# On Duncan's characterization of McKay's monstrous $E_8$

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**Abstract**

On Duncan's characterization of McKay's monstrous $E_8$

by

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McKay's Monstrous $E_8$ observation has provided further evidence, along with the evidence provided by the study of Monstrous Moonshine, that the Monster is intimately linked with a wide spectrum of other mathematical objects and, one might even say, with the natural organization of the universe. Although these links have been observed and facts about them proved, we have yet to understand exactly where and how they originate. We here review a set of conditions, due to Duncan, imposed on arithmetic subgroups of $PSL_2(\mathbb{R})$ that return McKay's Monstrous $E_8$ diagram. The purpose is to compare these with Conway, McKay and Sebbar's (CMS) conditions that return the complete set of Monstrous Moonshine groups in order to gain some insight on their meaning. By way of doing this review of Duncan's conditions, we will also review and elaborate on Conway's method for understanding groups like $\Gamma_0(N)$.

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# Introduction

A classification has been made of all finite simple groups [?] [?]. This classification consists the 5 following families, the first four being infinite.

(1) the cyclic groups $C_p$ for $p$ prime

(2) the alternating groups $A_n$ for $n \geq 5$

(3) the matrix groups $L_n(q)$, $O^{\pm}(q)$, $U_n(q)$ and $Sp_n(q)$, these are the simple groups derived from the $n$-dimensional linear, orthogonal, unitary and symplectic matrix groups over the field of order $q$

(4) the exceptional groups of Lie-type

(5) twenty-six "sporadic" groups

We are here interested in the largest of the sporadic simple groups. It is called the Monster group, or less colorfully the Fischer-Griess group. Although some facts about this group had already been proven assuming its existence, Griess first constructed it in 1982 in [?]. Having an order slightly larger than $8 \cdot 10^{53}$, the Monster is indeed a difficult beast to approach. It luckily has only 194 conjugacy classes, which makes its character table at least manageable.

Looking at this character table, McKay noticed that the coefficient of $q$ in the $q$-expansion (Fourier series expansion, written as a Laurent series in terms of $q = \exp(2\pi i\tau)$) of the $j$-function

$$j = q^{-1} + 744 + 196884q + 21493760q^2 + ...$$

is $196883 + 1$, the sum of two character degrees of the Monster. Thompson later found in [?] that for

$$j(\tau) = q^{-1} + \sum_{n=0}^{\infty} u_n q^n$$

we had

$$u_i = \sum_j a_{ij}\chi_j(1)$$

for $i \leq 5$ and $j \leq 7$, where $\chi_j(1)$ is the degree of the $j^{th}$ character and

$$a_{ij} = \begin{pmatrix} 1 & 1 & . & . & . & . & . \\ 1 & 1 & 1 & . & . & . & . \\ 2 & 2 & 1 & 1 & . & . & . \\ 3 & 3 & 1 & 2 & 1 & . & . \\ 4 & 5 & 3 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

Conway and Norton then conjectured in [?] that if one replaces the coefficients in the series with characters of so-called head representations, the functions obtained were the normalized generator of a genus zero function field arising from a group between $\Gamma_0(N)$ and its normalizer in $PSL_2(\mathbb{R})$. These normalizers that arise are, by the same token, naturally associated with the Monster's conjugacy classes. This correspondence between conjugacy classes, hauptmodules and modular functions is, amongst other things, what is known as monstrous moonshine.

The Monster has two conjugacy classes of involutions, namely $2A$ (short involutions) and $2B$ (long involutions), the short involutions being called transpositions. This notation for classes is from the ATLAS [?]. Character calculations show that the product of two transpositions lies in one of the classes $1A, 2A, 3A, 4A, 5A, 6A, 2B, 4B$ or $3C$, and McKay pointed out in [?] what seems to be a suggestive correspondence between these and the extended $E_8$ diagram (Figure ??).

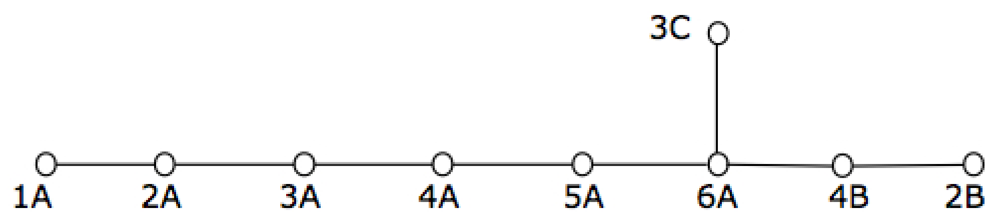



Figure 1: The $E_8$ diagram

The classes of ADE-type classifications are Dynkin diagrams. This type of classification appears in many disparate situations, sometimes showing evidence of a deeper link between them. A discussion of the ubiquity of all examples that are given here can be found in [?]. ADE diagrams (as well as other Dynkin diagrams) appear in the classification of Semisimple Lie algerbras over algebraically closed fields. They also come up in the classification of discrete subgroups of $SU(2)$, platonic solids, Weyl groups with discrete roots of equal length, representation of quivers as well as singularities of algebraic hypersurfaces with definite intersection form [?]. Through string theory, it is used to classify minimal models and certain quantum categories [?] and [?]. The $E_8$ diagram we obtain here, being the $E$ in $ADE$, is then one more example of this ubiquity.

Duncan, in [**?**], proposed a set of conditions that, when imposed on subgroups of $PSL_2(\mathbb{R})$, returned us with exactly the 9 groups corresponding to the 9 original classes, and in the same $E_8$ configuration as that observed by McKay. We here review this paper as well as place it in context. In Chapter **??**, we explain how one may understand groups like $\Gamma_0(N)$. Chapter **??** imposes Duncan's conditions. The first set of conditions returns McKay's 9 groups and the second arranges them in the $E_8$ diagram. The first set of conditions is very much reminiscent of the CMS conditions [**?**]. A discussion on generalizations, uniqueness and context of this occurrence follows in Chapter **??**.

# Chapter 1

# The action of $PGL_2^+(\mathbb{Q})$ and its subgroups on projective lattices

## 1.1 Lattices

We will follow Duncan's approach [?] to picking out the nine McKay groups. This approach makes use of projective lattices as geometrical objects and defines Gamma groups as certain subgroups of the group of isometries of this geometry. We will start by defining lattices, then projective lattices, in a way that makes the geometrical interpretation of them natural. For simplicity, we are considering the case of $V$ a vector space of dimension 2 over $\mathbb{Q}$. A more general case is handled in [?]. Let $\mathbb{F}^*$ and $\mathbb{F}^+$ denote all non-zero elements and all positive non-zero elements of an oriented field $\mathbb{F}$, respectively. Then for the usual definition of the determinant map, let

$$GL_2(\mathbb{Q}) = \{m \in M_2(\mathbb{Q}) \mid \det(m) \in \mathbb{Q}^*\}$$

$$GL_2^+(\mathbb{Q}) = \{m \in M_2(\mathbb{Q}) \mid \det(m) \in \mathbb{Q}^+\}$$

and

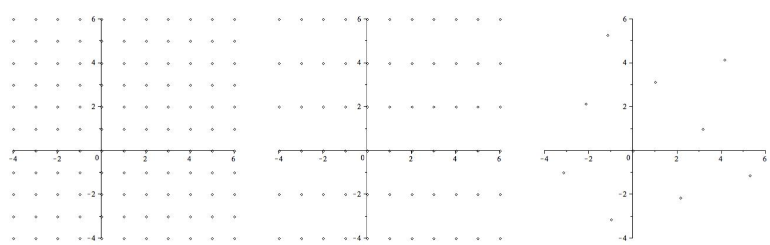
$$SL_2(\mathbb{Z}) = \{m \in M_2(\mathbb{Z}) \mid \det(m) = 1\}.$$

Notice that all three sets are groups under matrix multiplication.

For $q \in \mathbb{Q}^*$, let $g_q \in GL_2^+(\mathbb{Q})$ denote the element of $GL_2^+(\mathbb{Q})$ with $q$'s on the main diagonal and 0's everywhere else. This set $\{g_q \in GL_2^+(\mathbb{Q})\}$, with matrix multiplication, is isometric to $\mathbb{Q}^*$ as a multiplicative group. Furthermore, for $g \in GL_2^+(\mathbb{Q})$, $qg = g_q g = g g_q$, so that we will use simply $q$ (instead of $g_q$) to denote elements of both groups.

**Definition 1.1.1.** *A lattice $L$ in $V$ is an additive subgroup of $V$ such that $L$ is equivalent to $\mathbb{Z}^2$ as a $\mathbb{Z}$-module, and such that the linear span of $L$ is $V$.*

Some examples of lattices are given in figure **??**. In order to handle these lattices mathematically, we represent them as certain two by two matrices, which we will now define.



(a) $L_v$ (b) $L_w$ (c) $L_u$

Figure 1.1: Some examples of lattices. $L_v$ is defined by the basis $v$=[(1,0),(0,1)], $L_w$ by $w$=[(1,0),(0,2)] and $L_u$ by $u$=[(1,3.1416),(3.1416,1)].

Let $\mathcal{B}$ denote the set of all ordered bases of $V$. If one takes the map $\phi : \mathcal{B} \to M_2(\mathbb{Q})$ that maps $\mathbf{v} = [(v_{11}, v_{12}), (v_{21}, v_{22})]$ to the matrix $\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$, one recovers the usual bijection between $\mathcal{B}$ and $GL_2(\mathbb{Q})$. Let $\mathcal{B}^+$ denote the subset of ordered bases that map to matrices with positive determinants,

$$\mathcal{B}^+ := \{b \in \mathcal{B} | \phi(b) \in GL_2^+(\mathbb{Q})\},$$

and let us call these bases *positive*.

Let $\mathcal{L}$ denote the set of lattices in $V$. Let us introduce another map

$$\psi : \mathcal{B}^+ \mapsto \mathcal{L}$$

defined by taking $\psi(\mathbf{v})$ to be the lattice generated by $\mathbf{v}$. Let us denote $\psi(\mathbf{v})$ by $L_{\mathbf{v}}$. The map $\psi$ is surjective since, given a lattice $L \in \mathcal{L}$ and a set of generators for it, $v_1$ and $v_2 \in V$, the (oriented) set $[v_1, v_2]$ is a basis of $V$ by definition **??** and is thus an element of $\mathcal{B}$. The bases $[v_1, v_2]$ and $[-v_1, v_2]$ gives the same lattice under the map $\psi$ as in the proof of Lemma **??** and one of them must be positive.

We then have that the composition $\psi \circ \phi^{-1}$ maps elements of $GL_2^+(\mathbb{Q})$ onto $\mathcal{L}$ and through it we may say that elements of $GL_2^+(\mathbb{Q})$ generate lattices. However, two distinct positive bases (or indeed elements of $GL_2^+(\mathbb{Q})$) can generate the same lattice and we would like to adjust our set of generators to include a single generator for each lattice, making the map between generators and lattices bijective.

**Lemma 1.1.2.** *For* $\mathbf{v} = [v_1, v_2]$ *and* $\mathbf{w} = [w_1, w_2] \in \mathcal{B}^+$, $L_{\mathbf{v}} = L_{\mathbf{w}}$ *if and only if*

$$v_1 \textit{ and } v_2 \in L_{\mathbf{w}}$$

*and*

$$w_1 \textit{ and } w_2 \in L_{\mathbf{v}}$$

*Proof.* If $v_1$ and $v_2 \in L_{\mathbf{w}}$, then $L_{\mathbf{v}} \subseteq L_{\mathbf{w}}$. Similarly, If $w_1$ and $w_2 \in L_{\mathbf{v}}$, then $L_{\mathbf{v}} \subseteq L_{\mathbf{v}}$, so that $L_{\mathbf{v}} = L_{\mathbf{w}}$. Conversely, if $L_{\mathbf{v}} = L_{\mathbf{w}}$ then the set of their elements are also equal. □

Under which condition, then, do two elements of $GL_2^+(\mathbb{Q})$ generate the same lattice? If $L_{\mathbf{v}} = L_{\mathbf{w}}$, what is the relationship between $\phi(\mathbf{v})$ and $\phi(\mathbf{w})$?

**Lemma 1.1.3.** *For* $\mathbf{v}$ *and* $\mathbf{w}$ *as in Lemma* **??**, *the following are equivalent*

*(a)* $\phi(\mathbf{v}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \phi(\mathbf{w})$
*for some* $a, b, c, d \in \mathbb{Z}$

*(b)* $v_1 \in L_{\mathbf{w}}$ *and* $v_2 \in L_{\mathbf{w}}$

*(c)* $L_{\mathbf{v}} \subseteq L_{\mathbf{w}}$

*Proof.* developing $\phi(\mathbf{v})$ and $\phi(\mathbf{w})$ we get

$$(a) \qquad \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

which in turn gives us

$$v_1 = aw_1 + bw_2 \Leftrightarrow v_1 \in L_{\mathbf{w}}$$

and

$$v_2 = cw_1 + dw_2 \Leftrightarrow v_2 \in L_{\mathbf{w}}$$

giving us $(a) \Leftrightarrow (b)$.

If $v_1$ and $v_2 \in L_{\mathbf{w}}$, then $L_{\mathbf{v}} \subseteq L_{\mathbf{w}}$, the converse is also true, giving us $(b) \Leftrightarrow (c)$. □

**Proposition 1.1.4.** $L_{\mathbf{v}} = L_{\mathbf{w}}$ *if and only if*

$$\phi(\mathbf{v}) = g \cdot \phi(\mathbf{w})$$

*for* $g \in SL_2(\mathbb{Z})$.

*Proof.* For $g \in M_2(\mathbb{Z})$

$$\phi(\mathbf{v}) = g \cdot \phi(\mathbf{w}) \Leftrightarrow \phi(\mathbf{w}) = g^{-1} \cdot \phi(\mathbf{v}).$$

From Lemma **??**,

$$g \in M_2(\mathbb{Z}) \Leftrightarrow L_{\mathbf{v}} \subseteq L_{\mathbf{w}}$$

and

$$g^{-1} \in M_2(\mathbb{Z}) \Leftrightarrow L_{\mathbf{w}} \subseteq L_{\mathbf{v}}.$$

Furthermore, the inverse of an integer matrix $g$ is itself an integer matrix if and only if its determinant is $\pm 1$. We then have

$$L_{\mathbf{v}} = L_{\mathbf{w}} \Rightarrow g \in GL_2(\mathbb{Z}).$$

Since $\det(ab) = \det(a)\det(b)$, and since both $\mathbf{v}$ and $\mathbf{w}$ are positive bases, $g$ must have a positive determinant, and must therefore be an element of $SL_2(\mathbb{Z})$. □

**Corollary 1.1.5.** *For $h$ and $h' \in GL_2^+(\mathbb{Q})$, $g \in SL_2(\mathbb{Z})$,*

$$h = g \cdot h' \Leftrightarrow (\psi \circ \phi^{-1})h = (\psi \circ \phi^{-1})h'.$$

*Proof.* It is a restatement of proposition **??**, assuming that $\phi(\mathbf{v}) = h$ and $\phi(\mathbf{w}) = h'$, noticing that

$$(\psi \circ \phi^{-1})h = (\psi \circ \phi^{-1})\phi(\mathbf{v}) = L_{\mathbf{v}}.$$

□

For example, the lattices generated by the matrices $\psi(\mathbf{v}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\psi(\mathbf{w}) =$

$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ are the same since

$$\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and we notice that $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}^{-1} \in SL_2(\mathbb{Z})$.

