)|\;J_1(dx,dy)\\
&\quad\le C\int_{E\times E\backslash d}(\rho(x,y)\wedge 1)\;|v(y)|\;J_1(dx,dy)+C\int_{E\times E\backslash d}\rho^2(x,y)\;|v(y)|\;J(dx,dy)\\
&\quad< \infty,
\end{aligned}
$$

where the last inequality holds by (ρ.3) and (ρ.1). Thus $(u(y)-u(x))v(y)$ is integrable and therefore S.P.V. integrable w.r.t. J_1, which completes the proof. □

Remark 4.9. Theorem 4.8(i) can be slightly strengthened as follows.

Let $D_0\subset D_b^*$ be a special quasi-core. If $J_1(E\times\{v\neq 0\}\backslash d)<\infty$ for any $v\in D_0$, then $(u(y)-u(x))v(y)$ is S.P.V. integrable w.r.t. J and thus (4.3) holds for all $u\in D_b^*$ and $v\in D_0$.

5. Decomposition of quasi-regular (non-symmetric) Dirichlet form

Let $(\mathcal{E},D(\mathcal{E}))$ be as in Section 4. In this section, we assume further that the dual form $(\hat{\mathcal{E}},D(\mathcal{E}))$ $(\hat{\mathcal{E}}(u,v):=\mathcal{E}(v,u))$ satisfies the semi-Dirichlet property, i.e. $(\mathcal{E},D(\mathcal{E}))$ is a quasi-regular (non-symmetric) Dirichlet form. Let J,K (respectively, $\hat{J},\hat{K}$) be the σ-finite Borel measures obtained in Theorem 4.1 w.r.t. $(\mathcal{E},D(\mathcal{E}))$ (respectively, $(\hat{\mathcal{E}},D(\mathcal{E}))$) and $(\tilde{\mathcal{E}},D(\mathcal{E}))$ be the symmetric part of $(\mathcal{E},D(\mathcal{E}))$.

Proposition 5.1. *(i) Let D^* be specified by Proposition 4.4, then $D^*=\tilde{D}(\mathcal{E})$. Moreover, for* *any* $u \in D^*$,

$$\int_{E\times E\backslash d}(u(y)-u(x))^2 J(dx,dy)+\int_E u^2(x)K(dx)\le 2\mathcal{E}(u,u). \tag{5.1}$$

(ii) The metric ρ *in Theorem 4.5(i) can be constructed to satisfy* $(\rho.1)'$ *below.*

$(\rho.1)'$ $\int_{E\times E\backslash d}\rho^2(x,y)J(dx,dy)<\infty$.

Proof. (i) Note that $(\tilde{\mathcal{E}},D(\mathcal{E}))$ is a quasi-regular symmetric Dirichlet form on $L^2(E;m)$. By [DMS, Theorem 1.2], for $u,v\in D(\mathcal{E})_e$, the extended Dirichlet space of $(\mathcal{E},D(\mathcal{E}))$,

$$\tilde{\mathcal{E}}(u,v)=\tilde{\mathcal{E}}^c(u,v)+\int_{E\times E\backslash d}(\tilde{u}(y)-\tilde{u}(x))(\tilde{v}(y)-\tilde{v}(x))\tilde{J}(dx,dy)+\int_E\tilde{u}(x)\tilde{v}(x)\tilde{K}(dx), \tag{5.2}$$

where $\tilde{\mathcal{E}}^c$, $\tilde{J}$ and $\tilde{K}$ satisfy the following conditions:
(a) $(\tilde{\mathcal{E}}^c,D(\tilde{\mathcal{E}}^c))$ is a symmetric, nonnegative definite bilinear form with domain $D(\tilde{\mathcal{E}}^c)=D(\mathcal{E})_e$, such that $\tilde{\mathcal{E}}^c$ has the strong local property, i.e. $u\in I_q[v]\Rightarrow\tilde{\mathcal{E}}^c(u,v)=0$.
(b) $\tilde{J}$ is a σ-finite positive measure on $E\times E\backslash d$ and $\tilde{J}(N\times E\backslash d)=\tilde{J}(E\times N\backslash d)=0$ for any $\mathcal{E}$-exceptional set N.
(c) $\tilde{K}$ is a σ-finite positive measure on E, which charges no $\mathcal{E}$-exceptional sets.

