Rank Equalities Related to Generalized Inverses of Matrices and Their Applications

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This thesis is a summary of the work I accomplished in the past several years at Zhengzhou Technical School for Surveying and Mapping, Zhengzhou, China, and then at Concordia University. The whole work was mostly motivated by an earlier seminal paper of Marsaglia and Styan on rank equalities and inequalities of matrices published in Linear and Multilinear Algebra 2(1974), 269-292. I was attracted by the beauty and usefulness of the results in that paper, and deeply believed that they would certainly become a powerful tool for dealing with various problems in the theory of generalized inverses of matrices and its applications. Through several years' effort, I develop the results in the Marsaglia and Styans' paper into a complete method for establishing various equalities for ranks and generalized inverses of matrices. Parts of the thesis were published in [91], [92], [93], [94]. Other parts (as a joint paper with George P. H. Styan) were presented at the Seventh International Workshop on Matrices and Statistics (Fort Lauderdale, Florida, December 11–14, 1998). In addition, some parts of the thesis have recently been submitted for publication. They are Chapters 3, 4, 5 and 11.

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ABSTRACT

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by

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This thesis develops a general method for expressing ranks of matrix expressions that involve the Moore-Penrose inverse, the group inverse, the Drazin inverse, as well as the weighted Moore-Penrose inverse of matrices. Through this method we establish a variety of valuable rank equalities related to generalized inverses of matrices mentioned above. Using them, we characterize many matrix equalities in the theory of generalized inverses of matrices and their applications.

Contents

Chapter 1

Introduction and preliminaries

This is a comprehensive work on ranks of matrix expressions involving Moore-Penrose inverses, group inverses, Drazin inverses, as well as weighted Moore-Penrose inverses. In the theory of generalized inverses of matrices and their applications, there are numerous matrix expressions and equalities that involve these three kinds of generalized inverses of matrices. Now we propose such a problem: Let $p(A_1^\dagger, \cdots, A_k^\dagger)$ and $q(B_1^\dagger, \cdots, B_l^\dagger)$ be two matrix expressions involving Moore-Penrose inverses of matrices. Then determine necessary and sufficient conditions such that $p(A_1^\dagger, \cdots, A_k^\dagger) = q(B_1^\dagger, \cdots, B_l^\dagger)$ holds. A seemingly trivial condition for this equality to hold is apparently

$$\mathrm{rank}\,[\,p(A_1^\dagger, \cdots, A_k^\dagger) - q(B_1^\dagger, \cdots, B_l^\dagger)\,] = 0. \tag{1.1}$$

However, if we can reasonably find a formula for expressing the rank of the left-hand side of Eq.(1.1), then we can derive immediately from Eq.(1.1) nontrivial conditions for

$$p(A_1^\dagger, \cdots, A_k^\dagger) = q(B_1^\dagger, \cdots, B_l^\dagger)$$

to hold. This work has a far-reaching influence to many problems in the theory of generalized inverses of matrices and their applications. This consideration motivates us to make a thorough investigation to this work. In fact, the author has successfully used this idea to establish necessary and sufficient conditions such that $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$, $(ABC)^\dagger = (BC)^\dagger B(AB)^\dagger$, and $(A_1A_2\cdots A_k)^\dagger = A_k^\dagger \cdots A_2^\dagger A_1^\dagger$ (cf. Tian, 1992b, 1994). But the methods used in those papers are somewhat restricted and not applicable to various kind matrix expressions. In this thesis, we shall develop a general and complete method for establishing rank equalities for matrix expressions involving Moore-Penrose inverses, group inverses, Drazin inverses, as well as weighted Moore-Penrose inverses of matrices. Using these rank formulas, we shall characterize various equalities for generalized inverses of matrices, and then present their applications in the theory of generalized inverses of matrices.

The matrices considered in this paper are all over the complex number field $\mathcal{C}$. Let $A \in \mathcal{C}^{m\times n}$. We use A^*, $r(A)$ and $R(A)$ to stand for the conjugate transpose, the rank and the range (column space) of A, respectively.

It is well known that the Moore-Penrose inverse of matrix A is defined to be the unique solution X of the following four Penrose equations

$$(1)\quad AXA = A, \qquad (2)\quad XAX = X, \qquad (3)\quad (AX)^* = AX, \qquad (4)\quad (XA)^* = XA,$$

and is often denoted by $X = A^\dagger$. In addition, a matrix X that satisfies the first equation above is called an inner inverse of A, and often denoted by A^-. A matrix X that satisfies the second equation above is called an outer inverse of A, and often denoted by $A^{(2)}$. For simplicity, we use E_A and F_A to stand for the two projectors

$$E_A = I - AA^{\dagger} \qquad \text{and} \qquad F_A = I - A^{\dagger}A$$

induced by A. As to various basic properties concerning Moore-Penrose inverses of matrices, see, e.g., Ben-Israel and Greville (1980), Campbell and Meyer (1991), Rao and Mitra (1971).

Let $A \in \mathcal{C}^{m\times m}$ be given with $\operatorname{Ind} A = k$, the smallest positive integer such that $r(A^{k+1}) = r(A^k)$. The Drazin inverse of matrix A is defined to be the unique solution X of the following three equations

$$(1) \quad A^kXA = A^k, \qquad (2) \quad XAX = X, \qquad (3) \quad AX = XA,$$

and is often denoted by $X = A^D$. In particular, when $\operatorname{Ind} A = 1$, the Drazin inverse of matrix A is call the group inverse of A, and is often denoted by $A^{\#}$.

Let $A \in \mathcal{C}^{m\times n}$. The weighted Moore-Penrose inverse of $A \in \mathcal{C}^{m\times n}$ with respect to the two positive definite matrices $M \in \mathcal{C}^{m\times m}$ and $N \in \mathcal{C}^{n\times n}$ is defined to be the unique solution of the following four matrix equations

$$(1) \quad AXA = A, \qquad (2) \quad XAX = X, \qquad (3) \quad (MAX)^* = MAX,, \qquad (4) \quad (NXA)^* = NXA.$$

and this X is often denoted by $X = A^{\dagger}_{M,N}$. In particular, when $M = I_m$ and $N = I_n$, $A^{\dagger}_{M,N}$ is the conventional Moore-Penrose inverse $A^{\dagger}$ of A. Various basic properties concerning Drazin inverses, group inverses and Weighted Moore-Penrose inverses of matrices can be found in Ben-Israel and Greville (1980), Campbell and Meyer (1991), Rao and Mitra (1971).

It is well known that generalized inverses of matrices are a powerful tool for establishing various rank equalities on matrices. A seminal reference is the paper [56] by Marsaglia and Styan (1974). In that paper, some fundamental rank equalities and inequalities related to generalized inverses of matrices were established and a variety of consequences and applications of these rank equalities and inequalities were considered. Since then, the main results in that paper have widely been applying to dealing with various problems in the theory of generalized inverses of matrices and its applications. To some extent, this thesis could be regarded as a summary and extension of all work related to that remarkable paper.

We next list some key results in that paper, which will be intensively applied in this thesis.

Lemma 1.1 (Marsaglia and Styan, 1974). *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$, $C \in \mathcal{C}^{l\times n}$ and $D \in \mathcal{C}^{l\times k}$. Then*

$$r[\, A, \ B\,] = r(A) + r(\, B - AA^{\dagger}B\,) = r(B) + r(\, A - BB^{\dagger}A\,), \tag{1.2}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(\, C - CA^{\dagger}A\,) = r(C) + r(\, A - AC^{\dagger}C\,), \tag{1.3}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B)+r(C)+r(E_BAF_C) = r(B)+r(C)+r[\,(\, I_m - BB^{\dagger}\,)A(\, I_n - C^{\dagger}C\,)\,]. \tag{1.4}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r\begin{bmatrix} 0 & E_AB \\ CF_A & S_A \end{bmatrix}, \tag{1.5}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} + r[\, A,\ B\,] - r(A) + r[J(D)], \tag{1.6}$$

where $S_A = D - CA^{\dagger}B$ *is the Schur complement of* A *in* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, *and*

$$J(D) = [\, I - (CF_A)(CF_A)^{\dagger}\,]S_A[\, I - (E_AB)^{\dagger}(E_AB)\,],$$

called the rank complement of D *in* M. *In particular, if*

$$R(B) \subseteq R(A) \quad and \quad R(C^*) \subseteq R(A^*),$$

then

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r(\, D - CA^{\dagger}B\,). \tag{1.7}$$

The six rank equalities in Eqs.(1.2)—(1.7) *are also true when replacing* $A^{\dagger}$ *by* A^{-}.

Lemma 1.2 (Marsaglia and Styan, 1974). *Let* $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$, $C \in \mathcal{C}^{l\times n}$ *and* $D \in \mathcal{C}^{l\times k}$. *Then*

(a) $r[\, A,\ B\,] = r(A) + r(B) \iff R(A) \cap R(B) = \{0\} \iff R[(E_AB)^*] = R(B^*) \iff R[(E_BA)^*] = R(A^*)$.

(b) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C) \iff R(A^*) \cap R(C^*) = \{0\} \iff R(CF_A) = R(C) \iff R(AF_C) = R(A)$.

(c) $r[\, A,\ B\,] = r(A) \iff R(B) \subseteq R(A) \iff E_AB = 0$.

(d) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) \iff R(C^*) \subseteq R(A^*) \iff CF_A = 0$.

(e) $r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(A) + r(B) + r(C) \iff R(A) \cap R(B) = \{0\}$ *and* $R(A^*) \cap R(C^*) = \{0\}$.

(f) $r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) \iff R(B) \subseteq R(A)$ *and* $R(C^*) \subseteq R(A^*)$ *and* $D = CA^{\dagger}B$.

Lemma 1.3 (Marsaglia and Styan, 1974)(*rank cancellation rules*). *Let* $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$ *and* $C \in \mathcal{C}^{l\times n}$ *be given, and suppose that*

$$R(AQ) = R(A) \quad and \quad R[(PA)^*] = R(A^*).$$

Then

$$r[\, AQ,\ B\,] = r[\, A,\ B\,], \qquad r\begin{bmatrix} PA \\ C \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix}. \tag{1.8}$$

In particular,

$$r[\, AA^*,\ B\,] = r[\, A,\ B\,], \qquad r\begin{bmatrix} A^*A \\ C \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix}. \tag{1.9}$$

Lemma 1.4 (Marsaglia and Styan, 1974). *Let* $A,\ B \in \mathcal{C}^{m\times n}$. *Then*

(a) $r(\,A \pm B\,) \geq r\begin{bmatrix} A \\ B \end{bmatrix} + r[\,A,\ B\,] - r(A) - r(B).$

(b) *If* $R(A) \cap R(B) = \{0\}$, *then* $r(\,A + B\,) = r\begin{bmatrix} A \\ B \end{bmatrix}.$

(c) *If* $R(A^*) \cap R(B^*) = \{0\}$, *then* $r(\,A + B\,) = r[\,A,\ B\,]$.

(d) $r(\,A + B\,) = r(A) + r(B) \Longleftrightarrow r(\,A - B\,) = r(A) + r(B) \Longleftrightarrow R(A) \cap R(B) = \{0\}$, *and* $R(A^*) \cap R(B^*) = \{0\}$.

In addition, we shall also use in the sequel the following several basic rank formulas, which are either well known or easy to prove.

Lemma 1.5. *Let* $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{n\times m}$ *and* $N \in \mathcal{C}^{m\times m}$. *Then*

$$r(\,A - ABA\,) = r(A) + r(\,I_n - BA\,) - n = r(A) + r(\,I_m - AB\,) - m, \tag{1.10}$$

$$r(\,N \pm N^2\,) = r(N) + r(\,I_n \pm N\,) - m = r(A) + r(\,I_m - AB\,) - m, \tag{1.11}$$

$$r(\,I_m - N^2\,) = r(\,I_m + N\,) + r(\,I_m - N\,) - m. \tag{1.12}$$

Lemma 1.6. *Let* $A,\ B \in \mathcal{C}^{m\times n}$. *Then*

$$r\begin{bmatrix} A & B \\ B & A \end{bmatrix} = r(\,A + B\,) + r(\,A - B\,). \tag{1.13}$$

Proof. Follows from the following decomposition

$$\frac{1}{2}\begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}\begin{bmatrix} A & B \\ B & A \end{bmatrix}\begin{bmatrix} I_n & I_n \\ I_n & -I_m \end{bmatrix}\begin{bmatrix} A + B & 0 \\ 0 & A - B \end{bmatrix}. \qquad \square$$

Lemma 1.7 (Anderson and Styan, 1982). *Let* $A \in \mathcal{C}^{m\times m}$. *Then*

$$r(\,A - A^3\,) = r(\,A + A^2\,) + r(\,A - A^2\,) - r(A). \tag{1.14}$$

This thesis is divided into 17 chapters with over 240 theorems and corollaries. They organize as follows.

In Chapter 2, we establish several universal rank formulas for matrix expressions that involve Moore-Penrose inverses of matrices. These rank formulas will serve as a basic tool for developing the content in all the subsequent sections.

In Chapter 3, we present a set of rank formulas related to sums, differences and products of idempotent matrices. Based on them, we shall reveal a series of new and untrivial properties related idempotent matrices.

In Chapter 4, we extend the results in Chapter 3 to some matrix expressions that involve both idempotent matrices and general matrices. In addition, we shall also establish a group of new rank formulas related to involutory matrices and then consider their consequences.

In Chapter 5, we establish a set of rank formulas related to outer inverses of a matrix. Some of them will be applied in the subsequent chapters.

In Chapter 6, we examine various relationships between a matrix and its Moore-Penrose inverse using the rank equalities obtained in the preceding chapters. We also consider in the chapter how characterize some special types of matrices, such as, EP matrix, conjugate EP matrix, bi-EP matrix, star-dagger matrix, power-EP matrix, and so on.

In Chapter 7, we discuss various rank equalities for matrix expressions that involve two or more Moore-Penrose inverses, and then use them to characterize various matrix equalities that involve Moore-Penrose inverses.

In Chapter 8, we investigate various kind of reverse order laws for Moore-Penrose inverses of products of two or three matrices using the rank equalities established in the preceding chapters.

In Chapter 9, we investigate Moore-Penrose inverses of 2×2 block matrices, as well as $n \times n$ block matrices using the rank equalities established in the preceding chapters.

In Chapter 10, we investigate Moore-Penrose inverses of sums of matrices using the rank equalities established in the preceding chapters.

In Chapter 11, we study the relationships between Moore-Penrose inverses of block circulant matrices and sums of matrices. Based on them and the results in Chapter 9, we shall present a group of expressions for Moore-Penrose inverses of sums of matrices.

In Chapter 12, we present a group of formulas for expressing ranks of submatrices in the Moore-Penrose inverse of a matrix.

In Chapters 13—17, our work is concerned with rank equalities for Drazin inverses, group inverses, and weighted Moore-Penrose inverses of matrices and their applications. Various kinds of problems examined in Chapters 6—12 for Moore-Penrose inverses of matrices are almost considered in these five chapters for Drazin inverses, group inverses, and weighted Moore-Penrose inverses of matrices.

Chapter 2

Basic rank formulas

The first and most fundamental rank formula used in the sequel is given below.

Theorem 2.1. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$, $C \in \mathcal{C}^{l\times n}$ and $D \in \mathcal{C}^{l\times k}$ be given. Then the rank of the Schur complement $S_A = D - CA^{\dagger}B$ satisfies the equality*

$$r(D - CA^{\dagger}B) = r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} - r(A). \tag{2.1}$$

Proof. It is obvious that

$$R(A^*B) \subseteq R(A^*) = R(A^*AA^*), \quad \text{and} \quad R(AC^*) \subseteq R(A) = R(AA^*A).$$

Then it follows by Eq.(1.7) and a well-known basic property $A^*(A^*AA^*)^{\dagger}A^* = A^{\dagger}$(see Rao and Mitra. 1971, p. 69) that

$$r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} = r(A^*AA^*) + r[\, A - CA^*(A^*AA^*)^{\dagger}A^*B\,] = r(A) + r(\, D - CA^{\dagger}B\,),$$

establishing Eq(2.1). □

The significance of Eq.(2.1) is in that the rank of the Schur complement $S_A = D - CA^{\dagger}B$ can be evaluated by a block matrix formed by A, B, C and D in it, where no restrictions are imposed on S_A and no Moore-Penrose inverses appear in the right-hand side of Eq.(2.1). Thus Eq.(2.1) in fact provides us a powerful tool to express ranks of matrix expressions that involve Moore-Penrose inverses of matrices.

Eq.(2.1) can be extended to various general formulas. We next present some of them, which will widely be used in the sequel.

Theorem 2.2. *Let A_1, A_2, B_1, B_2, C_1, C_2 and D are matrices such that expression $D - C_1A_1^{\dagger}B_1 - C_2A_2^{\dagger}B_2$ is defined. Then*

$$r(D - C_1A_1^{\dagger}B_1 - C_2A_2^{\dagger}B_2) = r\begin{bmatrix} A_1^*A_1A_1^* & 0 & A_1^*B_1 \\ 0 & A_2^*A_2A_2^* & A_2^*B_2 \\ C_1A_1^* & C_2A_2^* & D \end{bmatrix} - r(A_1) - r(A_2). \tag{2.2}$$

In particular, if

$$R(B_1) \subseteq R(A_1), \quad R(C_1^*) \subseteq R(A_1^*), \quad R(B_2) \subseteq R(A_2) \quad \text{and} \quad R(C_2^*) \subseteq R(A_2^*),$$

then

$$r(\, D - C_1 A_1^{\dagger} B_1 - C_2 A_2^{\dagger} B_2 \,) = r \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D \end{bmatrix} - r(A_1) - r(A_2). \tag{2.3}$$

Proof. Let

$$C = [\, C_1,\ C_2 \,], \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}.$$

Then Eq.(2.1) can be written as Eq.(2.2), and Eq.(2.3) follows from Eq.(1.6). □

If the matrices in Eq.(2.2) satisfy certain conditions, the block matrix in Eq.(2.2) can easily be reduced to some simpler forms. Below are some of them.

Corollary 2.3. *Let $A \in \mathcal{C}^{m \times n}$, $B \in \mathcal{C}^{m \times k}$, $C \in \mathcal{C}^{l \times n}$ and $N \in \mathcal{C}^{k \times l}$ be given. Then*

$$r(\, N^{\dagger} - CA^{\dagger}B \,) = r \begin{bmatrix} AA^*A - A(BNC)^*A & AC^* & 0 \\ B^*A & 0 & N \\ 0 & N & 0 \end{bmatrix} - r(A) - r(N). \tag{2.4}$$

In particular, if

$$R(B^*A) \subseteq R(N) \quad \text{and} \quad R(CA^*) \subseteq R(N^*),$$

then

$$r(\, N^{\dagger} - CA^{\dagger}B \,) = r[\, AA^*A - A(BNC)^*A \,] + r(N) - r(A). \tag{2.5}$$

If

$$R(B^*A) \subseteq R(N), \ R(CA^*) \subseteq R(N^*), \ R(BN) \subseteq R(A) \quad \text{and} \quad R[(NC)^*] \subseteq R(A^*).$$

then

$$r(\, N^{\dagger} - CA^{\dagger}B \,) = r(\, A - BNC \,) + r(N) - r(A). \tag{2.6}$$

Theorem 2.4. *Let A_t, B_t, B_t, $C_t (t = 1,\ 2,\ \cdots,\ k)$ and D are matrices such that expression $D - C_1 A_1^{\dagger} B_1 - \cdots - C_k A_k^{\dagger} B_k$ is defined. Then*

$$r(\, D - C_1 A_1^{\dagger} B_1 - \cdots - C_k A_k^{\dagger} B_k \,) = r \begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} - r(A). \tag{2.7}$$

where $A = \mathrm{diag}(\, A_1,\ A_2,\ \cdots,\ A_k \,)$, $B^* = [\, B_1^*,\ B_2^*,\ \cdots,\ B_k^* \,]$ and $C = [\, C_1,\ C_2,\ \cdots,\ C_k \,]$.

Theorem 2.5. *Let A, B, C, D, P and Q are matrices such that expression $D - CP^{\dagger}AQ^{\dagger}B$ is defined. Then*

$$r(\, D - CP^{\dagger}AQ^{\dagger}B \,) = r \begin{bmatrix} P^*AQ^* & P^*PP^* & 0 \\ Q^*QQ^* & 0 & Q^*B \\ 0 & CP^* & -D \end{bmatrix} - r(P) - r(Q). \tag{2.8}$$

In particular, if

$$R(A) \subseteq R(P), \quad R(A^*) \subseteq R(Q^*), \quad R(B) \subseteq R(Q) \quad \text{and} \quad R(C^*) \subseteq R(P^*),$$

then

$$r(\, D - CP^\dagger AQ^\dagger B\,) = r\begin{bmatrix} A & P & 0 \\ Q & 0 & B \\ 0 & C & -D \end{bmatrix} - r(P) - r(Q). \tag{2.9}$$

Proof. Note that

$$\begin{aligned} r(\, D - CP^\dagger AQ^\dagger B\,) &= r\begin{bmatrix} A & AQ^\dagger B \\ CP^\dagger A & D \end{bmatrix} - r(A) \\ &= r\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix}^\dagger \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) - r(A). \end{aligned}$$

Applying Eq.(2.1) to it and then simplifying yields Eq.(2.8). Eq.(2.9) is derived from Eqs.(2.8) by the rank cancellation law Eq.(1.8). □

Theorem 2.6. *Suppose that the matrix expression $S = D - C_1P_1^\dagger A_1Q_1^\dagger B_1 - C_2P_2^\dagger A_2Q_2^\dagger B_2$ is defined. Then*

$$r(S) = r\begin{bmatrix} P_1^*A_1Q_1^* & 0 & P_1^*P_1P_1^* & 0 & 0 \\ 0 & P_2^*A_2Q_2^* & 0 & P_2^*P_2P_2^* & 0 \\ Q_1^*Q_1Q_1^* & 0 & 0 & 0 & Q_1^*B_1 \\ 0 & Q_2^*Q_2Q_2^* & 0 & 0 & Q_2^*B_2 \\ 0 & 0 & C_1P_1^* & C_2P_2^* & -D \end{bmatrix} - d. \tag{2.10}$$

where $d = r(P_1) + r(P_2) + r(Q_1) + r(Q_2)$. In particular, if

$$R(A_i) \subseteq R(P_i), \quad R(A_i^*) \subseteq R(Q_i^*), \quad R(B_i) \subseteq R(Q_i) \quad \text{and} \quad R(C_i^*) \subseteq R(P_i^*),$$

then

$$r(S) = r\begin{bmatrix} A_1 & 0 & P_1 & 0 & 0 \\ 0 & A_2 & 0 & P_2 & 0 \\ Q_1 & 0 & 0 & 0 & B_1 \\ 0 & Q_2 & 0 & 0 & B_2 \\ 0 & 0 & C_1 & C_2 & -D \end{bmatrix} - r(P_1) - r(Q_1) - r(P_2) - r(Q_2). \tag{2.11}$$

Moreover,

$$r(\, D^\dagger - CP^\dagger AQ^\dagger B\,) = r\begin{bmatrix} D^*DD^* & 0 & 0 & D^* \\ 0 & P^*AQ^* & P^*PP^* & 0 \\ 0 & Q^*QQ^* & 0 & Q^*B \\ D^* & 0 & CP^* & 0 \end{bmatrix} - r(P) - r(Q) - r(D). \tag{2.12}$$

Proof. Writing S as

$$S = D - [C_1,\ C_2] \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}^{\dagger} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}^{\dagger} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

and then applying Eq.(2.8) to it produce Eqs.(2.10). Eq.(2.11) is derived from Eqs.(2.10) by the rank cancellation law Eq.(1.8). Eq.(2.12) is a special case of Eq.(2.10). □

It is easy to see that a general rank formula for

$$D - C_1P_1^{\dagger}A_1Q_1^{\dagger}B_1 - C_2P_2^{\dagger}A_2Q_2^{\dagger}B_2 - \cdots - C_kP_2^{\dagger}A_kQ_k^{\dagger}B_k$$

can also be established by the similar method for deriving Eq.(2.10). As to some other general matrix expressions, such as

$$S_k = A_0P_1^{\dagger}A_1P_2^{\dagger}A_2 \cdots P_k^{\dagger}A_k$$

and their linear combinations. the formulas for expressing their ranks can also be established. However they are quite tedious in form. we do not intend to give them here.

Chapter 3

Rank equalities related to idempotent matrices

A square matrix A is said to be idempotent if $A^2 = A$. If we consider it as a matrix equation $A^2 = A$, then its general solution can be written as $A = V(V^2)^\dagger V$, where V is an arbitrary square matrix. This assertion can easily be verified. In fact, $A = V(V^2)^\dagger V$, apparently satisfies $A^2 = A$. Now for any matrix A with $A^2 = A$, we let $V = A$. Then $V(V^2)^\dagger V = A(A^2)^\dagger A = AA^\dagger A = A$. Thus $A = V(V^2)^\dagger V$ is indeed the general solution the idempotent equation $A^2 = A$. This fact clearly implies that any matrix expression that involves idempotent matrices could be regarded as a conventional matrix expression that involves Moore-Penroses inverses of matrices. Thus the formulas in Chapter 2 are all applicable to determine ranks of matrix expressions that involve idempotent matrices. However because of speciality of idempotent matrices, the rank equalities related to idempotent matrices can also be deduced by various elementary methods. The results in the chapter are originally derived by the rank formulas in Chapter 2, we later also find some elementary methods to establish them. So we only show these results in these elementary methods.

Theorem 3.1. *Let $P, Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then the difference $P - Q$ satisfies the rank equalities*

$$r(P - Q) = r\begin{bmatrix} P \\ Q \end{bmatrix} + r[P, Q] - r(P) - r(Q), \tag{3.1}$$

$$r(P - Q) = r(P - PQ) + r(PQ - Q), \tag{3.2}$$

$$r(P - Q) = r(P - QP) + r(QP - Q). \tag{3.3}$$

Proof. Let $M = \begin{bmatrix} -P & 0 & P \\ 0 & Q & Q \\ P & Q & 0 \end{bmatrix}$. Then it is easy to see by block elementary operations of matrices that

$$r(M) = r\begin{bmatrix} -P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & P - Q \end{bmatrix} = r(P) + r(Q) + r(P - Q).$$

On the other hand, note that $P^2 = P$ and $Q^2 = Q$. It is also easy to find by block elementary operations of matrices that

$$r(M) = r\begin{bmatrix} -P & 0 & P \\ -QP & 0 & Q \\ P & Q & 0 \end{bmatrix} = r\begin{bmatrix} 0 & 0 & P \\ 0 & 0 & Q \\ P & Q & 0 \end{bmatrix} = r\begin{bmatrix} P \\ Q \end{bmatrix} + r[\,P,\ Q\,].$$

Combining the above two equalities yields Eq.(3.1). Consequently applying Eqs.(1.2) and (1.3) to $[\,P,\ Q\,]$ and $\begin{bmatrix} P \\ Q \end{bmatrix}$ in Eq.(3.1) respectively yields

$$r[\,P,\ Q\,] = r(P) + r(\,Q - PQ\,), \tag{3.4}$$

$$r[\,P,\ Q\,] = r(Q) + r(\,P - QP\,), \tag{3.5}$$

$$r\begin{bmatrix} P \\ Q \end{bmatrix} = r(P) + r(\,Q - QP\,), \tag{3.6}$$

$$r\begin{bmatrix} P \\ Q \end{bmatrix} = r(Q) + r(\,P - PQ\,). \tag{3.7}$$

Putting Eqs.(3.4) and (3.7) in Eq.(3.1) produces Eq.(3.2), putting Eqs.(3.5) and (3.6) in Eq.(3.1) produces Eq.(3.3). $\square$

Corollary 3.2. *Let $P,\ Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then*

(a) $R(\,P - PQ\,) \cap R(\,PQ - Q\,) = \{0\}$ *and* $R[(\,P - PQ\,)^*] \cap R[(\,PQ - Q\,)^*] = \{0\}$.

(b) $R(\,P - QP\,) \cap R(\,QP - Q\,) = \{0\}$ *and* $R[(\,P - QP\,)^*] \cap R[(\,QP - Q\,)^*] = \{0\}$.

(c) *If* $PQ = 0$ *or* $QP = 0$, *then* $r(\,P - Q\,) = r(P) + r(Q)$, *i.e.,* $R(P) \cap R(Q) = \{0\}$ *and* $R(P^*) \cap R(Q^*) = \{0\}$.

(d) *If both P and Q are Hermitian idempotent, then* $r(\,P - Q\,) = 2r[\,P,\ Q\,] - r(P) - r(Q)$.

Proof. Parts (a) and (b) follows from applying Lemma 1.4(d) to Eqs.(3.2) and (3.3). Part (c) is a direct consequence of Eqs.(3.2) and (3.3). Part (d) follows from Eq.(3.1). $\square$

On the basis of Eq.(3.1), we can easily deduce a known result due to Hartwig and Styan (1987) on the rank subtractivity two idempotent matrices.

Corollary 3.3. *Let $P,\ Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then the following statements are equivalent:*

(a) $r(\,P - Q\,) = r(P) - r(Q)$, *i.e.,* $Q \leq_{rs} P$.

(b) $r\begin{bmatrix} P \\ Q \end{bmatrix} = r[\,P,\ Q\,] = r(P)$.

(c) $R(Q) \subseteq R(P)$ *and* $R(Q^*) \subseteq R(P^*)$.

(d) $PQ = QP = Q$.

(e) $PQP = Q$.

Proof. The equivalence of (a) and (b) follows immediately from applying Eq.(3.1). The equivalence of (b), (c), (d) and (e) can trivially be verified. □

In addition, from Eq.(3.1), we can immediately find the following result, which has been given in [36] by Gross and Trenkler.

Corollary 3.4. *Let $P,\ Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then the following statements are equivalent:*

(a) *The difference $P - Q$ is nonsingular.*

(b) $r\begin{bmatrix} P \\ Q \end{bmatrix} = r[\,P,\ Q\,] = r(P) + r(Q) = m$. (c) $R(P) \dotplus R(Q) = R(P^*) \dotplus R(Q^*) = \mathcal{C}^m$.

Proof. Follows directly from Eq.(3.1). □

Notice that if a matrix P is idempotent, the $I_m - P$ is also idempotent. Thus replacing P in Eq.(3.1) by $I_m - P$, we get the following.

Theorem 3.5. *Let $P,\ Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then the rank of $I_m - P - Q$ satisfies the equalities*

$$r(\,I_m - P - Q\,) = r(PQ) + r(QP) - r(P) - r(Q) + m, \tag{3.8}$$

$$r(\,I_m - P - Q\,) = r(\,I_m - P - Q + PQ\,) + r(PQ), \tag{3.9}$$

$$r(\,I_m - P - Q\,) = r(\,I_m - P - Q + QP\,) + r(QP). \tag{3.10}$$

Proof. Replacing P in Eq.(3.1) by $I_m - P$ yields

$$r(\,I_m - P - Q\,) = r\begin{bmatrix} I_m - P \\ Q \end{bmatrix} + r[\,I_m - P,\ Q\,] - r(\,I_m - P\,) - r(Q). \tag{3.11}$$

It follows by Eqs.(1.2) and (1.3) that

$$r[\,I_m - P,\ Q\,] = r(\,I_m - P\,) + r[\,Q - (\,I_m - P\,)Q\,] = m - r(P) + r(PQ),$$

and

$$r\begin{bmatrix} I_m - P \\ Q \end{bmatrix} = r(\,I_m - P\,) + r[\,Q - Q(\,I_m - P\,)\,] = m - r(P) + r(QP).$$

Putting them in Eq.(3.11) produces Eq.(3.8). On the other hand, replacing P in Eqs.(3.2) and (3.3) by $I_m - P$ produces

$$\begin{aligned} r[\,(I_m - P) - Q\,] &= r[\,(I_m - P) - (I_m - P)Q\,] + r[\,(I_m - P)Q - Q\,] \\ &= r(\,I_m - P - Q + PQ\,) + r(PQ), \end{aligned}$$

and

$$\begin{aligned} r[\,(I_m - P) - Q\,] &= r[\,(I_m - P) - Q(I_m - P)\,] + r[\,Q(I_m - P) - Q\,] \\ &= r(\,I_m - P - Q + QP\,) + r(QP), \end{aligned}$$

both of which are exactly Eqs.(3.9) and (3.10). $\square$

Corollary 3.6. *Let $P,\ Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then*

(a) $R(\, I_m - P - Q + PQ\,) \cap R(PQ) = \{0\}$ *and* $R[(\, I_m - P - Q + PQ\,)^*] \cap R[(PQ)^*] = \{0\}$.

(b) $R(\, I_m - P - Q + QP\,) \cap R(QP) = \{0\}$ *and* $R[(\, I_m - P - Q + QP\,)^*] \cap R[(QP)^*] = \{0\}$.

(c) $P + Q = I_m \iff PQ = QP = 0$ *and* $R(P) \oplus R(Q) = R(P^*) \oplus R(Q^*) = \mathcal{C}^m$.

(d) *If $PQ = QP = 0$, then $r(\, I_m - P - Q\,) = m - r(P) - r(Q)$.*

(e) *$I_m - P - Q$ is nonsingular if and only if $r(PQ) = r(QP) = r(P) = r(Q)$.*

(f) *If both P and Q are Hermitian idempotent, then $r(\, I_m - P - Q\,) = 2r(PQ) - r(P) - r(Q) + m$.*

Proof. Parts (a) and (b) follow from applying Lemma 1.4(d) to Eqs.(3.9) and (3.10). Note from Eqs.(3.8)—(3.10) that $P + Q = I_m$ is equivalent to $PQ = QP = 0$ and $r(P) + r(Q) = m$. This assertion is also equivalent to $PQ = QP = 0$ and $R(P) \oplus R(Q) = R(P^*) \oplus R(Q^*) = \mathcal{C}^m$, which is Part (c). Parts (d), (e) and (f) follow from Eq.(3.8). $\square$

As for the rank of sum of two idempotent matrices, we have the following several results.

Theorem 3.7. *Let $P,\ Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then the sum $P + Q$ satisfies the rank equalities*

$$r(\, P + Q\,) = r\begin{bmatrix} P & Q \\ Q & 0 \end{bmatrix} - r(Q) = r\begin{bmatrix} Q & P \\ P & 0 \end{bmatrix} - r(P), \tag{3.12}$$

$$r(\, P + Q\,) = r(\, P - PQ - QP + QPQ\,) + r(Q), \tag{3.13}$$

$$r(\, P + Q\,) = r(\, Q - PQ - QP + PQP\,) + r(P). \tag{3.14}$$

Proof. Let $M = \begin{bmatrix} P & 0 & P \\ 0 & Q & Q \\ P & Q & 0 \end{bmatrix}$. Then it is easy to see by block elementary operations of matrices that

$$r(M) = r\begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & -P-Q \end{bmatrix} = r(P) + r(Q) + r(\, P + Q\,).$$

On the other hand, note that $P^2 = P$ and $Q^2 = Q$. It is also easy to find by block elementary operations of matrices that

$$r(M) = r\begin{bmatrix} P & 0 & P \\ -QP & 0 & Q \\ P & Q & 0 \end{bmatrix} = r\begin{bmatrix} 2P & 0 & P \\ 0 & 0 & Q \\ P & Q & 0 \end{bmatrix} = r\begin{bmatrix} 2P & 0 & 0 \\ 0 & 0 & Q \\ 0 & Q & \frac{1}{2}P \end{bmatrix} = r\begin{bmatrix} P & Q \\ Q & 0 \end{bmatrix} + r(P).$$

and

$$r(M) = r\begin{bmatrix} 0 & -PQ & P \\ 0 & Q & Q \\ P & Q & 0 \end{bmatrix} = r\begin{bmatrix} 0 & 0 & P \\ 0 & 2Q & Q \\ P & Q & 0 \end{bmatrix} = r\begin{bmatrix} 0 & 0 & P \\ 0 & 2Q & 0 \\ P & 0 & \frac{1}{2}Q \end{bmatrix} = r\begin{bmatrix} Q & P \\ P & 0 \end{bmatrix} + r(Q).$$

The combination of the above three rank equalities yields the two equalities in Eq.(3.12). Consequently applying Eq.(1.4) to the two block matrices in Eq.(3.12) yields Eqs.(3.13) and (3.14), respectively. □

Corollary 3.8. *Let $P, Q \in \mathcal{C}^{m \times m}$ be two idempotent matrices.*

(a) *If $PQ = QP$, then*

$$r(P+Q) = r[P, Q] = r\begin{bmatrix} P \\ Q \end{bmatrix}, \tag{3.15}$$

or, equivalently,

$$R(Q) \subseteq R(P+Q) \quad and \quad R(Q^*) \subseteq R(P^*+Q^*). \tag{3.16}$$

(b) *If $R(Q) \subseteq R(P)$ or $R(Q^*) \subseteq R(P^*)$, then $r(P+Q) = r(P)$.*

Proof. If $PQ = QP$, then Eqs.(3.13) and (3.14) reduce to

$$r(P+Q) = r(P - PQ) + r(Q) = r(Q - PQ) + r(P).$$

Combining them with Eqs.(3.4) and (3.7) yields Eq.(3.15). The equivalence of Eqs.(3.15) and (3.16) follows from a simple fact that

$$r\begin{bmatrix} P \\ Q \end{bmatrix} = r\begin{bmatrix} P+Q \\ Q \end{bmatrix} \quad \text{and} \quad r[P, Q] = r[P+Q, Q],$$

as well as Lemma 1.2(c) and (d). The result in Part (b) follows immediately from Eq.(3.12). □

Corollary 3.9. *Let $P, Q \in \mathcal{C}^{m \times m}$ be two idempotent matrices. Then the following five statements are equivalent:*

(a) *The sum $P+Q$ is nonsingular.*

(b) $r\begin{bmatrix} P \\ Q \end{bmatrix} = m$ *and* $R\begin{bmatrix} P \\ Q \end{bmatrix} \cap R\begin{bmatrix} Q \\ 0 \end{bmatrix} = \{0\}$.

(c) $r[P, Q] = m$ *and* $R\begin{bmatrix} P^* \\ Q^* \end{bmatrix} \cap R\begin{bmatrix} Q^* \\ 0 \end{bmatrix} = \{0\}$.

(d) $r\begin{bmatrix} Q \\ P \end{bmatrix} = m$ *and* $R\begin{bmatrix} Q \\ P \end{bmatrix} \cap R\begin{bmatrix} P \\ 0 \end{bmatrix} = \{0\}$.

(e) $r[Q, P] = m$ *and* $R\begin{bmatrix} Q^* \\ P^* \end{bmatrix} \cap R\begin{bmatrix} P^* \\ 0 \end{bmatrix} = \{0\}$.

Proof. In light of Eq.(3.12), the sum $P+Q$ is nonsingular if and only if

$$r\begin{bmatrix} P & Q \\ Q & 0 \end{bmatrix} = r(Q) + m. \tag{3.17}$$

or equivalently

$$r\begin{bmatrix} Q & P \\ P & 0 \end{bmatrix} = r(P) + m. \tag{3.18}$$

Observe that

$$r\begin{bmatrix} P & Q \\ Q & 0 \end{bmatrix} \le r\begin{bmatrix} P \\ Q \end{bmatrix} + r\begin{bmatrix} Q \\ 0 \end{bmatrix} \le m + r(Q),$$

$$r\begin{bmatrix} P & Q \\ Q & 0 \end{bmatrix} \le r[P,\ Q] + r[Q,\ 0] \le m + r(Q),$$

$$r\begin{bmatrix} Q & P \\ P & 0 \end{bmatrix} \le r\begin{bmatrix} Q \\ P \end{bmatrix} + r\begin{bmatrix} P \\ 0 \end{bmatrix} \le m + r(P),$$

$$r\begin{bmatrix} Q & P \\ P & 0 \end{bmatrix} \le r[Q,\ P] + r[P,\ 0] \le m + r(P).$$

Combining them with Eqs.(3.17) and (3.18) yields the equivalence of Parts (a)—(e). □

Theorem 3.11. *Let $P,\ Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then*

(a) *The rank of $I_m + P - Q$ satisfies the equality*

$$r(\, I_m + P - Q\,) = r(QPQ) - r(Q) + m. \tag{3.19}$$

(b) *The rank of $2I_m - P - Q$ satisfies the two equalities*

$$r(\, 2I_m - P - Q\,) = r(\, Q - QPQ\,) - r(Q) + m. \tag{3.20}$$

$$r(\, 2I_m - P - Q\,) = r(\, P - PQP\,) - r(P) + m. \tag{3.21}$$

Proof. Replacing Q in Eq.(3.12) by the idempotent matrix $I_m - Q$ and applying Eq.(1.4) to it yields

$$\begin{aligned} r(\, I_m + P - Q\,) &= r\begin{bmatrix} P & I_m - Q \\ I_m - Q & 0 \end{bmatrix} - r(\, I_m - Q\,) \\ &= r(\, I_m - Q\,) + r[(\, I_m - (\, I_m - Q\,)\,)P(\, I_m - (\, I_m - Q\,)\,)] \\ &= m - r(Q) + r(QPQ), \end{aligned}$$

establishing Eq.(3.19). Further, replacing P and Q in Eq.(3.12) by $I_m - P$ and $I_m - Q$, we also by Eq.(1.4) find that

$$\begin{aligned} r(\, 2I_m - P - Q\,) &= r\begin{bmatrix} I_m - P & I_m - Q \\ I_m - Q & 0 \end{bmatrix} - r(\, I_m - Q\,) \\ &= r(\, I_m - Q\,) + r[(\, I_m - (\, I_m - Q\,))(\, I_m - P\,)(\, I_m - (\, I_m - Q\,)\,)] \\ &= m - r(Q) + r(\, Q - QPQ\,), \end{aligned}$$

establishing Eq.(3.20). Similarly, we can show Eq.(3.21). □

Corollary 3.11. *Let $P,\ Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices.*

(a) *If $R(P) \subseteq R(Q)$ and $R(P^*) \subseteq R(Q^*)$, then P and Q satisfy the two rank equalities*

$$r(I_m + P - Q) = m + r(P) - r(Q), \tag{3.22}$$

$$r(2I_m - P - Q) = m + r(Q - P) - r(Q). \tag{3.23}$$

(b) $I_m + P - Q$ *is nonsingular* $\iff r(QPQ) = r(Q)$.

(c) $2I_m - P - Q$ *is nonsingular* $\iff r(P - PQP) = r(P) \iff r(Q - QPQ) = r(Q)$.

(d) $Q - P = I_m \iff r(QPQ) + r(Q) = m$.

Proof. The two conditions $R(P) \subseteq R(Q)$ and $R(P^*) \subseteq R(Q^*)$ are equivalent to $QP = P = PQ$. In that case. Eq.(3.19) reduces to Eq.(3.22). Eqs.(3.20) and (3.21) reduce to Eq.(3.23). The results in Parts (a)—(c) are direct consequences of Eq.(3.19). $\square$

we next consider the rank of $PQ - QP$ for two idempotent matrices P and Q.

Theorem 3.12. *Let $P, Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then the difference $PQ - QP$ satisfies the five rank equalities*

$$r(PQ - QP) = r(P - Q) + r(I_m - P - Q) - m, \tag{3.24}$$

$$r(PQ - QP) = r(P - Q) + r(PQ) + r(QP) - r(P) - r(Q), \tag{3.25}$$

$$r(PQ - QP) = r\begin{bmatrix} P \\ Q \end{bmatrix} + r[P, Q] + r(PQ) + r(QP) - 2r(P) - 2r(Q), \tag{3.26}$$

$$r(PQ - QP) = r(P - PQ) + r(PQ - Q) + r(PQ) + r(QP) - r(P) - r(Q), \tag{3.27}$$

$$r(PQ - QP) = r(P - QP) + r(QP - Q) + r(PQ) + r(QP) - r(P) - r(Q). \tag{3.28}$$

In particular. if both P and Q are Hermitian idempotent, then

$$r(PQ - QP) = 2r[P, Q] + 2r(PQ) - 2r(P) - 2r(Q). \tag{3.29}$$

Proof. It is easy to verify that that $PQ - QP = (P - Q)(P + Q - I_m)$. Thus the rank of $PQ - QP$ can be expressed as

$$r(PQ - QP) = r[(P - Q)(P + Q - I_m)] = r\begin{bmatrix} I_m & P + Q - I_m \\ P - Q & 0 \end{bmatrix} - m. \tag{3.30}$$

On the other hand. it is easy to verify the factorization

$$\begin{bmatrix} I_m & P + Q - I_m \\ P - Q & 0 \end{bmatrix} = \begin{bmatrix} I_m & 2P - I_m \\ 0 & I_m \end{bmatrix} \begin{bmatrix} 0 & P + Q - I_m \\ P - Q & 0 \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 2Q - I_m & I_m \end{bmatrix}.$$

Hence

$$r\begin{bmatrix} I_m & P + Q - I_m \\ P - Q & 0 \end{bmatrix} = r(P - Q) + r(I_m - P - Q).$$

Putting it in Eq.(3.30) yields Eq.(3.24). Consequently putting Eq.(3.8) in Eq.(3.24) yields Eq.(3.25); putting Eq.(3.1) in Eq.(3.25) yields Eq.(3.26); putting Eqs.(3.2) and (3.3) respectively in Eq.(3.25) yields Eqs.(3.27) and (3.28). □

Corollary 3.13. *Let P, $Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then the following five statements are equivalent:*

(a) $PQ = QP$.

(b) $r(\,P - Q\,) + r(\,I_m - P - Q\,) = m$.

(c) $r(\,P - Q\,) = r(P) + r(Q) - r(PQ) - r(QP)$.

(d) $r(\,P - PQ\,) = r(P) - r(PQ)$ *and* $r(\,Q - PQ\,) = r(Q) - r(PQ)$, *i.e.*, $PQ \leq_{rs} P$ *and* $PQ \leq_{rs} Q$.

(e) $r(\,P - QP\,) = r(P) - r(QP)$ *and* $r(\,Q - QP\,) = r(Q) - r(QP)$, *i.e.*, $QP \leq_{rs} P$ *and* $QP \leq_{rs} Q$.

(f) $r\begin{bmatrix} P \\ Q \end{bmatrix} = r(P) + r(Q) - r(PQ)$ *and* $r[\,P,\ Q\,] = r(P) + r(Q) - r(QP)$.

(g) $r\begin{bmatrix} P \\ Q \end{bmatrix} = r(P) + r(Q) - r(QP)$ *and* $r[\,P,\ Q\,] = r(P) + r(Q) - r(PQ)$.

Proof. Follows immediately from Eqs.(3.24)—(3.28). □

Corollary 3.14. *Let P, $Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then the following three statements are equivalent:*

(a) $r(\,PQ - QP\,) = r(\,P - Q\,)$.

(b) $I_m - P - Q$ *is nonsingular.*

(c) $r(PQ) = r(QP) = r(P) = r(Q)$.

Proof. The equivalence of Parts (a) and (b) follows from Eq.(3.24). The equivalence of Parts (b) and (c) follows from Corollary 3.6(e). □

Corollary 3.15. *Let P, $Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then the following three statements are equivalent:*

(a) $PQ - QP$ *is nonsingular.*

(b) $P - Q$ *and* $I_m - P - Q$ *are nonsingular.*

(c) $R(P) \oplus R(Q) = R(P^*) \oplus R(Q^*) = \mathcal{C}^m$ *and* $r(PQ) = r(QP) = r(P) = r(Q)$ *hold.*

Proof. The equivalence of Parts (a) and (b) follows from Eq.(3.24). The equivalence of Parts (b) and (c) follows from Corollaries 3.4(e) and 3.6(e). □

A group of analogous rank equalities can also be derived for $PQ + QP$, where P and Q are two idempotent matrices P and Q.

Theorem 3.16. *Let P, $Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then $PQ + QP$ satisfies the rank equalities*

$$r(\,PQ + QP\,) = r(\,P + Q\,) + r(\,I_m - P - Q\,) - m, \tag{3.31}$$

$$r(PQ + QP) = r(P + Q) + r(PQ) + r(QP) - r(P) - r(Q). \tag{3.32}$$

$$r(PQ + QP) = r(P - PQ - QP + QPQ) + r(PQ) + r(QP) - r(P). \tag{3.33}$$

$$r(PQ + QP) = r(Q - PQ - QP + PQP) + r(PQ) + r(QP) - r(Q). \tag{3.34}$$

Proof. Note that $PQ + QP = (P + Q)^2 - (P + Q)$. Then applying Eq.(1.11) to it, we directly obtain Eq.(3.31). Consequently, putting Eq.(3.8) in Eq.(3.21) yields Eq.(3.32), putting Eqs.(3.13) and (3.14) respectively in (3.32) yields Eqs.(3.33) and (3.34). □

Corollary 3.17. *Let $P,\ Q \in \mathcal{C}^{m \times m}$ be two idempotent matrices. Then the following four statements are equivalent:*

(a) $r(PQ + QP) = r(P + Q)$.

(b) $I_m - P - Q$ *is nonsingular.*

(c) $r(PQ) = r(QP) = r(P) = r(Q)$.

(d) $r(PQ - QP) = r(P - Q)$.

Proof. The equivalence of Parts (a) and (b) follows from Eq.(3.31), and the equivalence of Parts (b)—(d) comes from Corollary 3.14. □

Corollary 3.18. *Let $P,\ Q \in \mathcal{C}^{m \times m}$ be two idempotent matrices. Then the following two statements are equivalent:*

(a) $PQ + QP$ *is nonsingular.*

(b) $P + Q$ *and* $I_m - P - Q$ *are nonsingular.*

Proof. Follows directly from Eq.(3.31). □

Combining the two rank equalities in Eqs.(3.24) and (3.31), we obtain the following.

Corollary 3.19. *Let $P,\ Q \in \mathcal{C}^{m \times m}$ be two idempotent matrices. Then both of them satisfy the following rank identity*

$$r(P + Q) + r(PQ - QP) = r(P - Q) + r(PQ + QP). \tag{3.35}$$

Thoerem 3.20. *Let $P,\ Q \in \mathcal{C}^{m \times m}$ be two idempotent matrices. Then*

$$r[(P - Q)^2 - (P - Q)] = r(I_m - P + Q) + r(P - Q) - m. \tag{3.36}$$

$$r[(P - Q)^2 - (P - Q)] = r(PQP) - r(P) + r(P - Q). \tag{3.37}$$

Proof. Eq.(3.36) is derived from Eq.(1.11). According to Eq.(3.19), we have $r(I_m - P + Q) = r(PQP) - r(P) + m$. Putting it in Eq.(3.36) yields Eq.(3.37). □

Corollary 3.21 (Hartwig and Styan, 1987). *Let $P,\ Q \in \mathcal{C}^{m \times m}$ be two idempotent matrices. Then the following five statements are equivalent:*

(a) $P - Q$ *is idempotent.*

(b) $r(\, I_m - P + Q\,) = m - r(\, P - Q\,)$.

(c) $r(\, P - Q\,) = r(P) - r(Q)$, *i.e.*, $Q \leq_{rs} P$.

(d) $R(Q) \subseteq R(P)$ *and* $R(Q^*) \subseteq R(P^*)$.

(e) $PQP = Q$.

Proof. The equivalence of Parts (a) and (b) follows immediately from Eq.(3.36), and the equivalence of Parts (c), (d) and (e) is from Corollary 3.3(d). The equivalence of Parts (a) and (e) follows from a direct matrix computation. □

In Chapter 4, we shall also establish a rank formula for $(\, P - Q\,)^3 - (\, P - Q\,)$ and consider tripotency of $P - Q$, where P, Q are two idempotent matrices.

Theorem 3.22. *Let P, $Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then $I_m - PQ$ satisfies the rank equalities*

$$r(\, I_m - PQ\,) = r(\, 2I_m - P - Q\,) = r[\,(\, I_m - P\,) + (\, I_m - Q\,)\,]. \tag{3.38}$$

Proof. According to Eq.(1.10) we have

$$r(\, I_m - PQ\,) = r(\, Q - QPQ\,) - r(Q) + m.$$

Consequently putting Eq.(3.20) in it yields Eq.(3.38). □

Corollary 3.23. *Let P, $Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then the sum $P + Q$ satisfies the rank identities*

$$r(\, P + Q\,) = r(\, P + Q - PQ\,) = r(\, P + Q - QP\,). \tag{3.39}$$

In particular, if $PQ = QP$, then

$$r(\, P + Q\,) = r(P) + r(Q) - r(PQ). \tag{3.40}$$

Proof. Replacing P and Q in Eq.(3.38) by two idempotent matrices $I_m - P$ and $I_m - Q$ immediately yields Eq.(3.39). If $PQ = QP$, then we know by Eqs.(3.13) and (3.14) that

$$r(\, P + Q\,) = r(\, P - PQ\,) + r(Q) = r(\, Q - QP\,) + r(P). \tag{3.41}$$

and by Corollary 3.13 we also know that

$$r(\, P - PQ\,) = r(P) - r(PQ) \quad \text{and} \quad r(\, Q - QP\,) = r(Q) - r(QP). \tag{3.42}$$

Putting Eq.(3.42) in Eq.(3.41) yields Eq.(3.40). □

Corollary 3.24. *Let P, $Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then*

$$r[\, PQ - (PQ)^2\,] = r(\, I_m - PQ\,) + r(PQ) - m = r(\, 2I_m - P - Q\,) + r(PQ) - m. \tag{3.43}$$

In particular, the following statements are equivalent:

(a) PQ *is idempotent.*

(b) $r(I_m - PQ) = m - r(PQ)$.

(c) $r(2I_m - P - Q) = m - r(PQ)$.

Proof. Applying Eq.(1.11) to $PQ - (PQ)^2$ gives the first equality in Eq.(3.43). The second one follows from Eq.(3.38). □

Corollary 3.25. *Let $P, Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then*

$$r(I_m - P - Q + PQ) = m - r(P) - r(Q) + r(QP).$$

Proof. This follows from replacing A in Eq.(1.4) by I_m. □

Notice that if a matrix A is idempotent, then A^* is also idempotent. Thus we can easily find the following.

Corollary 3.26. *Let $P \in \mathcal{C}^{m\times m}$ be an idempotent matrix. Then*

(a) $r(P - P^*) = 2r[P, P^*] - 2r(P)$.

(b) $r(I_m - P - P^*) = r(I_m + P - P^*) = m$.

(c) $r(P + P^*) = r[P, P^*]$, *i.e.,* $R(P) \subseteq R(P + P^*)$ *and* $R(P^*) \subseteq R(P + P^*)$.

(d) $r(PP^* - P^*P) = r(P - P^*)$.

