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A NOTE ON EXPONENTIAL STABILITY OF THE NONLINEAR FILTER FOR DENUMERABLE MARKOV CHAINS

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A note on exponential stability of the nonlinear filter for denumerable Markov chains

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Abstract

We study asymptotic stability of the optimal filter with respect to its initial conditions. We show that exponential stability of the nonlinear filter holds for a large class of denumerable Markov chains, including all finite Markov chains, under the assumption that the observation function is one-to-one and the observation noise is sufficiently small. Ergodicity of the signal process is not assumed.

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1. Introduction

In this note, we consider the stability problem of the nonlinear filter for discrete-time finite or countable state Markov chains. Let $(X_n)_{n\geq 0}$ be a time-homogeneous Markov chain on the denumerable space S with transition matrix $Q = (q_{ij})_{i,j\in S}$. Hereafter we suppose that $S = \mathbf{Z}$, the integer space, or $S = \{1, 2, \ldots, m\}$ for some $m \in \mathbf{N}$. The information about the *signal process* $(X_n)_{n\geq 0}$ is obtained through the *observation process* $(Y_n)_{n\geq 1}$ as follows.

$$Y_n = h(X_n) + \sigma V_n, \quad n \geq 1, \tag{1}$$

where h is a real-valued function defined on S, $(V_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with the density function g, and $\sigma > 0$ is a constant. For simplicity, $(V_n)_{n\geq 1}$ is assumed to be independent of $(X_n)_{n\geq 0}$. We suppose that $(X_n)_{n\geq 0}$, $(V_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ are all defined on the same probability space $(\Omega, \mathcal{F}, P)$.

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The optimal filtering problem consists of computing the conditional law of X_n given the σ-algebra $\mathcal{F}_n^Y := \sigma(Y_1, \ldots, Y_n)$, that is, computing $\pi_n := (\pi_n(i), i \in S)^*$, where $\pi_n(i)$ is defined by

$$\pi_n(i) := P\left(X_n = i | \mathcal{F}_n^Y\right), \quad i \in S.$$

We use D_n to denote the diagonal matrix with $D_n(i,i) := \Delta_n(i) := g((Y_n - h(i))/\sigma)$, $i \in S$. Define

$$\rho_n^\mu := D_n Q^* \rho_{n-1}^\mu, \quad n \geq 1, \tag{2}$$

where Q^* is the transpose of Q and $\rho_0^\mu = \mu = (\mu(i), i \in S)^*$ is the distribution of X_0. By Bayes' rule, we have that

$$\pi_n^\mu = \rho_n^\mu / \|\rho_n^\mu\|. \tag{3}$$

Hereafter, we use $\|\nu\| := \sum_{i \in S} |\nu(i)|$ to denote the total variation norm of a finite signed measure ν on S, and use π_n^μ instead of π_n to emphasize the dependence on the initial distribution μ. Note that (3) is well defined since $P(\|\rho_n^\mu\| = 0) = 0$.

Let $\mu' \neq \mu$ be another probability measure on S satisfying $\mu << \mu'$. We denote the corresponding solutions of (2) and (3) by $\rho_n^{\mu'}$ and $\pi_n^{\mu'}$, respectively. Note that (3) remains well defined P-a.s., since $P(\|\rho_n^{\mu'}\| = 0) = 0$. Denote by E the expectation with respect to P. Then, the filter (or the filtering process) is called *stable* if for any bounded function f on S it holds that

$$E|\pi_n^\mu(f) - \pi_n^{\mu'}(f)| := E\left|\int_S f(x)\pi_n^\mu(dx) - \int_S f(x)\pi_n^{\mu'}(dx)\right| \to 0 \text{ as } n \to \infty.$$

Since the actual initial distribution of the signal process $(X_n)_{n\geq 0}$ is rarely known, it is important to study the stability of the filter from an applications point of view.

