On the Mean Curvature Flow

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ABSTRACT

On the Mean Curvature Flow

Janine Bachrachas

We present a self-contained expository review on the mean curvature flow for smooth embedded hypersurfaces in the (n+1)-dimensional Euclidean space. We start by addressing the short time existence of solutions to the flow, followed by the long time existence in the case of compact convex hypersurfaces and entire graphs. Although the results presented here are part of the classical literature originated in the 80's, we derive all necessary calculations and gather the simplest possible approach in view of later developments of the area.

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Contents

Chapter 1

Notations and preliminaries

Let M denote a connected orientable differential manifold without boundary of dimension n. For each point $p \in M$ we have local coordinates $\{x^1, ..., x^n\}$.

The tangent vectors

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, ..., \left.\frac{\partial}{\partial x^n}\right|_p \right\}$$

form a basis of T_pM with dual basis

$$\left\{ \left.dx^1\right|_p, ..., \left.dx^n\right|_p \right\}.$$

Assume M can be smoothly embedded in $\mathbb{R}^{n+1}$ via the map

$$X : M \longrightarrow \mathbb{R}^{n+1}.$$

Let us denote by the same letter M the image of M under X.

Remark 1.1.

Notice that $M \subset \mathbb{R}^{n+1}$ is an immersed submanifold of codimension 1. These type of submanifolds are often called *hypersurfaces*. We will adopt this terminology.

We equip the manifold M with the Riemannian metric induced by $\mathbb{R}^{n+1}$. This is

$$g_{ij}(X(p)) = \langle \frac{\partial X}{\partial x^i}(p) \, , \frac{\partial X}{\partial x^j}(p) \rangle$$

for $p \in M$, where $\langle \cdot \, , \cdot \rangle$ denotes the canonical inner product in $\mathbb{R}^{n+1}$.

Notice that

$$\frac{\partial X}{\partial x^i}(p) = dX|_p \left(\frac{\partial}{\partial x^i} \right)$$

and that since dX is injective the vectors

$$\left\{ \frac{\partial X}{\partial x^i}(p) \mid 1 \leq i \leq n \right\}$$

form a basis of $T_{X(p)}M$.

Remark 1.2.

Throughout this thesis we shall be working with the *Levi-Civita connection.*

As usual, $g = (g_{ij})_{ij}$ denotes the metric tensor. Its inverse g^{-1} will be denoted with superscripts, $g^{-1} = (g^{ij})_{ij}$.

The *Second Fundamental Form* on M, $A = (h_{ij})$, is given by

$$h_{ij}(X(p)) = - \langle \frac{\partial^2 X}{\partial x^i \partial x^j}(p) \, , \nu \rangle$$

where ν is the outer unit normal vector at the point $X(p)$.

Notation 1.1.

We will be using the Einstein notation for sums. This is, we will not write the summation symbol capital sigma (Σ), and we will understand a sum is made over repeated indices, from 1 to n.

Definition 1.1 (Mean Curvature).

The *Mean Curvature* of M at p is the trace of the second fundamental form A, i.e.,

$$H = g^{ij} h_{ij}.$$

Definition 1.2.

The *Laplace-Beltrami* operator on (M, g) is the second order differential operator defined as

$$\Delta_g = g^{ij} \nabla_i \nabla_j.$$

Theorem 1.1 (Gauss-Weingarten Relations)

Let $X : M^n \longrightarrow \bar{M}^{n+1}$ *be an immersion. Assume that for* $p \in M$ *the tangent space* $T_{X(p)}\bar{M}$ *splits in*

$$T_{X(p)}\bar{M} = T_pM \oplus (T_pM)^{\perp},$$

which varies differentiably with p. *Then the following equations are valid for all* $1 \leq i,\ j \leq n$*:*

$$\frac{\partial^2 X}{\partial x^i \partial x^j} = \Gamma^k_{ij} \frac{\partial X}{\partial x^k} - h_{ij}\nu. \tag{1.1}$$

$$\frac{\partial \nu}{\partial x^j} = h_{jl} g^{lm} \frac{\partial X}{\partial x^m}. \tag{1.2}$$

Proposition 1.2 (Gauss and Coddazi-Mainardi equations)

For all $1 \leq i, j, k, l \leq n$ we have

$$\frac{\partial}{\partial x^k}\Gamma^l_{ij} - \frac{\partial}{\partial x^j}\Gamma^l_{ik} + \Gamma^r_{ij}\Gamma^l_{rk} - \Gamma^r_{ik}\Gamma^l_{rj} = (h_{ij}h_{km} - h_{ik}h_{jm})g^{ml} \tag{1.3}$$

$$\frac{\partial}{\partial x^k}h_{ij} - \frac{\partial}{\partial x^j}h_{ik} + \Gamma^r_{ij}h_{rk} - \Gamma^r_{ik}h_{rj} = 0 \tag{1.4}$$

Proposition 1.3 (Intertwining covariant derivatives)

Let ∇ denote the Levi-Civita connection on a Riemannian manifold M. Then the following differentiation rules hold for all $1 \leq i, j, k, l \leq n$:

For tangent vectors:

$$\nabla_i \nabla_j X^l - \nabla_j \nabla_i X^l = R^l_{ijk} X^k,$$

for cotangent vectors:

$$\nabla_i \nabla_j Y_k - \nabla_j \nabla_i Y_k = R_{ijkl} g^{lm} Y_m.$$

for $(2,0)$-tensors:

$$\nabla_i \nabla_j \alpha_{kl} - \nabla_j \nabla_i \alpha_{kl} = R_{ijkm} g^{mn} \alpha_{nl} + R_{ijlm} g^{mn} \alpha_{kn}.$$

Definition 1.3 (Mean Curvature Flow).

Assume the manifold M can be smoothly embedded into $\mathbb{R}^{n+1}$ via

$$X_0 : M \longrightarrow \mathbb{R}^{n+1}.$$

We say that the manifold M moves by Mean Curvature Flow (MCF) if there exists a family of smooth embeddings $\{X(\cdot, t) : M \longrightarrow \mathbb{R}^{n+1}\}$ which is a solution to the following differential equations on M

$$\begin{aligned} \frac{\partial}{\partial t} X(p,t) &= -H(p,t)\nu(p,t) \qquad p \in M, \\ X(p,0) &= X_0(p). \end{aligned}$$

Here $H(p,t)$ denotes the mean curvature of the manifold $M_t := X(M,t)$ at the point $X(p,t)$ and $\nu(p,t)$ the outer unit normal vector at that point.

Later in this thesis, we shall prove that solutions X to the MCF exist, given smooth initial data. If so, for each t such that a solution $X(p,t)$ exists, we have a submanifold of $\mathbb{R}^{n+1}$ that we will denote by

$$M_t := \{X(p,t) \in \mathbb{R}^{n+1} \mid p \in M\}.$$

Indeed, the local charts arise from the Inverse Function Theorem and charts on M.

Notice that the vectors

$$\left\{ \frac{\partial X}{\partial x^i}(p,t) \mid 1 \leq i \leq n \right\}$$

form a basis of $T_{X(p,t)}M$.

Remark 1.3.

The MCF equations can be written as

$$\frac{\partial}{\partial t}X(p,t) = \Delta_g X(p,t) \qquad p \in M,$$
$$X(p,0) = X_0(p).$$

where Δ_g is the Laplace-Beltrami operator on M_t. Indeed, using the Gauss-Weingarten relations, we compute

$$\Delta_g X = g^{ij}\left(\frac{\partial^2 X}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial X}{\partial x^k}\right) = -g^{ij}h_{ij}\nu = -H\nu$$

since $H = g^{ij}h_{ij}$.

Chapter 2

Short Time Existence

In this section we intend to prove the short time existence of solutions to the Mean curvature flow equations for smooth initial data.

We begin by analyzing the simple case when the manifold M is the *graph* of a function. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function of class C^∞ on $\mathbb{R}^n$. By the Implicit Function Theorem we know that the graph of f

$$M_0 = \{(x, f(x)) \in \mathbb{R}^{n+1} |\, x \in \mathbb{R}^n\}$$

is an embedded submanifold of $\mathbb{R}^{n+1}$, where the embedding is given by

$$X_0 : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1},$$

$$X_0(x) = (x, f(x)), \qquad \forall\, x = (x_1, ..., x_n) \in \mathbb{R}^n.$$

In a later section we will show that if the initial manifold is a graph, so are the evolving manifolds at later times. Hence, if a solution for the MCF with a graph as initial data exists, it is of the form

$$X(x,t) = (u(x,t), f(u(x,t),t)). \tag{2.1}$$

where

$$u(x,t) = (X^1(x,t), ..., X^n(x,t)),$$

the first n components of the position vector X.

Let us now compute the tangent vectors defined by equation 2.1.

$$\frac{\partial X}{\partial x_i} = (\frac{\partial u^1}{\partial x_i}, ..., \frac{\partial u^n}{\partial x_i}, \frac{\partial f}{\partial x_j}\frac{\partial u^j}{\partial x_i}) = (\frac{\partial u}{\partial x_i}, \nabla_x f \cdot \frac{\partial u}{\partial x_i}),$$

where $\cdot$ denotes the usual inner product in $\mathbb{R}^n$ and $\nabla_x f$ is the gradient of f with respect to the space variables $x = (x_1, ..., x_n) \in \mathbb{R}^n$.

Now, the outer unit normal vector at the point $X(x,t) \in M_t$ should be orthogonal to all vectors in the tangent space of M at $X(x,t)$. Then for every $1 \leq i \leq n$ we have

$$0 = \langle \nu\,, \frac{\partial X}{\partial x^i} \rangle = \langle \nu\,, \left(\frac{\partial u^1}{\partial x^i}, ..., \frac{\partial u^n}{\partial x^i}, \frac{\partial f}{\partial x^k}\frac{\partial X^k}{\partial x^i} \right) \rangle.$$

Set $\tilde{\nu} := (\nabla_x f, -1)$. We get that

$$
\begin{aligned}
\langle\tilde{\nu}\,,\frac{\partial X}{\partial x^i}\rangle &= \langle(\nabla_x f,-1)\,,\frac{\partial u}{\partial x^i},\frac{\partial f}{\partial x^k}\frac{\partial u^k}{\partial x^i}\rangle \\
&= \frac{\partial f}{\partial x^k}\frac{\partial u^k}{\partial x^i}-\frac{\partial f}{\partial x^k}\frac{\partial u^k}{\partial x^i} \\
&= 0
\end{aligned}
$$

so $\tilde{\nu}\in(T_{X(x,t)}M)^{\perp}$. Normalizing $\tilde{\nu}$ we obtain that the unit normal vector is

$$
\nu=\frac{1}{(1+|\nabla_x f|^2)^{1/2}}(\nabla_x f,-1).
$$

Notice that, in this setting, the MCF equations become

$$
\frac{\partial X}{\partial t}=(\frac{\partial u}{\partial t},\frac{\partial f}{\partial x^k}\frac{\partial u^k}{\partial t}+f_t). \tag{2.2}
$$

The second fundamental form is given by

$$
\begin{aligned}
h_{ij} &= -\langle\frac{\partial^2 X}{\partial x_i\partial x_j},\nu\rangle \\
&= -\frac{1}{(1+|\nabla_x f|^2)^{1/2}}\langle\frac{\partial^2 u}{\partial x^i\partial x^j},\frac{\partial^2 f}{\partial x^j\partial x^k}\frac{\partial u^k}{\partial x^i}+\frac{\partial f}{\partial x^k}\frac{\partial^2 u^k}{\partial x^i\partial x^j}\,,(\nabla_x f,-1)\rangle \\
&= (1+|\nabla_x f|^2)^{-1/2}\frac{\partial^2 f}{\partial x^j\partial x^k}\frac{\partial u^k}{\partial x^i},
\end{aligned} \tag{2.3}
$$

for $1\leq i,j\leq n$.

To proceed, we consider the scalar equivalent to the MCF equation. This is, the evolution equation obtained by taking inner product with ν on both sides of 2.2. Notice that, multiplying by ν we recover the equation

$$\left(\frac{\partial}{\partial t}X\right)^{\perp} = -H\nu$$

which is equivalent, up to tangential diffeomorphisms, to the MCF equations.

The right hand side of the scalar MCF equation becomes $-H$, while the left hand side is

$$\begin{aligned}\langle X_t, \nu\rangle &= (1+|\nabla_x f|^2)^{-1/2}\,\langle(\frac{\partial u}{\partial t}, \frac{\partial f}{\partial x^k}\frac{\partial u^k}{\partial t} + f_t)\,, (\nabla_x f, -1)\rangle \\ &= -(1+|\nabla_x f|^2)^{-1/2} f_t.\end{aligned}$$

Hence,

$$f_t = (\sqrt{1+|\nabla_x f|^2})\, H.$$

Since $H = g^{ij}h_{ij}$ and h_{ij} as in 2.3, the previous equation rewrites as

$$f_t = g^{ij} f_{ij}. \tag{2.4}$$

Notice that equation 2.4 is a linear second order parabolic differential equation for f, since the matrix of the metric g is positive definite. Therefore, we know by corollary A.2 that there exists a unique solution f, at least over a finite time interval.

Remark 2.1.

Another way to prove the short time existence of solutions in this case is observing that H equals

$$H = \operatorname{div}\left(\frac{\nabla_x f}{\sqrt{1+|\nabla_x f|^2}}\right) \tag{2.5}$$

Then we can rewrite the evolution of f as an equation in divergence form

$$f_t = \sqrt{1+|\nabla_x f|^2}\cdot \operatorname{div}\left(\frac{\nabla_x f}{\sqrt{1+|\nabla_x f|^2}}\right) \tag{2.6}$$

Consequently, by theorem A.3 we also get the short time existence and uniqueness of the solution.

□

In the general setting, the short time existence does not follow from the theory of quasi-linear parabolic PDE's. The problem is that the MCF equation will be, in general, degenerate. By remark 1.3 we know that

$$\frac{\partial X}{\partial t} = g^{ij}\left(\frac{\partial^2 X}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial X}{\partial x^k}\right).$$

Since the Christoffel Symbols are defined by

$$\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x^i}g_{jl} + \frac{\partial}{\partial x^j}g_{li} - \frac{\partial}{\partial x^l}g_{ij}\right)g^{lk},$$

we write

$$\begin{aligned}
\frac{\partial X}{\partial t} &= g^{ij}\frac{\partial^2 X}{\partial x^i \partial x^j} - \frac{1}{2}\left[\left(\frac{\partial}{\partial x^i}g_{jl} + \frac{\partial}{\partial x^j}g_{li} - \frac{\partial}{\partial x^l}g_{ij}\right)g^{lk}g^{ij}\right]\frac{\partial X}{\partial x^k} \\
&= g^{ij}\frac{\partial^2 X}{\partial x^i \partial x^j} - \frac{1}{2}\left[\left(\langle\frac{\partial^2 X}{\partial x^i \partial x^j}, \frac{\partial X}{\partial x^l}\rangle + \langle\frac{\partial^2 X}{\partial x^i \partial x^l}, \frac{\partial X}{\partial x^j}\rangle\right.\right. \\
&+ \langle\frac{\partial^2 X}{\partial x^i \partial x^j}, \frac{\partial X}{\partial x^l}\rangle + \langle\frac{\partial^2 X}{\partial x^j \partial x^l}, \frac{\partial X}{\partial x^i}\rangle \\
&- \left.\left.\langle\frac{\partial^2 X}{\partial x^i \partial x^l}, \frac{\partial X}{\partial x^j}\rangle - \langle\frac{\partial^2 X}{\partial x^j \partial x^l}, \frac{\partial X}{\partial x^i}\rangle\right)g^{lk}g^{ij}\right]\frac{\partial X}{\partial x^k},
\end{aligned}$$

which simplifies to

$$\begin{aligned}\frac{\partial X}{\partial t} &= g^{ij}\frac{\partial^2 X}{\partial x^i \partial x^j} - g^{ij}g^{kl}\langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \frac{\partial X}{\partial x^l}\rangle \frac{\partial X}{\partial x^k} \\ &= g^{ij}\frac{\partial^2 X}{\partial x^i \partial x^j} - g^{ij}g^{kl}\left[\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\frac{\partial X^\alpha}{\partial x^l}\right]\frac{\partial X}{\partial x^k}.\end{aligned}$$

This vectorial PDE reduces to now a system of parabolic PDE's for the components X^β of X, $1 \leq \beta \leq n+1$. If we consider the β-th component the field, we have

$$\begin{aligned}\frac{\partial X^\beta}{\partial t} &= g^{ij}\frac{\partial^2 X^\beta}{\partial x^i \partial x^j} - g^{ij}g^{kl}\left[\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\frac{\partial X^\alpha}{\partial x^l}\right]\frac{\partial X^\beta}{\partial x^k} \\ &= g^{ij}\left[1 - g^{kl}\frac{\partial X^\beta}{\partial x^l}\frac{\partial X^\beta}{\partial x^k}\right]\frac{\partial^2 X^\beta}{\partial x^i \partial x^j} - g^{ij}g^{kl}\sum_{\alpha \neq \beta}\left[\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\frac{\partial X^\alpha}{\partial x^l}\right]\frac{\partial X^\beta}{\partial x^k}.\end{aligned}$$

The evolution equation for X^β is therefore a quasi-linear second order equation, but only weakly parabolic as it is degenerate along the tangential directions satisfying

$$g^{kl}\frac{\partial X^\beta}{\partial x^l}\frac{\partial X^\beta}{\partial x^k} = 1.$$

Thus, short time existence of solution to the flow is not insured by the classical theory of parabolic equations. In order to prove short time existence we will make use of a technique which was first shown by *Denis De Turck*.

The De Turck Trick.

The idea behind this technique is to make a time dependent change of variables, so that the MCF equations in the new variables is a strictly parabolic equation. In this setting, we can apply the classical existence results for strictly parabolic PDE's and the original MCF will have solutions whenever the new one has.

Let $\tilde{X}$ be a reparametrization of X

$$X(p,t) = \tilde{X}(y(p,t),t)$$

where the map $(p,t) \mapsto y(p,t)$ is C^∞ with C^∞ inverse.