Proof. Parts (a) follows from Eq.(3.1). Part (b) follows from Eq.(3.8). Part (c) follows from Eq.(3.31). Part (d) follows from Eq.(3.24) and Part (b). □

The results in the preceding theorems and corollaries can easily be extended to matrices with properties $P^2 = \lambda P$ and $Q^2 = \mu Q$, where $\lambda \neq 0$ and $\mu \neq 0$. In fact, observe that

$$\left(\frac{1}{\lambda}P\right)^2 = \frac{1}{\lambda^2}P^2 = \frac{1}{\lambda}P, \qquad \left(\frac{1}{\mu}Q\right)^2 = \frac{1}{\mu^2}Q^2 = \frac{1}{\mu}Q.$$

Thus both P/λ and Q/μ are idempotent. In that case, applying the results in the previous theorems and corollaries, one may establish a variety of rank equalities and their consequences related to such kind of matrices. For example,

$$r(\mu P - \lambda Q) = r\begin{bmatrix} P \\ Q \end{bmatrix} + r[P, Q] - r(P) - r(Q),$$

$$r(\mu P + \lambda Q) = r\begin{bmatrix} P & Q \\ P & 0 \end{bmatrix} - r(Q) = r\begin{bmatrix} Q & P \\ P & 0 \end{bmatrix} - r(P),$$

$$r(\lambda\mu I_m - \mu P - \lambda Q) = r(PQ) + r(QP) - r(P) - r(Q) + m,$$

$$r(PQ - QP) = r(\mu P - \lambda Q) + r(\lambda\mu I_m - \mu P - \lambda Q) - m,$$

$$r(PQ + QP) = r(\mu P + \lambda Q) + r(\lambda\mu I_m - \mu P - \lambda Q) - m,$$

$$r(\lambda\mu I_m - PQ) = r(2\lambda\mu I_m - \mu P - \lambda Q),$$

and so on. We do not intend to present them in details.

Chapter 4

More on rank equalities related to idempotent matrices and their applications

The rank equalities in Chapter 3 can partially be extended to matrix expressions that involve idempotent matrices and general matrices. In addition, they can also be applied to establish rank equalities related to involutory matrices. The corresponding results are presented in this chapter.

Theorem 4.1. *Let $A \in C^{m\times n}$ be given. $P \in C^{m\times m}$ and $Q \in C^{n\times n}$ be two idempotent matrices. Then the difference $PA - AQ$ satisfies the two rank equalities*

$$r(PA - AQ) = r\begin{bmatrix} PA \\ Q \end{bmatrix} + r[\, AQ,\ P\,] - r(P) - r(Q). \tag{4.1}$$

$$r(PA - AQ) = r(PA - PAQ) + r(PAQ - AQ). \tag{4.2}$$

Proof. Let $M = \begin{bmatrix} -P & 0 & PA \\ 0 & Q & Q \\ P & AQ & 0 \end{bmatrix}$. Then it is easy to see by the block elementary operations of matrices that

$$r(M) = r\begin{bmatrix} -P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & PA - AQ \end{bmatrix} = r(P) + r(Q) + r(PA - AQ). \tag{4.3}$$

On the other hand, note that $P^2 = P$ and $Q^2 = Q$. It is also easy to find by block elementary operations of matrices that

$$r(M) = r\begin{bmatrix} 0 & PAQ & PA \\ 0 & Q & Q \\ P & AQ & 0 \end{bmatrix} = r\begin{bmatrix} 0 & 0 & PA \\ 0 & 0 & Q \\ P & AQ & 0 \end{bmatrix} = r\begin{bmatrix} PA \\ Q \end{bmatrix} + r[\, AQ,\ P\,]. \tag{4.4}$$

Combining Eqs.(4.3) and (4.4) yields Eq.(4.1). Consequently applying Eqs.(1.2) and (1.3) to $[\, AQ,\ P\,]$ and $\begin{bmatrix} PA \\ Q \end{bmatrix}$ in (4.1) respectively yields Eq.(4.2). $\square$

Corollary 4.2. *Let $A \in C^{m\times n}$ be given, $P \in C^{m\times m}$ and $Q \in C^{n\times n}$ be two idempotent matrices. Then*

(a) *$R(PA - PAQ) \cap R(PAQ - AQ) = \{0\}$ and $R[(PA - PAQ)^*] \cap R[(PAQ - AQ)^*] = \{0\}$.*

(b) *If $PAQ = 0$, then $r(PA - AQ) = r(PA) + r(AQ)$, or, equivalently $R(PA) \cap R(AQ) = \{0\}$ and $R[(PA)^*] \cap R[(AQ)^*] = \{0\}$.*

(c) *$PA = AQ \Longleftrightarrow PA(I - Q) = 0$ and $(I - P)AQ = 0 \Longleftrightarrow R(AQ) \subseteq R(P)$ and $R[(PA)^*] \subseteq R(Q^*)$.*

Proof. Part (a) follows from applying Lemma 1.4(d) to Eq.(4.2). Parts (b) and (c) are direct consequences of Eq.(4.2). □

Corollary 4.3. *Let A, P, $Q \in \mathcal{C}^{m\times m}$ be given with P, Q being two idempotent matrices. Then the following three statements are equivalent:*

(a) $PA - AQ$ *is nonsingular.*

(b) $r\begin{bmatrix} PA \\ Q \end{bmatrix} = r[\, AQ,\ P\,] = r(P) + r(Q) = m.$

(c) $r(PA) = r(P)$, $r(AQ) = r(Q)$ *and* $R(AQ) \oplus R(P) = R[(PA)^*] \oplus R(Q^*) = \mathcal{C}^m$.

Proof. Follows from Eq.(4.1). □

Based on Corollary 4.2(c), we find an interesting result on the general solution of a matrix equation.

Corollary 4.4. *Let $P \in \mathcal{C}^{m\times m}$ and $Q \in \mathcal{C}^{n\times n}$ be two idempotent matrices. Then the general solution of the matrix equation $PX = XQ$ can be written in the two forms*

$$X = PUQ + (\, I_m - P\,)V(\, I_m - Q\,), \tag{4.5}$$

$$X = PW + WQ - 2PWQ, \tag{4.6}$$

where U, V, $W \in \mathcal{C}^{m\times n}$ are arbitrary.

Proof. According to Corollary 4.2(c), the matrix equation $PX = XQ$ is equivalent to the pair of matrix equations

$$PX(\, I - Q\,) = 0 \quad \text{and} \quad (\, I - P\,)XQ = 0. \tag{4.7}$$

Solving the pair of equations, we can find that both Eq.(4.5) and Eq.(4.6) are the general solutions of $PX = XQ$. The process is somewhat tedious. Instead, we give here the verification. Putting (4.5) in PX and XQ, we get

$$PX = PUQ \quad \text{and} \quad XQ = PUQ.$$

Thus Eq.(4.5) is solution of $PX = XQ$. On the other hand, suppose that X_0 is a solution of $PX = XQ$ and let $U = V = X_0$ in Eq.(4.5). Then Eq.(4.5) becomes

$$X = PX_0Q + (\, I_m - P\,)X_0(\, I_m - Q\,) = PX_0Q + X_0 - PX_0 - X_0Q + PX_0Q = X_0.$$

which implies that any solution of $PX = XQ$ can be expressed by Eq.(4.5). Hence Eq.(4.5) is indeed the general solution of the equation $PX = XQ$. Similarly we can verify that Eq.(4.6) is also a general solution to $PX = XQ$. □

As one of the basic linear matrix equation, $AX = XB$ was examined (see, e.g., Hartwig [37], Parker [77], Slavova et al [88]). In general cases, the solution of $AX = XB$ can only be determined by the canonical forms of A and B. The result in Corollary 4.4 manifests that for idempotent matrices A and B, the general solution of $AX = XB$ can directly be written in A and B. Obviously, the result in Corollary

4.4 is also valid for an operator equation of the form $AX = XB$ when both A and B are idempotent operators.

Theorem 4.5. *Let $A \in \mathcal{C}^{m\times n}$ be given, $P \in \mathcal{C}^{m\times m}$ and $Q \in \mathcal{C}^{n\times n}$ be two idempotent matrices. Then the sum $PA + AQ$ satisfies the rank equalities*

$$r(\, PA + AQ\,) = r\begin{bmatrix} PA & AQ \\ Q & 0 \end{bmatrix} - r(Q) = r\begin{bmatrix} AQ & P \\ PA & 0 \end{bmatrix} - r(P). \tag{4.8}$$

$$r(\, PA + AQ\,) = r\begin{bmatrix} AQ - PAQ \\ PA \end{bmatrix} = r[\, PA - PAQ,\ AQ\,]. \tag{4.9}$$

Proof. Let $M = \begin{bmatrix} P & 0 & PA \\ 0 & Q & Q \\ P & AQ & 0 \end{bmatrix}$. Then it is easy to see by block elementary operations of matrices that

$$r(M) = r\begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & PA + AQ \end{bmatrix} = r(P) + r(Q) + r(\, PA + AQ\,).$$

On the other hand, note that $P^2 = P$ and $Q^2 = Q$. We also obtain by block elementary operations of matrices that

$$\begin{aligned} r(M) &= r\begin{bmatrix} P & -PAQ & PA \\ 0 & 0 & Q \\ P & AQ & 0 \end{bmatrix} \\ &= r\begin{bmatrix} 2P & 0 & PA \\ 0 & 0 & Q \\ P & AQ & 0 \end{bmatrix} = r\begin{bmatrix} 2P & 0 & 0 \\ 0 & 0 & Q \\ 0 & AQ & -\frac{1}{2}PA \end{bmatrix} = r(P) + r\begin{bmatrix} PA & AQ \\ Q & 0 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} r(M) &= r\begin{bmatrix} 0 & -PAQ & PA \\ 0 & Q & Q \\ P & AQ & 0 \end{bmatrix} \\ &= r\begin{bmatrix} 0 & 0 & PA \\ 0 & 2Q & Q \\ P & AQ & 0 \end{bmatrix} = r\begin{bmatrix} 0 & 0 & PA \\ 0 & 2Q & 0 \\ P & 0 & -\frac{1}{2}AQ \end{bmatrix} = r(Q) + r\begin{bmatrix} AQ & P \\ PA & 0 \end{bmatrix}. \end{aligned}$$

Combining the above three rank equalities for M yields Eq.(4.8). Consequently applying Eqs.(1.2) and (1.3) to the two block matrices in Eq.(4.8) yields Eq.(4.9). □

Corollary 4.6. *Let $A \in \mathcal{C}^{m\times n}$ be given, $P \in \mathcal{C}^{m\times m}$ and $Q \in \mathcal{C}^{n\times n}$ be two idempotent matrices.*

(a) *If $PAQ = 0$, then $r(\, PA + AQ\,) = r(PA) + r(AQ)$, or equivalently $R(PA) \cap R(AQ) = \{0\}$ and $R[(PA)^*] \cap R[(AQ)^*] = \{0\}$.*

(b) $PA + AQ = 0 \iff PA = 0$ *and* $AQ = 0$.

(c) *The general solution of the matrix equation* $PX + XQ = 0$ *is* $X = (I - P)U(I - Q)$, *where* $U \in \mathcal{C}^{m\times n}$ *is arbitrary.*

Proof. If $PAQ = 0$, then $r(PA - AQ) = r(PA) + r(AQ)$ by Theorem 4.1(b). Consequently $r(PA + AQ) = r(PA) + r(AQ)$ by Lemma 1.4(d). The result in Part (b) follows from Eq.(4.9). According to (b), the equation $PX + XQ = 0$ is equivalent to the pair of matrix equations $PX = 0$ and $XQ = 0$. According to Rao and Mitra (1971), and Mitra (1984), the common general solution of the pair of equation is exactly $X = (I - P)U(I - Q)$, where $U \in \mathcal{C}^{m\times n}$ is arbitrary. □

Corollary 4.7. *Let* A, P, $Q \in \mathcal{C}^{m\times m}$ *be given with* P, Q *being two idempotent matrices. Then the following five statements are equivalent:*

(a) *The sum* $PA + AQ$ *is nonsingular.*

(b) $r\begin{bmatrix} PA & AQ \\ Q & 0 \end{bmatrix} = m + r(Q)$.

(c) $r\begin{bmatrix} AQ & P \\ PA & 0 \end{bmatrix} = m + r(P)$.

(d) $r[\, PA,\ AQ\,] = m$ *and* $R\begin{bmatrix} (PA)^* \\ (AQ)^* \end{bmatrix} \cap R\begin{bmatrix} Q^* \\ 0 \end{bmatrix} = \{0\}$.

(e) $r\begin{bmatrix} AQ \\ PA \end{bmatrix} = m$ *and* $R\begin{bmatrix} AQ \\ PA \end{bmatrix} \cap R\begin{bmatrix} P \\ 0 \end{bmatrix} = \{0\}$.

Proof. Follows from Eq.(4.8). □

Theorem 4.8. *Let* $A \in \mathcal{C}^{m\times n}$ *be given,* $P \in \mathcal{C}^{m\times m}$ *and* $Q \in \mathcal{C}^{n\times n}$ *be two idempotent matrices. Then the rank of* $A - PA - AQ$ *satisfies the equalities*

$$r(A - PA - AQ) = r\begin{bmatrix} A & P \\ Q & 0 \end{bmatrix} + r(PAQ) - r(P) - r(Q) = r(A - PA - AQ + PAQ) + r(PAQ). \quad (4.10)$$

In particular,

(a) $PA + AQ = A \iff (I - P)A(I - Q) = 0$ *and* $PAQ = 0$.

(b) *The general solution of the matrix equation* $PX + XQ = X$ *is* $X = (I - P)UQ + V(I - Q)$, *where* U, $V \in \mathcal{C}^{m\times n}$ *are arbitrary.*

Proof. According to Eq.(4.1), we first find that

$$r(A - PA - AQ) = r[\,(I - P)A - AQ\,] = r\begin{bmatrix} (I - P)A \\ Q \end{bmatrix} + r[\, AQ,\ I - P\,] - r(I - P) - r(Q).$$

According to Eq.(1.2) and (1.3), we also get

$$r\begin{bmatrix} (I - P)A \\ Q \end{bmatrix} = r\begin{bmatrix} A & P \\ Q & 0 \end{bmatrix} - r(P), \quad \text{and} \quad r[\, AQ,\ I - P\,] = r(PAQ) + r(I - P).$$

Combining the above three yields the first equality in Eq.(4.10). Consequently applying Eq.(1.4) to the block matrix in it yields the second equality in Eq.(4.10). Part (a) is a direct consequence of Eq.(4.10). Part (a) follows from Corollary 4.4. □

If replacing P and Q in Theorem 4.5 by $I_m - P$ and $I_m - Q$, we can also obtain two rank equalities for $2A - PA - AQ$. For simplicity we omit them here.

Theorem 4.9. *Let $A \in \mathcal{C}^{m\times n}$ be given, $P \in \mathcal{C}^{m\times m}$ and $Q \in \mathcal{C}^{n\times n}$ be two idempotent matrices. Then the rank of $A - PAQ$ satisfies the equality*

$$r(\, A - PAQ\,) = r\begin{bmatrix} A & AQ & P \\ PA & 0 & 0 \\ Q & 0 & 0 \end{bmatrix} - r(P) - r(Q) = r\begin{bmatrix} (\, I - P\,)A(\, I - Q\,) & (\, I - P\,)AQ \\ PA(\, I - Q\,) & 0 \end{bmatrix}. \tag{4.11}$$

In particular,

(a) $PAQ = A \iff (\, I - P\,)A(\, I - Q\,) = 0, \ (\, I - P\,)AQ = 0$ *and* $PA(\, I - Q\,) = 0 \iff PA = A$ *and* $AQ = A$.

(b) *The general solution of the matrix equation $PXQ = X$ is $X = PUQ$, where $U \in \mathcal{C}^{m\times n}$ is arbitrary.*

Proof. Note that $P^2 = P$ and $Q^2 = Q$. It is easy to find that

$$\begin{aligned} r\begin{bmatrix} A & AQ & P \\ PA & 0 & 0 \\ Q & 0 & 0 \end{bmatrix} &= r\begin{bmatrix} A & 0 & P \\ 0 & -PAQ & -P \\ Q & -Q & 0 \end{bmatrix} \\ &= r\begin{bmatrix} A & 0 & P \\ -PAQ & 0 & -P \\ 0 & -Q & 0 \end{bmatrix} \\ &= r\begin{bmatrix} A - PAQ & 0 & 0 \\ 0 & 0 & -P \\ 0 & -Q & 0 \end{bmatrix} = r(\, A - PAQ\,) + r(P) + r(Q), \end{aligned}$$

as required for the first equality in Eq.(4.11). Consequently applying Eq.(1.4) to its left side yields the second one in Eq.(4.11). Part (a) is a direct consequence of Eq.(4.11). Part (b) can trivially be verified. □

□

Applying Eq.(4.1) to powers of difference of two idempotent matrices, we also find following several results.

Theorem 4.10. *Let $P,\ Q \in \mathcal{C}^{m\times m}$ be two idempotent matrices. Then*

(a) $(\, P - Q\,)^3$ *satisfies the two rank equalities*

$$r[\,(\, P - Q\,)^3\,] = r\begin{bmatrix} P - PQP \\ Q \end{bmatrix} + r[\, Q - QPQ,\ P\,] - r(P) - r(Q), \tag{4.12}$$

$$r[(P-Q)^3] = r[P - PQP - PQ + (PQ)^2] + r[Q - QPQ - PQ + (PQ)^2]. \tag{4.13}$$

In particular,

(b) *If* $(PQ)^2 = PQ$, *then*

$$r[(P-Q)^3] = r(P - PQP) + r(QPQ - Q). \tag{4.14}$$

(c) $r[(P-Q)^3] = r(P-Q)$, *i.e.*, $\mathrm{Ind}(P-Q) \leq 1$, *if and only if*

$$r\begin{bmatrix} P - PQP \\ Q \end{bmatrix} = r\begin{bmatrix} P \\ Q \end{bmatrix}, \quad \text{and} \quad r[Q - QPQ, \ P] = r[Q, \ P]. \tag{4.15}$$

or, equivalently,

$$R\left(\begin{bmatrix} P - PQP \\ Q \end{bmatrix}^*\right) = R\left(\begin{bmatrix} P \\ Q \end{bmatrix}^*\right), \quad \text{and} \quad R[Q - QPQ, \ P] = R[Q, \ P]. \tag{4.16}$$

(d) $(P-Q)^3 = 0 \Longleftrightarrow r\begin{bmatrix} P - PQP \\ Q \end{bmatrix} = r(Q)$ *and* $r[Q - QPQ, \ P] = r(P) \Longleftrightarrow R(Q - QPQ) \subseteq R(P)$ *and* $R[(P - PQP)^*] \subseteq R(Q^*)$.

Proof. Since $P^2 = P$ and $Q^2 = Q$, it is easy to verify that

$$(P-Q)^3 = P(I_m - QP) - (I_m - QP)Q. \tag{4.17}$$

Letting $A = I_m - QP$ and applying Eqs.(4.1) and (4.2) to Eq.(4.17) immediately yields Eq.(4.12) and (4.13). The results in Parts (b)—(d) are natural consequences of Eq.(4.13). □

Corollary 4.11. *Let* $P, \ Q \in \mathcal{C}^{m \times m}$ *be two idempotent matrices. Then*

$$r[(P-Q)^3 - (P-Q)] = r\begin{bmatrix} PQP \\ Q \end{bmatrix} + r[QPQ, \ P] - r(P) - r(Q). \tag{4.18}$$

In particular,

(a) $P - Q$ *is tripotent* $\Longleftrightarrow R(QPQ) \subseteq R(P)$ *and* $R[(PQP)^*] \subseteq R(Q^*)$.

(b) *If* $PQ = QP$, *then* $P - Q$ *is tripotent.*

Proof. Observe from Eq.(4.17) that

$$(P-Q)^3 - (P-Q) = -PQP + QPQ.$$

Applying Eq.(4.1) to it immediately yields Eq.(4.18). The results in Parts (b) and (c) are natural consequences of Eq.(4.18). □

Corollary 4.12. *A matrix* $A \in \mathcal{C}^{m \times m}$ *is tripotent if and only if it can factor as* $A = P - Q$, *where* P *and* Q *are two idempotent matrices with* $PQ = QP$.

Proof. The "if" part comes from Corollary 4.11(b). The " only if" part follows from a decomposition of A

$$A = \frac{1}{2}(A^2 + A) - \frac{1}{2}(A^2 - A),$$

where $P = \frac{1}{2}(A^2 + A)$ and $Q = \frac{1}{2}(A^2 - A)$ are two idempotent matrices with $PQ = QP$. $\square$

The rank equality (4.12) can be extended to the matrix $(P-Q)^5$, where both P and Q are idempotent. In fact, it is easy to verify

$$(P - Q)^5 = P(I_m - QP)^2 - (I_m - QP)^2 Q.$$

Hence by Eq.(4.1) it follows that

$$r[(P - Q)^5] = r \begin{bmatrix} P(I_m - QP)^2 \\ Q \end{bmatrix} + r[(I_m - QP)^2 Q, \ P] - r(P) - r(Q).$$

Moreover, the above work can also be extended to $(P - Q)^{2k+1} (k = 3, \ 4, \ \cdots)$, where both P and Q are idempotent.

Applying Eq.(4.1) to $PQ - QP$, where both P and Q are idempotent, we also obtain the following.

Corollary 4.13. *Let $P, \ Q \in C^{m \times m}$ be two idempotent matrices. Then*

$$r(PQ - QP) = r \begin{bmatrix} PQ \\ P \end{bmatrix} + r[QP, \ P] - 2r(P), \tag{4.19}$$

$$r(PQ - QP) = r \begin{bmatrix} QP \\ Q \end{bmatrix} + r[PQ, \ Q] - 2r(Q), \tag{4.20}$$

$$r(PQ - QP) = r(PQ - PQP) + r(PQP - QP), \tag{4.21}$$

$$r(PQ - QP) = r(PQ - QPQ) + r(QPQ - QP). \tag{4.22}$$

In particular, if both P and Q are Hermitian idempotent, then

$$r(PQ - QP) = 2r(PQ - PQP) = 2r(PQ - QPQ). \tag{4.23}$$

The rank equality (4.23) was proved by Bérubé, Hartwig and Styan(1993).

Corollary 4.14. *Let $P, \ Q \in C^{m \times m}$ be two idempotent matrices. Then*

$$r[(P - PQ) + \lambda(PQ - Q)] = r(P - Q) \tag{4.24}$$

holds for all $\lambda \in C$ with $\lambda \neq 0$. In particular,

$$r(P + Q - 2PQ) = r(P + Q - 2QP) = r(P - Q). \tag{4.25}$$

Proof. Observe that

$$(P - PQ) + \lambda(PQ - Q) = P(P + \lambda Q) - (P + \lambda Q)Q.$$

Thus it follows by Eq.(4.1) that

$$\begin{aligned} r[(P - PQ) + \lambda(PQ - Q)] &= r\begin{bmatrix} P(P + \lambda Q) \\ Q \end{bmatrix} + r[(P + \lambda Q)Q, \ P] - r(P) - r(Q) \\ &= r\begin{bmatrix} P \\ Q \end{bmatrix} + r[\lambda Q, \ P] - r(P) - r(Q) \\ &= r\begin{bmatrix} P \\ Q \end{bmatrix} + r[P, \ Q] - r(P) - r(Q). \end{aligned}$$

Contrasting it with Eq.(3.1) yields Eq.(4.24). Setting $\lambda = -1$ we have Eq.(4.25). □

Replacing P by $I_m - P$ in Eq.(4.24), we also obtain the following.

Corollary 4.15. *Let $P, Q \in \mathcal{C}^{m \times m}$ be two idempotent matrices. Then*

$$r(I_m - P - Q + \lambda PQ) = r(I_m - P - Q)$$

holds for all $\lambda \in \mathcal{C}$ with $\lambda \neq 1$.

In the remainder of this paper, we apply the results in Chapter 3 and this chapter to establish various rank equalities related to involutory matrices. A matrix A is said to be involutory if its square is identity, i.e., $A^2 = I$. As two special types of matrices, involutory matrices and idempotent matrices are closely linked. As a matter of fact, for any involutory matrix A, the two corresponding matrices $(I + A)/2$ and $(I - A)/2$ are idempotent. Conversely, for any idempotent matrix A, the two corresponding matrices $\pm(I - 2A)$ are involutory. Based on the basic fact, all the results in Chapter 3 and this chapter on idempotent matrices can dually be extended to involutory matrices. We next list some of them.

Theorem 4.16. *Let $A, B \in \mathcal{C}^{m \times m}$ be two involutory matrices. Then the rank of $A + B$ and $A - B$ satisfy the equalities*

$$r(A + B) = r\begin{bmatrix} I + A \\ I - B \end{bmatrix} + r[I + A, \ I - B] - r(I + A) - r(I - B). \tag{4.26}$$

$$r(A + B) = r[(I + A)(I + B)] + r[(I - A)(I - B)], \tag{4.27}$$

$$r(A - B) = r\begin{bmatrix} I + A \\ I + B \end{bmatrix} + r[I + A, \ I + B] - r(I + A) - r(I + B). \tag{4.28}$$

$$r(A - B) = r[(I + A)(I - B)] + r[(I - A)(I + B)]. \tag{4.29}$$

Proof. Notice that both $P = (I + A)/2$ and $Q = (I - B)/2$ are idempotent when A and B are involutory. In that case,

$$r(P - Q) = r\left[\frac{1}{2}(I + A) - \frac{1}{2}(I - B)\right] = r(A + B),$$

and

$$r\begin{bmatrix} P \\ Q \end{bmatrix} + r[P,\ Q] - r(P) - r(Q) = r\begin{bmatrix} I+A \\ I-B \end{bmatrix} + r[I+A,\ I-B] - r(I+A) - r(I-B).$$

Putting them in Eq.(3.1) produces Eq.(4.26). Furthermore we have

$$r(P - PQ) = r\left[(I+A)\left(I - \frac{1}{2}(I-B)\right)\right] = r[(I+A)(I+B)],$$

$$r(PQ - Q) = r\left[\left(\frac{1}{2}(I+A) - I\right)(I-B)\right] = r[(I-A)(I-B)],$$

Putting them in Eq.(3.2) yields Eq.(4.27). moreover, if B is involutory, then $-B$ is also involutory. Thus replacing B by $-B$ in Eqs.(4.26) and (4.27) yields Eqs.(4.28) and (4.29). □

Corollary 4.27. *Let $A,\ B \in \mathcal{C}^{m\times m}$ be two involutory matrices.*

(a) *If $(I+A)(I-B) = 0$ or $(I-B)(I+A) = 0$, then*

$$r(A+B) = r(I+A) + r(I-B). \tag{4.30}$$

(b) *If $(I+A)(I+B) = 0$ or $(I+B)(I+A) = 0$, then*

$$r(A-B) = r(I+A) + r(I+B). \tag{4.31}$$

Proof. The condition $(I+A)(I-B) = 0$ is equivalent to

$$I + A = B + BA \quad \text{and} \quad I - B = AB - A.$$

In that case,

$$(I+A)(I+B) = I + A + B + AB = 2(I+A),$$

and

$$(I-A)(I-B) = I - B - A + AB = 2(I-B).$$

Thus Eq.(4.27) reduces to (4.30). Similarly we show that under $(I-B)(I+A) = 0$, the rank equality (4.30) also holds. The result in Part (b) is obtained by replacing B in Part(a) by $-B$. □

Corollary 4.18. *Let $A,\ B \in \mathcal{C}^{m\times m}$ be two involutory matrices.*

(a) *The sum $A + B$ is nonsingular if and only if*

$$R(I+A) \cap R(I-B) = \{0\}, \qquad R(I+A^*) \cap R(I-B^*) = \{0\} \tag{4.32}$$

and

$$r(I+A) + r(I-B) = m. \tag{4.33}$$

(b) *The difference $A - B$ is nonsingular if and only if*

$$R(I+A) \cap R(I+B) = \{0\}, \qquad R(I+A^*) \cap R(I+B^*) = \{0\} \tag{4.34}$$

and

$$r(I+A)+r(I+B)=m. \tag{4.35}$$

Proof. Follows immediately from (4.26) and (4.27). $\square$

Theorem 4.19. *Let $A,\ B \in \mathcal{C}^{m\times m}$ be two involutory matrices. Then $A+B$ and $A-B$ satisfy the rank equalities*

$$r(A+B)=r[(I+A)(I+B)]+r[(I+B)(I+A)]-r(I+A)-r(I+B)+m. \tag{4.36}$$

$$r(A-B)=r[(I+A)(I-B)]+r[(I-B)(I+A)]-r(I+A)-r(I-B)+m. \tag{4.37}$$

Proof. Putting $P=(I+A)/2$ and $Q=(I-B)/2$ in Eq.(3.8) and simplifying yields Eq.(4.36). Replacing B by $-B$ in Eq.(4.36) yields Eq.(4.37). $\square$

The combination of Eq.(4.26) with Eq.(4.36) produces the following rank equality

$$r[(I+B)(I+A)]=r(I+B)+r(I+A)-m+r[(I-A)(I-B)]. \tag{4.38}$$

Theorem 4.20. *Let $A,\ B \in \mathcal{C}^{m\times m}$ be two involutory matrices. Then*

$$r(AB-BA)=r(A+B)+r(A-B)-m. \tag{4.39}$$

In particular,

$$AB=BA \Longleftrightarrow r(A+B)+r(A-B)=m. \tag{4.40}$$

Proof. Putting $P=(I+A)/2$ and $Q=(I-B)/2$ in Eq.(3.24) and simplifying yields Eq.(4.39). $\square$

Putting the given formulas in Eqs.(4.26)—(4.29), (4.36) and (4.37) in Eq.(4.39) may yield some other rank equalities for $AB-BA$. For simplicity, we do not list them.

Theorem 4.21. *Let $A,\ B \in \mathcal{C}^{m\times m}$ be two involutory matrices. Then*

$$r\left[\left(\frac{A+B}{2}\right)^2-\frac{A+B}{2}\right]=r(I-A-B)+r(A+B)-m. \tag{4.41}$$

and

$$r\left[\left(\frac{A-B}{2}\right)^2-\frac{A-B}{2}\right]=r(I-A+B)+r(A-B)-m. \tag{4.42}$$

In particular,

$$\frac{1}{2}(A+B) \text{ is idempotent} \Longleftrightarrow r(I-A-B)+r(A+B)=m \Longleftrightarrow r(A+B)=r(I+A)-r(I-B),$$

and

$$\frac{1}{2}(A-B) \text{ is idempotent} \Longleftrightarrow r(I-A+B)+r(A-B)=m \Longleftrightarrow r(A-B)=r(I+A)-r(I+B).$$

Proof. Putting $P = (I + A)/2$ and $Q = (I - B)/2$ in Eq.(3.36) and simplifying yields (4.41). $\square$

Theorem 4.22. *Let $A,\ B \in \mathcal{C}^{m\times m}$ be two involutory matrices. Then*

$$r(3I - A - B + AB) = r(2I - A - B). \tag{4.43}$$

Proof. Putting $P = (I + A)/2$ and $Q = (I - B)/2$ in Eq.(3.38) and simplifying yields (4.43). $\square$

Theorem 4.23. *Let $A \in \mathcal{C}^{m\times m}$ be an involutory matrix. Then*

(a) $r(A - A^*) = 2r[I + A,\ I + A^*] - 2r(I + A) = r[I - A,\ I - A^*] - 2r(I - A)$.

(b) $r(A + A^*) = m$.

(c) $r(AA^* - A^*A) = r(A - A^*)$.

Proof. Putting $P = (I + A)/2$ and $Q = (I - B)/2$ in Corollary 3.26 and simplifying yields the desired results. $\square$

Theorem 4.24. *Let $A \in \mathcal{C}^{m\times m}$ and $B \in \mathcal{C}^{n\times n}$ be two involutory matrices. $X \in \mathcal{C}^{m\times m}$. Then $AX - XB$ satisfies the rank equalities*

$$r(AX - XB) = r\begin{bmatrix} (I_m + A)X \\ I_n + B \end{bmatrix} + r[X(I_n + B),\ I_m + A] - r(I_m + A) - r(I_n + B), \tag{4.44}$$

$$r(AX - XB) = r[(I_m + A)X(I_n - B)] + r[(I_m - A)X(I_n + B)], \tag{4.45}$$

In particular,

$$AX = XB \Longleftrightarrow (I_m + A)X(I_n - B) = 0 \ \ and \ \ (I_m - A)X(I_n + B) = 0. \tag{4.46}$$

Proof. Putting $P = (I_m + A)/2$ and $Q = (I_n + B)/2$ in Eqs.(4.1) and (4.2) yields Eq.(4.44) and (4.45). The equivalence in Eq.(4.46) follows from Eq.(4.45). $\square$

Theorem 4.25. *Let $A \in \mathcal{C}^{m\times m}$ and $B \in \mathcal{C}^{n\times n}$ be two involutory matrices. Then the general solution of the matrix equation $AX = XB$ is*

$$X = V + AVB, \tag{4.47}$$

where $V \in \mathcal{C}^{m\times n}$ is arbitrary.

Proof. We only give the verification. Obviously the matrix X in Eq.(4.47) satisfies $AX = AV + VB$ and $XB = VB + AV$ Thus X is a solution of $AX = XB$. On the other hand, for any solution X_0 of $AX = XB$, let $V = X_0/2$ in Eq.(4.47), then we get $V = AX_0B = X_0$, that is, X_0 can be represented by Eq.(4.47). Thus Eq.(4.47) is the general solution of the matrix equation $AX = XB$. $\square$

Theorem 4.26. *Let $A \in \mathcal{C}^{m\times m}$ be an involutory matrix, and $X \in \mathcal{C}^{m\times m}$. Then*

(a) $AX - XA$ *satisfies the rank equalities*

$$r(AX - XA) = r\begin{bmatrix} (I+A)X \\ I+A \end{bmatrix} + r[X(I+A),\ I+A] - r(I+A) - r(I+A),$$
$$r(AX - XA) = r[(I+A)X(I-A)] + r[(I-A)X(I+A)],$$

In particular,

$$AX = XA \Longleftrightarrow (I+A)X(I-A) = 0 \ \text{ and } \ (I-A)X(I+A) = 0.$$

(b) *The general solution of the matrix equation* $AX = XA$ *is*

$$X = V + AVA,$$

where $V \in \mathcal{C}^{m\times m}$ *is arbitrary.*

Chapter 5

Rank equalities related to outer inverses of matrices

An outer inverse of a matrix A is the solution to the matrix equation $XAX = X$, and is often denoted by $X = A^{(2)}$. The collection of all outer inverses of A is often denoted by $A\{2\}$. Obviously, the Moore-Penrose inverse, the Drazin inverse, the group inverse, and the weighted Moore-Penrose inverse of a matrix are naturally outer inverses of the matrix. If outer inverse of a matrix is also an inner inverse the matrix, it is called a reflexive inner inverse of the matrix, and is often denoted by A_r^-. The The collection of all reflexive inner inverses of a matrix A is denoted by $A\{1, 2\}$. As one of important kinds of generalized inverses of matrices, outer inverses of matrices and their applications have well been examined in the literature (see, e.g., [10], [13], [30], [51], [76], [100], [101]). In this chapter, we shall establish several basic rank equalities related to differences and sums of outer inverses of a matrix, and then consider their various consequences. The results obtained in this chapter will also be applied in the subsequent chapters.

Theorem 5.1. *Let $A \in \mathcal{C}^{m\times n}$ be given, and $X_1,\ X_2 \in A\{2\}$. Then the difference $X_1 - X_2$ satisfies the following three rank equalities*

$$r(X_1 - X_2) = r\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + r[\, X_1,\ X_2\,] - r(X_1) - r(X_2), \tag{5.1}$$

$$r(X_1 - X_2) = r(X_1 - X_1AX_2) + r(X_1AX_2 - X_2), \tag{5.2}$$

$$r(X_1 - X_2) = r(X_1 - X_2AX_1) + r(X_2AX_1 - X_2). \tag{5.3}$$

Proof. Let $M = \begin{bmatrix} -X_1 & 0 & X_1 \\ 0 & X_2 & X_2 \\ X_1 & X_2 & 0 \end{bmatrix}$. Then it is easy to see by block elementary operations of matrices that

$$r(M) = r\begin{bmatrix} -X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_1 - X_2 \end{bmatrix} = r(X_1) + r(X_2) + r(X_1 - X_2). \tag{5.4}$$

On the other hand, note that $X_1AX_1 = X_1$ and $X_2AX_2 = X_2$. Thus

$$\begin{bmatrix} I_n & 0 & X_1A \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix}\begin{bmatrix} -X_1 & 0 & X_1 \\ 0 & X_2 & X_2 \\ X_1 & X_2 & 0 \end{bmatrix}\begin{bmatrix} I_m & 0 & 0 \\ 0 & I_m & 0 \\ 0 & -AX_2 & I_m \end{bmatrix} = \begin{bmatrix} 0 & 0 & X_1 \\ 0 & 0 & X_2 \\ X_1 & X_2 & 0 \end{bmatrix}.$$

which implies that

$$r(M) = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + r[\, X_1,\ X_2 \,]. \tag{5.5}$$

Combining Eqs.(5.4) and (5.5) yields Eq.(5.1). Consequently applying Eqs.(1.2) and (1.3) to the two block matrices in Eq.(5.1) respectively and noticing that $A \in X_1\{2\}$ and $A \in X_2\{2\}$, we can write Eq.(5.1) as Eqs.(5.2) and (5.3). $\square$

It is obvious that if $A = I_m$ in Theorem 5.1, then $X_1,\ X_2 \in I_m\{2\}$ are actually two idempotent matrices. In that case. Eqs.(5.1)—(5.3) reduce to the results in Theorem 3.1.

Corollary 5.2. *Let $A \in \mathcal{C}^{m\times n}$ be given. and $X_1,\ X_2 \in A\{2\}$. Then*

(a) $R(\, X_1 - X_1AX_2 \,) \cap R(\, X_1AX_2 - X_2 \,) = \{0\}$ *and* $R[(\, X_1 - X_1AX_2 \,)^*] \cap R[(\, X_1AX_2 - X_2 \,)^*] = \{0\}$.

(b) $R(\, X_1 - X_2AX_1 \,) \cap R(\, X_2AX_1 - X_2 \,) = \{0\}$ *and* $R[(\, X_1 - X_2AX_1 \,)^*] \cap R[(\, X_2AX_1 - X_2 \,)^*] = \{0\}$.

(c) *If* $X_1AX_2 = 0$ *or* $X_2AX_1 = 0$, *then* $r(\, X_1 - X_2 \,) = r(X_1) + r(X_2)$.

Proof. The results in Parts (a) and (b) follow immediately from applying Lemma 1.4(d) to Eqs.(5.2) and (5.3). Parts (c) is a direct consequence of Eqs.(5.2) and (5.3). $\square$

Corollary 5.3. *Let $A \in \mathcal{C}^{m\times n}$ be given, and $X_1,\ X_2 \in A\{2\}$. Then the following five statements are equivalent:*

(a) $r(\, X_1 - X_2 \,) = r(X_1) - r(X_2)$, *i.e.*, $X_2 \leq_{rs} X_1$.

(b) $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = r[\, X_1,\ X_2 \,] = r(X_1)$.

(c) $R(X_2) \subseteq R(X_1)$ *and* $R(X_2^*) \subseteq R(X_1^*)$.

(d) $X_1AX_2 = X_2$ *and* $X_2AX_1 = X_2$.

(e) $X_1AX_2AX_1 = X_2$.

Proof. The equivalence of Parts (a) and (b) follows directly from Eq.(5.1). The equivalence of Parts (b). (c) and (d) follows directly from Lemma 1.2(c) and (d). Combining the two equalities in Part (d) yields the equality in Part (e). Conversely, suppose that $X_1AX_2AX_1 = X_2$ holds. Pre- and post-multiplying X_1A and AX_1 to it yields $X_1AX_2AX_1 = X_1AX_2 = X_2AX_1$. Combining it with $X_1AX_2AX_1 = X_2$ yields the two rank equalities in Part (d). $\square$

Corollary 5.4. *Let $A \in \mathcal{C}^{m\times m}$ be given, and $X_1,\ X_2 \in A\{2\}$. Then the following three statements are equivalent:*

(a) *The difference $X_1 - X_2$ is nonsingular.*

(b) $r\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = r[\, X_1,\ X_2 \,] = r(X_1) + r(X_2) = m$.

(c) $R(X_1) \oplus R(X_2) = R(X_1^*) \oplus R(X_2^*) = \mathcal{C}^m$.

Proof. A trivial consequence of Eq.(5.1). $\square$

Corollary 5.5. *Let $A \in C^{m\times n}$ be given, and $X \in A\{2\}$. Then*

$$r(A - AXA) = r(A) - r(AXA), \quad i.e., \quad AXA \leq_{rs} A. \tag{5.6}$$

In particular,

$$AXA = A, \ i.e., \ X \in A\{1, 2\} \Longleftrightarrow r(A) = r(X). \tag{5.7}$$

Proof. It is easy to verify that both A and AXA are outer inverses of $A^{\dagger}$. Thus by Eq.(5.1) we obtain

$$r(A - AXA) = r\begin{bmatrix} A \\ AXA \end{bmatrix} + r[A, \ AXA] - r(A) - r(AXA) = r(A) - r(AXA),$$

the desired in Eq.(5.6). □

Corollary 5.6. *Let $A \in C^{m\times m}$ be given, and $X \in A\{2\}$. Then*

$$r(AX - XA) = r\begin{bmatrix} X \\ XA \end{bmatrix} + r[X, \ AX] - 2r(X) = r(XA - XA^2X) + r(XA^2X - AX). \tag{5.8}$$

In particular,

$$AX = XA \Longleftrightarrow R(AX) = R(X) \ \text{and} \ R[(AX)^*] = R(X^*). \tag{5.9}$$

Proof. It is easy to verify that both AX and XA are idempotent when $X \in A\{2\}$. Thus we find by Eqs.(3.1), (1.2) and (1.3) that

$$\begin{aligned} r(AX - XA) &= r\begin{bmatrix} AX \\ XA \end{bmatrix} + r[AX, \ XA] - r(AX) - r(XA) \\ &= r\begin{bmatrix} X \\ XA \end{bmatrix} + r[X, \ AX] - r(AX) - r(XA) \\ &= r(XA - XA^2X) + r(XA^2X - AX), \end{aligned}$$

as required for Eq.(5.8). Eq.(5.9) is a direct consequence of Eq.(5.8). □

Corollary 5.7. *Let $A \in C^{m\times n}$ be given, and $X_1, \ X_2 \in A\{2\}$. Then*

$$r(AX_1A - AX_2A) = r\begin{bmatrix} X_1A \\ X_2A \end{bmatrix} + r[AX_1, \ AX_2] - r(X_1) - r(X_2). \tag{5.10}$$

In particular,

$$AX_1A = AX_2A \Longleftrightarrow X_1AX_2AX_1 = X_1 \ \text{and} \ X_2AX_1AX_2 = X_2. \tag{5.11}$$

Proof. Notice that Both AX_1A and AX_2A are outer inverses of $A^\dagger$ when $X_1,\ X_2 \in A\{2\}$. Moreover, observe that $r(AX_1A) = r(AX_1) = r(X_1A) = r(X_1)$, and $r(AX_2A) = r(AX_2) = r(X_2A) = r(X_2)$. Thus it follows from Eq.(5.1) that

$$\begin{aligned} r(\, AX_1A - AX_2A\,) &= r\begin{bmatrix} AX_1A \\ AX_2A \end{bmatrix} + r[\, AX_1A,\ AX_2A\,] - r(AX_1A) - r(AX_2A) \\ &= r\begin{bmatrix} X_1A \\ X_2A \end{bmatrix} + r[\, AX_1,\ AX_2\,] - r(X_1) - r(X_2), \end{aligned}$$

as required for Eq.(5.10). The verification of Eq.(5.11) is trivial, hence is omitted. □

Corollary 5.8. *Let $A \in \mathcal{C}^{m\times n}$ be given, and $X_1,\ X_2 \in A\{2\}$. Then the following five statements are equivalent:*

(a) $r(\, AX_1A - AX_2A\,) = r(AX_1A) - r(AX_2A)$, *i.e.*, $AX_2A \leq_{rs} AX_1A$.

(b) $\begin{bmatrix} X_1A \\ X_2A \end{bmatrix} = r[\, AX_1,\ AX_2\,] = r(X_1)$.

(c) $R(AX_2) \subseteq R(AX_1)$ *and* $R[(X_2A)^*] \subseteq R[(X_1A)^*]$.

(d) $AX_1AX_2A = AX_2A$ *and* $AX_2AX_1A = AX_2A$.

(e) $AX_1AX_2AX_1A = AX_2A$.

Proof. Follows form Corollary 5.3 by noticing that Both AX_1A and AX_2A are outer inverses of $A^\dagger$ when $X_1,\ X_2 \in A\{2\}$. □

Theorem 5.9. *Let $A \in \mathcal{C}^{m\times n}$ be given, and $X_1,\ X_2 \in A\{2\}$. Then the sum $X_1 + X_2$ satisfies the rank equalities*

$$r(\, X_1 + X_2\,) = r\begin{bmatrix} X_1 & X_2 \\ X_2 & 0 \end{bmatrix} - r(X_2) = r\begin{bmatrix} X_2 & X_1 \\ X_1 & 0 \end{bmatrix} - r(X_1). \tag{5.11}$$

$$r(\, X_1 + X_2\,) = r[\,(\, I_n - X_2A\,)X_1(\, I_m - AX_2\,)\,] + r(X_2). \tag{5.12}$$

$$r(\, X_1 + X_2\,) = r[\,(\, I_n - X_1A\,)X_2(\, I_m - AX_1\,)\,] + r(X_1). \tag{5.13}$$

Proof. Let $M = \begin{bmatrix} X_1 & 0 & X_1 \\ 0 & X_2 & X_2 \\ X_1 & X_2 & 0 \end{bmatrix}$. Then it is easy to see by block elementary operations that

$$r(M) = r\begin{bmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & -(\, X_1 + X_2\,) \end{bmatrix} = r(X_1) + r(X_2) + r(\, X_1 + X_2\,). \tag{5.14}$$

On the other hand, note that $X_1AX_1 = X_1$ and $X_2AX_2 = X_2$. Thus

$$\begin{bmatrix} I_n & 0 & X_1A \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} X_1 & 0 & X_1 \\ 0 & X_2 & X_2 \\ X_1 & X_2 & 0 \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ 0 & I_m & 0 \\ 0 & -AX_2 & I_m \end{bmatrix} = \begin{bmatrix} 2X_1 & 0 & X_1 \\ 0 & 0 & X_2 \\ X_1 & X_2 & 0 \end{bmatrix},$$

which implies that

$$r(M) = r\begin{bmatrix} 2X_1 & 0 & X_1 \\ 0 & 0 & X_2 \\ X_1 & X_2 & 0 \end{bmatrix} = r\begin{bmatrix} 2X_1 & 0 & 0 \\ 0 & 0 & X_2 \\ 0 & X_2 & -\frac{1}{2}X_1 \end{bmatrix} = r\begin{bmatrix} X_1 & X_2 \\ X_2 & 0 \end{bmatrix} + r(X_1). \tag{5.15}$$

Combining Eqs.(5.14) and (5.15) yields the first equality in Eq.(5.11). By symmetry, we have the second equality in Eq.(5.15). Applying Eq.(1.3) to the two block matrices in Eq.(5.11), respectively, and noticing that $A \in \{X_1^-\}$ and $A \in \{X_2^-\}$, we then can write Eq.(5.11) as Eqs.(5.12) and (5.13). □

Corollary 5.10. *Let $A \in \mathcal{C}^{m\times n}$ be given, and $X_1,\ X_2 \in A\{2\}$.*

(a) *If $X_1AX_2 = X_2AX_1$, then* $r(\, X_1 + X_2\,) = r\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = r[\, X_1,\ X_2\,]$.

(b) *If $X_1AX_2 = X_2AX_1 = 0$, then* $r(\, X_1 + X_2\,) = r(X_1) + r(X_2)$.

Proof. Under $X_1AX_2 = X_2AX_1$, we find from Eqs.(5.12) and (5.13) that

$$r(\, X_1 + X_2\,) = r(\, X_1 - X_1AX_2\,) + r(X_2) = r(X_1) + r(\, X_1AX_2 - X_2\,).$$

Note by Eq.(1.2) and (1.3) that

$$r\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = r(\, X_1 - X_1AX_2\,) + r(X_2), \quad \text{and} \quad r[\, X_1,\ X_2\,] = r(X_1) + r(\, X_1AX_2 - X_2\,).$$

Thus we have the results in Part (a). Part (b) follows immediately from Eq.(5.12). □

Corollary 5.11. *Let $A \in \mathcal{C}^{m\times m}$ be given, and $X_1,\ X_2 \in A\{2\}$. Then the following five statements are equivalent:*

(a) *The sum $X_1 + X_2$ is nonsingular.*

(b) $r\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = m$ *and* $R\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \cap R\begin{bmatrix} X_2 \\ 0 \end{bmatrix} = \{0\}$.

(c) $r[\, X_1,\ X_2\,] = m$ *and* $R\begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} \cap R\begin{bmatrix} X_2^* \\ 0 \end{bmatrix} = \{0\}$.

(d) $r\begin{bmatrix} X_2 \\ X_1 \end{bmatrix} = m$ *and* $R\begin{bmatrix} X_2 \\ X_1 \end{bmatrix} \cap R\begin{bmatrix} X_1 \\ 0 \end{bmatrix} = \{0\}$.

(e) $r[\, X_2,\ X_1\,] = m$ *and* $R\begin{bmatrix} X_2^* \\ X_1^* \end{bmatrix} \cap R\begin{bmatrix} X_1^* \\ 0 \end{bmatrix} = \{0\}$.

Proof. Follows immediately from Eq.(5.11). □

Corollary 5.12. *Let $A \in \mathcal{C}^{m\times n}$ be given, and $X \in A\{2\}$. Then*

$$r(\, A + AXA\,) = r(A). \tag{5.16}$$

holds for all $X \in A\{2\}$.

Proof. Notice that Both A and AX_2A are outer inverses of $A^{\dagger}$ when $X \in A\{2\}$. Thus Eq.(5.16) follows from Eq.(5.11). □

Theorem 5.13. *Let $A \in \mathcal{C}^{m\times n}$ be given, and $X_1,\ X_2 \in A\{2\}$. Then the difference $X_1 - X_2$ satisfies the rank equalities*

$$r[\,(X_1 - X_2)A(X_1 - X_2) - (X_1 - X_2)\,] = r(\,I_m - AX_1 + AX_2\,) + r(\,X_1 - X_2\,) - m. \tag{5.17}$$

$$r[\,(X_1 - X_2)A(X_1 - X_2) - (X_1 - X_2)\,] = r(X_1AX_2AX_1\,) - r(X_1) + r(\,X_1 - X_2\,). \tag{5.18}$$

Proof. Letting $X = X_1 - X_2$ and applying Eq.(1.10) yields

$$r(\,XAX - X\,) = r(\,I_m - AX\,) + r(X) - m,$$

which is Eq.(5.17). Note that AX_1 and AX_2 are idempotent. It turns out by Eq.(3.19) that

$$r(\,I_m - AX_1 + AX_2\,) = r(AX_1AX_2AX_1) - r(AX_1) + m = r(X_1AX_2AX_1) - r(X_1) + m.$$

Putting it in Eq.(5.17) yields Eq.(5.18). □

Corollary 5.14. *Let $A \in \mathcal{C}^{m\times n}$ be given, and $X_1,\ X_2 \in A\{2\}$. Then the following five statements are equivalent:*

(a) $X_1 - X_2 \in A\{2\}$.

(b) $r(\,I_m - AX_1 + AX_2\,) = m - r(\,X_1 - X_2\,)$.

(c) $r(\,X_1 - X_2\,) = r(X_1) - r(X_2)$, *i.e.*, $X_2 \leq_{rs} X_1$.

(d) $R(X_2) \subseteq R(X_1)$ *and* $R(X_2^*) \subseteq R(X_1^*)$.

(e) $X_1AX_2AX_1 = X_2$.

Proof. The equivalence of Parts (a) and (b) follows immediately from Eq.(5.17). The equivalence of Parts (c), (d) and (e) is from Corollary 5.3. We next show the equivalence of Parts (a) and (e). It is easy to verify that

$$(\,X_1 - X_2\,)A(\,X_1 - X_2\,) - (\,X_1 - X_2\,) = -X_1AX_2 - X_2AX_1 + 2X_2.$$

Thus $X_1 - X_2 \in A\{2\}$ holds if and only if

$$X_1AX_2 + X_2AX_1 = 2X_2. \tag{5.19}$$

Pre- and post-multiplying X_1A and AX_1 to it, we get

$$X_1AX_2AX_1 = X_1AX_2 \quad \text{and} \quad X_1AX_2AX_1 = X_2AX_1. \tag{5.20}$$

Putting them in Eq.(5.19) yields Part (e). Conversely, if Part (e) holds, then Eq.(5.20) holds. Combining Part (e) with Eq.(5.20) leads to Eq.(5.19), which is equivalent to $X_1 - X_2 \in A\{2\}$. □

The problem considered in Corollary 5.14 could be regarded as an extension of the work in Corollary 3.21, which was examined by Getson and Hsuan (1988). In that monograph, they only gave a sufficient condition for $X_1 - X_2 \in A\{2\}$ to hold when $X_1,\ X_2 \in A\{2\}$. Our result in Corollary 5.14 is a complete conclusion on this problem.

Theorem 5.11. *Let $A \in \mathcal{C}^{m\times n}$ be given, and $X_1,\ X_2 \in A\{2\}$. Then the sum $X_1 + X_2$ satisfies the two rank equalities*

$$r[(X_1+X_2)A(X_1+X_2)-(X_1+X_2)] = r(I_m - AX_1 - AX_2) + r(X_1+X_2) - m. \tag{5.21}$$

$$r[(X_1+X_2)A(X_1+X_2)-(X_1+X_2)] = r(X_1AX_2) + r(X_2AX_1) + r(X_1+X_2) - r(X_1) - r(X_2). \tag{5.22}$$

Proof. Letting $X = X_1 + X_2$ and applying Eq.(1.10) to $XAX - X$ yields Eq.(5.21). Note that AX_1 and AX_2 are idempotent. It turns out by Eq.(3.8) that

$$r(I_m - AX_1 - AX_2) = r(X_1AX_2) + r(X_2AX_1) - r(X_1) - r(X_2) + m.$$

Putting it in Eq.(5.21) yields Eq.(5.22). □

Corollary 5.16. *Let $A \in \mathcal{C}^{m\times n}$ be given, and $X_1,\ X_2 \in A\{2\}$. Then the following four statements are equivalent:*

(a) $X_1 + X_2 \in A\{2\}$.

(b) $X_1AX_2 + X_2AX_1 = 0$.

(c) $r(I_m - AX_1 - AX_2) = m - r(X_1 + X_2)$.

(d) $X_1AX_2 = 0$, *and* $X_2AX_1 = 0$.

