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OPTIMAL REINSURANCE WITH GENERAL RISK FUNCTIONS

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Abstract. The paper studies the optimal reinsurance problem if the risk level is measured by a general risk function. Necessary and sufficient optimality conditions are given for a wide family of risk functions, including Deviation Measures, Expectation Bounded Risk Measures and Coherent Measures of Risk. Then the optimality conditions are used to verify whether the classical reinsurance contracts (quota-share, stop-loss) are optimal regardless of the risk function to be used, and the paper ends by particularizing the findings so as to study in detail two deviation measures and the Conditional Value at Risk.

Key Words. Optimal reinsurance, Risk measure and deviation measure, Optimality conditions.

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JEL Classification. G22, G11.

1. Introduction

General risk functions are becoming more and more important in insurance and finance. Since the paper of Artzner *et al.* (1999) introduced the axioms and properties of their "Coherent Measures of Risk", many authors have extended the discussion. The recent development of new markets (insurance or weather linked derivatives, commodity or energy/electricity derivatives, etc.) and products (inflation-linked bonds, equity indexes annuities, hedge funds, etc.), the necessity of managing new types of risk (credit risk, operational risk, etc.), the presence of asymmetries and fat tails, and the (often legal) obligation of providing initial capital requirements have made it rather convenient to overcome the variance as the most important risk measure and to introduce more general risk functions. Hence, it is not surprising that the recent literature presents many interesting contributions focusing on new methods for measuring risk levels. Among others, Goovaerts *et al.* (2004) have introduced the Consistent Risk Measures, also studied in Burgert and Rüschendorf (2006), Frittelli and Scandolo (2005) have analyzed Risk Measures for Stochastic Processes, and Rockafellar *et al.* (2006) have defined the Deviations and the Expectation Bounded Risk Measures.

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Many classical actuarial and financial problems have been revisited by using new risk functions. For example, Nakano (2004) draws on a Coherent Measure to price in incomplete markets, Konno *et al.* (2005) minimize the Absolute Deviation in a Portfolio Choice Problem, Mansini *et al.* (2007) deal with Portfolio Choice Problems and more complex measures, Alexander *et al.* (2006) compare the minimization of Value at Risk (VaR) and the Conditional Value at Risk ($CVaR$) for a portfolio of derivatives, and Schied (2007) deals with Optimal Investment with Convex Risk Measures.

The Optimal Reinsurance Problem is a classical issue in Actuarial Science. Usually,authors consider the primary (or ceding) company viewpoint. A common approach attempts to minimize some measure of the first insurer risk after reinsurance, restricted to some premium condition. A first paper was by Borch (1960), who proved that the stop loss reinsurance minimizes the variance of the retained loss if premiums are calculated following the expected value principle. A few years later Arrow (1963) also assumed the expected value principle and showed that the same stop loss reinsurance maximizes the expected utility of the terminal wealth of a risk-averse insurer. Since maximizing the utility $u(x)$ is equivalent to minimizing the loss $w(x) = -u(-x)$, both approaches aim to minimize some risk function.

The posterior research followed the ideas outlined in the foundational articles, trying to take into account more general risk measures and premium principles which often give optimal contracts other than stop loss. In recent years there have appeared some interesting articles devoted to this subject. For example, Kaluszka (2001) still takes the variance of the retained loss as the risk function to be minimized, but considers other premium principles like the standard deviation principle and the variance principle. Also, Gajec and Zagrodny (2004) consider more general symmetric and even asymmetric risk functions like the expected absolute deviation and the truncated variance of the retained loss, under the standard deviation premium principle. Young (1999) maximizes the expected utility of the final wealth under Wang's premium principle. Kaluszka (2005) studies reinsurance contracts with several different convex measures of the retained risk and also many convex premium principles (exponential, semi-deviation and semi-variance, Dutch, Wang and Gini principles, etc.). Other famous financial risk measures like the VaR or the Tail Value at Risk ($TVaR$) are also being considered. For example, Kaluszka (2005) uses the $TVaR$ as a premium principle and Cai and Tan (2007) calculate the optimal retention for a stop loss reinsurance by considering the VaR and Conditional Tail Expectation risk measures, under the expected value premium principle.[1]

[1] The $CVaR$ is also called $TVaR$, Expected Shortfall, Conditional Tail Expectation, etc., although for some discrete random variables there might be some slight differences among the definitions used by several authors.

This paper considers the expected value premium principle and deals with the Optimal Reinsurance Problem if risk levels are measured by modern risk functions. These risk functions include as particular cases every Deviation Measure, every Expectation Bounded Risk Measure, and most of the Coherent, Convex or Consistent Risk Measures. Thus it may be worth to point out the level of generality of the analysis, since a unified approach is developed that does not depend on the concrete risk function to be used.

The paper's outline is as follows. Section 2 will present our general Optimal Reinsurance Problem and the basic conditions and properties of the risk function ρ to be used. Since the risk function is not differentiable in general, the optimization problem is not differentiable either, and Section 3 will be devoted to overcome this caveat. Actually, the results of this section will play a critical role in the rest of the article. We will use the Representation Theorems of Risk Measures so as to transform the initial Optimal Reinsurance Problem in a minimax problem. Later, following an idea developed in Balbás and Romera (2007) and Balbás *et al.* (2008), the minimax problem is equivalent to a new linear (and therefore differentiable) problem in Banach spaces. In particular, the dual variable belongs to the set of probabilities on the Borel $\sigma-$algebra of the sub-gradient of ρ. Since this fact would provoke high degree of complexity when dealing with the optimality conditions of the linear problem, Theorem 3 is one of the most important results in Section 3 and the whole article, because it guarantees that the optimal dual solution will be a Dirac Delta, and thus we can leave the use of general probability measures in order to characterize the Optimal Reinsurance Contract. Section 3 ends by yielding necessary and sufficient optimality conditions. Special interest may merit Theorem 4, since it provides a Variational Principle that will often apply in the remaining sections.

Section 4 is devoted to verify whether the usual types of reinsurance satisfy the Optimality Conditions, with special focus on quota-share and stop-loss contracts. It will be shown that a quota-share reinsurance hardly will be optimal regardless of the risk function ρ, while a stop-loss reinsurance much more easily satisfies the Optimality Conditions. The main reason is that the optimality of stop-loss contracts is closely related to the existence of bang-bang-like solutions for the Variational Principle above.

Despite of the level of generality of the analysis it may be worth to study particular risk functions in detail, and this is the focus of Section 5. So, the Optimality Conditions will be tested if ρ equals the Standard Deviation, the Absolute Deviation and the Conditional Value at Risk. Obviously, the Optimality Conditions may be tested in detail for much more alternative risk measures (the measure of Wang, Wang, 2000, down side semi-deviations, etc.), but we had to make a decision. The Standard Deviation was selected because it has been very frequently used in Finance and Insurance, the Absolute Deviation has shown more adequate properties with respect to the Stochastic Dominance in presence of heavy tails and/or asymmetries (Ogryczak and Ruszczynski, 2002) and the Conditional Value at Risk is becoming more and more interesting in Finance and Insurance because it also respects the Stochastic Dominance (Ogryczak and Ruszczynski, 2002), provides information about the degree of risk in monetary terms (capital requirements, reserves, etc.), shows suitable analytic properties and is deeply known and understood by many practitioners. For the three risk functions we will find that the optimal strategy is closely related to a stop-loss-like reinsurance.

The last section of the paper points out the most important conclusions.