Suppose $\tilde{X}$ satisfies the equation

$$\frac{\partial \tilde{X}}{\partial t} = \Delta_g \tilde{X} + v^k \frac{\partial \tilde{X}}{\partial x^k} \tag{2.7}$$

for some v^k's such that 2.7 is strictly parabolic. We know that

$$\begin{aligned} \Delta_g X &= \frac{\partial X}{\partial t} = \frac{\partial \tilde{X}}{\partial t} + \frac{\partial \tilde{X}}{\partial x^k} \frac{dy^k}{dt} \\ &= \Delta_g \tilde{X} + \left(v^k + \frac{dy^k}{dt} \right) \frac{\partial \tilde{X}}{\partial x^k}. \end{aligned} \tag{2.8}$$

Therefore, the tangential components in 2.8 have to be zero. For this, we choose the parametrization y such that the coefficients of the tangential directions v^k's get cancelled. Imposing

$$\begin{cases} \dfrac{dy^k}{dt}(p,t) = -v^k(p,t), \\ y^k(p,0) = x^k, \end{cases}$$

we get that

$$\frac{\partial \tilde{X}}{\partial t} = g^{ij} \frac{\partial^2 \tilde{X}}{\partial x^i \partial x^j} + \left(v^k + g^{ij} \Gamma_{ij}^k \right) \frac{\partial \tilde{X}}{\partial x^k}.$$

Hence we may take

$$v^k = g^{ij}(\overline{\Gamma}_{ij}^k - \Gamma_{ij}^k)$$

where $\overline{\Gamma}_{ij}^k$ are the Christoffel symbols associated to a given fixed metric $\overline{g}$. Then 2.7 can be rewritten as

$$\frac{\partial \tilde{X}}{\partial t} = g^{ij}\frac{\partial^2 \tilde{X}}{\partial x^i \partial x^j} + g^{ij}\overline{\Gamma}_{ij}^k \frac{\partial \tilde{X}}{\partial x^k}.$$

This equation is strictly parabolic and we can apply the classical existence results.

Chapter 3

Evolution of Geometric Quantities

The main tool for studying geometric flows is to analyze how does the geometry change with the flow. Thus we derive evolution equations for the metric, curvature and second fundamental form.

Proposition 3.1 (Evolution of geometric quantities under the MCF)

1. $\dfrac{\partial g_{ij}}{\partial t} = -2Hh_{ij}$,

2. $\dfrac{\partial \nu}{\partial t} = \nabla_g H$,

3. $\dfrac{\partial h_{ij}}{\partial t} = \Delta_g h_{ij} - 2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij}$, *where* $|A|^2 = g^{ij}g^{kl}h_{ik}h_{jl}$ *is the norm of the second fundamental form,*

4. $\dfrac{\partial H}{\partial t} = \Delta H + |A|^2 H$,

5. $\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4$.

Proof :

1)

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial t} &= \frac{\partial}{\partial t}\langle\frac{\partial X}{\partial x^i},\frac{\partial X}{\partial x^j}\rangle = \langle\frac{\partial}{\partial x^i}\frac{\partial X}{\partial t},\frac{\partial X}{\partial x^j}\rangle + \langle\frac{\partial}{\partial x^j}\frac{\partial X}{\partial t},\frac{\partial X}{\partial x^i}\rangle \\
&= -\langle\frac{\partial H\nu}{\partial x^i},\frac{\partial X}{\partial x^j}\rangle - \langle\frac{\partial H\nu}{\partial x^j},\frac{\partial X}{\partial x^i}\rangle \\
&= -\frac{\partial H}{\partial x^i}\langle\nu,\frac{\partial X}{\partial x^j}\rangle - H\langle\frac{\partial \nu}{\partial x^i},\frac{\partial X}{\partial x^j}\rangle - \frac{\partial H}{\partial x^j}\langle\nu,\frac{\partial X}{\partial x^i}\rangle - H\langle\frac{\partial \nu}{\partial x^j},\frac{\partial X}{\partial x^i}\rangle \\
&= -H\langle\frac{\partial \nu}{\partial x^i},\frac{\partial X}{\partial x^j}\rangle - H\langle\frac{\partial \nu}{\partial x^j},\frac{\partial X}{\partial x^i}\rangle = 2H\langle\frac{\partial^2 X}{\partial x^i \partial x^j},\nu\rangle \\
&= -2Hh_{ij}.
\end{aligned}$$

□

We will resume the proof of proposition 3.1 after stating a few corollaries of 1) which will be used later in the proof.

Corollary 3.2

$$\frac{\partial g^{ij}}{\partial t} = 2Hg^{ik}h_{km}g^{mj}.$$

Proof:

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}(g_{ik}g^{kj}) = g^{kj}\frac{\partial g_{ik}}{\partial t} + g_{ik}\frac{\partial g^{kj}}{\partial t} \\
&= -2g^{kj}Hh_{ik} + g_{ik}\frac{\partial g^{kj}}{\partial t}.
\end{aligned}$$

So,

$$g_{il}\frac{\partial g^{lj}}{\partial t} = 2Hg^{kj}h_{ik} \tag{3.1}$$

where we changed the name of the summing index on the left hand side of the above equation. Now, we multiply equation 3.1 by g^{im} and sum over i to get

$$\delta_{ml}\frac{\partial g^{lj}}{\partial t} = 2Hg^{kj}h_{ik}g^{im}.$$

Thus

$$\frac{\partial g^{mj}}{\partial t} = 2Hg^{kj}h_{ik}g^{im},$$

which proves the corollary up to a change of indices.

□

Corollary 3.3

Let us denote by μ_t the volume form on (M_t, g), i.e.

$$\mu_t = \sqrt{\det g_{ij}(t)}.$$

Then,

$$\frac{\partial}{\partial t}\mu_t = -H^2\mu_t.$$

Proof:

For simplicity of notation we will drop the subindex t. Use the definition of determinant to compute

$$\begin{aligned}\frac{\partial\mu^2}{\partial t}=\frac{\partial}{\partial t}\det g_{ij} &= \frac{\partial}{\partial t}\sum_{\sigma\in S_n}\varepsilon(\sigma)g_{1\sigma(1)}\dots g_{n\sigma(n)} \\ &= \sum_i\sum_{\sigma\in S_n}\varepsilon(\sigma)g_{1\sigma(1)}\dots\frac{\partial g_{i\sigma(i)}}{\partial t}\dots g_{n\sigma(n)} \\ &= \sum_i\sum_{\sigma\in S_n}\varepsilon(\sigma)g_{1\sigma(1)}\dots(-2Hh_{i\sigma(i)})\dots g_{n\sigma(n)} \\ &= -2H\sum_{\sigma\in S_n}\varepsilon(\sigma)(\sum_i h_{i\sigma(i)}g^{i\sigma(i)})g_{1\sigma(1)}\dots g_{n\sigma(n)} \\ &= -2H(\sum_{\sigma\in S_n}\sum_i h_{i\sigma(i)}g^{i\sigma(i)})\sum_{\sigma\in S_n}\varepsilon(\sigma)g_{1\sigma(1)}\dots g_{n\sigma(n)} \\ &= -2H^2\det g_{ij}.\end{aligned}$$

□

For the proof of 2) in Proposition 3.1 we will need the following observations.

a) The vectors

$$\left\{g^{ij}\frac{\partial X}{\partial x^j}\ \middle|\ 1\le i\le n\right\}$$

form a basis of T_pM. Indeed,

$$0=\alpha_i g^{ij}\frac{\partial X}{\partial x^j}\ \Rightarrow\ \alpha_i g^{ij}=0,\ \forall\, j:\ 1\le j\le n.$$

This is equivalent to having

$$g^{-1}(\alpha_1,...,\alpha_n)^t=0$$

Since $g^{-1}=(g^{ij})_{ij}$, which is a positive definite matrix, $\alpha_i=0$, $\forall\ i:\ 1\le i\le n$.

□

$b)$ Notice that, for all $i, j, k: \ 1 \le i, \ j, \ k \le n$,

$$\langle g^{ij} \frac{\partial X}{\partial x^j}, \frac{\partial X}{\partial x^k} \rangle = g^{ij} g_{jk} = \delta_{ik}.$$

Thus if $v \in T_pM$, $v = a_i g^{ij} \dfrac{\partial X}{\partial x^j}$,

$$\langle a_i g^{ij} \frac{\partial X}{\partial x^j}, \frac{\partial X}{\partial x^k} \rangle = a_i \delta_{ik} = a_k,$$

which implies

$$v = \langle v, \frac{\partial X}{\partial x^i} \rangle g^{ij} \frac{\partial X}{\partial x^j}.$$

□

We now continue with the proof of Proposition 3.1.

2) Notice that $\dfrac{\partial \nu}{\partial t} \in T_pM$. Indeed,

$$1 = \langle \nu, \nu \rangle \ \Rightarrow \ 0 = \langle \nu, \frac{\partial \nu}{\partial t} \rangle,$$

so $\dfrac{\partial \nu}{\partial t}$ is orthogonal to ν and therefore lies in T_pM. Using part $b)$ of the previous remark, we write

$$\begin{aligned}\frac{\partial \nu}{\partial t} &= \langle \frac{\partial \nu}{\partial t}, \frac{\partial X}{\partial x^i}\rangle g^{ij}\frac{\partial X}{\partial x^j} = -\langle \nu, \frac{\partial}{\partial t}\frac{\partial X}{\partial x^i}\rangle g^{ij}\frac{\partial X}{\partial x^j} \\ &= \langle \nu, \frac{\partial}{\partial x^i}(H\nu)\rangle g^{ij}\frac{\partial X}{\partial x^j} = \frac{\partial H}{\partial x^i} g^{ij}\frac{\partial X}{\partial x^j} \\ &= \nabla_g H,\end{aligned}$$

where we used that

$$0 = \langle \nu, \frac{\partial X}{\partial x^i}\rangle \Rightarrow 0 = \frac{\partial}{\partial t}\langle \nu, \frac{\partial X}{\partial x^i}\rangle = \langle \frac{\partial \nu}{\partial t}, \frac{\partial X}{\partial x^i}\rangle + \langle \nu, \frac{\partial}{\partial t}\frac{\partial X}{\partial x^i}\rangle.$$

□

Lemma 3.4

$$\Delta h_{ij} = \nabla_i\nabla_j H + H h_{il} g^{lm} h_{mj} - |A|^2 h_{ij}.$$

Proof:

$$\begin{aligned}\Delta h_{ij} &= g^{mn}\nabla_m\nabla_n h_{ij} = g^{mn}\nabla_m\nabla_i h_{jn} \\ &= g^{mn}\nabla_i\nabla_m h_{jn} + R_{mijl} g^{ls} h_{sn} + R_{minl} g^{ls} h_{js},\end{aligned}$$

where we used the Coddazi equation 1.4 and the rules for intertwining derivatives. Now, denoting $h_k^l := g^{ls} h_{sk}$, we write

$$
\begin{aligned}
\Delta h_{ij} &= g^{mn}\nabla_i\nabla_m h_{jn} + R_{mijl}h^l_n + R_{minl}h^l_j \\
&= g^{mn}\nabla_i\nabla_j h_{mn} + g^{mn}(h_{mj}h_{il} - h_{ml}h_{ij})h^l_n + g^{mn}(h_{mn}h_{il} - h_{ml}h_{in})h^l_j \\
&= \nabla_i\nabla_j H + h^n_j h_{il}h^l_n - h^n_l h_{ij}h^l_n + Hh_{il}h^l_j - h^n_l h_{in}h^l_j
\end{aligned}
$$

Note that the second and last term get canceled and that $|A|^2 = h^k_n h^n_k$. Finally we get

$$
\Delta h_{ij} = \nabla_i\nabla_j H - |A|^2 h_{ij} + Hh_{il}g^{ls}h_{sj}
$$

□

3)

$$
\frac{\partial h_{ij}}{\partial t} = -\frac{\partial}{\partial t}\langle\frac{\partial^2 X}{\partial x^i\partial x^j}\,,\nu\rangle = \langle\frac{\partial^2 H\nu}{\partial x^i\partial x^j}\,,\nu\rangle - \langle\frac{\partial^2 X}{\partial x^i\partial x^j}\,,\frac{\partial\nu}{\partial t}\rangle.
$$

Now, on one hand we have that

$$
\begin{aligned}
\langle\frac{\partial^2 H\nu}{\partial x^i\partial x^j}\,,\nu\rangle &= \langle\frac{\partial}{\partial x^i}\left(\frac{\partial H}{\partial x^j}\nu + H(h_{jl}g^{lm}\frac{\partial X}{\partial x^m})\right)\,,\nu\rangle \\
&= \frac{\partial^2 H}{\partial x^i\partial x^j} + H\,\langle\frac{\partial}{\partial x^i}(h_{jl}g^{lm}\frac{\partial X}{\partial x^m})\,,\nu\rangle \\
&= \frac{\partial^2 H}{\partial x^i\partial x^j} + H\,\langle h_{jl}g^{lm}\frac{\partial^2 X}{\partial x^i\partial x^j}\,,\nu\rangle \\
&= \frac{\partial^2 H}{\partial x^i\partial x^j} - Hh_{jl}g^{lm}h_{im},
\end{aligned}
$$

where we used 1.2.

On the other hand,

$$\begin{aligned}
\langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \frac{\partial \nu}{\partial t} \rangle &= \langle \frac{\partial^2 X}{\partial x^i \partial x^j}, g^{lm} \frac{\partial H}{\partial x^l} \frac{\partial X}{\partial x^m} \rangle = \langle \Gamma^k_{ij} \frac{\partial X}{\partial x^k} - h_{ij} \nu , g^{lm} \frac{\partial H}{\partial x^l} \frac{\partial X}{\partial x^m} \rangle \\
&= \Gamma^k_{ij} \frac{\partial H}{\partial x^l} g^{lm} g_{km} \\
&= \Gamma^k_{ij} \frac{\partial H}{\partial x^k}.
\end{aligned}$$

Putting all the terms together we get

$$\begin{aligned}
\frac{\partial h_{ij}}{\partial t} &= \frac{\partial^2 H}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial H}{\partial x^k} - H h_{jl} g^{lm} h_{im} \\
&= \nabla_i \nabla_j H - H h_{jl} g^{lm} h_{im}. \qquad (3.2)
\end{aligned}$$

By lemma 3.4, we know that

$$\Delta h_{ij} - |A|^2 h_{ij} = \nabla_i \nabla_j H + H h_{il} g^{lm} h_{mj}. \qquad (3.3)$$

Therefore,

$$\frac{\partial h_{ij}}{\partial t} = \Delta h_{ij} - 2H h_{il} g^{lm} h_{mj} + |A|^2 h_{ij}.$$

□

4)

$$
\begin{aligned}
\frac{\partial H}{\partial t} &= \frac{\partial}{\partial t}(g^{ij}h_{ij}) = h_{ij}\frac{\partial g^{ij}}{\partial t} + g^{ij}\frac{\partial h_{ij}}{\partial t} \\
&= 2Hg^{ik}h_{km}g^{mj}h_{ij} + g^{ij}\nabla_i\nabla_j H - Hh_{jl}g^{lm}h_{im}g^{ij} \\
&= \Delta H + Hg^{ik}h_{km}g^{mj}h_{ij} \\
&= \Delta H + H|A|^2,
\end{aligned}
$$

where we used equation 3.2.

□

Notation 3.1.

Let us denote by $(\cdot\,,\,\cdot)$ the inner product for (2,0)-tensors on M. Explicitly, if T^i_{jk} and S^i_{jk} are (2,0)-tensors on M,

$$(T^i_{jk}\,,\,S^i_{jk}) = g_{il}g^{jm}g^{km}T^i_{jk}S^l_{mn}.$$

In particular, the norm of the second fundamental form is

$$|A|^2 = \,(h_{ij}\,,\,h_{ij}) = g^{ij}g^{kl}h_{ik}h_{jl}.$$

□

5)

$$\begin{aligned}
\frac{\partial}{\partial t}|A|^2 &= \frac{\partial}{\partial t}(g^{ik}g^{jl}h_{ij}h_{kl}) \\
&= \frac{\partial g^{ik}}{\partial t}g^{jl}h_{ij}h_{kl} + \frac{\partial g^{jl}}{\partial t}g^{ik}h_{ij}h_{kl} + \frac{\partial h_{ij}}{\partial t}g^{ik}g^{jl}h_{kl} + \frac{\partial h_{kl}}{\partial t}g^{ik}g^{jl}h_{ij} \\
&= 4Hg^{ik}h_{mn}g^{nk} \cdot g^{jl}h_{ij}h_{kl} \\
&+ 2g^{ik}g^{jl}(\Delta h_{ij} - 2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij})h_{kl} \\
&= 4Hg^{ik}h_{mn}g^{nk} \cdot g^{jl}h_{ij}h_{kl} + 2g^{ik}g^{jl}h_{kl}\Delta h_{ij} \\
&- 4Hh_{il}g^{lm}h_{mj} \cdot g^{ik}g^{jl}h_{kl} + 2|A|^2 g^{ik}g^{jl}h_{ij}h_{kl} \\
&= 2\,(h_{ij}\,,\,\Delta h_{ij}) + 2|A|^4.
\end{aligned}$$

Now we compute

$$\begin{aligned}
\Delta|A|^2 &= \Delta\,(h_{ij}\,,\,h_{ij}) = 2g^{mn}\nabla_m\,(h_{ij}\,,\,\nabla_n h_{ij}) \\
&= 2g^{mn}\,(\nabla_m h_{ij}\,,\,\nabla_n h_{ij}) + 2g^{mn}\,(h_{ij}\,,\,\nabla_m\nabla_n h_{ij}) \\
&= 2\,(\nabla_n h_{ij}\,,\,\nabla_n h_{ij}) + 2\,(h_{ij}\,,\,\Delta h_{ij}) \\
&= 2|\nabla A|^2 + 2\,(h_{ij}\,,\,\Delta h_{ij}).
\end{aligned}$$

Consequently,

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4.$$

□

Chapter 4

Compact Convex Hypersurfaces

Definition 4.1.

A hypersurface in $\mathbb{R}^{n+1}$ is said to be *strictly convex* if the second fundamental form of its embedding is everywhere positive definite.

Let M be an n-dimensional strictly convex compact hypersurface. Then, its Gauss map $\nu : M \longrightarrow S^n$, which assigns to each point p of M the unit outer normal vector at p, is a diffeomorphism. Then we can use $X = X(\nu^{-1}(z))$ to reparametrize the hypersurface as

$$X : S^n \longrightarrow \mathbb{R}^{n+1},$$

$$X(z) = r(z)z, \qquad \forall\, z \in S^n,$$

where $r = r(z)$ is a positive function. Let $\{u^1, ..., u^n\}$ be a system of local coordinates on the sphere S^n. As before, let g denote the metric on M induced by $\mathbb{R}^{n+1}$ and let $\bar{g}$ be the standard metric on the unit sphere, i.e., the metric on $S^n \subset \mathbb{R}^{n+1}$ induced by the ambient Euclidean space. We will compute the metric g in the coordinates $(u_1, ..., u_n)$.

We have, for $1 \le i \le n$,

$$\frac{\partial X}{\partial u_i} = r\frac{\partial z}{\partial u^i} + \frac{\partial r}{\partial u^i}z.$$

Hence,

$$\begin{aligned} g_{ij} &= \langle r\frac{\partial z}{\partial u^i} + \frac{\partial r}{\partial u^i}z\,, r\frac{\partial z}{\partial u^j} + \frac{\partial r}{\partial u^j}z\rangle \\ &= r^2\bar{g}_{ij} + \frac{\partial r}{\partial u^i}\frac{\partial r}{\partial u^j}, \end{aligned}$$

since $\langle\frac{\partial z}{\partial u^i}\,, z\rangle = 0$ and $|z| = 1$.

