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SMOOTHNESS OF DENSITY FUNCTION FOR RANDOM MAPS

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Abstract. A discrete-time random dynamical system is said to be a random map if one of a number of transformations is randomly selected and applied at each iteration of the process. Invariant densities of random maps describe the asymptotic properties of a random map. If the individual maps of a random map are piecewise onto and piecewise expanding then the random map satisfies Pelikan's average expanding condition and the random map has invariant densities. For individual maps, piecewise expanding and piecewise onto are sufficient to establish many important properties of the invariant densities, in particular, the fact that the densities inherit smoothness properties of individual maps. It is of interest to see if this property is transferred to random maps satisfying piecewise expanding and piecewise onto conditions. We show that if all the maps constituting the random map are piecewise expanding and piecewise onto, then the same result is true.

Keywords. Random maps; Frobenius-Perron operator; Density function; Smoothness; Absolutely continuous invariant measures.

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1 Introduction

One of the fundamental problems in ergodic theory is to describe the asymptotic behavior of trajectories defined by a dynamical system. The existence and properties of invariant measures for chaotic dynamical systems reflect their long time behavior and play an important role in understanding their chaotic nature. For a single transformation of an interval, much is known about the densities of the absolutely continuous invariant measures (acim). For example, it is known that the densities inherit smoothness properties from the map itself (Halfant [8] , Szewc [16]), that the supports consist of a finite union of intervals and that the densities are bounded below on their supports (Keller [9] and Kowalski [10]).

Random dynamical systems provide a useful framework for modeling and analyzing various physical, social and economic phenomena [5,14]. A random dynamical system of special interest is a random map where the process switches from one map to another according to fixed probabilities [13] or, more generally, position dependent probabilities [1, 3, 4]. Random maps have applications in the study of fractals [3], in modeling interference effects in quantum mechanics [5] and in computing metric entropy [15]. Random maps are also a convenient framework for modeling processes with randomly changing environment, e.g., the stock market. In [1, 2] random maps are used to replace the binomial model applied to determine option prices.

The existence and properties of invariant measures for random maps reflect their long time behavior and play an important role in understanding their chaotic nature. It is, therefore, important to establish properties of their absolutely continuous invariant measures. In particular, it is interesting if the density of an acim of a random map inherits the smoothness properties of the individual maps involved in the construction of the random map. In this paper we generalize to random maps results of Halfant [8], who proved that the density of an acim of a nonsingular map of an interval inherits the smoothness properties of the map itself. A random map is a far more complicated system than an individual deterministic chaotic map. Although the methods of exploring both are similar, an extra complication of a random choice of the acting map is involved on every step. Our main results are proven under the assumption that the individual maps used to construct the random map are piecewise onto and piecewise expanding. Our proof is, in a sense, more complete than the proof in [8] where the author leaves the main inductional step for a reader. So it is interesting even for the case of an individual chaotic map.

In Section 2 we present the notation and summarize the results we shall need in the sequel. In Section 3 we prove the main result.

Now, we state the main theorem (Theorem 1.1) of this paper. All definitions and notations are given later.

Theorem 1.1 *Let* $T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\}$ *be a random map such that* $\tau_1, \ldots, \tau_K \in \mathcal{T}_1(I)$ *are piecewise* $C^r, r \geq 2$*, piecewise onto and* T *satisfy conditions A and B in Section 2. Then,* $T-$*invariant density* f^* *is of class* C^{r-2} *and, for any* $s \leq r-2, (P_T^n \mathbf{1})^{(s)} \to f^{*(s)}$ *uniformly as* $n \to +\infty$.

2 Preliminaries

Let $I = [0,1] \subset \mathbb{R}$ and $(I, \mathcal{B}, \lambda)$ be a normalized measure space, where λ is the Lebesgue measure on I. Let $\tau : (I, \mathcal{B}, \lambda) \to (I, \mathcal{B}, \lambda)$ be a piecewise monotonic on a partition $\mathcal{P} = \{I_1, I_2, ..., I_q\}$ of I. That is, τ restricted to I_i is a monotonic function. Let $V(.)$ be the standard one dimensional variation of a function, and $BV(I)$ be the space of functions of bounded variation on I equipped with the norm $\| \,.\, \|_{BV} = V(.) + \| \,.\, \|_1$. In this paper we study the behavior of the systems which admit absolutely continuous invariant measure. Then, the asymptotic behavior of τ is given by a probability density function (pdf), f, of τ associated with the absolutely continuous invariant measure μ of τ. This is stated mathematically by the following equation:

$$\int_A f d\lambda = \int_{\tau^{-1}(A)} f d\lambda$$

for any (measurable) set $A \subset I$. The Frobenius-Perron operator, $P_\tau f$, defined by

$$\int_A P_\tau f d\lambda = \int_{\tau^{-1}(A)} f d\lambda$$

acts on the space of integrable functions and transforms a pdf into a pdf. If τ is piecewise smooth and piecewise differentiable on a partition of n subintervals, then we have the following representation for P_τ [4,10]:

$$P_\tau f(x) = \sum_{z \in \{\tau^{-1}(x)\}} \frac{f(z)}{|\tau'(z)|}, \tag{1}$$

where, for any x, the set $\{\tau^{-1}(x)\}$ consists of at most n points.