2. PRELIMINARIES AND NOTATIONS

Consider the probability space $(\Omega, \mathcal{F}, \mu)$ composed of the set of "states of the world" Ω, the $\sigma-$algebra $\mathcal{F}$ and the probability measure μ. Consider also a couple of conjugate numbers $p \in [1, \infty)$ and $q \in (1, \infty]$ (*i.e.*, $1/p + 1/q = 1$). As usual L^p (L^q) denotes the Banach space of $\mathbb{R}-$valued measurable functions y on Ω such that $E\left(|y|^p\right) < \infty$, $E\left(\right)$ representing the mathematical expectation ($E\left(|y|^q\right) < \infty$, or y essentially bounded if $q = \infty$). According to the Riesz Representation Theorem, we have that L^q is the dual space of L^p.

Fix the random variable $y_0 \in L^p_+ = \{y \in L^p : \mu\left(y \geq 0\right) = 1\}$ providing us with the final amount of money that a insurance company must pay at a future date T, and take a "Minimum Pure Premium"

$$0 \leq S \leq E\left(y_0\right) \tag{1}$$

that this company would like to withhold.

Let

$$\rho : L^p \longrightarrow \mathbb{R}$$

be the general risk function that the insurer uses in order to control the risk level of his final (at T) wealth. Denote by

$$\Delta_\rho = \left\{z \in L^q; -E\left(yz\right) \leq \rho\left(y\right),\ \forall y \in L^p\right\}. \tag{2}$$

The set Δ_ρ is obviously convex. We will assume that Δ_ρ is also $\sigma\left(L^q, L^p\right)-$compact and

$$\rho\left(y\right) = Max\ \left\{-E\left(yz\right) : z \in \Delta_\rho\right\} \tag{3}$$

holds for every $y \in L^p$. Actually, these are quite natural assumptions. Indeed, they are closely related to the Representation Theorems of Risk Measures stated in Rockafellar *et al.* (2006), where the authors consider $p = 2$. Following the ideas of the paper above, and bearing in mind the Representation Theorem 2.4.9 in Zalinescu (2002) for convex functions, it is easy to prove that the $\sigma\left(L^q, L^p\right)-$compactness of Δ_ρ and the fulfillment of (3) hold if:

a) ρ is a continuous Expectation Bounded Risk Measure in the sense of Rockafellar *et al.* (2006),[2] in which case

$$\Delta_\rho \subset \{z \in L^q; E(z) = 1\}$$

and $\rho(y) \geq -E(y)$ for every $y \in L^p$ (with strict inequality if y is not a constant or zero-variance random variable).

b) ρ is a continuous Deviation (or Deviation Measure) in the sense of Rockafellar *et al.* (2006),[3] in which case

$$\Delta_\rho \subset \{z \in L^q; E(z) = 0\}$$

and $\rho(y) \geq 0$ for every $y \in L^p$ (with strict inequality if y is not a constant or zero-variance random variable).

Particular interesting examples are the Conditional Value at Risk ($CVaR$) of Rockafellar *et al.* (2006), The Dual Power Transform (DPT) of Wang (2000), the Wang Measure (Wang, 2000), the $p-$deviation given by $\rho(y) = \left[E\left(|E(y) - y|^p\right)\right]^{1/p}$, or the downside $p-$semi-deviation given by $\rho(y) = \left[E\left(|Max\ \{E(y) - y, 0\}|^p\right)\right]^{1/p}$, amongst many other risk functions.[4]

Assumption 1. Henceforth we will assume that Δ_ρ is $\sigma(L^q, L^p)-$compact, (3) holds and $E()$ remains constant on Δ_ρ. If we denote $E(z) = \tilde{E} \in \mathbb{R}$ for every $z \in \Delta_\rho$ then we will also suppose that $\tilde{E} \geq 0$ and

$$\rho(y) \geq -\tilde{E}E(y) \tag{4}$$

holds for every $y \in L^p$. □

Proposition 1. *Under Assumption* 1 *the constant random variable* $z = \tilde{E}$ *a.s. belongs to* Δ_ρ.

Proof. It immediately follows from (2) and (4). □

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[2] Furthermore, if ρ is also coherent in the sense of Artzner *et al.* (1999) then

$$\Delta_\rho \subset L^q_+ = \{z \in L^q; \mu(z \geq 0) = 1\}.$$

[3] Moreover, according to Theorem 2.2.20 in Zalinescu (2002), ρ is continuous if and only if ρ is lower semi-continuous. The same equivalence holds if ρ is a Expectation Bounded Risk Measure.

[4] If ρ equals the Wang measure or the DPT (or other risk measures given by distorting functions) then see Cherney (2006) for further details about Δ_ρ.

Denote by $y \in L^p$ the total amount that the insurer will pay after a reinsurance contract, *i.e.*, $y_0 - y$ will denote the amount to be paid by the reinsurer. Then the insurer will choose y by solving the optimization problem

$$\begin{cases} Min\ \rho\left(kE\left(y\right)-y\right) \\ y \leq y_0 \\ S - E\left(y\right) \leq 0 \\ y \geq 0 \end{cases} \tag{5}$$

where $k \geq 1$ denotes the proportion of the Pure Premium that insurer and reinsurer use in order to price.

3. Optimal reinsurance: Primal and dual problems and optimality conditions

In general ρ will be non-differentiable and therefore so will be Problem (5). To overcome this caveat we will follow the method proposed in Balbás *et al.* (2008). So, bearing in mind (3), Problem (5) is equivalent to Problem

$$\begin{cases} Min\ \theta \\ \theta + E\left(\left(kE\left(y\right)-y\right)z\right) \geq 0, \quad \forall z \in \Delta_\rho \\ y \leq y_0 \\ S - E\left(y\right) \leq 0 \\ \theta \in \mathbb{R},\ y \geq 0 \end{cases} \tag{6}$$

in the sense that y solves (5) if and only if there exists $\theta \in \mathbb{R}$ such that (θ, y) solves (6), in which case

$$\theta = \rho\left(kE\left(y\right)-y\right)$$

holds. Since

$$E\left(\left(kE\left(y\right)-y\right)z\right) = kE\left(y\right)\tilde{E} - E\left(yz\right) = E\left(y\left(k\tilde{E}-z\right)\right)$$

(5) and (6) are equivalent to

$$\begin{cases} Min\ \theta \\ \theta + E\left(y\left(k\tilde{E}-z\right)\right) \geq 0, \quad \forall z \in \Delta_\rho \\ y \leq y_0 \\ S - E\left(y\right) \leq 0 \\ \theta \in \mathbb{R},\ y \geq 0 \end{cases} \tag{7}$$

Notice that (7) is more than differentiable because it is linear. Its first constraint is valued on the Banach space $\mathcal{C}\left(\Delta_\rho\right)$ of real-valued and continuous functions on the

($weak^*$) compact space Δ_ρ, whereas the second one is valued on L^p. Since their duals are $\mathcal{M}(\Delta_\rho)$ and L^q, $\mathcal{M}(\Delta_\rho)$ denoting the space of inner regular real valued $\sigma-$additive measures on the Borel $\sigma-$algebra of Δ_ρ (endowed with the $weak^*$ topology), the Lagrangian function

$$\mathcal{L} : \mathbb{R}\times L^p \times \mathcal{M}(\Delta_\rho) \times L^q \times \mathbb{R} \longrightarrow \mathbb{R}$$

becomes

$$\begin{array}{l} \mathcal{L}(\theta, y, \nu, \lambda, \tau) = \\ \theta \left(1 - \int_{\Delta_\rho} d\nu(z)\right) + \int_{\Delta_\rho} E\left[yz - k\tilde{E}y\right] d\nu(z) + E(y\lambda) - E(y_0\lambda) + S\tau - E(y)\tau. \end{array}$$

Following Luenberger (1969) the element $(\nu, \lambda, \tau) \in \mathcal{M}(\Delta_\rho) \times L^q \times \mathbb{R}$ is dual feasible if and only if it belongs to the non-negative cone $\mathcal{M}_+(\Delta_\rho) \times L^q_+ \times \mathbb{R}_+$ and

$$Inf\ \{\mathcal{L}(\theta, y, \nu, \lambda, \tau) : \theta \in \mathbb{R},\ y \in L^p_+\} > -\infty,$$

in which case the infimum above equals the dual objective on (ν, λ, τ). Hence, the dual problem of (7) becomes

$$\begin{cases} Max\ S\tau - E(y_0\lambda) \\ \int_{\Delta_\rho} E\left[y\left(z + \lambda - k\tilde{E} - \tau\right)\right] d\nu(z) \geq 0, \quad \forall y \in L^p_+ \\ \nu \in \mathcal{P}(\Delta_\rho),\ \lambda \in L^q_+,\ \tau \in \mathbb{R}_+ \end{cases} \tag{8}$$

$\mathcal{P}(\Delta_\rho)$ denoting the set composed of those elements in $\mathcal{M}(\Delta_\rho)$ that are probabilities.