Proof. The equivalence of Parts (a) and (b) follows immediately from expanding $(X_1 + X_2)A(X_1 + X_2) - (X_1 + X_2)$. The equivalence of (a) and (c) is from Eq.(5.21). We next show the equivalence of (b) and (d). Pre- and post-multiplying X_1A and AX_1 to $X_1AX_2 + X_2AX_1 = 0$, we get

$$X_1AX_2 + X_1AX_2AX_1 = 0, \quad \text{and} \quad X_1AX_2AX_1 + X_2AX_1 = 0.$$

which implies that $X_1AX_2 = X_2AX_1$. Putting them in Part (b) yields (d). Conversely, if Part (d) holds, then Part (b) naturally holds. □

Chapter 6

Rank equalities related to a matrix and its Moore-Penrose inverses

In this chapter, we shall establish a variety of rank equalities related to a matrix and its Moore-Penrose inverse, and then use them to characterize various specified matrices, such as, EP matrices, conjugate EP matrices, bi-EP matrices, star-dagger matrices, and so on.

As is well known, a matrix A is said to be EP (or Range-Hermitian) if $R(A) = R(A^*)$. EP matrices have some nice properties, meanwhile they are quite inclusive. Hermitian matrices and normal matrices are special cases of EP matrices. EP matrices and their applications have well be examined in the literature. One of the basic and nice properties related to a EP matrix A is $AA^\dagger = A^\dagger A$, see, e.g., Ben-Israel and Greville (1980), Campbell and Meyer (1991). This equality motivates us to consider the rank of $AA^\dagger - A^\dagger A$, as well as its various extensions.

Theorem 6.1. *Let $A \in \mathcal{C}^{m\times m}$ be given. Then the rank of $AA^\dagger - A^\dagger A$ satisfies the following rank equality*

$$r(\, AA^\dagger - A^\dagger A\,) = 2r[\,A, \ A^*\,] - 2r(A) = 2r(\,A - A^2A^\dagger\,) = 2r(\,A - A^\dagger A^2\,). \tag{6.1}$$

In particular,

(a) $AA^\dagger = A^\dagger A \Longleftrightarrow r[\,A, \ A^*\,] = r(A) \Longleftrightarrow A = A^2A^\dagger \Longleftrightarrow A = A^\dagger A^2 \Longleftrightarrow R(A) = R(A^*)$, *i.e., A is EP.*

(b) $AA^\dagger - A^\dagger A$ *is nonsingular* $\Longleftrightarrow r[\,A, \ A^*\,] = 2r(A) = m \Longleftrightarrow R(A) \dotplus R(A^*) = \mathcal{C}^m$.

Proof. Note that $AA^\dagger$ and $A^\dagger A$ are idempotent matrices. Then applying Eq.(3.1), we first obtain

$$r(\, AA^\dagger - A^\dagger A\,) = r\begin{bmatrix} AA^\dagger \\ A^\dagger A \end{bmatrix} + r[\,AA^\dagger, \ A^\dagger A\,] - r(AA^\dagger) - r(A^\dagger A). \tag{6.2}$$

Observe that $r(AA^\dagger) = r(A^\dagger A) = r(A)$, and

$$r\begin{bmatrix} AA^\dagger \\ A^\dagger A \end{bmatrix} = r\begin{bmatrix} A^\dagger \\ A \end{bmatrix} = r\begin{bmatrix} A^* \\ A \end{bmatrix}, \qquad r[\,AA^\dagger, \ A^\dagger A\,] = r[\,A, \ A^\dagger\,] = r[\,A, \ A^*\,].$$

Thus Eq.(6.2) reduces to the first rank equality in Eq.(6.1). Consequently applying Eq.(1.2) to $[\,A, \ A^*\,]$ in Eq.(6.1) yields the other two rank equalities in Eq.(6.1). The equivalences in Part (a) are well-known results on a EP matrix, which now is a direct consequence of Eq.(6.1). It remains to show Part (b). If $r[\,AA^\dagger - A^\dagger A\,] = m$, then $r[\,A, \ A^*\,] = r[\,AA^\dagger, \ A^\dagger A\,] = r[\,AA^\dagger - A^\dagger A, \ A^\dagger A\,] = m$. Putting it in Eq.(6.1), we obtain $2r(A) = m$. Conversely, if $r[\,A, \ A^*\,] = 2r(A) = m$, then we immediately have $r(\,AA^\dagger - A^\dagger A\,) = m$ by Eq.(6.1). Hence the first equivalence in Part (b) is true. The second equivalence is obvious. $\square$

Another group of rank equalities related to EP matrix is given below, which is motivated by a work of Campbell and Meyer (1975).

Theorem 6.2. *Let $A \in \mathcal{C}^{m\times m}$ be given. Then*

(a) $r[\, AA^{\dagger}(\, A + A^{\dagger}\,) - (\, A + A^{\dagger}\,)AA^{\dagger}\,] = 2r[\, A,\ A^{*}\,] - 2r(A)$.

(b) $r[\, A^{\dagger}A(\, A + A^{\dagger}\,) - (\, A + A^{\dagger}\,)A^{\dagger}A\,] = 2r[\, A,\ A^{*}\,] - 2r(A)$.

(c) $r[\, AA^{\dagger}(\, A + A^{*}\,) - (\, A + A^{*}\,)AA^{\dagger}\,] = 2r[\, A,\ A^{*}\,] - 2r(A)$.

(d) $r[\, A^{\dagger}A(\, A + A^{*}\,) - (\, A + A^{*}\,)A^{\dagger}A\,] = 2r[\, A,\ A^{*}\,] - 2r(A)$.

(e) *The following statements are equivalent(Campbell and Meyer, 1975):*

(1) A *is EP.*

(2) $AA^{\dagger}(\, A + A^{\dagger}\,) = (\, A + A^{\dagger}\,)AA^{\dagger}$.

(3) $A^{\dagger}A(\, A + A^{\dagger}\,) = (\, A + A^{\dagger}\,)A^{\dagger}A$.

(4) $AA^{\dagger}(\, A + A^{*}\,) = (\, A + A^{*}\,)AA^{\dagger}$.

(5) $A^{\dagger}A(\, A + A^{*}\,) = (\, A + A^{*}\,)A^{\dagger}A$.

Proof. Follows from Eq.(4.1) by noting that $AA^{\dagger}$ and $A^{\dagger}A$ are idempotent matrices. □

Replacing the matrices $(\, A + A^{\dagger}\,)$ and $(\, A + A^{*}\,)$ by $(\, A - A^{\dagger}\,)$ and $(\, A - A^{*}\,)$ in Theorem 6.2, the results are also true.

In an earlier paper by Meyer (1970) and a recent paper by Hartwig and Katz (1997), they established a necessary and sufficient condition for a block triangular matrix to be EP. Their work now can be extended to the following general settings.

Corollary 6.3. *Let $A \in \mathcal{C}^{m\times m}$, $B \in \mathcal{C}^{m\times k}$, and $D \in \mathcal{C}^{k\times k}$ be given, and let $M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$. Then*

$$r(\, MM^{\dagger} - M^{\dagger}M\,) = 2r\begin{bmatrix} A & A^{*} & B & 0 \\ 0 & B^{*} & D & D^{*} \end{bmatrix} - 2r(M). \tag{6.3}$$

In particular,

(a) *if both A and D are EP, then*

$$r(\, MM^{\dagger} - M^{\dagger}M\,) = 2r[\, A,\ B\,] + 2r\begin{bmatrix} B \\ D \end{bmatrix} - 2r\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}.$$

(b) *If $R(B) \subseteq R(A)$ and $R(B^{*}) \subseteq R(D^{*})$, then*

$$r(\, MM^{\dagger} - M^{\dagger}M\,) = 2r[\, A,\ A^{*}\,] + 2r[\, D,\ D^{*}\,] - 2r(A) - 2r(D).$$

(c) (Meyer, 1970, Hartwig and Katz, 1997) *M is EP if and only if A and D are EP, and $R(B) \subseteq R(A)$ and $R(B^{*}) \subseteq R(D^{*})$. In that case, $MM^{\dagger} = \begin{bmatrix} AA^{\dagger} & 0 \\ 0 & DD^{\dagger} \end{bmatrix}$.*

Proof. Follows immediately from Theorem 6.1 by putting M in it. □

Corollary 6.4. *Let*

$$M = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ & A_{22} & \cdots & A_{2k} \\ & & \ddots & \vdots \\ & & & A_{kk} \end{bmatrix} \in \mathcal{C}^{m\times n}, \qquad A_{ij} \in \mathcal{C}^{m_i \times m_j}$$

be given. Then M is EP if and only if $A_{11}, \ A_{22} \cdots, \ A_{nn}$ are EP, and

$$R(A_{ij}) \subseteq R(A_{ii}), \quad and \quad R(A_{ij}^*) \subseteq R(A_{jj}^*), \qquad i, \ j = 1, \ \cdots, \ n.$$

In that case, $MM^{\dagger} = \mathrm{diag}(\, A_{11}A_{11}^{\dagger}, \ A_{22}A_{22}^{\dagger}, \ \cdots, \ A_{nn}A_{nn}^{\dagger}\,)$.

Proof. Follows from Theorem 6.1(a) by putting M in it. □

A parallel concept to EP matrices is so-called conjugate EP matrices. A matrix A is said to be conjugate EP if $R(A) = R(A^T)$. If matrices considered are real, then EP matrices and conjugate EP matrices are identical. Much similar to EP matrices, conjugate EP matrices also have some nice properties. one of the basic and nice properties related to a conjugate EP matrix A is $AA^{\dagger} = \overline{A^{\dagger}A}$ (see the series work [58], [59], [60], [61], and [62] by Meenakshi and Indira). This equality motivates us to find the following results.

Theorem 6.5. *Let $A \in \mathcal{C}^{m\times m}$ be given. Then*

$$r(\, AA^{\dagger} - \overline{A^{\dagger}A}\,) = 2r[\, A, \ A^T\,] - 2r(A). \tag{6.4}$$

In particular,

(a) $AA^{\dagger} = \overline{A^{\dagger}A} \Longleftrightarrow r[\, A, \ A^T\,] = r(A) \Longleftrightarrow R(A) = R(A^T)$, *i.e., A is conjugate EP.*

(b) $AA^{\dagger} - \overline{A^{\dagger}A}$ *is nonsingular* $\Longleftrightarrow r[\, A, \ A^T\,] = 2r(A) = m \Longleftrightarrow R(A) \dotplus R(A^T) = \mathcal{C}^m$.

Proof. Since $A^{\dagger}A$ and $\overline{A^{\dagger}A}$ are idempotent, applying Eq.(3.1) to $AA^{\dagger} - \overline{A^{\dagger}A}$, we obtain

$$\begin{aligned} r(\, AA^{\dagger} - \overline{A^{\dagger}A}\,) &= r\begin{bmatrix} AA^{\dagger} \\ \overline{A^{\dagger}A} \end{bmatrix} + r[\, AA^{\dagger}, \ \overline{A^{\dagger}A}\,] - r(AA^{\dagger}) - r(\overline{A^{\dagger}A}) \\ &= r\begin{bmatrix} A^{\dagger} \\ \overline{A} \end{bmatrix} + r[\, A, \ \overline{A^{\dagger}}\,] - 2r(A) \\ &= r\begin{bmatrix} A^{*} \\ \overline{A} \end{bmatrix} + r[\, \overline{A}, \ A^{*}\,] - 2r(A) = 2r[\, A, \ A^T\,] - 2r(A), \end{aligned}$$

which is exactly Eq.(6.4). The results in Parts (a) and (b) follow immediately from Eq.(6.4). □

Corollary 6.6. *Let $A \in \mathcal{C}^{m\times m}$, $B \in \mathcal{C}^{m\times k}$, and $D \in \mathcal{C}^{k\times k}$ be given, and denote* $M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$. *Then*

$$r(\, MM^{\dagger} - \overline{M^{\dagger}M}\,) = 2r\begin{bmatrix} A & A^T & B & 0 \\ 0 & B^T & D & D^T \end{bmatrix} - 2r(M).$$

In particular, M is conjugate EP if and only if A and D are con-EP, and $R(B) \subseteq R(A)$ and $R(B^T) \subseteq R(D^T)$.

Proof. Follows from Theorem 6.5 by putting M in it. $\square$

The work in Theorem 6.1 can be extended to matrix expressions that involve the power of a matrix.

Theorem 6.7. *Let $A \in \mathcal{C}^{m \times m}$ be given and k be an integer with $k \geq 2$. Then*

$$r(A^kA^\dagger - A^\dagger A^k) = r\begin{bmatrix} A^k \\ A^* \end{bmatrix} + r[A^k, A^*] - 2r(A). \tag{6.5}$$

In particular,

(a) $A^kA^\dagger = A^\dagger A^k \Longleftrightarrow r\begin{bmatrix} A^k \\ A^* \end{bmatrix} = r[A^k, A^*] = r(A) \Longleftrightarrow R(A^k) \subseteq R(A^*)$ *and* $R[(A^k)^*] \subseteq R(A)$.

(b) $A^kA^\dagger - A^\dagger A^k$ *is nonsingular* $\Longleftrightarrow r\begin{bmatrix} A^k \\ A^* \end{bmatrix} = r[A^k, A^*] = 2r(A) = m \Longleftrightarrow r(A^k) = r(A)$ *and* $R(A) \oplus R(A^*) = \mathcal{C}^m$.

Proof. Writing $A^kA^\dagger - A^\dagger A^k = -[(A^\dagger A)A^{k-1} - A^{k-1}(AA^\dagger)]$ and applying Eq.(4.1) to it, we obtain

$$\begin{aligned} r(A^kA^\dagger - A^\dagger A^k) &= r\begin{bmatrix} A^\dagger A^k \\ AA^\dagger \end{bmatrix} + r[A^kA^\dagger, A^\dagger A] - r(AA^\dagger) - r(A^\dagger A) \\ &= r\begin{bmatrix} A^k \\ A^\dagger \end{bmatrix} + r[A^k, A^\dagger] - 2r(A) = r\begin{bmatrix} A^k \\ A^* \end{bmatrix} + r[A^k, A^*] - 2r(A), \end{aligned}$$

as required for Eq.(6.5). The results in Parts (a) and (b) follow immediately from Eq.(6.5). $\square$

From the result in Theorem 6.7(a) we can extend the concept of EP matrix to power case: A square matrix A is said to be *power-EP* if $R(A^k) \subseteq R(A^*)$ and $R[(A^k)^*] \subseteq R(A)$, where $k \geq 2$. It is believed that power-EP matrices, as a special type of matrices, might also have some interesting properties. But we do not intend to discuss power-EP matrices and the related topics in this thesis.

Theorem 6.8. *Let $A \in \mathcal{C}^{m \times m}$ be given and k be an integer with $k \geq 2$. Then*

(a) $r[A(A^k)^\dagger - (A^k)^\dagger A] = r\begin{bmatrix} A^k \\ A^kA^* \end{bmatrix} + r[A^k, A^*A^k] - 2r(A^k)$.

In particular,

(b) $A(A^k)^\dagger = (A^k)^\dagger A \Longleftrightarrow r\begin{bmatrix} A^k \\ A^kA^* \end{bmatrix} = r[A^k, A^*A^k] = r(A^k) \Longleftrightarrow R(A^k) = R(A^*A^k)$ *and* $R[(A^k)^*] = R[A(A^k)^*]$.

(c) $A(A^k)^\dagger = (A^k)^\dagger A \Longrightarrow A^kA^\dagger = A^\dagger A^k$.

Proof. It follows by Eq.(2.2) and block elementary operations that

$$r[A(A^k)^\dagger - (A^k)^\dagger A] = r\begin{bmatrix} (A^k)^*A^k(A^k)^* & 0 & (A^k)^* \\ 0 & -(A^k)^*A^k(A^k)^* & (A^k)^*A \\ A(A^k)^* & (A^k)^* & 0 \end{bmatrix} - 2r(A^k)$$

$$= r\begin{bmatrix} (A^k)^*A^k(A^k)^* & (A^k)^*A^{k-1}(A^k)^* & (A^k)^* \\ 0 & 0 & (A^k)^*A \\ A(A^k)^* & (A^k)^* & 0 \end{bmatrix} - 2r(A^k)$$

$$= r\begin{bmatrix} A^k \\ A^kA^* \end{bmatrix} + r[\, A^k, \ A^*A^k\,] - 2r(A^k),$$

establishing Part (a).Part (b) follows immediately from Part (a) . Combining Theorem 6.7(a) and 6.8(b) yields the implication in Part (c). □

A square matrix A is said to be bi-EP, if A and its Moore-Penrose inverse $A^\dagger$ satisfy $(AA^\dagger)(A^\dagger A) = (A^\dagger A)(AA^\dagger)$. This special type of matrices were examined by Campbell and Meyer (1975), Hartwig and Spindelböck (1983). Just as for EP matrices and conjugate EP matrices, bi-EP matrices can also be characterized by a rank equality.

Theorem 6.9. *Let $A \in \mathcal{C}^{m\times m}$ be given. Then*

$$r[\,(AA^\dagger)(A^\dagger A) - (A^\dagger A)(AA^\dagger)\,] = 2r[\, A, \ A^*\,] + 2r(A^2) - 4r(A). \tag{6.6}$$

and

$$r[\, A^2 - A^2(A^\dagger)^2A^2\,] = r[\, A, \ A^*\,] + r(A^2) - 2r(A). \tag{6.7}$$

In particular, the following four statements are equivalent:

(a) $(AA^\dagger)(A^\dagger A) = (A^\dagger A)(AA^\dagger)$, *i.e., A is bi-EP.*

(b) $(A^\dagger)^2 \in \{(A^2)^-\}$.

(c) $r[\, A, \ A^*\,] = 2r(A) - r(A^2)$.

(a) $\dim[R(A) \cap R(A^*)] = r(A^2)$.

Proof. Note that both $AA^\dagger$ and $A^\dagger A$ are Hermitian idempotent and $R(A^\dagger) = R(A^*)$. We have by Eq.(3.29) that

$$\begin{aligned} r[\,(AA^\dagger)(A^\dagger A) - (A^\dagger A)(AA^\dagger)\,] &= 2r[\, AA^\dagger, \ A^\dagger A\,] + 2r[\,(AA^\dagger)(A^\dagger A)\,] - 2r(AA^\dagger) - 2r(A^\dagger A) \\ &= 2r[\, A, \ A^*\,] + 2r(A^\dagger A^\dagger) - 4r(A) \\ &= 2r[\, A, \ A^*\,] + 2r(A^2) - 4r(A), \end{aligned}$$

establishing Eq.(6.6). Applying Eq.(2.8) and then the rank cancellation laws (1.8) to $A^2 - A^2(A^\dagger)^2A^2$, we then obtain

$$\begin{aligned} r[\, A^2 - A^2(A^\dagger)^2A^2\,] &= r\begin{bmatrix} A^*A^* & A^*AA^* & 0 \\ A^*AA^* & 0 & A^*A^2 \\ 0 & A^2A^* & -A^2 \end{bmatrix} - 2r(A) \\ &= r\begin{bmatrix} A^*A^* & A^*AA^* & 0 \\ A^*AA^* & A^*A^2A^* & 0 \\ 0 & 0 & -A^2 \end{bmatrix} - 2r(A) \end{aligned}$$

$$= r\begin{bmatrix} A^*A^* & A^*AA^* \\ A^*AA^* & A^*A^2A^* \end{bmatrix} + r(A^2) - 2r(A)$$

$$= r\begin{bmatrix} A^*A^* & A^*A \\ AA^* & A^2 \end{bmatrix} + r(A^2) - 2r(A)$$

$$= r([A,\ A^*]^*[A,\ A^*]) + r(A^2) - 2r(A)$$

$$= r[A,\ A^*] + r(A^2) - 2r(A),$$

as required for Eq.(6.7). The equivalence of (a)—(c) follows from Eqs.(6.6) and (6.7). The equivalence of (c) and (d) follows from a well-known rank equality $r[A,\ B] = r(A) + r(B) - \dim[R(A) \cap R(B)]$. □

The above work can also be extended to the conjugate case.

Theorem 6.10. *Let $A \in \mathcal{C}^{m\times m}$ be given. Then*

$$r[(AA^\dagger)(\overline{A^\dagger A}) - (\overline{A^\dagger A})(AA^\dagger)] = 2r[A,\ A^T] + 2r(A\overline{A}) - 4r(A).$$

In particular,

$$(AA^\dagger)(\overline{A^\dagger A}) = (\overline{A^\dagger A})(AA^\dagger) \Longleftrightarrow r[A.\ A^T] = 2r(A) - r(A\overline{A}).$$

Proof. Follows from Eq.(3.29) by noticing that both $AA^\dagger$ and $\overline{A^\dagger A}$ are idempotent. □.

A parallel concept to bi-EP matrix now can be introduced: A square matrix A is said to be *conjugate bi-EP* if $(AA^\dagger)(\overline{A^\dagger A}) = (\overline{A^\dagger A})(AA^\dagger)$. The properties and applications of this special type of matrices remain to further study.

We next consider rank equalities related to star-dagger matrices. A square matrix A is said to be star-dagger if $A^*A^\dagger = A^\dagger A^*$. This special types of matrices were proposed and examined by Hartwig and Spindelböck (1983), later by Meenakshi and Rajian (1988).

Theorem 6.11. *Let $A \in \mathcal{C}^{m\times m}$ be given. Then*

(a) $r(A^*A^\dagger - A^\dagger A^*) = r(AA^*A^2 - A^2A^*A)$.

(b) $r(AA^*A^\dagger A - AA^\dagger A^*A) = r(AA^*A^2 - A^2A^*A)$.

(c) $r(A^*A^\dagger - A^\dagger A^*) = r(AA^* - A^*A)$, *if A is EP.*

(d) *In particular, the following statement are equivalent (Hartwig and Spindelböck, 1983):*

 (1) $A^*A^\dagger = A^\dagger A^*$, *i.e., A is star-dagger.*

 (2) $AA^*A^\dagger A = AA^\dagger A^*A$.

 (3) $AA^*A^2 = A^2A^*A$.

Proof. We find by Eq.(2.2) that

$$r(A^*A^\dagger - A^\dagger A^*) = r\begin{bmatrix} -A^*AA^* & 0 & A^* \\ 0 & A^*AA^* & A^*A^* \\ A^*A^* & A^* & 0 \end{bmatrix} - 2r(A)$$

$$= r\begin{bmatrix} -AA^*A & 0 & A^2 \\ 0 & AA^*A & A \\ A & A^2 & 0 \end{bmatrix} - 2r(A)$$

$$= r\begin{bmatrix} 0 & AA^*A^2 - A^2A^*A & 0 \\ 0 & 0 & A \\ A & 0 & 0 \end{bmatrix} - 2r(A) = r(\, AA^*A^2 - A^2A^*A\,).$$

as required for Part (a). Similarly we can show Part (b). The result in Part (c) follows immediately from Part (a). and Part (d) follows from Parts (a) and (b). □

As pointed out by Hartwig and Spindelböck, 1983, star-dagger matrices are quite inclusive. Normal matrices. partial isometries, idempotent matrices, 2-nilpotent matrices. and so on are all special cases of star-dagger matrices. this assertion can be seen from the statement (3) in Theorem 6.11(d).

The results in Theorem 6.11 can be extended to general cases. Below are three of them. The proofs are much similar to that of Theorem 6.11, are therefore omitted.

Theorem 6.12. *Let $A \in \mathcal{C}^{m\times m}$ be given. Then*

(a) $r(\, A^*AA^*A^\dagger - A^\dagger A^*AA^*\,) = r[\,(AA^*)^2A^2 - A^2(A^*A)^2\,]$.

(b) $r[\,(AA^*)^2A^\dagger A - AA^\dagger(A^*A)^2\,] = r[\,(AA^*)^2A^2 - A^2(A^*A)^2\,]$.

(c) $A^*AA^*A^\dagger = A^\dagger A^*AA^* \iff (AA^*)^2A^\dagger A = AA^\dagger(A^*A)^2 \iff (AA^*)^2A^2 = A^2(A^*A)^2$.

Theorem 6.13. *Let $A \in \mathcal{C}^{m\times m}$ be given and k be an integer with $k \geq 2$. Then*

(a) $r[\,(A^*)^kA^\dagger - A^\dagger(A^*)^k\,] = r(\, AA^*A^{k+1} - A^{k+1}A^*A\,)$.

(b) $r[\,A(A^*)^kA^\dagger A - AA^\dagger(A^*)^kA\,] = r(\, AA^*A^{k+1} - A^{k+1}A^*A\,)$.

(c) $(A^*)^kA^\dagger = A^\dagger(A^*)^k \iff A(A^*)^kA^\dagger A = AA^\dagger(A^*)^kA \iff AA^*A^{k+1} = A^{k+1}A^*A$.

(d) *If $A^{k+1} = A$. or $A^{k+1} = 0$. or $AA^* = A^*A$. or $AA^*A = A$. then $(A^*)^kA^\dagger = A^\dagger(A^*)^k$ holds.*

Theorem 6.14. *Let $A \in \mathcal{C}^{m\times m}$ be given and k be an integer with $k \geq 2$. Then*

(a) $r[\,A^*(A^k)^\dagger - (A^k)^\dagger A^*\,] = r[\,A^k(A^k)^*A^{k+1} - A^{k+1}(A^k)^*A^k\,]$.

(b) $A^*(A^k)^\dagger = (A^k)^\dagger A^* \iff A^k(A^k)^*A^{k+1} = A^{k+1}(A^k)^*A^k$.

Next are several results on ranks of matrix expressions involving powers of the Moore-Penrose inverse of a matrix.

Theorem 6.15. *Let $A \in \mathcal{C}^{m\times m}$ be given. Then*

(a) $r[\,I_m \pm A^\dagger\,] = r(\,A^2 \pm AA^*A\,) - r(A) + m$.

(b) $r[\,I_m - (A^\dagger)^2\,] = r(\,A^2 + AA^*A\,) + r(\,A^2 - AA^*A\,) - 2r(A) + m$.

(c) $r[\,I_m \pm A^\dagger\,] = r(\,A \pm AA^*\,) - r(A) + m$, *if A is EP.*

(d) $r[\,I_m - (A^\dagger)^2\,] = r(\,A + AA^*\,) + r(\,A - AA^*\,) - 2r(A) + m$, *if A is EP.*

(e) $r[\,I_m \pm A^\dagger\,] = r(\,A \pm A^2\,) - r(A) + m$. *if A is Hermitian.*

(f) $r[\,I_m - (A^\dagger)^2\,] = r(\,A + A^2\,) + r(\,A - A^2\,) - 2r(A) + m$. *if A is Hermitian.*

Proof. By Eq.(2.1) we easily obtain

$$\begin{aligned} r(I_m - A^{\dagger}) &= r\begin{bmatrix} A^*AA^* & A^* \\ A^* & I_m \end{bmatrix} - r(A) \\ &= r\begin{bmatrix} A^*AA^* - A^*A^* & 0 \\ 0 & I_m \end{bmatrix} - r(A) = r(AA^*A - A^2) + m - r(A), \end{aligned}$$

and

$$\begin{aligned} r(I_m + A^{\dagger}) &= r\begin{bmatrix} -A^*AA^* & A^* \\ A^* & I_m \end{bmatrix} - r(A) \\ &= r\begin{bmatrix} -A^*AA^* - A^*A^* & 0 \\ 0 & I_m \end{bmatrix} - r(A) = r(AA^*A + A^2) + m - r(A). \end{aligned}$$

Both of the above are exactly Part (a). Next applying Eq.(1.12) to $I_m - (A^{\dagger})^2$ we obtain

$$\begin{aligned} r[\, I_m - (A^{\dagger})^2 \,] &= r(I_m + A^{\dagger}) + r(I_m - A^{\dagger}) - m \\ &= r(A^2 + AA^*A) + r(A^2 - AA^*A) - 2r(A) + m. \end{aligned}$$

establishing Part (b). The results in Parts (c)—(f) follow directly from Parts (a) and (b). □

Theorem 6.16. *Let $A \in \mathcal{C}^{m \times m}$ be given. Then*

(a) $r[\, A^{\dagger} \pm (A^{\dagger})^2 \,] = r(A^2 \pm AA^*A) = r[\, A \pm A(A^{\dagger})^2A \,]$.

(b) $r[\, A^{\dagger} - (A^{\dagger})^2 \,] = r(A - AA^*) = r(A - A^*A)$, *if A is EP.*

(c) $r[\, A^{\dagger} \pm (A^{\dagger})^2 \,] = r(A \pm A^2)$, *if A is Hermitian.*

(d) $r[\, A^{\dagger} - (A^{\dagger})^2 \,]$, *i.e.*, $(A^{\dagger})^2 \leq_{rs} A^{\dagger} = r(A^{\dagger}) - r[\,(A^{\dagger})^2\,] \Longleftrightarrow r(AA^*A - A^2) = r(A) - r(A^2)$, *i.e.*, $A^2 \leq_{rs} AA^*A$.

(e) $(A^{\dagger})^2 = A^{\dagger} \Longleftrightarrow AA^*A = A^2 \Longleftrightarrow A = (AA^{\dagger})(A^{\dagger}A) \Longleftrightarrow (A^{\dagger})^2 \in \{A^-\}$.

(f) $(A^{\dagger})^2 = A^{\dagger} \Longleftrightarrow AA^* = A$, *if A is EP.*

(g) $(A^{\dagger})^2 = A^{\dagger} \Longleftrightarrow A^2 = A$, *if A is Hermitian.*

Proof. It follows first from Eq.(1.11) that

$$r[\, A^{\dagger} - (A^{\dagger})^2 \,] = r(I_m - A^{\dagger}) + r(A) - m.$$

$$r[\, A^{\dagger} + (A^{\dagger})^2 \,] = r(I_m + A^{\dagger}) + r(A) - m.$$

Then we have the first two equalities in Part (a) by Theorem 6.15(a). Note that

$$A[\, A^{\dagger} \pm (A^{\dagger})^2 \,]A = A \pm A(A^{\dagger})^2A \quad \text{and} \quad A^{\dagger}[\, A \pm (A^{\dagger})^2A \,]A^{\dagger} = A^{\dagger} \pm (A^{\dagger})^2.$$

It follows that

$$r[\, A^{\dagger} \pm (A^{\dagger})^2 \,] = r[\, A \pm A(A^{\dagger})^2A \,].$$

Thus we have the second equality in Part (a). Parts (b)—(g) follow from Part (a). □

Theorem 6.17. *Let $A \in \mathcal{C}^{m\times m}$ be given. Then*

(a) $r[\,A^{\dagger} - (A^{\dagger})^3\,] = r(\,A^2 + AA^*A\,) + r(\,A^2 - AA^*A\,) - r(A)$.

(b) $r[\,A^{\dagger} - (A^{\dagger})^3\,] = r(\,A + AA^*\,) + r(\,A - AA^*\,) - r(A)$, *if A is EP.*

(c) $r[\,A^{\dagger} - (A^{\dagger})^3\,] = r(\,A + A^2\,) + r(\,A - A^2\,) - r(A) = r(\,A - A^3\,)$, *if A is Hermitian.*

(d) $(A^{\dagger})^3 = A^{\dagger} \Longleftrightarrow r(\,A^2 + AA^*A\,) + r(\,A^2 - AA^*A\,) = r(A) \Longleftrightarrow R(\,AA^*A + A^2\,) \cap R(\,AA^*A - A^2\,) = \{0\}$ *and* $R[(\,AA^*A + A^2\,)^*] \cap R[(\,AA^*A - A^2\,)^*] = \{0\}$.

Proof. Applying Anderson and Styans' rank equality (1.14)

$$r(\,N - N^3\,) = r(\,N + N^2\,) + r(\,N - N^2\,) - r(N)$$

to $A^{\dagger} - (A^{\dagger})^3$, we obtain

$$r[\,A^{\dagger} - (A^{\dagger})^3\,] = r[\,A^{\dagger} + (A^{\dagger})^2\,] + r[\,A^{\dagger} - (A^{\dagger})^2\,] - r(A).$$

Then putting Theorem 6.16(a) in it yields Part (a). The results in Parts (b)—(d) follow all from Part (a). $\square$

Theorem 6.18. *Let $A \in \mathcal{C}^{m\times n}$ be given. Then*

(a) $r(\,A^{\dagger} - A^*\,) = r(\,A - AA^*A\,)$.

(b) $r(\,A^{\dagger} - A^*AA^*\,) = r(\,A - AA^*AA^*A\,)$.

In particular,

(c) $A^{\dagger} = A^* \Longleftrightarrow AA^*A = A$, *i.e., A is partial isometry.*

(d) $A^{\dagger} = A^*AA^* \Longleftrightarrow AA^*AA^*A = A$.

Proof. Follows from Eq.(2.1). $\square$

Theorem 6.19. *Let $A \in \mathcal{C}^{m\times m}$ be idempotent. Then*

(a) $r(\,A - A^{\dagger}\,) = 2r[\,A,\ A^*\,] - 2r(A)$.

(b) $r(\,A^{\dagger} - AA^{\dagger}A^{\dagger}A\,) = 2r[\,A,\ A^*\,] - 2r(A)$.

(c) $r(\,A - AA^{\dagger}A^{\dagger}A\,) = r[\,A,\ A^*\,] - r(A)$.

(d) $A = A^{\dagger} \Longleftrightarrow A^{\dagger} = AA^{\dagger}A^{\dagger}A \Longleftrightarrow A = AA^{\dagger}A^{\dagger}A \Longleftrightarrow A = A^*$.

Proof. Note that $A,\ A^{\dagger} \in A\{2\}$ when A is idempotent. Thus we have by Eq.(5.1) that

$$\begin{aligned} r(\,A - A^{\dagger}\,) &= r\begin{bmatrix} A \\ A^{\dagger} \end{bmatrix} + r[\,A,\ A^{\dagger}\,] - r(A) - r(A^{\dagger}) \\ &= r\begin{bmatrix} A \\ A^* \end{bmatrix} + r[\,A,\ A^*\,] - 2r(A) = 2r[\,A,\ A^*\,] - 2r(A), \end{aligned}$$

establishing Part (a). Parts (b) and (c) follow from Eqs.(6.6) and (6.7). Part (d) is a direct consequence of Parts (a)—(c). $\square$

Theorem 6.20. *Let $A \in \mathcal{C}^{m\times m}$ be tripotent, that is, $A^3 = A$. Then*

(a) $r(A - A^{\dagger}) = 2r[A,\ A^{*}] - 2r(A)$.

(b) $r(A^2A^{\dagger} - A^{\dagger}A^2) = 2r[A,\ A^{*}] - 2r(A)$.

(c) $r[A(A^2)^{\dagger} - (A^2)^{\dagger}A] = 2r[A,\ A^{*}] - 2r(A)$.

(d) $r[(AA^{\dagger})(A^{\dagger}A) - (A^{\dagger}A)(AA^{\dagger})] = 2r[A,\ A^{*}] - 2r(A)$.

(e) $r[A^2 - A^2(A^{\dagger})^2A^2] = r[A,\ A^{*}] - r(A)$.

(f) $(A^{*})^2A^{\dagger} = A^{\dagger}(A^{*})^2$.

(g) *The following five statements are equivalent:*

(1) $A = A^{\dagger}$.

(2) $A^2A^{\dagger} = A^{\dagger}A^2$.

(3) $A(A^2)^{\dagger} = (A^2)^{\dagger}A$.

(4) $(AA^{\dagger})(A^{\dagger}A) = (A^{\dagger}A)(AA^{\dagger})$.

(5) $A^2 = A^2(A^{\dagger})^2A^2$.

(6) $R(A) = R(A^{*})$, *i.e., A is EP.*

Proof. Note that $A,\ A^{\dagger} \in A\{2\}$ when A is tripotent. Thus we have by Eq.(5.1) that

$$\begin{aligned} r(A - A^{\dagger}) &= r\begin{bmatrix} A \\ A^{\dagger} \end{bmatrix} + r[A,\ A^{\dagger}] - r(A) - r(A^{\dagger}) \\ &= r\begin{bmatrix} A \\ A^{*} \end{bmatrix} + r[A,\ A^{*}] - 2r(A) = 2r[A,\ A^{*}] - 2r(A), \end{aligned}$$

establishing Part (a). Parts (b)—(e) follow from Eqs.(6.5), (6.6), (6.7) and Theorem 6.8(a). Part (g) follows from Theorem 6.13(c). Part (f) is a direct consequence of Parts (a)—(e). □

The following result is motivated by a problem of Rao and Mitra (1971) on the nonsingularity of a matrix of the form $I + A - A^{\dagger}A$.

Theorem 6.21. *Let $A \in \mathcal{C}^{m\times m}$ and $1 \neq \lambda \in \mathcal{C}$ be given. Then*

(a) $r(I_m + A - A^{\dagger}A) = r(I_m + A - AA^{\dagger}) = r(A^2) - r(A) + m$.

(b) $r(I_m - A - A^{\dagger}A) = r(I_m - A - AA^{\dagger}) = r(A^2) - r(A) + m$.

(c) $r(\lambda I_m + A - A^{\dagger}A) = r(\lambda I_m + A - AA^{\dagger}) = r[(\lambda - 1)I_m + A]$.

(b) (Rao and Mitra, 1971) $I_m + A - A^{\dagger}A$ *is nonsingular* $\Longleftrightarrow$ $I_m + A - AA^{\dagger}$ *is nonsingular* $\Longleftrightarrow$ $r(A^2) = r(A)$.

Proof. Applying Eq.(2.1) and then Eq.(1.8), we find that

$$\begin{aligned} r(I_m - A - A^{\dagger}A) &= r\begin{bmatrix} A^{*}AA^{*} & A^{*}A \\ A^{*} & I_m + A \end{bmatrix} - r(A) \\ &= r\begin{bmatrix} AA^{*} & A \\ A^{*} & I_m + A \end{bmatrix} - r(A) \end{aligned}$$

$$
\begin{aligned}
&= r\begin{bmatrix} 0 & A \\ AA^* & I_m \end{bmatrix} - r(A) \\
&= r\begin{bmatrix} A^2A^* & 0 \\ 0 & I_m \end{bmatrix} - r(A) = m + r(A^2) - r(A).
\end{aligned}
$$

By symmetry, we also get $r(\, I_m + A - AA^\dagger\,) = m + r(A^2) - r(A)$. Both of them are the result in Part (a). Similarly we can show Part (c) by Eq.(2.1) and (1.11). Part (d) is a direct consequence of Part(a). □

Chapter 7

Rank equalities related to matrices and their Moore-Penrose inverses

We consider in this chapter ranks of various matrix expressions that involve two or more matrices and their Moore-Penrose inverses, and present their various consequences, which can reveal a series of intrinsic properties related to Moore-Penrose inverses of matrices. Most of the results obtained in this chapter are new and are not considered before.

Theorem 7.1. *Let A, $B \in \mathcal{C}^{m \times m}$ be given. Then*

$$r(AA^{\dagger}B - BA^{\dagger}A) = r\begin{bmatrix} A \\ A^*B \end{bmatrix} + r[A, \ BA^*] - 2r(A). \tag{7.1}$$

In particular,

(a) $AA^{\dagger}B = BA^{\dagger}A \Longleftrightarrow r\begin{bmatrix} A \\ A^*B \end{bmatrix} = r[A, \ BA^*] = r(A) \Longleftrightarrow R(BA^*) \subseteq R(A)$ *and* $R(B^*A) \subseteq R(A^*)$.

(b) $AA^{\dagger}B - BA^{\dagger}A$ *is nonsingular* $\Longleftrightarrow r\begin{bmatrix} A \\ A^*B \end{bmatrix} = r[A, \ BA^*] = 2r(A) = m \Longleftrightarrow R(A) \dotplus R(BA^*) = \mathcal{C}^m$ *and* $R(AB^*) = R(A) \Longleftrightarrow R(A^*) \oplus R(B^*A) = \mathcal{C}^m$ *and* $R(A^*B) = R(A^*)$.

Proof. Note that $AA^{\dagger}$ and $A^{\dagger}A$ are idempotent and $R(A^{\dagger}) = R(A^*)$. We have by Eq.(4.1) that

$$\begin{aligned} r(AA^{\dagger}B - BA^{\dagger}A) &= r\begin{bmatrix} AA^{\dagger}B \\ A^{\dagger}A \end{bmatrix} + r[BA^{\dagger}A, \ AA^{\dagger}] - r(AA^{\dagger}) - r(A^{\dagger}A) \\ &= r\begin{bmatrix} A^{\dagger}B \\ A \end{bmatrix} + r[BA^{\dagger}, \ A] - 2r(A) \\ &= r\begin{bmatrix} A^*B \\ A \end{bmatrix} + r[BA^*, \ A] - 2r(A), \end{aligned}$$

establishing Eq.(7.1). Parts (a) and (b) follow from it. $\square$

Clearly the results in Theorems 6.1 and 6.7 are special cases of the above theorem.

Theorem 7.2. *Let $A \in \mathcal{C}^{m \times n}$, $B \in \mathcal{C}^{m \times k}$ and $C \in \mathcal{C}^{l \times n}$ be given. Then*

(a) $r(AA^{\dagger} - BB^{\dagger}) = 2r[A, \ B] - r(A) - r(B)$.

(b) $r(A^{\dagger}A - C^{\dagger}C) = 2r\begin{bmatrix} A \\ C \end{bmatrix} - r(A) - r(C)$.

(c) $r(AA^{\dagger} + BB^{\dagger}) = r[A, \ B]$.

(d) $r(A^{\dagger}A + C^{\dagger}C) = r\begin{bmatrix} A \\ C \end{bmatrix}$.

In particular,

(e) $AA^{\dagger} = BB^{\dagger} \iff R(A) = R(B)$.

(f) $A^{\dagger}A = C^{\dagger}C \iff R(A^*) = R(C^*)$.

(g) $r(AA^{\dagger} - BB^{\dagger}) = r(AA^{\dagger}) - r(BB^{\dagger}) \iff R(B) \subseteq R(A)$.

(h) $r(A^{\dagger}A - C^{\dagger}C) = r(A^{\dagger}A) - r(C^{\dagger}C) \iff R(C^*) \subseteq R(A^*)$.

(i) $r(AA^{\dagger} - BB^{\dagger}) = m \iff r[A, \ B] = r(A) + r(B) = m \iff R(A) \oplus R(B) = \mathcal{C}^m$.

(j) $r(A^{\dagger}A - C^{\dagger}C) = n \iff r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C) = n \iff R(A^*) \oplus R(C^*) = \mathcal{C}^n$.

Proof. Note that $AA^{\dagger}$, $A^{\dagger}A$, $BB^{\dagger}$, and $C^{\dagger}C$ are all idempotent. Thus we can easily derive by Eqs.(3.1) and (3.12) the four rank equalities in Parts (a)—(d). The results in Parts (e)—(j) are direct consequences of Parts (a) and (b). □

Theorem 7.3. *Let $A \in \mathcal{C}^{m \times n}$, $B \in \mathcal{C}^{k \times m}$ be given. Then*

$$r(AA^{\dagger}B^{\dagger}B - B^{\dagger}BAA^{\dagger}) = 2r[A, \ B^*] + 2r(BA) - 2r(A) - 2r(B). \tag{7.2}$$

In particular,

$$(AA^{\dagger})(B^{\dagger}B) = (B^{\dagger}B)(AA^{\dagger}) \iff r[A, \ B^*] = r(A) + r(B) - r(BA) \iff \dim[R(A) \cap R(B^*)] = r(BA). \tag{7.3}$$

Proof. Note that $AA^{\dagger}$, $A^{\dagger}A$, $BB^{\dagger}$, and $B^{\dagger}B$ are Hermitian idempotent. Thus we find by Eq.(3.29) that

$$\begin{aligned} r[(AA^{\dagger})(B^{\dagger}B) - (B^{\dagger}B)(AA^{\dagger})] &= 2r[AA^{\dagger}, \ B^{\dagger}B] + 2r[(AA^{\dagger})(B^{\dagger}B)] - 2r(AA^{\dagger}) - 2r(B^{\dagger}B) \\ &= 2r[A, \ B^*] + 2r(BA) - 2r(A) - 2r(B), \end{aligned}$$

as required for Eq.(7.2). The results in Eq.(7.3) is a direct consequence of Eq.(7.2). □

Theorem 7.4. *Let $A \in \mathcal{C}^{m \times n}$, $B \in \mathcal{C}^{k \times m}$ be given. Then*

(a) $r[A^*(AA^* + BB^*)^{\dagger}A - A^{\dagger}A] = r(A) + r(B) - r[A, \ B]$.

(b) $r[A(A^*A + C^*C)^{\dagger}A^* - AA^{\dagger}] = r(A) + r(C) - r\begin{bmatrix} A \\ C \end{bmatrix}$.

(c) $r[A^*(AA^* + BB^*)^{\dagger}B] = r(A) + r(B) - r[A, \ B]$.

(d) $r[A(A^*A + C^*C)^{\dagger}C^*] = r(A) + r(C) - r\begin{bmatrix} A \\ C \end{bmatrix}$.

In particular,

(e) $A^*(AA^* + BB^*)^{\dagger}A = A^{\dagger}A \iff A^*(AA^* + BB^*)^{\dagger}B = 0 \iff r[A, \ B] = r(A) + r(B) \iff R(A) \cap R(B) = \{0\}$.

(f) $A(A^*A + C^*C)^\dagger A^* = AA^\dagger \iff A(A^*A + C^*C)^\dagger C^* = 0 \iff r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C) \iff$ $R(A^*) \cap R(C^*) = \{0\}$.

Proof. Parts (a) and (b) are derived from Eq.(2.2). Part(c) and (d) are derived from Eq.(2.1). The results in Parts (e) and (f) are direct consequences of Parts (a)—(d). □

Theorem 7.5. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$ and $C \in \mathcal{C}^{l\times n}$ be given. Then*

$$r(BB^\dagger A - AC^\dagger C) = r\begin{bmatrix} B^*A \\ C \end{bmatrix} + r[\, AC^*,\ B\,] - r(B) - r(C). \tag{7.4}$$

In particular,

(a) $BB^\dagger A = AC^\dagger C \iff r\begin{bmatrix} B^*A \\ C \end{bmatrix} = r(C)$ *and* $r[\, AC^*,\ B\,] = r(B) \iff R(AC^*) \subseteq R(B)$ *and* $R(A^*B) \subseteq$ $R(C^*)$.

(b) $BB^\dagger A - AC^\dagger C$ *is nonsingular* $\iff r\begin{bmatrix} B^*A \\ C \end{bmatrix} = r[\, AC^*,\ B\,] = r(B) + r(C) = m$.

Proof. Follows from Eq.(4.1) by noticing that both $BB^\dagger$ and $C^\dagger C$ are idempotent. □

It is well known that the matrix equation $BXC = A$ is solvable if and only if $BB^\dagger AC^\dagger C = A$ (see, e.g., Rao and Mitra, 1971). This lead us to consider the rank of $A - BB^\dagger AC^\dagger C$.

Theorem 7.6. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$ and $C \in \mathcal{C}^{l\times n}$ be given. Then*

$$r(A - BB^\dagger AC^\dagger C) = r\begin{bmatrix} A & AC^* & B \\ B^*A & 0 & 0 \\ C & 0 & 0 \end{bmatrix} - r(B) - r(C). \tag{7.5}$$

and

$$r(2A - BB^\dagger A - AC^\dagger C) = r\begin{bmatrix} A & AC^* & B \\ B^*A & 0 & 0 \\ C & 0 & 0 \end{bmatrix} - r(B) - r(C). \tag{7.6}$$

In particular,

$$BB^\dagger AC^\dagger C = A \iff BB^\dagger A + AC^\dagger C = 2A \iff R(A) \subseteq R(B) \ \ \textit{and} \ \ R(A^*) \subseteq R(C^*). \tag{7.7}$$

Proof. Applying Eq.(2.8) and the rank cancellation law (1.8) to $A - BB^\dagger AC^\dagger C$ produces

$$\begin{aligned} & r(A - BB^\dagger AC^\dagger C) \\ &= r\begin{bmatrix} B^*AC^* & B^*BB^* & 0 \\ C^*CC^* & 0 & C^*C \\ 0 & BB^* & -A \end{bmatrix} - r(B) - r(C) \end{aligned}$$

$$= r\begin{bmatrix} B^*AC^* & B^*B & 0 \\ CC^* & 0 & C \\ 0 & B & -A \end{bmatrix} - r(B) - r(C) = r\begin{bmatrix} 0 & 0 & B^*A \\ 0 & 0 & C \\ AC^* & B & -A \end{bmatrix} - r(B) - r(C).$$

as required for Eq.(7.5). In the same way we can show Eq.(7.6). The result in Eq.(7.7) is well known. □

Theorem 7.7. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$ and $C \in \mathcal{C}^{l\times n}$ be given, and let $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$. Then*

$$r(\, A - BB^\dagger A - AC^\dagger C\,) = r(M) + r(CA^*B) - r(B) - r(C). \tag{7.8}$$

that is, the block matrix M satisfies the rank equality

$$r(M) = r(B) + r(C) - r(CA^*B) + r(\, A - BB^\dagger A - AC^\dagger C\,). \tag{7.9}$$

Proof. Applying Eqs.(2.2) and (1.8) to $A - BB^\dagger A - AC^\dagger C$ yields

$$\begin{aligned} & r(\, A - BB^\dagger A - AC^\dagger C\,) \\ = \;& r\begin{bmatrix} B^*BB^* & 0 & B^*A \\ 0 & C^*CC^* & C^*C \\ BB^* & AC^* & A \end{bmatrix} - r(B) - r(C) \\ = \;& r\begin{bmatrix} B^*B & 0 & B^*A \\ 0 & CC^* & C \\ B & AC^* & A \end{bmatrix} - r(B) - r(C) \\ = \;& r\begin{bmatrix} 0 & -B^*AC^* & 0 \\ 0 & 0 & C \\ B & 0 & A \end{bmatrix} - r(B) - r(C) \\ = \;& r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + r(B^*AC^*) - r(B) - r(C), \end{aligned}$$

as required for Eq.(7.8). □

Theorem 7.8. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$ and $C \in \mathcal{C}^{l\times n}$ be given, and let $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$. Then*

$$r[\, A - A(E_BAF_C)^\dagger A\,] = r(A) + r(B) + r(C) - r(M), \tag{7.10}$$

that is, the block matrix M satisfies the rank equality

$$r(M) = r(A) + r(B) + r(C) - r[\, A - A(E_BAF_C)^\dagger A\,], \tag{7.11}$$

where $E_B = I - BB^\dagger$ and $F_C = I - C^\dagger C$. In particular,

$$(E_BAF_C)^\dagger \in \{A^-\} \tag{7.12}$$

holds if and only if

$$r(M) = r(A) + r(B) + r(C), \quad i.e., \quad R(A) \cap R(B) = \{0\} \text{ and } R(A^*) \cap R(C^*) = \{0\}. \tag{7.13}$$

Proof. Let $N = E_BAF_C$. Then it is easily to verify that $N^*NN^* = N^*AN^*$. In that case, applying Eqs.(2.1), and then (1.2) and (1.3) to $A - A(E_BAF_C)^\dagger A$ yields

$$\begin{aligned} r[\,A - A(E_BAF_C)^\dagger A\,] = r[\,A - AN^\dagger A\,] &= r\begin{bmatrix} N^*NN^* & N^*A \\ AN^* & A \end{bmatrix} - r(M) \\ &= r\begin{bmatrix} N^*NN^* - N^*AN^* & 0 \\ 0 & A \end{bmatrix} - r(N) \\ &= r\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} - r(N) \\ &= r(A) - r(N) \\ &= r(A) + r(B) + r(C) - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \end{aligned}$$

as required for Eq.(7.10). The equivalence of Eqs.(7.12) and (7.13) follows immediately from Eq.(7.11). □

It is known that for any B and C, the matrix $(E_BAF_C)^\dagger$ is always an outer inverse of A (Greville, 1974). Thus the rank formula (7.11) can also be derived from Eq.(5.6).

In the remainder of this chapter, we establish various rank equalities related to ranks of Moore-Penrose inverses of block matrices, and then present their consequences.

Theorem 7.9. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$ and $C \in \mathcal{C}^{l\times n}$ be given. Then*

(a) $r\left([\,A,\ B\,]^\dagger - \begin{bmatrix} A^\dagger \\ B^\dagger \end{bmatrix} \right) = r[\,AA^*B,\ BB^*A\,]$.

(b) $r\left(\begin{bmatrix} A \\ C \end{bmatrix}^\dagger - [\,A^\dagger,\ C^\dagger\,] \right) = r\begin{bmatrix} AC^*C \\ CA^*A \end{bmatrix}$.

(c) $r\left([\,A,\ B\,]^\dagger[\,A,\ B\,] - \begin{bmatrix} A^\dagger \\ B^\dagger \end{bmatrix}[\,A,\ B\,] \right) = r[\,AA^*B,\ BB^*A\,]$.

(d) $r\left(\begin{bmatrix} A \\ C \end{bmatrix}\begin{bmatrix} A \\ C \end{bmatrix}^\dagger - \begin{bmatrix} A \\ C \end{bmatrix}[\,A^\dagger,\ C^\dagger\,] \right) = r\begin{bmatrix} AC^*C \\ CA^*A \end{bmatrix}$.

In particular,

(e) $[\,A,\ B\,]^\dagger = \begin{bmatrix} A^\dagger \\ B^\dagger \end{bmatrix} \Longleftrightarrow [\,A,\ B\,]^\dagger[\,A,\ B\,] = \begin{bmatrix} A^\dagger \\ B^\dagger \end{bmatrix}[\,A,\ B\,] \Longleftrightarrow A^*B = 0.$

(f) $\begin{bmatrix} A \\ C \end{bmatrix}^{\dagger} = [\,A^{\dagger},\ C^{\dagger}\,] \Longleftrightarrow \begin{bmatrix} A \\ C \end{bmatrix}\begin{bmatrix} A \\ C \end{bmatrix}^{\dagger} = \begin{bmatrix} A \\ C \end{bmatrix}[\,A^{\dagger},\ C^{\dagger}\,] \Longleftrightarrow CA^{*} = 0.$

Proof. Let $M = [\,A,\ B\,]$. Then it follows by Eq.(2.7) that

$$
\begin{aligned}
& r\left([\,A,\ B\,]^{\dagger} - \begin{bmatrix} A^{\dagger} \\ B^{\dagger} \end{bmatrix} \right) \\
&= r\left([\,A,\ B\,]^{\dagger} - \begin{bmatrix} I \\ 0 \end{bmatrix} A^{\dagger} - \begin{bmatrix} 0 \\ I \end{bmatrix} B^{\dagger} \right) \\
&= r\begin{bmatrix} -M^{*}MM^{*} & 0 & 0 & M^{*} \\ 0 & A^{*}AA^{*} & 0 & A^{*} \\ 0 & 0 & B^{*}BB^{*} & B^{*} \\ \begin{bmatrix} A^{*} \\ B^{*} \end{bmatrix} & \begin{bmatrix} A^{*} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ B^{*} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix} - r(M) - r(A) - r(B) \\
&= r\begin{bmatrix} -M^{*}MM^{*} & 0 & 0 & M^{*} \\ -A^{*}AA^{*} & 0 & 0 & A^{*} \\ -B^{*}BB^{*} & 0 & 0 & B^{*} \\ 0 & A^{*} & 0 & 0 \\ 0 & 0 & B^{*} & 0 \end{bmatrix} - r(M) - r(A) - r(B) \\
&= r\begin{bmatrix} M^{*}MM^{*} & M^{*} \\ A^{*}AA^{*} & A^{*} \\ B^{*}BB^{*} & B^{*} \end{bmatrix} - r(M) \\
&= r\begin{bmatrix} MM^{*}M & AA^{*}A & BB^{*}B \\ M & A & B \end{bmatrix} - r(M) \\
&= r\begin{bmatrix} 0 & AA^{*}A - MM^{*}A & BB^{*}B - MM^{*}B \\ M & 0 & 0 \end{bmatrix} - r(M) \\
&= r[\,AA^{*}A - MM^{*}A,\ BB^{*}B - MM^{*}B\,] = r[\,AA^{*}B,\ BB^{*}A\,],
\end{aligned}
$$

as required in Part (a). Similarly, we can show Parts (b), (c) and (d). The results in Parts (e) and (f) follow immediately from Parts (a)—(d). □

A general result is given below, the proof is omitted.