Recently, there is an increasing interest in considering the stability problem of the nonlinear filter for Markov signals. The first result was obtained by Ocone and Pardoux (1996). They used results of Kunita (1971) to show that the optimal filter forgets its initial distribution in an L^p sense when the signal process is ergodic. However, the paper doesn't provide a rate of convergence and needs to be revised, in view of the recently spotted gap in the proof of Theorem 3.3 of Kunita (1971) (see Baxendale, Chigansky and Liptser (2004) and Budhiraja (2003)). Some new approaches based on, for instance, the Hilbert projective metric and the related Birkhoff's contraction coefficient (Atar and Zeitouni (1997a, b)), the semi-group techniques and Dobrushin ergodic coefficient (Del Moral and Guionnet (2001)), have been introduced to study the stability problem. Most of the known results, under different assumptions, lead to the stronger exponential convergence of the total variation norm (in this case, the filter is said to be *exponentially stable*):

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\pi_n^\mu - \pi_n^{\mu'}\| < 0 \qquad P - a.s. \tag{4}$$

We refer the interested readers to Atar and Zeitouni (1997a, b), Baxendale, Chigansky and Liptser (2004), Chigansky and Liptser (2004) and Chigansky (2005) for the recent results on nonlinear filtering for ergodic signals on finite or compact state spaces, and to Atar (1998), Budhiraja and Ocone (1999), LeGland and Oudjane (2003) and Stannat (2004a, b) for the cases of non-ergodic signals or noncompact state spaces.

In this note, under the assumption that h is one-to-one and σ is sufficiently small, we show that (4) holds for a large class of denumerable Markov chains $(X_n)_{n\geq 0}$, including all finite Markov chains. The main results, Theorems 2.1 and 2.2, and some examples are given in Section 2. The proofs of the main results are given in Section 3.

2. Results and examples

To simplify notation, in the sequel, we respectively denote $\rho_n^\mu, \pi_n^\mu, \rho_n^{\mu'}, \pi_n^{\mu'}$ by $\rho_n, \pi_n, \rho'_n, \pi'_n$.

Theorem 2.1. Let $(X_n)_{n\geq 0}$ be a Markov chain on $\mathbf{Z}$ with the transition matrix $Q = (q_{ij})_{i,j\in\mathbf{Z}}$ and the initial distribution μ. Define the observation process $(Y_n)_{n\geq 1}$ by (1). Suppose that the following conditions hold.
(i) There exists a sequence $\{a_k\}_{k=-\infty}^{\infty}$ of nonnegative real numbers such that

$$q_{i+k,i} \leq a_k \quad \text{for all} \quad i,k \in \mathbf{Z} \quad \text{and} \quad \sum_{k=-\infty}^{\infty} a_k < \infty.$$

(ii) Define

$$\Omega_X := \{(i_0, i_1, \ldots) : i_j \in \mathbf{Z}, q_{i_j,i_{j+1}} > 0, \forall j \geq 0 \text{ and } \mu(i_0) > 0\}. \tag{5}$$

Then, there exists a constant K such that for any path $(i_0, i_1, \ldots) \in \Omega_X$,

$$\liminf_{n\to\infty} \frac{1}{n} \left[\log \prod_{j=0}^{n-1} q_{i_j,i_{j+1}} \right] \geq K.$$

(iii) There is a constant $c > 0$ such that for any $i, j \in \mathbf{Z}, i \neq j$,

$$|h(i) - h(j)| \geq c.$$

(iv) The density function g is bounded on $\mathbf{R}$ satisfying $\int_{-\infty}^{\infty} |\log(g(x))| g(x)dx < \infty$ and $\lim_{|x|\to\infty} g(x) = 0$.
Then, for any probability measure μ' on $\mathbf{Z}$ satisfying $\mu << \mu'$, we have that

$$\lim_{\sigma\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \|\pi_n - \pi'_n\| < 0 \qquad P - a.s.$$

Theorem 2.2. Let $(X_n)_{n\geq 0}$ be a finite Markov chain on $\{1, \ldots, m\}$ with the initial distribution μ. Define the observation process $(Y_n)_{n\geq 1}$ by (1). Suppose that h is one-to-one and g satisfies condition (iv) of Theorem 2.1. Then, for any probability measure μ' on $\{1,\ldots,m\}$ satisfying $\mu << \mu'$, we have that

$$\lim_{\sigma\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\|\pi_n-\pi'_n\|<0 \qquad P-a.s.$$

Remark 2.3. (i) Suppose that there exists an integer $M>0$ such that for any $i\in\mathbf{Z}, \{j: q_{ij}>0\}\subseteq\{j:|j-i|\le M\}$. Define

$$a_k := \begin{cases} 1, & \text{if } |k|\le M,\\ 0, & \text{otherwise.}\end{cases}$$