$\mathcal{P}(\Delta_\rho)$ is convex, and the Alaoglu's Theorem easily leads to the compactness of $\mathcal{P}(\Delta_\rho)$ when endowed with the $\sigma(\mathcal{M}(\Delta_\rho), \mathcal{C}(\Delta_\rho))$ $-$topology (Horvàth, 1966, or Anderson and Nash, 1987). Besides, given $z \in \Delta_\rho$ we will denote by $\delta_z \in \mathcal{P}(\Delta_\rho)$ the usual Dirac delta that concentrates the mass on $\{z\}$, *i.e.*, $\delta_z(\{z\}) = 1$ and $\delta_z(\Delta_\rho \setminus \{z\}) = 0$. It is known that the set of extreme points of $\mathcal{P}(\Delta_\rho)$ is given by

$$ext(\mathcal{P}(\Delta_\rho)) = \{\delta_z; z \in \Delta_\rho\}, \tag{9}$$

though we will not have to draw on this result. The objective function in (8) does not depend on the variable $\nu \in \mathcal{P}(\Delta_\rho)$, which, along with (9), suggest that the solution of (8) could be achieved in $\{\delta_z; z \in \Delta_\rho\}$. Let us show that this guesstimate is correct.

Lemma 2. *(Mean Value Theorem). Let $\nu \in \mathcal{P}(\Delta_\rho)$. Then there exists $z_\nu \in \Delta_\rho$ such that*

$$\int_{\Delta_\rho} E(yz)\, d\nu(z) = E(yz_\nu) \tag{10}$$

holds for every $y \in L^p$.

Proof. Consider the linear function

$$L^p \ni y \longrightarrow \varphi(y) = \int_{\Delta_\rho} E(yz)\, d\nu(z) \in \mathbb{R}.$$

φ is clearly continuous because for every sequence $(y_n)_{n=1}^{\infty}$ converging to zero in L^p the sequence of functions

$$L^q \ni z \longrightarrow E(y_n z) \in \mathbb{R}$$

uniformly converges to zero on bounded sets of L^q, and, consequently,

$$\left(\int_{\Delta_\rho} E(y_n z)\, d\nu(z)\right)_{n=1}^{\infty}$$

converges to zero if Δ_ρ is bounded. Δ_ρ is bounded because it is $\sigma(L^q, L^p)$ −compact.

Since φ is linear and continuous the Riesz Representation Theorem guarantees the existence of $z_\nu \in L^q$ such that (10) holds. Thus, it only remains to show that $z_\nu \in \Delta_\rho$, *i.e.*, according to (2), we must prove the inequality

$$-E(yz_\nu) \le \rho(y), \ \forall y \in L^p.$$

(10) points out that it is sufficient to see

$$-\int_{\Delta_\rho} E(yz)\, d\nu(z) \le \rho(y), \ \forall y \in L^p.$$

For every $z \in \Delta_\rho$ one has that $-E(yz) \le \rho(y)$, $\forall y \in L^p$, and, therefore,

$$-\int_{\Delta_\rho} E(yz)\, d\nu(z) \le \int_{\Delta_\rho} \rho(y)\, d\nu(z) = \rho(y),$$

for every $y \in L^p$. □

Theorem 3. *If $(\nu, \lambda, \tau) \in \mathcal{P}(\Delta_\rho) \times L^q_+ \times \mathbb{R}_+$ solves (8) then there exists $z \in \Delta_\rho$ such that $(\delta_z, \lambda, \tau)$ solves (8).*

Proof. Consider (ν, λ, τ) solving (8) and take $z_\nu \in \Delta_\rho$ satisfying (10). Then, for every $y \in L^p_+$ we have that

$$\begin{aligned} 0 \ \le \ & \int_{\Delta_\rho} E\left[y\left(z + \lambda - k\tilde{E} - \tau\right)\right] d\nu(z) = \\ & E(yz_\nu) + E(y\lambda) - k\tilde{E}E(y) - \tau E(y). \end{aligned}$$

Thus

$$0 \leq \int_{\Delta_\rho} E\left[y\left(z+\lambda-k\tilde{E}-\tau\right)\right] d\delta_{z_\nu}(z),$$

holds for every $y \in L_+^p$, which proves that $(\delta_{z_\nu}, \lambda, \tau)$ is (8)-feasible. Thus, the result trivially follows because the objective values of (8) in (ν, λ, τ) and $(\delta_{z_\nu}, \lambda, \tau)$ are identical. □

Remark 1. *The latter theorem leads to significant consequences. In particular, we can consider the alternative and far simpler dual problem*

$$\begin{cases} Max\ S\tau - E(y_0\lambda) \\ z+\lambda-k\tilde{E}-\tau \geq 0 \\ z \in \Delta_\rho,\ \lambda \in L_+^q,\ \tau \in \mathbb{R}_+ \end{cases} \tag{11}$$

where $z \in \Delta_\rho$ *is playing the role of* $\nu \in \mathcal{P}(\Delta_\rho)$. *Indeed, notice that Theorem* 3 *guarantees that we only have to focus on the (8)-feasible solutions taking the form* $(\delta_z, \lambda, \tau)$ *for some* $z \in \Delta_\rho$, *and for such a feasible solution the first constraint of (8) is equivalent to the first constraint of (11).* □

Bearing in mind the latter remark the Karush-Kuhn-Tucker (or complementary slackness) conditions of (7) may be given by using (11) rather than (8). Then they are

$$\begin{cases} \theta + E\left(y\left(k\tilde{E}-z\right)\right) = 0 \\ \lambda(y_0 - y) = 0 \\ \tau(E(y) - S) = 0 \\ y\left(z+\lambda-k\tilde{E}-\tau\right) = 0 \\ \theta \in \mathbb{R},\ y \in L_+^p, z \in \Delta_\rho,\ \lambda \in L_+^q,\ \tau \in \mathbb{R}_+ \end{cases} \tag{12}$$

and are sufficient optimality conditions to solve both (7) and (11). In general, they are not necessary since we are dealing with infinite dimensional Banach spaces and the so called "duality gap" between (7) and (11) might arise, though several qualifications may be imposed so as to prevent this pathological situation (see Luenberger, 1969, or Anderson and Nash, 1987). Besides, as usual in Optimization Theory, the dual variable λ provides us with sensitivity of the optimal value θ with respect to the initial risk y_o, and τ gives the sensitivity of θ with respect to the withheld Pure Premium S.

Assumption 2. There is no duality gap between (7) and (8), *i.e.*, (12), along with the restrictions of (7) and (11), provide necessary and sufficient optimality conditions for both (7) and (11). □

Let us end this section with alternative optimality conditions implied by (12) that will often apply throughout the rest of the paper.