Definition 2.1 *Let $\mathcal{T}_0(I)$ denote the class of transformations $\tau : I \to I$ that satisfy the following conditions:*
(i) *τ is piecewise monotonic, i.e., there exists a partition $\mathcal{P} = \{I_i = [a_{i-1}, a_i], i = 1, 2, \ldots, q\}$ of I such that $\tau_i = \tau|I_i$ is C^1, and*

$$|\tau_i'(x)| \geq \alpha > 0, \tag{2}$$

for any i and for all $x \in (a_{i-1}, a_i)$;
(ii) *$g(x) = \frac{1}{|\tau_i'(x)|}$ is a function of bounded variation, where $\tau_i'(x)$ is the appropriate one-sided derivative at the end points of $\mathcal{J}$.*

We say that $\tau \in \mathcal{T}_1(I)$ if $\tau \in \mathcal{T}_0(I)$ and $\alpha > 1$ in condition (2), i.e., τ is piecewise expanding.

The following important result was established in [12](see also [4]):

Theorem 2.2 *Let τ be piecewise monotonic, piecewise C^2 map of an interval $I = [0, 1]$ into itself satisfying $\inf_{x \in I} |\tau'(x)| > 1$. Then,*

1. *If $f \in L^1([0, 1])$ is of bounded variation, $P_\tau f$ is also of bounded variation and*
   $V_0^1 P_\tau f \leq \alpha \parallel f \parallel + \beta V_0^1 f$ with $\alpha > 0$ and $\beta = \frac{2}{M}$ where $M = \inf |\tau'(x)|$.

2. *τ has an acim whose density f is of bounded variation and satisfies $P_\tau f = f$.*

Thus, the fixed points of the operator P_τ are the density functions of the acims for the map τ.

In Section 3, we will generalize the following theorem established in [8,4]

Theorem 2.3 *Let $\tau \in \mathcal{T}_1(I)$ be piecewise onto and piecewise $C^r, r \geq 2$. Then $\tau-$ invariant density f^* is of class C^{r-2} and, for any $s \leq r-2, (P_\tau^n \mathbf{1})^{(s)} \to {f^*}^{(s)}$ uniformly as $n \to +\infty$*

Random maps: Let $\tau_k : I \to I$, $k = 1, 2, \dots, K$ be piecewise monotonic on a common partition $\mathcal{P} = \{I_i = [a_{i-1}, a_i], i = 1, 2, \dots, N\}$ of I and nonsingular transformations. A random map T with constant probabilities is defined as

$$T = \{\tau_1, \tau_2, \dots, \tau_K; p_1, p_2, \dots, p_K\},$$

where $\{p_1, p_2, \dots, p_K\}$ is a set of constant probabilities on I. For any $x \in X$, $T(x) = \tau_k(x)$ with probability p_k and, for any non-negative integer N, $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \dots \circ \tau_{k_1}(x)$ with probability $\Pi_{j=1}^N p_{k_j}$. A measure μ is $T-$invariant if and only if it satisfies the following condition [13]:

$$\mu(E) = \sum_{k=1}^{K} p_k \mu(\tau_k^{-1}(E)), \tag{3}$$

for any $E \in \mathcal{B}$.

The Frobenius-Perron operator of T is given by [13]:

$$P_T f(x) = \sum_{k=1}^{K} p_k (P_{\tau_k} f)(x), \tag{4}$$

where P_{τ_k} is the Frobenius-Perron operator associated with the transformation τ_k. The properties of P_T resemble the properties of the traditional Frobenius-Perron operator [4].

Theorem 2.4 *([13]) Let $T = \{\tau_1, \tau_2, \dots, \tau_K; p_1, p_2, \dots, p_K\}$ be a random map, where $\tau_k \in \mathcal{T}_0(I)$, with the common partition $\mathcal{P} = \{I_1, I_2, \dots, I_q\}$, $k = 1, 2, \dots, K$. If, for all $x \in [0, 1]$, the following Pelikan's condition*

$$\sum_{k=1}^{K} \frac{p_k}{|\tau_k'(x)|} \leq \gamma < 1, \tag{5}$$

is satisfied, then for all $f \in L^1 = L^1([0, 1], \lambda)$:
(i) The limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} P_T^i(f) = f^* \text{ exists in } \mathrm{L}^1;$$

(ii) $P_T(f^*) = f^*$;
(iii) $V_{[0,1]}(f^*) \leq C \cdot \|f\|_1$, *for some constant* $C > 0$, *which is independent of* $f \in L^1$.

We consider random maps $T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\}$ satisfying conditions A and B below:

CONDITION A:

1. τ_k, $k = 1, 2, \ldots, K$, have a common defining partition $\mathcal{P} = \{I_i = (a_{i-1}, a_i), i = 1, 2, \ldots, q\}$ of I;
2. For each $i = 1, 2, \ldots, q$, $\tau_{k,i} = \tau_k|_{I_i}$, $k = 1, 2, \ldots, K$ has a C^2-extension to the closure $\overline{I_i}$ of I_i;
3. For each $i = 1, 2, \ldots, q$, $\tau_{k,i}$, $k = 1, 2, \ldots, K$ is strictly monotone on $\overline{I_i}$ and therefore determines a $1-1$ mapping of $\overline{I_i}$ onto some closed subinterval $\tau_k(\overline{I_i})$ of I.
4. For each $J \in \mathcal{P}$ and for each k, $k = 1, 2, \ldots, K$,
$$\tau_k(J) = I,$$
i.e., each τ_k is piecewise onto.