Then, g^{ij} is given by

$$g^{ij} = r^{-2}\left(\bar{g}^{ij} - \frac{\bar{\nabla}^i r\bar{\nabla}^j r}{r^2 + |\bar{\nabla} r|^2}\right)$$

where $\bar{\nabla}^i r = \bar{g}^{ij}\bar{\nabla}_j r$ and $\bar{\nabla} r$ denotes the gradient of r with respect to the round metric on the sphere. In local coordinates this is

$$\bar{\nabla} r = \bar{\nabla}^i r\frac{\partial}{\partial u^i}.$$

Lemma 4.1

The outer unit normal vector at $X(z)$ is

$$\nu = \frac{1}{\sqrt{r^2 + |\bar{\nabla} r|^2}}(rz - \bar{g}^{ik}\frac{\partial r}{\partial u^i}\frac{\partial z}{\partial u^k}). \tag{4.1}$$

Proof:

A basis for the tangent plane is given by the vectors

$$\frac{\partial X}{\partial u^i} = \frac{\partial r}{\partial u^i} z + r \frac{\partial z}{\partial u^i} \qquad 1 \leq i \leq n.$$

Let us find a vector $\tilde{\nu}$ orthogonal to all the vectors $\{\frac{\partial X}{\partial u^i} \mid 1 \leq i \leq n\} \subset T_x M$. We can write $\tilde{\nu}$ as a linear combination of the basis of $\mathbb{R}^{n+1}$

$$\{z, \frac{\partial z}{\partial u^1}, ..., \frac{\partial z}{\partial u^n}\}.$$

Thus,

$$\tilde{\nu} = a_i \frac{\partial z}{\partial u^i} + bz$$

We may choose, without any loss of generality, $b = r$ and evaluate

$$\langle a_i \frac{\partial z}{\partial u^i} + rz \, , \frac{\partial r}{\partial u^j} z + r \frac{\partial z}{\partial u^j} \rangle = a_i r \bar{g}_{ij} + r \frac{\partial r}{\partial u^j}.$$

We conclude that the a_i's are given by

$$a_i = -\frac{\partial r}{\partial u^j} \bar{g}^{ij}.$$

Thus,

$$\tilde{\nu} = rz - \bar{g}^{ij} \frac{\partial r}{\partial u^i} \frac{\partial z}{\partial u^j}.$$

Note that $|\tilde{\nu}| = \sqrt{r^2 + |\bar{\nabla} r|^2}$, so normalizing $\tilde{\nu}$, we get the normal vector as stated in 4.1.

□

Lemma 4.2

The coefficients of the second fundamental form at $X(z)$ are

$$h_{ij} = \frac{1}{\sqrt{r^2 + |\bar{\nabla} r|^2}} \left(-r \frac{\partial^2 r}{\partial u^i \partial^j} + 2 \frac{\partial r}{\partial u^i} \frac{\partial r}{\partial u^j} + r^2 \bar{g}_{ij} \right),$$

$1 \leq i,\, j \leq n.$

<u>Proof:</u>

$$\begin{aligned}
h_{ij} &= -\langle \frac{\partial^2 X}{\partial u^i \partial u^j}, \nu \rangle \\
&= -\frac{1}{\sqrt{r^2 + |\bar{\nabla} r|^2}} \langle \frac{\partial^2 r}{\partial u^i \partial u^j} z + \frac{\partial r}{\partial u^j} \frac{\partial z}{\partial u^i} + \frac{\partial r}{\partial u^i} \frac{\partial z}{\partial u^j} + r \frac{\partial^2 z}{\partial u^i \partial u^j}, rz - \bar{g}^{kl} \frac{\partial r}{\partial u^k} \frac{\partial z}{\partial u^l} \rangle \\
&= -\frac{1}{\sqrt{r^2 + |\bar{\nabla} r|^2}} \left(r \frac{\partial^2 r}{\partial u^i \partial u^j} + r^2 \langle \frac{\partial^2 z}{\partial u^i \partial u^j}, z \rangle \right. \\
&- \left. \bar{g}^{kl} \frac{\partial r}{\partial u^j} \frac{\partial r}{\partial u^k} \langle \frac{\partial z}{\partial u^i}, \frac{\partial z}{\partial u^l} \rangle - \bar{g}^{kl} \frac{\partial r}{\partial u^i} \frac{\partial r}{\partial u^k} \langle \frac{\partial z}{\partial u^j}, \frac{\partial z}{\partial u^l} \rangle \right) \\
&= -\frac{1}{\sqrt{r^2 + |\bar{\nabla} r|^2}} \left(r \frac{\partial^2 r}{\partial u^i \partial u^j} - r^2 \bar{g}_{ij} - \bar{g}_{li} \bar{g}^{kl} \frac{\partial r}{\partial u^j} \frac{\partial r}{\partial u^k} - \bar{g}_{jl} \bar{g}^{lk} \frac{\partial r}{\partial u^i} \frac{\partial r}{\partial u^k} \right) \\
&= \frac{1}{\sqrt{r^2 + |\bar{\nabla} r|^2}} \left(-r \frac{\partial^2 r}{\partial u^i \partial^j} + 2 \frac{\partial r}{\partial u^i} \frac{\partial r}{\partial u^j} + r^2 \bar{g}_{ij} \right)
\end{aligned}$$

where we have used that the metric and its inverse are symmetric tensors and that

$$\begin{aligned}
0 &= \frac{\partial}{\partial u^j} \langle \frac{\partial z}{\partial u^i}, z \rangle = \langle \frac{\partial^2 z}{\partial u^i \partial u^j}, z \rangle + \langle \frac{\partial z}{\partial u^i}, \frac{\partial z}{\partial u^j} \rangle \\
&= \langle \frac{\partial^2 z}{\partial u^i \partial u^j}, z \rangle + \bar{g}_{ij}.
\end{aligned}$$

□

We will now consider a smooth, strictly convex, compact hypersurface M and we let it evolve by the Mean Curvature Flow.

An important tool in the study of the evolution is the following maximum principle for tensors due to Richard Hamilton [4].

Before stating the theorem, let us introduce some terminology.

Definition 4.2.

Let M_{ij} be (2,0)-tensor on a manifold M. A *polynomial* in M_{ij} formed by g-contracting products of M_{ij} with itself, is a tensor of the form

$$f_0\delta_{ij} + f_1 M_{ij} + f_2 M_{ik} g^{kl} M_{lj} + \cdots + f_k M_{im_1} g^{m_1 m_2} M_{m_2 m_3} \cdots g^{m_{2(k-1)-1} m_{2(k-1)}} M_{m_{2(k-1)} j}.$$

where the f_i's are smooth functions on M.

Lemma 4.3 (Maximum principle for tensors)

Let $M_{ij} = M_{ij}(p,t)$ be a nonnegative definite symmetric (2,0)-tensor on M_t for each t. Suppose that

$$\frac{\partial M_{ij}}{\partial t} = \Delta M_{ij} + y^k \frac{\partial M_{ij}}{\partial x_k} + N_{ij}, \qquad 0 \le t \le T, \tag{4.2}$$

where $y = y^k \dfrac{\partial X}{\partial x^k}$ is a vector field and $N_{ij} = P(M_{ij}, g_{ij})$ is a polynomial in M_{ij} formed by g-contracting products of M_{ij} with itself. Assume that N_{ij} satisfies the null-vector condition, i.e., if $v \in T_p M_t$ is such that

$$M_{ij} v_j = 0,$$

then

$$N_{ij} v_i v_j \ge 0.$$

Then, if $M_{ij} \geq 0$ at time $t = 0$, it will remain nonnegative as long as the solution to 4.2 exists.

Proof:

Let

$$K = \max_{p \in M} |M_{ij}|,$$

where $|M_{ij}|^2 = (M_{ij}\,,\, M_{ij})$, as defined in the previous section (notation 3.1). Take now an arbitrary $\varepsilon :\ 0 < \varepsilon < 1$ and define

$$\tilde{M_{ij}} = M_{ij} + \varepsilon(\delta + t) g_{ij}.$$

Clearly $\tilde{M_{ij}} > 0$ at time $t = 0$. We claim that, for the time interval $[0, \delta]$, $\tilde{M_{ij}} > 0$ everywhere on the manifold. . Suppose that this is not true. Then there exists a first time $t_0 \in (0, \delta]$ and a point $x_0 \in M_{t_0}$, such that $\tilde{M_{ij}}$ has a null eigenvector. Namely $v \in T_{x_0} M_{t_0}$, that we can take to be with $|v| = 1$.

At the point (x_0, t_0), the tensor N_{ij} satisfies

$$\begin{aligned} N_{ij} v_i v_j &= \tilde{N}_{ij} v_i v_j + (N_{ij} - \tilde{N}_{ij}) v_i v_j \geq -|N_{ij} - \tilde{N}_{ij}| \geq -C_1 |M_{ij} - \tilde{M}_{ij}| \\ &\geq -2C\varepsilon\delta \end{aligned}$$

where $\tilde{N}_{ij} = P(\tilde{M}_{ij}, g_{ij})$ and $C = C(K)$ is a constant that depends on K, since P is a polynomial.

To proceed, extend v to a unitary vector field on a neighborhood U of x_0 by parallel transporting v along geodesics starting at x_0, during some time interval $[0, t_1]$.

Note that $\frac{Dv}{dt} = 0$ along any geodesic and $\nabla_{\frac{\partial}{\partial x^i}} v_i = 0$ on U.

Consider now the function defined by

$$f(x,t) = \tilde{M}_{ij} v_i v_j.$$

Then f satisfies $f(x,t) \geq 0$ for all $x \in M_t$ and $t: \ 0 \leq t \leq \delta$.

Moreover, since f has a minimum at (x_0, t_0),

$$\frac{\partial f}{\partial t}(x_0, t_0) \leq 0, \tag{4.3}$$

$$\frac{\partial f}{\partial x^k}(x_0, t_0) = 0 \qquad \text{and} \qquad \Delta f(x_0, t_0) \geq 0,$$

We conclude that, at (x_0, t_0),

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial M_{ij}}{\partial t} v_i v_j + \varepsilon = (\Delta M_{ij}) v_i v_j + (y^k \frac{\partial M_{ij}}{\partial x^k}) v_i v_j + N_{ij} v_i v_j + \varepsilon \\
&= \Delta f + y^k \frac{\partial f}{\partial x^k} + N_{ij} v_i v_j + \varepsilon \\
&\geq (1 - 2C\delta)\varepsilon.
\end{aligned}$$

Hence, choosing $\delta = \min 1/4C,\, T$, we get $\frac{\partial f}{\partial t}(x_0, t_0) > 0$, which contradicts 4.3. Therefore

$$M_{ij} + \varepsilon(\delta + t) g_{ij} = \tilde{M}_{ij} > 0.$$

Now, taking limit when $\varepsilon \to 0$ we get that $M_{ij} \geq 0$ for all $t \in [0, \delta]$. We can repeat the same argument to prove that $M_{ij} \geq 0$ for $t \in [\delta, 2\delta]$, so $M_{ij} \geq 0$ on $[0, 2\delta]$. After finite iterations, starting with $M_{ij}(\delta)$, we get the nonnegativity over all $[0, T]$.

□

Proposition 4.4

If the second fundamental form h_{ij} is nonnegative definite at $t = 0$, it remains so as long as the flow exists.

Proof:

We know that

$$\frac{\partial h_{ij}}{\partial t} = \Delta h_{ij} - 2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij}.$$

Let us define

$$M_{ij} := h_{ij},$$

$$N_{ij} := -2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij}.$$

Observe that if $h_{ij}v_j = 0$ then $(-2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij})v_j = 0$. Hence, by the maximum principle for tensors we obtain that $h_{ij} \geq 0$ for all t.

□

Next we will show that the strict convexity of a hypersurface is preserved by the mean curvature flow. The following lemma is due to Huisken [5].

Lemma 4.5 (Pinching estimate)

Suppose there exist two positive constants ε and β, with $0 < \varepsilon \leq \frac{1}{n} < \beta < 1$, such that

$$\varepsilon H g_{ij} \leq h_{ij} \leq \beta H g_{ij} \tag{4.4}$$

and

$$H > 0 \tag{4.5}$$

at time $t = 0$. *Then inequalities 4.4 and 4.5 also hold for any positive time where the flow exists.*

<u>Proof:</u>

We know that H evolves by

$$\frac{\partial H}{\partial t} = \Delta H + |A|^2 H,$$

so

$$\frac{\partial H}{\partial t} - \Delta H > 0.$$

Then, by the maximum principle for parabolic PDE's, we know that $H > 0$ for all times.

In order to prove the first inequality in 4.4, let

$$M_{ij} := \frac{h_{ij}}{H} - \varepsilon g_{ij},$$

$$y^k := \frac{2}{H} g^{kl} \frac{\partial H}{\partial x^l}$$

and

$$N_{ij} := 2\varepsilon H h_{ij} - 2 h_{im} g^{ml} h_{lj}.$$

We want to apply the maximum principle for tensors. Then $M_{ij} > 0$ for all times and the first inequality in 4.4 will be proven.

Notice that

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{h_{ij}}{H}\right) &= \frac{H\frac{\partial h_{ij}}{\partial t} - h_{ij}\frac{\partial H}{\partial t}}{H^2} \\
&= \frac{H\Delta h_{ij} - 2H^2 h_{il} g^{lm} h_{mj} + |A|^2 h_{ij} H - h_{ij}\Delta H - h_{ij}|A|^2 H}{H^2} \\
&= \frac{H\Delta h_{ij} - h_{ij}\Delta H}{H^2} - 2h_{im} g^{lm} h_{lj},
\end{aligned}
$$

and

$$
\nabla_l \left(\frac{h_{ij}}{H}\right) = \frac{H\nabla_l h_{ij} - h_{ij}\nabla_l H}{H^2},
$$

so

$$
\begin{aligned}
\Delta\left(\frac{h_{ij}}{H}\right) &= g^{kl}\nabla_k\nabla_l\left(\frac{h_{ij}}{H}\right) \\
&= g^{kl}\frac{\nabla_k H \nabla_l h_{ij} + H\nabla_k\nabla_l h_{ij} - \nabla_k h_{ij}\nabla_l H - h_{ij}\nabla_k\nabla_l H}{H^2} \\
&- g^{kl}\frac{2H\nabla_k H(H\nabla_l h_{ij} - h_{ij}\nabla_l H)}{H^4} \\
&= \frac{H\Delta h_{ij} - h_{ij}\Delta H}{H^2} - g^{kl}\frac{2\nabla_k H(H\nabla_l h_{ij} - h_{ij}\nabla_l H)}{H^3} \\
&= \frac{H\Delta h_{ij} - h_{ij}\Delta H}{H^2} - g^{kl}\frac{2}{H}\nabla_k H \nabla_l \left(\frac{h_{ij}}{H}\right).
\end{aligned}
$$

Then, on one hand we have

$$
\begin{aligned}
\frac{\partial M_{ij}}{\partial t} &= \frac{H\Delta h_{ij} - h_{ij}\Delta H}{H^2} - 2h_{im}g^{lm}h_{lj} + \varepsilon \frac{\partial g_{ij}}{\partial t} \\
&= \frac{H\Delta h_{ij} - h_{ij}\Delta H}{H^2} - 2h_{im}g^{lm}h_{lj} + 2\varepsilon H h_{ij}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\Delta M_{ij} + y^k \frac{\partial M_{ij}}{\partial x_k} + N_{ij} &= \frac{H\Delta h_{ij} - h_{ij}\Delta H}{H^2} - g^{kl}\frac{2}{H}\nabla_k H \nabla_l \left(\frac{h_{ij}}{H}\right) \\
&+ \frac{2}{H} g^{kl} \nabla_l H \left(\frac{H\nabla_k h_{ij} - h_{ij}\nabla_k H}{H^2} - \varepsilon \nabla_k g_{ij}\right) \\
&+ 2\varepsilon H h_{ij} - 2h_{im}g^{ml}h_{lj} \\
&= 2\varepsilon H h_{ij} - 2h_{im}g^{ml}h_{lj} \\
&= \frac{\partial M_{ij}}{\partial t},
\end{aligned}
$$

since $\nabla_k g_{ij} = 0$.

Then condition 4.2 in theorem 4.3 is satisfied for our choice of tensors. Let us check.

Suppose that for a tangent vector v we have

$$
M_{ij}v_j = 0 = \frac{h_{ij}v_j}{H} - \varepsilon g_{ij}v_j = \frac{h_{ij}v_j}{H} - \varepsilon v_i.
$$

Then,

$$
\begin{aligned}
N_{ij}v_iv_j &= 2\varepsilon H h_{ij}v_iv_j - 2h_{im}g^{ml}h_{lj}v_iv_j = 2\varepsilon H(H\varepsilon v_j)v_j - 2g^{ml}(H\varepsilon v_l)(H\varepsilon v_m) \\
&= 2\varepsilon^2 H^2 v_jv_j - 2\varepsilon H v_m v_m \\
&= 0.
\end{aligned}
$$

We are therefore in hypothesis of theorem 4.3, and our claim holds.

The analogous argument with

$$M'_{ij} = \beta g_{ij} - \frac{h_{ij}}{H}$$

proves the second inequality in equation 4.4.

□

We are now able to prove that a convex hypersurface evolving by the MCF will remain embedded and strictly convex for all times such that the flow exists.

Proposition 4.6 (Convexity is preserved)

Let M_t be a compact embedded hypersurface evolving by the Mean Curvature Flow. If $M = M_0$ is strictly convex, then M_t remains a strictly convex compact embedded manifold for $t > 0$.

Proof:

By lemma 4.5 we know that the mean curvature H and the second fundamental form h_{ij} will remain positive for all times. It remains to prove that M_t is an embedded hypersurface for $t > 0$. Suppose for some t_0 the manifold develops a self-intersection at $x_0 \in M_{t_0}$. Then by a theorem of Hadamard, near x_0 the second fundamental form h_{ij} cannot be positive, which contradicts that it should remain positive everywhere.

□

As a consequence of this proposition, the convex manifold M_t can be described as

$$X(z,t) = r(z,t)z, \qquad z \in S^n \tag{4.6}$$

for any $t \geq 0$ as long as the flow exists.

Hence, to understand the evolution of the manifolds M_t it is enough to know how the function $r(z,t)$ behaves in time. We therefore seek its evolution equation.