Theorem 4. *(Variational Principle). Suppose that $y \in L^p$ is (5)-feasible and satisfies $\mu(y > 0) = 1$. Then:*

a) y solves (5) if and only if there exist $\tau \in \mathbb{R}_+$ and $z^* \in \Delta_\rho$ such that

$$\begin{cases} E(yz^*) \geq E(yz), & \forall z \in \Delta_\rho \\ z^* \leq k\tilde{E} + \tau \\ \left(k\tilde{E} + \tau - z^*\right)(y_0 - y) = 0 \\ \tau (E(y) - S) = 0 \end{cases} \tag{13}$$

In such a case $\theta = E\left(y\left(z^* - k\tilde{E}\right)\right)$, y, $z = z^*$, $\lambda = k\tilde{E} + \tau - z^*$, and τ solve both (7) and (11) respectively.

b) If $\tilde{E} = 0$ and $E(y) > S$ then y solves (5) if and only if $0 \geq E(yz)$ for every $z \in \Delta_\rho$. In such a case $\theta = 0$, y, $z = 0$, $\lambda = 0$, and $\tau = 0$ solve both (7) and (11) respectively.

c) If ρ is a Deviation Measure and $E(y) > S$ then y solves (5) if and only if y is a constant (or zero variance) random variable.

Proof. *a*) Suppose that y solves (5), and take a primal solution (θ, y) and a dual one (z^*, λ, τ). The first constraint of (7) leads to

$$\theta + E\left(y\left(k\tilde{E} - z\right)\right) \geq 0$$

for every $z \in \Delta_\rho$, whereas the first constraint in (12) leads to

$$\theta + E\left(y\left(k\tilde{E} - z^*\right)\right) = 0.$$

Whence, the first condition in (13) becomes trivial. Besides, the fourth equation in (12) gives

$$\lambda = k\tilde{E} + \tau - z^*$$

and therefore $\lambda \geq 0$ implies the second condition in (13). Moreover, the latter expression and the second equation in (12) imply the third equation in (13), and the last equation in (13) comes from the third equation in (12).

Conversely, if y is (5)-feasible and satisfies $\mu(y > 0) = 1$ then it may be immediately verified that the proposed solution satisfies (12) along with the primal and the dual constraints.

b) If $\tilde{E} = 0$ and $E(y) > S$ then the last equation in (13) gives $\tau = 0$ and the second one gives $z^* \leq 0$. Since the expected value of z^* must vanish ($z^* \in \Delta_\rho$) we have that $z^* = 0$.[5] Thus the first condition of (13) trivially implies that $0 \geq E(yz)$ for every $z \in \Delta_\rho$.

Conversely, if $0 \geq E(yz)$ for every $z \in \Delta_\rho$ then it may be immediately verified that the proposed solution satisfies (12) along with the primal and the dual constraints.

c) If ρ is a Deviation Measure and $E(y) > S$ then $\tilde{E} = 0$, and Statement *b*) shows that the optimal value of (5) is $\theta = 0$ if y solves (5). Then $\rho(kE(y) - y) = 0$ implies that $kE(y) - y$ is constant and therefore so is y.

Conversely if y is constant then so is $kE(y) - y$ and therefore $\rho(kE(y) - y) = 0$, which implies that y solves (5) because ρ cannot attain negative values. □

Remark 2. *Notice that Theorem* 4*c*) *points out that for every deviation* ρ *the solution* y *of (5) will almost always satisfy* $E(y) = S$. *Indeed,* $E(y) > S$ *would provoke that* y *should be constant, and therefore it should equal its expectation. Then* $y_0 \geq y = E(y) > S$, *and consequently the minimum Pure Premium to be withheld should be less than a lower bound of the initial risk* y_0, *which hardly occurs in practice.* □

Remark 3. *Conditions (12) are necessary and sufficient and therefore they are a quite useful tool. Nevertheless, in practical examples and applications it might be difficult to find an explicit solution of the system generated by (12). Things become much easier if we are able to compute the solution of (7) or (11) by an alternative algorithm, since then (12) easily applies to solve the remaining problem. This is for instance the way followed in Balbás and Romera (2007) or Balbás et al. (2008), where the authors deal with infinite-dimensional linear programming in order to solve risk minimization problems associated with usual financial topics. Following the ideas of these authors we could develope an infinite-dimensional simplex-like algorithm so as to solve (11) under appropriate assumptions (see also Anderson and Nash, 1987), and then we could use (12) so as to solve (7). However, in this paper we will prefer to draw on the Variational Principle provided by the condition* $E(yz^*) \geq E(yz)$ *of Theorem* 4. *Indeed, the next two sections will show that it may be very useful in both theoretical approaches and practical situations. In particular, when dealing with practical applications it may yield an interesting relationship between the solution* y *of (7) and the solution* z^* *of (11), that may be found by solving the simple and frequently linear problem*

$$\begin{cases} Max\ E(yz) \\ z \in \Delta_\rho \end{cases} \tag{14}$$

Some illustrative examples will be studied in Section 5. □

[5] Recall that $z^* = 0 \in \Delta_\rho$ owing to Proposition 1.

4. Particular reinsurance contracts

This section will be devoted to verify whether the most important (or usual) reinsurance contracts solve Systems (12) or (13). In particular, we will focus on the full reinsurance ($y = 0$), the null reinsurance ($y = y_0$) and quota-share and/or stop-loss reinsurance contracts.

Proposition 5. *Suppose that $\mu(y_0 > 0) = 1$. If $\tilde{E} = 0$ then $y = 0$ solves (5) (or (7)) if and only if $S = 0$. If $\tilde{E} > 0$ then $y = 0$ solves (5) if and only if $S = 0$ and $k = 1$.*[6]

Proof. If $y = 0$ solves (5) then the primal constraint $E(y) = 0 \geq S$ and (1) lead to $S = 0$. Moreover, Conditions (12) become $\theta = 0$ and $\lambda = 0$. Therefore the constraints of (11) lead to

$$z \geq k\tilde{E} + \tau.$$

Consequently, taking the expectation, $\tilde{E} \geq k\tilde{E} + \tau$, which implies

$$0 \geq (1 - k)\tilde{E} \geq \tau \geq 0.$$

Thus all of them are equalities, and $\tilde{E} > 0$ leads to $k = 1$.

Conversely, if the proposed conditions hold then it is easy to see that $\theta = 0$, $y = 0$, $\lambda = 0$, $z = \tilde{E}$ and $\tau = 0$ satisfy System (12), and they are feasible due to Proposition 1. □

Proposition 6. *Suppose that $S = E(y_0)$. Then $y = y_0$ solves (5).*

Proof. Every (5)-feasible solution must satisfy $E(y) \geq S = E(y_0)$. Thus, $E(y_0 - y) = 0$ and $y_0 - y \geq 0$ imply $y_0 - y = 0$, *i.e.*, y_0 is the unique (5)-feasible random variable. □

Theorem 7. *Suppose that $S < E(y_0)$ and $\mu(y_0 > 0) = 1$. Then:*

a) $y = y_0$ solves (5) if and only if there exists $z^* \in \Delta_\rho$ such that

$$E(y_0 z^*) \geq E(y_0 z) \tag{15}$$

for every $z \in \Delta_\rho$ and

$$\mu\left(k\tilde{E} - z^* \geq 0\right) = 1. \tag{16}$$

In such a case $\theta = E\left(y_0\left(z^* - k\tilde{E}\right)\right)$, $y = y_0$, $z = z^*$, $\lambda = k\tilde{E} - z^*$ and $\tau = 0$ solve the primal and dual problem respectively.

[6] Recall that $\tilde{E} = 0$ holds for deviations and $\tilde{E} = 1$ holds for expectation bounded risk measures.

b) If $\tilde{E} = 0$ then $y = y_0$ is optimal if and only if $E(y_0 z) \leq 0$ for every $z \in \Delta_\rho$. In such a case $\theta = 0$, $y = y_0$, $\lambda = 0$, $z = 0$ and $\tau = 0$ solve the primal and dual problem respectively.

c) If ρ is a Deviation Measure then $y = y_0$ is optimal if and only if y_0 is constant (or zero-variance).