For each $n \geq 1$ we define
$$\Omega_n = \{\omega_n = (k_1, k_2, \ldots, k_n) : k_j \in \{1, 2, \ldots, K\}, j = 1, 2, \ldots, n\}.$$

For $\omega_{n-1} \in \Omega_{n-1}$, $\omega_{n-1} = (k_1, k_2, \ldots, k_{n-1})$ let $\mathcal{P}^{(n)}_{\omega_{n-1}}$ be the the common refinement of
$$\mathcal{P}, \tau_{k_1}^{-1}(\mathcal{P}), (\tau_{k_2} \circ \tau_{k_1})^{-1}(\mathcal{P}), \ldots, (\tau_{k_{n-1}} \circ \tau_{k_{n-2}} \circ \ldots \circ \tau_{k_2} \circ \tau_{k_1})^{-1}(\mathcal{P}),$$
$n \geq 1$, where $\sigma^{-1}\mathcal{P} = \{\sigma^{-1}(J) : J \in \mathcal{P}\}$. Then, $\mathcal{P}^{(n)}$ is the union of all $\mathcal{P}^{(n)}_{\omega_{n-1}}$, $\omega_{n-1} \in \Omega_{n-1}$. Let $I^{(n)}$ denote a generic element of $\mathcal{P}^{(n)}$. We have $I^{(n)} \in \mathcal{P}^{(n)}_{\omega_{n-1}}$, for some $\omega_{n-1} = (k_1, k_2, \ldots, k_{n-1})$ or $I^{(n)} = (\tau_{k_{n-1}} \circ \tau_{k_{n-2}} \circ \ldots \circ \tau_{k_2} \circ \tau_{k_1})^{-1} I_s$, for some $I_s \in \mathcal{P}$. We will write $I^{(n)} = I^{(n)}_{\omega_{n-1}}$. Let $T^{i-1}_{\omega_{i-1}} = \tau_{k_{i-1}} \circ \tau_{k_{i-2}} \circ \ldots \circ \tau_{k_2} \circ \tau_{k_1}$. Then, $T^{n-1}_{\omega_{n-1}}$ is well defined on $I^{(n)}_{\omega_{n-1}}$ and $T^{n-1}_{\omega_{n-1}}(I^{(n)}_{\omega_{n-1}}) = I_s$. Moreover $T^{i-1}_{\omega_{i-1}}(I^{(n)}_{\omega_{n-1}}) \in \mathcal{P}^{(n-i)}_{\omega_{n-i-1}}$, where $\omega_{n-i-1} = (k_{i+1}, k_{i+2}, \ldots, k_{n-1})$.

Two points x, y are in the same $I^{(n)}_{\omega_{n-1}}$ if and only if $T^i_{\omega_i}(x), T^i_{\omega_i}(y)$ lie in the same element of $\mathcal{P}$ for $0 \leq i \leq n-1$. $T^n_{\omega_n}$ has a C^2-extension to $\overline{I}^{(n+1)}_{\omega_n}$, also denoted by $T^n_{\omega_n}$, which maps $\overline{I}^{(n+1)}_{\omega_n}$ monotonically onto some $\overline{I_s} \in \mathcal{P}$. Let

$$M(\overline{I}^{(n+1)}_{\omega_n}) = \sup_{x,y \in I^{(n+1)}} |(T^n_{\omega_n})'(x)/(T^n_{\omega_n})'(y)|$$

and

$$M_n = \sup_{\omega_n \in \Omega_n} \sup_{I^{(n+1)}_{\omega_n} \in \mathcal{P}^{(n+1)}_{\omega_n}} M(\overline{I}^{(n+1)}_{\omega_n}). \tag{6}$$

CONDITION B: There exists an $\epsilon > 0$ and a positive integer $p \geq 1$ such that for all $\omega_p \in \Omega_p$, all $I^{(p+1)}_{\omega_p}$ and all $x \in I^{(p+1)}_{\omega_p}$ we have

$$|(T^p_{\omega_p})'(x)| > 1 + \epsilon.$$

For such random maps we conclude from the chain rule that

$$\alpha = \inf |(T^j_{\omega_j})'(x)| > 0, \tag{7}$$

where the infimum is taken over all $\omega_j \in \Omega_j$, all $I^{(j+1)}_{\omega_j}$ and all $x \in I^{(j+1)}_{\omega_j}$ for $0 \leq j \leq p$. Furthermore,

$$\inf_{x \in I^{(n+1)}_{\omega_n},\ I^{(n+1)}_{\omega_n} \in \mathcal{P}^{(n+1)}} |(T^n)'(x)| \geq \alpha(1+\epsilon)^{[n/p]} \geq \alpha(1+\epsilon)^{(n/p)-1}, \ \ n \geq 1. \tag{8}$$

For each $I^{(n+1)}$ there is an I_s such that $T^n_{\omega_n}(\overline{I}^{(n+1)}) = \overline{I_s}$. Thus by virtue of the mean value theorem, we know that there exists an $x \in \overline{I}^{(n+1)}$ such that $\lambda(I^{(n+1)}) = \lambda(I_s)/|(T^n_{\omega_n})'(x)|$. It follows from Condition B and (8) that

$$\lambda(I^{(n+1)}) \leq B\theta^n, \tag{9}$$

where $\theta = (1+\epsilon)^{-1/p} < 1$ and $B = \frac{(1+\epsilon)}{\alpha} \max_{I_s \in \mathcal{P}} \lambda(I_s)$.