From 4.6 we note that

$$\frac{\partial X}{\partial t} = \frac{\partial r}{\partial t} z + \bar{g}(\bar{\nabla} r, \frac{\partial z}{\partial t}) z + r \frac{\partial z}{\partial t}.$$

Moreover X satisfies the MCF equation $\frac{\partial X}{\partial t} = -H\nu$, and since

$$\nu = \frac{1}{\sqrt{r^2 + |\bar{\nabla} r|^2}} (rz - \bar{g}^{ik} \frac{\partial r}{\partial u^i} \frac{\partial z}{\partial u^k}),$$

we have

$$\frac{\partial r}{\partial t} z + \bar{g}(\bar{\nabla} r, \frac{\partial z}{\partial t}) z + r \frac{\partial z}{\partial t} = -\frac{1}{\sqrt{r^2 + |\bar{\nabla} r|^2}} H (rz - \bar{g}^{ik} \frac{\partial r}{\partial u^i} \frac{\partial z}{\partial u^k}). \tag{4.7}$$

Taking inner product with z in both sides of equation 4.7 we get

$$\frac{\partial r}{\partial t} = -\frac{1}{\sqrt{r^2 + |\bar{\nabla} r|^2}} rH - \bar{g}(\bar{\nabla} r, \frac{\partial z}{\partial t}) \tag{4.8}$$

Also, by taking inner product with $\frac{\partial z}{\partial u^j}$ in both sides of equation 4.7 we get

$$\begin{aligned} r \langle \frac{\partial z}{\partial t}, \frac{\partial z}{\partial u^j} \rangle &= \frac{H}{\sqrt{r^2 + |\bar{\nabla} r|^2}} \bar{g}^{ik} \bar{g}_{kj} \frac{\partial r}{\partial u^i} \\ &= \frac{H}{\sqrt{r^2 + |\bar{\nabla} r|^2}} \frac{\partial r}{\partial u^j}. \end{aligned}$$

Since we can write any $v \in T_p S^n$ as $v = \langle v, \frac{\partial z}{\partial u^i} \rangle \bar{g}^{ij} \frac{\partial z}{\partial u^j}$, we get that

$$\frac{\partial z}{\partial t} = \bar{g}^{ij}\frac{H}{r}\frac{1}{\sqrt{r^2+|\bar{\nabla} r|^2}}\frac{\partial r}{\partial u^j}\frac{\partial z}{\partial u^i}$$

and therefore

$$\begin{aligned}
\bar{g}(\bar{\nabla} r, \frac{\partial z}{\partial t}) &= \bar{g}^{ij}\frac{\partial r}{\partial u^j}\langle\frac{\partial z}{\partial t}, \frac{\partial z}{\partial u^i}\rangle \\
&= \bar{g}^{ij}\bar{g}^{ik}\frac{H}{r}\frac{1}{\sqrt{r^2+|\bar{\nabla} r|^2}}\frac{\partial r}{\partial u^j}\frac{\partial r}{\partial u^k}\langle\frac{\partial z}{\partial u^i}, \frac{\partial z}{\partial u^i}\rangle \\
&= \bar{g}^{ij}\bar{g}_{ii}\bar{g}^{ik}\frac{H}{r}\frac{1}{\sqrt{r^2+|\bar{\nabla} r|^2}}\frac{\partial r}{\partial u^j}\frac{\partial r}{\partial u^k} \\
&= |\bar{\nabla} r|^2\frac{H}{r}\frac{1}{\sqrt{r^2+|\bar{\nabla} r|^2}}.
\end{aligned}$$

Now we can rewrite equation 4.8 as

$$\begin{aligned}
\frac{\partial r}{\partial t} &= -\frac{rH}{\sqrt{r^2+|\bar{\nabla} r|^2}} - \frac{|\bar{\nabla} r|^2 H}{r\sqrt{r^2+|\bar{\nabla} r|^2}} \\
&= -\frac{(r^2+|\bar{\nabla} r|^2)H}{r\sqrt{r^2+|\bar{\nabla} r|^2}} \\
&= -\frac{H}{r}\sqrt{r^2+|\bar{\nabla} r|^2}.
\end{aligned}$$

Since $H = g^{ij}h_{ij}$,

$$\begin{aligned}
\frac{\partial r}{\partial t} = & - \quad r^{-2}\left(\bar{g}^{ij} - \frac{\bar{\nabla}^i r\bar{\nabla}^j r}{r^2+|\bar{\nabla} r|^2}\right)\frac{1}{\sqrt{r^2+|\bar{\nabla} r|^2}}\left(-r\frac{\partial^2 r}{\partial u^i\partial u^j}\right. \\
& + \quad \left. 2\frac{\partial r}{\partial u^i}\frac{\partial r}{\partial u^j} + r^2\bar{g}_{ij}\right)\frac{1}{r}\sqrt{r^2+|\bar{\nabla} r|^2},
\end{aligned}$$

and consequently

$$\frac{\partial r}{\partial t} = r^{-3}\left(\bar{g}^{ij} - \frac{\bar{\nabla}^i r \bar{\nabla}^j r}{r^2 + |\bar{\nabla} r|^2}\right)\left(r\frac{\partial^2 r}{\partial u^i \partial u^j} - 2\frac{\partial r}{\partial u^i}\frac{\partial r}{\partial u^j} - r^2 \bar{g}_{ij}\right). \tag{4.9}$$

□

4.1 Longtime existence and convergence results

To prove long-term existence of MCF for compact convex hypersurfaces we move now to a new set up. This is not the original argument of Huisken in [5], but rather a simpler argument due to Andrews [1].

Definition 4.3.

The *support function* of the convex hypersurface $X : M \longrightarrow \mathbb{R}^{n+1}$ is

$$S(z) = \langle z\,, X(\nu^{-1}(z))\rangle \qquad\qquad z \in S^n,$$

where $\nu^{-1} : S^n \longrightarrow M$ is the inverse of the Gauss map.

Remark 4.1.

We can represent the hypersurface M via the support function by

$$X(z) = S(z)z + \bar{\nabla} S(z),$$

where $\bar{\nabla}$ is the connection associated to the round metric $\bar{g}$ on S^n.

Proof:

Consider the basis of T_zS^n

$$\{z, \bar{\nabla}_1 z, ..., \bar{\nabla}_n z\}$$

In this basis, the normal component of the vector $X(z)$ is given by the support function while the tangent components are simply

$$\langle \bar{\nabla}_i z \, , X(z) \rangle = \bar{\nabla}_i S(z).$$

□

Proposition 4.7

The components of the second fundamental form of the hypersurface are, in terms of the support function, given by

$$h_{ij} = \bar{\nabla}_i \bar{\nabla}_j S + S \bar{g}_{ij},$$

for all $1 \leq i, \, j \leq n$.

Proof:

Note that

$$0 = \langle z \, , \bar{\nabla}_j X \rangle \;\Rightarrow\; \langle \bar{\nabla}_i z \, , \bar{\nabla}_j X \rangle = - \langle z \, , \bar{\nabla}_i \bar{\nabla}_j X \rangle.$$

Then,

$$\begin{aligned}
\bar{\nabla}_i \bar{\nabla}_j S &= \langle \bar{\nabla}_i \bar{\nabla}_j z \, , X(z) \rangle + \langle \bar{\nabla}_i z \, , \bar{\nabla}_j X(z) \rangle + \langle \bar{\nabla}_j z \, , \bar{\nabla}_i X(z) \rangle + \langle z \, , \bar{\nabla}_i \bar{\nabla}_j X(z) \rangle \\
&= \langle \bar{\nabla}_i \bar{\nabla}_j z \, , X(z) \rangle + \langle \bar{\nabla}_j z \, , \bar{\nabla}_i X(z) \rangle = -\bar{h}_{ij} \langle z \, , X \rangle - \langle z \, , \bar{\nabla}_j \bar{\nabla}_i X \rangle \\
&= -\bar{g}_{ij} S(z) + h_{ji} - \langle z \, , \bar{\Gamma}^k_{ij} \frac{\partial X}{\partial u^k} \rangle \\
&= -\bar{g}_{ij} S(z) + h_{ij},
\end{aligned}$$

where we used that $\bar{g}_{ij} = \bar{h}_{ij}$. Indeed,

$$0 = \langle z \, , \bar{\nabla}_j z \rangle \;\Rightarrow\; \bar{g}_{ij} = \langle \bar{\nabla}_i z \, , \bar{\nabla}_j z \rangle = - \langle z \, , \bar{\nabla}_i \bar{\nabla}_j z \rangle = \bar{h}_{ij}$$

$\square$

Definition 4.4.

Let M be a manifold and $\{\kappa_1, ..., \kappa_n\}$ its principal curvatures. The *principal radii* of M are given by

$$\{\frac{1}{\kappa_1}, ..., \frac{1}{\kappa_n}\}.$$

Lemma 4.8

The eigenvalues of h_{ij} with respect to the metric $\bar{g}$ are the principal radii of the hypersurface, i.e. the reciprocal of the principal curvatures.

<u>Proof:</u>

$$\begin{aligned} \bar{g}_{ij} &= \langle \frac{\partial z}{\partial x^i} \, , \frac{\partial z}{\partial x^j} \rangle = \langle h_{ik} g^{kl} \frac{\partial X}{\partial x^l} \, , h_{jm} g^{mn} \frac{\partial X}{\partial x^n} \rangle = h_{ik} h_{jm} g^{kl} g^{mn} g_{ln} \\ &= h_{ik} h_{jl} g^{kl}, \end{aligned}$$

where we used the Gauss-Weingarten relations 1.2 to differentiate $\frac{\partial z}{\partial x^k}$. Then,

$$\bar{g}^{ij} = (h^{-1})^{ik} (h^{-1})^{jl} g_{kl},$$

and therefore

$$\begin{aligned} h_{ij}\bar{g}^{jk} &= h_{ij}(h^{-1})^{js}(h^{-1})^{kl}g_{sl} \\ &= (h^{-1})^{kl}g_{il}. \end{aligned}$$

This means that the eigenvalues of h_{ij} with respect to $\bar{g}$ are the same as the eigenvalues of $(h^{-1})^{ij}$ with respect to g. The latter are the reciprocal of the eigenvalues of h_{ij} with respect to g.

□

Definition 4.5.

Let M be a compact convex manifold. The *width* of M at $p = \nu^{-1}(z)$ is given by

$$w(z) = S(z) + S(-z) \qquad \forall\, z \in S^n.$$

The next lemma is due to Andrews [1].

Lemma 4.9

Let M be an smooth compact hypersurface of $\mathbb{R}^{n+1}$. Suppose there exists a constant C such that

$$\kappa_{max}(x) \leq C\kappa_{min}(x) \qquad \forall x \in M.$$

Then

$$w_{max} \leq Cw_{min},$$

where $\kappa_{max}(x) = \max\{\kappa_1(x), ..., \kappa_n(x)\}$, $\kappa_{min}(x) = \min\{\kappa_1(x), ..., \kappa_n(x)\}$, $w_{max} = \max_{z\in S^n} w(x)$ and $w_{min} = \min_{z\in S^n} w(x)$.

Proof:

Let z_- and z_+ be such that

$$z_- \in \min\{w(x) \,|\, z \in S^n\},$$

$$z_+ \in \max\{w(x) \,|\, z \in S^n\}.$$

Let Σ be a totally geodesic sphere of dimension 2 in S^n containing z_- and z_+. We can parametrize the 2-sphere by local coordinates

$$(x_+^1, x_+^2) \mapsto (\cos x_+^2 \sin x_+^1,\ \sin x_+^2 \sin x_+^1,\ \cos x_+^1),$$

with $(x_+^1, x_+^2) \in [0, \pi] \times [0, \pi/2]$ and $x_+^1 = 0$ corresponding to z_+.
Note that

$$\bar{g}_{11} = |(\cos x_+^2 \cos x_+^2,\ \sin x_+^1 \cos x_+^2,\ -\sin x_+^1)|^2 = 1$$

and

$$\begin{aligned} \bar{g}_{12} &= (\cos x_+^2 \cos x_+^2,\ \sin x_+^1 \cos x_+^2,\ -\sin x_+^1) \cdot (-\sin x_+^2 \sin x_+^1,\ \cos x_+^2 \sin x_+^1,\ 0) \\ &= 0. \end{aligned}$$

Therefore

$$\begin{aligned} \bar{\nabla}_1 \bar{\nabla}_1 S &= \frac{\partial^2 S}{\partial (x_+^1)^2} - \Gamma_{11}^1 \frac{\partial S}{\partial x_+^1} - \Gamma_{11}^2 \frac{\partial S}{\partial x_+^2} \\ &= \frac{\partial^2 S}{\partial (x_+^1)^2}, \end{aligned}$$

since $\Gamma^1_{11} = \Gamma^2_{11} = 0$. Then we can write

$$\bar{\nabla}_1 \bar{\nabla}_1 S + \bar{g}_{11} S = \frac{\partial^2 S}{\partial (x^1_+)^2} + S. \tag{4.10}$$

Let B be the matrix

$$B := Hess_{\bar{\nabla}} S + \bar{g} S.$$

We know by lemma 4.8 that the eigenvalues of B are the reciprocal of the principal curvatures of M. Let us compute

$$\begin{aligned}
\int_\Sigma B(x^1_+, x^1_+) d\mu_\Sigma &= \int_0^{2\pi} \int_0^{\pi} (Hess_{\bar{\nabla}} S + \bar{g} S) \sin x^1_+ dx^1_+ dx^2_+ \\
&= \int_0^{2\pi} \int_0^{\pi} \left(\frac{\partial^2 S}{\partial (x^1_+)^2} + S \right) \sin x^1_+ dx^1_+ dx^2_+ \\
&= \int_0^{2\pi} \left(-S \cos x^1_+ \Big|_0^{2\pi} \right) dx^2_+ \\
&= 2\pi (S(z_+) + S(-z_+)).
\end{aligned}$$

Analogously, we would get

$$\int_\Sigma B(x^1_-, x^1_-) d\mu_\Sigma = 2\pi (S(z_-) + S(-z_-)).$$

By hypothesis, $\kappa_{max}(x) \leq C \kappa_{min}(x)$, therefore

$$\frac{1}{\kappa_{min}(x)} \leq C \frac{1}{\kappa_{max}(x)},$$

implying that

$$B(x^1_+, x^1_+) \leq C\, B(x^1_-, x^1_-). \tag{4.11}$$

Integrating 4.11 over Σ we get

$$(S(z_+) + S(-z_+)) \leq C(S(z_-) + S(-z_-)).$$

Thus,

$$w_{max} \leq Cw_{min}.$$

$\square$

Lemma 4.10 (Containment Principle)

Let M and N be two n-dimensional, strictly convex, compact embedded hypersurfaces in $\mathbb{R}^{n+1}$ via

$$X : M \longrightarrow \mathbb{R}^{n+1} \quad \textit{and} \quad Y : N \longrightarrow \mathbb{R}^{n+1}$$

respectively. Suppose $N \subsetneq M$ and we let them evolve by the mean curvature flow. Then, $N_t \subseteq M_t$ as long as both flows exist.

Proof:

Suppose t_0 is the first moment such that $M_{t_0} \cap N_{t_0} \neq \emptyset$, and let us consider a point x_0 in the intersection.

At the point (x_0, t_0) we have,

$$H_M(x_0, t_0) < H_N(x_0, t_0),$$

where H_M and H_N denote the mean curvature of M and N respectively. Let us also denote by ν_M and ν_N the outer unit normal vectors of M and N.

Then, since x_0 is a tangency point we have that

$$\nu_M(x_0, t_0) = \nu_N(x_0, t_0),$$

so the velocity vector v_M of $x_0 \in M_{t_0}$ is colinear with the velocity v_N at $x_0 \in N_{t_0}$, and their modulus verify

$$|v_M| = H_M(x_0, t_0) < H_N(x_0, t_0) = |v_N|.$$

Therefore the tangency point disappears immediately.

□

Theorem 4.11

The solution of the Mean Curvature Flow with initial data a smooth, strictly convex, compact hypersurface M exists on a maximal time interval $[0, \omega)$ with $\omega < +\infty$. Moreover, $X(\cdot, t)$ converges uniformly to a point in $\mathbb{R}^{n+1}$ as $t \to \omega$.

Proof:

Since M is compact, it is contained in a sphere of radius R. We know by equation 4.9 that the radius of the sphere evolves according to

$$r(t) = \sqrt{R^2 - 2nt}.$$

Therefore, the sphere collapses to a point at time $T = R^2/2n$. Since M is contained in the sphere the maximal interval of existence of the flow also has to be finite. To prove that M_t converges to a point, it is enough to show that the enclosed area tends to zero as $t \to \omega$. If it's not zero, there exists a small ball contained in M_t for all $t \in [0, \omega)$. Hence if we write

$$X(z,t) = r(z,t)z,$$

then $r(z,t)$ and $|\bar{\nabla} r|$ have uniform upper and lower bounds coming from the enclosing and enclosed spheres. Therefore equation 4.9 is uniformly parabolic, and the solution cannot be singular at time $t = \omega$. This contradicts that $[0, \omega)$ is maximal.

□

4.1.1 The normalized mean curvature flow

To understand the shape of M_t near the singularity, let us rescale the solution as

$$\tilde{X}(x,\tau) = \frac{1}{\sqrt{2(\omega - t)}} \left(X(x,t) - X(x,\omega)\right),$$

where $\tau = \tau(t)$ is a reparametrization of time. Explicitly,

$$\tau = -\frac{1}{2} \log\left(\frac{\omega - t}{\omega}\right).$$

Then

$$\frac{d\tau}{dt} = \frac{1}{2(\omega - t)},$$

and the evolution equation for $\tilde{X}$ is given by

$$\frac{\partial \tilde{X}(x,\tau)}{\partial \tau} = -\tilde{H}(x,\tau)\tilde{\nu}(x,\tau) + \tilde{X}(x,\tau).$$

Indeed,

$$\begin{aligned}
\frac{\partial \tilde{X}(x,\tau)}{\partial \tau} &= \frac{\partial \tilde{X}}{\partial t}\frac{dt}{d\tau} \\
&= \frac{1}{2(w-t)}\left[\frac{\partial X}{\partial t}\sqrt{2(w-t)} + \frac{1}{\sqrt{2(w-t)}}(X(x,t)-X(x,w))\right]\frac{dt}{d\tau} \\
&= \sqrt{2(w-t)}\ \frac{\partial X}{\partial t} + \tilde{X} = -\sqrt{2(w-t)}\ H\nu + \tilde{X} \\
&= -\tilde{H}\tilde{\nu} + \tilde{X}.
\end{aligned}$$

We will adopt the symbol $\tilde{M}_t$ to refer to the solution of the rescaled MCF.