Proof. *a)* and *b)*. The given conditions immediately follow from (13) since $S < E(y_0)$ leads to $\tau = 0$. Conversely, if (15) and (16) hold then it is immediate to verify that the proposed solution is feasible and satisfies (12).

c) It immediately follows from Theorem 4*c*. □

Next let us verify quota-share-like reinsurance contracts. Obviously, $y \in L^p$ and lying between 0 and y_0 is said to be a quota-share reinsurance if there exists $\alpha \in (0,1)$ such that $y = \alpha y_0$.[7]

Theorem 8. *Suppose that $\mu(y_0 > 0) = 1$.*

a) Suppose that $\tilde{E} > 0$. $y = \alpha y_0$ with $\alpha \in (0,1)$ is optimal if and only if $\alpha \geq \dfrac{S}{E(y_0)}$, $k = 1$ and $E(y_0 z) \leq \tilde{E} E(y_0)$ for every $z \in \Delta_\rho$. In such a case $\theta = 0$, $y = \alpha y_0$, $z = \tilde{E}$, $\lambda = 0$ and $\tau = 0$ solve the primal and the dual problem respectively.

b) Suppose that $\tilde{E} = 0$. $y = \alpha y_0$ with $\alpha \in (0,1)$ is optimal if and only if $\alpha \geq \dfrac{S}{E(y_0)}$ and $E(y_0 z) \leq 0$ for every $z \in \Delta_\rho$. In such a case $y = \alpha y_0$, $\theta = 0$, $z = 0$, $\lambda = 0$ and $\tau = 0$ solve the primal and the dual problem respectively.

c) If ρ is a Deviation Measure then $y = \alpha y_0$ with $\alpha \in (0,1)$ is optimal if and only if $\alpha \geq \dfrac{S}{E(y_0)}$ and y_0 is constant.

Proof. Suppose that $y = \alpha y_0$ with $\alpha \in (0,1)$ is optimal. Then the third condition of System (13) leads to

$$z^* = k\tilde{E} + \tau.$$

Computing the expectation in both sides one has $\tilde{E} = k\tilde{E} + \tau$ and $\tau = (1-k)\tilde{E}$. Since $1 - k \leq 0$ and $\tilde{E} \geq 0$ we have $\tau \leq 0$ and thus

$$\tau = 0$$

because the opposite inequality is imposed in the statement of Theorem 4.

[7] The extreme cases $\alpha = 0$ and $\alpha = 1$ have been excluded because they were analyzed in the discussion above.

Now, $z^* = k\tilde{E} + \tau$ gives

$$z^* = k\tilde{E}, \tag{17}$$

and the condition $\theta = E\left(y\left(k\tilde{E} - z^*\right)\right)$ of Theorem 4 implies

$$\theta = 0. \tag{18}$$

a) Suppose that $\tilde{E} > 0$ and $y = \alpha y_0$ with $\alpha \in (0,1)$ is optimal. Taking expectation in (17) we have $\tilde{E} = k\tilde{E}$ and therefore $k = 1$. $\alpha \geq \dfrac{S}{E(y_0)}$ must hold owing to the third constraint of (7), and (18), along with the first constraint of (13), imply that

$$\alpha E(y_0 z) \leq \alpha \tilde{E} E(y_0)$$

must hold for every $z \in \Delta_\rho$ because $z^* = k\tilde{E} = \tilde{E}$. Conversely, if the given conditions hold, then it is easy to verify that the provided solution is composed of feasible elements that satisfy (12).

b) Suppose that $\tilde{E} = 0$ and $y = \alpha y_0$ with $\alpha \in (0,1)$ is optimal. Then the given conditions may be proved by following a similar argument as in the proof above. Furthermore, the converse implication becomes trivial once more.

c) Suppose that ρ is a Deviation measure and $y = \alpha y_0$ with $\alpha \in (0,1)$ is optimal. Then $\alpha \geq \dfrac{S}{E(y_0)}$ follows from the third primal constraint, and (18) leads to $\alpha\rho(kE(y_0) - y_0) = 0$, which implies that $kE(y_0) - y_0$ is constant, and thus so is y_0. The converse is obvious because every Deviation Measure vanishes on zero-variance random variables and is positive for non-constant ones. □

Finally let us verify stop–loss-like reinsurance contracts. Obviously, $y \in L^p$ and lying between 0 and y_0 is said to be a stop-loss reinsurance if there exists $\alpha > 0$ such that

$$y = \begin{cases} y_0, & y_0 \leq \alpha \\ \alpha & otherwise \end{cases}. \tag{19}$$

Hereafter the random variable of (19) will be denoted by y_0^α. Obviously, since the null reinsurance has been already studied, without loss of generality we will assume in the remainder of this section that $\mu(y_0 > \alpha) > 0$.

Theorem 9. *Suppose that $\mu(y_0 > \alpha) > 0$, $S < E(y_0^\alpha)$ and $\mu(y_0 > 0) = 1$ hold. Denote $\Omega_\alpha = \{\omega \in \Omega; y_0(\omega) > \alpha\}$. Then:*

a) *y_0^α solves (5) if and only if there exists $z^* \in \Delta_\rho$ such that*

$$z^* \leq k\tilde{E}, \tag{20}$$

$$z^{*}(\omega) = k\tilde{E},\ \omega \in \Omega_{\alpha} \tag{21}$$

and

$$E\left(y_0^{\alpha} z^{*}\right) \geq E\left(y_0^{\alpha} z\right) \tag{22}$$

for every $z \in \Delta_{\rho}$. In such a case $\theta = E\left(y_0^{\alpha} z^{*}\right) - k\tilde{E}E\left(y_0^{\alpha}\right)$, y_0^{α}, z^{*}, $\lambda = k\tilde{E} - z^{*}$ and $\tau = 0$ solve both the primal and the dual problem respectively.

b) Suppose that $\tilde{E} = 0$. y_0^{α} solves (5) if and only if

$$E\left(y_0^{\alpha} z\right) \leq 0 \tag{23}$$

for every $z \in \Delta_{\rho}$. In such a case $\theta = 0$, y_0^{α}, $z = 0$, $\lambda = 0$ and $\tau = 0$ solve both the primal and the dual problem.

c) Suppose that ρ is a Deviation Measure. Then y_0^{α} solves (5) if and only if y_0^{α} is constant (zero-variance).

Proof. Suppose that y_0^{α} solves (5). Then the last condition in (13) gives $\tau = 0$ and (20) follows from the second constraint of (13). Furthermore, (21) immediately follows from the third condition of (13) and (22) is obvious.

a) It only remains to prove the converse implication, which is trivial since one only has to verify that the proposed solution satisfies (12).

b) According to (20) $z^{*} \leq 0$, which implies $z^{*} = 0$ because $E\left(z^{*}\right) = 0$ ($z^{*} \in \Delta_{\rho}$). Hence, (23) follows from (22), and the converse is immediate since one only needs to check the proposed solution in (12).

c) It is a trivial consequence of Theorem 4*c*. □

If we remove the assumption $S < E\left(y_0^{\alpha}\right)$ then things become a little bit more complex.

Theorem 10. *Suppose that $\mu\left(y_0 > \alpha\right) > 0$, $S = E\left(y_0^{\alpha}\right)$ and $\mu\left(y_0 > 0\right) = 1$ hold. Denote $\Omega_{\alpha} = \{\omega \in \Omega; y_0(\omega) > \alpha\}$. Then, y_0^{α} solves (5) if and only if there exist $z^{*} \in \Delta_{\rho}$ and $\tau \in \mathbb{R}_{+}$ such that*

$$z^{*} \leq k\tilde{E} + \tau,$$

$$z^{*}(\omega) = k\tilde{E} + \tau,\ \omega \in \Omega_{\alpha}$$

and

$$E\left(y_0^{\alpha} z^{*}\right) \geq E\left(y_0^{\alpha} z\right)$$

for every $z \in \Delta_{\rho}$. In such a case $\theta = E\left(y_0^{\alpha} z^{*}\right) - k\tilde{E}E\left(y_0^{\alpha}\right)$, y_0^{α}, z^{*}, $\lambda = k\tilde{E} + \tau - z^{*}$ and τ solve both the primal and the dual problem respectively. □

We will not give any proof of this result because it is absolutely analogous to the proof of the previous theorem.