3 Smoothness of invariant densities for random maps

Let $T = \{\tau_1, \tau_2, \dots, \tau_K; p_1, p_2, \dots, p_K\}$ be a random map such that $\tau_1, \tau_2, \dots, \tau_K \in \mathcal{T}_1(I)$ are piecewise $C^r, r \geq 2$, piecewise onto and T satisfies condition A and B in Section 2. Thus, by Pelikan's Theorem (Theorem 2.4) T has an absolutely continuous invariant probability measure μ with respect to Lebesgue measure. In this section we generalize Theorem 2.3 for a single transformation to a theorem on random map.

The proof of the main result (Theorem 1.1) proceed by following lemmas.

Lemma 3.1 *There exists $c > 0$ such that for all $N \geq 0$, all $\omega_N \in \Omega_N$ and all $I^{(N+1)}_{\omega_N} \in P^{(N+1)}$*

$$|(\frac{d}{dx}|(T^N)'(x)|)^{-1}| < c, \tag{10}$$

where

$$T^N(x) = T^N_{\omega_N}(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \ldots \circ \tau_{k_1}(x), \quad x \in I^{(N+1)}_{\omega_N}.$$

Proof: By (7), $(T^N)'(x)$ does not vanish on $\bar{I}^{(N+1)}$ and so $|(T^N)'(x)|$ is C^1 on $\bar{I}^{(N+1)}$. Using chain rule,

$$(T^m)'(x) = (\tau_{k_m})'(\tau_{k_{m-1}} \circ \tau_{k_{m-2}} \circ \ldots \circ \tau_{k_1}(x)) \cdots (\tau_{k_2})'(\tau_{k_1}(x)) \cdot \tau'_{k_1}(x)$$

and

$$(T^N)'(x) = (T^{N-m})'(T^m(x)) \cdot (T^m)'(x).$$

Logarithmic differentiation gives

$$\begin{aligned}
&\left|\left(\frac{d}{dx}|(T^N)'(x)|\right)^{-1}\right| = \left|(T^N)'(x)\right|^{-1}\left|\frac{d}{dx}\log|(T^N)'(x)|\right| \\
= \;& |(T^N)'(x)|^{-1}\left|\frac{d}{dx}\log|(\tau_{k_N})'(\tau_{k_{N-1}} \circ \tau_{k_{N-2}} \circ \ldots \circ \tau_{k_1}(x))|\right. \\
&\quad + \frac{d}{dx}\log|(\tau_{k_{N-1}})'(\tau_{k_{N-2}} \circ \tau_{k_{N-3}} \circ \ldots \circ \tau_{k_1}(x))| \\
&\quad \left. + \ldots + \frac{d}{dx}\log|(\tau_{k_2})'(\tau_{k_1}(x))| + \frac{d}{dx}\log|(\tau_{k_1})'(x)|\right| \\
= \;& |(T^N)'(x)|^{-1}\left|\sum_{m=0}^{N-1}\frac{d}{dx}\log(\tau_{k_{m+1}})'(T^m(x))\right| \\
= \;& |(T^N)'(x)|^{-1} \\
&\left|\frac{|(\tau_{k_N})''(\tau_{k_{N-1}} \circ \tau_{k_{N-2}} \circ \ldots \circ \tau_{k_1}(x))|(\tau_{k_{N-1}} \circ \tau_{k_{N-2}} \circ \ldots \circ \tau_{k_1})'(x)}{|(\tau_{k_N})'(\tau_{k_{N-1}} \circ \tau_{k_{N-2}} \circ \ldots \circ \tau_{k_1}(x))|}\right. \\
&+ \frac{|(\tau_{k_{N-1}})''(\tau_{k_{N-2}} \circ \tau_{k_{N-3}} \circ \ldots \circ \tau_{k_1}(x))|(\tau_{k_{N-2}} \circ \tau_{k_{N-3}} \circ \ldots \circ \tau_{k_1})'(x)}{|(\tau_{k_{N-1}})'(\tau_{k_{N-2}} \circ \tau_{k_{N-3}} \circ \ldots \circ \tau_{k_1}(x))|} \\
&\quad \left. + \ldots + \frac{|(\tau_{k_2})''(\tau_{k_1}(x))|(\tau_{k_1})'(x))}{|(\tau_{k_2})'(\tau_{k_1}(x))|} + \frac{|(\tau_{k_1})''(x)|}{|(\tau_{k_1})'(x)|}\right| \\
= \;& \sum_{m=0}^{N-1}\left|\frac{(\tau_{k_{m+1}})''(T^m(x))}{(\tau_{k_{m+1}})'(T^m(x))}\frac{(T^m)'(x)}{(T^N)'(x)}\right| \\
\leq \;& \max\{\sup\left|\frac{\tau_1''(x)}{\tau_1'(x)}\right|, \sup\left|\frac{\tau_2''(x)}{\tau_2'(x)}\right|, \ldots, \sup\left|\frac{\tau_K''(x)}{\tau_K'(x)}\right|\}\sum_{m=0}^{N-1}\left|\frac{(T^m)'(x)}{(T^N)'(x)}\right| \\
\leq \;& \max\{\sup\left|\frac{\tau_1''(x)}{\tau_1'(x)}\right|, \sup\left|\frac{\tau_2''(x)}{\tau_2'(x)}\right|, \ldots, \sup\left|\frac{\tau_K''(x)}{\tau_K'(x)}\right|\}
\end{aligned}$$