Then, condition (i) of Theorem 2.1 is satisfied.
(ii) If $\alpha := \inf\{q_{ij}:\ q_{ij}>0, i,j\in\mathbf{Z}\}>0$, then condition (ii) of Theorem 2.1 is satisfied with $K:=\log\alpha$.
(iii) The counterexample given in Kaijser (1975) (see also Example 5.1 of Chigansky and Liptser (2004)) indicates that if the observation function h is not one-to-one, then the filtering process can be unstable even if the signal process is ergodic and $\sigma=0$.
(iv) Condition (iv) of Theorem 2.1 is satisfied for a large class of functions such as the density functions of normal, uniform and Pareto random variables, etc.

Remark 2.4. Here, we recall some known results on the exponential stability of the nonlinear filter for discrete-time finite Markov chains. Let $(X_n)_{n\ge 0}$ be a finite Markov chain with the transition matrix Q. Suppose that the observation process $(Y_n)_{n\ge 1}$ is defined by (1). If $(X_n)_{n\ge 0}$ is *ergodic* and $(V_n)_{n\ge 1}$ is a sequence of i.i.d. standard Gaussian random variables, then
(i) If all the entries of Q are strictly positive, then the filter is exponentially stable (see Theorem 1.2 of Atar and Zeitouni (1997a)).
(ii) If σ is sufficiently small and there exists a state i such that $\{j: h(j)=h(i)\}$ consists of a single point, or if σ is sufficiently large, then the filter is exponentially stable (see Theorems 1.3 and 1.4 of Atar and Zeitouni (1997a)).
(iii) If h is one-to-one, then the filter is exponentially stable. This result also holds for more general observation noise $(V_n)_{n\ge 1}$ (see Lemma 4.1 of Chigansky (2005)).

In the following examples, all the filtering processes are exponentially stable under the assumption that h and g satisfy conditions (iii) and (iv) of Theorem 2.1.

Example 2.5. (Simple random walk) Let the Markov chain $(X_n)_{n\ge 0}$ in Theorem 2.1 be a simple random walk on $\mathbf{Z}$, i.e. $(X_n)_{n\ge 0}$ is a Markov chain on $\mathbf{Z}$ such that

$$q_{i,i+1}=p,\ q_{i,i-1}=1-p,\ \forall i\in\mathbf{Z},$$

where $0<p<1$ is a constant. Then conditions (i) and (ii) of Theorem 2.1 are satisfied. Note that the transition matrix Q doesn't satisfy the "non-mixing" condition in Chigansky and Liptser (2004). Since $(X_n)_{n\ge 0}$ is periodic and even transient if $p\neq\frac{1}{2}$, $(X_n)_{n\ge 0}$ is non-ergodic.

Example 2.6. Let $(X_n)_{n\geq 0}$ be the Markov chain on $\mathbf{Z}$ with the transition matrix $Q = (q_{ij})_{i,j\in\mathbf{Z}}$, where

$$q_{00} = 1,$$
$$q_{n0} = p^{|n|},\ q_{nn} = 1 - p^{|n|},\quad \forall n \neq 0,\ 0 < p < 1.$$

Then condition (i) of Theorem 2.1 is satisfied with the sequence $\{a_k := p^{|k|}, k \in \mathbf{Z}\}$. In the following we show that condition (ii) of Theorem 2.1 is also satisfied.

Let Ω_X be defined as in (5). For any path $(i_0, i_1, \ldots) \in \Omega_X$, there are two cases:
Case 1. There exists a $j_0 \in \mathbf{Z}_+$ such that $i_{j_0} = 0$. Then, by the definitions of Q and Ω_X, $i_j = 0, \forall j \geq j_0$ and thus $q_{i_j,i_{j+1}} = 1, \forall j \geq j_0$. Hence

$$\liminf_{n\to\infty} \frac{1}{n}\left[\log \prod_{j=0}^{n-1} q_{i_j,i_{j+1}}\right] = 0.$$

Case 2. There exist an $i \in \mathbf{Z}\backslash\{0\}$ such that $i_j = i, \forall j \geq 0$. Then

$$\liminf_{n\to\infty} \frac{1}{n}\left[\log \prod_{j=0}^{n-1} q_{i_j,i_{j+1}}\right] = \liminf_{n\to\infty} \frac{1}{n}\left[\log \prod_{j=0}^{n-1}(1 - p^{|i|})\right] = \log(1 - p^{|i|}) \geq \log(1 - p).$$

Set $K := \log(1 - p)$. Then condition (ii) of Theorem 2.1 is satisfied.