Following the proof of [DMS, Theorem 2.1], we find that $\tilde{J}=(J+\hat{J})/2$, $\tilde{K}=(K+\hat{K})/2$. Thus, for $u\in\tilde{D}(\mathcal{E})$, by (5.2),

$$\begin{aligned}
&\int_{E\times E\backslash d}(u(y)-u(x))^2J(dx,dy)+\int_E u^2(x)K(dx)\\
&\le 2\left[\int_{E\times E\backslash d}(u(y)-u(x))^2\frac{J+\hat{J}}{2}(dx,dy)+\int_E u^2(x)\frac{K+\hat{K}}{2}(dx)\right]\\
&= 2\left[\int_{E\times E\backslash d}(u(y)-u(x))^2\tilde{J}(dx,dy)+\int_E u^2(x)\tilde{K}(dx)\right]\\
&\le 2\tilde{\mathcal{E}}(u,u)\\
&= 2\mathcal{E}(u,u).
\end{aligned}$$

Therefore, $D^*=\tilde{D}(\mathcal{E})$ and (5.1) holds.
(ii) Let $D_2-D_2:=\{u_n|n\in\mathbf{N}\}$, Y and the metric $\bar{d}$ be as in the proof of Theorem 4.5(i). We define a metric ρ on E by

$$\rho(x,y)=\begin{cases}\bar{d}(x,y), & x,y\in E\backslash Y,\\ \infty, & x\in Y,y\in E\backslash Y \text{ or } y\in Y,x\in E\backslash Y,\\ \left(\sum_{n=1}^\infty 2^{-n}\frac{(u_n(x)-u_n(y))^2}{1+\|u_n\|_\infty+2\mathcal{E}(u_n,u_n)}\right)^{1/2}, & x,y\in Y.\end{cases} \tag{5.3}$$

By (5.1), (5.3) and property (a) of Theorem 4.1(i), one can easily check that ρ satisfies $(\rho.1)'$. $\square$

For $v\in\tilde{D}(\mathcal{E})$, we define

$$I_q^{(0)}(v):=\{u\in\tilde{D}(\mathcal{E})|u=0\ \mathcal{E}\text{-q.e. on a quasi open set containing }\mathrm{supp}_q[v]\}.$$

Combining the decompositions of $\mathcal{E}$ and $\hat{\mathcal{E}}$, we have the following theorem.

Theorem 5.2. *(i) Let ρ be a quasi-compatible metric satisfying $(\rho.1)'$. Then, for any $u, v \in D_b^*$ and any $\varepsilon > 0$, we have the following unique decomposition*

$$\begin{aligned} \mathcal{E}(u,v) &= \tilde{\mathcal{E}}^c(u,v) + \int_{E\times E\backslash d}(u(y)-u(x))(v(y)-v(x))J(dx,dy) + \int_E u(x)v(x)\tilde{K}(dx) \\ &\quad + \check{\mathcal{E}}^{\rho,\varepsilon}(u,v) + \int_{\rho(x,y)>\varepsilon}(u(y)v(x)-u(x)v(y))J(dx,dy), \end{aligned} \tag{5.4}$$

where $\tilde{\mathcal{E}}^c$ and $\tilde{K}$ are the same as in (5.2), $\check{\mathcal{E}}^{\rho,\varepsilon}$ is an anti-symmetric form satisfying

$$\check{\mathcal{E}}^{\rho,\varepsilon}(u,v) = \int_{\rho(x,y)<\varepsilon}(u(y)v(x)-u(x)v(y))J(dx,dy) \text{ for } u \in I_q^{(0)}(v) \text{ and } v \in I_q^{(0)}(u).$$

(ii) Let $u, v \in D^*$ be such that

$$(u(y)v(x)-u(x)v(y)) \textit{ is S.P.V. integrable w.r.t. } J. \tag{5.5}$$

Then

$$\begin{aligned} \mathcal{E}(u,v) &= \tilde{\mathcal{E}}^c(u,v) + \int_{E\times E\backslash d}(u(y)-u(x))(v(y)-v(x))J(dx,dy) + \int_E u(x)v(x)\tilde{K}(dx) \\ &\quad + \check{\mathcal{E}}^c(u,v) + S.P.V.\int_{E\times E\backslash d}(u(y)v(x)-u(x)v(y))J(dx,dy), \end{aligned} \tag{5.6}$$

where $\tilde{\mathcal{E}}^c, J$ and $\tilde{K}$ are the same as in (5.4), $\check{\mathcal{E}}^c$ is an anti-symmetric form satisfying the local property, i.e. if $u \in I_q^{(0)}(v)$ and $v \in I_q^{(0)}(u)$ then $\check{\mathcal{E}}^c(u,v) = 0$.