Theorem 7.10. *Let $A = [\,A_1,\ A_2,\ \cdots,\ A_k\,] \in \mathcal{C}^{m \times n}$ be given, and denote $M = \begin{bmatrix} A_1^{\dagger} \\ \vdots \\ A_k^{\dagger} \end{bmatrix}$. Then*

$$
r(\,A^{\dagger} - M\,) = r(\,A^{\dagger}A - MA\,) = r[\,N_1N_1^{*}A_1,\ N_2N_2^{*}A_2,\ \cdots,\ N_kN_k^{*}A_k\,], \tag{7.14}
$$

where $N_i = [\,A_1,\ \cdots\ A_{i-1},\ A_{i+1},\ \cdots,\ A_k\,]$, $i = 1,\ 2,\ \cdots,\ k$. In particular,

$$
A^{\dagger} = M \Longleftrightarrow A^{\dagger}A = MA \Longleftrightarrow A_iA_j^{*} = 0 \quad \textit{for all } i \neq j. \tag{7.15}
$$

Theorem 7.11. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$ and $C \in \mathcal{C}^{l\times n}$ be given. Then*

(a) $r([A,\ B][A,\ B]^\dagger - (AA^\dagger + BB^\dagger)) = r[A,\ B] + 2r(A^*B) - r(A) - r(B)$.

(b) $r\left(\begin{bmatrix} A \\ C \end{bmatrix}^\dagger \begin{bmatrix} A \\ C \end{bmatrix} - (A^\dagger A + C^\dagger C)\right) = r\begin{bmatrix} A \\ C \end{bmatrix} + 2r(CA^*) - r(A) - r(C)$.

In particular,

(c) $[A,\ B][A,\ B]^\dagger = AA^\dagger + BB^\dagger \Longleftrightarrow A^*B = 0 \Longleftrightarrow [A,\ B]^\dagger = \begin{bmatrix} A^\dagger \\ B^\dagger \end{bmatrix}$.

(d) $\begin{bmatrix} A \\ C \end{bmatrix}^\dagger \begin{bmatrix} A \\ C \end{bmatrix} = A^\dagger A + C^\dagger C \Longleftrightarrow CA^* = 0 \Longleftrightarrow \begin{bmatrix} A \\ C \end{bmatrix}^\dagger = [A^\dagger,\ C^\dagger]$.

Proof. Let $M = [A,\ B]$. Then it follows by Eqs.(2.7), (1.8) and block elementary operations of matrices that

$$
\begin{aligned}
& r(MM^\dagger - AA^\dagger - BB^\dagger) \\
= \ & r\begin{bmatrix} -M^*MM^* & 0 & 0 & M^* \\ 0 & A^*AA^* & 0 & A^* \\ 0 & 0 & B^*BB^* & B^* \\ MM^* & AA^* & BB^* & 0 \end{bmatrix} - r(M) - r(A) - r(B) \\
= \ & r\begin{bmatrix} -M^*M & 0 & 0 & M^* \\ 0 & A^*A & 0 & A^* \\ 0 & 0 & B^*B & B^* \\ M & A & B & 0 \end{bmatrix} - r(M) - r(A) - r(B) \\
= \ & r\begin{bmatrix} 0 & 0 & 0 & 0 & A^* \\ 0 & 0 & 0 & 0 & B^* \\ A^*A & A^*B & A^*A & 0 & A^* \\ B^*A & B^*B & 0 & B^*B & B^* \\ A & B & A & B & 0 \end{bmatrix} - r(M) - r(A) - r(B) \\
= \ & r\begin{bmatrix} 0 & 0 & 0 & 0 & A^* \\ 0 & 0 & 0 & 0 & B^* \\ 0 & 0 & 0 & -A^*B & 0 \\ 0 & 0 & -B^*A & 0 & 0 \\ A & B & 0 & 0 & 0 \end{bmatrix} - r(M) - r(A) - r(B) \\
= \ & r(M) + 2r(A^*B) - r(A) - r(B),
\end{aligned}
$$

as required in Part (a). In the same way, we can show Part (b). We know from Part (a) that

$$
MM^\dagger = AA^\dagger + BB^\dagger \Longleftrightarrow r[A,\ B] = r(A) + r(B) - 2r(A^*B). \tag{7.16}
$$

On the other hand, observe from Eq.(1.2) that

$$r[\,A,\ B\,] = r(A) + r(\,B - AA^\dagger B\,) \geq r(A) + r(B) - r(AA^\dagger B)$$
$$= r(A) + r(B) - r(A^*B) \geq r(A) + r(B) - 2r(A^*B).$$

Thus Eq.(7.16) is also equivalent to $A^*B = 0$. In the similar manner, we can show Part (d). □

A general result is given below, the proof is omitted.

Corollary 7.12. *Let $A = [\,A_1,\ A_2,\ \cdots,\ A_k\,] \in \mathcal{C}^{m\times n}$ be given. Then*

$$r[AA^\dagger - (\,A_1A_1^\dagger + \cdots + A_kA_k^\dagger\,)] = r\begin{bmatrix} 0 & A_1^*A_2 & \cdots & A_1^*A_k \\ A_2^*A_1 & 0 & \cdots & A_2^*A_k \\ \vdots & \vdots & \ddots & \vdots \\ A_k^*A_1 & A_k^*A_2 & \cdots & 0 \end{bmatrix} + r(A) - r(A_1) - \cdots - r(A_k). \quad (7.17)$$

In particular,

$$AA^\dagger = A_1A_1^\dagger + \cdots + A_kA_k^\dagger \Longleftrightarrow A_i^*A_j = 0, \quad for\ all\ i \neq j. \quad (7.18)$$

Theorem 7.13. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$ and $C \in \mathcal{C}^{l\times n}$ be given. Then*

(a) $$r\left(\,[\,A,\ B\,]^\dagger - \begin{bmatrix} (E_BA)^\dagger \\ (E_AB)^\dagger \end{bmatrix}\right) = r(A) + r(B) - r[\,A,\ B\,].$$

(b) $$r\left(\begin{bmatrix} A \\ C \end{bmatrix}^\dagger - [\,(AF_C)^\dagger,\ (CF_A)^\dagger\,]\right) = r(A) + r(C) - r\begin{bmatrix} A \\ C \end{bmatrix}.$$

In particular,

(c) $$[\,A,\ B\,]^\dagger = \begin{bmatrix} (E_BA)^\dagger \\ (E_AB)^\dagger \end{bmatrix} \Longleftrightarrow r[\,A,\ B\,] = r(A) + r(B) \Longleftrightarrow R(A) \cap R(B) = \{0\}.$$

(d) $$\begin{bmatrix} A \\ C \end{bmatrix}^\dagger = [\,(AF_C)^\dagger,\ (CF_A)^\dagger\,] \Longleftrightarrow r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C) \Longleftrightarrow R(A^*) \cap R(C^*) = \{0\}.$$

Proof. Let $M = [\,A,\ B\,]$. Then it follows by Eqs.(2.7) and (1.8) and block elementary operations of matrices that

$$\begin{aligned} & r\left(\,[\,A,\ B\,]^\dagger - \begin{bmatrix} (E_BA)^\dagger \\ (E_AB)^\dagger \end{bmatrix}\right) \\ = \ & r\left(\,[\,A,\ B\,]^\dagger - \begin{bmatrix} I \\ 0 \end{bmatrix}(E_BA)^\dagger - \begin{bmatrix} 0 \\ I \end{bmatrix}(E_AB)^\dagger\right) \\ = \ & r\begin{bmatrix} -M^*MM^* & 0 & 0 & M^* \\ 0 & (E_BA)^*(E_BA)(E_BA)^* & 0 & (E_BA)^* \\ 0 & 0 & (E_AB)^*(E_AB)(E_AB)^* & (E_AB)^* \\ \begin{bmatrix} A^* \\ B^* \end{bmatrix} & \begin{bmatrix} (E_BA)^* \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ (E_AB)^* \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
& -r(M) - r(E_BA) - r(E_AB) \\
= \; & r\begin{bmatrix} -M^*MM^* & 0 & 0 & M^* \\ 0 & (E_BA)^*A(E_BA)^* & 0 & (E_BA)^* \\ 0 & 0 & (E_AB)^*B(E_AB)^* & (E_AB)^* \\ A^* & (E_BA)^* & 0 & 0 \\ B^* & 0 & (E_AB)^* & 0 \end{bmatrix} \\
& -r(M) - r(E_BA) - r(E_AB) \\
= \; & r\begin{bmatrix} 0 & M^*A(E_BA)^* & M^*B(E_AB)^* & M^* \\ 0 & (E_BA)^*A(E_BA)^* & 0 & (E_BA)^* \\ 0 & 0 & (E_AB)^*B(E_AB)^* & (E_AB)^* \\ A^* & 0 & 0 & 0 \\ B^* & 0 & 0 & 0 \end{bmatrix} - r(M) - r(E_BA) - r(E_AB) \\
= \; & r\begin{bmatrix} M^*A(E_BA)^* & M^*B(E_AB)^* & M^* \\ (E_BA)^*A(E_BA)^* & 0 & (E_BA)^* \\ 0 & (E_AB)^*B(E_AB)^* & (E_AB)^* \end{bmatrix} - r(E_BA) - r(E_AB) \\
= \; & r\begin{bmatrix} 0 & 0 & M^* \\ 0 & 0 & (E_BA)^* \\ 0 & 0 & (E_AB)^* \end{bmatrix} - r(E_BA) - r(E_AB) \\
= \; & r[\,M,\ E_BA,\ E_AB\,] - r(E_BA) - r(E_AB) \\
= \; & r[\,A,\ B\,] - r(E_BA) - r(E_AB) = r(A) + r(B) - r[\,A,\ B\,],
\end{aligned}
$$

as required for Part (a). Similarly, we can show Part (b). The results in Parts (c) and (d) follow immediately from Parts (a) and (b). □

A general results is given below.

Theorem 7.14. *Let $A = [\,A_1,\ A_2,\ \cdots,\ A_k\,] \in \mathcal{C}^{m\times n}$ be given. Then*

$$
r\left([\,A_1,\ A_2,\ \cdots,\ A_k\,]^\dagger - \begin{bmatrix} (E_{N_1}A_1)^\dagger \\ (E_{N_2}A_2)^\dagger \\ \vdots \\ (E_{N_n}A_k)^\dagger \end{bmatrix} \right) = r(A_1) + r(A_2) + \cdots + r(A_k) - r(A). \tag{7.19}
$$

where $N_i = [\,A_1,\ \cdots,\ A_{i-1},\ A_{i+1}, \cdots,\ A_k\,]$, $i = 1,\ 2,\ \cdots,\ k$. In particular,

$$
[\,A_1,\ A_2,\ \cdots,\ A_k\,]^\dagger = \begin{bmatrix} (E_{N_1}A_1)^\dagger \\ (E_{N_2}A_2)^\dagger \\ \vdots \\ (E_{N_k}A_k)^\dagger \end{bmatrix} \Longleftrightarrow r(A) = r(A_1) + r(A_2) + \cdots + r(A_k). \tag{7.20}
$$

Theorem 7.15. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$ and $C \in \mathcal{C}^{l\times n}$ be given. Then*

(a) $r\left([A,\ B][A,\ B]^{\dagger} - A(E_BA)^{\dagger} - B(E_AB)^{\dagger}\right) = r(A) + r(B) - r[A,\ B]$.

(b) $r\left(\begin{bmatrix} A \\ C \end{bmatrix}^{\dagger}\begin{bmatrix} A \\ C \end{bmatrix} - (AF_C)^{\dagger}A - (CF_A)^{\dagger}C\right) = r(A) + r(C) - r\begin{bmatrix} A \\ C \end{bmatrix}$.

In particular,

(c) $[A,\ B][A,\ B]^{\dagger} = A(E_BA)^{\dagger} + A(E_AB)^{\dagger} \Longleftrightarrow R(A) \cap R(B) = \{0\}$.

(d) $\begin{bmatrix} A \\ C \end{bmatrix}^{\dagger}\begin{bmatrix} A \\ C \end{bmatrix} = (AF_C)^{\dagger}A + (CF_A)^{\dagger}C \Longleftrightarrow R(A^*) \cap R(C^*) = \{0\}$.

The proof of Theorem 7.15 is much similar to that of Theorem 7.13, hence omitted.

Theorem 7.16. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$ and $C \in \mathcal{C}^{l\times n}$ be given. Then*

(a) $r\left([A,\ B]^{\dagger}[A,\ B] - \begin{bmatrix} A^{\dagger}A & 0 \\ 0 & B^{\dagger}B \end{bmatrix}\right) = r(A) + r(B) - r[A,\ B]$.

(b) $r\left(\begin{bmatrix} A \\ C \end{bmatrix}\begin{bmatrix} A \\ C \end{bmatrix}^{\dagger} - \begin{bmatrix} AA^{\dagger} & 0 \\ 0 & CC^{\dagger} \end{bmatrix}\right) = r(A) + r(C) - r\begin{bmatrix} A \\ C \end{bmatrix}$.

In particular,

(c) $[A,\ B]^{\dagger}[A,\ B] = \begin{bmatrix} A^{\dagger}A & 0 \\ 0 & B^{\dagger}B \end{bmatrix} \Longleftrightarrow r[A,\ B] = r(A) + r(B) \Longleftrightarrow R(A) \cap R(B) = \{0\}$.

(d) $\begin{bmatrix} A \\ C \end{bmatrix}\begin{bmatrix} A \\ C \end{bmatrix}^{\dagger} = \begin{bmatrix} AA^{\dagger} & 0 \\ 0 & CC^{\dagger} \end{bmatrix} \Longleftrightarrow r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C) \Longleftrightarrow R(A^*) \cap R(C^*) = \{0\}$.

Proof. Let $M = [A,\ B]$ and $N = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Then we find by Theorem 7.2(b) that

$$\begin{aligned} & r\left([A,\ B]^{\dagger}[A,\ B] - \begin{bmatrix} A^{\dagger}A & 0 \\ 0 & B^{\dagger}B \end{bmatrix}\right) \\ = \ & r(M^{\dagger}M - N^{\dagger}N) \\ = \ & 2r\begin{bmatrix} M \\ N \end{bmatrix} - r(M) - r(N) \\ = \ & 2r\begin{bmatrix} A & B \\ A & 0 \\ 0 & B \end{bmatrix} - r[A,\ B] - r(A) - r(B) \\ = \ & r(A) + r(B) - r[A,\ B], \end{aligned}$$

as required for Part (a). Similarly we can show Part (b). Parts (c) and (d) are direct consequences of Parts (a) and (b).

□

A general result is given below, the proof is omitted.

Corollary 7.17. *Let $A = [A_1, A_2, \cdots, A_k] \in \mathcal{C}^{m\times n}$ be given. Then*

$$r[A^\dagger A - \mathrm{diag}(A_1^\dagger A_1, A_2^\dagger A_2, \cdots, A_k^\dagger A_k)] = r(A_1) + r(A_2) + \cdots + r(A_k) - r(A). \tag{7.21}$$

In particular,

$$A^\dagger A = \mathrm{diag}(A_1^\dagger A_1, A_2^\dagger A_2, \cdots, A_k^\dagger A_k) \Longleftrightarrow r(A) = r(A_1) + r(A_2) + \cdots + r(A_k). \tag{7.22}$$

Theorem 7.18. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$ and $C \in \mathcal{C}^{l\times n}$ be given, and let $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$. Then*

$$r\left(A - [A, 0]M^\dagger \begin{bmatrix} A \\ 0 \end{bmatrix}\right) = r(A) + r(B) + r(C) - r(M). \tag{7.23}$$

or alternatively

$$r(M) = r(A) + r(B) + r(C) - r\left(A - [A, 0]M^\dagger \begin{bmatrix} A \\ 0 \end{bmatrix}\right). \tag{7.24}$$

In particular,

$$-A[I, 0]\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^\dagger \begin{bmatrix} I \\ 0 \end{bmatrix} A = A \tag{7.25}$$

holds if and only if

$$r(M) = r(A) + r(B) + r(C), \quad i.e., \quad R(A) \cap R(B) = \{0\}, \text{ and } R(A^*) \cap R(C^*) = \{0\}. \tag{7.26}$$

Proof. It follows by Eq.(2.1) that

$$\begin{aligned}
r\left(A - [A, 0]M^\dagger \begin{bmatrix} A \\ 0 \end{bmatrix}\right) &= r\begin{bmatrix} M^*MM^* & M^*\begin{bmatrix} A \\ 0 \end{bmatrix} \\ [A, 0]M^* & A \end{bmatrix} - r(M) \\
&= r\begin{bmatrix} M^*MM^* - M^*\begin{bmatrix} I \\ 0 \end{bmatrix} A[I, 0]M^* & 0 \\ 0 & A \end{bmatrix} - r(M) \\
&= r\left(M^*\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} M^*\right) + r(A) - r(M) \\
&= r(A) + r(B) + r(C) - r(M),
\end{aligned}$$

as required for Eq.(7.23). $\square$

Theorem 7.19. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$ and $C \in \mathcal{C}^{l\times n}$ be given. Then*

$$r\left(A + [0, B]\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ C \end{bmatrix}\right) = r[A, B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{7.27}$$

or alternatively

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r[A,\ B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\left(A + [0,\ B]\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{\dagger}\begin{bmatrix} 0 \\ C \end{bmatrix}\right). \tag{7.28}$$

Proof. Let $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$. Then it follows by Eq.(2.1) and block elementary operation that

$$\begin{aligned}
& r\left(A + [0,\ B]M^{\dagger}\begin{bmatrix} 0 \\ C \end{bmatrix}\right) \\
= & \ r\begin{bmatrix} M^*MM^* & M^*\begin{bmatrix} 0 \\ C \end{bmatrix} \\ [0,\ B]M^* & -A \end{bmatrix} - r(M) \\
= & \ r\begin{bmatrix} M^*MM^* - M^*\begin{bmatrix} 0 \\ I \end{bmatrix}C[I,\ 0]M^* & M^*\begin{bmatrix} 0 \\ C \end{bmatrix} \\ [A,\ B]M^* & -A \end{bmatrix} - r(M) \\
= & \ r\begin{bmatrix} M^*MM^* - M^*\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}M^* & M^*\begin{bmatrix} A \\ C \end{bmatrix} \\ [A,\ B]M^* & -A \end{bmatrix} - r(M) \\
= & \ r\begin{bmatrix} 0 & M^*\begin{bmatrix} A \\ C \end{bmatrix} \\ [A,\ B]M^* & -A \end{bmatrix} - r(M) \\
= & \ r\begin{bmatrix} 0 & \begin{bmatrix} A \\ C \end{bmatrix} \\ [A,\ B] & -A \end{bmatrix} - r(M) = r[A,\ B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r(M).
\end{aligned}$$

as requird for Eq.(7.27). □

It is easy to derive from Eq.(1.6) that

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \geq r\begin{bmatrix} A \\ C \end{bmatrix} + r[A,\ B] - r(A).$$

Now replacing A by $A - BXC$ in the above inequality, where X is arbitrary, we obtain

$$r(A - BXC) \geq r[A,\ B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{7.29}$$

This rank inequality implies that the quantity in the right-hand side of Eq.(7.29) is a lower bound for the rank of $A - BXC$ with respect to the choice of X. Combining Eqs.(7.27) and (7.29), we immediately obtain

$$\min_X r(A - BXC) = r[A,\ B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \tag{7.30}$$

and a matrix satisfying Eq.(7.30) is given by

$$X = -[0,\ I_l]\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{\dagger}\begin{bmatrix} 0 \\ I_k \end{bmatrix}. \tag{7.31}$$

Theorem 7.20. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$ and $C \in \mathcal{C}^{l\times n}$ be given. Then*

(a) $r\left(\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^{\dagger} - \begin{bmatrix} AA^{\dagger} & 0 \\ 0 & DD^{\dagger} \end{bmatrix}\right) = r(D) - r(A) + 2r[A,\ B] - r\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}.$

(b) $r\left(\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^{\dagger}\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} - \begin{bmatrix} A^{\dagger}A & 0 \\ 0 & D^{\dagger}D \end{bmatrix}\right) = r(A) - r(D) + 2r\begin{bmatrix} B \\ D \end{bmatrix} - r\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}.$

In particular,

(c) $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^{\dagger} = \begin{bmatrix} AA^{\dagger} & 0 \\ 0 & DD^{\dagger} \end{bmatrix} \Longleftrightarrow R(B) \subseteq R(A).$

(d) $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^{\dagger}\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \begin{bmatrix} A^{\dagger}A & 0 \\ 0 & D^{\dagger}D \end{bmatrix} \Longleftrightarrow R(B^*) \subseteq R(D^*).$

Proof. Let $M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ and $N = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$. Then we find by Theorem 7.2(b) that

$$\begin{aligned} r(MM^{\dagger} - NN^{\dagger}) &= 2r[M,\ N] - r(M) - r(N) \\ &= 2r\begin{bmatrix} A & B & A & 0 \\ 0 & D & 0 & D \end{bmatrix} - r(M) - r(A) - r(D) \\ &= 2r[A,\ B] + r(D) - r(M) - r(A), \end{aligned}$$

as required for Part (a). Similarly we can show Part (b). Observe that

$$r(D) - r(A) + 2r[A,\ B] - r\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = (r[A,\ B] - r(A)) + \left(r(D) + r[A,\ B] - r\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}\right),$$

and

$$r(A) - r(D) + 2r\begin{bmatrix} B \\ D \end{bmatrix} - r\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \left(r\begin{bmatrix} B \\ D \end{bmatrix} - r(D)\right) + \left(r(A) + r\begin{bmatrix} B \\ D \end{bmatrix} - r\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}\right).$$

Thus (c) and (d) follow. $\square$

A general result is given below, the proof is omitted.

Theorem 7.21. *Let*

$$M = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ & A_{22} & \cdots & A_{2k} \\ & & \ddots & \vdots \\ & & & A_{kk} \end{bmatrix} \in \mathcal{C}^{m\times n}$$

be given, and let $A = \mathrm{diag}(\, A_{11},\ A_{22},\ \cdots,\ A_{kk}\,)$. *Then*

(a) $r\left[\, MM^{\dagger} - \mathrm{diag}(\, A_{11}A_{11}^{\dagger},\ \cdots,\ A_{kk}A_{kk}^{\dagger}\,)\right] = 2r[\, M,\ A\,] - r(M) - r(A).$

(b) $r\left[\, M^{\dagger}M - \mathrm{diag}(\, A_{11}^{\dagger}A_{11},\ \cdots,\ A_{kk}^{\dagger}A_{kk}\,)\right] = 2r\begin{bmatrix} M \\ A \end{bmatrix} - r(M) - r(A).$

In particular,

(c) $MM^{\dagger} = \mathrm{diag}(\, A_{11}A_{11}^{\dagger},\ \cdots,\ A_{kk}A_{kk}^{\dagger}\,) \Longleftrightarrow R(M) = R(A) \Longleftrightarrow R(A_{ij}) \subseteq R(A_{ii}),\ j = i+1, \cdots,\ k,\ i = 1, \cdots,\ k-1.$

(d) $M^{\dagger}M = \mathrm{diag}(\, A_{11}^{\dagger}A_{11},\ \cdots,\ A_{kk}^{\dagger}A_{kk}\,) \Longleftrightarrow R(M^{*}) = R(A^{*}) \Longleftrightarrow R(A_{ij}) \subseteq R(A_{jj}),\ j = 2, \cdots,\ k,\ i = 1, \cdots,\ j-1.$

Chapter 8

Reverse order laws for Moore-Penrose inverses of products of matrices

Reverse order laws for generalized inverses of products of matrices have been an attractive topic in the theory of generalized inverses of matrices, for these laws can reveal essential relationships between generalized inverses of products of matrices and generalized inverses of each matrix in the products. Various results on reverse order laws related to inner inverses, reflexive inner inverses, Moore-Penrose inverses, group inverses, Drazin inverses, and weighted Moore-Penrose inverses of products of matrices have widely been established by lot of authors (see, e.g., [7], [8], [20], [27], [28], [31], [32], [41], [86], [87], [90], [92], [94], [102], [103], [104]). In this chapter, we shall present some rank equalities related to products of Moore-Penrose inverses of matrices, and then derive from them various types of reverse order laws for Moore-Penrose inverses of products of matrices.

Theorem 8.1. *Let $A \in \mathcal{C}^{m\times n}$ and $B \in \mathcal{C}^{n\times p}$ be given. Then*

$$r(\, AB - ABB^{\dagger}A^{\dagger}AB \,) = r[\, A^*, \ B \,] + r(AB) - r(A) - r(B). \tag{8.1}$$

In particular, the following five statements are equivalent:

(a) $B^{\dagger}A^{\dagger} \in \{(AB)^-\}$, *i.e.*, $B^{\dagger}A^{\dagger}$ *is an inner inverse of* AB.

(b) $r[\, A^*, \ B \,] = r(A) + r(B) - r(AB)$.

(c) $\dim[R(A) \cap R(B^*)] = r(AB)$.

(d) $r(\, B - A^{\dagger}AB \,) = r(B) - r(A^{\dagger}AB)$, *i.e.*, $A^{\dagger}AB \leq_{rs} B$.

(e) $r(\, A - ABB^{\dagger} \,) = r(A) - r(ABB^{\dagger})$, *i.e.*, $ABB^{\dagger} \leq_{rs} A$.

Proof. Applying Eqs.(2.8) and (1.7) to $AB - ABB^{\dagger}A^{\dagger}AB$, we obtain

$$\begin{aligned} r(\, AB - ABB^{\dagger}A^{\dagger}AB \,) &= r\begin{bmatrix} B^*A^* & B^*BB^* & 0 \\ A^*AA^* & 0 & A^*AB \\ 0 & ABB^* & -AB \end{bmatrix} - r(A) - r(B) \\ &= r\begin{bmatrix} B^*A^* & B^*B & 0 \\ AA^* & 0 & AB \\ 0 & AB & -AB \end{bmatrix} - r(A) - r(B) \\ &= r\begin{bmatrix} B^*A^* & B^*B \\ AA^* & AB \end{bmatrix} + r(AB) - r(A) - r(B) \\ &= r\left(\begin{bmatrix} A \\ B^* \end{bmatrix} [\, A^*, \ B \,] \right) + r(AB) - r(A) - r(B) \end{aligned}$$

$$
\begin{aligned}
&= r([A^*,\ B]^*[A^*,\ B]) + r(AB) - r(A) - r(B) \\
&= r[A^*,\ B] + r(AB) - r(A) - r(B).
\end{aligned}
$$

Thus we have Eq.(8.1). The equivalence of Parts (a) and (b) follows immediately from Eq.(8.1). The equivalence of Parts (b) and (c) follows from the simple fact

$$r[A^*,\ B] = r(A) + r(B) - \dim[R(A^*) \cap R(B)].$$

The equivalence of Parts (b), (d) and (e) follows from Eqs.(1.2) and (1.3). $\square$

The rank formula (8.1) was established by Baksalary and Styan (1993, pp. 2) in an alternative form

$$r(AE_BF_AB) = r[A^*,\ B] + r(AB) - r(A) - r(B). \tag{8.1'}$$

Observe that

$$A(I - BB^\dagger)(I - A^\dagger A)B = -AB + ABB^\dagger A^\dagger AB.$$

Thus Eq.(8.1') is exactly Eq.(8.1). Some extensions and applications of Eq.(8.1') in mathematics statistics were also considered by Baksalary and Styan (1993). But in this thesis we only consider the application of Eq.(8.1) to the reverse order law $B^\dagger A^\dagger \in \{(AB)^-\}$. In addition, the results in Theorem 8.1 can also be extended to a product of n matrices. The corresponding results were presented by the author in [94].

Theorem 8.2. *Let $A \in \mathcal{C}^{m\times n}$ and $B \in \mathcal{C}^{n\times p}$ be given. Then*

(a) $r[(AB)(AB)^\dagger - (AB)(B^\dagger A^\dagger)] = r[B,\ A^*AB] - r(B) = r(A * AB - BB^\dagger A^*AB)$.

(b) $r[(AB)^\dagger(AB) - (B^\dagger A^\dagger)(AB)] = r\begin{bmatrix} A \\ ABB^* \end{bmatrix} - r(A) = r(ABB^* - ABB^*A^\dagger A)$.

(c) $(AB)(AB)^\dagger = (AB)(B^\dagger A^\dagger) \Longleftrightarrow A * AB = BB^\dagger A^*AB \Longleftrightarrow R(A^*AB) \subseteq R(B) \Longleftrightarrow B^\dagger A^\dagger \subseteq \{(AB)^{(1,2,3)}\}$.

(d) $(AB)^\dagger(AB) = (B^\dagger A^\dagger)(AB) \Longleftrightarrow ABB^* = ABB^*A^\dagger A \Longleftrightarrow R(BB^*A^*) \subseteq R(A^*) \Longleftrightarrow B^\dagger A^\dagger \subseteq \{(AB)^{(1,2,4)}\}$.

(e) *The following four statements are equivalent:*

(1) $(AB)^\dagger = B^\dagger A^\dagger$.

(2) $(AB)(AB)^\dagger = (AB)(B^\dagger A^\dagger)$ *and* $(AB)^\dagger(AB) = (B^\dagger A^\dagger)(AB)$.

(3) $A * AB = BB^\dagger A^*AB$ *and* $ABB^* = ABB^*A^\dagger A$.

(4) $R(A^*AB) \subseteq R(B)$ *and* $R(BB^*A^*) \subseteq R(A^*)$.

Proof. Let $N = AB$. Then by Eqs.(2.1), (2.7) and (1.8), it follows that

$$
\begin{aligned}
r(NN^\dagger - NB^\dagger A^\dagger) &= r\begin{bmatrix} N^*NN^* & N^* \\ NN^* & NB^\dagger A^\dagger \end{bmatrix} - r(N) \\
&= r\begin{bmatrix} 0 & N^* - N^*NB^\dagger A^\dagger \\ NN^* & 0 \end{bmatrix} - r(N) \\
&= r(N^* - N^*NB^\dagger A^\dagger)
\end{aligned}
$$

$$= r\begin{bmatrix} B^*A^* & B^*BB^* & 0 \\ A^*AA^* & 0 & A^* \\ 0 & N^*NB^* & -N^* \end{bmatrix} - r(A) - r(B)$$

$$= r\begin{bmatrix} B^*A^* & B^*B & 0 \\ 0 & 0 & A^* \\ B^*A^*AA^* & N^*N & 0 \end{bmatrix} - r(A) - r(B)$$

$$= r\begin{bmatrix} B^*A^* & B^*B \\ B^*A^*AA^* & N^*N \end{bmatrix} - r(B)$$

$$= r\begin{bmatrix} B^*B & B^*A^*AB \\ AB & AA^*AB \end{bmatrix} - r(B)$$

$$= r\left(\begin{bmatrix} B^* \\ A \end{bmatrix}[B,\ A^*AB]\right) - r(B) = r[B,\ A^*AB] - r(B).$$

as required for the first equality in Part (a). Applying Eq.(1.2) to it the block matrix in it yields the second equality in Part (a). Similarly, we can establish Part (b). The results in Parts (c) and (d) are direct consequences of Parts (a) and (b). The result in Part (e) follows directly from Parts (c) and (d). □

The result in Theorem 8.2(e) is well known, see, e.g., Arghiriade (1967), Rao and Mitra (1971), Ben-Israel and Greville (1980), Campbell and Meyer (1991). Now it can be regarded as direct consequences of some rank equalities related to Moore-Penrose inverses of products of two matrices. we next present another group rank equalities related to Moore-Penrose inverses of products of two matrices.

Theorem 8.3. *Let $A \in \mathcal{C}^{m\times n}$ and $B \in \mathcal{C}^{n\times p}$ be given. Then*

(a) $r[\,ABB^\dagger - (AB)(AB)^\dagger A\,] = r[\,B,\ A^*AB\,] - r(B)$.

(b) $r[\,A^\dagger AB - B(AB)^\dagger(AB)\,] = r\begin{bmatrix} A \\ ABB^* \end{bmatrix} - r(A)$.

(c) $r[\,A^*ABB^\dagger - BB^\dagger A^*A\,] = 2r[\,B,\ A^*AB\,] - 2r(B)$.

(d) $r[\,A^\dagger ABB^* - BB^*A^\dagger A\,] = 2r\begin{bmatrix} A \\ ABB^* \end{bmatrix} - 2r(A)$.

(e) $(AB)(AB)^\dagger A = ABB^\dagger \Longleftrightarrow A^*ABB^\dagger = BB^\dagger A^*A \Longleftrightarrow R(A^*AB) \subseteq R(B) \Longleftrightarrow B^\dagger A^\dagger \subseteq \{\,(AB)^{(1,2,3)}\,\}$.

(f) $A^\dagger AB = B(AB)^\dagger(AB) \Longleftrightarrow A^\dagger ABB^* = BB^*A^\dagger A \Longleftrightarrow R(BB^*A^*) \subseteq R(A^*) \Longleftrightarrow B^\dagger A^\dagger \subseteq \{\,(AB)^{(1,2,4)}\,\}$.

(e) *The following statements are equivalent:*

(1) $(AB)^\dagger = B^\dagger A^\dagger$.

(2) $(AB)(AB)^\dagger A = ABB^\dagger$ *and* $A^\dagger AB = B(AB)^\dagger(AB)$.

(3) $A^*ABB^\dagger = BB^\dagger A^*A$ *and* $A^\dagger ABB^* = BB^*A^\dagger A$.

Proof. We only show Part (b). Note that $AA^\dagger$ and $(AB)^\dagger(AB)$ are idempotent. We have by Eq.(3.1) that

$$\begin{aligned} & r[A^\dagger AB - B(AB)^\dagger(AB)] \\ = & \; r\begin{bmatrix} A^\dagger AB \\ (AB)^\dagger(AB) \end{bmatrix} + r[B(AB)^\dagger(AB), \; A^\dagger A] - r(A^\dagger A) - r[(AB)^\dagger(AB)] \\ = & \; r(AB) + r[B(AB)^*, \; A^*] - r(A) - r(AB) \\ = & \; r[BB^*A^*, \; A^*] - r(A), \end{aligned}$$

as required. □

The result in Theorem 8.3(e) was established by Greville (1966). We next consider ranks of matrix expressions involving Moore-Penrose inverses of products of three matrices, and then present their consequences related to reverse order laws.

Theorem 8.4. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{n\times p}$ and $C \in \mathcal{C}^{p\times q}$ be given, and let $M = ABC$. Then*

$$r[M - M(BC)^\dagger B(AB)^\dagger M] = r\left(\begin{bmatrix} (BC)^* \\ A \end{bmatrix} B[(AB)^*, \; C]\right) + r(M) - r(AB) - r(BC). \tag{8.2}$$

In particular,

$$(BC)^\dagger B(AB)^\dagger \in \{(ABC)^-\} \Longleftrightarrow r\left(\begin{bmatrix} (BC)^* \\ A \end{bmatrix} B[(AB)^*, \; C]\right) = r(AB) + r(BC) - r(M). \tag{8.3}$$

Proof. Applying Eqs.(2.8) and the rank cancellation law (1.8) to $M - M(BC)^\dagger B(AB)^\dagger M$, we obtain

$$\begin{aligned} & r[M - M(BC)^\dagger B(AB)^\dagger M] \\ = & \; r\begin{bmatrix} (BC)^*B(AB)^* & (BC)^*(BC)(BC)^* & 0 \\ (AB)^*(AB)(AB)^* & 0 & (AB)^*M \\ 0 & M(BC)^* & -M \end{bmatrix} - r(AB) - r(BC) \\ = & \; r\begin{bmatrix} (BC)^*B(AB)^* & (BC)^*(BC) & 0 \\ (AB)(AB)^* & 0 & M \\ 0 & M & -M \end{bmatrix} - r(AB) - r(BC) \\ = & \; r\begin{bmatrix} (BC)^*B(AB)^* & (BC)^*(BC) \\ (AB)(AB)^* & M \end{bmatrix} + r(M) - r(AB) - r(BC) \\ = & \; r\left(\begin{bmatrix} (BC)^* \\ A \end{bmatrix} B[(AB)^*, \; C]\right) + r(M) - r(AB) - r(BC). \end{aligned}$$

Thus we have Eqs.(8.2) and (8.3). □

Theorem 8.5. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{n\times p}$ and $C \in \mathcal{C}^{p\times q}$ be given, and let $M = ABC$. Then*

(a) *The rank of* $M^{\dagger} - (BC)^{\dagger}B(AB)^{\dagger}$ *is*

$$r[\,M^{\dagger} - (BC)^{\dagger}B(AB)^{\dagger}\,] = r\left(\begin{bmatrix} (BC)^{*} \\ M^{*}A \end{bmatrix} B[\,(AB)^{*},\ CM^{*}\,]\right) - r(M). \tag{8.4}$$

(b) *The following three statements are equivalent:*

(1) $(ABC)^{\dagger} = (BC)^{\dagger}B(AB)^{\dagger}$.

(2) $r\begin{bmatrix} MM^{*}M & M(BC)^{*}(BC) \\ (AB)(AB)^{*}M & ABB^{*}BC \end{bmatrix} = r(ABC)$.

(3) $ABB^{*}BC = AB(BCM^{\dagger}AB)^{*}BC$.

(c) *If* $r(ABC) = r(B)$, *then*

$$(ABC)^{\dagger} = (BC)^{\dagger}B(AB)^{\dagger} \quad \textit{and} \quad (ABC)^{\dagger} = (B^{\dagger}BC)^{\dagger}B^{\dagger}(ABB^{\dagger})^{\dagger}. \tag{8.5}$$

Proof. Applying Eq.(2.12) to $M^{\dagger} - (BC)^{\dagger}B(AB)^{\dagger}$, we obtain

$$\begin{aligned}
& r[\,M^{\dagger} - (BC)^{\dagger}B(AB)^{\dagger}\,] \\
= \ & r\begin{bmatrix} M^{*}MM^{*} & 0 & 0 & M^{*} \\ 0 & (BC)^{*}B(AB)^{*} & (BC)^{*}(BC)(BC)^{*} & 0 \\ 0 & (AB)^{*}(AB)(AB)^{*} & 0 & (AB)^{*} \\ M^{*} & 0 & (BC)^{*} & 0 \end{bmatrix} - r(M) - r(AB) - r(BC) \\
= \ & r\begin{bmatrix} M^{*}MM^{*} & -M^{*}(AB)(AB)^{*} & 0 & 0 \\ -(BC)^{*}(BC)M^{*} & (BC)^{*}B(AB)^{*} & 0 & 0 \\ 0 & 0 & 0 & (AB)^{*} \\ 0 & 0 & (BC)^{*} & 0 \end{bmatrix} - r(M) - r(AB) - r(BC) \\
= \ & r\begin{bmatrix} (BC)^{*}B(AB)^{*} & (BC)^{*}(BC)M^{*} \\ M^{*}(AB)(AB)^{*} & M^{*}MM^{*} \end{bmatrix} - r(M) \\
= \ & r\left(\begin{bmatrix} (BC)^{*} \\ M^{*}A \end{bmatrix} B[\,(AB)^{*},\ CM^{*}\,]\right) - r(M),
\end{aligned}$$

as required for Eq.(8.4). Then the equivalence of Statements (1) and (2) in Part (b) follows immediately from Eq.(8.4), and the equivalence of Statements (2) and (3) in Part (b) follows from Lemma 1.2(f). If $r(ABC) = r(B)$, then

$$r\begin{bmatrix} MM^{*}M & M(BC)^{*}(BC) \\ (AB)(AB)^{*}M & ABB^{*}BC \end{bmatrix} \geq r(MM^{*}M) = r(M).$$

On the other hand,

$$r\begin{bmatrix} MM^{*}M & M(BC)^{*}(BC) \\ (AB)(AB)^{*}M & ABB^{*}BC \end{bmatrix} = r\left(\begin{bmatrix} MC^{*} \\ AB \end{bmatrix} B^{*}[\,A^{*}M,\ BC\,]\right) \leq r(B) = r(M).$$

Thus we have

$$r\begin{bmatrix} MM^*M & M(BC)^*(BC) \\ (AB)(AB)^*M & ABB^*BC \end{bmatrix} = r(M).$$

Thus according to Statements (1) and (2) in Part (b), we know that the first equality in Eq.(8.5) is true. The second equality in Eq.(8.5) follows from writing $ABC = ABB^{\dagger}BC$ and then applying the first equality to it. □

Theorem 8.6. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{n\times p}$ and $C \in \mathcal{C}^{p\times q}$ be given, and let $M = ABC$. Then*

$$r[\,B^{\dagger} - (AB)^{\dagger}M(BC)^{\dagger}\,] = r\begin{bmatrix} M & (AB)(AB)^* \\ (BC)^*(BC) & (BC)^*B(AB)^* \end{bmatrix} + r(B) - r(AB) - r(BC). \tag{8.6}$$

In particular,

$$B^{\dagger} = (AB)^{\dagger}M(BC)^{\dagger} \iff r\begin{bmatrix} M & (AB)(AB)^* \\ (BC)^*(BC) & (BC)^*B(AB)^* \end{bmatrix} = r(AB) + r(BC) - r(B).$$

Proof. Applying Eq.(2.11) to $B^{\dagger} - (AB)^{\dagger}M(BC)^{\dagger}$, we obtain

$$\begin{aligned}
&r[\,B^{\dagger} - (AB)^{\dagger}M(BC)^{\dagger}\,] \\
&= r\begin{bmatrix} B^*BB^* & 0 & 0 & B^* \\ 0 & (AB)^*M(BC)^* & (AB)^*(AB)(AB)^* & 0 \\ 0 & (BC)^*(BC)(BC)^* & 0 & (BC)^* \\ B^* & 0 & (AB)^* & 0 \end{bmatrix} - r(B) - r(AB) - r(BC) \\
&= r\begin{bmatrix} 0 & 0 & 0 & B^* \\ 0 & (AB)^*M(BC)^* & (AB)^*(AB)(AB)^* & 0 \\ 0 & (BC)^*(BC)(BC)^* & (BC)^*B(AB)^* & 0 \\ B^* & 0 & 0 & 0 \end{bmatrix} - r(B) - r(AB) - r(BC) \\
&= r\begin{bmatrix} (AB)^*M(BC)^* & (AB)^*(AB)(AB)^* \\ (BC)^*(BC)(BC)^* & (BC)^*B(AB)^* \end{bmatrix} + r(B) - r(AB) - r(BC) \\
&= r\begin{bmatrix} M & (AB)(AB)^* \\ (BC)^*(BC) & (BC)^*B(AB)^* \end{bmatrix} + r(B) - r(AB) - r(BC),
\end{aligned}$$

as required for Eq.(8.6). □

Theorem 8.7. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{n\times p}$ and $C \in \mathcal{C}^{p\times q}$ be given. Then*

(a) $(ABC)^{\dagger} = (A^{\dagger}ABC)^{\dagger}B(ABCC^{\dagger})^{\dagger}$.

(b) $(ABC)^{\dagger} = [(AB)^{\dagger}ABC]^{\dagger}B^{\dagger}[ABC(BC)^{\dagger}]^{\dagger}$.

(c)(Cline, 1964) $(AB)^{\dagger} = (A^{\dagger}AB)^{\dagger}(ABB^{\dagger})^{\dagger}$.

(d) *If $R(C) \subseteq R[(AB)^*]$ and $R(A^*) \subseteq R(BC)$, then $(ABC)^{\dagger} = C^{\dagger}B^{\dagger}A^{\dagger}$.*

Proof. Write ABC as $ABC = A(A^\dagger ABCC^\dagger)C$. Then it is evident that

$$r(A^\dagger ABCC^\dagger) = r(ABC), \quad R[(ABCC^\dagger)^\dagger] \subseteq R(C), \quad \text{and} \quad R[((A^\dagger ABC)^\dagger)^*] \subseteq R(A^*).$$

Thus by Eq.(8.5), we find that

$$\begin{aligned}(ABC)^\dagger &= [\,A(A^\dagger ABCC^\dagger)C\,]^\dagger \\ &= (A^\dagger ABC)^\dagger A^\dagger ABCC^\dagger(ABCC^\dagger)^\dagger \\ &= (A^\dagger ABC)^\dagger B(ABCC^\dagger)^\dagger,\end{aligned}$$

as required for Part (a). On the other hand, we can write ABC as $ABC = (AB)B^\dagger(BC)$. Applying the equality in Part(a) to it yields

$$(ABC)^\dagger = [\,(AB)B^\dagger(BC)\,]^\dagger = [(AB)^\dagger ABC]^\dagger B^\dagger[ABC(BC)^\dagger]^\dagger.$$

as required for Part (b). Let B be identity matrix and replace C by B in the result in Part (a). Then we have the result in Part (c). The two conditions in Part (d) are equivalent to

$$(AB)^\dagger ABC = C, \quad \text{and} \quad ABC(BC)^\dagger = A.$$

In that case. the result in Part (b) reduces to the result in Part (d). □

In the remainder of this chapter we consider the relationship of $(ABC)^\dagger$ and the reverse order product $C^\dagger B^\dagger A^\dagger$, and present necessary and sufficient conditions for $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$ to hold. Some of the results were presented by the author in [91] and [93].

Lemma 8.8[93]. *Suppose that A_1, A_2, A_3, B_1 and B_2 satisfy the the following range inclusions*

$$R(B_i) \subseteq R(A_{i+1}), \quad \textit{and} \quad R(B_i^*) \subseteq R(A_i^*), \qquad i = 1,\ 2. \tag{8.7}$$

Then

$$\begin{bmatrix} 0 & 0 & A_1 \\ 0 & A_2 & B_1 \\ A_3 & B_2 & 0 \end{bmatrix}^\dagger = \begin{bmatrix} A_3^\dagger B_2 A_2^\dagger B_1 A_1^\dagger & -A_3^\dagger B_2 A_2^\dagger & A_3^\dagger \\ -A_2^\dagger B_1 A_1^\dagger & A_2^\dagger & 0 \\ A_1^\dagger & 0 & 0 \end{bmatrix}. \tag{8.8}$$

Proof. The range inclusions in Eq.(8.7) are equivalent to

$$A_{i+1}A_{i+1}^\dagger B_i = B_i, \quad \text{and} \quad B_i A_i^\dagger A_i = B_i, \qquad i = 1,\ 2.$$

In that case, it is easy to verify that the block matrix in the right-hand side of (8.8) and the given block matrix in the left-hand side of Eq.(8.8) satisfy the four Penrose equations. Thus Eq.(8.8) holds. □

Lemma 8.9. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{n\times p}$ and $C \in \mathcal{C}^{p\times q}$ be given. Then the product $C^\dagger B^\dagger A^\dagger$ can be written as*

$$C^\dagger B^\dagger A^\dagger = [\,I_q,\ 0,\ 0\,]\begin{bmatrix} 0 & 0 & AA^* \\ 0 & B^*BB^* & B^*A^* \\ C^*C & C^*B^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} I_m \\ 0 \\ 0 \end{bmatrix} := PJ^\dagger Q, \tag{8.9}$$

where the block matrices P, J and Q satisfy

$$r(J) = r(A) + r(B) + r(C), \quad R(QA) \subseteq R(J), \quad \text{and} \quad R[(AP)^*] \subseteq R(J^*). \tag{8.10}$$

Proof. Observe that

$$R(B^*A^*) \subseteq R(B^*BB^*), \;\; R(AB) \subseteq R(AA^*), \;\; R(C^*B^*) \subseteq R(C^*C), \;\; R(BC) \subseteq R(BB^*B).$$

as well as the three basic equalities on the Moore-Penrose inverse of a matrix

$$N^\dagger = N^*(N^*NN^*)^\dagger N^*, \qquad N^\dagger = (N^*N)^\dagger N^*, \qquad N^\dagger = N^*(NN^*)^\dagger.$$

Thus we find by Eq.(8.8) that

$$\begin{bmatrix} 0 & 0 & AA^* \\ 0 & B^*BB^* & B^*A \\ C^*C & C^*B^* & 0 \end{bmatrix}^\dagger = \begin{bmatrix} (C^*C)^\dagger C^*B^*(BB^*B)^\dagger B^*A^*(AA^*)^\dagger & * & * \\ * & * & 0 \\ * & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} C^\dagger B^\dagger A^\dagger & * & * \\ * & * & 0 \\ * & 0 & 0 \end{bmatrix}.$$

Hence we have Eq.(8.9). The properties in Eq.(8.10) are obvious. $\square$

Theorem 8.10. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{n\times p}$ and $C \in \mathcal{C}^{p\times q}$ be given and let $M = ABC$. Then*

$$r[\, M - M(C^\dagger B^\dagger A^\dagger)M \,] = r\begin{bmatrix} -M^* & 0 & C^*C \\ 0 & BB^*B & BC \\ AA^* & AB & 0 \end{bmatrix} - r(A) - r(B) - r(C) + r(M). \tag{8.11}$$

In particular,

$$C^\dagger B^\dagger A^\dagger \in \{(ABC)^-\} \Longleftrightarrow r\begin{bmatrix} -M^* & 0 & C^*C \\ 0 & BB^*B & BC \\ AA^* & AB & 0 \end{bmatrix} = r(A) + r(B) + r(C) - r(M). \tag{8.12}$$

Proof. It follows from Eqs.(1.7) and (8.9) that

$$\begin{aligned} r[\, M - M(C^\dagger B^\dagger A^\dagger)M \,] &= r(\, M - MPJ^\dagger QM \,) \\ &= \begin{bmatrix} J & QM \\ MP & M \end{bmatrix} - r(J) \\ &= \begin{bmatrix} J - QMP & 0 \\ 0 & M \end{bmatrix} - r(J) \\ &= r(\, J - QMP \,) + r(M) - r(J) \end{aligned}$$

$$= r\begin{bmatrix} -M & 0 & AA^* \\ 0 & B^*BB^* & B^*A^* \\ C^*C & C^*B^* & 0 \end{bmatrix} + r(M) - r(J)$$

$$= r\begin{bmatrix} -M^* & 0 & C^*C \\ 0 & BB^*B & BC \\ AA^* & AB & 0 \end{bmatrix} + r(M) - r(A) - r(B) - r(C),$$

as required for Eq.(8.11). □

Theorem 8.11. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{n\times p}$ and $C \in \mathcal{C}^{p\times q}$ be given and let $M = ABC$. Then*

$$r(\, M^\dagger - C^\dagger B^\dagger A^\dagger \,) = r\begin{bmatrix} -MM^*M & 0 & MC^*C \\ 0 & BB^*B & BC \\ AA^*M & AB & 0 \end{bmatrix} - r(B) - r(M). \tag{8.13}$$

In particular,

$$(ABC)^\dagger = C^\dagger B^\dagger A^\dagger \Longleftrightarrow \begin{bmatrix} -MM^*M & 0 & MC^*C \\ 0 & BB^*B & BC \\ AA^*M & AB & 0 \end{bmatrix} = r(B) + r(ABC). \tag{8.14}$$

Proof. Notice that

$$C^\dagger CM^\dagger AA^\dagger = M^\dagger \quad \text{and} \quad C^\dagger C(C^\dagger B^\dagger A^\dagger)AA^\dagger = C^\dagger B^\dagger A^\dagger.$$

We first get the following

$$\begin{aligned} r(\, M^\dagger - C^\dagger B^\dagger A^\dagger \,) &= r(\, CM^\dagger A - CC^\dagger B^\dagger A^\dagger A \,) \\ &= r(\, CM^\dagger A - CPJ^\dagger QA \,) \\ &= r(\, CM^*(M^*MM^*)^\dagger M^*A - CPJ^\dagger QA \,) \\ &= \left([\, CM^*,\ CP \,] \begin{bmatrix} -M^*MM^* & 0 \\ 0 & J \end{bmatrix}^\dagger \begin{bmatrix} M^*A \\ QA \end{bmatrix} \right). \end{aligned}$$

Observe from Eq.(8.10) that

$$R\begin{bmatrix} M^*A \\ QA \end{bmatrix} \subseteq R\begin{bmatrix} -M^*MM^* & 0 \\ 0 & J \end{bmatrix} \quad \text{and} \quad R([\, CM^*,\ CP \,]^*\,) \subseteq R\left(\begin{bmatrix} -M^*MM^* & 0 \\ 0 & J \end{bmatrix}^* \right).$$

Thus we find by Eq.(1.7) that ,

$$\begin{aligned} & r\left([\, CM^*,\ CP \,] \begin{bmatrix} -M^*MM^* & 0 \\ 0 & J \end{bmatrix}^\dagger \begin{bmatrix} M^*A \\ QA \end{bmatrix} \right) \\ &= r\begin{bmatrix} -MM^*M & 0 & M^*A \\ 0 & J & QA \\ CM^* & CP & 0 \end{bmatrix} - r\begin{bmatrix} -M^*MM^* & 0 \\ 0 & J \end{bmatrix} \end{aligned}$$

$$= r\begin{bmatrix} -MM^*M & 0 & 0 & 0 & M^*A \\ 0 & 0 & 0 & AA^* & A \\ 0 & 0 & B^*BB^* & B^*A^* & 0 \\ 0 & C^*C & C^*B^* & 0 & 0 \\ CM^* & C & 0 & 0 & 0 \end{bmatrix} - r(M) - r(J)$$

$$= r\begin{bmatrix} -MM^*M & 0 & 0 & 0 - M^*AA^* & 0 \\ 0 & 0 & 0 & 0 & A \\ 0 & 0 & B^*BB^* & B^*A^* & 0 \\ -C^*CM^* & 0 & C^*B^* & 0 & 0 \\ 0 & C & 0 & 0 & 0 \end{bmatrix} - r(M) - r(J)$$

$$= r\begin{bmatrix} -MM^*M & 0 & -M^*AA^* \\ 0 & B^*BB^* & B^*A^* \\ -C^*CM^* & C^*B^* & 0 \end{bmatrix} - r(M) - r(B).$$

The results in Eqs.(8.13) and (8.14) follow from it. □

Corollary 8.12. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{n\times p}$ and $C \in \mathcal{C}^{p\times q}$ be given, and suppose that*

$$R(B) \subseteq R(A^*) \ \ and \ \ R(B^*) \subseteq R(C). \tag{8.15}$$

Then

$$r(\, M^\dagger - C^\dagger B^\dagger A^\dagger\,) = r\begin{bmatrix} B \\ BCC^* \end{bmatrix} + r[\, B,\ A^*AB\,] - 2r(B). \tag{8.16}$$

In particular,

$$(ABC)^\dagger = C^\dagger B^\dagger A^\dagger \Longleftrightarrow R(A^*AB) \subseteq R(B) \ \ and \ \ R[(BCC^*)^*] \subseteq R(B^*). \tag{8.17}$$

Proof. Eq.(8.15) is equivalent to $A^\dagger AB = B$ and $BCC^\dagger = B$. Thus we can reduce Eq.(8.13) by block elementary operations to

$$\begin{aligned} r(\, M^\dagger - C^\dagger B^\dagger A^\dagger\,) &= r\begin{bmatrix} -BCM^*AB & 0 & BCC^* \\ 0 & BB^*B & B \\ A^*AB & B & 0 \end{bmatrix} - 2r(B) \\ &= r\begin{bmatrix} 0 & BCC^*B^* & BCC^* \\ 0 & BB^*B & B \\ A^*AB & B & 0 \end{bmatrix} - 2r(B) \\ &= r\begin{bmatrix} 0 & 0 & BCC^*B \\ 0 & 0 & B \\ A^*AB & B & 0 \end{bmatrix} - 2r(B) \\ &= r\begin{bmatrix} B \\ BCC^* \end{bmatrix} + r[\, B,\ A^*AB\,] - 2r(B). \qquad \square \end{aligned}$$

Corollary 8.13. *Let $B \in \mathcal{C}^{m\times n}$ be given, $A \in \mathcal{C}^{m\times m}$ and $C \in \mathcal{C}^{n\times n}$ be two invertible matrices. Let $M = ABC$. Then*

$$r[\,(ABC)^{\dagger} - C^{-1}B^{\dagger}A^{-1}\,] = r\begin{bmatrix} B \\ BCC^{*} \end{bmatrix} + r[\,B,\ A^{*}AB\,] - 2r(B), \tag{8.18}$$

and

$$r[\,(ABC)^{\dagger} - C^{-1}B^{\dagger}A^{-1}\,] = r\begin{bmatrix} M \\ MC^{*}C \end{bmatrix} + r[\,M,\ AA^{*}M\,] - 2r(M). \tag{8.19}$$

In particular,

$$(ABC)^{\dagger} = C^{-1}B^{\dagger}A^{-1} \Longleftrightarrow R(AA^{*}B) = R(B) \ \ and \ \ R(CC^{*}B^{*}) = R(B^{*}). \tag{8.20}$$

and

$$(ABC)^{\dagger} = C^{-1}B^{\dagger}A^{-1} \Longleftrightarrow R(AA^{*}M) = R(M) \ \ and \ \ R(C^{*}CM^{*}) = R(M^{*}). \tag{8.21}$$

Proof. Follows immediately from Corollary 8.12. □

Theorem 8.14. *Let $B \in \mathcal{C}^{m\times n}$ be given, $A \in \mathcal{C}^{m\times m}$ and $C \in \mathcal{C}^{n\times n}$ be two invertible matrices. Let $M = ABC$. Then*

(a) $r[\,MM^{\dagger} - ABB^{\dagger}A^{-1}\,] = r[\,B,\ A^{*}AB\,] - r(B)$.