Lemma 4.12

There exists a positive constant $\tilde{C}$ such that

$$\tilde{C}^{-1} \le \tilde{r}_{in} \le \tilde{r}_{out} \le \tilde{C} \qquad \forall\ \tau \ge 0.$$

Proof:

We know by lemma 4.9 that there exists a constant C such that

$$\tilde{r}_{out} \le \tilde{w}_{max} \le C\tilde{w}_{min} \le c(n)C\tilde{r}_{in},$$

where $c(n)$ is a constant depending on the dimension of the manifold (see [8]). Let us denote $\tilde{C} := c(n)C$.

By definition of r_{out}, the manifold M_t is enclosed by $S_{r_{out}(t)}(\xi)$, for some $\xi \in \mathbb{R}^{n+1}$. Since both $S_{r_{out}(t)}(\xi)$ and M_t are convex manifolds, $M_{t'}$ will be enclosed by the evolved sphere for $t' \in (t,\omega)$. The evolution equation for the radius of the sphere is, by 4.9,

$$\frac{\partial r}{\partial t} = nr^{-1},$$

since r does not depend on the point. Then,

$$\frac{r^2(t')}{2} = -n(t'-t) + \frac{r_{out}^2(t)}{2},$$

and therefore

$$0 \leq r_{out}(t') \leq r(t') = \sqrt{r_{out}^2(t) - 2n(t'-t)} \leq \sqrt{r_{out}(t) - 2(t'-t)},$$

which implies that $r_{out}(t) \geq 2(t'-t)$ for all $t' \in (t, \omega)$. Now, taking limit as $t' \to \omega$ we get $r_{out}(t) \geq 2(\omega - t)$. Hence we conclude that

$$\tilde{r}_{out}(\tau) \geq 1,$$

and since $\tilde{r}_{out} \leq \tilde{C}\tilde{r}_{in}$

$$\tilde{C}^{-1} \leq \tilde{r}_{in}.$$

Now, let us consider $S_{r_{in(t)}}(\xi')$ the biggest sphere enclosed by M_t. By the same argument

$$r(t') = \sqrt{r_{in}^2(t) - 2n(t'-t)} \leq r_{in}(t) \leq r_{in}(t'),$$

for all $t' \in (t, \omega)$. Again, taking limit as $t' \to \omega$ we get

$$r_{in}^2(t) - 2n(\omega - t) \leq r_{in}(\omega) = 0,$$

since M_t converges to a point as $t \to \omega$. Thus

$$\tilde{r}_{in}(\tau) \leq 1$$

and therefore

$$\tilde{r}_{out}(\tau) \leq \tilde{C}.$$

□

Lemma 4.13

There exists a positive constant C such that

$$\sup\{\tilde{H}(x,\tau) \mid x \in M, \tau \geq 0\} \leq C$$

Proof:

Let $t_0 \in (0,\omega)$, and suppose the biggest sphere contained in M_{t_0} is centered at $0 \in \mathbb{R}^{n+1}$. Then, for all $t \in [0,t_0]$

$$S(z,t) \geq r_{in}(t_0).$$

Note that

$$\begin{aligned} \frac{\partial S}{\partial t} &= \langle \frac{\partial z}{\partial t}, X(z)\rangle + \langle z, \frac{\partial X}{\partial t} + \frac{\partial X}{\partial x^k}\frac{\partial(\nu^{-1})^k}{\partial t}\rangle = \langle z, \frac{\partial X}{\partial t}\rangle \\ &= -H \end{aligned}$$

Using the calculations done in lemma 4.8 we can write the mean curvature H in terms of the metric in the sphere. Indeed,

$$\begin{aligned} H &= h_{ij}g^{ij} = h_{ij}(h^{-1})^{im}\bar{g}_{mn}(h^{-1})^{nj} = \delta_j^m \bar{g}_{mn}(h^{-1})^{nj} \\ &= \bar{g}_{mn}(h^{-1})^{nm}. \end{aligned}$$

Also, the same lemma and proposition 3.1 give us

$$\frac{\partial H}{\partial t} = g^{kl}(\bar{\nabla}_k \bar{\nabla}_l H + H\bar{g}_k l).$$

Let us consider the function $\Phi : S^n \times [0, t_0] \longrightarrow \mathbb{R}$ defined by

$$\Phi(z,t) = \frac{H(z,t)}{S(z,t) - a},$$

where $a = \frac{1}{2}r_{in}(t_0)$. Note that $S - a > 0$ for all $t \in [0, t_0]$.

Let $(z_1, t_1) \in S^n \times [0, t_0]$ be a point such that Φ achieves a maximum. Then at (z_1, t_1) we have

$$0 = \bar{\nabla}_i \Phi = \frac{\bar{\nabla}_i H}{S - a} - \frac{H\bar{\nabla}_i S}{(S-a)^2},$$

and therefore

$$\begin{aligned}
0 \quad \geq \quad & \bar{\nabla}_i\bar{\nabla}_j\Phi = \frac{\bar{\nabla}_i\bar{\nabla}_j H}{S-a} - \frac{\bar{\nabla}_j H\bar{\nabla}_i S}{(S-a)^2} - \frac{\bar{\nabla}_i H\bar{\nabla}_j S}{(S-a)^2} - \frac{H\bar{\nabla}_i\bar{\nabla}_j S}{(S-a)^2} + 2\frac{H\bar{\nabla}_j S\bar{\nabla}_i S}{(S-a)^3} \\
= \quad & \frac{\bar{\nabla}_i\bar{\nabla}_j H}{S-a} - \frac{\bar{\nabla}_i S}{S-a}\bar{\nabla}_j\Phi - \frac{\bar{\nabla}_j S}{S-a}\bar{\nabla}_i\Phi - \frac{H\bar{\nabla}_i\bar{\nabla}_j S}{(S-a)^2} \\
= \quad & \frac{\bar{\nabla}_i\bar{\nabla}_j H}{S-a} - \frac{H\bar{\nabla}_i\bar{\nabla}_j S}{(S-a)^2} = \frac{\bar{\nabla}_i\bar{\nabla}_j H}{S-a} + \frac{H\bar{g}_{ij}S - h_{ij}H}{(S-a)^2} \\
= \quad & \frac{\bar{\nabla}_i\bar{\nabla}_j H}{S-a} + \frac{H\bar{g}_{ij}(S-a) + aH\bar{g}_{ij} - h_{ij}H}{(S-a)^2} \\
= \quad & \frac{\bar{\nabla}_i\bar{\nabla}_j H + H\bar{g}_{ij}}{S-a} + \frac{aH\bar{g}_{ij} - h_{ij}H}{(S-a)^2}.
\end{aligned}$$

Thus,

$$\bar{\nabla}_i\bar{\nabla}_j H + H\bar{g}_{ij} \leq \frac{h_{ij}H - aH\bar{g}_{ij}}{S-a}. \tag{4.12}$$

Also,

$$
\begin{aligned}
0 &= \frac{\partial \Phi}{\partial t} = \frac{1}{S-a}\frac{\partial H}{\partial t} - \frac{H}{(S-a)^2}\frac{\partial S}{\partial t} \\
&= \frac{1}{S-a}\left(g^{ij}(\bar{\nabla}_i \bar{\nabla}_j H + H\bar{g}_{kl}) + \frac{H^2}{S-a}\right).
\end{aligned}
$$

So, using inequality 4.12,

$$
\begin{aligned}
0 &\leq g^{ij}\left(\frac{h_{ij}H - aH\bar{g}_{ij}}{S-a}\right) + \frac{H^2}{S-a} = \left(\frac{H^2 - aHg^{ij}\bar{g}_{ij}}{S-a}\right) + \frac{H^2}{S-a} \\
&= \frac{2H^2 - aH|A|^2}{S-a},
\end{aligned}
$$

since $g^{ij}\bar{g}_{ij} = g^{ij}h_{jk}g^{kl}h_{lj} = |A|^2$. Moreover, $H^2 \leq |A|^2$, thus

$$
H \leq \frac{2}{a} = \frac{4}{r_{in}(t_0)} \leq \frac{4C}{r_{out}(t_0)},
$$

due to lemma 4.9. Then using lemma 4.12 we conclude that, for $K = 4C\tilde{C}$ we have $H(z_1, t_1) \leq K$. Therefore, for all $z \in S^n$, $t \in [0, t_0]$ we have

$$
H(z,t) \leq K.
$$

Then, taking limit when $t_0 \to \omega$ we get the bound on all $S^n \times [0, \omega)$. Hence

$$
\tilde{H}(x,\tau) \leq K \qquad\qquad \forall\, x \in M,\ \tau \in [0, +\infty).
$$

□

To conclude that we have convergence to a point, we recall the following theorem, that can be found in [8].

Theorem 4.14 (Blaschke Selection Theorem)

Let $\{K_j\}_{j\in\mathbb{N}}$ be a sequence of compact convex sets of $\mathbb{R}^{n+1}$ which are contained in a bounded set. Then, there exists a subsequence $\{K_{j_k}\}_{k\in\mathbb{N}}$ and a compact convex set K in $\mathbb{R}^{n+1}$ such that K_{j_k} converges to K in the Hausdorff metric.

Theorem 4.15

Let $\tilde{M}_t$ be a smooth strictly convex, compact hypersurface embedded in R^{n+1} evolving by the normalized mean curvature flow. For any sequence of times $\{\tau_j\}_{j\in\mathbb{N}}$ such that $\tau_j \to +\infty$ there exists a subsequence $\{\tau_{j_k}\}_{k\in\mathbb{N}}$ such that $\{\tilde{M}_{\tau_{j_k}}\}_{k\in\mathbb{N}}$ converges to a smooth compact convex hypersurface $\tilde{M}_\infty$ in the Hausdorff metric.

<u>Proof:</u>

By definition, we know that for some $x_0 \in \mathbb{R}^{n+1}$

$$\tilde{M}_{\tau_0} \subseteq S_{out},$$

where $S_{out} = S_{\tilde{r}_{out}(\tau_0)}(x_0)$, the n-sphere of radius $\tilde{r}_{out}(\tau_0)$ centered at x_0. Then by Blaschke theorem, there exist a subsequence of times $\{\tau_{j_k}\}_{k\in\mathbb{N}}$ with $\tau_{j_k} \to +\infty$ such that $\{\tilde{M}_{\tau_{j_k}}\}_{k\in\mathbb{N}}$ converges to $\tilde{M}_\infty$ in the Hausdorff metric.

It remains to prove that $\tilde{M}_\infty$ is non-degenerate. For this, note that for all τ_{j_k} we have

$$\tilde{r}_{in}(\tau_{j_k}) \geq \tilde{C}^{-1},$$

by lemma 4.12. Therefore $\tilde{M}_\infty$ contains a sphere and since it is convex, it is non-degenerate.

□

Chapter 5

Evolution of entire graphs

Unless otherwise stated, the results of this section are due to Esker-Huisken [3].

Definition 5.1 (Entire graph).

An orientable manifold M^n embedded in $\mathbb{R}^{n+1}$ via $X : M^n \longrightarrow \mathbb{R}^{n+1}$ is said to be an *entire graph* if, once chosen a continuous normal vector field ν, there exists $\omega \in \mathbb{R}^{n+1}$ with $|\omega| = 1$ such that

$$\langle \nu|_{X(p)} \, , \omega \rangle > 0, \qquad \forall\, p \in M.$$

Remark 5.1.

Since the condition of being an entire graph is open, we can insure that the manifold will remain a graph for some small time interval $(0, \varepsilon)$, $\varepsilon > 0$. Later in this monograph we will be able to show that the graph condition is preserved as long as the flow exists.

Notation 5.1.

For simplicity of notation, given $p \in M$ with local coordinates $x = (x^1, ..., x^n) \in \mathbb{R}^n$, we will identify the point x in $\mathbb{R}^n$ with the point $X(p, t)$ in $\mathbb{R}^{n+1}$. It should be clear from the context which one we are referring to.

Definition 5.2.

A *backwards heat kernel* is a function $\rho : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ defined as

$$\rho(x,t) = [4\pi(t_0 - t)]^{-n/2} \exp\left(\frac{-|x_0 - x|^2}{4(t_0 - t)}\right),$$

where $(x_0, t_0) \in \mathbb{R}^{n+2}$ is an arbitrary fixed point, $x \in \mathbb{R}^{n+1}$ and $t < t_0$.

Lemma 5.1

Let M_t be an entire graph evolving by the mean curvature flow. Then, its backwards heat kernel satisfies the evolution equation

$$\frac{d\rho}{dt} = -\Delta\rho + \rho\left[\frac{\langle x_0 - x\,, -H\nu\rangle}{t_0 - t} - \frac{|(x_0 - x)^{\perp}|^2}{4(t_0 - t)^2}\right],$$

where the superscript $\perp$ denotes the normal component of the vector and Δ is the Laplace-Beltrami operator on M_t.

Proof:

We compute

$$\begin{aligned}
\frac{\partial}{\partial t}\rho &= \frac{n}{2}(4\pi)^{-n/2}(t_0 - t)^{-n/2-1} \exp\left(\frac{-|x_0 - x|^2}{4(t_0 - t)}\right) \\
&+ [4\pi(t_0 - t)]^{-n/2} \exp\left(\frac{-|x_0 - x|^2}{4(t_0 - t)}\right) \frac{\partial}{\partial t}\left(\frac{-|x_0 - x|^2}{4(t_0 - t)}\right) \\
&= \left[\frac{n}{2}\frac{1}{(t_0 - t)} + \frac{\partial}{\partial t}\left(\frac{-|x_0 - x|^2}{4(t_0 - t)}\right)\right]\rho \\
&= \left[\frac{n}{2}\frac{1}{(t_0 - t)} - \frac{1}{2}\frac{\langle x_0 - x\,, H\nu\rangle}{(t_0 - t)} - \frac{1}{4}\frac{|x_0 - x|^2}{(t_0 - t)^2}\right]\rho,
\end{aligned}$$

since

$$\frac{\partial x}{\partial t} = -H\nu.$$

Now we compute $\Delta\rho$ with respect to the evolving metric. We have

$$\begin{aligned}
\nabla_i \rho &= [4\pi(t_0-t)]^{-n/2} \exp\left(\frac{-|x_0-x|^2}{4(t_0-t)}\right) \frac{1}{4(t_0-t)} [2\, \langle x_0 - x\,, \frac{\partial x}{\partial x^i}\rangle] \\
&= \frac{1}{2(t_0-t)} \langle x_0 - x\,, \frac{\partial x}{\partial x^i}\rangle \rho
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 \rho}{\partial x^i \partial x^j} &= \frac{\partial}{\partial x^j}\left[\frac{1}{2(t_0-t)} \langle x_0 - x\,, \frac{\partial x}{\partial x^i}\rangle \rho\right] \\
&= \frac{\partial}{\partial x^j}\left[\frac{1}{2(t_0-t)} \langle x_0 - x\,, \frac{\partial x}{\partial x^i}\rangle\right] \rho + \frac{\partial \rho}{\partial x^j} \\
&= \frac{1}{2(t_0-t)} \left[\langle x_0 - x\,, \frac{\partial^2 x}{\partial x^i \partial x^j}\rangle - \langle \frac{\partial x}{\partial x^i}\,, \frac{\partial x}{\partial x^j}\rangle\right] \rho \\
&+ \frac{1}{4(t_0-t)^2} [\,\langle x_0 - x\,, \frac{\partial x}{\partial x^j}\rangle][\,\langle x_0 - x\,, \frac{\partial x}{\partial x^i}\rangle] \rho \\
&= \frac{1}{2(t_0-t)} \left[\langle x_0 - x\,, -h_{ij}\nu + \Gamma_{ij}^k \frac{\partial x}{\partial x^k}\rangle - g_{ij}\right] \rho \\
&+ \frac{1}{4(t_0-t)^2} [\,\langle x_0 - x\,, \frac{\partial x}{\partial x^j}\rangle][\,\langle x_0 - x\,, \frac{\partial x}{\partial x^i}\rangle] \rho.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Delta\rho &= g^{ij} \nabla_j \nabla_i \rho = g^{ij} \frac{\partial^2 \rho}{\partial x^i \partial x^j} - g^{ij} \frac{\partial \rho}{\partial x^k} \Gamma_{ij}^k \\
&= \rho \frac{1}{2} \frac{1}{(t_0-t)} [\,\langle x_0 - x\,, -H\nu\rangle - n] + \rho \frac{1}{4} \frac{1}{(t_0-t)^2} |(x_0-x)^T|^2 \\
&= \rho \left[\frac{1}{4} \frac{|(x_0-x)^T|^2}{(t_0-t)^2} - \frac{n}{2} \frac{1}{t_0-t} - \frac{1}{2} \frac{\langle x_0 - x\,, H\nu\rangle}{(t_0-t)}\right].
\end{aligned}$$

Hence we conclude that

$$\begin{aligned}\frac{\partial\rho}{\partial t}+\Delta\rho &= \rho\frac{\langle x_0-x\,,-H\nu\rangle}{(t_0-t)}-\frac{1}{4}\rho\left[\frac{|x_0-x|^2}{(t_0-t)^2}-\frac{|(x_0-x)^T|^2}{(t_0-t)^2}\right]\\ &= \rho\left[\frac{\langle x_0-x\,,-H\nu\rangle}{(t_0-t)}-\frac{1}{4}\frac{|(x_0-x)^\perp|^2}{(t_0-t)^2}\right],\end{aligned}$$

where the last equality holds by Pythagoras.

□

Theorem 5.2 ([6, Theorem 3.1])

Let M_t be an entire graph evolving by the mean curvature flow and let ρ be its backwards heat kernel.