Proposition **??** tells us that we want to consider the map induced by our map $\psi \circ \phi^{-1}$ on the coset space $SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q})$ if we want to make it bijective. For $h \in GL_2^+(\mathbb{Q})$, let

$$\Psi : SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q}) \mapsto \mathcal{L}$$

such that

$$\Psi(SL_2(\mathbb{Z}) \cdot h) = (\psi \circ \phi^{-1})(h).$$

**Theorem 1.1.6.** *Our map* $\Psi$ *is well defined and bijective.*

*Proof.* For $h' = u \cdot g$ where $g \in SL_2(\mathbb{Z})$,

$$\begin{aligned} \Psi(SL_2(\mathbb{Z}) \cdot h') &= (\psi \circ \phi^{-1})(h') \\ &= (\psi \circ \phi^{-1})(g \cdot h) \\ &= (\psi \circ \phi^{-1})(h) \qquad \text{corollary } \mathbf{??} \\ &= \Psi(SL_2(\mathbb{Z}) \cdot h), \end{aligned}$$

which shows that $\Psi$ is well-defined.

$\Psi$ is surjective, from the surjectivity of $\psi$, but it is also injective since by corollary **??**, the preimage (under $\Psi$) of a given lattice $L_{\mathbf{v}}$ is exactly the orbit $SL_2(\mathbb{Z}) \cdot \phi(\mathbf{v})$. □

The following diagram shows the relationships between the different objects involved in our construction of lattices and the maps between them. Let us denote the composition $\psi \circ \phi^{-1}$ by $\theta$ for the purposes of this diagram. Let it serve as a summary of what has been done in this section.

$$\begin{array}{ccc} GL_2^+(\mathbb{Q}) & \overset{\phi}{\longleftrightarrow} & \mathcal{B}^+ \\ \downarrow & \overset{\theta}{\searrow} & \downarrow \psi \\ SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q}) & \underset{\Psi}{\longleftrightarrow} & \mathcal{L} \end{array} \tag{1.1}$$

We now have defined lattices, as well as identified them with the coset space $SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q})$. We will postpone our choice of representatives of these cosets until we have defined another condition under which we will consider lattices equivalent.

## 1.2 Projective lattices

A projective lattice is defined as being the $\mathbb{Q}^*$-orbit of a lattice, indeed as

$$\mathbb{Q}^* \cdot L,$$

the action being defined as follows for $q \in \mathbb{Q}^*$ and $h \in GL_2^+(\mathbb{Q})$,

$$q \cdot (SL_2(\mathbb{Z}) \cdot h) := SL_2(\mathbb{Z}) \cdot (qh).$$

This action is well defined since if $g \in SL_2(\mathbb{Z})$ and $h' = gh$,

$$SL_2(\mathbb{Z}) \cdot (qh') = SL_2(\mathbb{Z}) \cdot (qgh) = SL_2(\mathbb{Z}) \cdot (qh),$$

$\mathbb{Q}^*$ being central in $GL_2^+(\mathbb{Q})$.

Recall that an action of a group $G$ on a set $X$ is a group action if it satisfies the following axioms:

$$g_1 g_2(x) = g_1(g_2 x)$$

for every $x \in X$ and $g_1, g_2 \in G$, and

$$ex = x$$

for the identity element $e \in G$. It is clear from our definition of the action above that $(q_1 q_2) \cdot h = q_1(q_2 \cdot h)$ and that $1 \cdot h = h$, for $q_1, q_2 \in \mathbb{Q}^*$ and $h$ as above, so that this action is indeed a group action.

Keeping this in mind, since we are interested here in projective lattices in geometric terms, we will offer another definition of projective lattices, one that goes through the geometric properties that justify their use. We will define a metric on our set of lattices through a notion of hyperdistance. In doing so, we will consider equivalent the lattices which are of distance zero from each other thus defining projective equivalence and projective lattices. We will also sow that this definition is equivalent to the one above.

First of all, our hyperdistance will depend on the notion of projective determinant.

**Definition 1.2.1.** *The projective determinant Pdet of a matrix* $g \in GL_2^+(\mathbb{Q})$ *is defined by*

$$Pdet : GL_2^+(\mathbb{Q}) \mapsto \mathbb{Z}^+$$

$$Pdet(g) := \det(\alpha \cdot g) = \alpha^2 \cdot \det(g)$$

*where* $\alpha \in \mathbb{Q}^*$ *is the smallest positive non-zero rational such that* $(\alpha \cdot g) \in M_2(\mathbb{Z})$

Such an $\alpha$ always exists since it is the unique fraction whose numerator is the lowest common multiples of the four denominators of the entries of $g$, and whose denominator is the greatest common divisor of the four numerators.

Most of the properties of our metric and of projective lattices actually stem from properties of this projective determinant.

**Lemma 1.2.2.** *Projective determinants are invariant under multiplication by an element of* $\mathbb{Q}^*$. *For* $q \in \mathbb{Q}^*$ *and* $g \in GL_2^+(\mathbb{Q})$,

$$Pdet(g) = Pdet(q \cdot g)$$

*Proof.* For $\alpha$ and $g$ as in Definition **??**, it is clear that $\frac{\alpha}{q} \cdot (q \cdot g) \in M_2(\mathbb{Z})$. Let us assume that there exists a $\beta > 0$, with $\beta < |\frac{\alpha}{q}|$ such that $\beta \cdot (q \cdot g) \in M_2(\mathbb{Z})$. We thus have $|q\beta| < \alpha$. But this would yield $\pm\beta \cdot (q \cdot g) = \pm(q\beta) \cdot g \in M_2(\mathbb{Z})$, where we choose + for a positive $q$ and - for a negative $q$, contradicting the minimality of $\alpha$. □

**Lemma 1.2.3.** *Projective determinants are invariant under taking inverses. For* $g \in GL_2^+(\mathbb{Q})$,

$$Pdet(g) = Pdet(g^{-1})$$

*Proof.* For $\alpha$ as in Definition **??** and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$, we have $g^{-1} = \frac{1}{\det(g)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. From Definition **??**, $Pdet(\det(g) \cdot g^{-1}) = Pdet(g)$. From Lemma **??**, we then have that $Pdet(g^{-1}) = Pdet(g)$. □

**Lemma 1.2.4.** *For integer matrices, projective determinants are always smaller than or equal to proper determinants. For* $g \in \Big(GL_2^+(\mathbb{Q}) \cap M_2(\mathbb{Z})\Big)$,

$$Pdet(g) \leq \det(g).$$

*Proof.* Here,

$$\alpha = \frac{1}{\text{greatest common divisor of the entries in } g} \leq 1$$

and use the fact that $Pdet(g) = \alpha^2 \cdot \det(g)$. $\square$

**Lemma 1.2.5.** *Projective determinants are invariant under a multiplication on the right or on the left by elements of $SL_2(\mathbb{Z})$. Let $h \in SL_2(\mathbb{Z})$, $g$ and $g' \in GL_2^+(\mathbb{Q})$. If $g' = g \cdot h$ or $g' = h \cdot g$ then*

$$Pdet(g) = Pdet(g').$$

*Proof.* Let us prove for the case of multiplication on the right. Let $\alpha$ be as in definition **??** and $\beta$ be the smallest positive non-zero rational such that $(\beta \cdot g') \in M_2(\mathbb{Z})$. Notice that

$$\alpha \cdot g' = (\alpha \cdot gh) \in M_2(\mathbb{Z}).$$

Suppose now that $\beta < \alpha$ then

$$\beta(gh) \in M_2(\mathbb{Z})$$

so

$$\beta g \in M_2(\mathbb{Z})h^{-1}$$

thus

$$\beta g \in M_2(\mathbb{Z}).$$

This contradicts the minimality of $\alpha$ and thus giving us $\beta = \alpha$. We also have

$$\det(g') = \det(gh) = \det(g')\det(h) = \det(g)$$

completing the proof. The argument for multiplication on the left is analogous. $\square$

To lighten notation, let us define an action on the left of positive bases by matrices in $GL_2^+(\mathbb{Q})$ in the following way

$$g \cdot \mathbf{v} := \phi^{-1}(g \cdot \phi(\mathbf{v}))$$

for $g \in GL_2^+(\mathbb{Q})$, and $\mathbf{v} \in \mathcal{B}^+$. We are acting on the left of positive bases in the same way we would act on their counterparts in $GL_2^+(\mathbb{Q})$. Checking the group action axioms we obtain

$$g \cdot (h \cdot \mathbf{v}) = \phi^{-1}(g \cdot \phi(\phi^{-1}(h \cdot \phi(\mathbf{v})))) = \phi^{-1}(g \cdot h \cdot \phi(\mathbf{v})) = gh \cdot \mathbf{v}$$

and

$$e \cdot \mathbf{v} = \mathbf{v}$$

for the identity $e \in GL_2^+(\mathbb{Q})$ and $h$ also in $GL_2^+(\mathbb{Q})$.

We can now define hyperdistance.

**Definition 1.2.6.** *The hyperdistance $\delta(L_{\mathbf{v}}, L_{\mathbf{w}})$ between two lattices $L_{\mathbf{v}}$ and $L_{\mathbf{w}}$ is the projective determinant of the unique element $g \in GL_2^+(\mathbb{Q})$ such that*

$$\mathbf{v} = g \cdot \mathbf{w}.$$

*Notice that $\delta(\cdot, \cdot) : \mathcal{L} \times \mathcal{L} \mapsto \mathbb{Z}^+$*

Such a $g$ always exists since, for any two $\phi(\mathbf{v})$ and $\phi(\mathbf{w}) \in GL_2^+(\mathbb{Q})$ we have

$$g = \phi(\mathbf{v}) \cdot (\phi(\mathbf{w}))^{-1}. \tag{1.2}$$

This $g$ is unique because if there was another element $h \in GL_2^+(\mathbb{Q})$ such that $\mathbf{v} = h \cdot \mathbf{w}$, then we would have $g^{-1}h\mathbf{w} = g^{-1}\mathbf{v} = g^{-1}g\mathbf{w} = \mathbf{w}$. This would mean that $g^{-1}h$ acts as the identity element in $GL_2^+(\mathbb{Q})$ which contradicts the fact that $h$ is distinct from $g$ since inverses are unique.

**Proposition 1.2.7.** *Hyperdistance is well defined.*

*Proof.* For $\mathbf{w}$ and $\mathbf{z}$ generating the same lattice we have $\mathbf{w} = h \cdot \mathbf{z}$ for some $h \in SL_2(\mathbb{Z})$

and

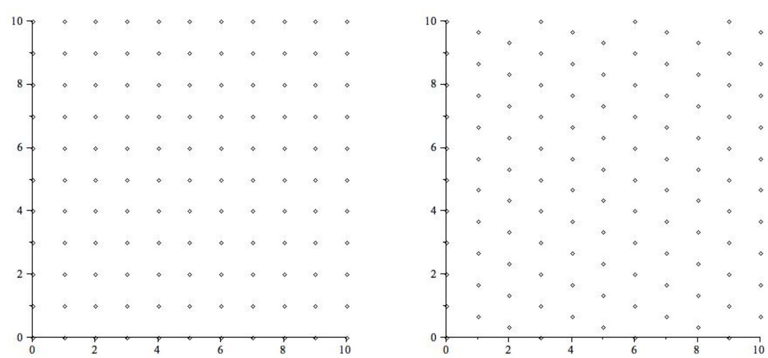
$$\begin{aligned}\delta(L_{\mathbf{v}}, L_{\mathbf{z}}) &= Pdet(\phi(\mathbf{v})\phi(\mathbf{z})^{-1}) \\ &= Pdet(\phi(\mathbf{v})\phi(\mathbf{w})^{-1}h^{-1}) \\ &= Pdet(\phi(\mathbf{v})\phi(\mathbf{w})^{-1}) \quad \text{Lemma } \mathbf{??} \\ &= \delta(L_{\mathbf{v}}, L_{\mathbf{w}}).\end{aligned}$$

A similar argument holds for the statement

$$\delta(L_{\mathbf{z}}, L_{\mathbf{v}}) = \delta(L_{\mathbf{w}}, L_{\mathbf{v}}).$$

□

Before moving on to prove some properties of hyperdistance, let us work out a simple example. Let $L_{\mathbf{v}}$ and $L_{\mathbf{w}}$ be generated by $\phi(\mathbf{v}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\phi(\mathbf{w}) = \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{pmatrix}$.




(a) $L_{\mathbf{v}}$ (b) $L_{\mathbf{w}}$

Figure 1.2: Two lattices and the hyperdistance between them

The element of $GL_2^+(\mathbb{Q})$ that maps the first generator to the other is $g = \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{pmatrix}$. To obtain the projective determinant of $g$ we multiply it by 3, the smallest rational that maps it to $M_2(\mathbb{Z})$. $3g = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$, the determinant of which, the hyperdistance between $L_{\mathbf{v}}$ and $L_{\mathbf{w}}$, is 9.

**Definition 1.2.8.** *A function $f : X \times X \mapsto \mathbb{R}$ is called a metric if it is non-negative, definite, symmetric and satisfies the triangle inequality.*

**Lemma 1.2.9.** *The function $d$ defined as the the logarithm of the hyperdistance, $d(L_{\mathbf{v}}, L_{\mathbf{w}}) := \log\Big(\delta(\phi(\mathbf{v}), \phi(\mathbf{w}))\Big)$, is non-negative, symmetric and satisfies the triangle inequality.*

*Proof.* Recall that the function $d(x, y)$ is non-negative if and only if $d(x, y) \geq 0$ for all $x$ and $y$ in its domain.

For any two $\phi(\mathbf{v})$ and $\phi(\mathbf{w}) \in GL_2^+(\mathbb{Q})$, since $GL_2^+(\mathbb{Q})$ is a group we have $g = \phi(\mathbf{v})\cdot\phi(\mathbf{w})^{-1} \in GL_2^+(\mathbb{Q})$, giving us that the determinant of $g$ is positive. Definition **??** assures us that the projective determinant of this $g$ will also be an integer. The log of a positive integer is non-negative, and thus so is $d$.

For $\mathbf{v} = g\cdot\mathbf{w}$, we have $g^{-1}\cdot\mathbf{v} = \mathbf{w}$. Corollary **??** tells us that $Pdet(g) = Pdet(g^{-1})$. This yields

$$\delta(L_{\mathbf{v}}, L_{\mathbf{w}}) = Pdet(g) = Pdet(g^{-1}) = \delta(L_{\mathbf{w}}, L_{\mathbf{v}}),$$

and thus the symmetry of $d$.

Thirdly, for $L_{\mathbf{v}}, L_{\mathbf{w}}, L_{\mathbf{z}} \in \mathcal{L}$, let us use the notation $g_{\mathbf{vw}}$ for the element in $GL_2^+(\mathbb{Q})$ such that

$$\mathbf{v} = g_{\mathbf{vw}}\mathbf{w}$$

and similarly for $g_{\mathbf{vz}}$ and so on. Then $\mathbf{z} = g_{\mathbf{zw}}g_{\mathbf{wv}}\mathbf{v}$, so that we know

$$Pdet(g_{\mathbf{zv}}) = Pdet(g_{\mathbf{zw}}g_{\mathbf{wv}}).$$

Let $\alpha_{\mathbf{vw}}$ be the smallest non-zero rational such that $\alpha_{\mathbf{vw}} g_{\mathbf{vw}} \in M_2(\mathbb{Z})$, and similarly for $\alpha_{\mathbf{zv}}$ and so on. Since $Pdet(g_{\mathbf{zw}} g_{\mathbf{wv}}) = Pdet\Big((\alpha_{\mathbf{zw}} g_{\mathbf{zw}})(\alpha_{\mathbf{wv}} g_{\mathbf{wv}})\Big)$ and that $(\alpha_{\mathbf{zw}} g_{\mathbf{zw}})(\alpha_{\mathbf{wv}} g_{\mathbf{wv}}) \in M_2(\mathbb{Z})$, using Lemma **??**,

$$Pdet(g_{\mathbf{zv}}) \leq \det(\alpha_{\mathbf{zw}} g_{\mathbf{zw}}) \cdot \det(\alpha_{\mathbf{wv}} g_{\mathbf{wv}}) = Pdet(g_{\mathbf{zw}}) \cdot Pdet(g_{\mathbf{wv}}).$$

Since projective determinants are positive, upon taking the logarithm of both sides of the inequation, one recovers the triangle inequality. □

There is very little missing for this "distance function" to be a metric. In fact, one need only prove definiteness, i.e. that if $d(L_{\mathbf{v}}, L_{\mathbf{w}}) = d(L_{\mathbf{w}}, L_{\mathbf{v}}) = 0$ then $L_{\mathbf{v}} = L_{\mathbf{w}}$. As it is, this is not the case. Take for example $L_{\mathbf{v}}$ and $L_{\mathbf{w}}$ generated by $\phi(\mathbf{v}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\phi(\mathbf{w}) = \begin{pmatrix} 6 & 4 \\ 2 & 2 \end{pmatrix}$. Then $d(L_{\mathbf{v}}, L_{\mathbf{w}}) = 0$ but $L_{\mathbf{v}} \neq L_{\mathbf{w}}$.