Remark 4. *Despite Theorem* 9 *seems to be more exhaustive than Theorem* 10*, condition* $S < E\left(y_0^{\alpha}\right)$ *is "more ambiguous" than* $S = E\left(y_0^{\alpha}\right)$*. Indeed there cannot be two different values of* α *satisfying the equality because if* $\alpha_1 < \alpha_2$*,* $\mu\left(y_0 > \alpha_2\right) > 0$ *and* $E\left(y_0^{\alpha_1}\right) = E\left(y_0^{\alpha_2}\right)$ *then* $E\left(y_0^{\alpha_2} - y_0^{\alpha_1}\right) = 0$ *and* $y_0^{\alpha_2} - y_0^{\alpha_1} \geq 0$ *imply* $y_0^{\alpha_2} - y_0^{\alpha_1} = 0$*. Whence,*

$$y_0^{\alpha_1}\left(\omega\right) = \alpha_1 < \alpha_2 = y_0^{\alpha_2}\left(\omega\right)$$

for $\omega \in \Omega_{\alpha_2}$ *implies* $\mu\left(y_0 > \alpha_2\right) = 0$*, against the assumptions.*

Consequently, it is also very easy to verify the conditions of Theorem 10 *in practice. One just needs to compute the unique* α *such that* $E\left(y_0^{\alpha}\right) = S$ *and then check the existence of* z^* *in* Δ_ρ*.* □

Theorem 8 has clarified that quota-share contracts will hardly be optimal. However, the next section will point out that stop-loss contracts are very often optimal or "almost optimal", *i.e.*, the conditions of Theorems 9 and 10 much more easily hold. The main reason will be that (14) frequently generates a linear problem with a Bang Bang solution (Luenberger, 1969), and this sort of solution will be closely related to the stop-loss reinsurance.

5. Particular risk functions

Until now all the previous results of the paper hold regardless of the risk function we are using. In this section we will analyze some important examples of risk function. In particular, we will focus on the Standard Deviation since, as said in the introduction, it is very used in the literature, the Absolute Deviation, since it has better properties with respect to the Second Order Stochastic Dominance if asymmetry and/or heavy tails are involved (Ogryczak and Ruszczynski, 1999), and the Conditional Value at Risk, since it is becoming a very well-known Coherent and Expectation Bounded Risk Measure that also respects the Stochastic Dominance (Ogryczak and Ruszczynski, 2002).

In general, the $p-$deviation

$$\sigma_p : L^p \longrightarrow \mathbb{R}$$

is defined by

$$\sigma_p\left(y\right) = \left(E\left(\left|y - E\left(y\right)\right|^p\right)\right)^{\frac{1}{p}} = \left\|y - E\left(y\right)\right\|_p.$$

Since L^q is the dual space of L^pit is known that

$$\begin{aligned}
\sigma_p\left(y\right) &= Max\ \left\{E\left(\left(y - E\left(y\right)\right)z\right); z \in L^q,\ \left\|z\right\|_q \leq 1\right\} \\
&= Max\ \left\{E(yz) - E\left(y\right)E\left(z\right); z \in L^q,\ \left\|z\right\|_q \leq 1\right\} \\
&= Max\ \left\{E\left(y\left(z - E\left(z\right)\right)\right); z \in L^q,\ \left\|z\right\|_q \leq 1\right\}.
\end{aligned}$$

Hence

$$\Delta_\rho = \left\{ z - E(z) ;\ z \in L^q,\ \ \|z\|_q \leq 1 \right\}. \tag{24}$$

Moreover, in the particular case $p = q = 2$, by using the properties of the orthogonal projection of Hilbert spaces it is easy to prove that

$$\Delta_2 = \left\{ z;\ z \in L^2,\ \ \|z\|_2 \leq 1,\ E(z) = 0 \right\} \tag{25}$$

(Rockafellar *et al.*, 2006).[8]

Theorem 11. *Suppose that $p = 2$, $\rho = \sigma_2$ and $\mu(y_0 > 0) = 1$.*

a) If $\mu(y_0 \geq S) = 1$ then (5) is solved by every constant function y such that $S \leq y \leq y_0$. If so, the optimal value of (5) vanishes.[9]

b) If $\mu(y_0 \geq S) < 1$ and $\alpha \in \mathbb{R}$ is such that $E(y_0^\alpha) = S$ then y_0^α solves (5).[10]

Proof. *a*) is obvious so let us prove *b*). We will prove that the properties of Theorem 4 are fulfilled. Take

$$\tau = \frac{\alpha - S}{\sigma_2(y_0^\alpha)} \tag{26}$$

and

$$z^* = \frac{y_0^\alpha - S}{\sigma_2(y_0^\alpha)}. \tag{27}$$

First of all notice that $\sigma_2(y_0^\alpha) > 0$ and therefore the definitions above are correct. Indeed, if y_0^α were constant then $E(y_0^\alpha) = S$ would imply $y_0^\alpha = S$, and consequently $y_0 \geq y_0^\alpha = S$, contradicting the assumptions.

Secondly, $\tau \geq 0$, since $\alpha < S$ would imply $y_0^\alpha \leq \alpha < S$, contradicting $E(y_0^\alpha) = S$. Thirdly, $z^* \in \Delta_2$. Indeed, according to (25) we must show that $E(z^*) = 0$ and $\|z^*\|_2^2 \leq 1$ hold. The first equality trivially follows from $E(y_0^\alpha) = S$, whereas the inequality is also satisfied because

$$\|z^*\|_2^2 = \left\| \frac{y_0^\alpha - S}{\sigma_2(y_0^\alpha)} \right\|_2^2 = \frac{\|y_0^\alpha - S\|_2^2}{\sigma_2(y_0^\alpha)^2} = \frac{\sigma_2(y_0^\alpha)^2}{\sigma_2(y_0^\alpha)^2} = 1. \tag{28}$$

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[8] Obviously

$$\Delta_2 \sqsupset \left\{ z;\ z \in L^2,\ \ \|z\|_2 \leq 1,\ E(z) = 0 \right\}$$

because $z = z - E(z)$ whenever $E(z) = 0$, and the opposite inclusion holds because for every z in the unit ball of L^2 we have that $E(z)$ and $z - E(z)$ are orthogonal, and therefore the Pithagorean Theorem leads to

$$1 \geq \|z\|_2^2 = \|E(z)\|_2^2 + \|z - E(z)\|_2^2 \geq \|z - E(z)\|_2^2.$$

[9] Notice that the assumed conditions imply that y is a stop-loss reinsurance.

[10] This result is closely related to that by Borch (1960).

Next let us prove that $E(y_0^\alpha z^*) \geq E(y_0^\alpha z)$ for every $z \in \Delta_2$. We will show that z^* solves the variational problem

$$\begin{array}{l} Max \ \int_\Omega y_0^\alpha z d\mu \\ \int_\Omega z d\mu = 0 \\ \int_\Omega z^2 d\mu \leq 1 \end{array}$$

Since this problem is obviously convex, it is sufficient to show that z^* satisfies the Karush-Kuhn-Tucker conditions (Luenberger, 1969), *i.e.*, we must state the existence of L_1, $L_2 \in \mathbb{R}$, $L_2 \geq 0$, such that $L_2\left(1 - \|z^*\|_2^2\right) = 0$ and

$$y_0^\alpha = L_1 + 2L_2 z^* = L_1 + 2L_2 \frac{y_0^\alpha - S}{\sigma_2\left(y_0^\alpha\right)}.$$

The first condition is obvious because (28) shows that $\|z^*\|_2^2 = 1$, and the second one clearly holds for $L_1 = S$ and $L_2 = \frac{1}{2}\sigma_2\left(y_0^\alpha\right)$.