(a) The following evolution equation is satisfied

$$\frac{d}{dt}\int_{M_t}\rho d\mu_t=-\int_{M_t}\rho\left|H\nu+\frac{1}{2(t_0-t)}(x_0-x)^\perp\right|^2 d\mu_t.$$

(b) More generally, for any function $f=f(x,t)$ on M_t we have

$$\frac{d}{dt}\int_{M_t}f\rho d\mu_t=\int_{M_t}\left(\frac{\partial f}{\partial t}-\Delta f\right)\rho d\mu_t-\int_{M_t}f\rho\left|H\nu+\frac{1}{2(t_0-t)}(x_0-x)^\perp\right|^2 d\mu_t.$$

Proof:

Without loss of generality we assume $x_0=0$ and set $\tau=t_0-t$. Then, we use the evolution equation given by lemma 5.1 to compute

$$
\begin{aligned}
\int_{M_t} \rho d\mu_t &= \int_{M_t} \frac{\partial \rho}{\partial t} d\mu_t - \int_{M_t} \rho\, H^2 d\mu_t \\
&= -\int_{M_t} \Delta \rho d\mu_t + \int_{M_t} \rho \left[\frac{\langle x\,, H\nu\rangle}{\tau} - \frac{|x^\perp|^2}{4\tau^2} \right] d\mu_t - \int_{M_t} \rho\, H^2 d\mu_t \\
&= \int_{M_t} \rho \left[\frac{\langle x^\perp\,, H\nu\rangle}{\tau} - \frac{|x^\perp|^2}{4\tau^2} \right] d\mu_t - \int_{M_t} \rho\, H^2 d\mu_t \\
&= \int_{M_t} \rho \left| H\nu - \frac{x^\perp}{2\tau} \right|^2 d\mu_t,
\end{aligned}
$$

where we used that the manifold M_t has no boundary, and therefore

$$
\int_{M_t} \Delta \rho \, d\mu_t = \int_{M_t} \rho \, \Delta(1) \, d\mu_t = 0.
$$

□

The argument to prove part (b) is analogous to the one used in part (a).

$$
\begin{aligned}
\int_{M_t} f\, \rho d\mu_t &= \int_{M_t} (\frac{\partial f}{\partial t} \rho + f \frac{\partial \rho}{\partial t}) d\mu_t - \int_{M_t} f\, \rho\, H^2 d\mu_t \\
&= \int_{M_t} (\frac{\partial f}{\partial t} \rho - f \Delta \rho) d\mu_t + \int_{M_t} f \rho \left[\frac{\langle x\,, H\nu\rangle}{\tau} - \frac{|x^\perp|^2}{4\tau^2} \right] d\mu_t \\
&- \int_{M_t} f \rho\, H^2 d\mu_t \\
&= \int_{M_t} (\frac{\partial f}{\partial t} \rho - \Delta f \rho) d\mu_t + \int_{M_t} f \rho \left[\frac{\langle x^\perp\,, H\nu\rangle}{\tau} - \frac{|x^\perp|^2}{4\tau^2} \right] d\mu_t \\
&- \int_{M_t} f \rho\, H^2 d\mu_t \\
&= \int_{M_t} (\frac{\partial f}{\partial t} \rho - \Delta f \rho) d\mu_t + \int_{M_t} f \rho \left| H\nu - \frac{x^\perp}{2\tau} \right|^2 d\mu_t,
\end{aligned}
$$

since

$$\int_{M_t} f\Delta\rho\, d\mu_t = \int_{M_t} \Delta f\, \rho\, d\mu_t.$$

$\square$

Corollary 5.3

Let $f = f(x,t)$ be a function, let $V = (x,t)$ be a vector field on M_t and $x = X(p,t) \in M_t$. Suppose that for some $t_1 > 0$,

$$s = \sup_{M\times[0,t_1]} |V| < \infty$$

and that the following condition is satisfied

$$\left(\frac{\partial}{\partial t} - \Delta\right) f \leq V \cdot \nabla f,$$

where $\nabla = \nabla_{g(t)}$ is the gradient on M_t. Then, we have

$$\sup_{M_t} f \leq \sup_{M_0} f, \qquad \forall t \in [0, t_1].$$

Proof:

Let

$$f_K = \max\{f - K, 0\}$$

where $K = \sup_{M_0} f$. Then f_K is piecewise differentiable so, weakly, we have

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right) f_K^2 &= 2f_K\frac{\partial f_K}{\partial t}-2f_K\Delta f_K-2|\nabla f_K|^2 = 2f_K\left(\frac{\partial}{\partial t}-\Delta\right) f_K - 2|\nabla f_K|^2 \\
&\leq 2f_K(V\cdot\nabla f_K)-2|\nabla f_K|^2 = (f_K V)\cdot 2\nabla f_K - 2|\nabla f_K|^2 \\
&\leq \frac{1}{2}f_K^2|V|^2+2|\nabla f_K|^2-2|\nabla f_K|^2 \\
&\leq \frac{1}{2}s^2 f_K^2,
\end{aligned}$$

where we used the arithmetic-geometric inequality, i.e.,

$$ab \leq \frac{1}{2}(a^2+b^2) \tag{5.1}$$

for all real numbers a and b.

Fix $t_0 \in \mathbb{R}$ with $0 < t_0 < t_1$. Using theorem 5.2 part b, we know that for $x \in M_t$ and $0 < t < t_0$ we have

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{M_t} f_K^2 \rho d\mu_t &= \int_{M_t}\left(\frac{\partial f_K^2}{\partial t}-\Delta f_K^2\right)\rho d\mu_t - \int_{M_t} f_K^2\rho\left|H\nu+\frac{1}{2(t_0-t)}(x_0-x)^{\perp}\right|^2 d\mu_t \\
&\leq \frac{1}{2}s^2\int_{M_t} f_K^2\rho d\mu_t,
\end{aligned}$$

for ρ defined around a point $(x_0, t_0) \in \mathbb{R}^{n+2}$ with $x_0 = X(p, t_0)$.

Therefore, comparing the integral

$$\int_{M_t} f_K^2 \rho d\mu_t$$

with the solution of the ODE

$$\begin{cases} y' = \frac{1}{2}s^2 y \\ y(0) = \int_{M_0} f_K^2 \rho d\mu_0 = 0 \end{cases}$$

we see that

$$0 \leq \int_{M_t} f_K^2 \rho d\mu_t \leq \left[\int_{M_0} f_K^2 \rho d\mu_0\right] e^{\frac{1}{2}s^2 t} = 0$$

since $f_K(x, 0) = 0$. Thus, f_K is zero on every M_t and it follows that

$$f(x,t) \leq \sup_{M_0} f$$

for every $0 < t < t_0$ and $x \in M_t$. Now taking the supremum over all $x \in M_t$ and all $0 \leq t \leq t_0$, we get

$$\sup_{M_t} f \leq \sup_{M_0} f.$$

for every $t \in [0,\, t_0]$. Taking the limit when $t_0 \to t_1$ yields the result.

□

5.1 A priori height estimates.

Definition 5.3 (Height).

Let t be fixed. The *height* of M_t with respect to the hyperplane

$$[\omega]^{\perp} = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} | \left\langle (x_1, ..., x_{n+1}), \omega \right\rangle = 0\}$$

is the real valued function

$$u(x,t) = \langle x\,,\omega\rangle, \qquad\qquad \forall\, x \in M_t.$$

Remark 5.2.

The definition of height suggests us that the manifold $M = M_0$ can be represented as the set of points $(x, u(x,0))$, $x \in [w]^{\perp}$ and u being the height. Subsequently, the manifolds M_t, $t > 0$, are determined by $u(x,t)$. Moreover, u satisfies

$$\frac{\partial u}{\partial t} = \sqrt{1 + |\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{1 + |\nabla u|^2}\right), \tag{5.2}$$

where the derivatives of u are taken in the directions perpendicular to ω.

□

Remark 5.3.

The function height satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right) u = 0$$

Indeed,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\langle X(p,t)\,,\omega\rangle = \langle \frac{\partial}{\partial t} X(p,t)\,,\omega\rangle = \langle \Delta X(p,t)\,,\omega\rangle = \Delta\,\langle X(p,t)\,,\omega\rangle \\ &= \Delta u. \end{aligned}$$

□

Lemma 5.4 *1. The function*

$$\eta_1(x,t) := |x|^2 + 2nt, \qquad x \in M_t,\ t > 0,$$

satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right)\eta_1 = 0.$$

2. The function

$$\eta_2(x,t) := 1 + |x|^2 - u^2 + 2nt, \qquad x \in M_t,\ t > 0,$$

satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right)\eta_2^p = -p(p-1)|\nabla\eta_2|^2\eta_2^{p-2} + 2p\eta_2^{p-1}|\nabla u|^2, \qquad \forall\, p \in \mathbb{R}.$$

<u>Proof:</u>

1)

$$\begin{aligned}\frac{\partial}{\partial t}\eta_1 &= \frac{\partial}{\partial t}\langle X\,, X\rangle + 2n = 2\,\langle\frac{\partial X}{\partial t}\,, X\rangle + 2n \\ &= 2n - 2\,\langle H\nu\,, X\rangle\end{aligned}$$

and

$$\begin{aligned}
\Delta\eta_1 = \Delta\langle X\,,X\rangle &= g^{ij}\left[\frac{\partial^2}{\partial x^i\partial x^j}\langle X\,,X\rangle - \Gamma_{ij}^k\frac{\partial}{\partial x^k}\langle X\,,X\rangle\right] \\
&= g^{ij}\left[2\frac{\partial}{\partial x^j}\langle\frac{\partial X}{\partial x^i}\,,X\rangle - 2\Gamma_{ij}^k\langle\frac{\partial X}{\partial x^k}\,,X\rangle\right] \\
&= 2g^{ij}\left[g_{ij} + \langle\frac{\partial^2 X}{\partial x^i\partial x^j}\,,X\rangle - \Gamma_{ij}^k\langle\frac{\partial X}{\partial x^k}\,,X\rangle\right] \\
&= 2n + 2\langle\Delta X\,,X\rangle \\
&= 2n - 2\langle H\nu\,,X\rangle.
\end{aligned}$$

Hence

$$\left(\frac{\partial}{\partial t} - \Delta\right)\eta_1 = 0.$$

□

2) Notice that

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)u^2 &= 2u\frac{\partial u}{\partial t} - 2u\Delta u - 2|\nabla u|^2 = 2u\left(\frac{\partial}{\partial t} - \Delta\right)u - 2|\nabla u|^2 \\
&= -2|\nabla u|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)\eta_2 &= \left(\frac{\partial}{\partial t} - \Delta\right)\eta_1 - \left(\frac{\partial}{\partial t} - \Delta\right)u^2 \\
&= 2|\nabla u|^2.
\end{aligned}$$

Now,

$$\frac{\partial}{\partial t}\eta_2^p = p\,\frac{\partial \eta_2}{\partial t}\,eta_2^{p-1}$$

and

$$\Delta\eta_2^p = p\eta_2^{p-1}\Delta\eta_2 + p(p-1)|\nabla\eta_2|^2\eta_2^{p-2},$$

thus

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\eta_2^p &= p\eta_2^{p-1}\left(\frac{\partial}{\partial t} - \Delta\right)\eta_2 - p(p-1)|\nabla\eta_2|^2\eta_2^{p-2} \\ &= 2p|\nabla u|^2\eta_2^{p-1} - p(p-1)|\nabla\eta_2|^2\eta_2^{p-2}.\end{aligned}$$

□

Proposition 5.5

If there exist a constant c_0 and a nonnegative real number p verifying

$$u^2(x,0) \le c_0(1+|x|^2 - u^2(x,0))^p \qquad \forall\, x \in M,$$

then

$$u^2(x,t) \le c_0(1+|x|^2 - u^2(x,t) + (2n+4(p-1))t)^p \qquad \forall\, t \ge 0,\ \forall x \in M_t.$$

Proof:

Let us consider the function

$$\eta(x,t) := 1 + |x|^2 - u^2 + (2n + 4(p-1))t.$$

We compute

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) u^2\eta^{-p} &= \eta^{-p}\left(\frac{\partial}{\partial t} - \Delta\right) u^2 + u^2\left(\frac{\partial}{\partial t} - \Delta\right)\eta^{-p} - 2\nabla(u^2)\cdot\nabla(\eta^{-p}) \\
&= -2\eta^{-p}|\nabla u|^2 - p(p+1)u^2|\nabla\eta|^2\eta^{-p-2} - 2u^2 p\eta^{-p-1}|\nabla u|^2 \\
&- 4(p-1)p\eta^{-p-1}u^2 - 4pu\eta^{-p-1}\nabla u\cdot\nabla\eta,
\end{aligned}$$

since the calculations done for η_2 also hold for η.

We observe that

$$\begin{aligned}
|4pu\eta^{-p-1}\nabla u\cdot\nabla\eta| &= |(2u\eta^{-p/2}\nabla u)\cdot(2pu\eta^{-p/2-1}\nabla\eta)| \\
&\leq 2\eta^{-p}|\nabla u|^2 + 2p^2u^2\eta^{-p-2}|\nabla\eta|^2.
\end{aligned}$$

Also,

$$\frac{\partial\eta}{\partial x^i} = 2\,\langle\frac{\partial x}{\partial x^i}\,,x\rangle - 2\,\langle\frac{\partial x}{\partial x^i}\,,\omega\rangle\,\langle x\,,\omega\rangle \tag{5.3}$$

$$= 2\,\langle\frac{\partial x}{\partial x^i}\,,x - \langle x\,,\omega\rangle\omega\rangle, \tag{5.4}$$

thus

$$
\begin{aligned}
|\nabla\eta|^2 &= g^{ii}\left(\frac{\partial\eta}{\partial x^i}\right)^2 = 4g^{ii}\left(\langle\frac{\partial x}{\partial x^i}\,, x - \langle x\,,\omega\rangle\omega\rangle\right)^2 \\
&\le 4g^{ii}g_{ii}|x - \langle x\,,w\rangle w|^2 = 4(\langle x\,,x\rangle - 2\langle x\,,w\rangle^2 + \langle x\,,w\rangle^2|w|^2) \\
&= 4(|x|^2 - u^2) \\
&\le 4\eta,
\end{aligned}
$$

since $n \ge 2$ and $p \ge 0$. Then

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)u^2\eta^{-p} &\le -2\eta^{-p}|\nabla u|^2 - p(p+1)u^2|\nabla\eta|^2\eta^{-p-2} - 2u^2p\eta^{-p-1}|\nabla u|^2 \\
&- 4p(p-1)u^2\eta^{-p-1} + 2\eta^{-p}|\nabla u|^2 + 2p^2u^2\eta^{-p-2}|\nabla\eta|^2 \\
&= p(p-1)u^2|\nabla\eta|^2\eta^{-p-2} - 2u^2p\eta^{-p-1}|\nabla u|^2 \\
&- 4p(p-1)u^2\eta^{-p-1} \\
&\le 4p(p-1)u^2\eta^{-p-1} - 2u^2p\eta^{-p-1}|\nabla u|^2 - 4p(p-1)u^2\eta^{-p-1} \\
&= -2u^2p\eta^{-p-1}|\nabla u|^2 \le 0.
\end{aligned}
$$

Now notice that

$$
\sup_{M_0} u^2\eta^{-p} \le \sup_{M_0} \frac{c_0(1+|x|^2-u^2)^p}{(1+|x|^2-u^2)^p} = c_0,
$$

so we can apply corollary 5.3 to the function $f = u^2\eta^{-p}$ and the vector field $V \equiv 0$. We conclude that

$$
\sup_{M_t} u^2\eta^{-p} \le \sup_{M_0} u^2\eta^{-p} \le c_0.
$$

Thus

$$u^2 \leq c_0 \eta^p,$$

which is the desired result.

$\square$

5.2 A priori gradient estimates.

In order to show that M_t remains a graph for all times, we want to estimate the quantity

$$\langle \nu \, , \omega \rangle$$

from below. Hence, let

$$v := \frac{1}{\langle \nu \, , \omega \rangle}.$$

We shall find a priori upper bounds for v.

Lemma 5.6

The function v satisfies the following evolution equation

$$\left(\frac{\partial}{\partial t} - \Delta \right) v = -|A|^2 v - 2v^{-1} |\nabla v|^2.$$

Proof:

We know that

$$\frac{\partial \nu}{\partial t} = \nabla H,$$

then

$$\frac{\partial v}{\partial t} = -v^2 \langle \frac{\partial \nu}{\partial t} \,, \omega \rangle = -v^2 \langle \nabla H \,, \omega \rangle.$$

To compute Δv let us consider $\{e_1, ..., e_n\}$ an orthonormal basis of T_pM in which the metric g at p is the identity. Then

$$\Delta = \sum_{i=1}^{n} \nabla_{e_i} \nabla_{e_i}.$$

To shorten notation we write $\nabla_i := \nabla_{e_i}$. Now,

$$\nabla_i v = -v^2 \langle \nabla_i \nu \,, w \rangle = -v^2 \langle h_{ij} e_j \,, w \rangle,$$

where the last equality holds because of the Weingarten equations 1.2. Then,

$$|\nabla v|^2 = \sum_{ij} v^4 (\langle h_{ij} e_j \,, \omega \rangle)^2.$$

For each $i \in \{1, ..., n\}$ we have

$$\begin{aligned} \nabla_i \nabla_i v &= \nabla_i (-v^2 \langle h_{ij} e_j \,, w \rangle) = 2v^3 (\langle h_{ij} e_j \,, \omega \rangle)^2 - v^2 \langle \nabla_i (h_{ij} e_j) \,, \omega \rangle \\ &= 2v^3 (\langle h_{ij} e_j \,, \omega \rangle)^2 - v^2 \langle \nabla_i (h_{ij}) e_j \,, \omega \rangle - v^2 \langle h_{ij} \nabla_i (e_j) \,, \omega \rangle \\ &= 2v^{-1} |\nabla v|^2 - v^2 \langle \nabla_j h_{ii} e_j \,, \omega \rangle + v^2 h_{ij} H \langle h_{ij} \nu \,, \omega \rangle. \end{aligned}$$

Therefore Δv is given by

$$\Delta v = 2v^{-1} |\nabla v|^2 - v^2 \langle \nabla H \,, \omega \rangle + |A|^2 v,$$

where we used the Codazzi equations $\nabla_k h_{ij} = \nabla_j h_{ki} = \nabla_i h_{jk}$ and that $H = \sum_{i=1}^n h_{ii}$.

Hence

$$\left(\frac{\partial}{\partial t} - \Delta\right) v = -|A|^2 v - 2v^{-1}|\nabla v|^2,$$

as claimed.

$\square$

Corollary 5.7

If v is bounded from above at $t = 0$, then it will remain bounded for all times by the same constant.

Proof:

Suppose that $v(x, 0) \leq b$, $\forall\ x \in M$. By the previous lemma we have

$$\frac{\partial v}{\partial t} = \Delta v - |A|^2 v - 2v^{-1}|\nabla v|^2.$$

Let

$$v_{max}(t) := \max_{M_t} v(x, t),$$

which exists, at least for some interval $[0, \varepsilon)$.

If the maximum on M_t is reached at a point $x^{(t)}$, then

$$\nabla v_{max}(t) = \nabla v(x^{(t)}, t) = 0$$

and

$$\Delta v_{max}(t) = \Delta v(x^{(t)}, t) \leq 0.$$

Therefore,

$$\frac{\partial}{\partial t} v_{max} \leq -|A|^2 v_{max} \leq 0,$$

possibly in the weak sense, as v_{max} may not be everywhere differentiable, but it is Lipschitz continuous. Thus, v_{max} is decreasing with respect to t. Since

$$v_{max}(0) \leq b,$$

we conclude that

$$v(x,t) \leq v_{max}(t) \leq b,$$

for all $x \in M_t$, $t > 0$ such that the flow exists.