We will solve this problem by saying that two lattices are equivalent if the distance between them is zero and applying our function to these equivalence classes instead, checking that this induced function is still well defined. In the last example, it amounts to formally changing the $\neq$ sign to a $=$. This will define projective equivalence and projective lattices.

**Definition 1.2.10.** *Two lattices are called projectively equivalent if the distance $d$ between them is 0.*

**Proposition 1.2.11.** *Projective equivalence defines a proper equivalence relation.*

*Proof.* Let us denote projective equivalence by $r$ such that the equivalence between two lattices $L_{\mathbf{v}}$ and $L_{\mathbf{w}}$ is denoted

$$L_{\mathbf{v}} r L_{\mathbf{w}}.$$

We must now show that this relation is reflective, symmetric and transitive.

It is clear that $\phi(\mathbf{v}) = I \cdot \phi(\mathbf{v})$, with $\log(Pdet(I)) = 0$, showing reflectivity.

Symmetry follows from symmetry of $d$.

Transitivity follows from the fact that $d$ satisfies the triangle inequality and is non-negative. If $d(L_{\mathbf{v}}, L_{\mathbf{w}}) = d(L_{\mathbf{w}}, L_{\mathbf{z}}) = 0$, then we have

$$0 \le d(L_{\mathbf{v}}, L_{\mathbf{z}}) \le d(L_{\mathbf{v}}, L_{\mathbf{w}}) + d(L_{\mathbf{w}}, L_{\mathbf{z}}) = 0$$

completing the proof. □

An example of two projectively equivalent lattices would be $L_{\mathbf{v}}$ generated by $\mathbf{v} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $L_{\mathbf{w}}$ by $\mathbf{w} = \begin{pmatrix} 3.1416 & 0 \\ 0 & 3.1416 \end{pmatrix}$ as they are scalar multiples of one another (using Lemma **??**).

**Definition 1.2.12.** *A class of projectively equivalent lattices is called a projective lattice. The set of all projective lattices is denoted by $P\mathcal{L}$.*

**Lemma 1.2.13.** *The function $D$ obtained by applying $d$ on projective lattices is well defined.*

*Proof.* Let $L_{\mathbf{v}}$, $L_{\mathbf{w}}$ and $L_{\mathbf{z}}$ be three lattices such that $L_{\mathbf{v}}$ and $L_{\mathbf{w}}$ are projectively equivalent. The triangle inequality gives us

$$d(L_{\mathbf{v}}, L_{\mathbf{z}}) \le d(L_{\mathbf{v}}, L_{\mathbf{w}}) + d(L_{\mathbf{w}}, L_{\mathbf{z}}) = d(L_{\mathbf{w}}, L_{\mathbf{z}})$$

but then we also have, because of symmetry

$$d(L_{\mathbf{w}}, L_{\mathbf{z}}) \le d(L_{\mathbf{v}}, L_{\mathbf{z}})$$

so that

$$d(L_{\mathbf{w}}, L_{\mathbf{z}}) = d(L_{\mathbf{v}}, L_{\mathbf{z}}).$$

□

**Corollary 1.2.14.** *$D$ is a metric.*

*Proof.* $D$ has already been shown to satisfy all four conditions of Definition **??**. □

When considering lattices, our generators, elements of $GL_2^+(\mathbb{Q})$, were regarded as equivalent under multiplication on the left by elements of $SL_2(\mathbb{Z})$. In other words, our map $\theta : GL_2^+(\mathbb{Q}) \to \mathcal{L}$ was invariant under a multiplication on the left by elements of $SL_2(\mathbb{Z})$. This allowed us to naturally induce a bijection on the cosets $SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q})$ from our map $\theta$. When considering projective lattices, we posed an additional condition under which generators would be considered equivalent: multiplication on the left by elements of $\mathbb{Q}^*$. In the same way, we should be able to use this to induce a bijection on the cosets $(\mathbb{Q}^* SL_2(\mathbb{Z})) \setminus GL_2^+(\mathbb{Q})$. To prove the existence of this bijection, we should show that $(\mathbb{Q}^* SL_2(\mathbb{Z}))$ is the kernel of some surjective map from $GL_2^+(\mathbb{Q})$ to $P\mathcal{L}$.

**Lemma 1.2.15.** *An element of $GL_2^+(\mathbb{Q})$ has projective determinant 1 if and only if it is in the subgroup $\mathbb{Q}^* SL_2(\mathbb{Z})$ generated by $\mathbb{Q}^*$ and $SL_2(\mathbb{Z})$.*

*Proof.* Notice that $\mathbb{Q}^*$ in central in $GL_2^+(\mathbb{Q})$ so that any element $m$ of the subgroup generated by $\mathbb{Q}^*$ and $SL_2(\mathbb{Z})$ can be written in the form $qh$, for $q \in \mathbb{Q}^*$ and $h \in SL_2(\mathbb{Z})$. This justifies the use of the notation $\mathbb{Q}^* SL_2(\mathbb{Z})$.

If an element $h$ of $SL_2(\mathbb{Z})$ had entries that all shared a common divisor $d \in \mathbb{Z}$ other than 1, then the determinant of $h$ would have to be at least $d^2$, contradicting the fact that $h$ was in $SL_2(\mathbb{Z})$ in the first place. The $\alpha$ in Definition **??** then must be 1. Since projective determinants are invariant under multiplication by elements of $\mathbb{Q}^*$ (Lemma **??**), $Pdet(m) = 1$

Conversely if $Pdet(g) = 1$, then for $\alpha$ as above, $\alpha g \in SL_2(\mathbb{Z})$ so that $g = \frac{1}{\alpha}h$, for $h$ as above, giving the statement that g must be in $\mathbb{Q}^* SL_2(\mathbb{Z})$. □

**Lemma 1.2.16.** *A lattice $L_{\mathbf{w}}$ is projectively equivalent to $L_{\mathbf{v}}$ if and only if it is generated by a (positive) basis in the orbit $(Q^* SL_2(\mathbb{Z})) \cdot \mathbf{v}$.*

*Proof.* Recall that two lattices are projectively equivalent if and only if the distance between them is 0, that is if and only if the hyperdistance between them is 1 (Definition **??**). We have already shown that $\mathbf{v}$ and $\mathbf{w}$ are related via

$$\mathbf{v} = g \cdot \mathbf{w}$$

for a unique $g$. Furthermore, since hyperdistance is well defined and thus does not depend on our choice of generators for $L_{\mathbf{v}}$ and $L_{\mathbf{w}}$, it is sufficient to prove

$$L_{\mathbf{v}} \sim L_{\mathbf{w}} \Leftrightarrow g \in \mathbb{Q}^* SL_2(\mathbb{Z}).$$

If $L_{\mathbf{v}} \sim L_{\mathbf{w}}$ then $Pdet(g) = 1$ and we can deduce from Lemma **??** that $g \in \mathbb{Q}^* SL_2(\mathbb{Z})$.

If $g \in \mathbb{Q}^* SL_2(\mathbb{Z})$ then we know from Lemma **??** that $Pdet(g) = 1$ and so that $L_{\mathbf{v}} \sim L_{\mathbf{w}}$. □

**Proposition 1.2.17.** *There is a bijection between the cosets $\mathbb{Q}^* SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q})$ and the space of projective lattices $P\mathcal{L}$.*

*Proof.* Let

$$\pi : \quad \mathbb{Q}^* SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q}) \longrightarrow P\mathcal{L}$$

be such that for $g \in GL_2^+(\mathbb{Q})$, and $\mathbf{v} = \phi^{-1}(g)$,

$$\pi\Big(\mathbb{Q}^* SL_2(\mathbb{Z}) \cdot g\Big) = \mathbb{Q}^* \cdot L_{\mathbf{v}}.$$

This is well defined from Lemma (**??**).

This map $\pi$ is surjective since our choice of g, and thus the image $\mathbb{Q}^* \cdot L_v$, is arbitrary, and from the fact that $GL_2^+(\mathbb{Q})$ maps to $\mathcal{L}$ surjectively through $\theta$ (recall Diagram (**??**)).

Furthermore, $\pi$ is injective since, form Lemma **??**), that is to say if and only if $\left(\mathbb{Q}^* SL_2(\mathbb{Z}) \cdot g\right)$ and $\left(\mathbb{Q}^* SL_2(\mathbb{Z}) \cdot g'\right)$ are the same coset.

$\pi$ is then a bijection and we have

$$\mathbb{Q}^* SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q}) \stackrel{\pi}{\longleftrightarrow} P\mathcal{L}.$$

□

We now have an identification of the space of projective lattices $P\mathcal{L}$ with the cosets $\mathbb{Q}^* SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q})$. In the next chapter, using this bijection, we will define a right action on projective lattices and thus define our "Gamma groups". Having achieved our goal of defining a nice representation of $P\mathcal{L}$, this section could logically conclude here. However, there is a second identification, one of $P\mathcal{L}$ with $PSL_2(\mathbb{Z}) \setminus PGL_2^+(\mathbb{Q})$, which we will define bellow, that might seem more elegant. Let us show that these two ways of looking at projective lattices are equivalent.

Let us first define $PSL_2(\mathbb{Z})$ and $PGL_2^+(\mathbb{Q})$, the projective special linear group and the projective positive linear group. Considering a subgroup of $GL_2^+(\mathbb{Q})$ projectively means that we are considering all non-zero rational multiples as equivalent. Notice that this is exactly how we defined projective lattices, so that our vocabulary is consistent. For $G$ a subgroup of $GL_2^+(\mathbb{F})$, where $\mathbb{F}$ is some field, its associated projective group $PG$ is defined as

$$PG := G/(\mathbb{F}^* \cap G).$$

Since $\mathbb{F}^*$ is normal in $GL_2^+(\mathbb{F})$, $(\mathbb{F}^* \cap G)$ is normal in $G$ so that $PG$ is well defined as a group. We then have that

$$PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/(\mathbb{Q}^* \cap SL_2(\mathbb{Z})) = SL_2(\mathbb{Z})/\{-1,1\} = SL_2(\mathbb{Z})/\langle -I\rangle$$

and

$$PGL_2^+(\mathbb{Q}) = GL_2^+(\mathbb{Q})/(\mathbb{Q}^* \cap GL_2^+(\mathbb{Q})) = GL_2^+(\mathbb{Q})/\mathbb{Q}^*.$$

$PSL_2(\mathbb{Z})$ and $PGL_2^+(\mathbb{Q})$ now being defined, $PSL_2(\mathbb{Z}) \setminus PGL_2^+(\mathbb{Q})$ is still nebulous, however, since $PSL_2(\mathbb{Z})$ is now no longer a subgroup $PGL_2^+(\mathbb{Q})$ as $\mathbb{Q}^*SL_2(\mathbb{Z})$ was a subgroup of $GL_2^+(\mathbb{Q})$. The definition of the coset space will require a little more work.

**Lemma 1.2.18.** *Our notion of projective determinant induces well defined functions on $PGL_2^+(\mathbb{Q})$ as well as on $\mathbb{Q}^*SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q})$. We will speak indiscriminately of projective determinants whether we are applying the function to $GL_2^+(\mathbb{Q})$, $PGL_2^+(\mathbb{Q})$ or $\mathbb{Q}^*SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q})$.*

*Proof.* The result is clear from the fact that projective determinants are invariant under multiplication by elements of $\mathbb{Q}^*$ (Lemma **??**) and multiplication on the left by elements of $\mathrm{SL}_2(\mathbb{Z})$ (Lemma **??**). □

As a note, when considering the projective linear group $PGL_2^+(\mathbb{Q})$, we can see the multiplication by the $\alpha$ of the definition of projective determinant (**??**) to be a choice of representatives for the cosets of $PGL_2^+(\mathbb{Q})$.

To show that $\mathbb{Q}^*SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q})$ is equivalent to $PSL_2(\mathbb{Z}) \setminus PGL_2^+(\mathbb{Q})$ (which has yet to be defined), we show that there exists a bijection between both of them and $G_1 \setminus PGL_2^+(\mathbb{Q})$, where $G_1 = \Big(\mathbb{Q}^*SL_2(\mathbb{Z})\Big)/\mathbb{Q}^*$.

Consider the following group homomorphism.

$$GL_2^+(\mathbb{Q}) \xrightarrow{\gamma} PGL_2^+(\mathbb{Q})$$

The map $\gamma$ is the natural projection homomorphism of $GL_2^+(\mathbb{Q})$ into $GL_2^+(\mathbb{Q})/\mathbb{Q}^*$, $\mathbb{Q}^*$ being normal in $GL_2^+(\mathbb{Q})$. Consider $\gamma(SL_2(\mathbb{Z}))$, the image of $SL_2(\mathbb{Z}) \subset GL_2^+(\mathbb{Q})$ in $PGL_2^+(\mathbb{Q})$ under this homomorphism. The pre-image of this subgroup $\gamma(SL_2(\mathbb{Z}))$ of $PGL_2^+(\mathbb{Q})$ is $\mathbb{Q}^*SL_2(\mathbb{Z})$, since $\mathbb{Q}^*$ is, by definition, the kernel of $\gamma$.

$$(\gamma^{-1} \circ \gamma)SL_2(\mathbb{Z}) = \mathbb{Q}^*SL_2(\mathbb{Z}).$$

**Lemma 1.2.19.** *$G_1$, which we have just shown to be $\gamma(SL_2(\mathbb{Z}))$, is isomorphic to $PSL_2(\mathbb{Z})$.*

*Proof.* For $G$ a group, $A$ and $B$ subgroups and $A \leq N_G(B)$, the Diamond Isomorphism Theorem [**?**] states that

$$AB/B \cong A/A \cap B,$$

which proves the result for $G = GL_2^+(\mathbb{Q})$, $A = SL_2(\mathbb{Z})$ and $B = \mathbb{Q}^*$, recalling that $\mathbb{Q}^*$ is central is $GL_2^+(\mathbb{Q})$. □

$G_1$ acts naturally on $PGL_2^+(\mathbb{Q})$. Through the isomorphism of the previous Lemma **??**, we can also let $PSL_2(\mathbb{Z})$ act on $PGL_2^+(\mathbb{Q})$. We can then define the orbit space $PSL_2(\mathbb{Z}) \setminus PGL_2^+(\mathbb{Q})$ through that action and through a bijection of sets between $PSL_2(\mathbb{Z}) \setminus PGL_2^+(\mathbb{Q})$ and $G_1 \setminus PGL_2^+(\mathbb{Q})$. Let us now show that there is also a bijection between this latter coset space and the coset space $\mathbb{Q}^*SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q})$.