(b) $r[\,M^{\dagger}M - C^{-1}B^{\dagger}BC\,] = r\begin{bmatrix} B \\ BCC^{*} \end{bmatrix} - r(B)$.

(c) $MM^{\dagger} = ABB^{\dagger}A^{-1} \Longleftrightarrow R(A^{*}AB) = R(B)$.

(d) $M^{\dagger}M = ABB^{\dagger}A^{-1} \Longleftrightarrow R(CC^{*}B^{*}) = R(B^{*})$.

Proof. Note that

$$r(\,MM^{\dagger} - ABB^{\dagger}A^{-1}\,) = r(\,MM^{\dagger}A - ABB^{\dagger}\,),$$

and

$$r(\,M^{\dagger}M - C^{-1}B^{\dagger}BC\,) = r(\,CM^{\dagger}M - BB^{\dagger}C\,).$$

Applying Eq.(4.1) to both of them yields the results in the theorem. □

Chapter 9

Moore-Penrose inverses of block matrices

In this chapter we wish to establish some rank equalities related to the factorizations of 2×2 block matrices and then deduce from them various expressions of Moore-Penrose inverses for 2×2 block matrices. as well as $m \times n$ block matrices. Most of the results in this chapter appears in the author recent paper [95].

Theorem 9.1. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$ and $C \in \mathcal{C}^{l\times n}$ be given. and factor $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ as*

$$M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} I_m & E_BAC^\dagger \\ 0 & I_l \end{bmatrix} \begin{bmatrix} E_BAF_C & B \\ C & 0 \end{bmatrix} \begin{bmatrix} I_n & B^\dagger A \\ 0 & I_k \end{bmatrix} := PNQ, \tag{9.1}$$

where $E_B = I_m - BB^\dagger$ and $F_C = I_n - C^\dagger C$. Then

(a) *The rank of $M^\dagger - Q^{-1}N^\dagger P^{-1}$ satisfies the equality*

$$r(\, M^\dagger - Q^{-1}N^\dagger P^{-1}\,) = r\begin{bmatrix} A \\ C \end{bmatrix} + r[\, A,\ B\,] + r(B) + r(C) - 2r(M). \tag{9.2}$$

(b) *The Moore-Penrose inverse of M can be expressed as $M^\dagger = Q^{-1}N^\dagger P^{-1}$. that is.*

$$M^\dagger = \begin{bmatrix} (E_BAF_C)^\dagger & C^\dagger - (E_BAF_C)^\dagger AC^\dagger \\ B^\dagger - B^\dagger A(E_BAF_C)^\dagger & -B^\dagger AC^\dagger + B^\dagger A(E_BAF_C)^\dagger AC^\dagger \end{bmatrix} \tag{9.3}$$

holds if and only if A. B and C satisfy the rank additivity condition

$$r(M) = r\begin{bmatrix} A \\ C \end{bmatrix} + r(B) = r[\, A,\ B\,] + r(C), \tag{9.4}$$

or equivalently

$$R\begin{bmatrix} A \\ C \end{bmatrix} \cap \begin{bmatrix} B \\ 0 \end{bmatrix} = \{0\} \ \ and \ \ R([\, A,\ B\,]^*) \cap R([\, C.\ 0\,]^*) = \{0\}. \tag{9.5}$$

Proof. It follows first by Eqs.(9.1) and (8.19) that

$$r(\, M^\dagger - Q^{-1}N^\dagger P^{-1}\,) = r\begin{bmatrix} M \\ MQ^*Q \end{bmatrix} + r[\, M.\ PP^*M\,] - 2r(M). \tag{9.6}$$

The ranks of the two block matrices in (9.6) can simplify to

$$r[\, M,\ PP^*M\,] \ = \ r\left[\begin{bmatrix} A & B \\ C & 0 \end{bmatrix},\ \begin{bmatrix} I_m & E_BAC^\dagger \\ 0 & I_l \end{bmatrix}\begin{bmatrix} I_m & 0 \\ (E_BAC^\dagger)^* & I_l \end{bmatrix}\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\right]$$

$$
\begin{aligned}
&= r\begin{bmatrix} A & B & A+(E_BAC^\dagger)(E_BAC^\dagger)^*A+E_BAC^\dagger C & B \\ C & 0 & C+(E_BAC^\dagger)^*A & 0 \end{bmatrix} \\
&= r\begin{bmatrix} A & B & AC^\dagger(E_BAC^\dagger)^*A+AC^\dagger C \\ C & 0 & (E_BAC^\dagger)^*A \end{bmatrix} \\
&= r\begin{bmatrix} A & B & AC^\dagger C \\ C & 0 & 0 \end{bmatrix} \\
&= r\begin{bmatrix} A & B & 0 \\ C & 0 & -C \end{bmatrix} = r[\,A,\ B\,]+r(C),
\end{aligned}
$$

and

$$
\begin{aligned}
r\begin{bmatrix} M \\ MQ^*Q \end{bmatrix} &= r\begin{bmatrix} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \\ \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} I_n & (B^\dagger A)^* \\ 0 & I_k \end{bmatrix}\begin{bmatrix} I_n & 0 \\ B^\dagger A & I_k \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} A & B \\ C & 0 \\ A+A(B^\dagger A)^*(B^\dagger A)+BB^\dagger A & B+A(B^\dagger A)^* \\ C+C(B^\dagger A)^*B^\dagger A & C(B^\dagger A)^* \end{bmatrix} \\
&= \begin{bmatrix} A & B \\ C & 0 \\ A+A(B^\dagger A)^*(B^\dagger A) & A(B^\dagger A)^* \\ C(B^\dagger A)^*B^\dagger A & C(B^\dagger A)^* \end{bmatrix} \\
&= \begin{bmatrix} A & B \\ C & 0 \\ A & 0 \\ 0 & 0 \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r(B).
\end{aligned}
$$

Putting both of them in Eq.(9.6) yields Eq.(9.2). Notice that $(E_BAF_C)^\dagger B^\dagger = 0$ and $C^\dagger(E_BAF_C)^\dagger = 0$ always hold. Then it is easy to verify that

$$
N^\dagger = \begin{bmatrix} E_BAF_C & B \\ C & 0 \end{bmatrix}^\dagger = \begin{bmatrix} (E_BAF_C)^\dagger & C^\dagger \\ B^\dagger & 0 \end{bmatrix}.
$$

Putting it in Eq.(9.2) and letting the right-hand of Eq.(9.2) be zero, we get Eqs.(9.3)—(9.5). □

Corollary 9.2. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$ and $C \in \mathcal{C}^{l\times n}$ be given. If*

$$
R(A)\cap R(B) = \{0\} \quad \textit{and} \quad R(A^*)\cap R(C^*) = \{0\}, \tag{9.7}
$$

then

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{\dagger} = \begin{bmatrix} (E_BAF_C)^{\dagger} & C^{\dagger} - (E_BAF_C)^{\dagger}AC^{\dagger} \\ B^{\dagger} - B^{\dagger}A(E_BAF_C)^{\dagger} & 0 \end{bmatrix}. \tag{9.8}$$

Proof. Under Eq.(9.7), the rank equality (9.4) naturally holds. In that case, we know by Theorem 7.8 that $A(E_BAF_C)^{\dagger}A = A$. Thus Eq.(9.3) reduces to Eq.(9.8). □

Theorem 9.3. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$ and $C \in \mathcal{C}^{l\times n}$ be given. Then*

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{\dagger} = \begin{bmatrix} A - YC & B \\ C & 0 \end{bmatrix}^{\dagger} \begin{bmatrix} A - BX - YC & B \\ C & 0 \end{bmatrix} \begin{bmatrix} A - BX & B \\ C & 0 \end{bmatrix}^{\dagger}, \tag{9.9}$$

where X and Y are arbitrary. In particular,

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{\dagger} = \left[[AF_C,\ B]^{\dagger},\ \begin{bmatrix} C^{\dagger} \\ 0 \end{bmatrix} \right] \begin{bmatrix} A - BB^{\dagger} - ACC^{\dagger} & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} E_BA \\ C \end{bmatrix}^{\dagger} \\ [B^{\dagger},\ 0] \end{bmatrix}. \tag{9.10}$$

Proof. It is easy to verify that

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} I_m & Y \\ 0 & I_l \end{bmatrix} \begin{bmatrix} A - BX - YC & B \\ C & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ X & I_k \end{bmatrix} := PNQ,$$

where P and Q are nonsingular. Then we have by Eq.(8.5) that

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{\dagger} = (NQ)^{\dagger}N(PN)^{\dagger}.$$

Written in an explicit form, it is Eq.(9.9). Now let $X = B^{\dagger}A$ and $Y = AC^{\dagger}$ in Eq.(9.9). Then Eq.(9.9) becomes

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{\dagger} = \begin{bmatrix} AF_C & B \\ C & 0 \end{bmatrix}^{\dagger} \begin{bmatrix} A - BB^{\dagger}A - AC^{\dagger}C & B \\ C & 0 \end{bmatrix} \begin{bmatrix} E_BA & B \\ C & 0 \end{bmatrix}^{\dagger}.$$

Note that

$$[AF_C,\ B][C,\ 0]^* = 0, \quad \text{and} \quad \begin{bmatrix} B \\ 0 \end{bmatrix}^* \begin{bmatrix} E_BA \\ C \end{bmatrix} = 0.$$

Then it follows by Theorem 7.9(e) and (f) that

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{\dagger} = [[AF_C,\ B]^{\dagger},\ [C,\ 0]^{\dagger}],$$

and

$$\begin{bmatrix} E_BA & B \\ C & 0 \end{bmatrix}^{\dagger} = \begin{bmatrix} \begin{bmatrix} E_BA \\ C \end{bmatrix}^{\dagger} \\ \begin{bmatrix} B \\ 0 \end{bmatrix}^{\dagger} \end{bmatrix}.$$

Thus we have Eq.(9.10). $\square$

Theorem 9.4. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$, $C \in \mathcal{C}^{l\times n}$ and $D \in \mathcal{C}^{l\times n}$ be given, and factor $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ as*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ CA^\dagger & I_l \end{bmatrix} \begin{bmatrix} A & E_AB \\ CF_A & S_A \end{bmatrix} \begin{bmatrix} I_n & A^\dagger B \\ 0 & I_k \end{bmatrix} := PNQ. \tag{9.11}$$

where $S_A = D - CA^\dagger B$. Then

(a) *The rank of $M^\dagger - Q^{-1}N^\dagger P^{-1}$ satisfy the equality*

$$r(\, M^\dagger - Q^{-1}N^\dagger P^{-1}\,) = r\begin{bmatrix} A & 0 \\ 0 & C \\ B & D \end{bmatrix} + r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} - 2r(M). \tag{9.12}$$

(a) *The Moore-Penrose inverse of M in Eq.(9.11) can be expressed as $M^\dagger = Q^{-1}N^\dagger P^{-1}$, that is,*

$$M^\dagger = \begin{bmatrix} I_n & -A^\dagger B \\ 0 & I_k \end{bmatrix} \begin{bmatrix} A & E_AB \\ CF_A & S_A \end{bmatrix}^\dagger \begin{bmatrix} I_m & 0 \\ -CA^\dagger & I_l \end{bmatrix} \tag{9.13}$$

holds if and only if A, B, C and D satisfy

$$r\begin{bmatrix} A & 0 \\ 0 & C \\ B & D \end{bmatrix} = r(M) \quad and \quad r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix} = r(M), \tag{9.14}$$

or equivalently

$$R\begin{bmatrix} A \\ 0 \end{bmatrix} \subseteq R\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad and \quad R\begin{bmatrix} A^* \\ 0 \end{bmatrix} \subseteq R\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}. \tag{9.15}$$

Proof. It follows by Eqs.(9.11) and (8.19) that

$$r(\, M^\dagger - Q^{-1}N^\dagger P^{-1}\,) = r\begin{bmatrix} M \\ MQ^*Q \end{bmatrix} + r[\, M,\ PP^*M\,] - 2r(M).$$

The ranks of the two block matrices in it can reduce to

$$\begin{aligned} & r[\, M,\ PP^*M\,] \\ = \ & r\left[\begin{bmatrix} A & B \\ C & D \end{bmatrix},\ \begin{bmatrix} I & 0 \\ CA^\dagger & I \end{bmatrix} \begin{bmatrix} I & (CA^\dagger)^* \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right] \\ = \ & r\begin{bmatrix} A & B & A + (CA^\dagger)^*C & B + (CA^\dagger)^*D \\ C & D & CA^\dagger A + (CA^\dagger)(CA^\dagger)^*C + C & CA^\dagger B + (CA^\dagger)(CA^\dagger)^*D + D \end{bmatrix} \\ = \ & r\begin{bmatrix} A & B & (CA^\dagger)^*C & (CA^\dagger)^*D \\ C & D & (CA^\dagger)(CA^\dagger)^*C + C & CA^\dagger B + (CA^\dagger)(CA^\dagger)^*D \end{bmatrix} \end{aligned}$$

$$= r\begin{bmatrix} A & B & 0 & 0 \\ C & D & C & CA^\dagger B \end{bmatrix}$$

$$= r\begin{bmatrix} A & B & 0 \\ C & D & C \end{bmatrix} = r\begin{bmatrix} A & 0 & B \\ 0 & C & D \end{bmatrix}.$$

Similarly we can get

$$r\begin{bmatrix} M \\ MQ^*Q \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \\ C & D \end{bmatrix}.$$

Thus we have Eq.(9.12). Eqs.(9.13)—(9.15) are direct consequences of Eq.(9.12). □

Corollary 9.5. *Let $A \in C^{m\times n}$, $B \in C^{m\times k}$ and $C \in C^{l\times n}$ be given, and factor $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ as*

$$M = \begin{bmatrix} I_m & 0 \\ CA^\dagger & I_l \end{bmatrix} \begin{bmatrix} A & E_A B \\ CF_A & -CA^\dagger B \end{bmatrix} \begin{bmatrix} I_n & A^\dagger B \\ 0 & I_k \end{bmatrix} := PNQ, \tag{9.16}$$

where $S_A = D - CA^\dagger B$. Then

$$r[\, M^\dagger - Q^{-1}N^\dagger P^{-1} \,] = r\begin{bmatrix} A \\ C \end{bmatrix} + r[\, A,\ B\,] + r(B) + r(C) - 2r(M). \tag{9.17}$$

In particular, the Moore-Penrose inverse of M in (9.16) can be expressed as

$$M^\dagger = Q^{-1}N^\dagger P^{-1} = \begin{bmatrix} I_n & -A^\dagger B \\ 0 & I_k \end{bmatrix} \begin{bmatrix} A & E_A B \\ CF_A & -CA^\dagger B \end{bmatrix}^\dagger \begin{bmatrix} I_m & 0 \\ -CA^\dagger & I_l \end{bmatrix}, \tag{9.18}$$

if and only if A, B and C satisfy the following rank additivity condition

$$r(M) = r\begin{bmatrix} A \\ C \end{bmatrix} + r(B) = r[\, A,\ B\,] + r(C). \tag{9.19}$$

Clearly N in Eq.(9.11) can be written as

$$N = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & E_A B \\ CF_A & S_A \end{bmatrix} = N_1 + N_2. \tag{9.20}$$

Then it is easy to verify that

$$N^\dagger = \begin{bmatrix} A^\dagger & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & E_A B \\ CF_A & S_A \end{bmatrix}^\dagger = N_1^\dagger + N_2^\dagger. \tag{9.21}$$

Thus if we can find $N_2^\dagger$, then we can give the expression of $N^\dagger$ in Eq.(9.21). This consideration motivates us to find the following set of results on Moore-Penrose inverses of block matrices.

Lemma 9.6. *Let $A \in C^{m\times n}$, $B \in C^{m\times k}$, $C \in C^{l\times n}$ and $D \in C^{l\times k}$ be given. Then the rank additivity condition*

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r\begin{bmatrix} B \\ D \end{bmatrix} = r[A,\ B] + r[C,\ D] \tag{9.22}$$

is equivalent to the two range inclusions

$$R\begin{bmatrix} A \\ 0 \end{bmatrix} \subseteq R\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad R\begin{bmatrix} A^* \\ 0 \end{bmatrix} \subseteq R\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}. \tag{9.23}$$

and the rank additivity condition

$$r\begin{bmatrix} 0 & E_AB \\ CF_A & S_A \end{bmatrix} = r\begin{bmatrix} E_AB \\ S_A \end{bmatrix} + r(CF_A) = r[CF_A,\ S_A] + r(E_AB). \tag{9.24}$$

where $S_A = D - CA^\dagger B$.

Proof. Let

$$V_1 = \begin{bmatrix} A \\ C \end{bmatrix}, \quad V_2 = \begin{bmatrix} B \\ D \end{bmatrix}, \quad W_1 = [A,\ B], \quad W_2 = [C,\ D]. \tag{9.25}$$

Then Eq.(9.22) is equivalent to

$$R(V_1) \cap R(V_2) = \{0\} \qquad \text{and} \qquad R(W_1^*) \cap R(W_2^*) = \{0\}. \tag{9.26}$$

In that case, we easily find

$$r\begin{bmatrix} W_1 & A \\ W_2 & 0 \end{bmatrix} = r[W_1,\ A] + r[W_2,\ 0] = r(W_1) + r(W_2) = r(M).$$

and

$$r\begin{bmatrix} V_1 & V_2 \\ A & 0 \end{bmatrix} = r\begin{bmatrix} V_1 \\ A \end{bmatrix} + r\begin{bmatrix} V_2 \\ 0 \end{bmatrix} = r(V_1) + r(V_2) = r(M).$$

both of which are equivalent to the two inclusions in Eq.(9.23). On the other hand, observe that

$$\begin{bmatrix} E_AB \\ S_A \end{bmatrix} = \begin{bmatrix} B - AA^\dagger B \\ D - CA^\dagger B \end{bmatrix} = \begin{bmatrix} B \\ D \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix} A^\dagger B = V_2 - V_1A^\dagger B.$$

and

$$\begin{aligned} [CF_A,\ S_A] = [C - CAA^\dagger,\ D - CA^\dagger B] &= [C,\ D] - CA^\dagger[A,\ B] \\ &= W_2 - CA^\dagger W_1. \end{aligned}$$

Thus according to Eq.(9.26) and Lemma 1.4(b) and (c), we find that

$$r\begin{bmatrix} E_AB \\ S_A \end{bmatrix} = r(V_2 - V_1A^\dagger B) = r\begin{bmatrix} V_2 \\ V_1A^\dagger B \end{bmatrix} = r(V_2), \tag{9.27}$$

and

$$r[\,CF_A,\ S_A\,] = r(\,W_2 - CA^{\dagger}W_1\,) = r[\,W_2,\ CA^{\dagger}W_1\,] = r(W_2). \tag{9.28}$$

¿From both of them and the rank formulas in Eqs.(1.2), (1.3), (1.5), (9.23), (9.27) and (9.28), we derive the following two equalities

$$r\begin{bmatrix} 0 & E_AB \\ CF_A & S_A \end{bmatrix} = r(M) - r(A) = r(V_1) + r(V_2) - r(A) = r\begin{bmatrix} E_AB \\ S_A \end{bmatrix} + r(CF_A),$$

and

$$r\begin{bmatrix} 0 & E_AB \\ CF_A & S_A \end{bmatrix} = r(M) - r(A) = r(W_1) + r(W_2) - r(A) = r[\,CF_A,\ S_A\,] + r(E_AB).$$

Both of them are exactly the rank additivity condition Eq.(9.24). Conversely, adding $r(A)$ to the three sides of Eq.(9.24) and then applying Eqs.(1.2), (1.3) and (1.5) to the corresponding result we first obtain

$$r\begin{bmatrix} A & E_AB \\ CF_A & S_A \end{bmatrix} = r\begin{bmatrix} A \\ CF_A \end{bmatrix} + r\begin{bmatrix} E_AB \\ S_A \end{bmatrix} = r[\,A,\ E_AB\,] + r[\,CF_A,\ S_A\,]. \tag{9.29}$$

On the other hand, the two inclusions in Eq.(9.23) are also equivalent to

$$r(M) = r\begin{bmatrix} A & B & A \\ C & D & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & D & C \end{bmatrix}, \quad \text{and} \quad r(M) = r\begin{bmatrix} A & B \\ C & D \\ A & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \\ 0 & B \end{bmatrix}.$$

Applying Eq.(1.5) to the right-hand sides of the above two equalities and then combining them with Eq.(9.29), we find

$$r(M) = r\begin{bmatrix} A & E_AB & 0 \\ CF_A & S_A & C \end{bmatrix} = r[\,A,\ E_AB\,] + r[\,CF_A,\ S_A,\ C\,] = r[\,A,\ B\,] + r[\,C,\ D\,],$$

and

$$r(M) = r\begin{bmatrix} A & E_AB \\ CF_A & S_A \\ 0 & B \end{bmatrix} = r\begin{bmatrix} A \\ CF_A \end{bmatrix} + r\begin{bmatrix} E_AB \\ S_A \\ B \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r\begin{bmatrix} B \\ D \end{bmatrix}.$$

Both of them are exactly Eq.(9.22). □

Similarly we can establish the following.

Lemma 9.7. *The rank additivity condition* (9.22) *is equivalent to the following four conditions*

$$R\begin{bmatrix} A \\ 0 \end{bmatrix} \subseteq R\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad R\begin{bmatrix} A^* \\ 0 \end{bmatrix} \subseteq R\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}, \tag{9.30}$$

and

$$r\begin{bmatrix} S_D & BF_D \\ E_DC & 0 \end{bmatrix} = r\begin{bmatrix} S_D \\ E_DC \end{bmatrix} + r(BF_D) = r[\,S_D,\ BF_D\,] + r(E_DC). \tag{9.31}$$

where $S_D = A - BD^\dagger C$.

Theorem 9.8. *Suppose that the block matrix M in Eq.(9.11) satisfies the rank additivity condition* (9.22), *then the Moore-Penrose inverse of M can be expressed in the two forms*

$$M^\dagger = \begin{bmatrix} H_1 - H_2CA^\dagger - A^\dagger BH_3 + A^\dagger BJ^\dagger(D)CA^\dagger & H_2 - A^\dagger BJ^\dagger(D) \\ H_3 - J^\dagger(D)CA^\dagger & J^\dagger(D) \end{bmatrix}. \tag{9.32}$$

and

$$M^\dagger = \begin{bmatrix} J^\dagger(A) & J^\dagger(C) \\ J^\dagger(B) & J^\dagger(D) \end{bmatrix} = \begin{bmatrix} (E_{B_2}S_DF_{C_2})^\dagger & (E_{D_2}S_BF_{A_1})^\dagger \\ (E_{A_2}S_CF_{D_1})^\dagger & (E_{C_1}S_AF_{B_1})^\dagger \end{bmatrix}. \tag{9.33}$$

where

$$S_A = D - CA^\dagger B, \quad S_B = C - DB^\dagger A, \quad S_C = B - AC^\dagger D, \quad S_D = A - BD^\dagger C,$$

$$A_1 = E_BA, \qquad A_2 = AF_C, \qquad B_1 = E_AB. \quad B_2 = BF_D.$$

$$C_1 = CF_A, \qquad C_2 = E_DC, \qquad D_1 = E_CD. \quad D_2 = DF_B.$$

$$H_1 = A^\dagger + C_1^\dagger(\, S_AJ^\dagger(D)S_A - S_A\,)B_1^\dagger,$$

$$H_2 = C_1^\dagger[\, I - S_AJ^\dagger(D)\,], \qquad H_3 = [\, I - J^\dagger(D)S_A\,]B_1^\dagger.$$

Proof. Lemma 9.6 shows that the rank additivity condition in Eq.(9.22) is equivalent to Eqs.(9.23) and (9.24). It follows from Theorem 9.4 that under Eq.(9.23), the Moore-Penrose inverse of M can be expressed as Eq.(9.13). On the other hand, It follows from Theorem 9.1 that under Eq.(9.24) the Moore-Penrose inverse of N_2 in Eq.(9.20) can be written as

$$N_2^\dagger = \begin{bmatrix} C_1^\dagger[S_AJ^\dagger(D)S_A - S_A]B_1^\dagger & C_1^\dagger - C_1^\dagger S_AJ^\dagger(D) \\ B_1^\dagger - J^\dagger(D)S_AB_1^\dagger & J^\dagger(D) \end{bmatrix}. \tag{9.34}$$

where $B_1 = E_AB$. $C_1 = CF_A$ and $J(D) = E_{C_1}S_AF_{B_1}$. Now substituting Eq.(9.34) into Eq.(9.21) and then Eq.(9.21) into Eq.(9.13), we get

$$M^\dagger = Q^{-1}N^\dagger P^{-1} = Q^{-1}\begin{bmatrix} A^\dagger + C_1^\dagger[\, S_AJ^\dagger(D)S_A - S_A\,]B_1^\dagger & C_1^\dagger - C_1^\dagger S_AJ^\dagger(D) \\ B_1^\dagger - J^\dagger(D)S_AB_1^\dagger & J^\dagger(D) \end{bmatrix}P^{-1}. \tag{9.35}$$

Written in a 2×2 block matrix, Eq.(9.35) is Eq.(9.32). In the same way, we can also decompose M in Eq.(9.11) into the other three forms.

$$M = \begin{bmatrix} I_m & 0 \\ CB^\dagger & I_l \end{bmatrix}\begin{bmatrix} E_BA & B \\ S_B & DF_B \end{bmatrix}\begin{bmatrix} I_n & 0 \\ B^\dagger A & I_k \end{bmatrix}.$$

$$M = \begin{bmatrix} I_m & AC^\dagger \\ 0 & I_l \end{bmatrix}\begin{bmatrix} AF_C & S_C \\ C & E_CD \end{bmatrix}\begin{bmatrix} I_n & C^\dagger D \\ 0 & I_k \end{bmatrix},$$

and

$$M = \begin{bmatrix} I_m & BD^\dagger \\ 0 & I_l \end{bmatrix} \begin{bmatrix} S_D & BF_D \\ E_DC & D \end{bmatrix} \begin{bmatrix} I_n & 0 \\ D^\dagger C & I_k \end{bmatrix}.$$

Based on the above decompositions of M we can also find that under Eq.(9.22) the Moore-Penrose inverse of M can also be expressed as

$$M^\dagger = \begin{bmatrix} * & J^\dagger(C) \\ * & * \end{bmatrix} = \begin{bmatrix} * & * \\ J^\dagger(B) & * \end{bmatrix} = \begin{bmatrix} J^\dagger(A) & * \\ * & * \end{bmatrix}. \tag{9.36}$$

Finally from the uniqueness of the Moore-Penrose inverse of a matrix and the expressions in Eqs.(9.32) and (9.36), we obtain Eq.(9.33). □

Some fundamental properties on the Moore-Penrose inverse of M in Eq.(9.11) can be derive from Eqs.(9.32) and (9.33).

Corollary 9.9. *Denote the Moore-Penrose inverse of M in Eq.(9.11) by*

$$M^\dagger = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}, \tag{9.37}$$

where G_1, G_2, G_3 and G_4 are $n \times m$, $n \times l$, $k \times m$ and $k \times l$ matrices, respectively. If M in Eq.(9.11) satisfies the rank additivity condition (9.22), then the submatrices in M and $M^\dagger$ satisfy the rank equalities

$$r(G_1) = r(V_1) + r(W_1) - r(M) + r(D), \tag{9.38}$$

$$r(G_2) = r(V_1) + r(W_2) - r(M) + r(B), \tag{9.39}$$

$$r(G_3) = r(V_2) + r(W_1) - r(M) + r(C), \tag{9.40}$$

$$r(G_4) = r(V_2) + r(W_2) - r(M) + r(A), \tag{9.41}$$

$$r(G_1) + r(G_4) = r(A) + r(D), \qquad r(G_2) + r(G_3) = r(B) + r(C), \tag{9.42}$$

where V_1, V_2, W_1 and W_2 are defined in (9.25). Moreover, the products of $MM^\dagger$ and $M^\dagger M$ have the forms

$$MM^\dagger = \begin{bmatrix} W_1W_1^\dagger & 0 \\ 0 & W_2W_2^\dagger \end{bmatrix}, \qquad M^\dagger M = \begin{bmatrix} V_1^\dagger V_1 & 0 \\ 0 & V_2^\dagger V_2 \end{bmatrix}. \tag{9.43}$$

Proof. The four rank equalities in Eqs.(9.38)—(9.42) can directly be derived from the expression in Eq.(9.33) for $M^\dagger$ and the rank formula (1.6). The two equalities in Eq.(9.42) come from the sums of Eqs.(9.38) and (9.41), Eqs.(9.39) and (9.40), respectively. The two results in Eq.(9.43) are derived from Eq.(9.22) and Theorem 7.16(c) and (d). □

The rank additivity condition Eq.(9.22) is a quite weak restriction to a 2×2 block matrix. As matter of fact, any matrix satisfies a rank additivity condition as in Eq.(9.22) when its rows and columns are properly permuted. We next present a group of consequences of Theorem 9.8.

Corollary 9.10. *If the block matrix M in Eq.*(9.11) *satisfies Eq.*(9.23) *and the following two conditions*

$$R(C_1) \cap R(S_A) = \{0\} \quad and \quad R(B_1^*) \cap R(S_A^*) = \{0\}, \tag{9.44}$$

then the Moore-Penrose inverse of M can be expressed as

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^\dagger &= Q^{-1} \begin{bmatrix} A^\dagger & C_1^\dagger - C_1^\dagger S_A J^\dagger(D) \\ B_1^\dagger - J^\dagger(D) S_A B_1^\dagger & J^\dagger(D) \end{bmatrix} P^{-1} \\ &= \begin{bmatrix} A^\dagger - H_2 C A^\dagger - A^\dagger B H_3 + A^\dagger B J^\dagger(D) C A^\dagger & H_2 - A^\dagger B J^\dagger(D) \\ H_3 - J^\dagger(D) C A^\dagger & J^\dagger(D) \end{bmatrix}, \end{aligned}$$

where C_1, B_1, H_2, H_3 and $J(D)$ are as in (9.32), *P and Q are as in* (9.13).

Proof. The conditions in Eq.(9.44) imply that the block matrix N_2 in Eq.(9.20) satisfies the following rank additivity condition

$$r(N_2) = r(E_A B) + r(C F_A) + r(S_A),$$

which is a special case of Eq.(9.24). On the other hand, under Eq.(9.44) if follow by Theorem 7.7 that $S_A J^\dagger(D) S_A = S_A$. Thus Eq.(9.35) reduces to the desired result in the corollary. □

Corollary 9.11. *If the block matrix M in Eq.*(9.11) *satisfies Eq.*(9.23) *and the two conditions*

$$R(B S_A^*) \subseteq R(A) \quad and \quad R(C^* S_A) \subseteq R(A^*), \tag{9.45}$$

then the Moore-Penrose inverse of M can be expressed as

$$\begin{aligned} M^\dagger &= \begin{bmatrix} I_n & -A^\dagger B \\ 0 & I_k \end{bmatrix} \begin{bmatrix} A^\dagger & (C F_A)^\dagger \\ (E_A B)^\dagger & S_A^\dagger \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -C A^\dagger & I_l \end{bmatrix} \\ &= \begin{bmatrix} A^\dagger - A^\dagger B (E_A B)^\dagger - (C F_A)^\dagger C A^\dagger + A^\dagger B S_A^\dagger C A^\dagger & (C F_A)^\dagger - A^\dagger B S_A^\dagger \\ (E_A B)^\dagger - S_A^\dagger C A^\dagger & S_A^\dagger \end{bmatrix}, \end{aligned}$$

where $S_A = D - C A^\dagger B$.

Proof. Clearly Eq.(9.45) are equivalent to $(E_A B) S_A^* = 0$ and $S_A^*(C F_A) = 0$. In that case, Eq.(9.24) is satisfied, and $N^\dagger = \begin{bmatrix} A^\dagger & (C F_A)^\dagger \\ (E_A B)^\dagger & S_A^\dagger \end{bmatrix}$ in Eq.(9.35). □

Corollary 9.12 (Chen and Zhou, 1991). *If the block matrix M in* (9.11) *satisfies the following four conditions*

$$R(B) \subseteq R(A), \qquad R(C^*) \subseteq R(A^*), \qquad R(C) \subseteq R(S_A), \qquad R(B^*) \subseteq R(S_A^*), \tag{9.46}$$

then the Moore-Penrose inverse of M can be expressed as

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^\dagger &= \begin{bmatrix} I_n & -A^\dagger B \\ 0 & I_k \end{bmatrix} \begin{bmatrix} A^\dagger & 0 \\ 0 & S_A^\dagger \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -C A^\dagger & I_l \end{bmatrix} \\ &= \begin{bmatrix} A^\dagger + A^\dagger B S_A^\dagger C A^\dagger & -A^\dagger B S_A^\dagger \\ -S_A^\dagger C A^\dagger & S_A^\dagger \end{bmatrix}, \end{aligned}$$

where $S_A = D - CA^\dagger B$.

Proof. It is easy to verify that under the conditions in Eq.(9.46), the rank of M satisfies the rank additivity condition (9.22). In that case, $N^\dagger = \begin{bmatrix} A^\dagger & 0 \\ 0 & S_A^\dagger \end{bmatrix}$ in Eq.(9.35). □

Corollary 9.13. *If the block matrix M in* (9.11) *satisfies the four conditions*

$$R(A) \cap R(B) = \{0\}, \quad R(A^*) \cap R(C^*) = \{0\}, \quad R(D) \subseteq R(C), \quad R(D^*) \subseteq R(B^*), \tag{9.47}$$

then the Moore-Penrose inverse of M can be expressed as

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^\dagger &= \begin{bmatrix} I_n & -A^\dagger B \\ 0 & I_k \end{bmatrix} \begin{bmatrix} A^\dagger - C_1^\dagger S_A B_1^\dagger & C_1^\dagger \\ B_1^\dagger & 0 \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -CA^\dagger & I_l \end{bmatrix} \\ &= \begin{bmatrix} A^\dagger - A^\dagger B B_1^\dagger - C_1^\dagger C A^\dagger - C_1^\dagger S_A B_1^\dagger & C_1^\dagger \\ B_1^\dagger & 0 \end{bmatrix}, \end{aligned}$$

where $S_A = D - CA^\dagger B$, $B_1 = E_A B$ *and* $C_1 = CF_A$.

Proof. It is not difficult to verify by Eq.(1.5) that under Eq.(9.47) the rank of M satisfies Eq.(9.22). In that case, $J(D) = 0$ and $N^\dagger = \begin{bmatrix} A^\dagger - C_1^\dagger S_A B_1^\dagger & C_1^\dagger \\ B_1^\dagger & 0 \end{bmatrix}$ in Eq.(9.35). □

Corollary 9.14. *If the block matrix M in* (9.11) *satisfies the four conditions*

$$R(A) \cap R(B) = \{0\}, \qquad R(A^*) \cap R(C^*) = \{0\}, \tag{9.48}$$

$$R(S_A) \subseteq N(C^*), \qquad R(S_A^*) \subseteq N(B), \tag{9.49}$$

then the Moore-Penrose inverse of M can be expressed as

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^\dagger &= \begin{bmatrix} I_n & -A^\dagger B \\ 0 & I_k \end{bmatrix} \begin{bmatrix} A^\dagger & (CF_A)^\dagger \\ (E_A B)^\dagger & S_A^\dagger \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -CA^\dagger & I_l \end{bmatrix} \\ &= \begin{bmatrix} A^\dagger - A^\dagger B (E_A B)^\dagger - (CF_A)^\dagger C A^\dagger & (CF_A)^\dagger \\ (E_A B)^\dagger & S_A^\dagger \end{bmatrix}, \end{aligned}$$

where $S_A = D - CA^\dagger B$.

Proof. Clearly Eq.(9.49) is equivalent to $C^\dagger S_A = 0$ and $S_A B^\dagger = 0$, as well as $S_A^\dagger C = 0$ and $BS_A^\dagger = 0$. From them and (9.48), we also find

$$(CF_A)^\dagger S_A = 0 \quad and \quad S_A (E_A B)^\dagger = 0. \tag{9.50}$$

Combining Eqs.(9.48) and (9.50) shows that M satisfies Eqs.(9.23) and (9.24). In that case,

$$N^\dagger = \begin{bmatrix} A^\dagger & (CF_A)^\dagger \\ (E_A B)^\dagger & S_A^\dagger \end{bmatrix}$$

in Eq.(9.35). $\square$

Corollary 9.15. *If the block matrix M in Eq.(9.11) satisfies the rank additivity condition*

$$r(M_1) = r(A) + r(B) + r(C) + r(D), \tag{9.51}$$

then the Moore-Penrose inverse of M can be expressed as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\dagger} = \begin{bmatrix} (E_B A F_C)^{\dagger} & (E_D C F_A)^{\dagger} \\ (E_A B F_D)^{\dagger} & (E_C D F_B)^{\dagger} \end{bmatrix}. \tag{9.52}$$

Proof. Obviously Eq.(9.51) is a special case of Eq.(9.22). On the other hand, Eq.(9.51) is also equivalent to the following four conditions

$$R(A) \cap R(B) = \{0\}, \quad R(C) \cap R(D) = \{0\}, \quad R(A^*) \cap R(C^*) = \{0\}, \quad R(B^*) \cap R(D) = \{0\}.$$

In that case,

$$R(A_1^*) = R(A^*), \qquad R(A_2) = R(A), \qquad R(A_1^*) = R(A^*), \qquad R(B_2) = R(B),$$

$$R(C_1) = R(A), \qquad R(C_2^*) = R(C^*), \qquad R(D_1^*) = R(D^*), \qquad R(A_2) = R(D),$$

by Lemma 1.2(a) and (b). Then it turns out by Theorem 7.2(c) and (d) that

$$A_1^{\dagger} A_1 = A^{\dagger} A, \qquad A_2 A_2^{\dagger} = AA^{\dagger}, \qquad B_1^{\dagger} B_1 = B^{\dagger} B, \qquad B_2 B_2^{\dagger} = BB^{\dagger},$$

$$C_1 C_1^{\dagger} = CC^{\dagger}, \qquad C_2^{\dagger} C_2 = C^{\dagger} C, \qquad D_1^{\dagger} D_1 = D^{\dagger} D, \qquad D_2 D_2^{\dagger} = DD^{\dagger}.$$

Thus Eq.(9.33) reduces to Eq.(9.52) $\square$

Corollary 9.16. *If the block matrix M in (9.11) satisfies $r(M) = r(A) + r(D)$ and*

$$R(B) \subseteq R(A), \quad R(C) \subseteq R(D), \quad R(C^*) \subseteq R(A^*), \quad R(B^*) \subseteq R(D^*),$$

then the Moore-Penrose inverse of M can be expressed as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\dagger} = \begin{bmatrix} (A - BD^{\dagger}C)^{\dagger} & -A^{\dagger}B(D - CA^{\dagger}B)^{\dagger} \\ -(D - CA^{\dagger}B)^{\dagger}CA^{\dagger} & (D - CA^{\dagger}B)^{\dagger} \end{bmatrix}.$$

Corollary 9.17. *If the block matrix M in Eq.(9.11) satisfies $r(M) = r(A) + r(D)$ and the following four conditions*

$$R(A) = R(B), \quad R(C) = R(D), \quad R(A^*) = R(C^*), \quad R(B^*) = R(D^*),$$

then the Moore-Penrose inverse of M can be expressed as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\dagger} = \begin{bmatrix} S_D^{\dagger} & S_B^{\dagger} \\ S_C^{\dagger} & S_A^{\dagger} \end{bmatrix} = \begin{bmatrix} (A - BD^{\dagger}C)^{\dagger} & (C - DB^{\dagger}A)^{\dagger} \\ (B - AC^{\dagger}D)^{\dagger} & (D - CA^{\dagger}B)^{\dagger} \end{bmatrix}.$$

The above two corollaries can directly be derived from Eqs.(9.32) and (9.33), the proofs are omitted here.

Without much effort, we can extend the results in Theorem 9.8 to $m \times n$ block matrices when they satisfy rank additivity conditions.

Let

$$M = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \tag{9.53}$$

be an $m \times n$ block matrix , where A_{ij} is an $s_i \times t_j$ matrix ($1 \le i \le m$, $1 \le j \le n$), and suppose that M satisfies the following rank additivity condition

$$r(M) = r(W_1) + r(W_2) + \cdots + r(W_m) = r(V_1) + r(V_2) + \cdots + r(V_n). \tag{9.54}$$

where

$$W_i = [\, A_{i1},\ A_{i2},\ \cdots,\ A_{in} \,], \qquad V_j = \begin{bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{mj} \end{bmatrix}, \qquad 1 \le i \le m,\ 1 \le j \le n. \tag{9.55}$$

For convenience of representation, we adopt the following notation. Let $M = (A_{ij})$ be given in Eq.(9.53), where $A_{ij} \in \mathcal{C}^{s_i \times t_j}$, $1 \le i \le m$, $1 \le j \le n$, and $\sum_{i=1}^{m} s_i = s$, $\sum_{i=1}^{n} t_i = t$. For each A_{ij} in M we associate three block matrices as follows

$$B_{ij} = [\, A_{i1},\ \cdots,\ A_{i,j-1},\ A_{i,j+1},\ \cdots,\ A_{in} \,], \tag{9.56}$$

$$C_{ij}^{*} = [\, A_{1j}^{*},\ \cdots,\ A_{i-1,j}^{*},\ A_{i+1,j}^{*},\ \cdots,\ A_{mj}^{*} \,], \tag{9.57}$$

$$D_{ij} = \begin{bmatrix} A_{11} & \cdots & A_{1,j-1} & A_{1,j+1} & \cdots & A_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{i-1,1} & \cdots & A_{i-1,j-1} & A_{i-1,j+1} & \cdots & A_{i-1,n} \\ A_{i+1,1} & \cdots & A_{i+1,j-1} & A_{i+1,j+1} & \cdots & A_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{m1} & \cdots & A_{m,j-1} & A_{m,j+1} & \cdots & A_{mn} \end{bmatrix}. \tag{9.58}$$

The symbol $J(A_{ij})$ stands for

$$J(A_{ij}) = E_{\alpha_{ij}} S_{D_{ij}} F_{\beta_{ij}}, \qquad 1 \le i \le m,\ 1 \le j \le n, \tag{9.59}$$

where $\alpha_{ij} = B_{ij} F_{D_{ij}}$, $\beta_{ij} = E_{D_{ij}} C_{ij}$ and $S_{D_{ij}} = A_{ij} - B_{ij} D_{ij}^{\dagger} C_{ij}$ is the Schur complement of D_{ij} in M. We call the matrix $J(A_{ij})$ the ***rank complement*** of A_{ij} in M. Besides we partition the Moore-Penrose inverse of M in Eq.(9.53) into the form

$$M^{\dagger} = \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1m} \\ G_{21} & G_{22} & \cdots & G_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ G_{n1} & G_{n2} & \cdots & G_{nm} \end{bmatrix}. \tag{9.60}$$

where G_{ij} is a $t_i \times s_j$ matrix, $1 \le i \le n$, $1 \le j \le m$.

Next we build two groups of block permutation matrices as follows

$$P_1 = I_s, \qquad P_i = \begin{bmatrix} 0 & & & I_{s_i} & & & \\ I_{s_1} & \ddots & & & & & \\ & \ddots & \ddots & & & & \\ & & I_{s_{i-1}} & 0 & & & \\ & & & & I_{s_{i+1}} & & \\ & & & & & \ddots & \\ & & & & & & I_{s_m} \end{bmatrix}. \tag{9.61}$$

$$Q_1 = I_s, \qquad Q_j = \begin{bmatrix} 0 & I_{t_1} & & & & & \\ & \ddots & \ddots & & & & \\ & & \ddots & I_{t_{j-1}} & & & \\ I_{t_j} & & & 0 & & & \\ & & & & I_{t_{j+1}} & & \\ & & & & & \ddots & \\ & & & & & & I_{t_n} \end{bmatrix}. \tag{9.62}$$

where $2 \le i \le m$, $2 \le j \le n$. Applying Eqs.(9.61) and (9.62) to M in Eq.(9.53) and $M^{\dagger}$ in Eq.(9.60) we have the following two groups of expressions

$$P_i M Q_j = \begin{bmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{bmatrix}, \qquad 1 \le i \le m,\ 1 \le j \le n. \tag{9.63}$$

and

$$Q_j^T M^{\dagger} P_i^T = \begin{bmatrix} G_{ji} & * \\ * & * \end{bmatrix}, \qquad 1 \le i \le m,\ 1 \le j \le n. \tag{9.64}$$

These two equalities show that we can use two block permutation matrices to permute A_{ij} in M and the corresponding block G_{ji} in $M^{\dagger}$ to the upper left corners of M and $M^{\dagger}$, respectively. Observe that P_i and Q_j in Eqs.(9.61) and (9.62) are all orthogonal matrices. The Moore-Penrose inverse of $P_i M Q_j$ in Eq.(9.63) can be expressed as $(P_i M Q_j)^{\dagger} = Q_j^T M^{\dagger} P_i^T$. Combining Eq.(9.63) with Eq.(9.64), we have the following simple result

$$\begin{bmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{bmatrix}^{\dagger} = \begin{bmatrix} G_{ji} & * \\ * & * \end{bmatrix}, \qquad 1 \le i \le m,\ 1 \le j \le n. \tag{9.65}$$

If the block matrix M in Eq.(9.53) satisfies the rank additivity condition (9.54), then the 2×2 block matrix on the right-hand side of Eq.(9.63) naturally satisfies the following rank additivity condition

$$r\begin{bmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{bmatrix} = r\begin{bmatrix} A_{ij} \\ C_{ij} \end{bmatrix} + r\begin{bmatrix} B_{ij} \\ D_{ij} \end{bmatrix} = r[\, A_{ij},\ B_{ij}\,] + r[\, C_{ij},\ D_{ij}\,], \tag{9.66}$$

where $1 \leq i \leq m$, $1 \leq j \leq n$. Hence combining Eqs.(9.65) and (9.66) with Theorems 9.8 and 9.9, we obtain the following general result.

Theorem 9.18. *Suppose that the $m \times n$ block matrix M in Eq.(9.53) satisfies the rank additivity condition (9.54). Then*

(a) *The Moore-Penrose inverse of M can be expressed as*

$$M^{\dagger} = \begin{bmatrix} J^{\dagger}(A_{11}) & J^{\dagger}(A_{21}) & \cdots & J^{\dagger}(A_{m1}) \\ J^{\dagger}(A_{12}) & J^{\dagger}(A_{22}) & \cdots & J^{\dagger}(A_{m2}) \\ \cdots & \cdots & \cdots & \cdots \\ J^{\dagger}(A_{1n}) & J^{\dagger}(A_{2n}) & \cdots & J^{\dagger}(A_{mn}) \end{bmatrix}, \tag{9.67}$$

where $J(A_{ij})$ is defined in Eq.(9.59).

(b) *The rank of the block entry $G_{ji} = J^{\dagger}(A_{ij})$ in $M^{\dagger}$ is*

$$r(G_{ji}) = r[J(A_{ij})] = r(W_i) + r(V_j) - r(M) + r(D_{ij}), \tag{9.68}$$

where $1 \leq i \leq m$, $1 \leq j \leq n$, W_i and V_j are defined in Eq.(9.55).

(c) *$MM^{\dagger}$ and $M^{\dagger}M$ are two block diagonal matrices*

$$MM^{\dagger} = \mathrm{diag}(\, W_1W_1^{\dagger},\ W_2W_2^{\dagger},\ \cdots,\ W_mW_m^{\dagger}\,), \tag{9.69}$$

$$M^{\dagger}M = \mathrm{diag}(\, V_1^{\dagger}V_1,\ V_2^{\dagger}V_2,\ \cdots,\ V_n^{\dagger}V_n\,), \tag{9.70}$$

written in explicit forms, Eqs.(9.69) and (9.70) are equivalent to

$$A_{i1}G_{1j} + A_{i2}G_{2j} + \cdots + A_{in}G_{nj} = \begin{cases} W_iW_i^{\dagger} & i = j \\ 0 & i \neq j \end{cases} \qquad i,\ j = 1,\ 2,\ \cdots,\ m.$$

$$G_{i1}A_{1j} + G_{i2}A_{2j} + \cdots + G_{im}A_{mj} = \begin{cases} V_i^{\dagger}V_i & i = j \\ 0 & i \neq j \end{cases} \qquad i,\ j = 1,\ 2,\ \cdots,\ n.$$

In addition to the expression given in Eq.(9.67) for $M^{\dagger}$, we can also derive some other expressions for G_{ij} in $M^{\dagger}$ from Eq.(9.32). But they are quite complicated in form, so we omit them here.

Various consequences can be derived from Eq.(9.67) when the submatrices in M satisfies some additional conditions, or M has some particular patterns, such as triangular forms, circulant forms and tridiagonal forms. Here we only give one consequences.

Corollary 9.19. *If the block matrix M in Eq.*(9.53) *satisfies the following rank additivity condition*

$$r(M) = \sum_{i=1}^{m}\sum_{j=1}^{n} r(A_{ij}), \tag{9.71}$$

then the Moore-Penrose inverse of M can be expressed as

$$M^{\dagger} = \begin{bmatrix} (E_{B_{11}}A_{11}F_{C_{11}})^{\dagger} & \cdots & (E_{B_{m1}}A_{m1}F_{C_{m1}})^{\dagger} \\ \vdots & & \vdots \\ (E_{B_{1n}}A_{1n}F_{C_{1n}})^{\dagger} & \cdots & (E_{B_{mn}}A_{mn}F_{C_{mn}})^{\dagger} \end{bmatrix}, \tag{9.72}$$

where B_{ij} and C_{ij} are defined in Eqs.(9.56) *and* (9.57).

Proof. In fact, Eq.(9.71) is equivalent to

$$R(A_{ij}) \cap R(B_{ij}) = \{0\}, \qquad R(A_{ij}^{*}) \cap R(C_{ij}^{*}) = \{0\}, \qquad 1 \leq i \leq m, \ 1 \leq j \leq n,$$

$$R(C_{ij}) \cap R(D_{ij}) = \{0\}, \qquad R(B_{ij}^{*}) \cap R(D_{ij}^{*}) = \{0\}, \qquad 1 \leq i \leq m, \ 1 \leq j \leq n.$$

We can from them $J(A_{ij}) = E_{B_{ij}}A_{ij}F_{C_{ij}}$. Putting them in Eq.(9.67) yields Eq.(9.72). $\square$

Chapter 10

Rank equalities related to Moore-Penrose inverses of sums of matrices

In this chapter, we establish rank equalities related to Moore-Penrose inverses of sums of matrices and consider their various consequences.

Theorem 10.1. *Let $A,\ B \in \mathcal{C}^{m\times n}$ be given and let $N = A + B$. Then*

$$r[\, N - N(\, A^{\dagger} + B^{\dagger}\,)N\,] = r(N) + r(AB^{*}A) + r(BA^{*}B) - r(A) - r(B). \tag{10.1}$$

In particular,

$$A^{\dagger} + B^{\dagger} \in \{(A+B)^{-}\} \Longleftrightarrow r(\, A + B\,) = r(A) + r(B) - r(AB^{*}A) - r(BA^{*}B). \tag{10.2}$$

Proof. It follows by Eq.(2.2) and block elementary operations that

$$\begin{aligned} r[\, N - N(\, A^{\dagger} + B^{\dagger}\,)N\,] &= r\begin{bmatrix} A^{*}AA^{*} & 0 & A^{*}N \\ 0 & B^{*}BB^{*} & B^{*}N \\ NA^{*} & NB^{*} & N \end{bmatrix} - r(A) - r(B) \\ &= r\begin{bmatrix} -A^{*}BA^{*} & 0 & 0 \\ 0 & -B^{*}AB^{*} & 0 \\ 0 & 0 & N \end{bmatrix} - r(A) - r(B) \\ &= r(N) + r(AB^{*}A) + r(BA^{*}B) - r(A) - r(B). \end{aligned}$$

Thus we have Eqs.(10.1) and (10.2). $\square$

A general result given below.