$$\sum_{m=0}^{N-1} |(T^{(N-m)})'(T^m(x))|^{-1}.$$

Using (9), we obtain

$$\begin{aligned}
&\left|\left(\frac{d}{dx}|(T^N)'(x)|\right)^{-1}\right| \\
&\leq \max\{\sup\left|\frac{\tau_1''(x)}{\tau_1'(x)}\right|, \sup\left|\frac{\tau_2''(x)}{\tau_2'(x)}\right|, \ldots, \sup\left|\frac{\tau_K''(x)}{\tau_K'(x)}\right|\} \sum_{m=0}^{N-1} \frac{1+\epsilon}{\alpha}\theta^{N-m} \\
&\leq \max\{\sup\left|\frac{\tau_1''(x)}{\tau_1'(x)}\right|, \sup\left|\frac{\tau_2''(x)}{\tau_2'(x)}\right|, \ldots, \sup\left|\frac{\tau_K''(x)}{\tau_K'(x)}\right|\} \\
&\qquad \frac{1+\epsilon}{\alpha}\sum_{m=0}^{N-1}\theta^{N-m} < \infty.
\end{aligned}$$

□

Lemma 3.2 *There exists $M > 0$ such that for all $N \geq 0$*

$$M_N \leq M,$$

where M_N is as in (6).

Proof: Let us fix N, $\omega_N \in \Omega_N$ and $I^{(N+1)}_{\omega_N}$. We will skip the ω_N subscript. For $x, y \in I^{(N+1)}$, we get by monotonicity of T^N on $I^{(N+1)}$ and Lemma 3.1

$$\begin{aligned}
\log\left|\frac{(T^N)'(x)}{(T^N)'(y)}\right| &= \log\frac{(T^N)'(x)}{(T^N)'(y)} = \int_y^x \frac{(T^N)''(t)}{(T^N)'(t)}dt \\
&= -\int_y^x (T^N)'(t)\frac{d}{dt}\frac{1}{(T^N)'(t)}dt \\
&\leq c\cdot|\int_y^x (T^N)'(t)dt| \leq c\cdot\lambda(T^N(\bar{I}^{(N+1)})) \leq c.
\end{aligned}$$

Setting $M = e^c$ completes the proof.
□

Now, we prove the main result (Theorem 1.1) of this paper. We split the proof into a number of lemmas.

Lemma 3.3 *Let $T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\}$ be a random map such that T satisfies conditions of Theorem 1.1 with $r = 2$. Then the $T-$invariant density f^* is a uniform limit of $\{P_T^n\mathbf{1}\}_{n\geq 0}$ and continuous.*

Proof: Consider the sequence $f_n = P_T^n\mathbf{1}, n = 1, 2, \ldots$. We have

$$f_0(x) = \mathbf{1}(x)$$

$$\begin{aligned} f_1(x) &= P_T\mathbf{1}(x) = \sum_{k=1}^{K} p_k P_{\tau_k}\mathbf{1}(x) \\ &= \sum_{k=1}^{K} p_k \sum_{i=1}^{q} \frac{1}{|\tau_k'(\tau_{k,i}^{-1}(x))|}. \end{aligned}$$

Since $\tau_k, k = 1, 2, \ldots, K$ is piecewise onto and $\tau'_{k,i} = \tau'_k|_{I_i}, k = 1, 2, \ldots, K, i = 1, 2, \ldots, q$ are continuous, $f_1(x)$ is continuous.

$$\begin{aligned} f_2(x) &= P_T^2\mathbf{1}(x) = P_T\left(P_T\mathbf{1}(x)\right) = P_T\left(\sum_{k=1}^{K} p_k P_{\tau_k}\mathbf{1}(x)\right) \\ &= \sum_{k=1}^{K} p_k P_T\left(P_{\tau_k}\mathbf{1}(x)\right) \\ &= \sum_{k=1}^{K} p_k \sum_{l=1}^{K} p_l P_{\tau_l}\left(P_{\tau_k}\mathbf{1}(x)\right) \\ &= \sum_{k=1}^{K} p_k \sum_{l=1}^{K} p_l \sum_{j=1}^{q}\sum_{i=1}^{q} \frac{1}{|\tau_l'(\tau_{l,j}^{-1}(\tau_{k,i}^{-1})(x))||\tau_k'(\tau_{k,i}^{-1}(x))|}, \end{aligned}$$

where $\tau'_{k,i} = \tau'_k|_{I_i}, \tau'_{l,j} = \tau'_l|_{I_j}, k, l = 1, 2, \ldots K,$ and i, j = 1, 2, . . . , q. Thus, we have

$$f_2(x) = \sum_{1\le k_1,k_2\le K}\ \sum_{1\le i_1,i_2\le q} \frac{p_{k_1}p_{k_2}}{|\tau'_{k_2}(\tau^{-1}_{k_2,i_2}(\tau^{-1}_{k_1,i_1})(x))||\tau'_{k_1}(\tau^{-1}_{k_1,i_1}(x))|},$$