**Lemma 1.2.20.** *There exists a bijection between $G_1 \setminus PGL_2^+(\mathbb{Q})$ and $\mathbb{Q}^*SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q})$.*

*Proof.* Let $p$ be a element of $\mathbb{Q}^*SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q})$.

$$p = \mathbb{Q}^*SL_2(\mathbb{Z}) \cdot g,$$

for some $g \in GL_2^+(\mathbb{Q})$. Take the map

$$\pi : \mathbb{Q}^* SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q}) \to G_1 \setminus PGL_2^+(\mathbb{Q})$$

such that

$$\pi(p) = G_1 \cdot (g \cdot \mathbb{Q}^*).$$

$\pi$ is surjective since the choice of $g$ is arbitrary. It is also injective since for any $q \in \mathbb{Q}^*$ and $s \in SL_2(\mathbb{Z})$

$$\pi(qsg) = G_1 \cdot (qsg \cdot \mathbb{Q}^*) = G_1 \cdot (sg \cdot \mathbb{Q}^*) = \pi(g).$$

$\pi$ is then a suitable bijection. □

Here is what we now have, and we may call it the conclusion of this section.

$$P\mathcal{L} \leftrightarrow \mathbb{Q}^* SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q}) \leftrightarrow PSL_2(\mathbb{Z}) \setminus PGL_2^+(\mathbb{Q})$$

Let us also say that an element $g \in GL_2^+(\mathbb{Q})$ *generates* the projective lattice $\mathbb{Q}^* SL_2(\mathbb{Z}) \cdot g$.

## 1.3 Trees

The reason why we insisted on regarding projective lattices as geometrical objects is that we will be acting on the space of projective lattices by isometries. In this section, we will elaborate diagrams that render these isometries clearly and make their use quite natural.

In these diagrams, the vertices will be projective lattices and the edges going from one to the other will be labeled with the hyperdistance between them. For example, take $\mathbf{v}$ and $\mathbf{v}' \in \mathcal{B}+$. Then we have

$$L_{\mathbf{v}} \xrightarrow{\delta(L_{\mathbf{v}}, L_{\mathbf{v}'})} L_{\mathbf{v}'}.$$

This example is not quite correct, however, since the vertices are labeled with lattices and not projective lattices. The problem is that we have yet to define a labeling scheme, or indeed a canonical representative, for our projective lattices. As it pertains to the above example, we could say that we are considering $L_{\mathbf{v}}$ and $L_{\mathbf{v}'}$ projectively (that is, considering their $\mathbb{Q}^*$ orbits), but we would still like to choose particular representatives for projective lattices to avoid confusion and not consider two lattices as different when they are really equivalent.

So let us come to the elaboration of a particular representative for each set of equivalent lattices. We will call this representative the *name* of the projective lattice, following [**?**]. We will show that each projective lattice has a unique generator of the form $\begin{pmatrix} M & \frac{l}{h} \\ 0 & 1 \end{pmatrix}$ where $M > 1$ and $0 \leq l < h$. The constructive proof of the uniqueness and existence of this name is given in the following algorithm. It closely resembles the proof in Duncan's paper [**?**]. The idea is to act on the left on elements of $GL_2^+(\mathbb{Q})$ with elements of $\mathbb{Q}^*$ and $SL_2(\mathbb{Z})$, so that each successive image is projectively equivalent to its preimage. Recall that elements of $GL_2^+(\mathbb{Q})$ are projectively equivalent if and only if the are related via a multiplication on the left an element of $\mathbb{Q}^* SL_2(\mathbb{Z})$ (Lemma **??**). Let $g \in GL_2^+(\mathbb{Q})$ such that

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

for $a, b, c$ and $d \in \mathbb{Q}$ and note that $\det(g) = ad - bc > 0$ so that either $a \neq 0$ or $c \neq 0$. This $g$ is the generator of the projective lattice we wish to name.

(1) We first multiply $g$ by the smallest non-zero rational $\alpha$ such that $\alpha \cdot g = g' \in M_2(\mathbb{Z})$.

$$g' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

(2) We now want to make the bottom left entry zero. This can be done by multiplying on the left by a matrix of the form

$$\pm \begin{pmatrix} s & t \\ -x & y \end{pmatrix},$$

where $x$ and $y$ are $c'$ and $a'$ divided by their greatest common divisor. Since $x$ and $y$ are coprime, the Euclidean algorithm assures us that there exist positive integers $s$ and $t$ such that the determinant of this matrix will be 1, so that it is an element of $SL_2(\mathbb{Z})$. This gives us

$$g'' = \begin{pmatrix} a'' & b'' \\ 0 & d'' \end{pmatrix} \in M_2(\mathbb{Z})$$

Where, upon choosing $+$ or $-$ above, $a''$ and $d'' > 0$.

(3) We can then reduce $b''$ modulo $d''$ by multiplying on the left by

$$\begin{pmatrix} 1 & -\lfloor \frac{b''}{d''} \rfloor \\ 0 & 1 \end{pmatrix}$$

(4) The last step consists in dividing the resulting matrix by $d''$ to obtain the form

$$\begin{pmatrix} M & \frac{l}{h} \\ 0 & 1 \end{pmatrix}.$$

with $M > 0$ and $0 \leq l < h$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} M & \frac{l}{h} \\ 0 & 1 \end{pmatrix}.$$

Any further action on the left by some $q \in \mathbb{Q}$ would make the bottom right entry different than 1. Also, an further action by some element of $SL_2(\mathbb{Z})$ that leaves the bottom row invariant would be of the form

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

but in order to keep $0 \leq \frac{l}{h} < 1$, $n$ must be 0, proving uniqueness.

Since projective lattices are defined by the two numbers $M$ and $\frac{l}{h}$, we use $L_{M,\frac{l}{h}}$ to denote them. Let us also denote by $g_{M,\frac{l}{h}}$ or equivalently by $\begin{bmatrix} M & \frac{l}{h} \\ 0 & 1 \end{bmatrix}$ the coset of $\begin{pmatrix} M & \frac{l}{h} \\ 0 & 1 \end{pmatrix}$ in $PGL_2^+(\mathbb{Q})$.

$$L_{M,\frac{l}{h}} := \mathbb{Q}^* SL_2(\mathbb{Z}) \cdot \begin{pmatrix} M & \frac{l}{h} \\ 0 & 1 \end{pmatrix}.$$

$$g_{M,\frac{l}{h}} = \begin{bmatrix} M & \frac{l}{h} \\ 0 & 1 \end{bmatrix} := \mathbb{Q}^* \cdot \begin{pmatrix} M & \frac{l}{h} \\ 0 & 1 \end{pmatrix}$$

When $l = 0$, the second subscript is omitted to give $L_M$.

Notice that the name of any projective lattice generated by elements $G_1$ is $L_1$. We will call it our *distinguished* lattice.

We now have a labeling scheme for the vertices of our diagrams, so that our previous example can be rectified to give

$$L_{M,\frac{l}{h}} \overset{\delta(L_{M,\frac{l}{h}}, L_{M',\frac{l'}{h'}})}{\overline{\qquad\qquad\qquad}} L_{M',\frac{l'}{h'}}$$

Here is a more specific example

$$L_1 \overset{3}{\overline{\qquad\qquad}} L_3.$$

**Definition 1.3.1.** *Along the lines of [?], we will refer to the diagram that includes all projective lattices and all edges as the Big Picture.*

Before defining isometries on the Big Picture, it would be best to introduce some of its interesting smaller substructures.

**Definition 1.3.2.** *A diagram in which only vertices whose hyperdistance from $L_1$ is equal to some power of a prime $p$ and only the edges labeled by this prime $p$ are included is called the hyperdisance-p tree. The fact that the obtained diagrams are indeed tree diagrams is proven later in Corollary* **??**

**Definition 1.3.3.** *The hypercircle of radius $n$ centered at $L_{M,\frac{l}{h}}$ is the set of projective lattices whose hyperdistance from $L_{M,\frac{l}{h}}$ is $n$. It is denoted $HC_n(M, \frac{l}{h})$.*

Figure **??** is an more elaborate example of diagram of projective lattices. The labels on the edges have been omitted, it is understood that they would all be 3. To compute all the elements of the hypercircle of radius 9 centered at $L_1$, as in figure **??**, one acts on $L_1$ on the left with all the elements given in Lemma **??**, names them, then repeats the process with the resulting matrices.

We can now make diagrams that represent the geometry of our space of projective lattices. We have also defined some simple substructures. Let us now define an action on the Big Picture, check that it is an isometry, and start to explore how our substructures behave under it.

Let us define an action on the right of our projective lattices by elements of $GL_2^+(\mathbb{Q})$. For $a$ and $g \in GL_2^+(\mathbb{Q})$ and $L =\in \mathcal{L}$ such that $L = (\mathbb{Q}^* SL_2(\mathbb{Z}))g$, let

$$L \cdot b := (\mathbb{Q}^* SL_2(\mathbb{Z}))(gb). \tag{1.3}$$

**Lemma 1.3.4.** *The action defined in equation* **??** *is well defined.*

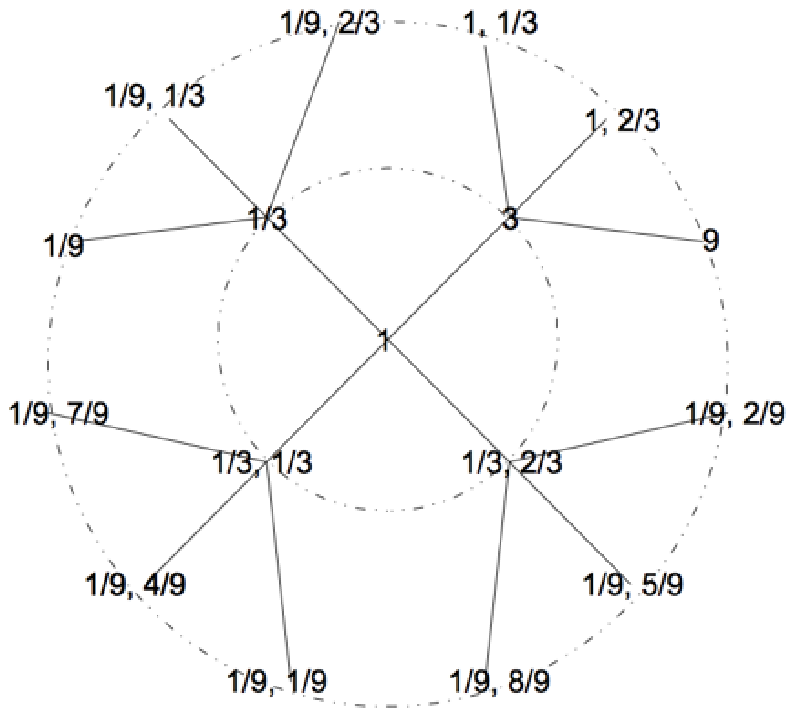



Figure 1.3: Part of the 3-branch tree. More specifically, the (9|1)-ball. Dotted circles have been added to represent the hypercircles of radii 3 and 9 centered at $L_1$. This vocabulary and notation was defined by Conway in [**?**].

*Proof.* Let

$$b : H \setminus G \to H \setminus G$$

such that

$$b(H \cdot g) \mapsto H \cdot (gb).$$

Then for $hg = g'$

$$b(H \cdot g') = H \cdot (g'b) = H \cdot (hgb) = H \cdot (gb) = b(H \cdot g).$$

This concludes the proof using $H = \mathbb{Q}^* SL_2(\mathbb{Z})$, $G = GL_2^+(\mathbb{Q})$ and $b \in GL_2^+(\mathbb{Q})$. $\square$

**Lemma 1.3.5.** *The action defined in Equation* **??** *generalizes to an action on the right by elements of* $PGL_2^+(\mathbb{Q})$.

*Proof.* For group a $G$ acting on a set $X$, if a subgroup $H$ of $G$ acts trivially on $X$, then $H$ is normal in $G$. Furthermore, one can act on $X$ with $G/H$ and the coset $H \cdot g$ will act as $g$ on $X$. Use $H = \mathbb{Q}^* SL_2(\mathbb{Z})$, $G = GL_2^+(\mathbb{Q})$ and $X = P\mathcal{L}$. □

**Lemma 1.3.6.** *The action defined in equation* **??** *preserves hyperdistance.*

*Proof.* Take $L_{M,\frac{l}{h}}$ and $L_{M',\frac{l'}{h'}} \in P\mathcal{L}$. Notice that $D(L_{M,\frac{l}{h}}, L_{M',\frac{l'}{h'}}) = \log(Pdet(g_{M,\frac{l}{h}} g^{-1}_{M',\frac{l'}{h'}}))$ (Recall from Definition **??** and Lemma **??** that, where our action **??** is on the right, distance $D$ is computed from an action on the left). Act on the right of $L_{M,\frac{l}{h}}$ and $L_{M',\frac{l'}{h'}}$ with some $b \in GL_2^+(\mathbb{Q})$.

$$\begin{aligned} D(L_{M,\frac{l}{h}}b, L_{M',\frac{l'}{h'}}b) &= \log\left[Pdet\left((g_{M,\frac{l}{h}}b)(g^{-1}_{M',\frac{l'}{h'}}b)^{-1})\right)\right] \\ &= \log\left[Pdet\left(g_{M,\frac{l}{h}}bb^{-1}g^{-1}_{M',\frac{l'}{h'}}\right)\right] \\ &= \log\left[Pdet\left(g_{M,\frac{l}{h}}g^{-1}_{M',\frac{l'}{h'}}\right)\right] \\ &= D(L_{M,\frac{l}{h}}, L_{M',\frac{l'}{h'}}) \end{aligned}$$

□

How does this action affect the Big Picture? How does it affect hyperdistance-$p$ trees? Lemma **??** assures us that the action is an isometry of the Big Picture. It does not, however, preserve our hyperdistance-$p$ trees. Take, for example, the action of $g_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ on the part of the 3-branch tree depicted in Figure **??**. It results in Figure **??**, in which all projective lattices are still of hyperdistance 3 from each other, but none are hyperdistance 3 from $L_1$. In fact, we have changed the center of the hypercircles from $L_1$ to $L_2$.

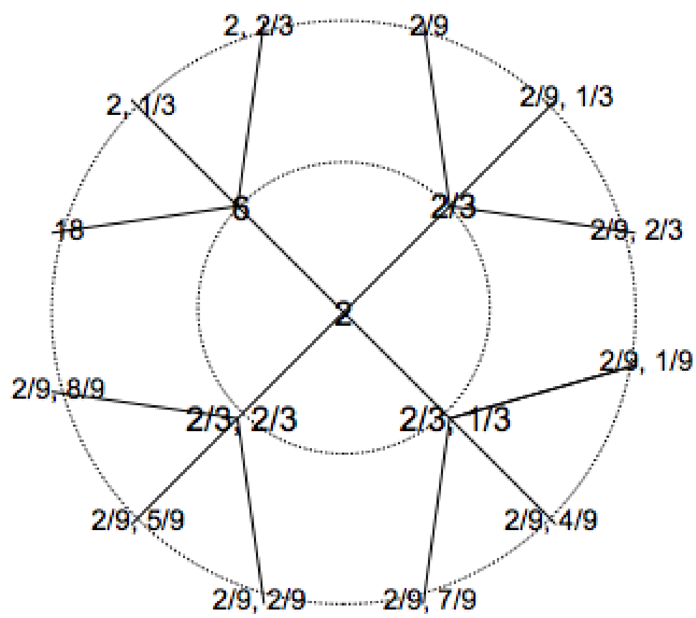



Figure 1.4: The image of part of the 3-branch tree of Figure **??** when multiplied on the right by $g_2$.

We may now ask what are the subgroups of $GL_2^+(\mathbb{Q})$ that do preserve hyperdistance-$p$ trees. The task of answering this question is greatly simplified by the fact that our action on the right is doubly transitive. Double transitivity is defined as the existence of an element of $GL_2^+(\mathbb{Q})$ that acts on a pair of projective lattices that are hyperdistant $N$ to any other pair that have the same hyperditnace between each other. This allows us to consider the pair $L_1$ and $L_N$ as a general case.