It only remains to verify that $z^* \leq \tau$ and $z^* = \tau$ if $y_0^\alpha < y_0$ (*i.e.*, if $y_0^\alpha = \alpha$) but both expressions trivially follow from (26) and (27). □

Remark 5. *In the proof of the theorem above we have provided the values of τ and z^* without any previous computation, and then we have checked that z^* solves the variational problem (14). However, in practice the process will be different, that is, we will have to solve (14) in order to establish the relationship between the primal solution y and the dual one z^*. For this reason we will follow this second way in order to study the Absolute Deviation and the Conditional Value at Risk, despite the exposition will be a little bit more tedious.* □

Remark 6. *Let us consider now Problem (5) with $\rho = \sigma_1$, and suppose that $\mu(y_0 > 0) = 1$. Then (24) obviously implies that*

$$\Delta_1 = \left\{z - E(z);\ z \in L^\infty, -1 \leq z \leq 1\right\}.$$

Then the Variational Principle of Theorem 4 and Remark 3 leads to the linear optimization problem

$$\left\{\begin{array}{l} Max \ E\left(y\left(z - E(z)\right)\right) = \int_\Omega yz d\mu - E(y) \int_\Omega z d\mu \\ z \leq 1 \\ z \geq -1 \\ z \in L^\infty \end{array}\right.$$

It is easy to verify that the problem above satisfies the Slater Qualification (Luenberger, 1969), so the Karush-Kuhn-Tucker conditions become necessary and sufficient. Furthermore, the dual space of L^∞ contains L^1 and is composed of those *finitely-additive measures on the Borel $\sigma-$algebra of Ω being $\mu-$continuous and having finite variation (Horvàth, 1966). Thus, the Karush-Kuhn-Tucker conditions lead to the existence of two such a measures $m_1 \geq 0$ and $m_2 \geq 0$ such that*

$$\left\{ \begin{array}{l} y = E(y) + m_1 - m_2 \\ \int_\Omega (1 - z)\, dm_1 = 0 \\ \int_\Omega (1 + z)\, dm_2 = 0 \end{array} \right. \tag{29}$$

The second and third conditions lead to $z = 1$ whenever $m_1 \neq 0$ and $z = -1$ whenever $m_2 \neq 0$. Thus, there is a measurable partition $\Omega = A \cup B \cup C$ such that

$$\left\{ \begin{array}{lll} z = -1, & m_1 = 0, & \omega \in A \\ -1 \leq z \leq 1, & m_1 = m_2 = 0, & \omega \in B \\ z = 1, & m_2 = 0, & \omega \in C \end{array} \right.$$

Consequently, the first equality in (29) gives

$$\left\{ \begin{array}{ll} m_2 = E(y) - y, & \omega \in A \\ m_1 = y - E(y) & \omega \in C \end{array} \right.$$

and therefore $m_i \in L^1$, $i = 1, 2$, because they vanish out of the indicated sets. Summarizing

$$\left\{ \begin{array}{llll} z = -1, & m_1 = 0, & y = E(y) - m_2 \leq E(y), & \omega \in A \\ -1 < z < 1, & m_1 = m_2 = 0, & y = E(y), & \omega \in B \\ z = 1, & m_2 = 0, & y = E(y) + m_1 \geq E(y), & \omega \in C \end{array} \right. \tag{30}$$

The remaining conditions in Theorem 4 impose the existence of $\tau \geq 0$ such that

$$\left\{ \begin{array}{l} z \leq E(z) + \tau \\ z = E(z) + \tau, \quad if\ y < y_0 \end{array} \right.$$

Therefore the maximum of z will be $E(z) + \tau$.[11] *Since $z \leq 1$ there are to cases to consider:*

Case 1. *$E(z) + \tau < 1$. In such a case C is void (or a null set) and (30) implies that $y \leq E(y)$. Thus $y = E(y)$ has to be constant. Then $y_0 \geq y = E(y) \geq S$ imply that S has been chosen as a lower bound of the initial risk y_0 and thus every feasible and constant random variable y solves (5) because it makes σ_1 vanish.*

Case 2. *$E(z) + \tau = 1$. According to Theorem 4c we can also impose $E(y) = S$ since otherwise the solution of (5) would be constant and we would be in the scenario*

[11] Suppose at the moment that y_0 is not the solution of (5). According to Proposition 6 and Theorem 7, this case only appears in the trivial cases $S = E(y_0)$ or $S < E(y_0)$ and y_0 constant.

of Case 1*.Hence, bearing in mind (30),* $E(z)+\tau=1$ *and* $E(y)=S$ *there must be a measurable set* C *such that*

$$\begin{cases} C\subset\{\omega;\ y_0\geq S\} & \\ y=y_0, & \omega\notin C \\ S\leq y\leq y_0, & \omega\in C \\ E(y)=S & \end{cases} \tag{31}$$

and these are the necessary and sufficient conditions so as to guarantee that y *is optimal, since the requirements of Theorem* 4 *are fulfilled by taking* $z^*=z-E(z)$ *with* $z=\chi_C-\chi_{\Omega\setminus C}$ *and* $\tau=1-E(z)$*, which is non negative because* $-1\leq z\leq 1$ *and satisfies*

$$z^*=z-E(z)\leq 1-E(z)=\tau$$

with equality on C *because* $z=1$.[12]

In order to summarize we will provide a formal statement reflecting the findings above. □

Theorem 12. *Suppose that* $p=1$, $\rho=\sigma_1$ *and* $\mu(y_0>0)=1$.

a) If $\mu(y_0\geq S)=1$ *then (5) is solved by every constant function* y *such that* $S\leq y\leq y_0$*. If so, the optimal value of (5) vanishes.*

b) If $\mu(y_0\geq S)<1$ *and* y *is (5)-feasible then* y *solves (5) if and only if there exists a measurable set* C *such that (31) holds.*

c) If $\mu(y_0\geq S)<1$ *and* $\alpha\in\mathbb{R}$ *is such that* $E(y_0^\alpha)=S$ *then* y_0^α *solves (5).*

Proof. *a*) is obvious and *b*) may be proved by checking that the elements z^* and τ in the remark above make the conditions of Theorem 4 hold. To prove *c*) one must show that the conditions of *b*) are respected by y_0^α. It is sufficient to see that

$$\{\omega;\ y_0\geq\alpha\}\subset\{\omega;\ y_0\geq S\}$$

which is trivial if one shows that $\alpha\geq S$. But $\alpha<S$ would imply the contradiction $y_0^\alpha\leq\alpha<S=E(y_0^\alpha)$. □

Remark 7. *Suppose now that* $\rho=CVaR_{\mu_0}$, $\mu_0\in(0,1)$ *being the level of confidence.*[13] *In such a case Rockafellar et al. (2006) has stated that*

$$\Delta_\rho=\left\{z\in L^\infty;\ 0\leq z\leq\frac{1}{\mu_0},\ E(z)=1\right\}. \tag{32}$$

[12] As usual, χ_C and $\chi_{\Omega\setminus C}$ are the characteristic functions of C and its complementary.

[13] In order to simplify the exposition we will assume that $k<1/\mu_0$ and that the distribution of y_0 is continuous. The rest of cases may be also analized but the exposition is much more tedious. Furthermore both restrictions are quite natural. In particular, regarding the first one, k will never be in practice higher than 2 or 3, and μ_0 will never be more than 5%, *i.e.*, $1/\mu_0$ will be 20 at least.