□

Proposition 5.8

Suppose there exist a constant c_1 and a nonnegative number p such that

$$v(x,0) \leq c_1(1 + |x|^2 - u^2(x,0))^p, \qquad \forall\, x \in M_0.$$

Then, v satisfies

$$v(x,t) \leq c_1(1 + |x|^2 - u^2(x,t) + 2nt)^p \qquad \forall\, x \in M_t,$$

$\forall\, t \geq 0$ *for which the flow exists.*

<u>Proof:</u>

The argument is analogous to the one used to prove lemma 5.5

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) v\eta_2^{-p} &= -|A|^2 v\eta_2^{-p} - 2v^{-1}|\nabla v|^2 \eta_2^{-p} - p(p+1)v|\nabla \eta_2|^2 \eta^{-p-2} \\
&- 2pv\eta_2^{-p-1}|\nabla u|^2 - 2p\eta_2^{-p-1}\nabla v \cdot \nabla \eta_2
\end{aligned}
$$

and

$$
\begin{aligned}
|2p\eta_2^{-p-1}\nabla v \cdot \nabla \eta_2| &= |(2v^{-1/2}|\nabla v|\eta_2^{-p/2})(pv^{1/2}\eta_2^{-p/2-1}|\nabla \eta_2|)| \\
&\leq 2v^{-1}|\nabla v|^2 \eta_2^{-p} + \frac{1}{2}p^2 v\eta_2^{-p-2}|\nabla \eta_2|^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) v\eta_2^{-p} &\leq -|A|^2 v\eta_2^{-p} - \left(\frac{1}{2}p^2 + p\right) v|\nabla \eta_2|^2 \eta^{-p-2} - 2p\eta_2^{-p-1}|\nabla u|^2 \\
&\leq 0,
\end{aligned}
$$

as $p \geq 0$. We now apply corollary 5.3 for $f = v\eta_2^{-p}$ and $V = 0$ to conclude that

$$
\sup_{M_t} v\eta_2^{-p} \leq c_1,
$$

and the result follows.

□

5.3 Curvature estimates and longtime existence.

In this section we will show that as long as M_t remains an entire graph with bounded gradient v, the curvature remains bounded as well. In fact, the a priori bounds to the second fundamental form M_t are essential for proving longtime existence to the mean curvature flow with smooth initial data. However, in order to get the estimates and longtime existence for the flow, extra hypothesis are needed.

Assumption. We will assume that the manifold has *linear growth*, i.e. there exists a constant $c_1 \geq 1$ such that

$$v \leq c_1$$

at every point in M_0. Proposition 5.8 assures that this bound will hold for all times.

Lemma 5.9

The following differential inequality is satisfied during the evolution of an entire graph under the mean curvature flow

$$\left(\frac{\partial}{\partial t} - \Delta\right)|A|^2 v^2 \leq -2v^{-1}\nabla v \cdot \nabla(|A|^2 v^2).$$

Proof:

Using lemma 5.6, we can compute

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) v^2 &= 2v\frac{\partial}{\partial t} - 2v\Delta v - 2|\nabla v|^2 = 2v(-|A|^2 v - 2v^{-1}|\nabla v|^2) - 2|\nabla v|^2 \\ &= -2|A|^2 v^2 - 6|\nabla v|^2.\end{aligned}$$

By proposition 3.1 we know how $|A|^2$, the norm of the second fundamental form, evolves in time. Hence

$$\left(\frac{\partial}{\partial t} - \Delta\right)|A|^2 = -2|\nabla A|^2 + 2|A|^4 \leq -2|\nabla|A||^2 + 2|A|^4.$$

Therefore

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)(|A|^2 v^2) &= v^2\left(\frac{\partial}{\partial t} - \Delta\right)|A|^2 + |A|^2\left(\frac{\partial}{\partial t} - \Delta\right)v^2 - 2\nabla(|A|^2)\cdot\nabla(v^2) \\
&\leq -2|\nabla|A||^2 v^2 - 6|\nabla v|^2||A|^2 - 2\nabla(|A|^2)\cdot\nabla(v^2).
\end{aligned}$$

Now,

$$\begin{aligned}
-2\nabla(|A|^2)\cdot\nabla(v^2) &= -\nabla(|A|^2)\cdot\nabla(v^2) - 4v|A|(\nabla|A|)\nabla v \\
&= -v^{-2}\nabla(v^2)\cdot\nabla(|A|^2v^2) + v^{-2}|\nabla(v^2)|^2|A|^2 - 4v|A|(\nabla|A|)\nabla v \\
&= -2v^{-1}\nabla v\cdot\nabla(|A|^2v^2) + 4|\nabla v|^2|A|^2 - 4v|A|(\nabla|A|)\nabla v \\
&\leq -2v^{-1}\nabla v\cdot\nabla(|A|^2v^2) + 4|\nabla v||A|^2 + 2v^2|\nabla|A||^2 + 2|\nabla v|^2|A|^2 \\
&= -2v^{-1}\nabla v\cdot\nabla(|A|^2v^2) + 6|\nabla v|^2|A|^2 + 2v^2|\nabla|A||^2,
\end{aligned}$$

where we estimated $-4v|A|(\nabla|A|)\nabla v$ with the arithmetic-geometric inequality. We can use the last inequality to conclude that

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)(|A|^2 v^2) &\leq -2|\nabla|A||^2 v^2 - 6|\nabla v|^2||A|^2 - 2\nabla(|A|^2)\cdot\nabla(v^2) \\
&\leq -2v^{-1}\nabla v\cdot\nabla(|A|^2v^2).
\end{aligned}$$

□

Proposition 5.10

If M evolves by the Mean Curvature Flow with bounded gradient and bounded curvature for each M_t, then

$$\sup_{M_t} |A|^2 v^2 \leq \sup_{M_0} |A|^2 v^2.$$

Proof:

Let $f = v^2|A|^2$ and $V = -2v^{-1}\nabla v$. Note that

$$v^{-1}|\nabla v| \leq |A|v.$$

Indeed, if $\{e_1, ..., e_n\}$ is an orthonormal basis of T_pM in which the metric at p is $g_{ij} = \delta_{ij}$, then

$$\nabla_i v = -v^2 \langle \nabla_i \nu \, , w \rangle = -v^2 \langle h_{ik} e_k \, , w \rangle$$

so

$$|\nabla_i v| \leq \sum_k |v^2||h_{ik}|$$

and

$$|\nabla v|^2 = \sum_i |\nabla_i v|^2 = v^2 \sum_{i,k} h_{ik}^2 = v^2|A|^2.$$

Then, V is bounded. Hence we apply corollary 5.3 to conclude the estimate.

□

Proceeding as in [4, §13], we can get the following lemma.

Lemma 5.11

For any nonnegative integer m we have

$$\frac{\partial}{\partial t}|\nabla^m A|^2 = - 2|\nabla^{m+1}A|^2 + |\nabla^m A|^2 + C(m,n) \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A, \tag{5.5}$$

where $\nabla^i A * \nabla^j A$ denotes any linear combination of the tensors $\nabla^i A$ and $\nabla^j A$. In particular, we have

$$\frac{\partial}{\partial t}|\nabla^m A|^2 \leq - 2|\nabla^{m+1}A|^2 + |\nabla^m A|^2 + C(m,n) \sum_{i+j+k=m} |\nabla^i A|\,|\nabla^j A|\,|\nabla^k A|\,|\nabla^m A|. \tag{5.6}$$

The preceding lemma allows us to derive uniform a priori estimates for derivatives of any order of the second fundamental form.

Proposition 5.12

Let M be an entire graph evolving by the mean curvature flow. Suppose that v, $|A|^2$, $|\nabla A|^2$, ..., $|\nabla^m A|^2$ are bounded on each M_t. Then for all $t \geq 0$

$$\sup_{M_t} |\nabla^m A| \leq C,$$

where C depends on m, n, c_1 and $\sup_{M_0} |\nabla^j A|$ for $j : 0 \leq j \leq m$.

For a proof of this proposition we refer to [6, Proposition 2.3].

The following proposition provides another proof of proposition 5.12 for any time interval $[0, \varepsilon)$, $\varepsilon > 0$, by giving interior estimates. Also, it shows that, asymptotically, the graph will flatten out at infinity.

Proposition 5.13

Let M be an entire graph evolving by the mean curvature flow. Suppose that M_t satisfies a linear growth $v \leq c_1$. Then there exists a constant $C = C(m, n, c_1)$ such that

$$t^{m+1}|\nabla^m A|^2 \leq C, \tag{5.7}$$

uniformly on each M_t.

<u>Proof:</u>

We'll prove this theorem by induction on m. Let us set $m = 0$ and compute the following

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)(2t|A|^2v^2 + v^2) &= 2|A|^2v^2 + 2t\left(\frac{\partial}{\partial t} - \Delta\right)(|A|^2v^2) \\
&+ 2v\frac{\partial v}{\partial t} - 2v\Delta v - 4|\nabla v|^2 \\
&\leq 2|A|^2v^2 - 2v^{-1}\nabla v \cdot \nabla(2t|A|^2v^2) \\
&+ 2v\left(\frac{\partial}{\partial t} - \Delta\right)v - 2|\nabla v|^2 \\
&= 2|A|^2v^2 - 2v^{-1}\nabla v \cdot \nabla(2t|A|^2v^2) \\
&- 2|A|^2v^2 - 6|\nabla v|^2 \\
&\leq -2v^{-1}\nabla v \cdot \nabla(2t|A|^2v^2),
\end{aligned}$$

where we used lemmas 5.6 and 5.9. Then, we can use corollary 5.3 with

$$f := 2t|A|^2v^2 + v^2,$$

$$V := -2v^{-1}\nabla v,$$

to conclude that

$$\sup_{M_t}(2t|A|^2v^2 + v^2) \leq \sup_{M_0} v^2 \leq c_1^2.$$

By 5.11, we know that for any $l \in \mathbb{N}$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)(t^{l+1}|\nabla^l A|^2) &\leq -2t^{l+1}|\nabla^{l+1}A|^2 + (l+1)t^l|\nabla^l A|^2 \\ &+ C(l,n)\, t^{l+1} \sum_{i+j+k=l} |\nabla^i A||\nabla^j A||\nabla^k A||\nabla^l A|. \quad (5.8)\end{aligned}$$

Suppose equation 5.7 holds up to $m-1$, and let us use it to estimate the following

$$\begin{aligned} t^{l+1} \sum_{i+j+k=l} & |\nabla^i A||\nabla^j A||\nabla^k A||\nabla^l A| \\ \leq & \; t^{l+1} \sum_{i+j+k=l} \sqrt{C(i)C(j)}t^{-i/2-j/2-1}|\nabla^k A||\nabla^l A| \\ \leq & \; C_1\, t^l \sum_{k\leq l} t^{k/2-l/2}|\nabla^k A||\nabla^l A| \\ = & \; C_1\, t^{l/2}|\nabla^l A| \sum_{k\leq l} t^{k/2}|\nabla^k A| \\ \leq & \; C_2 \sum_{k\leq l} t^k|\nabla^k A|^2. \quad (5.9)\end{aligned}$$

Putting together equations 5.8 and 5.9 we get that for all $l \leq m$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)(t^{l+1}|\nabla^l A|^2) &\leq -2t^{l+1}|\nabla^{l+1}A|^2 + (l+1)t^l|\nabla^l A|^2 + C_2 \sum_{k\leq l} t^{k/2}|t^k|\nabla^k A|^2 \\ &\leq -2t^{l+1}|\nabla^{l+1}A|^2 + C_3 \sum_{k\leq l} t^{k/2}|\nabla^k A|^2.\end{aligned}$$

Note that we can choose a constant k_1 such that

$$
\begin{aligned}
(\frac{\partial}{\partial t} &- \Delta)(t^{m+1}|\nabla^m A|^2 + k_1\, t^m|\nabla^{m-1}A|^2) = \\
&- 2t^{m+1}|\nabla^{m+1}A|^2 + C_3\sum_{k\leq m} t^{k/2}|\nabla^k A|^2 \\
&- 2k_1 t^m|\nabla^m A|^2 + k_1 C_3 \sum_{k\leq m-1} t^{k/2}|\nabla^k A|^2 \\
&\leq C_3\sum_{k\leq m} t^{k/2}|\nabla^k A|^2 - 2k_1 t^m|\nabla^m A|^2 + k_1 C_3 \sum_{k\leq m-1} t^{k/2}|\nabla^k A|^2 \\
&\leq C_3\sum_{k\leq m-1} t^{k/2}|\nabla^k A|^2 + k_1 C_3 \sum_{k\leq m-1} t^{k/2}|\nabla^k A|^2 \\
&\leq C_4\sum_{k\leq m-1} t^{k/2}|\nabla^k A|^2.
\end{aligned}
$$

Analogously we can get constants $k_2, ..., k_{m+1}$ such that

$$
\begin{aligned}
(\frac{\partial}{\partial t} &- \Delta)(t^{m+1}|\nabla^m A|^2 + k_1\, t^m|\nabla^{m-1}A|^2 + k_2\, t^{m-1}|\nabla^{m-2}A|^2 + ... \\
&+ ... + k_{m-1}\, t^2|\nabla A|^2 + k_m\, t|A|^2 + k_{m+1}v^2) \\
&\leq 0.
\end{aligned}
$$

Then, using once more corollary 5.3 with $V \equiv 0$ we get

$$
\begin{aligned}
t^{m+1}|\nabla^m A|^2 &+ k_1\, t^m|\nabla^{m-1}A|^2 + k_2\, t^{m-1}|\nabla^{m-2}A|^2 + ... \\
&+ ... + k_{m-1}\, t^2|\nabla A|^2 + k_m\, t|A|^2 + k_{m+1}v^2
\end{aligned}
$$

is uniformly bounded. Since the last $m+1$ terms are bounded by the induction hypothesis, we get that

$$
t^{m+1}|\nabla^m A|^2
$$

is uniformly bounded.

□

Theorem 5.14 (Longtime Existence)

Let M be a smooth entire graph with linear growth. Then, the Mean Curvature flow with initial data $M_0 = M$ has smooth solution for all $t \geq 0$.

Proof:

Let $[0,T)$ be the maximal interval of existence of the mean curvature flow with initial condition M. Suppose $T < +\infty$. By corollary 5.7 we know that M_t remains an entire graph for all $t \in [0,T)$ and so it is

$$\lim_{t \to T^-} M_t.$$

By corollary 5.10, we also know that the norm of the second fundamental form $|A|^2$ is bounded by initial conditions. Therefore H, the trace of the second fundamental form also remains bounded for all $t \in [0,T)$.

Then, Proposition 5.13 gives us bounds for all derivatives of the second fundamental form, so we can conclude that

$$M_T := \lim_{t \to T^-} M_t$$

is also a smooth entire graph with bounded Mean Curvature. Hence we can restart the flow with initial condition M_T, and will a priori defined on an interval $[T,T')$. Therefore the initial flow is defined on $[0,T')$ with $T' > T$, contradicting that $[0,T)$ was a maximal time interval. Thus, $T = +\infty$.

□

Appendix A

Generalities on Parabolic Equations

The well-known theory of quasi-linear parabolic equations is going to aid us when proving the existence and uniqueness of flows. In this section we recall the definitions and results which we will need. This section is an extract of [7].

Definition A.1.

Let $\Omega \subset \mathbb{R}^n$ be an open set. Let us consider the following *Linear second-order* differential equation on $\Omega \times (0, T)$:

$$L(x, t, (\partial_i)_i, (\partial_{ij})_{ij}, \frac{\partial}{\partial t})u = u_t - a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + a_i(x,t)\frac{\partial u}{\partial x_i} + a(x,t)u = f(x,t) \quad \text{(A.1)}$$

Where $u = u(x,t)$ and $f = f(x,t)$ are functions on $\Omega \times (0, T)$.

The operator L defined in A.1 is said to be *Uniformly Parabolic* in $\Omega \times (0,T)$ if there exist positive real numbers ν, μ such that

$$\nu|\xi|^2 \leq a_{ij}(x,t)\xi_i\xi_j \leq \mu|\xi|^2,$$

$\forall\, 1 \leq i,\, j \leq n$, $\forall\,(x,t) \in \Omega \times (0,T)$, $\forall \xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$.

Definition A.2.

Let us denote by Q a *quasi-linear second order* differential operator in $\Omega \subset \mathbb{R}^n$. Explicitly,

$$\begin{aligned} Q(x,t,(\partial_i)_i,(\partial_{ij})_{ij},\frac{\partial}{\partial t})u = \\ = \frac{\partial u}{\partial t} - a_{ij}(x,t,u,u_x)\frac{\partial^2 u}{\partial x_i \partial x_j} + a(x,t,u,u_x) \end{aligned} \tag{A.2}$$

The quasi-linear operator Q is said to be *Uniformly Parabolic* if there exist real positive non-increasing continuous functions ν, μ of τ, $\tau \geq 0$, such that

$$\nu(|u|)|\xi|^2 \leq a_{ij}(x,t,u,w)\xi_i\xi_j \leq \mu(|u|)|\xi|^2,$$

$\forall\, 1 \leq i,\, j \leq n$, $\forall\,(x,t) \in \Omega \times (0,T)$, $\forall\, w = (w_1, ..., w_n)$, $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$, $\forall\, u \in \mathbb{R}$.

Remark A.1.

A special type of quasilinear equations are those with principal part in divergence form. This is an equation of the form

$$Q(x,t,(\partial_i)_i,(\partial_{ij})_{ij},\frac{\partial}{\partial t})u = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x^i}a_i(x,t,u,u_x) + a(x,t,u,u_x). \tag{A.3}$$

The equation A.3 can also be written as

$$Q(x,t,(\partial_i)_i,(\partial_{ij})_{ij},\frac{\partial}{\partial t})u = \frac{\partial u}{\partial t} - \frac{\partial a_i(x,t,u,u_x)}{\partial u_{x^j}} u_{x^i} u_{x^j} + A(x,t,u,u_x), \qquad \text{(A.4)}$$

where

$$A(x,t,u,u_x) = a(x,t,u,u_x) - \frac{\partial a_i(x,t,u,u_x)}{\partial u} u_{x^i} - \frac{\partial a_i(x,t,u,u_x)}{\partial x^i}.$$

A.1 Hölder Spaces.

Let $\Omega \subset \mathbb{R}^{n+1}$ be an bounded and connected open set and let us denote $\Omega^T := \Omega \times (0,\, T)$.

Let u a bounded and continuous function on $\overline{\Omega}$. For $\gamma \in (0,\, 1)$ we define the γ-th Hölder seminorm by

$$[u]_{\Omega}^{\gamma} = \sup_{x,y\in\Omega,\, x\neq y} \frac{u(x)-u(y)}{|x-y|^{\gamma}}. \qquad \text{(A.5)}$$

A function is said to be γ-Hölder continuous is the supremum in A.5 is finite.
For $l \in \mathbb{R}$, define the norm

$$\|u\|_{\Omega}^{l} = \sum_{|\alpha|\leq[l]} \|D^{\alpha}u\|_{\infty} + \sum_{\alpha=[l]} [D^{\alpha}u]_{\Omega}^{l-[l]}, \qquad \text{(A.6)}$$

where

$$\|u\|_{\infty} = \sup_{x\in\Omega} |u(x)|.$$

The Hölder space $H^r(\overline{\Omega})$ is given by

$$\{u \in C^0(\Omega) \mid D^{\alpha}u \text{ exists and is continuous in } \overline{\Omega},\ \forall\ 1 \leq |\alpha| \leq [l],\ \text{and } \|u\|_{\Omega}^{l} < \infty\}.$$

Thus, the space $H^l(\overline{\Omega})$ consists of those functions u that are $[l]$-times continuously differentiable on $\overline{\Omega}$ and whose derivatives of order $[l]$ are Holder continuous with exponent $\gamma = l - [l]$.