**Lemma 1.3.7.** *The action defined in equation* **??** *is transitive.*

*Proof.* To map $L_{M,\frac{l}{h}}$ to $L_{M',\frac{l'}{h'}}$, act on it with $(g^{-1}_{M,\frac{l}{h}} g_{M',\frac{l'}{h'}})$. □

The next Lemma tells us how many edges to draw from vertices in a particular hyperdistance-$p$ tree.

**Lemma 1.3.8.** *For $p$ prime, there are $p+1$ elements $g \in \mathbb{Q}^* SL_2(\mathbb{Z}) \setminus GL_2^+(\mathbb{Q})$ such that $Pdet(g) = p$. They are the orbits under $\mathbb{Q}^* SL_2(\mathbb{Z})$ of*

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{p} & \frac{k}{p} \\ 0 & 1 \end{pmatrix},$$

*For $0 \leq k < p$.*

*Proof.* For any element $g \in GL_2^+(\mathbb{Q})$, its orbit $\mathbb{Q}^* \cdot SL_2(\mathbb{Z}) \cdot g$ has the same projective determinant by Lemmas **??** and **??**. We may then restrict ourselves to matrices of our form $\begin{pmatrix} M & \frac{l}{h} \\ 0 & 1 \end{pmatrix}$, since all others have such an equivalent. Given that the lower right entry is 1, the $\alpha$ of definition **??** must be bigger than 1. So we have that $Pdet(g) \geq \det(g)$. We then have two cases, either $M$ is or isn't an integer.

For the case that $M$ is an integer, $M = p$ is the only possibility since $p$ is prime.

For the case that $M$ is not an integer, consider the $\alpha$ of definition **??**. The product of the diagonal entries of $\alpha g$ must be $p$. Since $p$ is prime, it makes $\frac{1}{p}$ the only choice for the top left entry, and gives us $\alpha = p$. The denominator of the top right entry must then be a proper divisor of $\alpha$, which makes it $p$. Its numerator can then be any number $k$ such that $0 \leq k < p$. □

**Corollary 1.3.9.** *For $p$ prime, the valence of the hyperdistance-$p$ tree is $p+1$.*

*Proof.* Since our action on the right is transitive and is an isometry, it suffices to prove that the valence of our $L_1$ projective lattice in the hyperdistance-$p$ tree is $p+1$. To compute the hyperdistance between a projective lattice $L$ and $L_1$, you need only compute the projective determinant of a generator of $L$. The matrices of Lemma **??** then generate the only projective lattices that are distance $p$ away from $L_1$. □

**Lemma 1.3.10.** *$\mathbb{Q}^* SL_2(\mathbb{Z}) \subset GL_2^+(\mathbb{Q})$ acts trivially on $L_1$.*

*Proof.* For $h \in \mathbb{Q}^* SL_2$ and $e \in GL_2^+(\mathbb{Q})$ the identity element,

$$\begin{aligned} L_1 \cdot h &= (\mathbb{Q}^* SL_2(\mathbb{Z}))(eh) \\ &= (\mathbb{Q}^* SL_2(\mathbb{Z}))(he) \\ &= (\mathbb{Q}^* SL_2(\mathbb{Z}))(h^{-1}he) \\ &= (\mathbb{Q}^* SL_2(\mathbb{Z}))(e) \\ &= L_1 \end{aligned}$$

□

**Corollary 1.3.11.** *$G_1 \subset PGL_2^+(\mathbb{Q})$ acts trivially on $L_1$.*

*Proof.* Recall $G_1 = (\mathbb{Q}^* SL_2(\mathbb{Z}))/\mathbb{Q}^*$ from the previous section. The corollary follows from Lemmas **??** and **??**. □

**Corollary 1.3.12.** *$\mathbb{Q}^* SL_2(\mathbb{Z})$ preserves hypercircles centered at $L_1$.*

*Proof.* Follows directly from the facts that $\mathbb{Q}^* SL_2(\mathbb{Z})$ preserves hyperdistance and acts trivially on $L_1$ (Lemmas **??** and **??**). □

**Lemma 1.3.13.** *$\mathbb{Q}^* SL_2(\mathbb{Z})$ acts transitively on hypercircles centered at $L_1$.*

*Proof.* Take any projective lattice $L_{M,\frac{l}{h}}$ on the hypercircle of radius $N$ centered at $L_1$. Multiply $g_{M,\frac{l}{h}}$ by the smallest positive rational $\alpha$ such that $\alpha \cdot g_{M,\frac{l}{h}} \in M_2(\mathbb{Z})$. Notice that $\alpha \cdot g_{M,\frac{l}{h}}$ also generates $L_{M,\frac{l}{h}}$. It is a standard theorem that all integer matrices can be diagonalized through multiplication on the left and right by elementary integer matrices (a proof is offered in [?]). In two dimensions, elementary integer matrices either have determinant $\pm 1$. Furthermore, the entries $d_{i,i}$ of this diagonalized matrix can be chosen such that $d_{i,i}$ is non-negative and divides $d_{i+1,i+1}$ for all $i$. In our case it means that

$$h_1 \cdot (\alpha \cdot g_{M,\frac{l}{h}}) \cdot h_2^{-1} = g_d$$

for some $h_1$, $h_2 \in GL_2(\mathbb{Z})$ with determinant $\pm 1$ and some diagonal integer matrix $g_d$ with the same determinant as $\alpha \cdot g_{M,\frac{l}{h}}$. The determinants of $h_1$ and $h_2$ are then either both 1 or both -1. Notice that

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g_d \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = g_d$$

so that, should we obtain some $h_1$ and $h_2$ that both have determinant -1, we can use $h_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} h_1$ and $h_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} h_2$ to diagonalize our generator instead, where the latter matrices have determinant 1.

So we either have

$$h_1 \cdot g_{M,\frac{l}{h}} \cdot h_2^{-1} = g_d$$

for some $h_1$, $h_2 \in SL_2(\mathbb{Z})$, or

$$h_3 \cdot g_{M,\frac{l}{h}} \cdot h_4^{-1} = g_d$$

for some $h_3$, $h_4 \in SL_2(\mathbb{Z})$. Since actions on the right by element of $SL_2(\mathbb{Z})$ preserve hypercircles centered at $L_1$, and since actions on the left by elements of $SL_2(\mathbb{Z})$ fix all projective lattices, either

$$L_{M,\frac{l}{h}} \cdot h_2^{-1} = (\mathbb{Q}^* SL_2(\mathbb{Z})) \cdot (g_{M,\frac{l}{h}} \cdot h_2^{-1}) = (\mathbb{Q}^* SL_2(\mathbb{Z})) \cdot (h_1 \cdot g_{M,\frac{l}{h}} \cdot h_2^{-1}) = (\mathbb{Q}^* SL_2(\mathbb{Z})) \cdot g_d$$

or

$$L_{M,\frac{l}{h}} \cdot h_4^{-1} = (\mathbb{Q}^* SL_2(\mathbb{Z})) \cdot (g_{M,\frac{l}{h}} \cdot h_4^{-1}) = (\mathbb{Q}^* SL_2(\mathbb{Z})) \cdot (h_3 \cdot g_{M,\frac{l}{h}} \cdot h_4^{-1}) = (\mathbb{Q}^* SL_2(\mathbb{Z})) \cdot g_d$$

Since actions on the the right of projective lattices by elements of $SL_2(\mathbb{Z})$ preserve hypercircles centered at $L_1$, $g_d = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}$. The name of $g_d$ is then $g_{\frac{1}{N}}$ and

$$(\mathbb{Q}^* SL_2(\mathbb{Z})) \cdot g_d = L_{\frac{1}{N}}.$$

Thus every projective lattice in the hypercircle of radius N centered at $L_1$ can be mapped to $L_{\frac{1}{N}}$ by multiplication on the right by an element of $SL_2(\mathbb{Z})$.

Since the action on the right by $SL_2(\mathbb{Z})$ is a group action and thus has inverses, $L_{\frac{1}{N}}$ can be mapped to any other projective lattice on its hypercircle centered at $L_1$ and the proof is complete.

□

**Lemma 1.3.14.** *The action of $GL_2^+(\mathbb{Q})$ on the metric space of projective lattices is doubly transitive.*

*Proof.* Take any pair of projective lattices $L_{M,\frac{l}{h}}$ and $L_{M',\frac{l'}{h'}}$ such that the hyperdistance between them is $N$. Act on $L_{M,\frac{l}{h}}$ with the element that maps it to $L_1$. Let us call this element $g$. From the fact that the action is an isometry (Lemma **??**), it follows that $L_{M',\frac{l'}{h'}}$ is mapped to an element $L'$ of the hypercircle of radius $N$ centered at $L_1$. Since $\mathbb{Q}^* SL_2(\mathbb{Z})$ acts trivially on $L_1$ and transitively on hypercircles centered at $L_1$ (Lemmas **??** and **??**), it follows that there exists an element $h$ of $\mathbb{Q}^* SL_2(\mathbb{Z})$ that maps $L'$ to $L_N$ while acting trivially on $L_1$. We then have that $gh$ acts on

$$L_{M,\frac{l}{h}} \overset{N}{\text{———}} L_{M',\frac{l'}{h'}}$$

to give

$$L_1 \overset{N}{\text{———}} L_N.$$

From the fact that our action on the right is indeed a group action, we have that every element has an inverse. Thus if any such pair $L_{M,\frac{l}{h}}$ and $L_{M',\frac{l'}{h'}}$ can be mapped to $L_1$ and $L_N$, we also have that $L_1$ and $L_N$ can be mapped to any pair between which the hyperdistance is $N$, proving double transitivity. □

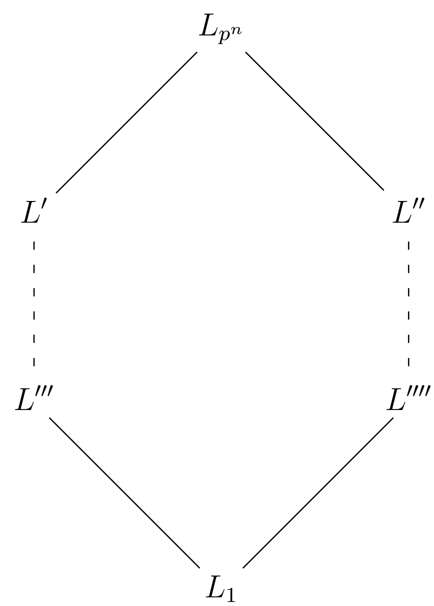
In a hyperdistance-$p$ tree, let us define a *path* from one projective lattice to another as a sequence of directed edges and vertices that connect them. Let us further define a *next step* from one projective lattice to another to be the next projective lattice encountered along a path connecting one to the other. Next steps depend on a choice of path.

**Lemma 1.3.15.** *For $p$ prime, on the hyperdistance-$p$ tree, there exists a unique next step that brings you closer to any projective lattice from another.*

*Proof.* We will prove that it is true for $L_N = L_{p^n}$ and $L_1$. From Lemma **??**, the projective lattices that are of distance $p$ from $L_{p^n}$ are $L_{p^{n-1}}$, $L_{p^{n+1}}$ and $L_{p^n,\frac{k}{p}}$, for all $0 < k < p$. The closer next step is the one that brings us to $L_{p^{n-1}}$, which is $p^{n-1}$-far from $L_1$, the other projective lattices all being $p^{n+1}$-far from $L_1$. □

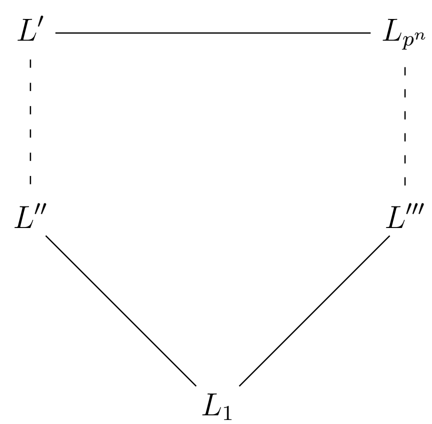
**Corollary 1.3.16.** *For $p$ prime, hyperdistance-$p$ trees are indeed tree diagrams.*

*Proof.* What we need to prove is that there are no loops in the diagram. Let us assume the existence of a loop. This loop can either be comprised of an even or an odd number of edges. In the even case,




there must be some projective lattice farthest from $L_1$ (here $L_{p^n}$) such there are (at least) two closer next steps from it to $L_1$. This contradicts the unique closer next step Lemma (Lemma **??**). There can then be no even loops.

The odd case is handled in a similar way.



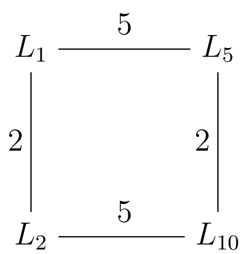

In this odd loop, there are now two projective lattices that have the same distance $p^n$ from $L_1$. As we have already shown, however, of all the projective lattices that are distance $p$ from $L_{p^n}$, one is $p^{n-1}$-far from $L_1$ and all others are $p^{n+1}$-far from it. Thus there can be no two adjacent projective lattices that have the same hyperdistance from $L_1$ and there can be no odd loops.

□

**Corollary 1.3.17.** *For $p$ prime, on the hyperdistance-$p$ tree, there is a unique directed path between two projective lattices.*

*Proof.* If there were two directed paths, they would necessarily form a loop, contradicting the previous no-loop Lemma. □

For $L_1$ and $L_N$ when $N$ is not a prime power, there is no unique closer next step. For example, considering the paths from $L_1$ to $L_{10}$. We have




so that there are two closer steps, one to $L_2$ and one to $L_5$, and the two paths form a loop. One can see, though, that any path between $L_1$ and $L_N$ where we consider the prime decomposition $N = p_1^{n_1} p_2^{n_2} ... p_k^{n_k}$, decomposes also into its $p_1, p_2, ..., p_k$ parts so that all directed paths comprised of successive closer next steps have the same length $l = n_1 p_1 + n_2 p_2 + ... + n_k p_k$ which is equal to the hyperdistance between the two projective lattices.

**Definition 1.3.18.** *The graph of all edges and projective lattices that are included in all paths comprised of successive closer next steps from $L_{M,\frac{l}{h}}$ to $L_{N,\frac{j}{f}}$ is called the $(M, \frac{l}{h} \mid N, \frac{j}{f})$-thread.*

We can now see that the previous example was in fact the $(1 \mid 10)$-thread.

**Corollary 1.3.19.** *For any* $M, N \in \mathbb{N}^*$ *such that* $1 < M < N$*, if* $M \mid N$ *then* $L_1$ *is not in the* $(M \mid N) - thread$*.*

*Proof.* Since $D(L_1, L_M) = M$, $D(L_1, L_N) = N$ and $D(L_M, L_N) = \frac{M}{N}$, and since all directed paths comprised of successive closer next steps have the same length, which is equal to the hyperdistance, a path comprised of successive closer next steps from $L_M$ by $L_1$ to $L_N$ would have length $M + N$ which is larger than $\frac{M}{N}$. This contradicts the fact that all paths comprised of successive closer next steps in a thread have the same length. □

In Section **??**, we finished the definition of a geometrical space $P\mathcal{L}$. In this Section **??**, we elaborated diagrams that allow us to visualize this space. We have also introduced an isometry of $P\mathcal{L}$ and interesting substructures, the hyperdistance-$p$ trees. We will now look at certain subroups of $GL_2^+(\mathbb{Q})$ that stabilize these trees.

## 1.4 Gamma groups as the stabilizers of certain families of projective lattices

The $E_8$ diagram that we will obtain in Section **??** will be elaborated partly through a set of restrictions that we will impose on subgroups of $PGL_2^+(\mathbb{Q})$. The trees will facilitate computations related to these restrictions as well as allow us to visualize them. We are beginning to restrict our set now.

Recall that $g_{M,\frac{l}{h}}$ denotes the the coset of $\begin{pmatrix} M & \frac{l}{h} \\ 0 & 1 \end{pmatrix} \in PGL_2^+(\mathbb{Q})$.