Example 2.7. Let $(X_n)_{n\geq 0}$ be the Markov chain on $\mathbf{Z}$ with the transition matrix $Q = (q_{ij})_{i,j\in\mathbf{Z}}$, where

$$q_{00} = 1,$$
$$q_{n,n-1} = p,\ q_{n,n-2} = p^2, \ldots, q_{n1} = p^{n-1},\ q_{n0} = p^n,\ q_{nn} = 1 - \sum_{i=1}^{n} p^i,$$
$$q_{-n,-j} = q_{n\,j},\ \forall n \geq 1,\ 0 \leq j \leq n,\ 0 < p < \frac{1}{2}.$$

Similar to Example 2.6, one can show that conditions (i) and (ii) of Theorem 2.1 are satisfied.

3. Proofs

Proof of Theorem 2.1. By Lemma 2 of Atar and Zeitouni (1997b), we have that

$$\|\pi_n - \pi'_n\| \leq \frac{\|\rho_n \wedge \rho'_n\|}{\|\rho_n\|\|\rho'_n\|}, \tag{6}$$

where $\rho_n \wedge \rho'_n$ is the matrix with entries

$$(\rho_n \wedge \rho'_n)(i,j) = \rho_n(i)\rho'_n(j) - \rho_n(j)\rho'_n(i),\quad \forall i,j \in \mathbf{Z}$$

and

$$\|\rho_n \wedge \rho'_n\| := \sum_{i,j\in\mathbf{Z}} (\rho_n \wedge \rho'_n)(i,j) = \sum_{i\neq j} (\rho_n \wedge \rho'_n)(i,j).$$

First, we show that

$$\lim_{\sigma\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \|\rho_n \wedge \rho'_n\| = -\infty. \tag{7}$$

By (2), we have that

$$\rho_n(i) = \left(\sum_{k=-\infty}^{\infty} \rho_{n-1}(i+k) q_{i+k,i}\right) g\left(\frac{Y_n - h(i)}{\sigma}\right).$$

Then

$$\begin{aligned}
&(\rho_n \wedge \rho'_n)(i,j) = \rho_n(i)\rho'_n(j) - \rho_n(j)\rho'_n(i) \\
&= \left(\sum_{k=-\infty}^{\infty} \rho_{n-1}(i+k) q_{i+k,i}\right)\left(\sum_{l=-\infty}^{\infty} \rho'_{n-1}(j+l) q_{j+l,j}\right) g\left(\frac{Y_n - h(i)}{\sigma}\right) g\left(\frac{Y_n - h(j)}{\sigma}\right) \\
&\quad - \left(\sum_{l=-\infty}^{\infty} \rho_{n-1}(j+l) q_{j+l,j}\right)\left(\sum_{k=-\infty}^{\infty} \rho'_{n-1}(i+k) q_{i+k,i}\right) g\left(\frac{Y_n - h(j)}{\sigma}\right) g\left(\frac{Y_n - h(i)}{\sigma}\right) \\
&= g\left(\frac{Y_n - h(i)}{\sigma}\right) g\left(\frac{Y_n - h(j)}{\sigma}\right) \left[\sum_{k,l=-\infty}^{\infty} q_{i+k,i} q_{j+l,j} \rho_{n-1}(i+k) \rho'_{n-1}(j+l)\right. \\
&\quad \left. - \sum_{l,k=-\infty}^{\infty} q_{j+l,j} q_{i+k,i} \rho_{n-1}(j+l) \rho'_{n-1}(i+k)\right] \\
&= g\left(\frac{Y_n - h(i)}{\sigma}\right) g\left(\frac{Y_n - h(j)}{\sigma}\right) \sum_{k,l=-\infty}^{\infty} q_{i+k,i} q_{j+l,j} (\rho_{n-1} \wedge \rho'_{n-1})(i+k, j+l).
\end{aligned} \tag{8}$$