Theorem 10.2. *Let $A_1,\ A_2,\ \cdots,\ A_k \in \mathcal{C}^{m\times n}$ be given and let $A = A_1 + A_2 + \cdots + A_k$, $X = A_1^{\dagger} + A_2^{\dagger} + \cdots + A_k^{\dagger}$. Then*

$$r(\, A - AXA\,) = r(\, DD^{*}D - PA^{*}Q\,) - r(D) + r(A), \tag{10.3}$$

where

$$D = \mathrm{diag}(\, A_1,\ A_2,\ \cdots,\ A_k\,),\quad P^{*} = [\, A_1^{*},\ A_2^{*},\ \cdots,\ A_k^{*}\,],\quad Q = [\, A_1,\ A_2,\ \cdots,\ A_k\,].$$

In particular,

$$X \in \{A^{-}\} \Longleftrightarrow r(\, DD^{*}D - PA^{*}Q\,) = r(D) - r(A), \quad i.e.,\quad PA^{*}Q \leq_{rs} DD^{*}D. \tag{10.4}$$

Proof. Let $P_1 = [\, I_n, \ \cdots, \ I_n \,]$ and $Q_1 = [\, I_m, \ \cdots, \ I_m \,]^*$. Then $X = P_1 D^\dagger Q_1$. In that case, it follows by Eq.(2.1) that

$$
\begin{aligned}
r(\, A - AXA \,) &= r(\, A - AP_1 D^\dagger Q_1 A \,) \\
&= r\begin{bmatrix} D^*DD^* & D^*Q_1 A \\ AP_1 D^* & A \end{bmatrix} - r(D) \\
&= r\begin{bmatrix} D^*DD^* - D^*Q_1 AP_1 D^* & 0 \\ 0 & A \end{bmatrix} - r(D) \\
&= r(\, D^*DD^* - D^*Q_1 AP_1 D^* \,) + r(A) - r(D) \\
&= r(\, DD^*D - DP_1^* A^* Q_1^* D \,) + r(A) - r(D)
\end{aligned}
$$

as required for Eq.(10.3). □

Theorem 10.3. *Let $A, \ B \in \mathcal{C}^{m\times n}$ be given and let $N = A + B$. Then*

$$
r(\, N^\dagger - A^\dagger - B^\dagger \,) = r\begin{bmatrix} -NN^*N & 0 & 0 & N \\ 0 & AA^*A & 0 & A \\ 0 & 0 & BB^*B & B \\ N & A & B & 0 \end{bmatrix} - r(N) - r(A) - r(B). \tag{10.5}
$$

In particular,

$$
N^\dagger = A^\dagger + B^\dagger \Longleftrightarrow r\begin{bmatrix} -NN^*N & 0 & 0 & N \\ 0 & AA^*A & 0 & A \\ 0 & 0 & BB^*B & B \\ N & A & B & 0 \end{bmatrix} = r(N) + r(A) + r(B). \tag{10.6}
$$

Proof. Follows immediately from Eq.(2.7). □

It is well known that for any two nonsingular matrices A and B, there always is $A(\, A^{-1} + B^{-1} \,)B = A + B$. Now for Moore-Penrose inverses of matrices we have the following.

Theorem 10.4. *Let $A, \ B \in \mathcal{C}^{m\times n}$ be given. Then*

$$
r[\, A + B - A(\, A^\dagger + B^\dagger \,)B \,] = r\begin{bmatrix} A \\ B \end{bmatrix} + r[\, A, \ B \,] - r(A) - r(B), \tag{10.7}
$$

and

$$
r[\, A^\dagger + B^\dagger - A^\dagger(A + B)B^\dagger \,] = r\begin{bmatrix} A \\ B \end{bmatrix} + r[\, A, \ B \,] - r(A) - r(B). \tag{10.8}
$$

In particular,

$$
A(\, A^\dagger + B^\dagger \,)B = A + B \Longleftrightarrow A^\dagger(A + B)B^\dagger = A^\dagger + B^\dagger \Longleftrightarrow R(A) = R(B) \ \ \text{and} \ \ R(A^*) = R(B^*). \tag{10.9}
$$

Proof. Writing

$$A + B - A(A^\dagger + B^\dagger)B = A + B - [A,\ A]\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^\dagger \begin{bmatrix} B \\ B \end{bmatrix}$$

and then applying Eq.(2.1) to it produce Eq.(10.7). Replacing A and B in Eq.(10.7) respectively by $A^\dagger$ and $B^\dagger$ leads to Eq.(10.8). The equivalences in Eq.(10.9) follow from Eqs.(10.7) and (10.8). □

Theorem 10.5. *Let $A,\ B \in \mathcal{C}^{m\times n}$ be given and let $N = A + B$. Then*

$$r[\,N - N(\,(E_BAF_B)^\dagger + (E_ABF_A)^\dagger\,)N\,] = r(N) + 2r(A) + 2r(B) - r\begin{bmatrix} A & B \\ B & 0 \end{bmatrix} - r\begin{bmatrix} B & A \\ A & 0 \end{bmatrix}. \tag{10.10}$$

In particular,

$$(E_BAF_B)^\dagger + (E_ABF_A)^\dagger \in \{(A+B)^-\} \iff r(\,A + B\,) = r(E_BAF_B) + r(E_ABF_A). \tag{10.11}$$

Proof. Let $P = E_BAF_B$ and $Q = E_ABF_A$. Then it is easy to verify that

$$P^*B = 0, \quad BP^* = 0, \quad Q^*A = 0, \quad AQ^* = 0$$

and

$$P^*PP^* = P^*AP^*, \qquad Q^*QQ^* = Q^*BQ^*.$$

Thus we find by Eq.(2.2) that

$$\begin{aligned} & r[\,N - N(\,P^\dagger + Q^\dagger\,)N\,] \\ &= r\begin{bmatrix} P^*PP^* & 0 & P^*N \\ 0 & Q^*QQ^* & Q^*N \\ NP^* & NQ^* & N \end{bmatrix} - r(P) - r(Q) \\ &= r\begin{bmatrix} P^*AP^* & 0 & P^*A \\ 0 & Q^*BQ^* & Q^*B \\ AP^* & BQ^* & A+B \end{bmatrix} - r(P) - r(Q) \\ &= r\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A+B \end{bmatrix} - r(P) - r(Q) = r(N) - r(P) - r(Q), \end{aligned}$$

where

$$r(P) = r\begin{bmatrix} A & B \\ B & 0 \end{bmatrix} - 2r(B), \quad r(Q) = r\begin{bmatrix} B & A \\ A & 0 \end{bmatrix} - 2r(A).$$

by Eq.(1.4). Thus we have Eqs.(10.10) and (10.11). □

Theorem 10.6. *Let $A,\ B \in \mathcal{C}^{m\times n}$ be given. Then*

(a) $r((A+B)(A+B)^{\dagger} - [A,\ B][A,\ B]^{\dagger}) = r[A,\ B] - r(A+B)$.

(b) $r\left((A+B)^{\dagger}(A+B) - \begin{bmatrix} A \\ B \end{bmatrix}^{\dagger} \begin{bmatrix} A \\ B \end{bmatrix} \right) = r\begin{bmatrix} A \\ B \end{bmatrix} - r(A+B)$.

In particular,

(c) $(A+B)(A+B)^{\dagger} = [A,\ B][A,\ B]^{\dagger} \Longleftrightarrow r[A,\ B] = r(A+B) \Longleftrightarrow R(A) \subseteq R(A+B)$ *and* $R(B) \subseteq R(A+B)$.

(d) $(A+B)^{\dagger}(A+B) = \begin{bmatrix} A \\ B \end{bmatrix}^{\dagger} \begin{bmatrix} A \\ B \end{bmatrix} \Longleftrightarrow r\begin{bmatrix} A \\ B \end{bmatrix} = r(A+B) \Longleftrightarrow R(A^*) \subseteq R(A^*+B^*)$ *and* $R(B^*) \subseteq R(A^*+B^*)$.

Proof. Let $N = A+B$ and $M = [A,\ B]$. Then it follows from Theorem 7.2(a) that

$$\begin{aligned} r(NN^{\dagger} - MM^{\dagger}) &= 2r[N,\ M] - r(N) - r(M) \\ &= 2r[A+B,\ A,\ B] - r(A+B) - r(A) - r(B) \\ &= r[A,\ B] - r(A) - r(B), \end{aligned}$$

as required for Part (a). Similarly we have Part (b). □

In general we have the following.

Theorem 10.7. *Let* $A_1,\ A_2,\ \cdots,\ A_k \in \mathcal{C}^{m\times n}$ *be given and let* $A = A_1 + A_2 + \cdots + A_k$. $M = [A_1,\ A_2, \cdots,\ A_k]$ *and* $N^* = [A_1^*,\ A_2^*, \cdots,\ A_k^*]$. *Then*

(a) $r(AA^{\dagger} - MM^{\dagger}) = r(M) - r(A)$.

(b) $r(A^{\dagger}A - N^{\dagger}N) = r(N) - r(A)$.

(c) $AA^{\dagger} = MM^{\dagger} \Longleftrightarrow r(M) = r(A) \Longleftrightarrow R(A_i) \subseteq R(M),\ i = 1,\ 2,\ \cdots,\ k$.

(b) $A^{\dagger}A = N^{\dagger}N \Longleftrightarrow r(N) = r(A) \Longleftrightarrow R(A_i^*) \subseteq R(N^*),\ i = 1,\ 2,\ \cdots,\ k$.

Theorem 10.8. *Let* $A,\ B \in \mathcal{C}^{m\times n}$ *be given and let* $N = A+B$. *Then*

(a) $r(AN^{\dagger}B) = r(NA^*) + r(B^*N) - r(N)$.

(b) $r(AN^{\dagger}B) = r(A) + r(B) - r(N)$, *if* $R(A^*) \subseteq R(N^*)$ *and* $R(B) \subseteq R(N)$.

(c) $r(AN^{\dagger}B - BN^{\dagger}A) = r\begin{bmatrix} N \\ NA^* \end{bmatrix} + r[N,\ AN^*] - 2r(N)$.

In particular,

(d) $AN^{\dagger}B = 0 \Longleftrightarrow r(NA^*) + r(B^*N) = r(N)$.

(e) $AN^{\dagger}B = BN^{\dagger}A \Longleftrightarrow R(AN^*) \subseteq R(N)$ *and* $R(A^*N) \subseteq R(N^*)$.

(f) $AN^{\dagger}B = BN^{\dagger}A$, *if* $R(A) \subseteq R(N)$ *and* $R(A^*) \subseteq R(N^*)$.

Proof. It follows by Eq.(2.1) that

$$r(AN^{\dagger}B) = r\begin{bmatrix} N^*NN^* & N^*B \\ AN^* & 0 \end{bmatrix} - r(N)$$

$$= r\begin{bmatrix} N^*AN^*+N^*BN^* & N^*B \\ AN^* & 0 \end{bmatrix} - r(N)$$

$$= r\begin{bmatrix} 0 & N^*B \\ AN^* & 0 \end{bmatrix} - r(N) = r(NA^*) + r(B^*N) - r(N),$$

as required for Part (a). Under $R(A^*) \subseteq R(N^*)$ and $R(B) \subseteq R(N)$, we know that $r(NA^*) = r(A)$ and $r(B^*N) = r(B)$. Thus we have Part (b). Similarly it follows by Eq.(2.1) that

$$\begin{aligned}
& r(AN^\dagger B - BN^\dagger A) \\
&= r\left([A,\ B]\begin{bmatrix} N & 0 \\ 0 & -N \end{bmatrix}^\dagger \begin{bmatrix} B \\ A \end{bmatrix}\right) \\
&= r\begin{bmatrix} N^*NN^* & 0 & N^*B \\ 0 & -N^*NN^* & N^*A \\ AN^* & BN^* & 0 \end{bmatrix} - 2r(N) \\
&= r\begin{bmatrix} N^*AN^* & N^*BN^* & N^*B \\ -N^*AN^* & -N^*BN^* & N^*A \\ AN^* & BN^* & 0 \end{bmatrix} - 2r(N) \\
&= r\begin{bmatrix} 0 & 0 & N^*B \\ 0 & 0 & N^*A \\ AN^* & BN^* & 0 \end{bmatrix} - 2r(N) \\
&= r\begin{bmatrix} N^*B \\ N^*A \end{bmatrix} + r[AN^*,\ BN^*] - 2r(N) \\
&= r\begin{bmatrix} N^*N \\ N^*A \end{bmatrix} + r[AN^*,\ NN^*] - 2r(N) = r\begin{bmatrix} N \\ N^*A \end{bmatrix} + r[AN^*,\ N] - 2r(N).
\end{aligned}$$

As required for Part (c). □

It is well known that if $R(A^*) \subseteq R(N^*)$ and $R(B) \subseteq R(N)$, the product $A(A + B)^\dagger B$ is called the parallel sum of A and B and denoted by $P(A,\ B)$. The results in Theorem 10.8(b) and (f) show that if A and B are parallel summable, then

$$r[\, P(A,\ B)\,] = r(A) + r(B) - r(A + B) \quad \text{and} \quad P(A,\ B) = P(B,\ A).$$

These two properties were obtained by Rao and Mitra (1971) with a different method.

The following three theorems are derived directly from Eq.(2.1). Their proofs are omitted here.

Theorem 10.9. *Let $A,\ B \in \mathcal{C}^{m\times n}$ be given. Then*

$$r\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} - \begin{bmatrix} A \\ B \end{bmatrix}(A + B)^\dagger [A,\ B] \right) = r(A) + r(B) - r(A + B). \tag{10.12}$$

In particular,

$$\begin{bmatrix} A \\ B \end{bmatrix} (A+B)^{\dagger}[A,\ B] = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \Longleftrightarrow r(A+B) = r(A) + r(B). \tag{10.13}$$

The equivalence in Eq.(10.13) was established by Marsaglia and Styan(1974).

Theorem 10.10. *Let $A_1,\ A_2,\ \cdots,\ A_k \in \mathcal{C}^{m\times n}$ be given and denote*

$$A = \mathrm{diag}(A_1,\ A_2,\ \cdots,\ A_k), \qquad N = A_1 + A_2 + \cdots + A_k.$$

Then

$$r\left(A - \begin{bmatrix} A_1 \\ \vdots \\ A_k \end{bmatrix} N^{\dagger}[A_1,\ \cdots,\ A_k] \right) = r(A_1) + \cdots + r(A_k) - r(N). \tag{10.14}$$

In particular,

$$\begin{bmatrix} A_1 \\ \vdots \\ A_k \end{bmatrix} N^{\dagger}[A_1,\ \cdots,\ A_k] = A \Longleftrightarrow r(N) = r(A_1) + \cdots + r(A_k). \tag{10.15}$$

The equivalence in Eq.(10.14) was established by Marsaglia and Styan(1974).

Theorem 10.11. *Let $A,\ B \in \mathcal{C}^{m\times n}$ be given and let $N = A + B$. Then*

(a) $r(A - AN^{\dagger}A) = r(NB^{*}N) + r(A) - r(N)$.

(b) $r(A - AN^{\dagger}A) = r(A) + r(B) - r(N)$, *if* $R(A) \subseteq R(N)$ *and* $R(A^{*}) \subseteq R(N^{*})$.

(c) $N^{\dagger} \in \{A^{-}\} \Longleftrightarrow r(NB^{*}N) = r(N) - r(A)$.

(d) $N^{\dagger} \in \{A^{-}\}$ *if* $r(N) = r(A) + r(B)$.

In the remainder of this chapter, we present a set of results related to expressions of Moore-Penrose inverses of Schur complements. These results have appeared in the author's rescent paper[95].

Theorem 10.12. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$, $C \in \mathcal{C}^{l\times n}$ and $D \in \mathcal{C}^{l\times n}$ be given. Then the rank additivity condition*

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r\begin{bmatrix} B \\ D \end{bmatrix} = r[A,\ B] + r[C,\ D], \tag{10.16}$$

then the following inversion formula holds

$$\begin{aligned}(E_{B_2}S_DF_{C_2})^{\dagger} &= A^{\dagger} + A^{\dagger}BJ^{\dagger}(D)CA^{\dagger} + C_1^{\dagger}[S_AJ^{\dagger}(D)S_A - S_A]B_1^{\dagger} \\ &\quad - A^{\dagger}B[I - J^{\dagger}(D)S_A]B_1^{\dagger} - C_1^{\dagger}[I - S_AJ^{\dagger}(D)]CA^{\dagger},\end{aligned} \tag{10.17}$$

where

$$S_A = D - CA^{\dagger}B, \qquad S_D = A - BD^{\dagger}C, \qquad J(D) = E_{C_1}S_AF_{B_1},$$

$$B_1 = E_AB, \qquad B_2 = BF_D, \qquad C_1 = CF_A, \qquad C_2 = E_DC.$$

Proof. Follows immediately from the two expressions of $M^\dagger$ in Theorem 9.8. □

The results given below are all the special cases of the general formula Eq.(10.17).

Corollary 10.13. *If A, B, C and D satisfy*

$$R\begin{bmatrix} A \\ 0 \end{bmatrix} \subseteq R\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad \textit{and} \qquad R\begin{bmatrix} A^* \\ 0 \end{bmatrix} \subseteq R\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}, \tag{10.18}$$

and the following two conditions

$$R(CS_D^*) \subseteq R(D), \qquad R(B^*S_D) \subseteq R(D^*), \tag{10.19}$$

or more specifically satisfy the four conditions

$$R(C) \subseteq R(D), \quad R(B^*) \subseteq R(D^*), \quad R(B) \subseteq R(S_D), \quad R(C^*) \subseteq R(S_D^*). \tag{10.20}$$

then the Moore-Penrose inverse of the Schur complement $S_D = A - BD^\dagger C$ satisfies the inversion formula

$$\begin{aligned}(A - BD^\dagger C)^\dagger = A^\dagger + A^\dagger BJ^\dagger(D)CA^\dagger + C_1^\dagger[S_AJ^\dagger(D)S_A - S_A]B_1^\dagger \\ - A^\dagger B[I - J^\dagger(D)S_A]B_1^\dagger - C_1^\dagger[I - S_AJ^\dagger(D)]CA^\dagger,\end{aligned} \tag{10.21}$$

where S_A, B_1, C_1 and $J(D)$ are defined in (10.17).

Proof. It is obvious that Eq.(10.19) is equivalent to $(E_DC)S_D^* = 0$ and $S_D^*(BF_D) = 0$, or equivalently

$$S_D(E_DC)^\dagger = 0 \qquad \text{and} \qquad (BF_D)^\dagger S_D = 0. \tag{10.22}$$

These two equalities clearly imply that S_D, E_DC and BF_D satisfy Eq.(9.31). Hence by Lemma 9.7, we know that under Eqs.(10.18) and (10.19), A, B, C and D naturally satisfy Eq.(10.16). Now substituting Eq.(10.22) into the left-hand side of Eq.(10.17) yields $J^\dagger(A) = (A - BD^\dagger C)^\dagger$. Hence Eq.(10.17) becomes Eq.(10.21). Observe that Eq.(10.20) is a special case of Eq.(10.19), hence Eq.(10.21) is also true under (10.20). □

Corollary 10.14. *If A, B, C and D satisfy Eqs.*(10.18), (10.19) *and the following two conditions*

$$R(CF_A) \cap R(S_A) = \{0\} \quad \textit{and} \quad R[(E_AB)^*] \cap R(S_A^*) = \{0\}. \tag{10.23}$$

then

$$(A - BD^\dagger C)^\dagger = A^\dagger + A^\dagger BJ^\dagger(D)CA^\dagger - A^\dagger B[I - J^\dagger(D)S_A]B_1^\dagger - C_1^\dagger[I - S_AJ^\dagger(D)]CA^\dagger, \tag{10.24}$$

where S_A, B_1, C_1 and $J(D)$ are defined in Eq.(10.17).

Proof. According to Theorem 7.7, the two conditions in Eq.(10.21) implies that $S_AJ^\dagger(D)S_A = S_A$. Hence Eq.(10.21) is simplified Eq.to (10.24). □

Corollary 10.15. *If A, B, C and D satisfy Eqs.*(10.18), (10.19) *and the following two conditions*

$$R(BS_A^*) \subseteq R(A) \quad \text{and} \quad R(C^*S_A) \subseteq R(A^*), \tag{10.25}$$

then

$$(A - BD^\dagger C)^\dagger = A^\dagger + A^\dagger BS_A^\dagger CA^\dagger - A^\dagger B(E_AB)^\dagger - (CF_A)^\dagger CA^\dagger, \tag{10.26}$$

where $S_A = D - CA^\dagger B$.

Proof. Clearly, Eq.(10.25) is equivalent to $(E_AB)S_A^* = 0$ and $S_A^*(CF_A) = 0$, which can also equivalently be expressed as $S_A(E_AB)^\dagger = 0$ and $(CF_A)^\dagger S_A = 0$. In that case, $J(D) = E_{C_1}S_AF_{B_1} = S_A$. Hence Eq.(10.21) is simplified to Eq.(10.26). □

Corollary 10.16. *If A, B, C and D satisfy Eqs.*(10.18), (10.19) *and the following two conditions*

$$R(B) \subseteq R(A) \quad \text{and} \quad R(C^*) \subseteq R(A^*), \tag{10.27}$$

then

$$(A - BD^\dagger C)^\dagger = A^\dagger + A^\dagger B(D - CA^\dagger B)^\dagger CA^\dagger. \tag{10.28}$$

Proof. The two inclusions in Eq.(10.27) are equivalent to $E_AB = 0$ and $CF_A = 0$. Substituting them into Eq.(10.21) yields Eq.(10.28). □

Corollary 10.17. *If A, B, C and D satisfy the following four conditions*

$$R(A) \cap R(B) = \{0\}, \qquad R(A^*) \cap R(C^*) = \{0\}, \tag{10.29}$$

$$R(C) = R(D), \qquad R(B^*) = R(D^*), \tag{10.30}$$

then

$$(A - BD^\dagger C)^\dagger = A^\dagger - A^\dagger B(E_AB)^\dagger - (CF_A)^\dagger CA^\dagger + (CF_A)^\dagger S_A(E_AB)^\dagger. \tag{10.31}$$

Proof. Under Eqs.(10.29) and (10.30), A, B, C and D naturally satisfy the rank additivity condition in Eq.(10.16). Besides, from (10.29), (10.30) and Theorem 7.2 (c) and (d) we can derive

$$B_1^\dagger B_1 = B^\dagger B, \quad C_1C_1^\dagger = CC^\dagger, \quad B_2 = 0, \quad C_2 = 0, \quad J(D) = 0.$$

Substituting them into Eq.(10.17) yields Eq.(10.31). □

If D is invertible, or $D = I$, or $B = C = -D$, then the inversion formula (10.17) can reduce to some other simpler forms. For simplicity, we do not list them here.

Chapter 11

Moore-Penrose inverses of block circulant matrices and sums of matrices

Let C be a circulant matrix over the complex number field $\mathcal{C}$ with the form

$$C = \begin{bmatrix} a_0 & a_1 & \cdots & a_{k-1} \\ a_{k-1} & a_0 & \cdots & a_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{bmatrix}. \tag{11.1}$$

Then it is a well known result (see, e.g., Davis, 1979) that C satisfies the following factorization equality

$$U^*CU = \mathrm{diag}(\,\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_k\,), \tag{11.2}$$

where U is a unitary matrix of the form

$$U = (u_{pq})_{k \times k}, \qquad u_{pq} = \frac{1}{\sqrt{k}}\omega^{(p-1)(q-1)}, \qquad \omega^k = 1, \ \ and \ \ \omega \neq 1, \tag{11.3}$$

and

$$\lambda_t = a_0 + a_1\omega^{(t-1)} + a_2(\omega^{(t-1)})^2 + \cdots + a_{k-1}(\omega^{(t-1)})^{k-1}, \quad t = 1, \ \cdots, \ k. \tag{11.4}$$

It is evident that the entries in the first row and first column of U are all $1/\sqrt{k}$, and

$$\lambda_1 = a_0 + a_1 + \cdots + a_{k-1}. \tag{11.5}$$

Observe that U in Eq.(11.3) has no relation with a_0—a_{k-1} in Eq.(11.1). Thus Eq.(11.2) can directly be extended to block circulant matrix as follows.

Lemma 11.1. *Let*

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_k \\ A_k & A_1 & \cdots & A_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix} \tag{11.6}$$

be a block circulant matrix over the complex number field $\mathcal{C}$, where $A_t \in \mathcal{C}^{m\times n}$, $t = 1, \ \cdots, \ k$. Then A satisfies the following factorization equality

$$U_m^*AU_n = \mathrm{diag}(\,J_1,\ J_2,\ \cdots,\ J_k\,), \tag{11.7}$$

where U_r and U_s are two block unitary matrices

$$U_m = (u_{pq}I_m)_{k\times k}, \qquad U_n = (u_{pq}I_n)_{k\times k}, \tag{11.8}$$

u_{pq} *is as in Eq.*(11.3), *meanwhile*

$$J_t = A_1 + A_2\omega^{(t-1)} + A_3(\omega^{(t-1)})^2 + \cdots + A_k(\omega^{(t-1)})^{k-1}, \qquad t = 1, \cdots, k. \tag{11.9}$$

Especially, the block entries in the first block rows and first block columns of U_m and U_n are all scalar products of $1/\sqrt{k}$ with identity matrices, and J_1 is

$$J_1 = A_1 + A_2 + \cdots + A_k. \tag{11.10}$$

Observe that J_1 in Eq.(11.7) is the sum of $A_1, A_2, \cdots, A_k$. Thus Eq.(11.7) implies that the sum $\sum_{t=1}^{k} A_t$ is closely linked to its corresponding block circulant matrix through a unitary factorization equality. Recall a fundamental fact in the theory of generalized inverses of matrices (see, e.g., Rao and Mitra, 1971) that

$$(PAQ)^{\dagger} = Q^{*}A^{\dagger}P^{*}, \qquad \text{if } P \text{ and } Q \text{ are unitary}. \tag{11.11}$$

Then from Eq.(11.7) we can directly find the following.

Lemma 11.2. *Let A be given in Eq.*(11.6), *U_r and U_s be given in Eq.*(11.8). *Then the Moore-Penrose inverse of A satisfies*

$$U_n^{*}A^{\dagger}U_m = \mathrm{diag}(\, J_1^{\dagger}, \ J_2^{\dagger}, \ \cdots, \ J_k^{\dagger}\,). \tag{11.12}$$

Proof. Since U_m and U_n in Eq.(11.7) are unitary, we find by Eq.(11.11) that

$$(U_m^{*}AU_n)^{\dagger} = U_n^{*}A^{\dagger}U_m.$$

On the other hand, it is easily seen that

$$[\,\mathrm{diag}(\, J_1, \ J_2, \ \cdots, \ J_k\,)\,]^{\dagger} = \mathrm{diag}(\, J_1^{\dagger}, \ J_2^{\dagger}, \ \cdots, \ J_k^{\dagger}\,).$$

Thus Eq.(11.12) follows. □

A main consequence of the above result is presented below.

Theorem 11.3(Tian, 1992b, 1998a). *Let $A_1, A_2, \cdots, A_k \in \mathcal{C}^{m\times n}$. Then the Moore-Penrose inverse of their sum satisfies*

$$(\, A_1 + A_2 + \cdots + A_k\,)^{\dagger} = \frac{1}{k}[\, I_n, \ I_n, \ \cdots, \ I_n\,] \begin{bmatrix} A_1 & A_2 & \cdots & A_k \\ A_k & A_1 & \cdots & A_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}^{\dagger} \begin{bmatrix} I_m \\ I_m \\ \vdots \\ I_m \end{bmatrix}. \tag{11.13}$$

In particular, if the block circulant matrix in it is nonsingular, then

$$(\, A_1 + A_2 + \cdots + A_k\,)^{-1} = \frac{1}{k}[\, I_m, \ I_m, \ \cdots, \ I_m\,] \begin{bmatrix} A_1 & A_2 & \cdots & A_k \\ A_k & A_1 & \cdots & A_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}^{-1} \begin{bmatrix} I_m \\ I_m \\ \vdots \\ I_m \end{bmatrix}. \tag{11.14}$$

Proof. Pre-multiplying $[I_n, 0, \cdots, 0]$ and post-multiplying $[I_m, 0, \cdots, 0]^T$ on the both sides of Eq.(11.12) immediately yield Eq.(11.13). □

It is easily seen that combining Eq.(11.13) with the results in Chapter 9 may produce lots of formulas for the Moore-Penrose inverses of matrix sums. We start with the simplest case—the Moore-Penrose inverse of sum of two matrices.

Let A and B be two $m \times n$ matrices. Then according to Eq.(11.13) we have

$$(A + B)^{\dagger} = \frac{1}{2}[I_n, \ I_n] \begin{bmatrix} A & B \\ B & A \end{bmatrix}^{\dagger} \begin{bmatrix} I_m \\ I_m \end{bmatrix}. \tag{11.15}$$

As a special case of Eq.(11.15), if we replace $A + B$ in Eq.(11.15) by a complex matrix $A + iB$, where both A and B are real matrices, then Eq.(11.15) becomes the equality

$$(A + iB)^{\dagger} = \frac{1}{2}[I_n, \ I_n] \begin{bmatrix} A & iB \\ iB & A \end{bmatrix}^{\dagger} \begin{bmatrix} I_m \\ I_m \end{bmatrix} = \frac{1}{2}[I_n, \ iI_n] \begin{bmatrix} A & -B \\ B & A \end{bmatrix}^{\dagger} \begin{bmatrix} I_m \\ -iI_m \end{bmatrix}. \tag{11.16}$$

Now applying Theorems 9.8 and 9.9 to Eqs.(11.15) and (11.16) we find the following two results.

Theorem 11.4 (Tian, 1998a). *Let A and B be two $m \times n$ complex matrices, and suppose that they satisfy the rank additivity condition*

$$r\begin{bmatrix} A & B \\ B & A \end{bmatrix} = r\begin{bmatrix} A \\ B \end{bmatrix} + r\begin{bmatrix} B \\ A \end{bmatrix} = r[A, \ B] + r[B, \ A]. \tag{11.17}$$

or alternatively

$$R(A) \subseteq R(A \pm B) \quad and \quad R(A^*) \subseteq R(A^* \pm B^*). \tag{11.18}$$

Then

(a) *The Moore-Penrose inverse of $A + B$ can be expressed as*

$$(A + B)^{\dagger} = J^{\dagger}(A) + J^{\dagger}(B) = (E_{B_2} S_A F_{B_1})^{\dagger} + (E_{A_2} S_B F_{A_1})^{\dagger}. \tag{11.19}$$

where $J(A)$ and $J(B)$ are, respectively, the rank complements of A and B in $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$, *and*

$$S_A = A - BA^{\dagger}B, \quad S_B = B - AB^{\dagger}A, \quad A_1 = E_B A, \quad A_2 = AF_B, \quad B_1 = E_A B, \quad B_2 = BF_A.$$

(b) *The matrices A, B, and the two terms $G_1 = J^{\dagger}(A)$ and $G_2 = J^{\dagger}(B)$ in the right-hand side of Eq.(11.19) satisfy the following several equalities*

$$r(G_1) = r(A), \qquad r(G_2) = r(B),$$

$$(A + B)(A + B)^{\dagger} = AG_1 + BG_2, \qquad (A + B)^{\dagger}(A + B) = G_1 A + G_2 B.$$

$$AG_2 + BG_1 = 0, \qquad G_2A + G_1B = 0.$$

Proof. The equivalence of Eqs.(11.17) and (1.18) is derived from Eq.(1.13). We know from Theorem 9.8 that under the condition (11.17), the Moore-Penrose inverse of $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$ can be expressed as

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}^{\dagger} = \begin{bmatrix} J^{\dagger}(A) & J^{\dagger}(B) \\ J^{\dagger}(B) & J^{\dagger}(A) \end{bmatrix} = \begin{bmatrix} (E_{B_2}S_AF_{B_1})^{\dagger} & (E_{A_2}S_BF_{A_1})^{\dagger} \\ (E_{A_2}S_BF_{A_1})^{\dagger} & (E_{B_2}S_AF_{B_1})^{\dagger} \end{bmatrix}.$$

Then putting it in Eq.(11.15) immediately yields Eq.(11.19). The results in Part (b) are derived from Theorem 9.9. □

Theorem 11.5 (Tian, 1998a). *Let $A + iB$ be an $m \times n$ complex matrix, where A and B are two real matrices, and suppose that A and B satisfy*

$$r\begin{bmatrix} A & -B \\ B & A \end{bmatrix} = r\begin{bmatrix} A \\ B \end{bmatrix} + r\begin{bmatrix} -B \\ A \end{bmatrix} = r[\, A, \ -B\,] + r[\, B, \ A\,]. \tag{11.20}$$

or, equivalently

$$R(A) \subseteq R(A \pm iB) \ \ and \ \ R(A^*) \subseteq R(A^T \pm iB^T). \tag{11.21}$$

Then the Moore-Penrose inverse of $A + iB$ can be expressed as

$$(A + iB)^{\dagger} = G_1 - iG_2 = [\, E_{B_2}(A + BA^{\dagger}B)F_{B_1}\,]^{\dagger} - i[\, E_{A_2}(B + AB^{\dagger}A)F_{A_1}\,]^{\dagger}, \tag{11.22}$$

where $A_1 = E_BA$, $A_2 = AF_B$, $B_1 = E_AB$ and $B_2 = BF_A$.

Proof. Follows directly from Theorem 11.4. □

Corollary 11.6 (Tian, 1998a). *Suppose that $A + iB$ is a nonsingular complex matrix, where A and B are real.*

(a) *If both A and B are nonsingular, then*

$$(A + iB)^{-1} = (A + BA^{-1}B)^{-1} - i(B + AB^{-1}A)^{-1}.$$

(b) *If both $R(A) \cap R(B) = \{0\}$ and $R(A^*) \cap R(B^*) = \{0\}$, then*

$$(A + iB)^{-1} = (E_BAF_B)^{\dagger} - i(E_ABF_A)^{\dagger}.$$

(c) *Let $A = \lambda I_m$, where λ is a real number such that $\lambda I_m + iB$ is nonsingular, then*

$$(\lambda I_m + iB)^{-1} = \lambda(\lambda^2 I_m + B^2)^{-1} - i(\lambda^2 B + B^{\dagger}B^3B^{\dagger})^{\dagger}.$$

Proof. Follows directly from Theorem 11.4. □

We next turn our attention to the Moore-Penrose inverse of the sum of k matrices. and give some general formulas.

Theorem 11.7 (Tian, 1998a). *Let $A_1, A_2, \cdots, A_k \in \mathcal{C}^{m\times n}$ be given. If they satisfy the following rank additivity condition*

$$r(A) = kr[A_1, \cdots, A_k] = kr[A_1^*, \cdots, A_k^*], \tag{11.23}$$

where A is the circulant block matrix defined in Eq.(11.6), then

(a) *The Moore-Penrose inverse of the sum $\sum_{i=1}^k A_i$ can be expressed as*

$$(A_1 + A_2 + \cdots + A_k)^\dagger = J^\dagger(A_1) + J^\dagger(A_2) + \cdots + J^\dagger(A_k). \tag{11.24}$$

where $J(A_i)$ is the rank complement of $A_i (1 \le i \le k)$ in A.

(b) *The rank of $J(A_i)$ is*

$$r[J(A_i)] = r[A_1, \cdots, A_k] + r[A_1^*, \cdots, A_k^*] - r(A) + r(D_i). \tag{11.25}$$

where $1 \le i \le k$. D_i is the $(k-1)\times(k-1)$ block matrix resulting from the deletion of the first block row and ith block column of A.

(c) *$A_1, A_2, \cdots, A_k$ and $J^\dagger(A_1), J^\dagger(A_2), \cdots, J^\dagger(A_k)$ satisfy the following two equalities*

$$(A_1 + \cdots + A_k)(A_1 + \cdots + A_k)^\dagger = A_1J^\dagger(A_1) + \cdots + A_kJ^\dagger(A_k).$$

$$(A_1 + \cdots + A_k)^\dagger(A_1 + \cdots + A_k) = J^\dagger(A_1)A_1 + \cdots + J^\dagger(A_k)A_k.$$

Proof. Follows from the combination of Theorem 9.18 with the equality in (11.13). □

Corollary 11.8 (Tian, 1998a). *Let $A_1, A_2, \cdots, A_k \in \mathcal{C}^{m\times n}$. If they satisfy the following rank additivity condition*

$$r(A_1 + A_2 + \cdots + A_k) = r(A_1) + r(A_2) + \cdots + r(A_k). \tag{11.26}$$

then

$$(A_1 + A_2 + \cdots + A_k)^\dagger = (E_{\alpha_1}A_1F_{\beta_1})^\dagger + (E_{\alpha_2}A_2F_{\beta_2})^\dagger + \cdots + (E_{\alpha_k}A_kF_{\beta_k})^\dagger. \tag{11.27}$$

where α_i and β_i are

$$\alpha_i = [A_1, \cdots A_{i-1}, A_{i+1}, \cdots, A_i], \quad \beta_i = \begin{bmatrix} A_1 \\ \vdots \\ A_{i-1} \\ A_{i+1} \\ \vdots \\ A_k \end{bmatrix}, \quad i = 1, 2, \cdots, k.$$

Proof. We first show that under the condition (11.17) the rank of the circulant matrix A in Eq.(11.6) is

$$r(A) = k[\, r(A_1) + r(A_2) + \cdots + r(A_k)\,]. \tag{11.28}$$

According to Eq.(11.7), we see that

$$r(A) = r(J_1) + r(J_2) + \cdots + r(J_k).$$

Under Eq.(11.26), the ranks of all J_i are the same, that is,

$$r(J_i) = r(A_1) + r(A_2) + \cdots + r(A_k) \quad i = 1,\ 2,\ \cdots,\ k.$$

Thus we have Eq.(11.28). In that case, applying the result in Corollary 9.19 to the circulant block matrix A in Eq.(11.13) produces the equality (11.27). □

At the end of this chapter, we should point out that the formulas on Moore-Penrose inverses of sums of matrices given in this chapter and those on Moore-Penrose inverses of block matrices given in Chapter 9 are, in fact, a group of dual results. That is to say, not only can we derive Moore-Penrose inverses of sums of matrices from Moore-Penrose inverses of block matrices, but also we can make a contrary derivation. For simplicity, here we only illustrate this assertion by a 2×2 block matrix. In fact, for any 2×2 block matrix can factor as

$$M = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} = N_1 + N_2.$$

If M satisfies the rank additivity condition (9.22), then N_1 and N_2 satisfy

$$r\begin{bmatrix} N_1 & N_2 \\ N_2 & N_1 \end{bmatrix} = r\begin{bmatrix} A & 0 & 0 & B \\ 0 & D & C & 0 \\ 0 & B & A & 0 \\ C & 0 & 0 & D \end{bmatrix} = 2r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = 2r\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = 2r[\, N_1,\ N_2\,].$$

Hence by Theorem 11.4, we have

$$M_1^\dagger = (\, N_1 + N_2\,)^\dagger = J^\dagger(N_1) + J^\dagger(N_2), \tag{11.29}$$

where $J(N_1)$ and $J(N_2)$ are, respectively, the rank complements of N_1 and N_2 in $\begin{bmatrix} N_1 & N_2 \\ N_2 & N_1 \end{bmatrix}$. Written in an explicit form, Eq.(11.29) is exactly the formula (9.33).

In addition, we shall mention another interesting fact that Eq.(11.16) can be extended to any real quaternion matrix of the form $A = A_0 + iA_1 + jA_2 + kA_3$, where A_0—A_4 are real $m \times n$ matrices and $i^2 = j^2 = k^2 = -1,\ ijk = -1$, as follows:

$$(\, A_0 + iA_1 + iA_2 + kA_3\,)^\dagger = \frac{1}{2}[\, I_n,\ jI_n\,] \begin{bmatrix} A_0 + iA_1 & -(\, A_2 + iA_3\,) \\ A_2 - iA_3 & A_0 - iA_1 \end{bmatrix}^\dagger \begin{bmatrix} I_m \\ -jI_m \end{bmatrix}, \tag{11.30}$$

and

$$(A_0 + iA_1 + iA_2 + kA_3)^{\dagger} = \frac{1}{4}[\, I_n,\ iI_n,\ jI_n,\ kI_n \,] \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & A_3 & -A_2 \\ A_2 & -A_3 & A_0 & A_1 \\ A_3 & A_2 & -A_1 & A_0 \end{bmatrix}^{\dagger} \begin{bmatrix} I_m \\ -iI_m \\ -jI_m \\ -kI_m \end{bmatrix}. \tag{11.31}$$

Their proofs will be given in the author's paper [97]. Furthermore, this work can even be extended to matrices over any 2^n-dimensional real Clifford algebras through a set of universal similarity factorization equalities established in the author's recent paper [96].

Chapter 12

Rank equalities for submatrices in Moore-Penrose inverses

Let

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{12.1}$$

be a 2×2 block matrix over $\mathcal{C}$, where $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times k}$, $C \in \mathcal{C}^{l\times n}$, $D \in \mathcal{C}^{l\times k}$, and let

$$V_1 = \begin{bmatrix} A \\ C \end{bmatrix}, \qquad V_2 = \begin{bmatrix} B \\ D \end{bmatrix}, \qquad W_1 = [\, A, \ B \,], \qquad W_2 = [\, C, \ D \,]. \tag{12.2}$$

Moreover, partition the Moore-Penrose inverse of M as

$$M^{\dagger} = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}, \tag{12.3}$$

where $G_1 \in \mathcal{C}^{n\times m}$. As is well known, the expressions of the submatrices G_1—G_4 in Eq.(12.3) are quite complicated if there are no restrictions on the blocks in M (see, e.g., Hung and Markham, 1975, Miao, 1990). In that case, it is hard to find properties of submatrices in $M^{\dagger}$. In the present chapter, we consider a simpler problem—what is the ranks of submatrices in $M^{\dagger}$, when M is arbitrarily given? This problem was examined by Robinson (1987) and Tian (1992c). In this chapter, we shall give this problem a new discussion.

Theorem 12.1. *Let M and $M^{\dagger}$ be given by Eq.(12.1) and (12.3). Then*

$$r(G_1) = r\begin{bmatrix} V_2D^*W_2 & V_1 \\ W_1 & 0 \end{bmatrix} - r(M), \quad r(G_2) = r\begin{bmatrix} V_2B^*W_1 & V_1 \\ W_2 & 0 \end{bmatrix} - r(M), \tag{12.4}$$

$$r(G_3) = r\begin{bmatrix} V_1C^*W_2 & V_2 \\ W_1 & 0 \end{bmatrix} - r(M), \quad r(G_4) = r\begin{bmatrix} V_1A^*W_1 & V_2 \\ W_2 & 0 \end{bmatrix} - r(M), \tag{12.5}$$

where V_1, V_2, W_1 and W_2 are defined in Eq.(12.2).

Proof. We only show the first equality in Eq.(12.4). In fact G_1 in Eq.(12.3) can be written as

$$G_1 = [\, I_n, \ 0 \,]M^{\dagger}\begin{bmatrix} I_m \\ 0 \end{bmatrix} = PM^{\dagger}Q. \tag{12.6}$$

Then applying Eq.(2.1) to it we find

$$r(G_1) \ = \ r\begin{bmatrix} M^*MM^* & M^*Q \\ PM^* & 0 \end{bmatrix} - r(M)$$

$$= r\begin{bmatrix} MM^*M & MP^* \\ Q^*M & 0 \end{bmatrix} - r(M)$$

$$= r\begin{bmatrix} [V_1,\ V_2]M^*\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} & V_1 \\ W_1 & 0 \end{bmatrix} - r(M)$$

$$= r\begin{bmatrix} [0,\ V_2]M^*\begin{bmatrix} 0 \\ W_2 \end{bmatrix} & V_1 \\ W_1 & 0 \end{bmatrix} - r(M) = r\begin{bmatrix} V_2D^*W_2 & V_1 \\ W_1 & 0 \end{bmatrix} - r(M),$$

establishing the first equality in Eq.(12.4). □

Corollary 12.2. *Let M and $M^\dagger$ be given by Eqs.*(12.1) *and* (12.3). *If*

$$r(M) = r(V_1) + r(V_2), \quad i.e., \quad R(V_1) \cap R(V_2) = \{0\}, \tag{12.7}$$

then

$$r(G_1) = r\begin{bmatrix} A & B \\ D^*C & D^*D \end{bmatrix} - r\begin{bmatrix} B \\ D \end{bmatrix}, \quad r(G_2) = r\begin{bmatrix} B^*A & B^*B \\ C & D \end{bmatrix} - r\begin{bmatrix} B \\ D \end{bmatrix}, \tag{12.8}$$

$$r(G_3) = r\begin{bmatrix} A & B \\ C^*C & C^*D \end{bmatrix} - r\begin{bmatrix} A \\ C \end{bmatrix}, \quad r(G_4) = r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} - r\begin{bmatrix} A \\ C \end{bmatrix}. \tag{12.9}$$

Proof. Under Eq.(12.7), we also know that $R(V_1) \cap R(V_2D^*W_2) = \{0\}$. Thus the first equality in Eq.(12.4) becomes

$$\begin{aligned} r(G_1) &= r\begin{bmatrix} V_2D^*W_2 & V_1 \\ W_1 & 0 \end{bmatrix} - r(M) \\ &= r\begin{bmatrix} V_2D^*W_2 \\ W_1 \end{bmatrix} + r(V_1) - r(M) \\ &= r\begin{bmatrix} D^*W_2 \\ W_1 \end{bmatrix} - r(V_2) \\ &= r\begin{bmatrix} W_1 \\ D^*W_2 \end{bmatrix} - r(V_2). \end{aligned}$$

establishing the first one in Eq.(12.8). Similarly, we can show the other three in Eqs.(12.8) and (12.9). □

Similarly, we have the following.

Corollary 12.3. *Let M and $M^\dagger$ be given by Eqs.*(12.1) *and* (12.3). *If*

$$r(M) = r(W_1) + r(W_2), \quad i.e., \quad R(W_1^*) \cap R(W_2^*) = \{0\}, \tag{12.10}$$

then

$$r(G_1) = r\begin{bmatrix} A & BD^* \\ C & DD^* \end{bmatrix} - r[C,\ D], \quad r(G_2) = r\begin{bmatrix} A & BB^* \\ C & DB^* \end{bmatrix} - r[A,\ B], \tag{12.11}$$

$$r(G_3) = r\begin{bmatrix} AC^* & B \\ CC^* & D \end{bmatrix} - r[C,\ D], \quad r(G_4) = r\begin{bmatrix} AA^* & B \\ CA^* & D \end{bmatrix} - r[A,\ B]. \tag{12.12}$$

Combining the above two corollaries, we obtain the following, which is previously shown in Corollary 9.9.

Corollary 12.4. *Let M and $M^\dagger$ be given by Eqs.*(12.1) *and* (12.3). *If M satisfies the rank additivity condition*

$$r(M) = r(V_1) + r(V_2) = r(W_1) + r(W_2), \tag{12.13}$$

then

$$r(G_1) = r(D) + r(V_1) + r(W_1) - r(M), \tag{12.14}$$

$$r(G_2) = r(B) + r(V_1) + r(W_2) - r(M), \tag{12.15}$$

$$r(G_3) = r(C) + r(V_2) + r(W_1) - r(M), \tag{12.16}$$

$$r(G_4) = r(A) + r(V_2) + r(W_2) - r(M). \tag{12.17}$$

Proof. We only show Eq.(12.14). Under Eq.(12.13), we find that

$$r\begin{bmatrix} V_2D^*W_2 & V_1 \\ W_1 & 0 \end{bmatrix} = r(V_2D^*W_2) + r(V_1) + r(W_1).$$

where

$$r(V_2D^*W_2) = r\begin{bmatrix} BD^*C & BD^*D \\ DD^*C & DD^*D \end{bmatrix} = r(D).$$

Thus the first equality in Eq.(12.4) reduces to Eq.(12.14). □

Corollary 12.5. *Let M and $M^\dagger$ be given by Eqs.*(12.1) *and* (12.3). *If M satisfies the rank additivity condition*

$$r(M) = r(A) + r(B) + r(C) + r(D), \tag{12.18}$$

then

$$r(G_1) = r(A), \quad r(G_2) = r(C), \quad r(G_3) = r(B), \quad r(G_4) = r(D). \tag{12.19}$$

Proof. Follows directly from Eqs.(12.14)—(12.17). □

Corollary 12.6. *Let M and $M^\dagger$ be given by Eqs.*(12.1) *and* (12.3). *If*

$$r(M) = r(V_1), \quad i.e., \quad R(V_2) \subseteq R(V_1), \tag{12.20}$$

then

$$r(G_1) = r(A), \qquad r(G_2) = r(C). \tag{12.21}$$

$$r(G_3) = r\begin{bmatrix} V_1C^*C & V_2 \\ A & 0 \end{bmatrix} - r(V_1), \qquad r(G_4) = r\begin{bmatrix} V_1A^*A & V_2 \\ C & 0 \end{bmatrix} - r(V_1). \tag{12.22}$$

Proof. The inclusion in Eq.(12.20) implies that

$$R(V_2D^*W_2) \subseteq R(V_1), \quad R(V_2D^*W_1) \subseteq R(V_1), \quad R(B) \subseteq R(A), \quad R(D) \subseteq R(C).$$

Thus the two rank equalities in Eq.(12.4) become

$$r(G_1) = r\begin{bmatrix} V_2D^*W_2 & V_1 \\ W_1 & 0 \end{bmatrix} - r(M) = r(V_1) + r(W_1) - r(M) = r(W_1) = r(A),$$

$$r(G_2) = r\begin{bmatrix} V_2B^*W_1 & V_1 \\ W_2 & 0 \end{bmatrix} - r(M) = r(V_1) + r(W_2) - r(M) = r(W_2) = r(C).$$

and the two rank equalities in Eq.(12.5) become

$$\begin{aligned} r(G_3) &= r\begin{bmatrix} V_1C^*W_2 & V_2 \\ W_1 & 0 \end{bmatrix} - r(M) \\ &= r\begin{bmatrix} V_1C^*C & V_1C^*D & V_2 \\ A & B & 0 \end{bmatrix} - r(M) = r\begin{bmatrix} V_1C^*C & V_2 \\ A & 0 \end{bmatrix} - r(M). \end{aligned}$$

$$\begin{aligned} r(G_4) &= r\begin{bmatrix} V_1A^*W_1 & V_2 \\ W_2 & 0 \end{bmatrix} - r(M) \\ &= r\begin{bmatrix} V_1A^*A & V_1A^*B & V_2 \\ C & D & 0 \end{bmatrix} - r(M) = r\begin{bmatrix} V_1A^*A & V_2 \\ C & 0 \end{bmatrix} - r(M). \end{aligned}$$

Hence we have Eqs.(12.21) and (12.22). □

Similarly, we have the following.

Corollary 12.7. *Let M and $M^\dagger$ be given by Eqs.*(12.1) *and* (12.3). *If*

$$r(M) = r(W_1), \quad i.e., \quad R(W_2^*) \subseteq R(W_1^*), \tag{12.23}$$

then

$$r(G_1) = r(A), \qquad r(G_3) = r(B), \tag{12.24}$$

$$r(G_2) = r\begin{bmatrix} BB^*W_1 & A \\ W_2 & 0 \end{bmatrix} - r(W_1), \qquad r(G_4) = r\begin{bmatrix} AA^*W_1 & B \\ W_2 & 0 \end{bmatrix} - r(W_1). \tag{12.25}$$

Combining the above two corollaries, we obtain the following.

Corollary 12.8. *Let M and $M^\dagger$ be given by Eqs.*(12.1) *and* (12.3). *If*

$$r(M) = r(A), \tag{12.26}$$

then

$$r(G_1) = r(A), \qquad r(G_2) = r(C), \qquad r(G_3) = r(B). \tag{12.27}$$

and

$$r(G_4) = r\begin{bmatrix} AA^*A & B \\ C & 0 \end{bmatrix} - r(A). \tag{12.28}$$

Proof. Clearly Eq.(12.26) implies that $r(M) = r(V_1) = r(W_1)$. Thus we have Eq.(12.27) by Corollaries 12.6 and 12.7. On the other hand, Eq.(12.26) is also equivalent to $AA^\dagger B = B$, $CA^\dagger A = C$ and $D = CA^\dagger B$ by Eq.(1.5). Hence

$$\begin{aligned} r(G_4) &= r\begin{bmatrix} V_1A^*W_1 & V_2 \\ W_2 & 0 \end{bmatrix} - r(M) \\ &= r\begin{bmatrix} AA^*W_1 & B \\ CA^*W_1 & D \\ W_2 & 0 \end{bmatrix} - r(A) \\ &= r\begin{bmatrix} AA^*W_1 & B \\ W_2 & 0 \end{bmatrix} - r(W_1) \\ &= r\begin{bmatrix} AA^*A & AA^*B & B \\ C & D & 0 \end{bmatrix} - r(A) = r\begin{bmatrix} AA^*A & B \\ C & 0 \end{bmatrix} - r(A), \end{aligned}$$

which is Eq.(12.28). □

Next we list a group of rank inequalities derived from Eqs.(12.4) and (12.5).

Corollary 12.9. *Let M and $M^\dagger$ be given by Eqs.*(12.1) *and* (12.2). *Then the rank of G_1 in $M^\dagger$ satisfies the rank inequalities*

$$r(G_1) \le r(D) + r[A,\ B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r(M), \tag{12.29}$$

$$r(G_1) \ge r[A,\ B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r(M), \tag{12.30}$$

$$r(G_1) \ge r(D) - r[C,\ D] - r\begin{bmatrix} B \\ D \end{bmatrix} + r(M). \tag{12.31}$$

Proof. Observe that

$$r(V_1) + r(W_1) \le r\begin{bmatrix} V_2D^*W_2 & V_1 \\ W_1 & 0 \end{bmatrix} \le r(D) + r(V_1) + r(W_1).$$

Putting them in the first rank equality in Eq.(12.4), we obtain Eqs.(12.29) and (12.30). To show Eq.(12.31), we need the following rank equality

$$r(CA^{\dagger}B) \geq r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r\begin{bmatrix} A \\ C \end{bmatrix} - r[\,A,\ B\,] + r(A), \tag{12.32}$$

which is derived from Eq.(1.6). Now applying Eq.(12.32) to $PM^{\dagger}Q$ in Eq.(12.6), we obtain

$$\begin{aligned} r(G_1) = r(PM^{\dagger}Q) \quad &\geq \quad r\begin{bmatrix} M & Q \\ P & 0 \end{bmatrix} - r\begin{bmatrix} M \\ P \end{bmatrix} - r[\,M,\ Q\,] + r(M) \\ &= \quad r(D) - r[\,C,\ D\,] - r\begin{bmatrix} B \\ D \end{bmatrix} + r(M), \end{aligned}$$

which is Eq.(12.31). $\square$

Rank inequalities for the block entries G_2, G_3 and G_4 in Eq.(12.3) can also be derived in the similar way shown above. Finally let $D = 0$ in Eq.(12.1). Then the results in Eqs.(12.4) and (12.5) can be simplified to the following.