Let us restrict our set of projective lattices to *numberlike* lattices, those whose variable $l$ is 0, and for which $M \in \mathbb{N}$.

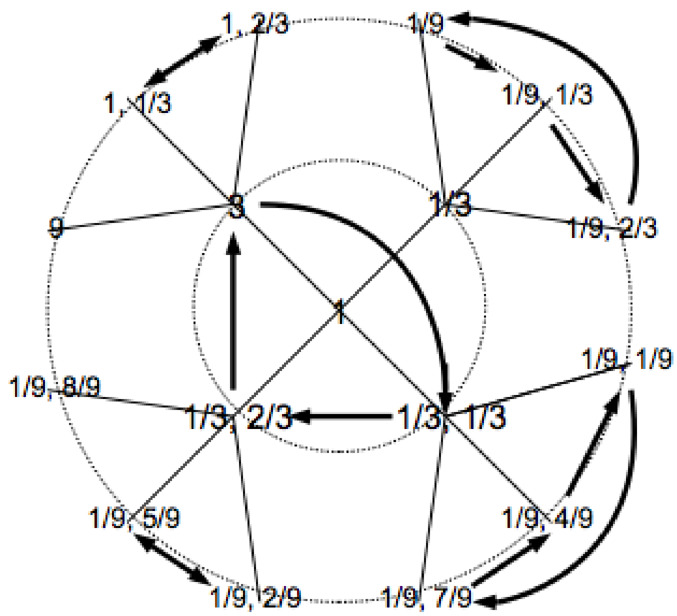



Figure 1.5: The action of $\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \in G_1$ on the part of the 3-branch tree in Figure **??**.

**Definition 1.4.1.** *Gamma groups*

*(1)* $G_N := \{g \in PGL_2^+(\mathbb{Q}) \mid L_N \cdot g = L_N\}$

*Notice that $G_1$ defined this way is the same as the $G_1$ defined in Section* **??**. *$G_N$ is obtained from $G_1$ in the following way:*

$$G_N = g_N^{-1} G_1 g_N.$$

*(2)* $\Gamma := PSL_2(\mathbb{Z})$

$\Gamma = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$

*We have shown in Section* **??** *that this group is isomorphic to $G_1$.*

*(3)* $\Gamma_0(N) := \{g \in PGL_2^+(\mathbb{Q}) \mid g \in G_1 \cap G_N\}$

$$\Gamma_0(N) = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid g \in \Gamma, c \mod N = 0 \right\}$$

*(4)* $\Gamma_0(X|Y) := \{g \in PGL_2^+(\mathbb{Q}) \mid g \in G_X \cap G_Y\}$

*This group is obtained by conjugating* $\Gamma_0(N)$ *by an element of* $GL_2^+(\mathbb{Q})$ *that takes* $L_N$ *to* $L_X$ *and* $L_1$ *to* $L_Y$*, where* $N$ *is the hyperdistance between* $L_X$ *and* $L_Y$*.*

*(5)* $\Gamma_0(N)+ := \left\{ g \in PGL_2^+(\mathbb{Q}) \mid g \text{ preserves the set } \{L_1, L_N\} \right\}.$

*For* $N$ *prime,* $\Gamma_0(N)+ = \left\langle \Gamma_0(N), \begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix} \right\rangle,$

*where* $\begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}$ *is called the Frick involution of order* $N$*.*

*For other N,* $\Gamma_0(N)+ = \left\{ \Gamma_0(N), W_e(N) \mid e \| N \right\},$

*where* $W_e(N) = \left\{ \begin{bmatrix} ae & b \\ cN & de \end{bmatrix} | a, b, c, d \in \mathbb{Z}, ade^2 - bcN = e \right\}.$

*(6)* $\Gamma_0(X \mid Y)+ := \left\{ g \in PGL_2^+(\mathbb{Q}) \mid g \text{ preserves the set } \{L_X, L_Y\} \right\},$
*it is obtained from* $\Gamma_0(N)+$ *as* $\Gamma_0(X \mid Y)$ *was obtained from* $\Gamma_0(N)$*.*

*(7)* $\Gamma_0^{(h)}(\frac{N}{h}|h)+$*, a subgroup of order* $h$ *in* $\Gamma_0(\frac{N}{h}|h)$*, where* $h$ *is the largest divisor of* 24 *such that* $h^2|N$*. This subgroup is defined in [**?**], generally. We will define here only the groups that are of relevance to our results.*

$\Gamma_0^{(2)}(4|2)+ := \langle \Gamma_0(8), (T^4)^t T^{\frac{1}{2}}, W_8 \rangle$

$\Gamma_0^{(3)}(3|3) := \langle \Gamma_0(9), (T^3)^t T^{\frac{1}{3}}, T - \frac{1}{3}(T^3)^t T^{-\frac{1}{3}} \rangle,$

*where $T^q$ denotes the coset of $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$ in $PGL_2^+(\mathbb{Q})$.*

**Theorem 1.4.2.** *The projective lattices that are fixed by $\Gamma_0(X|Y)$ are those whose hyperdistance from the $(X|Y)$-thread divides 24. Recall from Definition* **??** *that the $(X|Y)$-thread is the set of projective lattices that constitute the unique directed path between $L_X$ and $L_Y$.*

*Proof.* A proof of this theorem is given by Conway in [**?**]. □

**Definition 1.4.3.** *Two subgroups $A$ and $B$ of a group $G$ are called commensurable if their intersection has finite index in each of them.*

$GL_2^+(\mathbb{Q})$ and $PGL_2^+(\mathbb{Q})$ embed naturally in $PSL_2(\mathbb{R})$. For $g \in GL_2^+(\mathbb{Q})$ or $\mathbb{Q}^*g \in PGL_2^+(\mathbb{Q})$, there exists a unique $q \in \mathbb{Q}^+$ such that $\det(qg) = \det(-qg) = 1$. For $f : PGL_2^+(\mathbb{Q}) \mapsto PSL_2(\mathbb{R})$, take

$$f(\mathbb{Q}^*g) := \langle -I \rangle qg. \tag{1.4}$$

This map $f$ is a homomorphism. If $\det(q_1g_1) = \det(q_2g_2) = 1$ then $\det(q_1g_1q_2g_2) = 1$ and $f(\mathbb{Q}^*g_1 \cdot \mathbb{Q}^*g_2) = \langle -I \rangle q_1g_1q_2g_2 = f(\mathbb{Q}^*g_1)f(\mathbb{Q}^*g_2)$. The composition of $f$ with the natural map that takes elements of $GL_2^+(\mathbb{Q})$ to $PGL_2^+(\mathbb{Q})$ gives the required homomorphism from $GL_2^+(\mathbb{Q})$ to $PSL_2(\mathbb{R})$. Since $f(PGL_2^+(\mathbb{Q})) \cong PGL_2^+(\mathbb{Q})$, we will denote by $PGL_2^+(\mathbb{Q})$ the image of $PGL_2^+(\mathbb{Q})$ in $PSL_2(\mathbb{R})$, it should be clear from the context which of the two objects we are dealing with.

Notice that, since $SL_2(\mathbb{Z})$ embeds naturally in $SL_2(\mathbb{R})$ and

$$\mathbb{Q}^* \cap SL_2(\mathbb{Z}) = \mathbb{R}^* \cap SL_2(\mathbb{R}) = \langle -I \rangle,$$

$PSL_2(\mathbb{Z})$ also embeds naturally in $PSL_2(\mathbb{R})$.

Elements of $PSL_2(\mathbb{R})$ act transitively from the left on points of the upper-half complex plane (with infinity) through the associated Moebius transformation. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R})$ and $z \in \mathbb{C}^+$,

$$gz = \frac{az+b}{cz+d}.$$

We can then have our Gamma groups act on the real projective line through the function $f$.

**Definition 1.4.4.** *A subgroup of $PSL_2(\mathbb{R})$ is called arithmetic if it is commensurable with $\Gamma = PSL_2(\mathbb{Z})$. Let $Ar$ denote the set of all arithmetic groups.*

**Definition 1.4.5.** *An element $g$ of $PSL_2(\mathbb{R})$ is called parabolic if $g = \langle -I \rangle \cdot h$ for some $h \in SL_2(\mathbb{R})$ such that $tr(h) = \pm 2$.*

**Definition 1.4.6.** *Let $C^+$ denote the upper-half complex plane with infinity. A cusp of a subgroup $H$ of $PSL_2(\mathbb{R})$ is the orbit of a point $z$ of $\mathbb{C}^+$ such that $hz = z$ for some parabolic element $h \in H$.*

**Corollary 1.4.7.** *A point $z$ such that $hz = z$ for some parabolic element $h \in PSL_2(\mathbb{R})$ is either real or infinity.*

*Proof.* We have the following system of equations

$$\frac{az+b}{cz+d} = z$$

$$a + d = \pm 2$$

$$ad - bc = 1,$$

which yields

$$z = \frac{ac-1}{c^2},$$

which must be real for $a$ and $c$ real. □

**Lemma 1.4.8.** *Two commensurable subgroups of $PSL_2(\mathbb{R})$ have the same set of cusps.*

*Proof.* A proof of this standard theorem is included in [**?**]. □

**Lemma 1.4.9.** *$\Gamma$ has a single cusp, it is the rational projective line, denoted $P^1(\mathbb{Q})$.*

**Lemma 1.4.10.** *If $H$ is an arithmetic subgroup of $PSL_2(\mathbb{R})$, then $H$ is contained in the image under $f$ of $PGL_2^+(\mathbb{Q})$ in $PSL_2(\mathbb{R})$ (Equation* **??***).*

*Proof.* Lemmas **??** and **??** give us that $H$ stabilizes $P^1(\mathbb{Q})$ since $PSL_2(\mathbb{Z})$ does so. This action can be determined by its action on 1, 0 and $\infty$. Since these three points are part of the rational projective line, they are sent to other elements of $P(\mathbb{Q})$, which shows that any element of $H$ can be represented by a rational matrix. □

Particularly, Lemma **??** allows us to act from the right on $P\mathcal{L}$ with arithmetic subgroups of $PSL_2(\mathbb{R})$.

**Lemma 1.4.11.** *$G_N$ acts transitively on $P^1(\mathbb{Q})$ and thus has a single cusp.*

*Proof.* $G_N$ is $G_1^g$ for some $g \in PGL_2^+(\mathbb{Q})$ and both $PGL_2^+(\mathbb{Q})$ and $G_1$ act transitively on the rational projective line. □

**Theorem 1.4.12.** *(Helling [**?**])*

*The maximal discrete arithmetic subgroups of $PSL_2(\mathbb{R})$ are the conjugates of $\Gamma_0(N)^+$ for square-free $N$.*

**Definition 1.4.13.** *Let $Fix(\infty)$ denote the subgroup of $PSL_2(\mathbb{R})$ that fixes $\infty$. Notice that*

$$Fix(\infty) = \left\{ \mathbb{R}^* \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{R}, a > 0 \right\}.$$

*For $H$ a subgroup of $PSL_2(\mathbb{R})$, let $Fix_H(\infty)$ denote the intersection $H \cap Fix(\infty)$.*

Let $T^A$ denote the coset containing $\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}$ in $PSL_2(\mathbb{R})$. Notice that

$$\text{Fix}_{PGL_2^+(\mathbb{Q})}(\infty) = \{T^A \mid A \in \mathbb{Q}\}.$$

**Lemma 1.4.14.** *For $H$ an arithmetic group, $Fix_H(\infty)$ is contained in $\{T^A \mid A \in \mathbb{Q}\}$ and is an infinite cyclic group.*

*Proof.* Let $h \in \text{Fix}_H(\infty)$. Then $h$ is the coset containing $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ for some $a \in \mathbb{R}^+$, $b \in \mathbb{R}$. Since $H$ is arithmetic, $h^n \in PSL_2(\mathbb{Z})$ for some $n \in \mathbb{N}$. In particular, this means that $a^n = 1$ and so that $a = 1$. Furthermore, $b^n \in \mathbb{Z}$ so that $b \in \mathbb{Q}$, proving that $h \in \{f(T^A) \mid A \in \mathbb{Q}\}$.

Then $\text{Fix}_H(\infty)$ is some discrete subgroup of $\mathbb{Q}$ and is thus an infinite cyclic group.

□

The conditions that we will impose on Gamma groups to restrict them to the nine that are of concern to us make use of the notion of *width*.

**Definition 1.4.15.** *Let $H_0$ be an arithmetic group. For $H_0 \cdot x \in H_0 \backslash P^1(\mathbb{Q})$, let $g \in G_1$ be such that $x = g \cdot \infty$. Since $H_0$ is arithmetic, so is $H_0^g$. Let $T^A$ be the generator of (the infinite cyclic group) $H_0^g \cap Fix_{G_1}(\infty)$. Then let*

$$w_{G_1} : (Ar \times P^1(\mathbb{Q})) \mapsto \mathbb{N}^*$$

*denote the width of $H_0$ at $x$ with respects to $G_1$ and let*

$$w_{G_1}(H_0, x) = A.$$

**Lemma 1.4.16.** *For $H_0$ and $x$ as in Definition ??, the width of $H_0$ at $x$ with respects to $G_1$ is well defined.*

*Proof.* If there exist $g$ and $g' \in G_1$ such that $g \cdot \infty = g' \cdot \infty = x$, then $g^{-1}g \cdot \infty = \infty$ so that $g^{-1}g = T^B$ for some $B \in \mathbb{Q}$ (Lemma ??). We have $H_0^{g'} = (H_0^g)^{T^B}$, so that $H_0^{g'} \cap Fix_{G_1}(\infty) = H_0^g \cap Fix_{G_1}(\infty)$. This shows that the value of the width of $H_0$ at $x$ does not depend on our choice of $g$. □

The action of $PGL_2^+(\mathbb{Q})$ from the right on projective lattices is well suited to compute the width of a specific set of arithmetic groups.

**Lemma 1.4.17.** *Let $\mathcal{O}$ be an orbit of the action of $G_1$ on $P\mathcal{L}$. This is always a hypercircle centered at $L_1$. Pick $L_0 \in \mathcal{O}$ and let $H_0$ be the subgroup of $H$ that fixes of $L_0$. The width of $H_0$ at infinity is the size of the orbit of $L_0$ under the action of $Fix_{G_1}(\infty)$.*

$$w_{G_1}(H_0 \cdot \infty) = \#(L_0 \cdot Fix_{G_1}(\infty)).$$

*Proof.* Recall that $PGL_2^+(\mathbb{Q})$ acts from the right on $P\mathcal{L}$ and from the left on $P^1(\mathbb{Q})$. $\mathrm{Fix}_{G_1}(\infty) = T^1$. The size of the orbit of $L_0$ under $\mathrm{Fix}_{G_1}(\infty)$ is just the smallest $n \in \mathbb{N}^*$ such that $L_0 \cdot T^n = L_0$. Notice however that $L_0 \cdot T^n = L_0$ if and only if $T^n \in H_0$. Since $T^n \in \mathrm{Fix}_{G_1}(\infty)$, it is also $\in H_0 \cap \mathrm{Fix}_{G_1}(\infty)$. Since $n$ is the smallest number with this property, $\langle T^n \rangle = H_0 \cap \mathrm{Fix}_{G_1}(\infty) \Leftrightarrow n = w_{G_1}(H_0 \cdot \infty)$, concluding the proof. □

Another condition that we will impose in Section ?? will require some knowledge of the normalizers of $\Gamma_0(N)$.

**Theorem 1.4.18.** *(Atkin-Lehner)*

*The normalizer of $\Gamma_0(N)$ in $PSL_2(\mathbb{R})$ is the group $\Gamma_0(\frac{N}{h}|h)+$ where $h$ is the largest divisor of 24 for which $h^2|N$.*

*Proof.* A proof of this important result appears in [**?**]. □

# Chapter 2

# Obtaining the correspondence

## 2.1 The 3+1 conditions that must be satisfied for a group $G$ to be included

Let us call our potential candidate $G$. Let $I_G^\Gamma$ denote the index of $G \cap \Gamma$ in $\Gamma$ and $I_\Gamma^G$ the index of $G \cap \Gamma$ in $G$. The conditions are as follows.

