The Range Time for Jump Diffusion with Two-sided Exponential Jumps

Di Xu

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By: **Di Xu**

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Signed by the final examining committee:

Jose Garrido ______________________ Examiner

Cody Hyndman ______________________ Examiner

Xiaowen Zhou ______________________ Thesis Supervisor

Approved by Jose Garrido ______________________
Chair of Department or Graduate Program Director

__________2011_____ Brian Lewis ______________________
Dean of Faculty

ABSTRACT

The range time for jump diffusion with two-sided exponential jumps

Di Xu

The first passage time of a stochastic process with respect to set A is the time until the stochastic process first enters A. In this thesis, we study the first passage times for a doubly exponential jump diffusion process, which consists of a continuous part given by a Brownian motion with drift and a jump part given by a compound Poisson process with jump sizes following a double exponential distribution. The Laplace transform of the two-sided first passage time for a set $[-a, b]^c$, $a, b > 0$ is found in this thesis.

The range time is another topic that has been studied in several papers, especially for Brownian motion. The first range time is the first time when the range of a stochastic process reaches a given level. In this thesis, by using solutions of the two-sided exit problem, we find the Laplace transform for the first range time and the distribution of the value of the process at the first range time.

Chapter 1

Introduction

The range process $\{R_t; t \geq 0\}$ of a stochastic process $\{X_t; t \geq 0\}$ started at 0, is defined by

$$R_t = \overline{X}_t - \underline{X}_t, \qquad t \geq 0,$$

where $\overline{X}_t := \sup_{s \leq t} X_s$ and $\underline{X}_t := \inf_{s \leq t} X_s$. As a result, the proceess $\{R_t; t \geq 0\}$ is increasing and vanishes at $t = 0$. We denote its inverse, which is called the first range time for a certain range $r > 0$, by

$$T = T_r = \inf\{t \geq 0 : R_t \geq r\}.$$

Note that $P\{T < \infty\} = 1$ for most interesting Markov process. Intuitively, T is the first time when the range of the process X reaches r. The range interval at time T is $[\underline{X}_T, \overline{X}_T]$. Sometimes, the range time is also called cover time. One boundary of the range interval is visited at time T, and the other boundary is first visited at some time $S < T$, where

$$S = \inf\{s \leq T : X_s = \overline{X}_T + \underline{X}_T - X_T\} \qquad .$$

Throughout this thesis, the above notation will be preserved.

Let $\{X_k\}$ be a sequence of mutually independent random variables with expectation 0 and variance 1. Define $\mathrm{S}_n = \sum_{i=1}^{n} X_i$, $M_n = \max_{1 \le i \le n} S_i$, $m_n = \min_{1 \le i \le n} S_i$. The random variable $R_n = M_n - m_n$ is called the range of the cumulative sum S_n. For the Bachelier-Wiener process, Feller (1951) shows that the density function of the range R_n is given by

$$\delta(n; r) = 8 \sum_{k=1}^{\infty} (-1)^{k-1} k^2 \phi(\frac{kr}{\sqrt{t}}),$$

where $\phi(x)$ stands for the normal density function with zero mean and unit variance. To derive this, Feller (1951) starts from the density function for the event that $\{S_n = x, M_n \le v, m_n \ge -u\}$ and obtains the density function $f(n; u, v)$ for the event $\{M_n \le v, m_n \ge -u\}$. The density function of R_n is obtained by $\delta(n; r) = \int_0^r f(n;\ u,\ r-u)du$. To eliminate the trend when $E(X_k) \ne 0$, Feller (1951) replaces the random variables S_k by

$$S_k^* = S_k - \frac{kS_n}{n}, \quad (k = 1, 2, ..., n),$$

and defines the corresponding variables M_n^*, m_n^* and R_n^* by analogy. The adjusted range has a greater sampling stability and it can be used even when the means do not vanish. The density function of the range R_n^* is given by

$$\delta(n; r) = re''(r) + \sum_{k=2}^{\infty} \{2k(k-1)[e'((k-1)r) - e'(kr)] + (k-1)^2 re''((k-1)r) + k^2 re''(kr)\},$$

where $e(x) = \exp(-2x^2/t)$.

In Imhof (1985), the process $\{X_t\}$ stands for either the standard Brownian motion or the three dimensional Bessel process. Define the first hitting time at level y as $\tau(y) = \inf\{t, X(t) = y\}$. The density function for the range time T_r is shown to be

$$P(T_r \in dt) = 2(\partial/\partial r) Q_t(r/2, r/2, r)dt, \qquad t > 0,$$

where $Q_t(x,y,z)dy = P_x(X(t) \in dy, \tau(0) \wedge \tau(z) > t)$, $0 < x \wedge y \leq x \vee y < z$. Furthermore, the density for $T_{r_1} - T_r$ can be obtained by conditioning for instance on $X(T_r) > 0$, for $0 < r < r_1$, as follows,

$$P(T_{r_1} - T_r \in dt) = (\partial/\partial r_1 \{Q_t((r_1-r)/2, (r_1-r)/2, r_1) + Q_t((r_1-r)/2, (r_1+r)/2, r_1)\}dt.$$

Imhof (1985) also shows that for Brownian motion $\{X_t\}$, the process $\{|X(S+t) - X(S)|, 0 \leq t \leq T - S\}$ is a three-dimensional Bessel process up to time $\tau(r)$.