Thoerem 12.10. *Let*

$$M_1 = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \tag{12.33}$$

and denote the Moore-Penrose inverse of M as

$$M_1^{\dagger} = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}, \tag{12.34}$$

where $G_1 \in \mathcal{C}^{n\times m}$. Then

$$r(G_1) = r\begin{bmatrix} A \\ C \end{bmatrix} + r[\,A,\ B\,] - r(M_1), \qquad r(G_2) = r(C), \qquad r(G_3) = r(B). \tag{12.35}$$

$$r(G_4) = r\begin{bmatrix} AA^{*}A & AA^{*}B & B \\ CA^{*}A & CA^{*}B & 0 \\ C & 0 & 0 \end{bmatrix} - r(M_1). \tag{12.36}$$

Various consequences of Eqs.(12.35) and (12.36) can also be derived. But we omit them here.

Chapter 13

Ranks of matrix expressions that involve Drazin inverses

As one of the important types of generalized inverses of matrices, the Drazin inverses and their applications have well been examined in the literature. Having established so many rank equalities in the preceding chapters, one might naturally consider how to extend that work from Moore-Penrose inverses to Drazin inverses. To do this, we only need to use a basic identity on the Drazin inverse of a matrix $A^D = A^k(A^{2k+1})^\dagger A^k$ (see, e.g., Campbell and Meyer, 1979). In that case, the rank formulas obtained in the preceding chapters can all be applied to establish various rank equalities for matrix expressions involving Drazin inverses of matrices.

Theorem 13.1. *Let $A \in \mathcal{C}^{m\times m}$ with* $\mathrm{Ind}(A) = k$. *Then*

(a) $r(\, I_m \pm A^D\,) = r(\, A^{k+1} \pm A^k\,) - r(A^k) + m.$

(b) $r[\, I_m - (A^D)^2\,] = r(\, A^{k+1} + A^k\,) + r(\, A^{k+1} - A^k\,) - 2r(A^k) + m.$

Proof. Observe that $R(A^k) = R(A^{2k+1})$ and $R[(A^k)^*] = R[(A^{2k+1})^*]$. Thus applying Eq.(1.7) to $I_m - A^D = I_m - A^k(A^{2k+1})^\dagger A^k$ yields

$$\begin{aligned} r(\, I_m - A^D\,) &= r[\, I_m - A^k(A^{2k+1})^\dagger A^k\,] \\ &= r\begin{bmatrix} A^{2k+1} & A^k \\ A^k & I_m \end{bmatrix} - r(A^{2k+1}) \\ &= r\begin{bmatrix} A^{2k+1} - A^{2k} & 0 \\ 0 & I_m \end{bmatrix} - r(A^k) \\ &= r(\, A^{2k+1} - A^{2k}\,) + m - r(A^k) = r(\, A^{k+1} - A^k\,) + m - r(A^k). \end{aligned}$$

Similarly we can find $r(\, I_m + A^D\,) = r(\, A^{k+1} + A^k\,) - r(A^k) + m$. Note by Eq.(1.12) that

$$r[\, I_m - (A^D)^2\,] = r(\, I_m + A^D\,) + r(\, I_m - A^D\,) - m.$$

Thus Part (b) follows from Part (a). $\square$

Theorem 13.2. *Let $A \in \mathcal{C}^{m\times m}$ with* $\mathrm{Ind}(A) = k$. *Then*

(a) $r(\, A - AA^D\,) = r(\, A - A^DA\,) = r(\, A^{k+1} - A^k\,) + r(A) - r(A^k).$

(b) $r(\, A - AA^DA\,) = r(A) - r(A^k).$

(c) $AA^D = A^DA = A \iff A^2 = A.$

(d) $AA^DA = A \iff \mathrm{Ind}(A) \leq 1.$

The results in Theorem 13.2(d) is well known, see, e.g., Campbell and Meyer(1979).

Proof. Applying Eq.(1.6) to $r(A - AA^D)$ yields

$$\begin{aligned} r(A - AA^D) &= r[A - A^{k+1}(A^{2k+1})^{\dagger}A^k] \\ &= r\begin{bmatrix} A^{2k+1} & A^{k+1} \\ A^k & A \end{bmatrix} - r(A^{2k+1}) \\ &= r\begin{bmatrix} A^{2k+1} - A^{2k} & 0 \\ 0 & A \end{bmatrix} - r(A^k) \\ &= r(A^{2k+1} - A^{2k}) + r(A) - r(A^k) = r(A^{k+1} - A^k) + r(A) - r(A^k), \end{aligned}$$

as required for Part (a). Notice that A^D is an outer inverse of A. Thus it follow by Eq.(5.6) that

$$r(A - AA^DA) = r(A) - r(A^D) = r(A) - r(A^k),$$

as required for Part (b). The results in Parts (c) and (d) follow from Parts (a) and (b). $\square$

Theorem 13.3. *Let $A \in \mathcal{C}^{m\times m}$ with* $\mathrm{Ind}(A) = k$.

(a) $r(A - A^D) = r(A^{k+2} - A^k) + r(A) - r(A^k)$.

(b) $r(A - A^D) = r(A) - r(A^D)$, *i.e.*, $A^D \leq_{rs} A \Longleftrightarrow A^{k+2} = A^k$.

(c) $A^D = A \Longleftrightarrow A^3 = A$.

The results in Theorem 13.3(d) is well known.

Proof. Applying Eq.(1.6) to $A - A^D$ yields

$$\begin{aligned} r(A - A^D) &= r[A - A^k(A^{2k+1})^{\dagger}A^k] \\ &= r\begin{bmatrix} A^{2k+1} & A^k \\ A^k & A \end{bmatrix} - r(A^{2k+1}) \\ &= r\begin{bmatrix} A^{2k+1} - A^{2k-1} & 0 \\ 0 & A \end{bmatrix} - r(A^k) = r(A^{k+2} - A^k) + r(A) - r(A^k), \end{aligned}$$

as required for Part(a). The results in Parts (b) and (c) follow immediately from it. $\square$

Similarly, we can establish the following two.

Theorem 13.4. *Let $A \in \mathcal{C}^{m\times m}$ with* $\mathrm{Ind}(A) = k$.

(a) *If* $k \geq 2$, *then* $r(A^2 - A^D) = r(A^{k+3} - A^k) + r(A^2) - r(A^k)$.

(b) *If* $k = 2$, *then* $r(A^2 - A^D) = r(A^5 - A^2)$.

(c) *If* $k = 1$, *then* $r(A^2 - A^D) = r(A^4 - A)$.

(d) $A^2 = A^D \Longleftrightarrow A^4 = A$ when $k = 1$.

(e) $A^2 = A^D \Longleftrightarrow A^5 = A^2$ when $k = 2$.

The two equivalence relations in Theorem 13.4(d) and (e) were obtained by Grass and Trenkler(1997) when they considered generalized and hypergeneralized projectors.

Theorem 13.5. *Let $A \in \mathcal{C}^{m\times m}$ with* $\mathrm{Ind}(A) = k$.

(a) *If $k \geq 3$, then* $r(\, A^3 - A^D\,) = r(\, A^{k+4} - A^k\,) + r(A^3) - r(A^k)$.

(b) *If $k = 3$, then* $r(\, A^3 - A^D\,) = r(\, A^7 - A^3\,)$.

(c) *If $k = 2$, then* $r(\, A^3 - A^D\,) = r(\, A^6 - A^2\,)$.

(d) *If $k = 1$, then* $r(\, A^3 - A^D\,) = r(\, A^5 - A\,)$.

(e) $A^3 = A^D \iff A^7 = A^3$ *when* $k = 3$.

(f) $A^3 = A^D \iff A^6 = A^2$ *when* $k = 2$.

(g) $A^3 = A^D \iff A^5 = A$ *when* $k = 1$.

A square matrix A is said to be quasi -idempotent if $A^{k+1} = A^k$ for some positive integer k. In a recent paper by Mitra, 1996, quasi-idempotent matrices and the related topics are well examined. The results given below reveals a new aspect on quasi-idempotent matrices.

Theorem 13.6. *Let $A \in \mathcal{C}^{m\times m}$ with* $\mathrm{Ind}(A) = k$. *Then*

(a) $r[\,A^D \pm (A^D)^2\,] = r(\,A^{k+1} \pm A^k\,)$.

(b) $(A^D)^2 = A^D$, *i.e., A^D is idempotent* $\iff A^{k+1} = A^k$, *i.e., A is quasi-idempotent.*

Proof. By Eq.(2.3) and $(A^D)^2 = (A^2)^D$ we find that

$$
\begin{aligned}
& r[\,A^D - (A^D)^2\,] \\
&= r[\,A^k(A^{2k+1})^\dagger A^k - A^{2k}(A^{4k+2})^\dagger A^{2k}\,] \\
&= r\begin{bmatrix} -A^{2k+1} & 0 & A^k \\ 0 & A^{4k+2} & A^{2k} \\ A^k & A^{2k} & 0 \end{bmatrix} - r(A^{2k+1}) - r(A^{4k+2}) \\
&= r\begin{bmatrix} -A^{2k+1} & 0 & A^k \\ 0 & A^{2k+2} & A^k \\ A^k & A^k & 0 \end{bmatrix} - 2r(A^k) \\
&= r\begin{bmatrix} 0 & 0 & A^k \\ 0 & A^{2k+2} - A^{2k+1} & 0 \\ A^k & 0 & 0 \end{bmatrix} - 2r(A^k) = r(\,A^{2k+2} - A^{2k+1}\,) = r(\,A^{k+1} - A^k\,).
\end{aligned}
$$

Similarly, we can obtain $r[\,A^D + (A^D)^2\,] = r(\,A^{k+1} + A^k\,)$. The result in Part (b) follows immediately from Part (a). □

A square matrix A is said to be quasi-idempotent if $A^{k+1} = A^k$ for some positive integer k. In a recent paper by Mitra(1996), quasi-idempotent matrices and the related topics were well examined. The results given below reveal a new aspect on quasi-idempotent matrices.

Theorem 13.7. *Let $A \in \mathcal{C}^{m\times m}$ with* $\mathrm{Ind}(A) = k$. *Then*

(a) $r[\,A^D - (A^D)^3\,] = r(\,A^{k+1} + A^k\,) + r(\,A^{k+1} - A^k\,) - r(A^k)$.

(b) *The following three statements are equivalent:*

(1) $(A^D)^3 = A^D$, *i.e.*, A^D *is tripotent.*

(2) $r(A^{k+1} + A^k) + r(A^{k+1} - A^k) = r(A^k)$.

(3) $R(A^{k+1} + A^k) \cap R(A^{k+1} - A^k) = \{0\}$ *and* $R[(A^{k+1} + A^k)^*] \cap R[(A^{k+1} - A^k)^*] = \{0\}$.

Proof. Applying Eq.(1.14) and Theorem 6.3(a) to $A^D - (A^D)^3$ yields

$$\begin{aligned} r[A^D - (A^D)^3] &= r[A^D + (A^D)^2] + r[A^D - (A^D)^2] - r(A^D) \\ &= r(A^{k+1} + A^k) + r(A^{k+1} - A^k) - r(A^k). \end{aligned}$$

as required for Part (a). The equivalence in Part (b) follows directly from (a). □

Theorem 13.8. *Let* $A \in \mathcal{C}^{m\times m}$ *with* $\mathrm{Ind}(A) = k$. *Then*

(a) $r[AA^D - (AA^D)^*] = 2r[A^k, (A^k)^*] - 2r(A^k)$.

(b) $AA^D = (AA^D)^* \iff R(A^k) = R[(A^k)^*]$ *i.e.*, A^k *is EP.*

Proof. Note that both AA^D and $(AA^D)^*$ are idempotent. It follows from Eq.(3.1) that

$$\begin{aligned} r[AA^D - (AA^D)^*] &= r\begin{bmatrix} AA^D \\ (AA^D)^* \end{bmatrix} - r[AA^D, (AA^D)^*] - r(AA^D) - r[(AA^D)^*] \\ &= 2r[AA^D, (AA^D)^*] - 2r(A^D) \\ &= 2r[A^k, (A^k)^*] - 2r(A^k), \end{aligned}$$

as required for Part (a). The result in Part (b) follows immediately from Part (a). □

Theorem 13.9. *Let* $A \in \mathcal{C}^{m\times m}$ *with* $\mathrm{Ind}(A) = k$. *Then*

(a) $r(A^\dagger - A^D) = r\begin{bmatrix} A^k \\ A^* \end{bmatrix} + r[A^k, A^*] - r(A^k) - r(A)$.

(b) $r(A^\dagger - A^D) = r(A^\dagger) - r(A^D)$, *i.e.*, $A^D \leq_{rs} A^\dagger \iff R(A^k) \subseteq R(A^*)$ *and* $R[(A^k)^*] \subseteq r(A)$, *i.e.*, A *is power* $-$ *EP.*

(c) *If* $\mathrm{Ind}(A) = 1$, *then* $r(A^\dagger - A^\#) = 2r[A, A^*] - 2r(A)$.

(d) $A^\dagger = A^\# \iff R(A^*) = R(A)$, *i.e.*, A *is EP.*

Proof. Since both $A^\dagger$ and A^D are outer inverses of A, it follows from Eq.(5.1) that

$$\begin{aligned} r(A^\dagger - A^D) &= r\begin{bmatrix} A^\dagger \\ A^D \end{bmatrix} + r[A^\dagger, A^D] - r(A^\dagger) - r(A^D) \\ &= r\begin{bmatrix} A^* \\ A^k \end{bmatrix} + r[A^*, A^k] - r(A) - r(A^k), \end{aligned}$$

as required for Part (a). The results in Part (b)—(d) follows immediately from Part (a). □

Theorem 13.10. *Let* $A \in \mathcal{C}^{m\times m}$ *with* $\mathrm{Ind}(A) = k$. *Then*

(a) $r(AA^\dagger - AA^D) = r\begin{bmatrix} A^k \\ A^* \end{bmatrix} - r(A^k)$.

(b) $r(A^{\dagger}A - A^{D}A) = r[A^k,\ A^*] - r(A^k)$.

(c) $r(AA^{\dagger} - AA^{D}) = r(AA^{\dagger}) - r(AA^{D}) \iff R[(A^k)^*] \subseteq R(A)$.

(d) $r(A^{\dagger}A - A^{D}A) = r(A^{\dagger}A) - r(A^{D}A) \iff R(A^k) \subseteq R(A^*)$.

(e) $r[A^k(A^k)^{\dagger} - AA^{D}] = r\begin{bmatrix} A^k \\ (A^*)^k \end{bmatrix} - r(A^k)$.

(f) $r[(A^k)^{\dagger}A^k - A^{D}A] = r[A^k,\ (A^*)^k] - r(A^k)$.

(g) $A^k(A^k)^{\dagger} = AA^{D} \iff (A^k)^{\dagger}A^k = A^{D}A \iff A^k$ *is EP.*

Proof. Note that both $AA^{\dagger}$ and AA^{D} are idempotent. Then it follows by Eq.(3.1) that

$$\begin{aligned} r(AA^{\dagger} - AA^{D}) &= r\begin{bmatrix} AA^{\dagger} \\ AA^{D} \end{bmatrix} + r[AA^{\dagger},\ AA^{D}] - r(AA^{\dagger}) - r(AA^{D}) \\ &= r\begin{bmatrix} A^* \\ A^k \end{bmatrix} + r[A,\ A^k] - r(A) - r(A^k) = r\begin{bmatrix} A^* \\ A^k \end{bmatrix} - r(A^k). \end{aligned}$$

as required for Part (a). Similarly we can get (b). The result in Part (c) follows immediately from Parts (a) and (b). similarly we can show Parts (e)—(g). □

Theorem 13.11. *Let* $A \in C^{m\times m}$ *with* Ind$(A) = k$. *Then*

(a) $r(A^{\dagger}A^{D} - A^{D}A^{\dagger}) = r\begin{bmatrix} A^k \\ A^* \end{bmatrix} + r[A^k,\ A^*] - 2r(A) = r(A^{\dagger}A^k - A^kA^{\dagger})$.

(b) $A^{\dagger}A^{D} = A^{D}A^{\dagger} \iff R(A^k) \subseteq R(A^*)$ *and* $R[(A^k)^*] \subseteq R(A)$ *i.e.. A is power* $-$ *EP* $\iff A^{\dagger}A^k = A^kA^{\dagger}$.

Proof. Applying Eq.(2.2) to $A^{\dagger}A^{D} - A^{D}A^{\dagger}$ yields

$$\begin{aligned} r(A^{\dagger}A^{D} - A^{D}A^{\dagger}) &= r\begin{bmatrix} -A^*AA^* & 0 & A^*A^{D} \\ 0 & A^*AA^* & A^* \\ A^* & A^{D}A^* & 0 \end{bmatrix} - 2r(A) \\ &= r\begin{bmatrix} -A^*AA^* & -A^*A^{D}AA^* & A^*A^{D} \\ 0 & 0 & A^* \\ A^* & A^{D}A^* & 0 \end{bmatrix} - 2r(A) \\ &= r\begin{bmatrix} 0 & 0 & A^*A^{D} \\ 0 & 0 & A^* \\ A^* & A^{D}A^* & 0 \end{bmatrix} - 2r(A) \\ &= r\begin{bmatrix} A^{D} \\ A^* \end{bmatrix} + r[A^{D},\ A^*] - 2r(A) = r\begin{bmatrix} A^k \\ A^* \end{bmatrix} + r[A^k,\ A^*] - 2r(A). \end{aligned}$$

The second one in Part (a) follows from Theorem 6.4. The result in Part (b) follows immediately from Part (a). □

Theorem 13.12. *Let* $A \in C^{m\times m}$ *with* Ind$(A) = k$. *Then*

(a) $r(A^*A^D - A^DA^*) = r\begin{bmatrix} A^k(AA^* - A^*A)A^k & 0 & A^kA^* \\ 0 & 0 & A^k \\ A^*A^k & A^k & 0 \end{bmatrix} - 2r(A^k).$

(b) $r(A^*A^D - A^DA^*) = r(A^{k+1}A^*A^k - A^kA^*A^{k+1})$, *if* $R(A^*A^k) \subseteq R(A^k)$ *and* $R[A(A^k)^*] \subseteq R[(A^k)^*]$.

(c) $r(A^*A^D - A^DA^*) = r\begin{bmatrix} A^kA^* \\ A^k \end{bmatrix} + r[\, A^k,\ A^*A^k\,] - 2r(A^k)$, *if* $A^{k+1}A^*A^k = A^kA^*A^{k+1}$.

(d) $A^*A^D = A^DA^* \Longleftrightarrow R(A^*A^k) \subseteq R(A^k)$, $R[A(A^k)^*] \subseteq R[(A^k)^*]$ *and* $A^{k+1}A^*A^k = A^kA^*A^{k+1}$.

Proof. Applying Eq.(2.3) to $A^*A^D - A^DA^*$ yields

$$\begin{aligned} r(A^*A^D - A^DA^*) &= r[\, A^*A^k(A^{2k+1})^\dagger A^k - A^k(A^{2k+1})^\dagger A^kA^*\,] \\ &= r\begin{bmatrix} -A^{2k+1} & 0 & A^k \\ 0 & A^{2k+1} & A^kA^* \\ A^*A^k & A^k & 0 \end{bmatrix} - 2r(A^{2k+1}) \\ &= r\begin{bmatrix} -A^{2k+1} & 0 & A^k \\ -A^{k+1}A^*A^k & 0 & A^kA^* \\ A^*A^k & A^k & 0 \end{bmatrix} - 2r(A^k) \\ &= r\begin{bmatrix} 0 & 0 & A^k \\ A^kA^*A^{k+1} - A^{k+1}A^*A^k & 0 & A^kA^* \\ A^*A^k & A^k & 0 \end{bmatrix} - 2r(A^k) \\ &= r\begin{bmatrix} A^k(AA^* - A^*A)A^k & 0 & A^kA^* \\ 0 & 0 & A^k \\ A^*A^k & A^k & 0 \end{bmatrix} - 2r(A^k). \end{aligned}$$

as required for Part (a). Parts(b) and (c) follow from Part (a). Next applying Lemma 1.2(f) to the rank equality in Part (a) yields Part (d). □

Theorem 13.13. *Let* $A \in \mathcal{C}^{m\times m}$ *with* $\mathrm{Ind}(A) = k$. *Then*

(a) $r[\, A - (A^D)^\dagger\,] = r\begin{bmatrix} A & A^k & (A^k)^* \\ A^k & 0 & 0 \\ (A^k)^* & 0 & 0 \end{bmatrix} - 2r(A^k).$

(b) $r[\, A - (A^D)^\dagger\,] = r(A) - r(A^k)$, *if* A^k *is EP.*

(c) $(A^D)^\dagger = A \Longleftrightarrow A$ *is EP.*

Proof. According to Cline's identity $(A^D)^\dagger = (A^k)^\dagger A^{2k+1}(A^k)^\dagger$(see [10] and [22]), we find by Eq.(2.8) that

$$\begin{aligned} r[\, A - (A^D)^\dagger\,] &= r[\, A - (A^k)^\dagger A^{2k+1}(A^k)^\dagger\,] \\ &= r\begin{bmatrix} (A^k)^*A^{2k+1}(A^k)^* & (A^k)^*A^k(A^k)^* & 0 \\ (A^k)^*A^k(A^k)^* & 0 & (A^k)^* \\ 0 & (A^k)^* & -A \end{bmatrix} - 2r(A^k) \end{aligned}$$

$$= r\begin{bmatrix} A^{2k+1} & A^k(A^k)^* & 0 \\ (A^k)^*A^k & 0 & (A^k)^* \\ 0 & (A^k)^* & -A \end{bmatrix} - 2r(A^k)$$

$$= r\begin{bmatrix} A^{2k+1} & 0 & A^{k+1} \\ (A^k)^*A^k & 0 & (A^k)^* \\ 0 & (A^k)^* & -A \end{bmatrix} - 2r(A^k)$$

$$= r\begin{bmatrix} 0 & 0 & A^{k+1} \\ 0 & 0 & (A^k)^* \\ A^{k+1} & (A^k)^* & -A \end{bmatrix} - 2r(A^k)$$

$$= r\begin{bmatrix} A & A^k & (A^k)^* \\ A^k & 0 & 0 \\ (A^k)^* & 0 & 0 \end{bmatrix} - 2r(A^k).$$

as required for Part (a). The results in Part (b) and (c) follows immediately from Part (a). □

Theorem 13.14. *Let* $A \in \mathcal{C}^{m\times m}$ *with* $\operatorname{Ind}(A) = k$. *Then*

(a) $r[\,AA^DA - (A^D)^\dagger\,] = r\begin{bmatrix} A^k \\ (A^k)^* \end{bmatrix} + r[\,A^k,\ (A^k)^*\,] - 2r(A^k).$

(b) $(A^D)^\dagger = AA^DA \Longleftrightarrow A^k$ *is EP.*

Proof. It is easy to verify that both AA^DA and $(A^D)^\dagger$ are outer inverses of A^D. In that case it follows from Eq.(5.1) that

$$\begin{aligned} r[\,AA^DA - (A^D)^\dagger\,] &= r\begin{bmatrix} AA^DA \\ (A^D)^\dagger \end{bmatrix} + r[\,AA^DA,\ (A^D)^\dagger\,] - r(AA^DA) - r[(A^D)^\dagger] \\ &= r\begin{bmatrix} A^D \\ (A^D)^* \end{bmatrix} + r[\,A^D,\ (A^D)^*\,] - 2r(A^k) \\ &= r\begin{bmatrix} A^k \\ (A^k)^* \end{bmatrix} + r[\,A^k,\ (A^k)^*\,] - 2r(A^k), \end{aligned}$$

as required for Part (a). The result in Part (b) follows immediately from Part (a). □

Theorem 13.15. *Let* $A,\ B \in \mathcal{C}^{m\times m}$ *with* $\operatorname{Ind}(A) = k$ *and* $\operatorname{Ind}(B) = l$. *Then*

(a) $r(\,AA^D - BB^D\,) = r\begin{bmatrix} A^k \\ B^l \end{bmatrix} + r[\,A^k,\ B^l\,] - r(A^k) - r(B^l).$

(b) $AA^D = BB^D \Longleftrightarrow R(A^k) = R(B^l)$ *and* $R[(A^k)^*] = R[(B^l)^*]$.

(c) $r(\,AA^D\ -\ BB^D\,)$ *is nonsingular* $\Longleftrightarrow r\begin{bmatrix} A^k \\ B^l \end{bmatrix} = r[\,A^k,\ B^l\,] = r(A^k) + r(B^l) = m \Longleftrightarrow$ $R(A^k) \oplus R(B^l) = R[(A^k)^*] \oplus R[(B^l)^*] = \mathcal{C}^m$.

Proof. Note that both AA^D and BB^D are idempotent. Then it follows from Eq.(3.1) that

$$\begin{aligned} r(AA^D - BB^D) &= r\begin{bmatrix} AA^D \\ BB^D \end{bmatrix} + r[AA^D, BB^D] - r(AA^D) - r(BB^D) \\ &= r\begin{bmatrix} A^D \\ B^D \end{bmatrix} + r[A^D, B^D] - r(A^D) - r(B^D) \\ &= r\begin{bmatrix} A^k \\ B^l \end{bmatrix} + r[A^k, B^l] - r(A^k) - r(B^l), \end{aligned}$$

as required for Part (a). The results in Parts (b) and (c) follow immediately from Part (a). $\square$

Theorem 13.16. *Let $A, B \in \mathcal{C}^{m\times m}$ with* $\mathrm{Ind}(A) = k$. *Then*

(a) $r(AA^DB - BA^DA) = r\begin{bmatrix} A^k \\ A^kB \end{bmatrix} + r[A^k, BA^k] - 2r(A^k)$.

(b) $r(A^DAA^\dagger - A^\dagger AA^D) = r\begin{bmatrix} A^k \\ A^* \end{bmatrix} + r[A^k, A^*] - 2r(A) = r(A^DA^\dagger - A^\dagger A^D)$.

In particular,

(c) $AA^DB = BA^DA \Longleftrightarrow R(BA^k) = R(A^k)$ *and* $R[(A^kB)^*] = R[(A^k)^*]$.

(d) $A^DAA^\dagger = A^\dagger AA^D \Longleftrightarrow A^DA^\dagger = A^\dagger A^D \Longleftrightarrow R(A^k) \subseteq R(A^*)$ *and* $R[(A^k)^*] \subseteq R(A)$.

Proof. Note $AA^D = A^DA$ is idempotent. It follows by Eq.(4.1) that

$$\begin{aligned} r(AA^DB - BA^DA) &= r\begin{bmatrix} AA^DB \\ A^DA \end{bmatrix} + r[BA^DA, AA^D] - r(AA^D) - r(A^DA) \\ &= r\begin{bmatrix} A^DB \\ A^D \end{bmatrix} + r[BA^D, A^D] - 2r(A^D) \\ &= r\begin{bmatrix} A^kB \\ A^k \end{bmatrix} + r[BA^k, A^k] - 2r(A^k). \end{aligned}$$

Thus we have Parts (a). Replacing B by $A^\dagger$ in Part (a) and simplifying it yields the first equality in Part (b). The second equality in Part (b) follows from Theorem 13.5(a). $\square$

Theorem 13.17. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times m}$ with* $\mathrm{Ind}(B) = k$ *and $C \in \mathcal{C}^{n\times n}$ with* $\mathrm{Ind}(C) = l$. *Then*

(a) $r(BB^DA - AC^DC) = r\begin{bmatrix} B^kA \\ C^l \end{bmatrix} + r[AC^l, B^k] - r(B^k) - r(C^l)$.

(b) $BB^DA = AC^DC \Longleftrightarrow R(AC^l) \subseteq R(B^k)$ *and* $R[(B^kA)^*] \subseteq R[(C^l)^*]$.

Proof. Note that both BB^D and C^DC are idempotent. Then it follows from Eq.(4.1) that

$$r(BB^DA - AC^DC) = r\begin{bmatrix} BB^DA \\ C^DC \end{bmatrix} + r[AC^DC, BB^D] - r(BB^D) - r(C^DC)$$

$$= r\begin{bmatrix} B^DA \\ C^D \end{bmatrix} + r[\,AC^D,\ B^D\,] - r(B^D) - r(C^D)$$

$$= r\begin{bmatrix} B^kA \\ C^l \end{bmatrix} + r[\,AC^l,\ B^k\,] - r(B^k) - r(C^l),$$

as required for Part (a). $\square$

Theorem 13.18. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{m\times m}$ with* $\mathrm{Ind}(B) = k$ *and $C \in \mathcal{C}^{n\times n}$ with* $\mathrm{Ind}(C) = l$. *Then*

(a) $r[\,A,\ B^k\,] = r(B^k) + r(\,A - BB^DA\,)$.

(b) $r\begin{bmatrix} A \\ C^l \end{bmatrix} = r(C^l) + r(\,A - AC^DC\,)$.

(c) $r\begin{bmatrix} A & B^k \\ C^l & 0 \end{bmatrix} = r(B^k) + r(C^l) + r[\,(\,I_m - BB^D\,)A(\,I_n - C^DC\,)\,]$.

Proof. Applying Eq.(1.7) to $A - BB^DA$ yields

$$\begin{aligned} r(\,A - BB^DA\,) &= r[\,A - B^{k+1}(B^{2k+1})^{\dagger}B^kA\,] \\ &= r\begin{bmatrix} B^{2k+1} & B^kA \\ B^{k+1} & A \end{bmatrix} - r(B^{2k+1}) \\ &= r\begin{bmatrix} 0 & 0 \\ B^{k+1} & A \end{bmatrix} - r(B^k) = r[\,A,\ B^k\,] - r(B^k), \end{aligned}$$

as required for Part (a). Similarly we can show Parts (b) and (c). $\square$

Theorem 13.19. *Let $A,\ B \in \mathcal{C}^{m\times m}$ with* $\mathrm{Ind}(A) = k$ *and* $\mathrm{Ind}(B) = l$. *Then*

(a) $r(\,AB - ABB^DA^DAB\,) = r\begin{bmatrix} A^{2k} & A^kB^l \\ B^lA^k & B^{2l} \end{bmatrix} + r(AB) - r(A^k) - r(B^l)$.

(b) $B^DA^D \in \{(AB)^-\} \Longleftrightarrow r\begin{bmatrix} A^{2k} & A^kB^l \\ B^lA^k & B^{2l} \end{bmatrix} = r(A^k) + r(B^l) - r(AB)$.

Proof. It follows by Eq.(2.9) that

$$\begin{aligned} r(\,AB - ABB^DA^DAB\,) &= r[\,AB - AB^{k+1}(B^{2k+1})^{\dagger}B^lA^k(A^{2k+1})^{\dagger}A^{k+1}B\,] \\ &= r\begin{bmatrix} B^lA^k & B^{2l+1} & 0 \\ A^{2k+1} & 0 & A^{k+1}B \\ 0 & AB^{l+1} & -AB \end{bmatrix} - r(A^{2k+1}) - r(B^{2l+1}) \\ &= r\begin{bmatrix} B^lA^k & B^{2l+1} & 0 \\ A^{2k+1} & A^{k+1}B^{l+1} & 0 \\ 0 & 0 & -AB \end{bmatrix} - r(A^k) - r(B^l) \\ &= r\begin{bmatrix} B^lA^k & B^{2l+l} \\ A^{2k+1} & A^kB^{l+l} \end{bmatrix} + r(AB) - r(A^k) - r(B^l) \end{aligned}$$

$$= r\begin{bmatrix} B^lA^k & B^{2l} \\ A^{2k} & A^kB^l \end{bmatrix} + r(AB) - r(A^k) - r(B^l).$$

Thus we have Parts (a) and (b). $\square$

Theorem 13.20. *Let A, $B \in \mathcal{C}^{m\times m}$ with* $\mathrm{Ind}(A) = k$ *and* $\mathrm{Ind}(B) = l$. *Then*

(a) $r(\,AA^DB^DB - BB^DA^DA\,) = r\begin{bmatrix} A^k \\ B^l \end{bmatrix} + r[\,A^k,\ B^l\,] + r(A^kB^l) + r(B^lA^k) - 2r(A^k) - 2r(B^l)$.

(b) $AA^DB^DB = BB^DA^DA \Longleftrightarrow r\begin{bmatrix} A^k \\ B^l \end{bmatrix} = r(A^k) + r(B^l) - r(A^kB^l)$ *and* $r[\,A^k,\ B^l\,] = r(A^k) + r(B^l) - r(B^lA^k)$.

Proof. Note that both $AA^D = A^DA$ and $BB^D = B^DB$ are idempotent. Then it follows by Eq.(3.26) that

$$\begin{aligned} & r(\,AA^DB^DB - BB^DA^DA\,) \\ &= r\begin{bmatrix} AA^D \\ BB^D \end{bmatrix} + r[\,A^DA,\ B^DB\,] + r(AA^DB^DB) + r(BB^DA^DA) - 2r(AA^D) - 2r(BB^D) \\ &= r\begin{bmatrix} A^k \\ B^l \end{bmatrix} + r[\,A^k,\ B^l\,] + r(A^kB^l) + r(B^lA^k) - 2r(A^k) - 2r(B^l). \end{aligned}$$

as required for Part (a). $\square$

Theorem 13.21. *Let A, $B \in \mathcal{C}^{m\times m}$ with* $\mathrm{Ind}(\,A + B\,) = k$ *and denote $N = A + B$. Then*

(a) $r(AN^DB) = r(AN^k) + r(N^kB) - r(N^k)$.

(b) $r(\,AN^DB\,) = r(A) + r(B) - r(N^k)$, *if* $R(B) \subseteq R(N^k)$ *and* $R(A^*) \subseteq R[(N^k)^*]$.

(c) $r(\,AN^DB - BN^DA\,) = r\begin{bmatrix} N^k \\ N^kB \end{bmatrix} + r[\,N^k,\ BN^k\,] - 2r(N^k)$.

(d) $AN^DB = BN^DA \Longleftrightarrow R(BN^k) \subseteq R(N^k)$ *and* $R[(N^kB)^*] \subseteq R[(N^k)^*]$.

Proof. It follows by Eq.(1.7), that

$$\begin{aligned} r(AN^DB) &= r[\,AN^k(N^{2k+1})^{\dagger}N^kB\,] \\ &= r\begin{bmatrix} N^{2k+1} & N^kB \\ AN^k & 0 \end{bmatrix} - r(N^{2k+1}) \\ &= r\begin{bmatrix} 0 & N^kB \\ AN^k & 0 \end{bmatrix} - r(N^k) = r(AN^k) + r(N^kB) - r(N^k), \end{aligned}$$

which is the first equality in Part (a). The second equality in Part (a) follows from $r(AN^k) = r(AN^D)$, $r(N^kB) = r(N^DB)$ and $r(N^D) = r(N^k)$. Under $R(B) \subseteq R(N^k)$ and $R(A^*) \subseteq R[(N^k)^*]$, it follows that $r(AN^k) = r(A)$ and $r(N^kB) = r(B)$. Thus Part (a) becomes Part (b). Next applying Eq.(2.3) to $AN^DB - BN^DA$

yields

$$
\begin{aligned}
& r(AN^DB - BN^DA) \\
= \;& r[AN^k(N^{2k+1})^\dagger N^kB - BN^k(N^{2k+1})^\dagger N^kA] \\
= \;& r\begin{bmatrix} -N^{2k+1} & 0 & N^kB \\ 0 & N^{2k+1} & N^kA \\ AN^k & BN^k & 0 \end{bmatrix} - 2r(N^{2k+1}) \\
= \;& r\begin{bmatrix} -N^kAN^k & -N^kBN^k & N^kB \\ N^kAN^k & N^kBN^k & N^kA \\ AN^k & BN^k & 0 \end{bmatrix} - 2r(N^k) \\
= \;& r\begin{bmatrix} 0 & 0 & N^kB \\ 0 & 0 & N^kA \\ AN^k & BN^k & 0 \end{bmatrix} - 2r(N^k) \\
= \;& r\begin{bmatrix} N^kA \\ N^kB \end{bmatrix} + r[AN^k, \; BN^k] - 2r(N^k) = r\begin{bmatrix} N^k \\ N^kB \end{bmatrix} + r[N^k, \; BN^k] - 2r(N^k).
\end{aligned}
$$

Thus we have Parts (c) and (d). $\square$

Theorem 13.22. *Let $A, \; B \in \mathcal{C}^{m\times m}$ be given, and let $N = A + B$ with* $\mathrm{Ind}(A+B) = k$. *Then*

(a) $r\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} - \begin{bmatrix} A \\ B \end{bmatrix}(A+B)^D[A, \; B]\right) = r(A) + r(B) - r(N^k)$.

(b) $\begin{bmatrix} A \\ B \end{bmatrix}(A+B)^D[A, \; B] = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \Longleftrightarrow \mathrm{Ind}(A+B) \leq 1$ *and* $r(A+B) = r(A) + r(B)$.

Proof. It follows by Eq.(1.7) that

$$
\begin{aligned}
& r\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} - \begin{bmatrix} A \\ B \end{bmatrix} N^D[A, \; B]\right) \\
= \;& r\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} - \begin{bmatrix} A \\ B \end{bmatrix} N^k(N^{2k+1})^\dagger N^k[A, \; B]\right) \\
= \;& r\begin{bmatrix} N^{2k+1} & N^kA & N^kB \\ AN^k & A & 0 \\ BN^k & 0 & B \end{bmatrix} - r(N^k) \\
= \;& r\begin{bmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{bmatrix} - r(N^k) = r(A) + r(B) - r(N^k),
\end{aligned}
$$

which is exactly Part (a). Note that $r(N^k) \leq r(N) = r(A+B) \leq r(A) + r(B)$. Thus $r(N^k) = r(A) + r(B)$ is equivalent to $\mathrm{Ind}(N) \leq 1$ and $r(N) = r(A) + r(B)$. $\square$

In general we have the following.

Theorem 13.23. *Let $A_1, A_2, \cdots, A_k \in \mathcal{C}^{m\times m}$ with* $\mathrm{Ind}(N) = k$, *where* $N = A_1 + A_2 + \cdots + A_k$, *and denote* $A = \mathrm{diag}(A_1, A_2, \cdots, A_k)$. *Then*

(a) $$r\left(A - \begin{bmatrix} A_1 \\ \vdots \\ A_k \end{bmatrix} N^D[A_1, \cdots, A_k] \right) = r(A_1) + \cdots + r(A_k) - r(N^k).$$

(b) $$\begin{bmatrix} A_1 \\ \vdots \\ A_k \end{bmatrix} N^D[A_1, \cdots, A_k] = A \iff \mathrm{Ind}(N) \leq 1 \textit{ and } r(N) = r(A_1) + \cdots + r(A_k).$$

Theorem 13.24. *Let $A_1, A_2, \cdots, A_k \in \mathcal{C}^{m\times m}$. Then the Drazin inverse of their sum satisfies the following equality*

$$(A_1 + A_2 + \cdots + A_k)^D = \frac{1}{k}[I_m, I_m, \cdots, I_m] \begin{bmatrix} A_1 & A_2 & \cdots & A_k \\ A_k & A_1 & \cdots & A_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}^D \begin{bmatrix} I_m \\ I_m \\ \vdots \\ I_m \end{bmatrix}. \tag{13.1}$$

Proof. Since A_i are square, Eq.(11.7) can be written as

$$U_m^* A U_m = \mathrm{diag}(J_1, J_2, \cdots, J_k,).$$

In that case, it is easy to verify that

$$(U_m^* A U_m)^D = U_m^* A^D U_m,$$

and

$$[\mathrm{diag}(J_1, J_2, \cdots, J_k)]^D = \mathrm{diag}(J_1^D, J_2^D, \cdots, J_k^D)$$

Thus we have

$$\begin{aligned} J_1^D &= [I_m, 0, \cdots, 0]U_m^* A^D U_m [I_m, 0, \cdots, 0]^T \\ &= \frac{1}{k}[I_m, I_m, \cdots, I_m]U_m^* A^D U_m [I_m, I_m, \cdots, I_m]^T. \end{aligned}$$

which is Eq.(13.1). □

Chapter 14

Rank equalities for submatrices in Drazin inverses

Let

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{14.1}$$

be a square block matrix over $\mathcal{C}$, where $A \in \mathcal{C}^{m\times m}$ and $D \in \mathcal{C}^{n\times n}$,

$$V_1 = \begin{bmatrix} A \\ C \end{bmatrix}, \quad V_2 = \begin{bmatrix} B \\ D \end{bmatrix}, \quad W_1 = [\, A,\ B \,], \quad W_2 = [\, C,\ D \,]. \tag{14.2}$$

and partition the Drazin inverse of M as

$$M^D = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}, \tag{14.3}$$

where $G_1 \in \mathcal{C}^{m\times m}$. It is, in general, quite difficult to give the expression of G_1—G_4. In this chapter we consider a simpler problem—the ranks of the submatrices G_1—G_4 in Eq.(14.3).

Theorem 14.1. *Let M and M^D be given by Eqs.(14.1) and Eq.(14.3) with $\mathrm{Ind}(M) \geq 1$. Then the ranks of G_1—G_4 in Eq.(14.3) can be determined by the following formulas*

$$r(G_1) = r\begin{bmatrix} M^k J_1 M^k & M^{k-1}V_1 \\ W_1 M^{k-1} & 0 \end{bmatrix} - r(M^k), \tag{14.4}$$

$$r(G_2) = r\begin{bmatrix} M^k J_2 M^k & M^{k-1}V_2 \\ W_1 M^{k-1} & 0 \end{bmatrix} - r(M^k), \tag{14.5}$$

$$r(G_3) = r\begin{bmatrix} M^k J_3 M^k & M^{k-1}V_1 \\ W_2 M^{k-1} & 0 \end{bmatrix} - r(M^k), \tag{14.6}$$

$$r(G_4) = r\begin{bmatrix} M^k J_4 M^k & M^{k-1}V_2 \\ W_2 M^{k-1} & 0 \end{bmatrix} - r(M^k), \tag{14.7}$$

where V_1, V_2, W_1 and W_2 are defined in Eq.(14.2), and

$$J_1 = \begin{bmatrix} -A & 0 \\ 0 & D \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & B \\ -C & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & -B \\ C & 0 \end{bmatrix}, \quad J_4 = \begin{bmatrix} A & 0 \\ 0 & -D \end{bmatrix}. \tag{14.8}$$

Proof. We only show Eq.(14.4). In fact G_1 in Eq.(14.3) can be written as

$$G_1 = [I_m,\ 0]M^D \begin{bmatrix} I_m \\ 0 \end{bmatrix} = P_1 M^D Q_1 = P_1 M^k (M^{2k+1})^{\dagger} M^k Q_1.$$

where $P_1 = [I_m,\ 0]$ and $Q_1 = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$. Then it follows by Eq.(1.6) and block elementary operations that

$$\begin{aligned} r(G_1) &= r\begin{bmatrix} M^{2k+1} & M^k Q_1 \\ P_1 M^k & 0 \end{bmatrix} - r(M^{2k+1}) \\ &= r\begin{bmatrix} M^{2k+1} - M^k Q_1 P_1 M M^k - M^k M Q_1 P_1 M^k & M^k Q_1 \\ P_1 M^k & 0 \end{bmatrix} - r(M^k) \\ &= r\begin{bmatrix} M^k(M - Q_1 P_1 M - M Q_1 P_1)M^k & M^k Q_1 \\ P_1 M^k & 0 \end{bmatrix} - r(M^k) \\ &= r\begin{bmatrix} M^k J_1 M^k & M^{k-1} V_1 \\ W_1 M^{k-1} & 0 \end{bmatrix} - r(M^k), \end{aligned}$$

which is exactly the equality (14.4). □

The further simplification of Eqs.(14.4)—(14.7) are quite difficult, because the powers of M appear in them. However if M in Eq.(14.1) satisfies some additional conditions, the four rank equalities in Eqs.(14.4)—(14.7) can reduce to simpler forms. We next present some of them. The first one is related to the well-known result on the Drazin inverse of an upper triangular block matrix (see Campbell and Meyer, 1979).

$$\begin{bmatrix} A & B \\ 0 & N \end{bmatrix}^D = \begin{bmatrix} A^D & X \\ 0 & N^D \end{bmatrix}, \tag{14.9}$$

where

$$X = (A^D)^2 \left[\sum_{i=0}^{l-1} (A^D)^i B N^i \right] (I_n - N^D N) + (I_m - A A^D) \left[\sum_{i=0}^{k-1} A^i B (N^D)^i \right] (N^D)^2 - A^D B N^D, \tag{14.10}$$

and $\mathrm{Ind}(A) = k$, $\mathrm{Ind}(N) = l$.

Theorem 14.2. *The rank of the submatrix X in Eq.(14.9) is*

$$r(X) = r\begin{bmatrix} A^k & P_t(B) & 0 \\ 0 & A^t B N^t & P_t(B) \\ 0 & 0 & N^l \end{bmatrix} - r\begin{bmatrix} A^k & P_t(B) \\ 0 & N^l \end{bmatrix}, \tag{14.11}$$

where $t = \mathrm{Ind}\begin{bmatrix} A & B \\ 0 & N \end{bmatrix}$, $P_t(B) = \sum_{i=0}^{t-1} A^{t-i-1} B N^i$. *In particular if* $A^k B N^l = 0$, *then*

$$r(X) = r[\,A^k,\ P_t(B)\,] + r\begin{bmatrix} P_t(B) \\ N^l \end{bmatrix} - r\begin{bmatrix} A^k & P_t(B) \\ 0 & N^l \end{bmatrix}. \tag{14.12}$$

In particular if $R[P_t(B)] \subseteq R(A^k)$ *and* $R[(P_t(B))^*] \subseteq R[(N^l)^*]$, *then* $r(X) = r(A^kBC^l)$.

Proof. It is easy to verify that

$$M^t = \begin{bmatrix} A & B \\ 0 & N \end{bmatrix}^t = \begin{bmatrix} A^t & P_t(B) \\ 0 & N^t \end{bmatrix}, \quad \text{and} \quad P_{2t+1}(B) = A^{t+1}P_t(B) + P_t(B)N^{t+1} + A^tBN^t.$$

Then applying Eq.(1.7) to $X = [\, I_m,\ 0\,] \begin{bmatrix} A^D & X \\ 0 & N^D \end{bmatrix} \begin{bmatrix} 0 \\ I_n \end{bmatrix} = P_1M^t(M^{2t+1})^\dagger M^tQ_2$, we find that

$$\begin{aligned} r(X) &= r\begin{bmatrix} M^{2t+1} & M^tQ_2 \\ P_1M^t & 0 \end{bmatrix} - r(M^{2k+1}) \\ &= r\begin{bmatrix} A^{2t+1} & P_{2t+1}(B) & P_t(B) \\ 0 & N^{2t+1} & N^t \\ A^t & P_t(B) & 0 \end{bmatrix} - r(M^k) \\ &= r\begin{bmatrix} 0 & A^tBN^t & P_t(B) \\ 0 & 0 & N^t \\ A^t & P_t(B) & 0 \end{bmatrix} - r(M^k) \\ &= r\begin{bmatrix} A^k & P_t(B) & 0 \\ 0 & A^tBD^t & P_t(B) \\ 0 & 0 & N^l \end{bmatrix} - r\begin{bmatrix} A^k & P_t(B) \\ 0 & N^l \end{bmatrix}. \end{aligned}$$

Thus we have the desired results. $\square$

Theorem 14.3. *Let* M *be given by Eq.*(14.1) *with* $\mathrm{Ind}(M) = 1$. *Then the ranks of* G_1 —G_4 *in the group inverse of* M *in Eq.*(14.3) *can be expressed as*

$$r(G_1) = r\begin{bmatrix} V_2DW_2 & V_1 \\ W_1 & 0 \end{bmatrix} - r(M), \qquad r(G_2) = r\begin{bmatrix} V_1BW_2 & V_2 \\ W_1 & 0 \end{bmatrix} - r(M), \tag{14.13}$$

$$r(G_3) = r\begin{bmatrix} V_2CW_1 & V_1 \\ W_2 & 0 \end{bmatrix} - r(M), \qquad r(G_4) = r\begin{bmatrix} V_1AW_1 & V_2 \\ W_2 & 0 \end{bmatrix} - r(M), \tag{14.14}$$

where V_1, V_2, W_1 *and* W_2 *are defined in Eq.*(14.2).

Proof. Note that $M^\# = M(M^3)^\dagger M$ when $\mathrm{Ind}(M) = 1$. Thus G_1 in Eq.(14.13) can be written as $G_1 = W_1(M^3)^\dagger V_1$. In that case it follows by Eq.(1.7) that

$$\begin{aligned} r(G_1) &= r\begin{bmatrix} M^3 & V_1 \\ W_1 & 0 \end{bmatrix} - r(M^3) \\ &= r\begin{bmatrix} [0,\ V_2]M\begin{bmatrix} 0 \\ W_2 \end{bmatrix} & V_1 \\ W_1 & 0 \end{bmatrix} - r(M) = r\begin{bmatrix} V_2DW_2 & V_1 \\ W_1 & 0 \end{bmatrix} - r(M). \end{aligned}$$

In the same manner we can show the other three in Eqs.(14.13) and (14.14). $\square$

Corollary 14.4. *Let M be given by Eq.(14.1) with* $\mathrm{Ind}(M) = 1$.

(a) *If M satisfies the rank additivity condition*

$$r(M) = r(V_1) + r(V_2) = r(W_1) + r(W_2),$$

then the ranks of G_1 —G_4 in the group inverse of M in Eq.(14.3) can be expressed as

$$r(G_1) = r(V_1) + r(W_1) + r(V_2DW_2) - r(M),$$

$$r(G_2) = r(V_2) + r(W_1) + r(V_1BW_2) - r(M),$$

$$r(G_3) = r(V_2) + r(W_1) + r(V_2CW_1) - r(M),$$

$$r(G_4) = r(V_2) + r(W_2) + r(V_1AW_1) - r(M).$$

(b) *If M satisfies the rank additivity condition*

$$r(M) = r(A) + r(B) + r(C) + r(D),$$

then the ranks of G_1 —G_4 in the group inverse of M in Eq.(14.3) satisfy

$$r(G_1) = r(A) - r(D) + r(V_2DW_2), \qquad r(G_2) = r(B) - r(C) + r(V_1BW_2),$$

$$r(G_3) = r(C) - r(B) + r(V_2CW_1), \qquad r(G_2) = r(D) - r(A) + r(V_1AW_1),$$

where V_1, V_2, W_1 and W_2 are defined in Eq.(14.2).

In addition, we have some inequalities on ranks of submatrices in the group inverse of a block matrix.

Corollary 14.5. *Let M be given by Eq.(14.1) with* $\mathrm{Ind}(M) = 1$. *Then the ranks of the matrices G_1 —G_4 in (14.3) satisfy the following rank inequalities*

(a) $r(G_1) \geq r(V_1) + r(W_1) - r(M)$.

(b) $r(G_1) \leq r(V_1) + r(W_1) + r(D) - r(M)$.

(c) $r(G_2) \geq r(V_2) + r(W_1) - r(M)$.

(d) $r(G_2) \leq r(V_2) + r(W_1) + r(B) - r(M)$.

(e) $r(G_3) \geq r(V_1) + r(W_2) - r(M)$.

(f) $r(G_3) \leq r(V_1) + r(W_2) + r(C) - r(M)$.

(g) $r(G_4) \geq r(V_2) + r(W_2) - r(M)$.

(h) $r(G_4) \leq r(V_2) + r(W_2) + r(A) - r(M)$.

Proof. Follows from Eqs.(14.13) and (14.14). $\square$.

Chapter 15

Reverse order laws for Drazin inverses of products of matrices

In this chapter we consider reverse order laws for Drazin inverses of products of matrices. We will give necessary and sufficient conditions for $(ABC)^D = C^D B^D A^D$ to hold and then present some of its consequences.

Lemma 15.1. *Let A, $X \in \mathcal{C}^{m\times m}$ with $\operatorname{Ind}(A) = k$. Then $X = A^D$ if and only if*

$$A^{k+1}X = A^k, \qquad XA^{k+1} = A^k, \qquad \textit{and} \qquad r(X) = r(A^k). \tag{15.1}$$

Proof. Follows from the definition of the Drazin inverse of a matrix. □

Lemma 15.2. *Let A, B, $C \in \mathcal{C}^{m\times m}$ with $\operatorname{Ind}(A) = k_1$, $\operatorname{Ind}(B) = k_2$ and $\operatorname{Ind}(C) = k_3$. Then the product $C^D B^D A^D$ of the Drazin inverses of A, B, and C can be expressed in the form*

$$C^D B^D A^D = [C^{k_3},\ 0,\ 0]\begin{bmatrix} 0 & 0 & A^{2k_1+1} \\ 0 & B^{2k_2+1} & B^{k_2}A^{k_1} \\ C^{2k_3+1} & C^{k_3}B^{k_2} & 0 \end{bmatrix}^{\dagger}\begin{bmatrix} A^{k_1} \\ 0 \\ 0 \end{bmatrix} := PN^{\dagger}Q. \tag{15.2}$$

where P, N and Q satisfy the three properties

$$R(Q) \subseteq R(N), \qquad R(P^*) \subseteq R(N^*), \qquad r(N) = r(A^{k_1}) + r(B^{k_2}) + r(C^{k_3}). \tag{15.3}$$

Proof. It is easy to verify that the 3×3 block matrix N in Eq.(15.2) satisfies the conditions in Lemma 8.8. Hence it follows by Eq.(8.8) that

$$N^{\dagger} = \begin{bmatrix} (C^{2k_3+1})^{\dagger}C^{k_3}B^{k_2}(B^{2k_2+1})^{\dagger}B^{k_2}A^{k_1}(A^{2k_1+1})^{\dagger}A^{k_1} & * & * \\ * & * & 0 \\ * & 0 & 0 \end{bmatrix}. \tag{15.4}$$

Thus we have Eq.(15.2). The three properties in Eq.(15.3) follows from the structure of N. □

The main results of the chapter are the following two.