Analogously we can define Hölder spaces $H^{l,l/2}(\overline{\Omega^T})$. They consist on continuous functions u on $\overline{\Omega^T}$ having all the derivatives of the form $D_t^\alpha D_x^\beta$ with $2\alpha + \beta \leq l$ and having a finite norm

$$\|u\|_\Omega^l = \sum_{2|\alpha|+|\beta|\leq[l]} \|D_t^\alpha D_x^\beta u\|_\infty + \sum_{2\alpha+\beta=[l]} [D_t^\alpha D_x^\beta u]_{x,\Omega_T}^{l-[l]} + \sum_{0<l-2\alpha+\beta<2} [D_t^\alpha D_x^\beta u]_{t,\Omega_T}^{(l-2\alpha+\beta)/2},$$

where

$$[u]_{x,\Omega_T}^\gamma = \sup_{(x,t),(y,t)\in\Omega_T,\, x\neq y} \frac{u(x,t)-u(y,t)}{|x-y|^\gamma}.$$

$$[u]_{t,\Omega_T}^\gamma = \sup_{(x,t),(x,\tau)\in\Omega_T,\, t\neq\tau} \frac{u(x,t)-u(y,t)}{|t-\tau|^\gamma}.$$

A.2 The Cauchy Problem

Theorem A.1

Let $L(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t})$ be a parabolic operator on $\mathbb{R}^n \times (0,\, 1)$ with coefficients in $H^{\alpha,\alpha/2}(\overline{\Omega^T})$ with $\alpha < 1$. Then, there exists a fundamental solution for the equation $L \equiv 0$. This is, a bounded function

$$Z : \mathbb{R}^n \times \mathbb{R}^n \times (0,\, T) \times (0,\, T) \longrightarrow \mathbb{R}$$

such that

$$L(x,t,(\partial_i)_i,(\partial_{ij})_{ij},\frac{\partial}{\partial t})Z(x,\xi,t,\tau) = \delta(x-\xi)\delta(t-\tau)$$

where $\delta(x-y)=1$ if $x=y$ and $\delta(x-y)=0$ if $x \neq y$.

Corollary A.2

There exist a unique solution to the Cauchy problem

$$\begin{cases} Lu(x,t) = f(x,t) & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = \varphi(x) & \text{in } \mathbb{R}^n, \end{cases}$$

where f is a Hölder continuous function on $\mathbb{R}^n \times (0,T)$ and $\varphi(x)$ is a continuous function on $\mathbb{R}^n$.

Theorem A.3

Let Q be a quasilinear operator with principal part in divergence form on $\mathbb{R}^n \times [0,T]$. Suppose that on any $\Omega \subset \mathbb{R}^n$ bounded

1. *For $(x,t) \in \overline{\Omega_T}$ there exist positive constants b_1, b_2 independent of the dimension n of $\mathbb{R}^n$ such that*

   $$\frac{\partial a_i(x,t,u,p)}{\partial p_j}\xi_i\xi_j \geq 0, \qquad \forall\, x,\ u,\ p,\ \text{and } t \in (0,T]$$

   and

   $$A(x,t,u,0)u \geq -b_1u^2 - b_2$$

   or

   $$A(x,t,u,0)u \geq -\Phi(|u|)|u| - b_2, \qquad \int_0^\infty \frac{d\tau}{\Phi(\tau)} = \infty \qquad \Phi > 0.$$

   Here Φ is such that if $v = \Phi(w,\tau)$ then v is a solution to the following

$$v_t - \left[\tau \frac{\partial a_i(x,t,w,w_x)}{\partial w_{x^j}} + (1-\tau)\delta_i^j\right] v_{x^i x^j} + \tau A(x,t,w,w_x) - (1-\tau)(\psi_t - \Delta\psi)$$

$$v|_{\Gamma_T} = \psi|_{\Gamma_T} \qquad 0 \le \tau \le 1.$$

Then if u^τ *satisfies*

$$u^\tau = \Phi(u^\tau, \tau)$$

Then there exist constants M, M_1 *such that*

$$\max_{\overline{\Omega_T}} |u^\tau| \le M \qquad \tau \in [0,1]. \tag{A.7}$$

$$\max_{\overline{\Omega_T}} |u^\tau_x| \le M_1 \qquad \tau \in [0,1]. \tag{A.8}$$

2. *Let* $(x,t) \in \overline{\Omega_T}$ *and* u *such that* $|u| \le M$, *where the bound* M *comes from equation A.7. For arbitrary* p *the functions* $a_i(x,t,u,p)$ *and* $a(x,t,u,p)$ *are continuous, the* $a_i(x,t,u,p)$ *are differentiable with respect to* x, u *and* p, *and that the following inequalities are satisfied*

$$\nu\xi^2 \le \frac{\partial a_i(x,t,u,p)}{\partial p_j}\xi_i\xi_j \le \mu\xi^2, \qquad \nu > 0,$$

$$\sum_{i=1}^{n}\left(|a_i| + \left|\frac{\partial a_i}{\partial u}\right|\right)(1+|p|) + \sum_{i,j=1}^{n}\left|\frac{\partial a_i}{\partial x^j}\right| + |a| \le \mu(1+|p|)^2.$$

3. *For* $(x,t) \in \overline{\Omega_T}$ $|u| \le M$ *and* $|p| \le M_1$, *where the bounds come from equations A.7 and A.8 respectively. The functions* a_i, a, $\frac{\partial a_i}{\partial p_j}$, $\frac{\partial a_i}{\partial u}$ *and* $\frac{\partial a_i}{\partial x^i}$ *are continuous*

and satisfy a Hölder condition in x, t, u and p with exponents β, $\beta/2$, β and β respectively.

Then, there exists a solution to the Cauchy problem

$$\begin{cases} Qu(x,t) = 0 & \text{in } \mathbb{R}^n \times [0,T], \\ u(x,0) = \psi_0(x) & \text{in } \mathbb{R}^n, \end{cases}$$

where $\psi_0 \in H^{2+\beta}(\Omega)$ and is such that $\max_{\mathbb{R}^n} |\psi_0(x)| < \infty$.

Moreover, for every $\Omega \subset \mathbb{R}^n$ the solutions belong to $H^{2+\beta,\,1+\beta/2}(\overline{\Omega_T})$.

Theorem A.4

Let Q be a quasilinear operator with principal part in divergence form on $\mathbb{R}^n \times [0,T]$. Suppose that all the hypothesis in theorem A.3 are verified. Suppose as well that the functions $a_{ij}(x,t,u,p)$ and $A(x,t,u,p)$ are differentiable with respect to u and p and verify

$$\max_{(x,t)\in\mathbb{R}^n\times[0,T],\,|u,p|\le N} \left| \frac{\partial a_{ij}(x,t,u,p)}{\partial u}, \frac{\partial a_{ij}(x,t,u,p)}{\partial p}, \frac{\partial A(x,t,u,p)}{\partial p} \right| \le \mu_1(N),$$

$$\min_{(x,t)\in\mathbb{R}^n\times[0,T],\,|u,p|\le N} \frac{\partial A(x,t,u,p)}{\partial u} \ge -\mu_2(N),$$

for any N and constants μ_1 and μ_2 depending on N. Then, there exists at most one bounded function $u(x,t)$ with bounded first and second derivatives which is a solution to the Cauchy problem

$$\begin{cases} Qu(x,t) = 0 & \text{in } \mathbb{R}^n \times [0,T], \\ u(x,0) = \psi_0(x) & \text{in } \mathbb{R}^n, \end{cases}$$

where $\psi_0 \in H^{2+\beta}(\Omega)$ *and is such that* $\max_{\mathbb{R}^n} |\psi_0(x)| < \infty$.

Appendix B

The Linearization of the MCF equation

Let M be a connected n-dimensional manifold embedded into $\mathbb{R}^{n+1}$ via the map

$$X : M \longrightarrow \mathbb{R}^{n+1}.$$

As before, let us denote by ν the outer unit normal vector field of the hypersurface $X(M)$.

Let be Y be a fixed vector field in $\mathbb{R}^{n+1}$. For $s > 0$ small real number let

$$X^s = X + sY$$

and denote by

$$\phi = \langle Y\,, \nu \rangle.$$

Assume that X^s satisfies the mean curvature flow equation, i.e

$$\frac{\partial X^s}{\partial t} = -H^s \nu^s.$$

We will seek for the evolution satisfied by ϕ. Note that

$$\phi_t = \langle Y_t \,, \nu\rangle + \langle Y \,, \nu_t\rangle,$$

$$\frac{\partial \phi}{\partial x^i} = \langle \frac{\partial Y}{\partial x^i} \,, \nu\rangle + \langle Y \,, \frac{\partial \nu}{\partial x^i}\rangle,$$

$$\begin{aligned} \Delta\phi &= \langle \Delta Y \,, \nu\rangle + 2g^{ij}\,\langle \frac{\partial Y}{\partial x^i} \,, \frac{\partial \nu}{\partial x^j}\rangle + \langle Y \,, \Delta\nu\rangle \\ &= \langle \Delta Y \,, \nu\rangle + 2g^{ij}h_{jl}g^{lm}\,\langle \frac{\partial Y}{\partial x^i} \,, \frac{\partial X}{\partial x^m}\rangle + \langle Y \,, \Delta\nu\rangle, \end{aligned} \qquad \text{(B.1)}$$

and

$$\begin{aligned} g^s_{ij} &= \langle \frac{\partial X^s}{\partial x^i} \,, \frac{\partial X^s}{\partial x^j}\rangle \\ &= \langle \frac{\partial X}{\partial x^i} \,, \frac{\partial X}{\partial x^j}\rangle + s\,\langle \frac{\partial X}{\partial x^i} \,, \frac{\partial Y}{\partial x^j}\rangle + s\,\langle \frac{\partial Y}{\partial x^i} \,, \frac{\partial X}{\partial x^j}\rangle + s^2\,\langle \frac{\partial Y}{\partial x^i} \,, \frac{\partial Y}{\partial x^j}\rangle. \end{aligned}$$

Therefore

$$\delta g_{ij} = \frac{\partial}{\partial s}(g^s_{ij})\Big|_{s=0} = \langle \frac{\partial X}{\partial x^i} \,, \frac{\partial Y}{\partial x^j}\rangle + \langle \frac{\partial Y}{\partial x^i} \,, \frac{\partial X}{\partial x^j}\rangle$$

and equation B.1 rewrites as

$$\Delta\phi = \langle \Delta Y \,, \nu\rangle + g^{ij}h_{jl}g^{lm}\delta g_{im} + \langle Y \,, \Delta\nu\rangle.$$

Now, $g^{ik}g_{kj} = \delta^i_j$ so

$$(\delta g^{ik})g_{kj} + g^{ik}\delta g_{kj} = 0$$

$$(\delta g^{ik})\delta_k^l + g^{ik}(\delta g_{kj})g^{jl} = 0$$

Thus

$$\delta g^{il} = -g^{ik}(\delta g_{kj})g^{jl}. \tag{B.2}$$

Let ν^s be the outward normal vector to the s-hypersurface, whose position vector is X^s. Since $\langle \nu^s , \nu^s \rangle = 1$,

$$\begin{aligned} 0 &= \delta \langle \nu^s , \nu^s \rangle = \frac{\partial}{\partial s} \langle \nu^s , \nu^s \rangle \Big|_{s=0} \\ &= 2 \langle \frac{\partial}{\partial s} \nu^s , \nu^s \rangle \Big|_{s=0} = 2 \langle \delta\nu , \nu \rangle. \end{aligned}$$

Therefore $\delta\nu$ has no normal components and can be written as

$$\delta\nu = (\delta\nu)_j \frac{\partial X}{\partial x^j}.$$

Now, $0 = \langle \nu^s , \frac{\partial X^s}{\partial x^i} \rangle$ so

$$\begin{aligned} 0 &= \langle \frac{\partial \nu^s}{\partial s} , \frac{\partial X^s}{\partial x^i} \rangle + \langle \nu^s , \frac{\partial}{\partial s} \frac{\partial X^s}{\partial x^i} \rangle \\ &= \langle \frac{\partial \nu^s}{\partial s} , \frac{\partial X^s}{\partial x^i} \rangle + \langle \nu^s , \frac{\partial Y}{\partial x^i} \rangle \end{aligned}$$

and, evaluating at $s = 0$, we obtain

$$\begin{aligned} 0 &= \langle \delta\nu , \frac{\partial X}{\partial x^i} \rangle + \langle \nu , \frac{\partial Y}{\partial x^i} \rangle \\ &= (\delta\nu)_j g_{ij} + \langle \nu , \frac{\partial Y}{\partial x^i} \rangle \end{aligned}$$

thus

$$(\delta\nu)_j = -g^{ij} \langle \frac{\partial Y}{\partial x^i}, \nu \rangle. \tag{B.3}$$

On the other hand,

$$\begin{aligned} \delta h_{ij} &= -\langle \frac{\partial^2 Y}{\partial x^i \partial x^j}, \nu \rangle - \langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \delta\nu \rangle \\ &= -\langle \frac{\partial^2 Y}{\partial x^i \partial x^j}, \nu \rangle + g^{kl} \langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \frac{\partial X}{\partial x^k} \rangle \langle \frac{\partial Y}{\partial x^l}, \nu \rangle \end{aligned}$$

Since

$$\delta H = (\delta h_{ij}) g^{ij} + h_{ij} \delta g^{ij}$$

we get

$$\begin{aligned} \delta H &= -g^{ij} \langle \frac{\partial^2 Y}{\partial x^i \partial x^j}, \nu \rangle + g^{ij} g^{kl} \langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \frac{\partial X}{\partial x^k} \rangle \langle \frac{\partial Y}{\partial x^l}, \nu \rangle - h_{ij} g^{ik} (\delta g_{kl}) g^{lj} \\ &= -g^{ij} \langle \frac{\partial^2 Y}{\partial x^i \partial x^j}, \nu \rangle - g^{ij} g^{kl} g_{mk} \Gamma^m_{ij} \langle \frac{\partial Y}{\partial x^l}, \nu \rangle - h_{ij} g^{ik} (\delta g_{kl}) g^{lj} \\ &= -g^{ij} \langle \frac{\partial^2 Y}{\partial x^i \partial x^j}, \nu \rangle - g^{ij} \Gamma^l_{ij} \langle \frac{\partial Y}{\partial x^l}, \nu \rangle - h_{ij} g^{ik} (\delta g_{kl}) g^{lj} \\ &= -\langle \Delta Y, \nu \rangle - h_{ij} g^{ik} (\delta g_{kl}) g^{lj} \end{aligned}$$

and

$$\delta H = -\langle \Delta Y, \nu \rangle - h_{ij} g^{ik} (\delta g_{kl}) g^{lj} \tag{B.4}$$

Let $B := h_{ij} g^{ik} (\delta g_{kl}) g^{lj}$.

Since

$$Y_t = -\delta H\nu - H\delta\nu,$$

then

$$\langle Y_t \,, \nu\rangle = -\langle \delta H\nu \,, \nu\rangle - H\,\langle \delta\nu \,, \nu\rangle = -\delta H.$$

Now we compute

$$\begin{aligned}
\Delta\phi - \phi_t &= \langle \Delta Y \,, \nu\rangle + B + \langle Y \,, \Delta\nu\rangle - \langle Y_t \,, \nu\rangle - \langle Y \,, \nu_t\rangle \\
&= \langle \Delta Y \,, \nu\rangle + B + \langle Y \,, \Delta\nu\rangle + \delta H - \langle Y \,, \nabla H\rangle \\
&= \langle \Delta Y \,, \nu\rangle + B + \langle Y \,, \Delta\nu\rangle - \langle \Delta Y \,, \nu\rangle - B - \langle Y \,, \nabla H\rangle \\
&= \langle Y \,, \Delta\nu - \nabla H\rangle,
\end{aligned}$$

so

$$\phi_t = \Delta\phi + \langle Y \,, \nabla H - \Delta\nu\rangle. \tag{B.5}$$

Lemma B.1

$$\Delta\nu = \nabla H - |A|^2\nu,$$

Proof:

Let us consider local coordinates around $p \in M$ such that the metric g at p is the identity. We compute

$$\langle \frac{\partial\nu}{\partial x^i} \,, \frac{\partial X}{\partial x^j}\rangle = h_{ik}g_{kj} = h_{ij}.$$

Then, differentiating on both sides we get

$$\begin{aligned}
\frac{\partial}{\partial x^k} h_{ij} &= \langle \frac{\partial^2 \nu}{\partial x^i \partial x^k}, \frac{\partial X}{\partial x^j} \rangle + \langle \frac{\partial \nu}{\partial x^i}, \frac{\partial^2 X}{\partial x^j \partial x^k} \rangle \\
&= \langle \frac{\partial^2 \nu}{\partial x^i \partial x^k}, \frac{\partial X}{\partial x^j} \rangle + \langle \frac{\partial \nu}{\partial x^i}, -h_{jk}\nu \rangle \\
&= \langle \frac{\partial^2 \nu}{\partial x^i \partial x^k}, \frac{\partial X}{\partial x^j} \rangle.
\end{aligned}$$

So the tangent component of $\Delta\nu$ in the direction of $\frac{\partial X}{\partial x^j}$ is

$$\langle \frac{\partial^2 \nu}{\partial x^i \partial x^i}, \frac{\partial X}{\partial x^j} \rangle = \frac{\partial}{\partial x^i} h_{ij} = \frac{\partial}{\partial x^j} h_{ii} = \frac{\partial H}{\partial x^j} = \langle \nabla H, \frac{\partial X}{\partial x^j} \rangle.$$

Now, since

$$\langle \frac{\partial \nu}{\partial x^k}, \nu \rangle = 0,$$

it follows that

$$\langle \frac{\partial^2 \nu}{\partial x^k \partial x^k}, \nu \rangle = -\langle \frac{\partial \nu}{\partial x^k}, \frac{\partial \nu}{\partial x^k} \rangle = -\langle h_{kl} \frac{\partial X}{\partial x^l}, h_{km} \frac{\partial X}{\partial x^m} \rangle = -|A|^2,$$

and the lemma is proven.

□

Therefore we conclude that the linearization of the MCF is

$$\phi_t = \Delta\phi + |A|^2\phi. \tag{B.6}$$

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