In Vallois (1995), X_t considers a $\mathbb{R}$-valued Brownian motion. In the paper the Brownian trajectory is decomposed from extremes, via the inverse of the range process. The main result is an intrinsic decomposition which takes its values in some subset $\mathbb{U}$ of $C([0,\infty), \mathbb{R}_+)$ and indexed by $\Sigma = \{t \leq 0; X(T)X(S) < 0\}$ where $C([0,\infty), \mathbb{R}_+)$ denotes the set of all continuous functions. By splitting the initial point process into two independent $\mathbb{U}$-valued compound Poisson processes, it also gives a new proof of the last result in Imhof (1985). Further, the author considers the process $\{M_t\}$ defined by $M_t = \sqrt{2} X_{T_{\sqrt{t}}}$, where M is a square integrable martingale. M is connected to parabolic martingales, which allows to show that M has a chaotic property representation.

As in Vallois (1996), let $\{X_n\}$ denotes a Bernoulli random walk started at 0 with $P(X_{n+1} - X_n = 1) = p$ and $P(X_{n+1} - X_n = -1) = q$, $p + q = 1$. The generating function of the first range time T_r is denoted by $G(r)$ (i.e. $G(r) = E(r^{T_r})$), for every r in $[-1, 1]$,

$$G(r) = \frac{N(r)}{D(r)},$$

where

$$
\begin{aligned}
&c = q/p,\\
&\phi(r) = 1 - 4r^2pq,\\
&N(r) = \sqrt{\phi}\{2\sqrt{\phi} - r[q(1+c^n)(\alpha_1^{n+1} - \alpha_2^{n+1}) - (p+qc^n)(\alpha_1^n - \alpha_2^n)]\},\\
&D(r) = (r-1)[qrc^n(\alpha_1^{2n+1} + \alpha_2^{2n+1}) - 1].
\end{aligned}
$$

It is shown that the generating function is a rational function, which allows to compute the distributions for T_r and R_r. Moreover, by letting $n \to \infty$, the asymptotic behavior of T_n is investigated. At the end of this paper, two explicit results are proved for non-symmetric case $p \neq q$: T_n/n converges in probability to $1/|p-q|$ and $(T_n - n/|p-q|)/\sqrt{n})$ converges in distribution to normal distribution $\mathcal{N}(0, 4pq/|p-q|^3)$.

Based on the results in Vallois (1996), Chong et al. (2000) present a result on cover times for the asymmetric random walk and also random variables connected with the coverage of a Brownian motion. A simple asymmetric random walk on the integer points $\mathbb{Z}$ is stopped when the range first reaches a given magnitude m. The number of steps until this happens is called the cover time. Let S_n be the position of the random walk at time n. Suppose that the simple random walk with $S_0 = 0$ is stopped when its range is m. It is stopped at step

$$N = \min\{n : \max_{0 \le i \le n} S_i - \min_{0 \le i \le n} S_i = m\}.$$

The coverage of the walk is an interval consisting of $m+1$ consecutive integers with S_N being the point where the walk has reached a range of m. The other end point of the coverage interval is first visited at some step $M < N$, whence

$$S_M = 1_{(S_N < 0)}(S_N + m) + 1_{(S_N > 0)}(S_N - m).$$

The following theorems are about the results for a simple random walk.

Theorem 1.1. *(Chong et al. (2000)) With* $\eta(s) := \operatorname{arccos}h(1/(s\sqrt{4pq}))$ *and* $\varsigma(z) := \ln\sqrt{p/q} + \ln z$*, we have, for* $z > 0$ *and* $x, y \neq 0$*,*

$$\begin{aligned}
&E(x^M y^{N-M} z^{S_N}) \\
&= \Bigg(\frac{s[(m/2)\{\eta(x)+\varsigma(x)\}]s[((m+1)/2)\{\eta(x)+\varsigma(x)\}]}{s[\frac{1}{2}\{\eta(x)+\varsigma(x)\}]s[m\eta(x)]} \\
&\quad + \frac{s[(m/2)\{\eta(x)-\varsigma(x)\}]s[((m+1)/2)\{\eta(x)-\varsigma(x)\}]}{s[\frac{1}{2}\{\eta(x)-\varsigma(x)\}]s[m\eta(x)]} \Bigg) \frac{s[\eta(y)]}{s[(m+1)\eta(y)]} \\
&= \Bigg(\frac{s[(m+1)\eta(x)]c[m\varsigma(z)] - s[m\eta(x)]c[(m+1)\varsigma(z)] - s[\eta(x)]}{\{c[\eta(x)] - c[\varsigma(z)]\}s[m\eta(x)]s[(m+1)\eta(y)]} \Bigg) s[\eta(y)].
\end{aligned}$$

where $s[x] := \sinh(x) = \frac{1}{2}(e^x - e^{-x})$ and $c[x] := \cosh(x) = \frac{1}{2}(e^x + e^{-x})$.