In addition, Proposition 6 enables us to suppose that $S<E(y_0)$ holds.

Consequently, if $\mu(y>0)=1$ *and we would like to check whether* y *solves (5) then Theorem* 4 *and Remark* 3 *suggest to solve the linear optimization problem*

$$\begin{cases} Max\ E(yz)=\int_\Omega yzd\mu \\ z\le 1/\mu_0 \\ z\ge 0 \\ \int_\Omega zd\mu=1 \\ z\in L^\infty \end{cases}.$$

Once again it is easy to verify the fulfillment of the Slater Qualification, and the Karush-Kuhn-Tucker conditions become

$$\begin{cases} y=L+m_1-m_2 \\ \int_\Omega (1/\mu_0-z)\,dm_1=0 \\ \int_\Omega zdm_2=0 \\ L\in \mathbb{R},\ m_1\ge 0,\ m_1\ge 0 \end{cases}.$$

As in the previous case we can find a partition $\Omega=A\cup B\cup C$ *such that*

$$\begin{cases} z=0, & m_1=0, & y=L-m_2\le L, & \omega\in A \\ 0<z<1/\mu_0, & m_1=m_2=0, & y=L, & \omega\in B \\ z=1/\mu_0, & m_2=0, & y=L+m_1\ge L, & \omega\in C \end{cases} \tag{33}$$

and m_1 *and* m_2 *become random variables of* L^1. *Besides, Theorem* 4 *implies the existence of* $\tau\ge 0$ *such that* $z\le k+\tau$, $z=k+\tau$ *whenever* $y<y_0$, *and* $\tau(E(y)-S)=0$. *Let us consider three possible scenarios:*

Case 1, $k+\tau=1/\mu_0$. *Then* $\tau=1/\mu_0-k>0$ *and therefore* $E(y)=S$. *Moreover* $\mu(B)=0$ *since otherwise* $z<k+\tau$ *on* B *leads to* $y_0=y=L$ *on* B *and* y_0 *cannot be constant with positive probability because its distribution is continuous. Thus, let us remove* B *from (33).* $E(z)=1$ *implies that* $\mu(C)=\mu_0$, *so the necessary and sufficient conditions guaranteeing that* y *solves (5) will be:* $E(y)=S$ *and there exists a measurable set* C *such that* $\mu(C)=\mu_0$, $y=y_0$ *out of* C, *and* $y_0(\omega_C)\ge y_0(\omega)$ *whenever* $\omega_C\in C$ *and* $\omega\notin C$. *In particular, if* α *is such that* $E(y_0^\alpha)=S$ *then* y_0^α *satisfies the conditions of this case if and only if* $\mu(y_0>\alpha)\le\mu_0$, *since in such a case we can extend* $\{y_0>\alpha\}$ *to a set* $C=\{y_0>\alpha'\}$ $(\alpha'\le\alpha)$ *such that* $\mu(y_0>\alpha)=\mu_0$ *(recall that* y_0 *has continuous distribution).*

Case 2, $k+\tau<1/\mu_0$. *In this second scenario* C *has null probability, so let us remove it in (33). Then (33) clearly points out that* $y=y_0^L$, *since* $z=0<k+\tau$ *on* A *leads to* $y=y_0^L$ *on* A. *Notice that three more requirements must be satisfied. Firstly,* $E\left(y_0^L\right)\ge S$, *secondly,* $E(z)=1$ *implies* $\mu(B)(k+\tau)\ge 1$, *and thus*

$$1/\mu(B)\le k+\tau<1/\mu_0$$

which also leads to $\mu_0 < \mu(B)$. *Furthermore* z *must be constant (and equal* $k+\tau$*) on* B, *because otherwise* $y_0 = y_0^L = L$ *on a subset of* B *with positive probability, and being the distribution of* y_0 *continuous we have* $\mu(y_0 = L) = 0$. *Then,* $\mu(B)(k+\tau) = 1$ *implies* $0 \le \tau = 1/\mu(B) - k$, *and* $1/\mu(B) \ge k$ *must hold.Thirdly,* $\tau\left(E\left(y_0^L\right) - S\right) = 0$ *provokes* $E\left(y_0^L\right) = S$ *or* $1/\mu(B) = k$.

Case 3, $k + \tau > 1/\mu_0$. *Then (33) shows that* $z = k + \tau$ *never holds, and then* $y = y_0$. *On the other hand* $\tau > 1/\mu_0 - k > 0$ *implies* $S = E(y) = E(y_0)$, *contradicting the assumptions.* □

Theorem 13. *Suppose that* $S < E(y_0)$, *the distribution* y_0 *of is continuous,* $p = 1$, $\rho = CVaR_{\mu_0}$ *with* $0 < \mu_0 < 1$, $1/\mu_0 > k$ *and* y *is a feasible solution such that* $\mu(y > 0) = 1$. *Then,* y *solves (5) if and only if at least one of the following assertions hold:*

a) $E(y) = S$ *and there exists a measurable set* C *such that* $\mu(C) = \mu_0$, $y = y_0$ *out of* C, *and*

$$y_0(\omega_C) \ge y(\omega_C) \ge y(\omega) = y_0(\omega)$$

whenever $\omega_C \in C$ *and* $\omega \notin C$. *In such a case the optimal value of (5) is* $E(y(z-k))$ *where* $z = \chi_C - \chi_{\Omega \setminus C}$.

b) $y = y_0^\alpha$ *and* α *is such that* $E(y_0^\alpha) = S$ *and* $\mu(y_0 > \alpha) \le \mu_0$. *In such a case the optimal value of (5) is* $E(y(z-k))$ *where* $z = \chi_C - \chi_{\Omega \setminus C}$, C *being a set of the form* $C = \{\omega;\ y_0 > \alpha'\}$ *for some* $\alpha' \le a$ *and such that* $\mu(C) = \mu_0$.

c) $y = y_0^\alpha$, $E(y_0^\alpha) = S$, $\dfrac{1}{\mu(y_0 > \alpha)} \ge k$ *and* $\mu_0 < \mu(y_0 > \alpha)$. *If so the optimal value of (5) is* $E(y(z-k))$ *where* $z = \dfrac{1}{\mu(y_0 > \alpha)}\chi_{\mu(y_0 > \alpha)}$.

d) $y = y_0^\alpha$, $E(y_0^\alpha) > S$, $\dfrac{1}{\mu(y_0 > \alpha)} = k$ *and* $\mu_0 < \mu(y_0 > \alpha)$. *If so the optimal value of (5) is* $E(y(z-k))$ *where* $z = \dfrac{1}{\mu(y_0 > \alpha)}\chi_{\mu(y_0 > \alpha)}$.

Proof. It follows from the previous remark and the equality $\theta = E\left(y\left(z^* - k\tilde{E}\right)\right)$ of Theorem 4. □

6. Conclusions

The Optimal Reinsurance Problem is a classical topic in Actuarial Theory and has been studied under different assumptions and by using different criteria to compute the insurer risk level. Besides, General Risk Functions are becoming very important in Finance and Insurance, and many classical problems have been revisited by taking into account this recent approach. This article has shown that the Optimal Reinsurance Problem may be analyzed by drawing on General Risk Functions such as Deviation Measures, Expectation Bounded Measures of Risk or Coherent Measures of Risk, among others. A unified approach has been presented, in the sense that the findings are general enough and do not depend on the concrete Risk Function to be used. Necessary and sufficient Optimality Conditions have been provided. These conditions have been used so as to study the most important types of reinsurance in practice, pointing out that quota-share-like reinsurance contracts hardly can be optimal. Furthermore, three important concrete risk measures have been analyzed in detail, with special focus on the Conditional Value at Risk since this Coherent and Expectation Bounded Risk Measure is becoming more and more used in Finance and Insurance due to its interesting properties. □

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