and again f_2 is continuous. In general, we have

$$\begin{aligned} f_n(x) = &\sum_{1\le k_1,k_2,\ldots,k_n\le K}\ \sum_{1\le i_1,i_2,\ldots,i_n\le q} \\ &\left[\frac{p_{k_1}p_{k_2}\cdots p_{k_n}}{|\tau'_{k_n}(\tau^{-1}_{k_n,i_n}(\tau^{-1}_{k_{n-1},i_{n-1}}(\ldots(\tau^{-1}_{k_2,i_2}(\tau^{-1}_{k_1,i_1}(x)))\ldots))))|}\right. \\ &\qquad \left.\frac{1}{|\tau'_{k_{n-1}}(\tau^{-1}_{k_{n-1},i_{n-1}}(\ldots(\tau^{-1}_{k_2,i_2}(\tau^{-1}_{k_1,i_1}(x)))\ldots))|\ldots|\tau'_{k_1}(\tau^{-1}_{k_1,i_1}(x))|}\right] \\ = &\sum_{1\le k_1,k_2,\ldots,k_n\le K}\ \sum_{1\le i_1,i_2,\ldots,i_n\le q} \\ &\qquad \left[\frac{p_{k_1}p_{k_2}\cdots p_{k_n}}{|(\tau_{k_n,i_n}\circ\tau_{k_{n-1},i_{n-1}}\circ\ldots\circ\tau_{k_2,i_2}\circ\tau_{k_1,i_1})'(\phi_{n,\mathbf{k}_n,\mathbf{j}_n}(x))|}\right] \end{aligned}$$

where f_n is continuous and

$$\phi_{n,\mathbf{k}_n,\mathbf{j}_n} = (\tau_{k_n,i_n}\circ\tau_{k_{n-1},i_{n-1}}\circ\ldots\circ\tau_{k_2,i_2}\circ\tau_{k_1,i_1})^{-1},$$

with $\mathbf{j}_n = (i_n, i_{n-1}, \dots, i_2, i_1) \in \{1, 2 \dots, q\}^n$, $\mathbf{k}_n = (k_n, k_{n-1}, \dots, k_2, k_1) \in \{1, 2, \dots, K\}^n$.

We want to show that f_n's are uniformly bounded and equicontinuous. We have,

$$f_n(x) = \sum_{\mathbf{k}_n} \sum_{\mathbf{j}_n} p_{k_1} \cdot p_{k_2} \cdots p_{k_n} |\phi'_{n,\mathbf{k}_n,\mathbf{j}_n}(x)|. \tag{11}$$

In Lemma 3.2 we have proved that there exists a constant $M > 1$ such that for any $n \geq 1$, any $\mathbf{k}$, any $\mathbf{j}$ and any $x \in I$,

$$\frac{1}{M} < \frac{\sup |\phi'_{n,\mathbf{k},\mathbf{j}}(x)|}{\inf |\phi'_{n,\mathbf{k},\mathbf{j}}(x)|} < M. \tag{12}$$

We can apply inequality (12) to equation (11) to obtain

$$\frac{1}{M} < \frac{\sup f_n(x)}{\inf f_n(x)} < M. \tag{13}$$

Since f_n is a density function of normalized measure, we get

$$\frac{1}{M} < f_n(x) < M. \tag{14}$$

Next, we show that f_n is equicontinuous. It is easy to see that for $l \geq 0$,

$$f_{n+l}(x) = \sum_{\mathbf{k}_n} \sum_{\mathbf{j}_n} p_{k_1} p_{k_2} \dots p_{k_n} f_l(\phi_{n,\mathbf{k}_n,\mathbf{j}_n}(x)) |\phi'_{n,\mathbf{k}_n,\mathbf{j}_n}(x)|. \tag{15}$$

Differentiating both sides we obtain,

$$|f'_{n+l}(x)| \leq \sum_{\mathbf{k}_n} \sum_{\mathbf{j}_n} p_{k_1} p_{k_2} \dots p_{k_n} \left[f'_l(\phi_{n,\mathbf{k}_n,\mathbf{j}_n}(x)) |(\phi'_{n,\mathbf{k}_n,\mathbf{j}_n}(x))^2| \right.$$
$$\left. + f_l(\phi_{n,\mathbf{k}_n,\mathbf{j}_n}(x)) |\phi''_{n,\mathbf{k}_n,\mathbf{j}_n}(x)| \right]. \tag{16}$$

Now, using (8) and (14), we have

$$\sum_{\mathbf{k}_n} \sum_{\mathbf{j}_n} p_{k_1} p_{k_2} \dots p_{k_n} |(\phi'_{n,\mathbf{k}_n,\mathbf{j}_n}(x))^2|$$
$$\leq \sup_{\mathbf{k}_n} \sup_{\mathbf{j}_n} |\phi'_{n,\mathbf{k}_n,\mathbf{j}_n}(x)| \sum_{\mathbf{k}} \sum_{\mathbf{j}} p_{k_1} p_{k_2} \dots p_{k_n} |\phi'_{n,\mathbf{k}_n,\mathbf{j}_n}(x)| \leq \frac{(1+\epsilon)M}{\alpha(1+\epsilon)^{n/p}}.$$

Thus, for n_0 big enough, we have

$$\sup_{x \in I} \sum_{\mathbf{k}_{n_0}} \sum_{\mathbf{j}_{n_0}} p_{k_1} p_{k_2} \dots p_{k_{n_0}} |(\phi'_{n_0,\mathbf{k}_{n_0},\mathbf{j}_{n_0}}(x))^2| \leq \theta < 1. \tag{17}$$