Denote the supremum norm of g by $||g||_\infty$. Then, we obtain from (8), conditions (i) and (iii) that

$$\begin{aligned}
&\|\rho_n \wedge \rho'_n\| \\
&= \sum_{i\neq j} |(\rho_n \wedge \rho'_n)(i,j)| \\
&= \sum_{i\neq j} g\left(\frac{Y_n - h(i)}{\sigma}\right) g\left(\frac{Y_n - h(j)}{\sigma}\right) \left|\sum_{k,l=-\infty}^{\infty} q_{i+k,i} q_{j+l,j} (\rho_{n-1} \wedge \rho'_{n-1})(i+k, j+l)\right| \\
&\leq \sum_{i\neq j} g\left(\frac{Y_n - h(i)}{\sigma}\right) g\left(\frac{Y_n - h(j)}{\sigma}\right) \sum_{k,l=-\infty}^{\infty} a_k\, a_l |(\rho_{n-1} \wedge \rho'_{n-1})(i+k, j+l)|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k,l=-\infty}^{\infty} a_k\, a_l \sum_{i\neq j} g\left(\frac{Y_n - h(i)}{\sigma}\right) g\left(\frac{Y_n - h(j)}{\sigma}\right) |(\rho_{n-1}\wedge\rho'_{n-1})(i+k, j+l)| \\
&\leq ||g||_\infty \left(\sup_{|x|\geq\frac{c}{2\sigma}} |g(x)|\right)\left(\sum_{k=-\infty}^{\infty} a_k\right)^2 \|\rho_{n-1}\wedge\rho'_{n-1}\|.
\end{aligned}
$$

By induction, we get

$$
\|\rho_n\wedge\rho'_n\| \leq \left(||g||_\infty \left(\sup_{|x|\geq\frac{c}{2\sigma}} |g(x)|\right)\left(\sum_{k=-\infty}^{\infty} a_k\right)^2\right)^n \|\rho_0\wedge\rho'_0\|. \tag{9}
$$

It follows from (9) and condition (iv) that

$$
\begin{aligned}
&\lim_{\sigma\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\|\rho_n\wedge\rho'_n\| \\
&\leq \lim_{\sigma\to 0}\limsup_{n\to\infty}\left[\log\left(||g||_\infty\left(\sum_{k=-\infty}^{\infty} a_k\right)^2\right) + \log\left(\sup_{|x|\geq\frac{c}{2\sigma}}|g(x)|\right) + \frac{1}{n}\log\|\rho_0\wedge\rho'_0\|\right] \\
&= -\infty.
\end{aligned}
$$

Next, we show that

$$
\limsup_{n\to\infty}\left(-\frac{1}{n}\log\|\rho_n\|\right) \leq -K + \int_{-\infty}^{\infty} |\log(g(x))| g(x)dx \qquad P-a.s. \tag{10}
$$

For $n \geq 1$ and $i \in \mathbf{Z}$, we have that

$$
\rho_n(i) = \sum_{i_0,\ldots,i_{n-1}} \rho_0(i_0) q_{i_0,i_1}\cdots q_{i_{n-1},i}\triangle_1(i_1)\cdots\triangle_{n-1}(i_{n-1})\triangle_n(i).
$$

Then

$$
\|\rho_n\| = \sum_{i\in\mathbf{Z}}\rho_n(i) \geq \rho_n(X_n) \geq \rho_0(X_0)q_{X_0,X_1}\cdots q_{X_{n-1},X_n}\triangle_1(X_1)\cdots\triangle_n(X_n), \tag{11}
$$

where

$$
\triangle_j(X_j) = g(V_j), \quad 1 \leq j \leq n. \tag{12}
$$

Let Ω_X be defined as in (5). Then, one finds that $P\circ X_\cdot^{-1}(\Omega_X) = 1$. Thus, we obtain by (11), (12), condition (ii) and the law of large numbers that P-a.s.,

$$
\begin{aligned}
\liminf_{n\to\infty}\frac{1}{n}\log\|\rho_n\| &\geq \liminf_{n\to\infty}\frac{1}{n}\left[\log\prod_{j=0}^{n-1} q_{X_j,X_{j+1}}\right] + \liminf_{n\to\infty}\frac{1}{n}\log\left(\prod_{j=1}^{n} g(V_j)\right) \\
&\geq K - \int_{-\infty}^{\infty} |\log(g(x))| g(x)dx.
\end{aligned}
$$

Hence (10) holds.