Theorem 15.3. *Let A, B, $C \in \mathcal{C}^{m\times m}$ with $\operatorname{Ind}(A) = k_1$, $\operatorname{Ind}(B) = k_2$ and $\operatorname{Ind}(C) = k_3$, and denote $M = ABC$ with $\operatorname{Ind}(M) = t$. Then the reverse order law $(ABC)^D = C^D B^D A^D$ holds if and only if A, B*

and C satisfy the three rank equalities

$$r\begin{bmatrix} 0 & 0 & A^{2k_1+1} & A^{k_1} \\ 0 & B^{2k_2+1} & B^{k_2}A^{k_1} & 0 \\ C^{2k_3+1} & C^{k_3}B^{k_2} & 0 & 0 \\ M^{t+1}C^{k_3} & & 0 & M^t \end{bmatrix} = r(A^{k_1}) + r(B^{k_2}) + r(C^{k_3}), \tag{15.5}$$

$$r\begin{bmatrix} 0 & 0 & A^{2k_1+1} & A^{k_1}M^{t+1} \\ 0 & B^{2k_2+1} & B^{k_2}A^{k_1} & 0 \\ C^{2k_3+1} & C^{k_3}B^{k_2} & 0 & 0 \\ C^{k_3} & & 0 & M^t \end{bmatrix} = r(A^{k_1}) + r(B^{k_2}) + r(C^{k_3}). \tag{15.6}$$

$$r\begin{bmatrix} B^{2k_2+1} & B^{k_2}A^{k_1} \\ C^{k_3}B^{k_2} & 0 \end{bmatrix} = r(B^{k_2}) + r(M^t). \tag{15.7}$$

Proof. Let $X = C^DB^DA^D$. Then by definition of the Drazin inverse, $X = M^D$ if and only if $M^{t+1}X = M^t$, $XM^{t+1} = M^t$ and $r(X) = r(M^t)$, which are equivalent to

$$r(\, M^k - M^{k+1}X\,) = 0, \qquad r(\, M^k - XM^{k+1}\,) = 0 \ \text{ and } \ r(X) = r(M^t). \tag{15.8}$$

Replacing X in Eq.(15.8) by $X = PN^{\dagger}Q$ in Eq.(15.2) and applying Eq.(1.7) them, we find that

$$r(\, M^t - M^{t+1}X\,) = r(\, M^t - M^{t+1}PN^{\dagger}Q\,) = r\begin{bmatrix} N & Q \\ M^{t+1}P & M^t \end{bmatrix} - r(N).$$

$$r(\, M^t - XM^{t+1}\,) = r(\, M^t - PN^{\dagger}QM^{t+1}\,) = r\begin{bmatrix} N & QM^{t+1} \\ P & M^t \end{bmatrix} - r(N),$$

$$r(X) = r(PN^{\dagger}QM^{t+1}) = r\begin{bmatrix} N & Q \\ P & 0 \end{bmatrix} - r(N).$$

Putting them in Eq.(15.8), we obtain Eqs.(15.5)—(15.7). □

Theorem 15.4. *Let A, B, $C \in \mathcal{C}^{m\times m}$ with $\mathrm{Ind}(A) = k_1$, $\mathrm{Ind}(B) = k_2$ and $\mathrm{Ind}(C) = k_3$, and let $M = ABC$ with $\mathrm{Ind}(M) = t$. Then the reverse order law $(ABC)^D = C^DB^DA^D$ holds if and only if A, B and C satisfy the following rank equality*

$$r\begin{bmatrix} 0 & 0 & A^{2k_1+1} & A^{k_1} & 0 \\ 0 & B^{2k_2+1} & B^{k_2}A^{k_1} & 0 & 0 \\ C^{2k_3+1} & C^{k_3}B^{k_2} & 0 & 0 & 0 \\ C^{k_3} & 0 & 0 & 0 & M^t \\ 0 & 0 & 0 & M^t & M^{2t+1} \end{bmatrix} = r(A^{k_1}) + r(B^{k_2}) + r(C^{k_3}) - r(M^t). \tag{15.9}$$

Proof. Applying Eq.(2.3) to $(ABC)^D - C^D B^D A^D = M^t(M^{2t+1})^\dagger M^t - PN^\dagger Q$, we find that

$$\begin{aligned} r[(ABC)^D - C^D B^D A^D] &= r[PN^\dagger Q - M^t(M^{2t+1})^\dagger M^t] \\ &= r\begin{bmatrix} N & 0 & Q \\ 0 & -M^{2t+1} & M^t \\ P & M^t & 0 \end{bmatrix} - r(N) - r(M^t) \\ &= r\begin{bmatrix} N & 0 & Q \\ 0 & 0 & M^t \\ P & M^t & -M^{2t+1} \end{bmatrix} - r(N) - r(M^t). \end{aligned}$$

Thus Eq.(15.9) follows by putting P, N and Q in it. $\square$

We next give some particular cases of the above two theorems.

Corollary 15.5. *Let $A, B, C \in \mathcal{C}^{m\times m}$ with $\operatorname{Ind}(B) = k$ and $\operatorname{Ind}(ABC) = t$, where A and C are nonsingular. Then*

(a) $r[(ABC)^D - C^{-1}B^DA^{-1}] = r\begin{bmatrix} B^k \\ (ABC)^tA \end{bmatrix} + r[B^k, \ C(ABC)^t] - r(B^k) - r[(ABC)^t]$.

(b) $(ABC)^D = C^{-1}B^DA^{-1} \Longleftrightarrow R[C(ABC)^t] = R(B^k)$ *and* $R\{[(ABC)^tA]^*\} = R[(B^k)^*]$.

Proof. It is easy to verify that both $(ABC)^D$ and $C^{-1}B^DA^{-1}$ are outer inverses of ABC. Thus it follows from Eq.(5.1) that

$$\begin{aligned} & r[(ABC)^D - C^{-1}B^DA^{-1}] \\ &= r\begin{bmatrix} (ABC)^D \\ C^{-1}B^DA^{-1} \end{bmatrix} + r[(ABC)^D, \ C^{-1}B^DA^{-1}] - r[(ABC)^D] - r(B^D) \\ &= r\begin{bmatrix} (ABC)^tA \\ B^k \end{bmatrix} + r[C(ABC)^t, \ B^k] - r[(ABC)^t] - r(B^k), \end{aligned}$$

as required for Part (a). Notice that

$$r\begin{bmatrix} B^k \\ (ABC)^tA \end{bmatrix} \geq r(B^k), \quad r\begin{bmatrix} B^k \\ (ABC)^tA \end{bmatrix} \geq r[(ABC)^t].$$

and

$$r[B^t, \ C(ABC)^t] \geq r(B^k), \quad r[B^t, \ C(ABC)^t] \geq r[(ABC)^t].$$

Then Part (b) follows from part (a). $\square$

Corollary 15.6. *Let $A, B, C \in \mathcal{C}^{m\times m}$ with $\operatorname{Ind}(A) = k_1$, $\operatorname{Ind}(B) = k_2$ and $\operatorname{Ind}(C) = k_3$, and let $M = ABC$ with $\operatorname{Ind}(M) = t$. Moreover suppose that*

$$AB = BA, \qquad AC = CA, \qquad BC = CB. \tag{15.10}$$

Then the reverse order law $(ABC)^D = C^DB^DA^D$ holds if and only if A, B and C satisfy Eq.(15.7).

Proof. It is not difficult to verify that under Eq.(15.10), the two rank equalities in Eqs.(15.5) and (15.6) become two identities. Thus, Eq.(15.7) becomes a necessary and sufficient condition for $(ABC)^D = C^D B^D A^D$ to hold. □

Corollary 15.7. *Let* $A,\ B \in \mathcal{C}^{m\times m}$ *with* $\mathrm{Ind}(A) = k$, $\mathrm{Ind}(B) = l$ *and* $\mathrm{Ind}(AB) = t$. *Then the following three are equivalent:*

(a) $(AB)^D = B^D A^D$.

(b) $$r\begin{bmatrix} 0 & A^{2k+1} & A^k & 0 \\ B^{2l+1} & B^l A^k & 0 & 0 \\ B^l & 0 & 0 & (AB)^t \\ 0 & 0 & (AB)^t & (AB)^{2t+1} \end{bmatrix} = r(A^k) + r(B^l) - r[(AB)^t].$$

(c) *The three rank equalities are all satisfied*

$$r[(AB)^t] = r(B^l A^k).$$

$$r\begin{bmatrix} 0 & A^{2k+1} & A^k \\ B^{2l+1} & B^l A^k & 0 \\ (AB)^{t+1} B^l & 0 & -(AB)^t \end{bmatrix} = r(A^k) + r(B^l).$$

$$r\begin{bmatrix} 0 & A^{2k+1} & A^k (AB)^{t+1} \\ B^{2l+1} & B^l A^k & 0 \\ B^l & 0 & -(AB)^t \end{bmatrix} = r(A^k) + r(B^l).$$

Proof. Letting $C = I_m$ in Eq.(15.9) results in Part (b), and letting $B = I_m$ and replacing C by B in Theorem 15.4 result in Part (c). □

Chapter 16

Ranks equalities related to weighted Moore-Penrose inverses

The weighted Moore-Penrose inverse of a matrix $A \in \mathcal{C}^{m\times n}$ with respect to two positive definite matrices $M \in \mathcal{C}^{m\times m}$ and $N \in \mathcal{C}^{n\times n}$ is defined to be the unique solution of the following four matrix equations

$$AXA = A, \qquad XAX = X, \qquad (MAX)^* = MAX, \qquad (NXA)^* = NXA, \tag{16.1}$$

and this X is often denoted by $X = A^{\dagger}_{M,N}$. In particular, when $M = I_m$ and $N = I_n$, $A^{\dagger}_{M,N}$ is the standard Moore-Penrose inverse $A^{\dagger}$ of A. As is well known (see, e.g., Rao and Mitra, 1971), the weighted Moore-Penrose inverse $A^{\dagger}_{M,N}$ of A can be written as a matrix expressions involving a standard Moore-Penrose inverse as follows

$$A^{\dagger}_{M,N} = N^{-\frac{1}{2}}(M^{\frac{1}{2}}AN^{-\frac{1}{2}})^{\dagger}M^{\frac{1}{2}}, \tag{16.2}$$

where $M^{\frac{1}{2}}$ and $N^{\frac{1}{2}}$ are the positive definite square roots of M and N, respectively. According to Eq.(16.2), it is easy to verify that

$$R(A^{\dagger}_{M,N}) = R(N^{-1}A^*), \quad \text{and} \quad R[(A^{\dagger}_{M,N})^*] = R(MA). \tag{16.2}$$

Based on these basic facts and the rank formulas in Chapters 2—5, we now can establish various rank equalities related to weighted Moore-Penrose inverses of matrices, and the consider their various consequences.

Theorem 16.1. *Let $A \in \mathcal{C}^{m\times n}$ be given, $M \in \mathcal{C}^{m\times m}$ and $N \in \mathcal{C}^{n\times n}$ be two positive definite matrices. Then*

(a) $r(\, A^{\dagger} - A^{\dagger}_{M,N}\,) = r\begin{bmatrix} A \\ AN \end{bmatrix} + r[\, A, \ MA\,] - 2r(A)$.

(b) $r(\, A^{\dagger} - A^{\dagger}_{M,I}\,) = r[\, A, \ MA\,] - r(A)$.

(c) $r(\, A^{\dagger} - A^{\dagger}_{I,N}\,) = r\begin{bmatrix} A \\ AN \end{bmatrix} - r(A)$.

(d) $A^{\dagger}_{M,N} = A^{\dagger} \Longleftrightarrow R(MA) = R(A)$ *and* $R[(AN)^*] = R(A^*)$.

Proof. Note that $A^{\dagger}$ and $A^{\dagger}_{M,N}$ are outer inverses of A. Thus it follows from Eq.(5.1) that

$$\begin{aligned} r(\, A^{\dagger} - A^{\dagger}_{M,N}\,) &= r\begin{bmatrix} A^{\dagger} \\ A^{\dagger}_{M,N} \end{bmatrix} + r[\, A^{\dagger}, \ A^{\dagger}_{M,N}\,] - r(A^{\dagger}) - r(A^{\dagger}_{M,N}) \\ &= r\begin{bmatrix} A^* \\ (MA)^* \end{bmatrix} + r[\, A^*, \ N^{-1}A^*\,] - 2r(A) \end{aligned}$$

$$= r\begin{bmatrix} A \\ AN \end{bmatrix} + r[A,\ MA] - 2r(A).$$

Parts (a)—(c) follow immediately from it. □

Theorem 16.2. *Let $A \in \mathcal{C}^{m\times n}$ be given, $M \in \mathcal{C}^{m\times m}$ and $N \in \mathcal{C}^{n\times n}$ be two positive definite matrices. Then*

(a) $r(AA^{\dagger}_{M,N} - AA^{\dagger}) = r[A,\ MA] - r(A)$.

(b) $r(A^{\dagger}_{M,N}A - A^{\dagger}A) = r\begin{bmatrix} A \\ AN \end{bmatrix} - r(A)$.

(c) $AA^{\dagger}_{M,N} = AA^{\dagger} \iff R(MA) = R(A)$.

(d) $A^{\dagger}_{M,N}A = A^{\dagger}A \iff R[(AN)^*] = R(A^*)$.

Proof. Note that both $AA^{\dagger}$ and $AA^{\dagger}_{M,N}$ are idempotent. It follows from Eq.(3.1) that

$$\begin{aligned} r(AA^{\dagger} - AA^{\dagger}_{M,N}) &= r\begin{bmatrix} AA^{\dagger} \\ AA^{\dagger}_{M,N} \end{bmatrix} + r[AA^{\dagger},\ AA^{\dagger}_{M,N}] - r(AA^{\dagger}) - r(AA^{\dagger}_{M,N}) \\ &= r\begin{bmatrix} A^* \\ (MA)^* \end{bmatrix} + r[A,\ A] - 2r(A) \\ &= r[A,\ MA] - r(A), \end{aligned}$$

as required for Part (a). Similarly we can show Part (b). □

Theorem 16.3. *Let $A \in \mathcal{C}^{m\times m}$ be given, and $M,\ N \in \mathcal{C}^{m\times m}$ be two positive definite matrices. Then*

(a) $r(AA^{\dagger}_{M,N} - A^{\dagger}_{M,N}A) = r[A^*,\ MA] + r[A^*,\ NA] - 2r(A)$.

(b) $AA^{\dagger}_{M,N} = A^{\dagger}_{M,N}A \iff R(MA) = R(NA) = R(A^*) \iff$ *both MA and NA are EP.*

Proof. Note that both $AA^{\dagger}$ and $AA^{\dagger}_{M,N}$ are idempotent. It follows by Eq.(3.1) that

$$\begin{aligned} r(AA^{\dagger}_{M,N} - A^{\dagger}_{M,N}A) &= r\begin{bmatrix} AA^{\dagger}_{M,N} \\ A^{\dagger}_{M,N}A \end{bmatrix} + r[AA^{\dagger}_{M,N},\ A^{\dagger}_{M,N}A] - r(AA^{\dagger}_{M,N}) - r(A^{\dagger}_{M,N}A) \\ &= r\begin{bmatrix} A^{\dagger}_{M,N} \\ A \end{bmatrix} + r[A,\ A^{\dagger}_{M,N}] - 2r(A) \\ &= r\begin{bmatrix} (MA)^* \\ A \end{bmatrix} + r[A,\ N^{-1}A^*] - 2r(A). \end{aligned}$$

as required for Part (a). Part(b) follows immediately from Part (a). □

Based on the result in Theorem 16.3(b), we can extend the concept of EP matrix to weighted case: A square matrix A is said to be *weighted EP* if both MA and NA are EP, where both M and N are two positive definite matrices. It is expected that weighted EP matrix would have some nice properties. But we do not intend to go further along this direction in the thesis.

Theorem 16.4. *Let $A \in \mathcal{C}^{m\times m}$ be given, and $M,\ N \in \mathcal{C}^{m\times m}$ be two positive definite matrices. Then*

(a) $r(\, AA^{\dagger}_{M,N} - \overline{A^{\dagger}_{M,N}A}\,) = r[\, A^T,\ MA\,] + r[\, A^T,\ N^TA\,] - 2r(A)$.

(b) $AA^{\dagger}_{M,N} = \overline{A^{\dagger}_{M,N}A} \Longleftrightarrow R(MA) = R(N^TA) = R(A^T) \Longleftrightarrow$ *both MA and N^TA are EP.*

Proof. Follows from Eq.(3.1) by noting that both $AA^{\dagger}_{M,N}$ and $\overline{AA^{\dagger}_{M,N}}$ are idempotent. $\square$

Theorem 16.5. *Let $A \in \mathcal{C}^{m\times m}$ be given with* $\mathrm{Ind}(A) = 1$, *and $M,\ N \in \mathcal{C}^{m\times m}$ be two positive definite matrices. Then*

(a) $r(\, A^{\dagger}_{M,N} - A^{\#}\,) = r[\, A^*,\ MA\,] + r[\, A^*,\ NA\,] - 2r(A)$.

(b) $A^{\dagger}_{M,N} = A^{\#} \Longleftrightarrow R(MA) = R(NA) = R(A^*)$, *i.e., A is weighted EP.*

Proof. Note that both $A^{\dagger}$ and $A^{\#}$ are outer inverses of A. It follows by Eq.(5.1) that

$$\begin{aligned} r(\, A^{\dagger}_{M,N} - A^{\#}\,) &= r\begin{bmatrix} A^{\dagger}_{M,N} \\ A^{\#} \end{bmatrix} + r[\, A^{\dagger}_{M,N},\ A^{\#}\,] - r(A^{\dagger}_{M,N}) - r(A^{\#}) \\ &= r\begin{bmatrix} (MA)^* \\ A \end{bmatrix} + r[\, N^{-1}A^*,\ A\,] - 2r(A) \\ &= r[\, A^*,\ MA\,] + r[\, A^*,\ NA\,] - 2r(A), \end{aligned}$$

as required for Part (a). $\square$

Theorem 16.6. *Let $A \in \mathcal{C}^{m\times m}$ be given with* $\mathrm{Ind}(A) = 1$, *and $M,\ N \in \mathcal{C}^{m\times m}$ be two positive definite matrices. Then*

(a) $r(\, AA^{\dagger}_{M,N} - AA^{\#}\,) = r[\, A^*,\ MA\,] - r(A)$.

(b) $r(\, A^{\dagger}_{M,N}A - A^{\#}A\,) = r[\, A^*,\ NA\,] - r(A)$.

(c) $r(\, A^{\dagger}_{M,N} - A^{\#}\,) = r(\, AA^{\dagger}_{M,N} - AA^{\#}\,) + r(\, A^{\dagger}_{M,N}A - A^{\#}A\,)$.

In particular,

(d) $AA^{\dagger}_{M,N} = AA^{\#} \Longleftrightarrow R(MA) = R(A^*)$, *i.e., MA is EP.*

(e) $A^{\dagger}_{M,N}A = A^{\#}A \Longleftrightarrow R(NA) = R(A^*)$, *i.e., NA is EP.*

(f) $A^{\dagger}_{M,N} = A^{\#} \Longleftrightarrow AA^{\dagger}_{M,N} = AA^{\#}$ *and* $A^{\dagger}_{M,N}A = A^{\#}A$.

Proof. Note that both $AA^{\dagger}$ and $AA^{\#}$ are idempotent. It follows from Eq.(5.1) that

$$\begin{aligned} r(\, AA^{\dagger}_{M,N} - AA^{\#}\,) &= r\begin{bmatrix} AA^{\dagger}_{M,N} \\ AA^{\#} \end{bmatrix} + r[\, AA^{\dagger}_{M,N},\ AA^{\#}\,] - r(AA^{\dagger}_{M,N}) - r(AA^{\#}) \\ &= r\begin{bmatrix} A^{\dagger}_{M,N} \\ A^{\#} \end{bmatrix} + r[\, A,\ A\,] - 2r(A) \\ &= r\begin{bmatrix} (MA)^* \\ A^* \end{bmatrix} = r[\, A^*,\ MA\,] - r(A), \end{aligned}$$

as required for Part (a). $\square$

Theorem 16.7. *Let $A \in \mathcal{C}^{m\times m}$ be given with* $\mathrm{Ind}(A) = k$, *and* $M,\ N \in \mathcal{C}^{m\times m}$ *be two positive definite matrices. Then*

(a) $r(\, A^{\dagger}_{M,N} - A^D\,) = r\begin{bmatrix} A^kM^{-1} \\ A^* \end{bmatrix} + r[\, NA^k,\ A^*\,] - r(A) - r(A^k)$.

(b) $r(\, A^{\dagger}_{M,N} - A^D\,) = r(A^{\dagger}_{M,N}) - r(A^D) \iff R(NA^k) \subseteq r(A^*)$ *and* $R[(A^kM^{-1})^*] \subseteq r(A)$.

Proof. Note that both $A^{\dagger}_{M,N}$ and A^D are outer inverses of A. It follows by Eq.(5.1) that

$$\begin{aligned} r(\, A^{\dagger}_{M,N} - A^D\,) &= r\begin{bmatrix} A^{\dagger}_{M,N} \\ A^D \end{bmatrix} + r[\, A^{\dagger}_{M,N},\ A^D\,] - r(A^{\dagger}_{M,N}) - r(A^D) \\ &= r\begin{bmatrix} (MA)^* \\ A^k \end{bmatrix} + r[\, N^{-1}A^*,\ A^k\,] - 2r(A) \\ &= r\begin{bmatrix} A^* \\ A^kM^{-1} \end{bmatrix} + r[\, A^*,\ NA^k\,] - 2r(A), \end{aligned}$$

as required for Part (a). □

Theorem 16.8. *Let $A \in \mathcal{C}^{m\times m}$ be given with* $\mathrm{Ind}(A) = k$, *and* $M,\ N \in \mathcal{C}^{m\times m}$ *be two positive definite matrices. Then*

(a) $r(\, AA^{\dagger}_{M,N} - AA^D\,) = r\begin{bmatrix} A^kM^{-1} \\ A^* \end{bmatrix} - r(A^k)$.

(b) $r(\, A^{\dagger}_{M,N}A - A^DA\,) = r[\, NA^k,\ A^*\,] - r(A^k)$.

(c) $r(\, A^{\dagger}_{M,N} - A^D\,) = r(\, AA^{\dagger}_{M,N} - AA^D\,) + (\, A^{\dagger}_{M,N}A - A^DA\,) + r(A^k) - r(A)$.

Proof. Note that both $AA^{\dagger}_{M,N}$ and AA^D are idempotent. It follows from Eq.(5.1) that

$$\begin{aligned} r(\, AA^{\dagger}_{M,N} - AA^D\,) &= r\begin{bmatrix} AA^{\dagger}_{M,N} \\ AA^D \end{bmatrix} + r[\, AA^{\dagger}_{M,N},\ AA^D\,] - r(AA^{\dagger}_{M,N}) - r(AA^D) \\ &= r\begin{bmatrix} A^{\dagger}_{M,N} \\ A^D \end{bmatrix} + r[\, A,\ A^D\,] - r(A) - r(A^k) \\ &= r\begin{bmatrix} (MA)^* \\ A^k \end{bmatrix} - r(A) = r\begin{bmatrix} A^* \\ A^kM^{-1} \end{bmatrix} - r(A) \end{aligned}$$

as required for Part (a). Similarly we can show Part (b). Combining Theorem 16.6(a) and Theorem 16.7(a) yields Part (c). □

Theorem 16.9. *Let $A \in \mathcal{C}^{m\times n}$ be given, $M,\ N \in \mathcal{C}^{m\times m}$ be two positive definite matrices. Then*

(a) $r(\, A^{\dagger}_{M,N}A^k - A^kA^{\dagger}_{M,N}\,) = r\begin{bmatrix} A^k \\ A^*M \end{bmatrix} + r[\, A^k,\ N^{-1}A^*\,] - 2r(A)$.

(b) $A^{\dagger}_{M,N}A^k = A^kA^{\dagger}_{M,N} \iff R(A^k) \subseteq R(N^{-1}A^*)$ *and* $R[(A^k)^*] \subseteq R(MA)$.

Proof. Follows from Eq.(4.1). □

Based on the result in Theorem 16.3(b), we can extend the concept of power-EP matrix to weighted case: A square matrix A is said to be *weighted power-EP* if both $R(A^k) \subseteq R(N^{-1}A^*)$ and $R[(A^k)^*] \subseteq R(MA)$ hold, where both M and N are positive definite matrices.

Theorem 16.10. *Let $A \in \mathcal{C}^{m\times m}$ be given with* $\mathrm{Ind}(A) = k$, *and $M,\ N \in \mathcal{C}^{m\times m}$ be two positive definite matrices. Then*

(a) $r(\, A^{\dagger}_{M,N}A^D - A^DA^{\dagger}_{M,N}\,) = r\begin{bmatrix} A^k \\ A^*M \end{bmatrix} + r[\, A^k,\ N^{-1}A^*\,] - 2r(A) = r(\, A^{\dagger}_{M,N}A^k - A^kA^{\dagger}_{M,N}\,).$

(b) $A^{\dagger}_{M,N}A^D = A^DA^{\dagger}_{M,N} \Longleftrightarrow R(A^k) \subseteq R(N^{-1}A^*)$ *and* $R[(A^k)^*] \subseteq R(MA)$, *i.e., A is weighted power-EP.*

Proof. Follows from Eq.(4.1). □

Theorem 16.11. *Let $A \in \mathcal{C}^{m\times n}$ be given, $M,\ S \in \mathcal{C}^{m\times m}$ and $N,\ T \in \mathcal{C}^{n\times n}$ be four positive definite matrices. Then*

(a) $r(\, A^{\dagger}_{M,N} - A^{\dagger}_{S,T}\,) = r\begin{bmatrix} AN^{-1} \\ AT^{-1} \end{bmatrix} + r[\, MA,\ SA\,] - 2r(A).$

(b) $A^{\dagger}_{M,N} = A^{\dagger}_{S,T} \Longleftrightarrow R(MA) = R(SA)$ *and* $R[(AN^{-1})^*] = R[(AT^{-1})^*]$.

Proof. Note that both $A^{\dagger}_{M,N}$ and $A^{\dagger}_{P,Q}$ are outer inverses of A. Thus it follows by Eq.(5.1) that

$$\begin{aligned} r(\, A^{\dagger}_{M,N} - A^{\dagger}_{S,T}\,) &= r\begin{bmatrix} A^{\dagger}_{M,N} \\ A^{\dagger}_{S,T} \end{bmatrix} + r[\, A^{\dagger}_{M,N},\ A^{\dagger}_{S,T}\,] - r(A^{\dagger}_{M,N}) - r(A^{\dagger}_{S,T}) \\ &= r\begin{bmatrix} (MA)^* \\ (SA)^* \end{bmatrix} + r[\, N^{-1}A^*,\ T^{-1}A^*\,] - 2r(A) \\ &= r\begin{bmatrix} AN^{-1} \\ AT^{-1} \end{bmatrix} + r[\, MA,\ SA\,] - 2r(A), \end{aligned}$$

establishing Part (a). □

Theorem 16.12. *Let $A \in \mathcal{C}^{m\times n}$ be given, $M,\ S \in \mathcal{C}^{m\times m}$, $N,\ T \in \mathcal{C}^{n\times n}$ be four positive definite matrices. Then*

(a) $r(\, AA^{\dagger}_{M,N} - AA^{\dagger}_{S,T}\,) = r[\, MA,\ SA\,] - 2r(A).$

(b) $r(\, A^{\dagger}_{M,N}A - A^{\dagger}_{S,T}A\,) = r\begin{bmatrix} AN^{-1} \\ AT^{-1} \end{bmatrix} - r(A).$

(c) $r(\, A^{\dagger}_{M,N} - A^{\dagger}_{S,T}\,) = r(\, AA^{\dagger}_{M,N} - AA^{\dagger}_{S,T}\,) + r(\, A^{\dagger}_{M,N}A - A^{\dagger}_{S,T}A\,).$

Proof. Follows from Eq.(3.1) by noticing that $AA^{\dagger}_{M,N}$, $A^{\dagger}_{M,N}A$, $AA^{\dagger}_{S,T}$ and $A^{\dagger}_{S,T}A$ are idempotent matrices. □

Theorem 16.13. *Let $A \in \mathcal{C}^{m\times m}$ be an idempotent or tripotent matrix, and $M,\ N \in \mathcal{C}^{m\times m}$ be two positive definite matrices. Then*

(a) $r(\, A - A^{\dagger}_{M,N}\,) = r[\, A^*,\ MA\,] + r[\, A^*,\ NA\,] - 2r(A).$

(b) $A = A^{\dagger}_{M,N} \Longleftrightarrow R(MA) = R(NA) = R(A^{*})$, *i.e., A is weighted EP.*

Proof. Note that $A,\ A^{\dagger}_{M,N} \in A\{2\}$ when A is idempotent or tripotent. It follows by Eq.(5.1) that

$$\begin{aligned}
r(A - A^{\dagger}_{M,N}) &= r\begin{bmatrix} A \\ A^{\dagger}_{M,N} \end{bmatrix} + r[\, A,\ A^{\dagger}_{M,N}\,] - r(A) - r(A^{\dagger}_{M,N}) \\
&= r\begin{bmatrix} A \\ (MA)^{*} \end{bmatrix} + r[\, A,\ N^{-1}A^{*}\,] - 2r(A) \\
&= r[\, A^{*},\ MA\,] + r[\, A^{*},\ NA\,] - r(A) - r(A^{k}).
\end{aligned}$$

as required for Part (a). □

Theorem 16.14. *Let $A,\ B \in \mathcal{C}^{m \times m}$ be given, $M,\ N \in \mathcal{C}^{m \times m}$ be two positive definite matrices. Then*

(a) $r(AA^{\dagger}_{M,N}B - BA^{\dagger}_{M,N}A) = r\begin{bmatrix} A \\ A^{*}MB \end{bmatrix} + r[\, A,\ BN^{-1}A^{*}\,] - 2r(A).$

(b) $r(A^{\dagger}_{M,N}AB - BAA^{\dagger}_{M,N}) = r\begin{bmatrix} AB \\ A^{*}M \end{bmatrix} + r[\, BA,\ N^{-1}A^{*}\,] - 2r(A).$

(c) $AA^{\dagger}_{M,N}B = BA^{\dagger}_{M,N}A \Longleftrightarrow R(BN^{-1}A^{*}) \subseteq R(A)$ *and* $R(B^{*}MA) \subseteq R(A^{*})$.

(d) $A^{\dagger}_{M,N}AB = BAA^{\dagger}_{M,N} \Longleftrightarrow R(BA) \subseteq R(N^{-1}A^{*})$ *and* $R[(AB)^{*}] \subseteq R(MA)$.

Proof. Follows from Eq.(4.1). □

Theorem 16.15. *Let $A \in \mathcal{C}^{m \times m}$ be given with* $\mathrm{Ind}(A) = 1$. *$P,\ Q \in \mathcal{C}^{m \times m}$ be two nonsingular matrices. Then*

(a) $r[\,(PAQ)^{\dagger} - Q^{-1}A^{\#}P^{-1}\,] = r\begin{bmatrix} A \\ A^{*}P^{*}P \end{bmatrix} + r[\, A,\ QQ^{*}A^{*}\,] - 2r(A).$

(b) $(PAQ)^{\dagger} = Q^{-1}A^{\#}P^{-1} \Longleftrightarrow R(QQ^{*}A^{*}) = R(A)$ *and* $R(P^{*}PA) = R(A^{*})$.

Proof. It is easy to verify that both $(PAQ)^{\dagger}$ and $Q^{-1}A^{\#}P^{-1}$ are outer inverses of PAQ. Thus it follows by Eq.(5.1) that

$$\begin{aligned}
& r[\,(PAQ)^{\dagger} - Q^{-1}A^{\#}P^{-1}\,] \\
&= r\begin{bmatrix} (PAQ)^{\dagger} \\ Q^{-1}A^{\#}P^{-1} \end{bmatrix} + r[\,(PAQ)^{\dagger},\ Q^{-1}A^{\#}P^{-1}\,] - r[(PAQ)^{\dagger}] - r[Q^{-1}A^{\#}P^{-1}] \\
&= r\begin{bmatrix} (PAQ)^{*}P \\ A^{\#} \end{bmatrix} + r[\,Q(PAQ)^{*},\ A^{\#}\,] - 2r(A) \\
&= r\begin{bmatrix} A^{*}P^{*}P \\ A \end{bmatrix} + r[\,QQ^{*}A^{*},\ A\,] - 2r(A),
\end{aligned}$$

establishing Part (a) and then Part (a). □

Theorem 16.16. *Let $A \in \mathcal{C}^{m \times m}$ be given, $M,\ N \in \mathcal{C}^{m \times m}$ be two positive definite matrices. Then*

(a) $r[\,(PAQ)^{\dagger} - Q^{-1}A^{\dagger}_{M,N}P^{-1}\,] = r\begin{bmatrix} A \\ AQQ^{*}N \end{bmatrix} + r[\,A,\ M^{-1}P^{*}PA^{*}\,] - 2r(A).$

(b) $(PAQ)^{\dagger} = Q^{-1}A^{\dagger}_{M,N}P^{-1} \Longleftrightarrow R(M^{-1}P^{*}PA^{*}) = R(A)$ *and* $R(NQQ^{*}A^{*}) = R(A^{*}).$

Proof. It is easy to verify that both $(PAQ)^{\dagger}$ and $Q^{-1}A^{\dagger}_{M,N}P^{-1}$ are outer inverses of PAQ. Thus it follows by Eq.(5.1) that

$$
\begin{aligned}
& r[\,(PAQ)^{\dagger} - Q^{-1}A^{\dagger}_{M,N}P^{-1}\,] \\
&= r\begin{bmatrix} (PAQ)^{\dagger} \\ Q^{-1}A^{\dagger}_{M,N}P^{-1} \end{bmatrix} + r[\,(PAQ)^{\dagger},\ Q^{-1}A^{\dagger}_{M,N}P^{-1}\,] - r[(PAQ)^{\dagger}] - r[Q^{-1}A^{\dagger}_{M,N}P^{-1}] \\
&= r\begin{bmatrix} (PAQ)^{*}P \\ A^{\dagger}_{M,N} \end{bmatrix} + r[\,Q(PAQ)^{*},\ A^{\dagger}_{M,N}\,] - 2r(A) \\
&= r\begin{bmatrix} A^{*}P^{*}P \\ (MA)^{*} \end{bmatrix} + r[\,QQ^{*}A^{*},\ N^{-1}A^{*}\,] - 2r(A) \\
&= r\begin{bmatrix} A \\ AQQ^{*}N \end{bmatrix} + r[\,A,\ M^{-1}P^{*}PA\,] - 2r(A),
\end{aligned}
$$

establishing Part (a). $\square$

Chapter 17

Reverse order laws for weighted Moore-Penrose inverses of products of matrices

Just as for Moore-Penrose inverses and Drazin inverses of products of matrices, we can also consider reverse order laws for weighted Moore-Penrose inverses of products of matrices. Noticing the basic fact in Eq.(16.2), we can easily extend the results in Chapter 8 to weighted Moore-Penrose inverses of products of matrices.

Theorem 17.1. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{n\times k}$, and $C \in \mathcal{C}^{k\times l}$ be given and let $J = ABC$. Let $M \in \mathcal{C}^{m\times m}$, $N \in \mathcal{C}^{l\times l}$, $P \in \mathcal{C}^{n\times n}$, and $Q \in \mathcal{C}^{k\times k}$ be four positive definite matrices. Then the following three statements are equivalent:*

(a) $(ABC)^{\dagger}_{M,N} = C^{\dagger}_{Q,N}B^{\dagger}_{P,Q}A^{\dagger}_{M,P}$.

(b) $(M^{\frac{1}{2}}ABCN^{-\frac{1}{2}})^{\dagger} = (Q^{\frac{1}{2}}CN^{-\frac{1}{2}})^{\dagger}(P^{\frac{1}{2}}BQ^{-\frac{1}{2}})^{\dagger}(M^{\frac{1}{2}}AP^{-\frac{1}{2}})^{\dagger}$.

(c) $r\begin{bmatrix} BQ^{-1}B^{*}PB & 0 & BC \\ 0 & -JN^{-1}J^{*}MJ & JN^{-1}C^{*}QC \\ AB & AP^{-1}A^{*}MJ & 0 \end{bmatrix} = r(B) + r(J)$.

Proof. The equivalence of Part (a) and Part (b) follows directly from applying Eq.(16.2) to the both sides of $(ABC)^{\dagger}_{M,N} = C^{\dagger}_{Q,N}B^{\dagger}_{P,Q}A^{\dagger}_{M,P}$ and simplifying. Observe that the left-hand side of Part (b) can also be written as

$$(M^{\frac{1}{2}}ABCN^{-\frac{1}{2}})^{\dagger} = [\,(M^{\frac{1}{2}}AP^{-\frac{1}{2}})(P^{\frac{1}{2}}BQ^{-\frac{1}{2}})(Q^{\frac{1}{2}}CN^{-\frac{1}{2}})\,]^{\dagger}.$$

In that case, we see by Lemma 1.1 that Part (b) holds if and only if

$$r\begin{bmatrix} B_1B_1^{*}B_1 & 0 & B_1C_1 \\ 0 & -J_1J_1^{*}J_1 & J_1C_1^{*}C_1 \\ A_1B_1 & A_1A_1^{*}J_1 & 0 \end{bmatrix} = r(B_1) + r(J_1),$$

where

$$A_1 = M^{\frac{1}{2}}AP^{-\frac{1}{2}}, \quad B_1 = P^{\frac{1}{2}}BQ^{-\frac{1}{2}}, \quad C_1 = Q^{\frac{1}{2}}CN^{-\frac{1}{2}}, \quad J_1 = M^{\frac{1}{2}}ABCN^{-\frac{1}{2}}.$$

Simplifying this rank equality by the given condition that M, N, P and Q are positive definite, we obtain the rank equality in Part (c). □

Corollary 17.2. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{n\times k}$, and $C \in \mathcal{C}^{k\times l}$ be given and let $J = ABC$. Let $P \in \mathcal{C}^{n\times n}$ and $Q \in \mathcal{C}^{k\times k}$ be two positive definite matrices. Then the following three statements are equivalent:*

(a) $(ABC)^{\dagger} = C^{\dagger}_{Q,I}B^{\dagger}_{P,Q}A^{\dagger}_{I,P}$.

(b) $(ABC)^{\dagger} = (Q^{\frac{1}{2}}C)^{\dagger}(P^{\frac{1}{2}}BQ^{-\frac{1}{2}})^{\dagger}(AP^{-\frac{1}{2}})^{\dagger}$.

$$(c) \quad r\begin{bmatrix} BQ^{-1}B^*PB & 0 & BC \\ 0 & -JJ^*J & JC^*QC \\ AB & AP^{-1}A^*J & 0 \end{bmatrix} = r(B) + r(J).$$

Proof. Follows from Theorem 2.1 by setting M and N as identity matrices. □

Corollary 17.3. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{n\times k}$, and $C \in \mathcal{C}^{k\times l}$ be given and denote $J = ABC$. Let $M \in \mathcal{C}^{m\times m}$, $N \in \mathcal{C}^{l\times l}$ be two positive definite matrices. Then the following three statements are equivalent:*

(a) $(ABC)^{\dagger}_{M,N} = C^{\dagger}_{I,N}B^{\dagger}A^{\dagger}_{M,I}$.

(b) $(M^{\frac{1}{2}}ABCN^{-\frac{1}{2}})^{\dagger} = (CN^{-\frac{1}{2}})^{\dagger}B^{\dagger}(M^{\frac{1}{2}}A)^{\dagger}$.

$$(c) \quad r\begin{bmatrix} BB^*B & 0 & BC \\ 0 & -JN^{-1}J^*MJ & JN^{-1}C^*C \\ AB & AA^*MJ & 0 \end{bmatrix} = r(B) + r(J).$$

Proof. Follows from Theorem 17.1 by setting P and Q as identity matrices. □

Corollary 17.4. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{n\times k}$, and $C \in \mathcal{C}^{k\times l}$ be given with $r(A) = n$ and $r(C) = k$. Let $M \in \mathcal{C}^{m\times m}$, $N \in \mathcal{C}^{l\times l}$, $P \in \mathcal{C}^{n\times n}$, and $Q \in \mathcal{C}^{k\times k}$ be four positive definite matrices. Then the following two statements are equivalent:*

(a) $(ABC)^{\dagger}_{M,N} = C^{\dagger}_{Q,N}B^{\dagger}_{P,Q}A^{\dagger}_{M,P}$.

(b) $R(\,P^{-1}A^*MAB\,) \subseteq R(B)$ *and* $R[\,(BCN^{-1}C^*Q)^*\,] \subseteq R(B^*)$.

Proof. The given condition $r(A) = n$ and $r(C) = k$ is equivalent to $A^{\dagger}A = I_n$, $CC^{\dagger} = I_k$, and $r(ABC) = r(B)$. In that case, we can show by block elementary operations that

$$\begin{bmatrix} BQ^{-1}B^*PB & 0 & BC \\ 0 & -JN^{-1}J^*MJ & JN^{-1}C^*QC \\ AB & AP^{-1}A^*MJ & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & B \\ 0 & 0 & BCN^{-1}C^*Q \\ B & P^{-1}A^*MAB & 0 \end{bmatrix}$$

are equivalent, the detailed is omitted here. This result implies that

$$\begin{aligned} & r\begin{bmatrix} BQ^{-1}B^*PB & 0 & BC \\ 0 & -JN^{-1}J^*MJ & JN^{-1}C^*QC \\ AB & AP^{-1}A^*MJ & 0 \end{bmatrix} \\ = \ & r\begin{bmatrix} 0 & 0 & B \\ 0 & 0 & BCN^{-1}C^*Q \\ B & P^{-1}A^*MAB & 0 \end{bmatrix} \\ = \ & r\begin{bmatrix} B \\ BCN^{-1}C^*Q \end{bmatrix} + r[\,B,\ P^{-1}A^*MAB\,]. \end{aligned}$$

Thus under the given condition of this corollary, Part (c) of Theorem 17.1 reduces to

$$r\begin{bmatrix} B \\ BCN^{-1}C^*Q \end{bmatrix} + r[\,B,\ P^{-1}A^*MAB\,] = 2r(B),$$

which is obviously equivalent to Part (c) of this corollary. □

Corollary 17.5. *Let $A \in C^{m\times m}$, $B \in C^{m\times n}$, and $C \in C^{n\times n}$ be given with A and C nonsingular. Let M, $P \in C^{m\times m}$ and N, $Q \in C^{n\times n}$ be four positive definite Hermitian matrices. Then*

(a) $(ABC)^{\dagger}_{M,N} = C^{-1}B^{\dagger}_{P,Q}A^{-1} \iff R(P^{-1}A^{*}MAB) = R(B)$ *and* $R[(BCN^{-1}C^{*}Q)^{*}] = R(B^{*})$.

(b) $(ABC)^{\dagger}_{M,N} = C^{-1}B^{\dagger}A^{-1} \iff R(A^{*}MAB) = R(B)$ *and* $R[(BCN^{-1}C^{*})^{*}] = R(B^{*})$.

(c) $(ABC)^{\dagger} = C^{-1}B^{\dagger}_{P,Q}A^{-1} \iff R(P^{-1}A^{*}AB) = R(B)$ *and* $R[(BCC^{*}Q)^{*}] = R(B^{*})$.

In particular, the following two identities hold

$$(ABC)^{\dagger}_{M,N} = C^{-1}B^{\dagger}_{(A^{*}MA),\,(CN^{-1}C^{*})^{-1}}A^{-1}. \tag{17.1}$$

$$(ABC)^{\dagger} = C^{-1}B^{\dagger}_{(A^{*}A),\,(CC^{*})^{-1}}A^{-1}. \tag{17.2}$$

Proof. Let A and C be nonsingular matrices in Corollary 17.4. We can obtain Part (a) of this corollary. Parts (a) and (b) are special cases of Part (a). The equality (17.1) follows from Part (a) by setting $P = A^{*}MA$ and $Q = (CN^{-1}C^{*})^{-1}$. □

Theorem 17.6. *Let $A \in C^{m\times n}$, $B \in C^{n\times k}$, and $C \in C^{k\times l}$ be given and denote $J = ABC$. Let $M \in C^{m\times m}$, $N \in C^{l\times l}$, $P \in C^{n\times n}$, and $Q \in C^{k\times k}$ be four positive definite matrices. Then the following two statements are equivalent:*

(a) $(ABC)^{\dagger}_{M,N} = (BC)^{\dagger}_{P,N}B(AB)^{\dagger}_{M,Q}$.

(b) $r\begin{bmatrix} B^{*} & (AB)^{*}MJ & B^{*}PBC \\ JN^{-1}(BC)^{*} & 0 & 0 \\ ABQ^{-1}B^{*} & 0 & 0 \end{bmatrix} = r(B) + r(J).$

Proof. Write ABC as $ABC = (AB)B^{\dagger}_{P,Q}(BC)$ and notice that $(B^{\dagger}_{P,Q})^{\dagger}_{Q,P} = B$. Then by Theorem 17.1, we know that

$$\begin{aligned}(ABC)^{\dagger}_{M,N} &= [(AB)B^{\dagger}_{P,Q}(BC)]^{\dagger}_{M,N} \\ &= (BC)^{\dagger}_{P,N}(B^{\dagger}_{P,Q})^{\dagger}_{Q,P}(AB)^{\dagger}_{M,Q} = (BC)^{\dagger}_{P,N}B(AB)^{\dagger}_{M,Q}\end{aligned}$$

holds if and only if

$$r\begin{bmatrix} B^{\dagger}_{P,Q}P^{-1}(B^{\dagger}_{P,Q})^{*}QB^{\dagger}_{P,Q} & 0 & B^{\dagger}_{P,Q}BC \\ 0 & -JN^{-1}J^{*}MJ & JN^{-1}(BC)^{*}P(BC) \\ ABB^{\dagger}_{P,Q} & ABQ^{-1}(AB)^{*}MJ & 0 \end{bmatrix} = r(B^{\dagger}_{P,Q}) + r(J). \tag{17.3}$$

Note by Eq.(1.5) that

$$B^{\dagger}_{P,Q}P^{-1}(B^{\dagger}_{P,Q})^{*}QB^{\dagger}_{P,Q} = Q^{-\frac{1}{2}}(P^{\frac{1}{2}}BQ^{-\frac{1}{2}})^{\dagger}[(P^{\frac{1}{2}}BQ^{-\frac{1}{2}})^{\dagger}]^{*}(P^{\frac{1}{2}}AQ^{-\frac{1}{2}})^{\dagger}P^{\frac{1}{2}}.$$

Thus by block elementary operations, we can deduce that Eq.(17.3) is equivalent to Part (c) of the theorem. The details are omitted. □

Corollary 17.7. *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{n\times k}$, and $C \in \mathcal{C}^{k\times l}$ be given and denote $J = ABC$. Let $M \in \mathcal{C}^{m\times m}$, $N \in \mathcal{C}^{l\times l}$, $P \in \mathcal{C}^{n\times n}$, and $Q \in \mathcal{C}^{k\times k}$ be four positive definite matrices. If*

$$r(ABC) = r(B), \tag{17.4}$$

then the weighted Moore-Penrose inverse of the product ABC satisfies the following two equalities

$$(ABC)^{\dagger}_{M,N} = (BC)^{\dagger}_{P,N}B(AB)^{\dagger}_{M,Q}, \tag{17.5}$$

and

$$(ABC)^{\dagger}_{M,N} = (B^{\dagger}_{P,Q}BC)^{\dagger}_{Q,N}B^{\dagger}_{P,Q}(ABB^{\dagger}_{P,Q})^{\dagger}_{M,Q}. \tag{17.6}$$

Proof. Under Eq.(17.4), we know that

$$r(AB) = r(BC) = r(B),$$

which is equivalent to

$$R(BC) = R(B), \quad \textit{and} \quad R(B^*A^*) = R(B^*).$$

Based on them we further obtain

$$R(B^*PBC) = R(B^*PB) = R[\,(B^*P^{\frac{1}{2}})(B^*P^{\frac{1}{2}})^*\,] = R(B^*P^{\frac{1}{2}}) = R(B^*),$$

and

$$R(BQ^{-1}B^*A^*) = R(BQ^{-1}B^*) = R[\,(BQ^{-\frac{1}{2}})(BQ^{-\frac{1}{2}})^*\,] = R(BQ^{-\frac{1}{2}}) = R(B).$$

Under these two conditions, the left-hand side of Part (b) in Theorem 17.6 reduces to $2r(B)$. Thus Part (b) in Theorem 17.6 is indentity under Eq.(17.4). Therefore we have Eq.(17.5) under Eq.(17.4). Consequently writing ABC as $ABC = (AB)B^{\dagger}_{P,Q}(BC)$ and applying Eq.(17.5) to it yields Eq.(17.6). □

Some applications of Corollary 17.7 are given below.

Corollary 17.8. *Let $A,\ B \in \mathcal{C}^{m\times n}$ be given, $M \in \mathcal{C}^{m\times m}$, $N \in \mathcal{C}^{n\times n}$, $P \in \mathcal{C}^{2m\times 2m}$, and $Q \in \mathcal{C}^{2n\times 2n}$ be four positive definite matrices. If A and B satisfy the rank additivity condition*

$$r(\,A + B\,) = r(A) + r(B), \tag{17.7}$$

then the weighted Moore-Penrose of $A + B$ satisfies the two equalities

$$(\,A + B\,)^{\dagger}_{M,N} = \begin{bmatrix} A \\ B \end{bmatrix}^{\dagger}_{P,N} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} [\,A,\ B\,]^{\dagger}_{M,Q}, \tag{17.8}$$

$$(\,A + B\,)^{\dagger}_{M,N} = \begin{bmatrix} A^{\dagger}_{M,N}A \\ B^{\dagger}_{M,N}B \end{bmatrix}^{\dagger}_{Q,N} \begin{bmatrix} A^{\dagger}_{M,N} & 0 \\ 0 & B^{\dagger}_{M,N}A \end{bmatrix} [\,AA^{\dagger}_{M,N},\ BB^{\dagger}_{M,N}\,]^{\dagger}_{M,P}. \tag{17.9}$$

Proof. Write $A+B$ as

$$A+B=[\,I_m,\ I_m\,]\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} I_n \\ I_n \end{bmatrix}:=UDV.$$

Then the condition Eq.(17.7) is equivalent to $r(UDV)=r(D)$. Thus it turns out that

$$(UDV)^{\dagger}_{M,N}=(DV)^{\dagger}_{P,N}D(UD)^{\dagger}_{M,Q},$$

which is exactly Eq.(17.8). Next write $A+B$ as

$$A+B=[\,A,\ B\,]\begin{bmatrix} A^{\dagger}_{M,N} & 0 \\ 0 & B^{\dagger}_{M,N} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}:=U_1D_1V_1.$$

Then the condition Eq.(17.7) is also equivalent to $r(U_1D_1V_1)=r(D_1)$. Thus it follows by Eq.(17.5) that $(U_1D_1V_1)^{\dagger}_{M,N}=(D_1V_1)^{\dagger}_{Q,N}D_1(U_1D_1)^{\dagger}_{M,P}$, which is exactly Eq.(17.9). □

A generalization of Corollary 17.8 is presented below, the proof is omitted.

Corollary 17.9. *Let $A_1,\ \cdots,\ A_k\in\mathcal{C}^{m\times n}$ be given, and let $M\in\mathcal{C}^{m\times m}$, $N\in\mathcal{C}^{n\times n}$, $P\in\mathcal{C}^{km\times km}$, and $Q\in\mathcal{C}^{kn\times kn}$ be four positive definite Hermitian matrices. If*

$$r(\,A_1+\cdots+A_k\,)=r(A_1)+\cdots+r(A_k), \tag{17.10}$$

then the weighted Moore-Penrose inverse of the sum satisfies the following two equalities

$$(\,A_1+\cdots+A_k\,)^{\dagger}_{M,N}=\begin{bmatrix} A_1 \\ \vdots \\ A_k \end{bmatrix}^{\dagger}_{P,N}\begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{bmatrix}[\,A_1,\ \cdots,\ A_k\,]^{\dagger}_{M,Q}. \tag{17.11}$$

$$\left(\sum_{t=1}^{k}A_t\right)^{\dagger}_{M,N}=$$

$$\begin{bmatrix} (A_1)^{\dagger}_{M,N}A_1 \\ \vdots \\ (A_k)^{\dagger}_{M,N}A_k \end{bmatrix}^{\dagger}_{Q,N}\begin{bmatrix} (A_1)^{\dagger}_{M,N} & & \\ & \ddots & \\ & & (A_k)^{\dagger}_{M,N} \end{bmatrix}[\,A_1(A_1)^{\dagger}_{M,N},\ \cdots,\ A_k(A_k)^{\dagger}_{M,N}\,]^{\dagger}_{M,P}. \tag{17.12}$$

Corollary 17.10. *Let $A\in\mathcal{C}^{m\times n}$, $B\in\mathcal{C}^{m\times k}$, $C\in\mathcal{C}^{l\times n}$, $A\in\mathcal{C}^{l\times k}$ be given. M, $P\in\mathcal{C}^{(m+l)\times(m+l)}$, N, $Q\in\mathcal{C}^{(n+k)\times(n+k)}$ be four positive definite matrices. If*

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix}=r(A), \tag{17.13}$$

or equivalently $AA^{\dagger}B=B$, $CA^{\dagger}A=C$ and $D=CA^{\dagger}B$, then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\dagger}_{M,N}=\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^{\dagger}_{P,N}\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}^{\dagger}_{M,Q}. \tag{17.14}$$

In particular,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\dagger}_{M,N} = [A,\ B]^{\dagger}_{I,N} A \begin{bmatrix} A \\ C \end{bmatrix}^{\dagger}_{M,I}. \tag{17.15}$$

Proof. Under Eq.(17.13), we see that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ CA^{\dagger} & I_l \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_n & A^{\dagger}B \\ 0 & I_k \end{bmatrix} := ULV.$$

Thus by Corollary 17.6, we obtain

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\dagger}_{M,N} = (LV)^{\dagger}_{P,N} L (UL)^{\dagger}_{M,Q},$$

which is exactly Eq.(17.14). When $P = I_{m+l}$ and $Q = I_{n+k}$, we have

$$\begin{aligned} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^{\dagger}_{I,N} \begin{bmatrix} I_m \\ 0 \end{bmatrix} &= N^{-\frac{1}{2}} \left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} N^{-\frac{1}{2}} \right)^{\dagger} \begin{bmatrix} I_m \\ 0 \end{bmatrix} \\ &= N^{-\frac{1}{2}} \begin{bmatrix} ([A,\ B]N^{-\frac{1}{2}})^{\dagger} \\ [0,\ 0]^{\dagger} \end{bmatrix} \begin{bmatrix} I_m \\ 0 \end{bmatrix} \\ &= N^{-\frac{1}{2}}([A,\ B]N^{-\frac{1}{2}})^{\dagger} = [A,\ B]^{\dagger}_{I,N}. \end{aligned}$$

Similarly we can deduce

$$[I_n,\ 0] \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}^{\dagger}_{M,I} = \begin{bmatrix} A \\ C \end{bmatrix}^{\dagger}_{M,I}.$$

Putting both of them in Eq.(17.14) yields Eq.(17.15). □

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