Let

$$d = \sup_{x \in I} \sum_{\mathbf{k}_{n_0}} \sum_{\mathbf{j}_{n_0}} p_{k_1} p_{k_2} \dots p_{k_{n_0}} |\phi''_{n_0,\mathbf{k}_{n_0},\mathbf{j}_{n_0}}(x)| < +\infty$$
$$B_n = \sup_{x \in I} |f'_n(x)|, n = 0, 1, 2, \dots .$$

Then (16) with $n = n_0$ implies

$$B_{l+n_0} \leq B_l\theta + Md, l = 0, 1, 2, \ldots.$$

Thus,

$$B_{l+2n_0} \leq B_{l+n_0}\theta + Md \leq (B_l\theta + Md)\theta + Md,$$

and

$$B_{l+mn_0} \leq B_l\theta^{m-1} + Md(1 + \theta + \ldots \theta^{(m-1)}) \leq B_l + Md\frac{1}{1-\theta} = \bar{B}_l,$$

for $m = 1, 2, \ldots$. Thus the sequence $\{B_n\}_{n=0}^{\infty}$ is bounded by $\max\{\bar{B}_0, \bar{B}_1, \ldots, \bar{B}_{n_0-1}\}$. We have proved that the sequence $\{f_n\}$ is uniformly bounded and equicontinuous. By the Ascoli-Arzela theorem it contains a subsequence $\{f_{n_l}\}_{l\geq 0}$ uniformly convergent to a continuous function g. By our assumption, the random map has a unique invariant density f^*. Thus $\{f_{n_l}\}_{l\geq 0}$ converges to f^* in L^1. Thus, $g = f^*$ and $f_{n_l} \to f^*$ uniformly, as $l \to +\infty$. Since this argument applies to any subsequence of $\{f_n\}_{n\geq 0}$, the entire sequence converges uniformly to f^*, which is continuous. □

The following lemma can be proved by induction:

Lemma 3.4 *Let $F(x) = f(\phi(x))\phi'(x), x \in I$. Then*

$$F^{(s+1)} = f^{(s+1)}(\phi)(\phi')^{s+2} + \sum_{i\leq s} f^{(i)}(\phi)\left[P_{s,i}(\phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(s+2)})\right],$$

where $P_{s,i}$ is a polynomial of order $i+1$, $i = 0, 1, \ldots, s$.

Proof of Theorem 1.1:

We proceed by induction. The first step, i.e., uniform boundedness of $(P_T^n\mathbf{1})^{(1)}$ and uniform convergence $P_T^n\mathbf{1} \to f^*$, has been proved in Lemma 3.3. Let us assume that $r \geq 3$ and that $\{(P_T^n\mathbf{1})^{(j)}\}_{n\geq 0}$ is uniformly bounded for $j = 0, 1, \ldots, s \leq r-2, (P_T^n\mathbf{1})^{(j)} \to f^{*(j)}$ uniformly as $n \to +\infty$ for $j = 0, 1, \ldots, s-1$. We will show that the same is true for $s+1$. Let $f_n = P_T^n\mathbf{1}, n = 0, 1, \ldots$. Using Lemma 3.4 and formula (15), we can write

$$f_{n+l}^{(s+1)}(x) =$$
$$\sum_{\mathbf{k}_n}\sum_{\mathbf{j}_n} p_{k_1}p_{k_2}\cdots p_{k_n}\Bigg[f_l^{(s+1)}(\phi_{n,\mathbf{k}_n,\mathbf{j}_n}(x))(|\phi'_{n,\mathbf{k}_n,\mathbf{j}_n}(x)|)^{s+1}$$
$$+\sum_{i=0}^{s} f_l^{(i)}(\phi_{n,\mathbf{k},\mathbf{j}}(x))\left[P_{s,i}(\phi_{n,\mathbf{k}_n,\mathbf{j}_n}^{(1)}(x), \phi_{n,\mathbf{k}_n,\mathbf{j}_n}^{(2)}(x), \ldots, \phi_{n,\mathbf{k}_n,\mathbf{j}_n}^{(s+2)}(x))\right]\Bigg]$$

For n_0 of formula (17), we have

$$\sum_{\mathbf{k}_{n_0}}\sum_{\mathbf{j}_{n_0}} p_{k_1}p_{k_2}\cdots p_{k_{n_0}}(|\phi'_{n_0,\mathbf{k}_{n_0},\mathbf{j}_{n_0}}(x)|)^{(s+1)} \tag{18}$$
$$\leq \left(\frac{(1+\epsilon)}{\alpha(1+\epsilon)^{n_0/p}}\right)^s M = \theta_{s+1} < 1.$$

By the inductive assumption, $f_l^{(i)}$ are uniformly bounded for $i = 0, 1, \ldots, s$. Also $\phi^{(i)}_{n_0,\mathbf{k}_{n_0},\mathbf{j}_{n_0}}$, $i = 1, 2, \ldots s+2$ are bounded. Thus, we can find a constant $A_{s+1} > 0$ such that

$$\Bigg|\sum_{\mathbf{k}_{n_0}}\sum_{\mathbf{j}_{n_0}} p_{k_1}p_{k_2}\cdots p_{k_{n_0}} \sum_{i=0}^{s} f_l^{(i)}(\phi_{n_0,\mathbf{k}_{n_0},\mathbf{j}_{n_0}}(x)) \left[P_{s,i}(\phi^{(1)}_{n_0,\mathbf{k}_{n_0},\mathbf{j}_{n_0}}(x), \ldots, \phi^{(s+2)}_{n_0,\mathbf{k}_{n_0},\mathbf{j}_{n_0}}(x))\right]\Bigg| \le A_{s+1} \tag{19}$$

for all $l \ge 0$ and $x \in I$. If we denote $D_n^{(s+1)} = \sup_{x\in I} |f_n^{(s+1)}(x)|$, we obtain

$$D^{(s+1)}_{n_0+l} \le D_l^{(s+1)}\theta_{s+1} + A_{s+1}$$

for all $l = 0, 1, \ldots$.