Note that

$$\|\rho'_n\| = \sum_{i\in\mathbf{Z}} \rho'_n(i) \geq \rho'_n(X_n) \geq \rho'_0(X_0) q_{X_0,X_1} \cdots q_{X_{n-1},X_n} \triangle_1(X_1) \cdots \triangle_n(X_n).$$

Define

$$\Omega'_X := \{(i_0, i_1, \ldots) : i_j \in \mathbf{Z}, q_{i_j,i_{j+1}} > 0, \forall j \geq 0 \text{ and } \mu'(i_0) > 0\}.$$

Since $\mu << \mu'$, $P \circ X_\cdot^{-1}(\Omega'_X) = 1$. Similar to (10), one can show that

$$\limsup_{n\to\infty} \left(-\frac{1}{n} \log \|\rho'_n\| \right) \leq -K + \int_{-\infty}^{\infty} |\log(g(x))| g(x) dx \qquad P-a.s. \tag{13}$$

By (6), (7), (10) and (13), we find that P-a.s.,

$$\begin{aligned} \lim_{\sigma\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \|\pi_n - \pi'_n\| &\leq \lim_{\sigma\to 0} \limsup_{n\to\infty} \frac{1}{n} (\log \|\rho_n \wedge \rho'_n\| - \log \|\rho_n\| - \log \|\rho'_n\|) \\ &= -\infty, \end{aligned}$$

which completes the proof. □

Proof of Theorem 2.2. Define $c := \inf\{|h(i) - h(j)| : i, j \in \{1, \ldots, m\}, i \neq j\}$. Since h is assumed to be one-to-one, $c > 0$. For $i, j \in \mathbf{Z}$, define

$$\bar{q}_{ij} = \begin{cases} q_{ij}, & \text{if } i, j \in \{1, \ldots, m\}, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, define $\rho_n(i) = 0$ for $i \in \mathbf{Z}\backslash\{1, \ldots, m\}$ and $n \in \mathbf{Z}_+$. Then, for $i = 1, \ldots, m$, we have that

$$\rho_n(i) = \left(\sum_{k=-m+1}^{m-1} \rho_{n-1}(i+k)\, \bar{q}_{i+k,i} \right) g\left(\frac{Y_n - h(i)}{\sigma} \right).$$

Similar to (8), for any $i, j \in \{1, \ldots, m\}$, we have that

$$\begin{aligned} &(\rho_n \wedge \rho'_n)(i,j) \\ &= g\left(\frac{Y_n - h(i)}{\sigma} \right) g\left(\frac{Y_n - h(j)}{\sigma} \right) \left[\sum_{k,l=-m+1}^{m-1} \bar{q}_{i+k,i} \bar{q}_{j+l,j} (\rho_{n-1} \wedge \rho'_{n-1})(i+k, j+l) \right]. \end{aligned}$$

Then

$$\begin{aligned} \|\rho_n \wedge \rho'_n\| &\leq \sum_{i\neq j} g\left(\frac{Y_n - h(i)}{\sigma} \right) g\left(\frac{Y_n - h(j)}{\sigma} \right) \left[\sum_{k,l=-m+1}^{m-1} \left| (\rho_{n-1} \wedge \rho'_{n-1})(i+k, j+l) \right| \right] \\ &\leq m(m-1) \|g\|_\infty \left(\sup_{|x|\geq \frac{c}{2\sigma}} |g(x)| \right) \|\rho_{n-1} \wedge \rho'_{n-1}\|. \end{aligned}$$

Similar to (5), define

$$\Omega_X := \{(i_0, i_1, \ldots) : \ i_j \in \{1, \ldots, m\}, q_{i_j,i_{j+1}} > 0, \forall j \geq 0 \text{ and } \mu(i_0) > 0\}.$$

Then, one finds that $P \circ X_\cdot^{-1}(\Omega_X) = 1$. Define $\alpha := \inf\{q_{ij} : q_{ij} > 0, i, j \in \{1, \ldots, m\}\}$. Then $\alpha > 0$. Thus, for any path $(i_0, i_1, \ldots) \in \Omega_X$,

$$\liminf_{n\to\infty} \frac{1}{n} \left[\log \prod_{j=0}^{n-1} q_{i_j,i_{j+1}} \right] \geq \log \alpha.$$

The remainder of the proof is similar to that of Theorem 2.1, we omit the details here. □

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