Proof. (i) Note that $\hat{J}(dx,dy) = J(dy,dx)$ and $\tilde{J} = (J+\hat{J})/2$, one finds that

$$\begin{aligned} &\int_{E\times E\backslash d}(u(y)-u(x))(v(y)-v(x))J(dx,dy) \\ &\qquad = \int_{E\times E\backslash d}(u(y)-u(x))(v(y)-v(x))\tilde{J}(dx,dy). \end{aligned} \tag{5.7}$$

For $u, v \in D_b^*$, we have

$$\begin{aligned} &\int_{\rho(x,y)>\varepsilon}|(u(y)-u(x))v(y)|J(dx,dy) \\ &\quad \le \left(\int_{\rho(x,y)>\varepsilon}(u(y)-u(x))^2J(dx,dy)\right)^{1/2}\cdot\left(\int_{\rho(x,y)>\varepsilon}v(y)^2J(dx,dy)\right)^{1/2} \\ &\quad \le \left(\int_{E\times E\backslash d}(u(y)-u(x))^2J(dx,dy)\right)^{1/2}\cdot\left(\left(\frac{\|v\|_\infty}{\varepsilon}\right)^2\int_{E\times E\backslash d}\rho(x,y)^2J(dx,dy)\right)^{1/2} \\ &\quad < \infty, \end{aligned} \tag{5.8}$$

where (5.1) and $(\rho.1)'$ are used to obtain the last inequality. Since $u(y)v(x) - u(x)v(y) = (u(y) - u(x))v(y) - (v(y) - v(x))u(y)$, we obtain from (5.8) that for any $u, v \in D_b^*$ and $\varepsilon > 0$, $(u(y)v(x) - u(x)v(y))$ is integrable w.r.t. J on $\{(x, y) \in E \times E\backslash d | \rho(x, y) > \varepsilon\}$. For $u, v \in D_b^*$, set

$$\check{\mathcal{E}}^{\rho,\varepsilon}(u, v) := \mathcal{E}(u, v) - \tilde{\mathcal{E}}(u, v) - \int_{\rho(x,y)>\varepsilon} (u(y)v(x) - u(x)v(y))J(dx, dy). \tag{5.9}$$

By (5.2), (5.7) and (5.9), we obtain (5.4). The anti-symmetry of $\check{\mathcal{E}}^{\rho,\varepsilon}$ follows from (5.9). The uniqueness of decomposition (5.4) can be proved by virtue of the uniqueness of the classical Beurling-Deny formula for symmetric Dirichlet forms using the local-compactification (cf. the uniqueness part of Theorem 4.1(i)).
(ii) If $(u(y)v(x) - u(x)v(y))$ is S.P.V. integrable w.r.t. J, then one obtains (5.6) by simply setting

$$\check{\mathcal{E}}^{c}(u, v) := \mathcal{E}(u, v) - \tilde{\mathcal{E}}(u, v) - S.P.V. \int_{E\times E\backslash d} (u(y)v(x) - u(x)v(y))J(dx, dy). \tag{5.10}$$

The anti-symmetry of $\check{\mathcal{E}}^c$ follows from (5.10).

If $u \in I_q^{(0)}(v)$ and $v \in I_q^{(0)}(u)$, then by Theorem 4.1(i),

$$\begin{aligned} \mathcal{E}(u, v) &= \int_{E\times E\backslash d} 2(u(y) - u(x))v(y)J(dx, dy) + \int_E u(y)v(y)K(dy) \\ &= -2\int_{E\times E\backslash d} u(x)v(y)J(dx, dy) \end{aligned} \tag{5.11}$$

and

$$\mathcal{E}(v, u) = -2\int_{E\times E\backslash d} v(x)u(y)J(dx, dy).$$

It follows that

$$\tilde{\mathcal{E}}(u, v) = -\int_{E\times E\backslash d} (u(x)v(y) + v(x)u(y))J(dx, dy). \tag{5.12}$$

By (5.10)-(5.12), we obtain $\check{\mathcal{E}}^c(u, v) = 0$, which completes the proof. □

Remark 5.3. (i) If both $(u(y) - u(x))v(y)$ and $(v(y) - v(x))u(y)$ are S.P.V. integrable w.r.t. J, then (5.5) is fulfilled.
(ii) In [Bl, (9.2)], the author gave a representation which is essentially the same as (5.6) for regular (non-symmetric) Dirichlet forms but without introducing the notion of S.P.V. integral and the crucial condition (5.5). We point out that condition (5.5) cannot be dropped and refer the interested readers to [HMS] for a counterexample.

Theorem 5.4. *Let $J = J_0 + J_1$ be as in Theorem 4.8.*
(i) If $J_1(E\times E\backslash d) < \infty$, then (5.5) is fulfilled and thus decomposition (5.6) holds for all $u, v \in D_b^*$. In particular, if J is symmetric then (5.6) holds for all $u, v \in D_b^*$.

(ii) If we can find a quasi-compatible metric ρ satisfying $(\rho.1)'$, $(\rho.2)$ and $(\rho.3)$, then decomposition (5.6) holds for all $u, v \in \tilde{D}$, where $\tilde{D}$ is specified by Theorem 4.5(i).

Proof. (i) is clear. By Remark 5.3(i), assertion (ii) follows directly from Theorem 4.8(ii) and Theorem 5.2(ii). □

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