1. $G$ has width 1 at $\infty$

2. there is some $N$ such that $G$ contains and normalizes $\Gamma_0(N)$, and the quotient $G/\Gamma_0(N)$ is a group of exponent 1 or 2

3. $I_G^\Gamma \leq 12$ and $I_G^\Gamma / I_\Gamma^G \leq 3$

Notice that condition 2 implies that $G$ is arithmetic.

Looking first at condition 2. From Theorem **??** we have

$$\Gamma_0(N) \trianglelefteq G \subseteq \Gamma_0\left(\frac{N}{h}|h\right)+, \tag{2.1}$$

for $h$ the largest divisor of 24 such that $h^2|N$. To combine this with condition 3, we will need the following Lemma.

**Lemma 2.1.1.** *For the intersection of* $\Gamma_0(\frac{N}{h}|h)+$ *with* $\Gamma$ *we have*

$$\Gamma_0\left(\frac{N}{h}|h\right)+\cap\Gamma=\Gamma_0\left(\frac{N}{h}\right)$$

*Proof.* Recall that $h^2|N$ so that $h|\frac{N}{h}$ and $L_h$ is contained in the path from $L_1$ to $L_{\frac{N}{h}}$.

$$\Gamma_0\left(\frac{N}{h}|h\right)+\cap\Gamma=\left\{g\in PGL_2^+(\mathbb{Q})\mid g \text{ preserves the set } \{L_{\frac{N}{h}},L_h\} \text{ and fixes } L_1\right\}$$

The sets $W_e$ that make up the "+" in $\Gamma_0(\frac{N}{h}|h)+$ do not have projective determinant 1 so that they do not fix $L_1$ and are not be in the intersection. We then have

$$\Gamma_0\left(\frac{N}{h}|h\right)+\cap\Gamma=\Gamma_0\left(\frac{N}{h}|h\right)\cap\Gamma.$$

In terms of generators, the intersection is as follows. We are looking at the image of these classes in $PGL_2^+(\mathbb{Q})$

$$\left\{\begin{pmatrix}a_1 & b_1/h\\ c_1h & d_1\end{pmatrix}\right\}\cap\left\{\begin{pmatrix}a_2 & b_2/\frac{N}{h}\\ c_2\frac{N}{h} & d_2\end{pmatrix}\right\}\cap\left\{\begin{pmatrix}a_3 & b_3\\ c_3 & d_3\end{pmatrix}\right\},$$

where $a_n,b_n,c_n,d_n\in\mathbb{Z}$ and where $a_nd_n-b_nc_n=1$. Since

$$\left\{\begin{pmatrix}a_2 & b_2/\frac{N}{h}\\ c_2\frac{N}{h} & d_2\end{pmatrix}\right\}\cap\left\{\begin{pmatrix}a_3 & b_3\\ c_3 & d_3\end{pmatrix}\right\}\subset\left\{\begin{pmatrix}a_1 & b_1/h\\ c_1h & d_1\end{pmatrix}\right\}\cap\left\{\begin{pmatrix}a_3 & b_3\\ c_3 & d_3\end{pmatrix}\right\},$$

the intersection simplifies further to

$$\left\{\begin{pmatrix}a_2 & b_2/\frac{N}{h}\\ c_2\frac{N}{h} & d_2\end{pmatrix}\right\}\cap\left\{\begin{pmatrix}a_3 & b_3\\ c_3 & d_3\end{pmatrix}\right\},$$

which is exactly $\Gamma_0(N/h)$.

This proof is easily visualized using trees. We are looking for the joint stabilizer of the set $\{L_{\frac{N}{h}}, L_h\}$ and $L_1$. We have the following configuration

$$L_1 \text{ ———— } L_h \text{ ———— } L_{\frac{N}{h}}.$$

Evidently, the elements of $\Gamma_0(\frac{N}{h}|h)+$ that maps $L_h$ to $L_{\frac{N}{h}}$ and vice versa (the "+" elements) do not stabilize $L_1$ and the stabilizer of the path from $L_1$ to $L_{\frac{N}{h}}$ is $\Gamma_0(N/h)$. □

Considering now the intersection of G with $\Gamma$ as in condition 3, we have

$$\Gamma_0(N) \subseteq G \cap \Gamma \subseteq \Gamma_0(N/h) \tag{2.2}$$

where $h$ is the largest divisor of 24 such that $h^2|N$.

Looking now at the first part of condition 3. Considering the $\Gamma_0(N/h)$ that contains the intersection $G \cap \Gamma$, we see that $\frac{N}{h} + 1 \leq I^{\Gamma}_{\Gamma_0(N/h)} \leq I^{\Gamma}_{G}$.

**Theorem 2.1.2.** *The index $I^{\Gamma}_{\Gamma_0(N)}$ of $\Gamma_0(N)$ in $\Gamma$ is the conjugate Euler function*

$$\phi_+(N) = N \prod_{p|N} (1 + \frac{1}{p})$$

*Proof.* A proof of this classical result is offered in [?]. □

**Corollary 2.1.3.** *The possible values of $\frac{N}{h}$ in equations* **??** *and* **??** *are*

$$\frac{N}{h} \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11\}.$$

*Proof.* These are the values of $\frac{N}{h}$ for which $\phi_+(\frac{N}{h}) \leq 12$, following condition 3. □

This already provides us with a finite list of supergroups $\Gamma_0\left(\frac{N}{h}|h\right)+$. For $\frac{N}{h} \in \{1,2,3,4,5,6,7,8,9,11\}$ such that $h$ is the largest divisor of 24 and $h^2|N$, we have

$$(N,h) \in C := \{(1,1),(2,1),(3,1),(4,2),(5,1),(6,1),(7,1),\\ (8,1),(8,2),(9,1),(9,3),(11,1),(16,4),(36,6),(64,8)\}. \quad (2.3)$$

Giving us that $G$ is contained in one of the following

$$D := \{\Gamma,\Gamma_0(2)+,\Gamma_0(3)+,\Gamma_0(4)+,\Gamma(2),\Gamma_0(5)+,\Gamma_0(6)+,\Gamma_0(7)+,\Gamma_0(8)+,\\ \Gamma_0(4|2)+,\Gamma_0(9)+,\Gamma(3),\Gamma_0(11)+,\Gamma(4),\Gamma(6),\Gamma(8)\}. \quad (2.4)$$

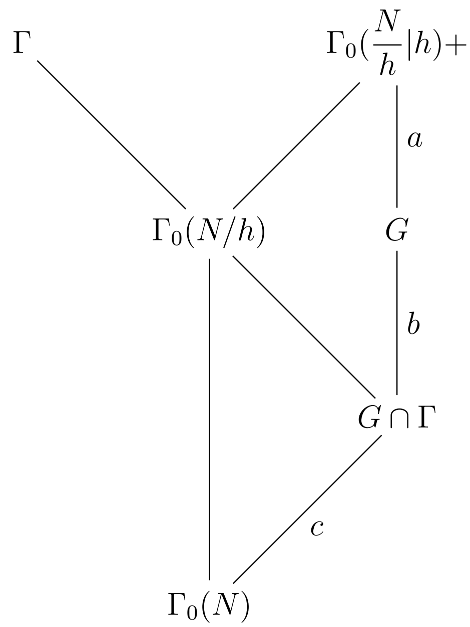
Recalling Equation **??**,

$$\Gamma_0(N) \trianglelefteq G \subseteq \Gamma_0\left(\frac{N}{h}|h\right)+,$$

we now have a finite set $C$ of $(N,h)$ to consider. These are the same $(N,h)$ also used in computations related to Equation **??**

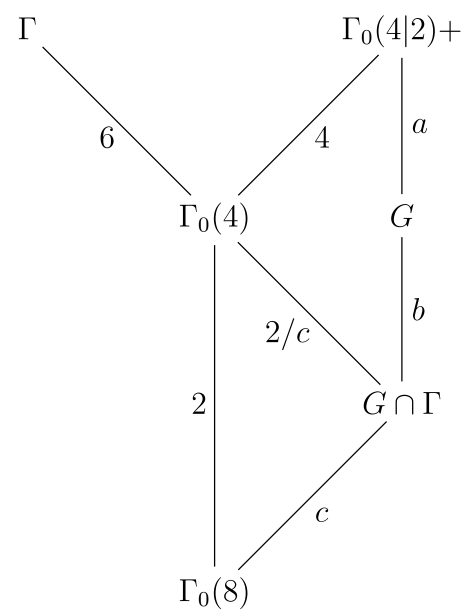
$$\Gamma_0(N) \subseteq G \cap \Gamma \subseteq \Gamma_0(N/h).$$

The computations necessary for determining the groups that satisfy the three conditions are given in detail by Duncan in [**?**]. Using a different notation than the one which we have used here, he begins by giving a diagram illustrating the two previous inclusions.




(2.5)

We will give an example of one of the more involved cases here. Let $N = 8$ and $h = 2$ ($h$ could also be 4 for $N = 8$). We then have




Since $\Gamma_0(4|2)+$ does not have width 1 at $\infty$, $G \neq \Gamma_0(4|2)+$. This gives us $a \geq 2$. From the second part of condition 3 we get $I_G^\Gamma / I_\Gamma^G = 12/bc \leq 3$, which in turn gives us $bc \geq 4$. Along with $abc = 8$, this yields $a = 2$ and $bc = 4$. In order to obtain

$G$, we will investigate the actions of $\Gamma_0(4|2)+$ and $\Gamma_0(8)$ on projective lattices. More precisely, on the union of the hypercircles of radius 2 centered at $L_2$ and $L_4$. Let $S = HC_2(2) \cup HC_2(4)$. Let $\lambda$ denote the natural map from $PGL_2^+(\mathbb{Q})$ to $Sym(S)$. The

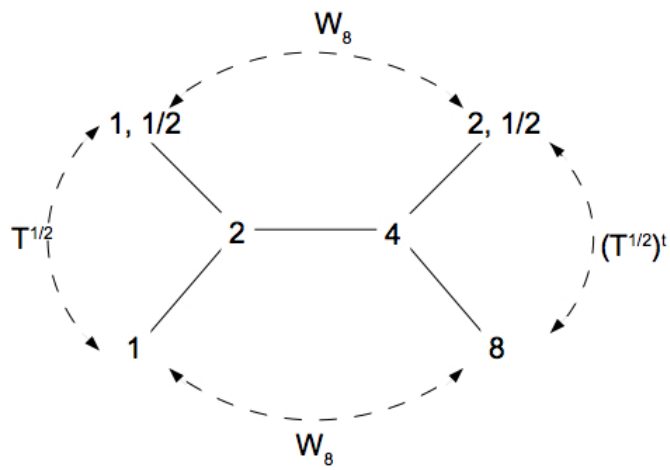



Figure 2.1: S, the projective lattices fixed by $\Gamma_0(8)$.

image of $\Gamma_0(4|2)+$ in $Sym(S)$ is obtained from figure **??**. It is the dihedral group of order 8. Since $\Gamma_0(8) \subset \Gamma_0(4|2)+$ fixes every element of $S$, and since it has index 8 in $\Gamma_0(4|2)+$, we know that it is the kernel of $\lambda$. We have

$$\begin{array}{ccc} \Gamma_0(4|2)+ & & \Gamma_0(4|2)+/\Gamma_0(8) \\ \Big| \, 2 & & \Big| \\ G & \Rightarrow & \lambda(G) \\ \Big| \, 4 & & \Big| \\ \Gamma_0(8) & & Id \end{array}$$

Our group $G$ has index 2 in $\Gamma_0(4|2)+$ and so $\lambda(G)$ must have order 4. Let $r$ and $s$ denote the two generators of $D_8$ such that $D_8 = \langle r, s \mid r^4 = s^2 = Id, srs = r^{-1} \rangle$. $D_8$

has two subgroups of order 4: $\langle r\rangle$ and $\langle r^2, s\rangle$.

$\langle r\rangle$ has exponent 4 and so does not satisfy condition 2.

$\langle r^2, s\rangle$, which is denoted $\Gamma_0^{(2)}(4|2)+$, satisfies all three conditions.

**Theorem 2.1.4.** *The groups that satisfy all three conditions are the elements of the following set*

$$\mathbb{S} := \{\Gamma, \Gamma_0(2), \Gamma_0(2)+, \Gamma_0(3)+, \Gamma_0(4)+, \Gamma_0(5)+, \Gamma_0(6)+, \Gamma_0^{(2)}(4|2)+, \Gamma_0^{(3)}(3|3)\}$$

*Proof.* A full proof of this theorem appears in [?]. □

## 2.2 The correspondence proper

We have the set $\mathbb{S}$, the set of groups that constitute the vertices of our first $E_8$ diagram (Diagram **??**). We still have to include some conditions to arrange them in the $E_8$ configuration.

Let us recall that

$$\Gamma_0(N) \trianglelefteq G \subseteq \Gamma_0\left(\frac{N}{h}|h\right)+, \tag{2.6}$$

for $h$ as in Equation (**??**), and that $I_M^N$ denotes the index of $M \cap N$ in $N$.

**Definition 2.2.1.** *For $G \in \mathbb{S}$, let $N_G$ be the smallest $N$ such that*

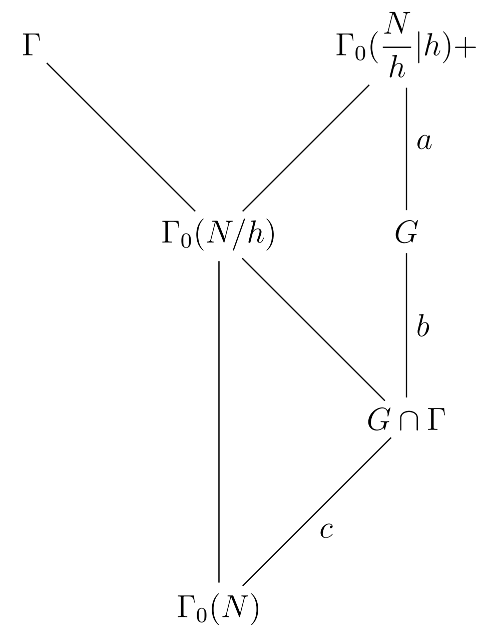
$$\Gamma_0(N) \trianglelefteq G \subseteq \Gamma_0(\frac{N}{h}|h)+,$$

*for some $h$ as in Equation (**??**).*

*Furthermore, let $a_G$ be the largest divisor of 24 such that $a_G^2$ divides $N_G$ and $I_G^{\Gamma_0(\frac{N_G}{a_G})|a_G)} = a_G$.*

**Lemma 2.2.2.** *For $G \neq \Gamma_0(4)+$, $a_G = h$ for the $h$ in equation (**??**). Also, $a_{\Gamma_0(4)+} = 1$.*

*Proof.* Recall Diagram (**??**),




which is how we got our set $\mathbb{S}$ in the first place. It gives us potential candidates $N_G = N$ and $a_G = h$ for all elements of $\mathbb{S}$. The condition that $I_G^{\Gamma_0(\frac{N_G}{h}|h)} = h$ is satisfied all cases.

One then has to check that this $N$ is the smallest number with the required properties. This is trivially true for all groups of the form $\Gamma_0(N)$ and $\Gamma_0(N)+$, for square-free $N$.