As in Lemma 3.3 this implies that the sequence $\{f_n^{(s+1)}\}$ is uniformly bounded. Thus, the sequence $\{f_n^{(s)}\}$ is uniformly bounded and equicontinuous. By the Ascoli-Arzela theorem it contains a subsequence $\{f_{n_l}\}_{l\ge 0}$ convergent uniformly to a continuous function g. Since $f_{n_l}^{(s-1)} \to (f^*)^{(s-1)}$ uniformly as $l \to \infty, g = (f^*)^{(s)}$. Since this argument applies to any subsequence of $\{f_n^{(s)}\}_{n\ge 0}$, the entire sequence $\{f_n^{(s)}\}_{n\ge 0}$, converges uniformly to $(f^*)^{(s)}$ which is continuous. This completes the proof of the theorem.

Remark: If we omit the assumption **A** (4), then the same reasoning proves the smoothness of the density function on the subintervals of the set

$$I \setminus \bigcup_{n\ge 1} T^n(\{a_0, a_1, \ldots, a_q\}),$$

where $T^n(\{a_0, a_1, \ldots, a_q\}) = \bigcup_{\omega_n\in\Omega_n} T^n_{\omega_n}(\{a_0, a_1, \ldots, a_q\})$.

Example: In [12] Lasota and Rusek created a model of oil drill operation using eventually piecewise expanding maps of an interval (see [4, Chapter 13] for a detailed description). The map τ models the process of drill jumping up and falling back down. The more uniform is the invariant density of τ the more uniform is the tear of the drill, so knowing the invariant density is of practical importance. Since the parameters of drill movement are measured only with a certain accuracy and they vary during the operation it is more realistic to model the system with a random map. Instead of considering just one map τ we will consider a family of approximations τ_i to τ applied at random on each step of the iteration.

For Froude number $\Lambda = 3$ we approximate τ by three eventually piecewise expanding, piecewise onto maps τ_1, τ_2 and τ_3. We define τ_i by formula (20) using constants: $a_1 = -1.40$, $a_2 = -1.41$, $a_3 = -1.39$, $b_1 = 6.5888$, $b_2 = -8.7850$, $b_3 = 11.7134$,$e_1 = 10$, $e_2 = 11$, $e_3 = 12$. The graph of τ_1 is shown in Figure 1 a). The graphs of the others are indistinguishable

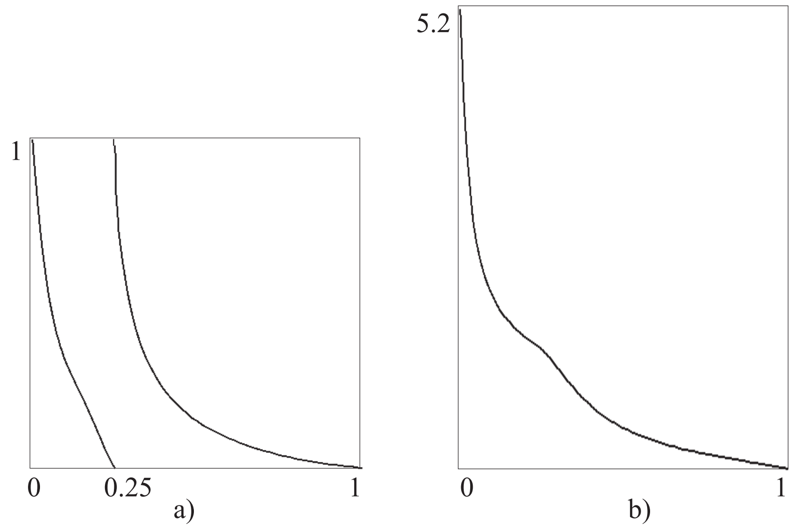

Figure 1: Map τ_1 and an approximation to the invariant density of random map T.

from the this one at the precision we can show. We define random map $T = \{\tau_1, \tau_2, \tau_3; p_1, p_2, p_3\}$, with $p_1 = 0.5$, $p_2 = 0.25$, $p_3 = 0.25$. We calculated densities $f_n = P_T^n \mathbf{1}$ numerically using Maple 11. f_5 is shown in Figure 1 b). The Maple 11 program is available on request.

$$\begin{aligned} &\tau_i(x) = \qquad (20)\\ &a_i(x-0.25) + (30.6667a_i + 66.3382)(x-0.25)^2\\ &+(240a_i + 562.5680)(x-0.25)^3 + (533.3333a_i + 1444.8607)(x-0.25)^4 \ ,\\ &\qquad \text{for} \ \ 0 \le x \le 0.25 \ ;\\ &(1 + b_i(x-1)^{e_i}) \left(0.9(x-1) - 0.17(x-1)^2 + \frac{3}{2}(1 - \sqrt{1 - \frac{4}{3}(1-x)}\right) \ ,\\ &\qquad \text{for} \ \ 0.25 < x \le 1 \ . \end{aligned}$$

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