For $G = \Gamma^{(3)}(3|3)$, $N = 9$ and $h = 3$. We have to check that $N_{\Gamma^{(3)}(3|3)} \notin \{1, 3\}$. $N_{\Gamma_0^{(3)}(3|3)} \neq 1$ since $\Gamma_0(\frac{1}{1}|1)+ = \Gamma$ and $\Gamma_0^{(3)}(3|3) \nsubseteq \Gamma$. Also, $N_{\Gamma_0^{(3)}(3|3)} \neq 3$ since $\Gamma_0(\frac{3}{1}|1)+ = \Gamma_0(3)+$ and $\Gamma_0^{(3)}(3|3) \nsubseteq \Gamma_0(3)+$.

For $G = \Gamma_0^{(2)}(4|2)+$, $N_{\Gamma_0^{(2)}(4|2)+} \notin \{1, 2, 4\}$ for reasons analogous to the ones for $N_{\Gamma^{(3)}(3)} \notin \{1, 3\}$.

For $G = \Gamma_0(4)+$, $(N_{\Gamma_0(4)+}, a_{\Gamma_0(4)+}) \notin \{(2, 1), (4, 2)\}$ for analogous reasons. We have, however, for $N_{\Gamma_0(4)+} = 4$ and $a_{\Gamma_0(4)+} = 1$

$$\Gamma_0(4) \trianglelefteq \Gamma_0(4)+ \subseteq \Gamma_0(4)+,$$

and $I_{\Gamma_0(4)+}^{\Gamma_0(4)+} = 1$. $\square$

**Definition 2.2.3.** *For $G \in \mathbb{S}$, define the subgroup $H \subset G$ as follows*

$$H := G \cap \Gamma_0(\frac{N_G}{a_G}|a_G).$$

**Lemma 2.2.4.** *For the subgroups $H$ of Definition **??**, we have*

$$H \trianglelefteq G$$

*and $G/H$ is a group of exponent 1 or 2.*

*Proof.* For $G \neq \Gamma_0(4)+$, the Lemma follows from condition 2 imposed on Gamma groups at the beginning of Section **??**. The case of $G = \Gamma_0(4)+$ is trivially true. □

**Definition 2.2.5.** *For $G$ an arithmetic subgroup of $\Gamma$ such that $\Gamma(N) \subset G$ for some $N$, the usual definition of level is such that $Lev(G)$ is the smallest $N$ with this property. For $G \in \mathbb{S}$, Duncan defines a normalized level $Lev_0(G)$, such that*

$$Lev_0(G) := \frac{Lev(G)}{a_G}.$$

**Definition 2.2.6.** *For $G \in \mathbb{S}$, let the valency of $G$ $val(G) := m + 1$ where $m$ satisfies*

$$G/H \cong (\mathbb{Z}/2)^m.$$

**Definition 2.2.7.** *Let us say that a group $G \in \mathbb{S}$ is faithful (to $\Gamma_0(2)$) if $I_G^{\Gamma_0(2)+} \leq 2$. Let us denote the set of faithful groups by $\mathbb{S}_1$ and the set of the remaining groups $\mathbb{S} \setminus \mathbb{S}_1$ by $\mathbb{S}_0$.*

We have

$$\mathbb{S}_1 = \{\Gamma_0(2)+, \Gamma_0(4)+, \Gamma_0(6)+, \Gamma_0(2)\}$$

and

$$\mathbb{S}_0 = \{\Gamma, \Gamma_0(3)+, \Gamma_0(5)+, \Gamma_0^{(3)}(3|3), \Gamma_0^{(2)}(4|2)+\}.$$

**Theorem 2.2.8.** *There is a unique graph with vertex set $\mathbb{S}$ that satisfies the following properties.*

1. *The valence of $G \in \mathbb{S}$ is $val(G)$.*

2. *The identity $2 \cdot Lev_0(G) = \sum_{G' \in adj(G)} Lev_0(G')$ holds for all $G \in \mathbb{S}$ where $adj(G)$ denotes the set of vertices that are adjacent to $G$.*

3. *If $G \in \mathbb{S}_1$ then $adj(G) \subset \mathbb{S}_0$.*

*This graph, represented in Figure* **??**, *in the more standard notation for the discrete*

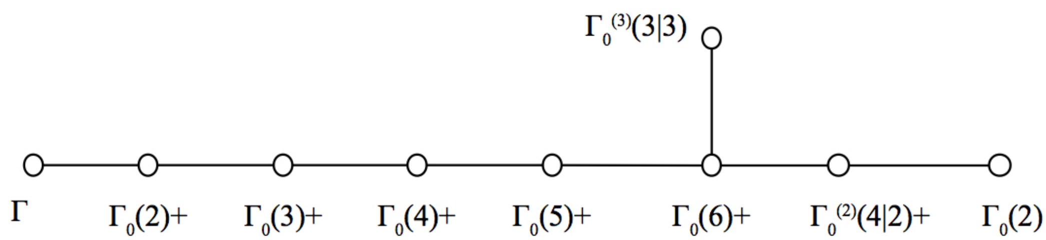



Figure 2.2: The second $E_8$ diagram

*subgroups of Monstrous Moonshine, is exactly Figure* **??**.

# Chapter 3

# Comments, questions and conclusion

## 3.1 A third way of obtaining the correspondence

So far, we have obtained the $E_8$ diagram in two different ways. In the first, McKay's way, the number associated with a particular conjugacy class serves as the usual label for a vertex in Dynkin Diagrarms, that is, such that it equals half the sum of the labels of the vertices adjacent to it. One also notices that each of the three different arms of the diagram have a single letter associated with them, the intersection of which is labeled $A$. In the second way, Duncan's way, while the same numbers appear suggestively in the names of our Gamma groups, we had to define valence, a normalized level as well as faithfulness to obtain the $E_8$ diagram.

Cummins and Duncan described in [**?**] a third way of obtaining the $E_8$ diagram. The 9 classes which are involved in McKay's Monstrous $E_8$ correspondence naturally correspond to 9 conjugacy classes in the Mathieu group $M_{24}$. They arise also as the product of short involutions in $M_{24}$. They showed that, in this case, the $E_8$ correspondence may be recovered using characteristics of these classes construed as multiplicative eta-products of weight at least 4. They obtained the following diagram (??).

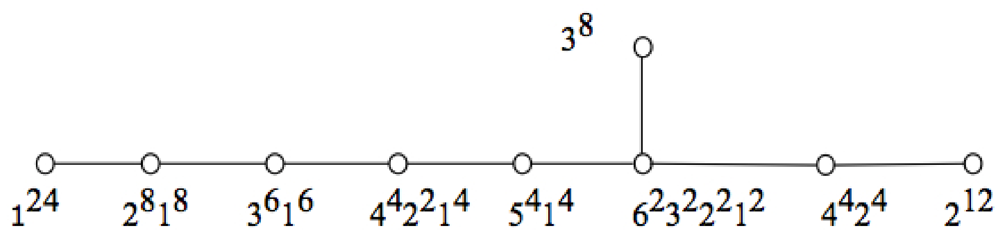



Figure 3.1: The third $E_8$ diagram

The third diagram is obtained in a way reminiscent of the second, in which faithfulness is replaced by a notion of parity. One also notices that the natural number labelling vertices in Dynkin diagrams appears suggestively in this third analogue. The third $E_8$ correspondence is quite natural since $M_{24}$, as a subgroup of the Monster, shares the class of short involutions whose products give us the other 9. Part of the interest of this third correspondence lies in the fact that it was obtained from the same objects in a novel way. On may hope that this confluence of different ways to obtain the correspondence might offer some insight into the deeper truths in play.

## 3.2 An observation about other conjugacy classes of the Monster

In all the correspondences obtained so far, the usual label for vertices in Dynkin diagrams appeared suggestively. In McKay's original correspondence it was the number in the ATLAS notation, in Duncan's, it was the $N/h$ in the $\Gamma_0^{(h)}(\frac{N}{h}|h)$, where $h$ is equal to 1 in most cases. In Cummins' and Duncan's third correspondence, the largest order of the elements in the $M_{24}$ classes. In all three cases, it was the number, in normal script, that appeared leftmost in the labels. This fact is what seems to have urged the search for proper mathematics justifying the (already obtained) organization in the first place.

The labels in Dynkin diagrams are normalized such that the first one equals 1. Taking any multiple of them, one would recover the same rule that they must equal half of the sum of indices adjacent to them.

In light of this, we here look for classes the have one of the original 9 as some prime power. Let us note that powers of classes represent multiples in the ATLAS notation. Recall

$$\mathbb{S} = \{1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A\}.$$

Then, for

$$S = \{7A, 14A, 14B, 21A, 21C, 28A, 28B, 35A, 42A\}$$

we have $S^7 = \mathbb{S}$, where $S^n = \{g^n \mid g \in S\}$.

We want to find all $S$ and $p$ such that $S^p = \mathbb{S}$.

$p \geq 17$ is ruled out since there are no classes labeled $6 \cdot 17 = 102$ or higher.

13 is ruled out since there are no classes labeled $6 \cdot 13 = 65$.

11 doesn't work because there is only one class labeled $4 \cdot 11 = 44$, $(44A)$, and it has $4A$ as a power, and not $4B$.

5 doesn't work because there is only one class labeled $5 \cdot 5 = 25$ $(25A)$ and it has $5B$ as a power, and not $5A$.

3 is ruled out because the classes labeled $6 \cdot 3 = 18$ have either $6E$ or $6D$ as powers,

and never $6A$.

That leaves us with powers 7 and 2, which both have solutions.

For $p = 7$ there are all combinations of

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 7A | 14A | 14B | 21A | 21C | 28B | 28A | 35A | 42A |
| 7B | | 14C | 21B | | | | | |

which yields 8 combinations.

And for $p = 2$,

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2A | 4B | 4A | 6A | 6F | 8B | 8C | 10A | 12C |
| 2B | | 4C | 6C | | | | 10B | |
| | | 4D | | | | | | |

which yields 24 possible combinations.

There is one notable the fact, however. The class $1A$ is a power of all other classes. One might then add the condition that all the classes in the set $S$ have as a power the first class in $S$. Let $\mathbb{S}_i$ for $1 \leq i \leq 9$ denote the numbers in the original set $\mathbb{S}$. Let $\{S_i^{\mathbb{S}_i}\}$ denote the set of all $\mathbb{S}_i$th powers of the elements of $S$. We have, for example

$$\{S_i^{\mathbb{S}_i}\} = \{7A^1, 14A^2, 14B^2, 21A^3, 21C^3, 28A^4, 28B^4, 35A^5, 42A^6\} = \{7A\}.$$

This is the only $S$, other than $\mathbb{S}$ itself, such that it has the property that the set $\{S_i^{\mathbb{S}_i}\}$ has only one element. It is notable, also, that the $E_8$ diagram that we would obtain should we place $S_i$ where $\mathbb{S}_i$ was placed would associate the same letter to every arm of the diagram.

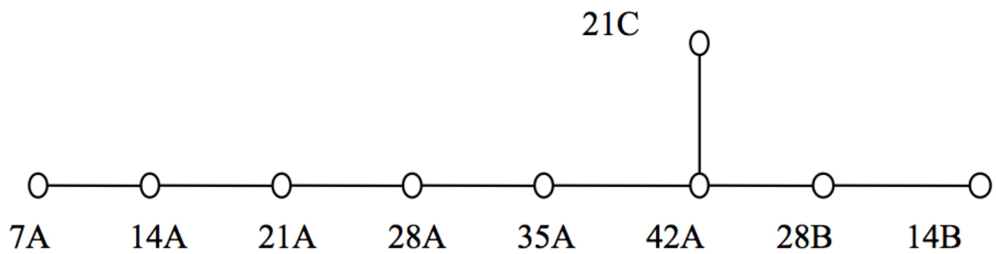



Figure 3.2: The fourth $E_8$ diagram

## 3.3 The CMS conditions, a discussion on context

Conway, McKay and Sebbar describe their main result in [?] as 'simple necessary and sufficient conditions for a discrete subgroup of $PSL_2(\mathbb{R})$ to be the invariance group of a Monstrous Moonshine function'.

**Theorem 3.3.1.** *(CMS)*

*A modular function occurs in Moonshine if and only if its invariance group*

1. *is genus zero,*
2. *has the form $\Gamma_0^{(h)}(n|h) + e, f, g, \ldots$ ,*
3. *its quotient by $\Gamma_0(nh)$ is a group of exponent 2, and*
4. *each cusp can be mapped to $\infty$ by an element of $PSL_2(\mathbb{R})$ that conjugates the group to one containing $\Gamma_0(nh)$*

The $+e, f, g, \ldots$ of CMS condition 2 are the $W_e(N)$ elements discussed when we defined Gamma groups (beginning of Section (**??**)). The $+$ notation is consistent. Also discussed in Section (**??**) is the form $\Gamma_0^{(h)}(n|h) + e, f, g, \ldots$, a subgroup of order $h$ in $\Gamma_0(n|h) + e, f, g, \ldots$.

CMS condition 2 returns a infinite list of groups. Imposing that these have genus zero, CMS condition 1, restricts this list to 213 groups, including the 171 Moonshine groups.

The next condition was imposed noticing that, while $\Gamma_0^{(3)}(3|3)$ is a Moonshine group, $\Gamma_0^{(2)}(2|2)$ is not and while $\Gamma_0^{(3)}(3|3)/\Gamma_0(9)$ has exponent 2, $\Gamma_0^{(2)}(2|2)/\Gamma_0(4)$ does not. It was then observed that all Moonshine groups possessed this property. This condition restricted the set to 173 groups.

All Moonshine groups have width 1 at $\infty$. CMS condition 4, closely related to our width 1 condition, filters out the two extra groups called the ghost elements.

Through the computations we did to find one of our groups in Section (**??**), we are able to see how our conditions work and how they might relate to the CMS conditions. While CMS imposes that the groups be of the form $\Gamma_0^{(h)}(n|h)+e, f, g, \ldots$, the exponent 2 condition was, in our case, enough to rule out a subgroup of $\Gamma_0(4|2)+$ that was not of that form. A subgroup of index 3 in $\Gamma_0(3)$ was ruled out for similar reasons. Our width condition sufficed when the more exclusive CMS condition 4 was needed to weed out the ghost elements.

Looking at the list of Moonshine groups, our first index condition restricts us to groups that are 'close' to $\Gamma$, i.e. to groups such that $N$ is small (Corollary (**??**)). Our second index condition prevents groups of the form $\Gamma_0(N)$ for $N > 2$, from making the list. $\Gamma_0^{(2)}(4|2)$ is also ruled out because of the second index condition.

Since our conditions only returned groups of genus zero, that condition was not necessary to obtain our 9 groups.

It seems as though the conditions that we used, Duncan's conditions, were modelled on the CMS conditions. By restraining our set with the index condition, we were able to drop CMS condition 4 and replace it with our width condition. These two were enough to narrow the set down to groups of the form $\Gamma_0^{(h)}(n|h)+e,f,g,...$, so that this condition was dropped as well. Then, as groups of the form $\Gamma_0^{(h)}(n|h)+e,f,g,...$ with small enough $N$ have genus zero, that condition was not needed either. Our condition 2 is identical with CMS condition 3.

## 3.4 Conclusion

Monstrous Moonshine seems to suggest that the Monster, its associated modular functions and modular groups are in fact different ways to approach the same mathematical truth. One is reminded of the story of the three blind men describing an elephant in three very different ways because one touched only the tail, the other touched the leg and the third the belly. It is then very natural, once one has made an observation about one of the objects, to try and find its analogue in the others. It serves as a way to study the underlying truth that these objects seem to be a part of.

The first part of this thesis elaborates a tool with which to understand Gamma groups. We then moved to reviewing Duncan's conditions that return the 9 McKay Monstrous $E_8$ groups. Because these conditions form a more restrictive set that includes the CMS conditions, one may hope, in future developments, that we will gain some insight into how and why these two sets of conditions are related to the Monster and to the $E_8$ diagram by trying to further modify these sets. Gannon offers in [**?**] a review of the latest advancements in what he calls this *conceptual gap*. No comments are made there about the $E_8$ correspondence, it is exclusively about the Monster.

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