Theorem 1.2. *(Chong et al. (2000)) Let* $\varsigma := \ln\sqrt{p/q}$*. The distribution of the stopping point is, for* $p \neq q$*,*

$$\begin{aligned}
P(S_N = k) &= e^{\varsigma k} \frac{s[|k|\,\varsigma]s[\varsigma]}{s[m\varsigma]s[(m+1)\varsigma]} \\
&= p^{m+(k-|k|)/2} q^{m-(k+|k|)/2} \frac{(p^{|k|} - q^{|k|})(p-q)}{(p^m - q^m)(p^{m+1} - q^{m+1})}.
\end{aligned}$$

Let the process X_t be a Brownian motion with drift μ, then the Laplace transform of (S, T, X_T) is shown as follows.

Theorem 1.3. *(Chong et al. (2000)) For a Brownian motion with drift* μ *and variance* $\sigma^2 > 0$*, we have, for* $t, u \geq 0$ *and all* v*, with* $s^n[x] := \sinh^n(x)$*,* $\varepsilon := r\mu/\sigma^2$ *and* $\alpha(x) := r/\sigma\sqrt{\mu^2/\sigma^2 + 2x}$*,*

$$E(e^{-uS-t(T-S)-vX_T}) = \frac{2\alpha(t)}{s[\alpha(t)]} \left(\frac{s^2[\frac{1}{2}(\alpha(u)+\varepsilon-rv)]}{(\alpha(u)+\varepsilon-rv)s[\alpha(u)]} + \frac{s^2[\frac{1}{2}(\alpha(u)-\varepsilon+rv)]}{(\alpha(u)-\varepsilon+rv)s[\alpha(u)]} \right).$$

In Vallois and Tanré (2003), the authors follow a similar path as Vallois (1996). Instead of considering the simple random walk, the process in this paper is a Brownian motion with drift μ.

In Vallois and Salminen (2005), the first range time of a linear diffusion on $\mathbb{R}$ is considered. Inspired by the observation that the exponentially randomized range time has the same law as a similarly randomized first exit time from an interval, the authors study a large family of non negative two-dimensional random variables (X, X'). They also explain the Markovian structure of the Brownian local time process when stopped at an exponentially randomized first range time.

So far, we have seen that most of the results on the first range time are derived for some specific processes, such as simple random walk or Brownian motion. In Ren (2006), a different and simple approach is used to derive a result of a joint Laplace transform on the first range time for general diffusion processes. It is shown that problems on the first range time could be eventually reduced to solving an ordinary differential equation.

In this thesis we present the most comprehensive results to date on first range time for a two-sided jump diffusion process. The process we consider is a standard Brownian motion with drift plus an independent compound Poisson process. The process starts from 0 and we are interested in the distribution of the time when the difference of the running maximum and running minimum first reaches level r. Our idea is to divide the interval $[0, r]$ into 2^n parts and consider the problem on each small interval and approximate the range time by the sum of two first passage times. The first passage time approximates S and the second passage time approximates $T - S$. Then we apply results of the two-sided exit times for a jump diffusion process. We first find the joint probability generating function of the range time and the value of the process at the range time. We also find the distribution of the value of the process at the range time.

The outline of this thesis is as follows. Chapter 2 introduces some basic concepts and preliminary results on the exit problems for jump diffusion processes. Chapter 3 presents the Laplace transform of the two-sided exit time from a finite interval for the jump diffusion process. In this chapter we also generalize a result in Kou and Wang (2003). Chapter 4 derives the Laplace transform for the first range time by using the two-sided exit problem results in Chapter 3 and some approximations. The distribution of the process at the first range time is also obtained in this chapter.

Chapter 2

Preliminaries

Let $W_t^{(\mu)} = \mu t + W_t$ be a standard, one-dimensional Brownian motion with drift μ. Let $H = H_{a,b} = \min\{t, W_t^{(\mu)} \notin (a,b)\}$ be the first-exit time. Define $E(X;A)$ as $E(X1_{\{A\}})$. From Borodin and Salminen (2002), we have the following conclusion:

$$E_x\left(e^{-\alpha H}; W_H^{(\mu)} = a\right) = e^{\mu(a-x)} \frac{\cosh((b-x)\sqrt{2\alpha + \mu^2})}{\cosh((b-a)\sqrt{2\alpha + \mu^2})}$$

$$E_x\left(e^{-\alpha H}; W_H^{(\mu)} = b\right) = e^{\mu(b-x)} \frac{\cosh((x-a)\sqrt{2\alpha + \mu^2})}{\cosh((b-a)\sqrt{2\alpha + \mu^2})}$$

Our interest is focused on jump diffusion processes, which are widely used, for example, in finance to model asset prices(stock, bond, currency, etc.). Two examples are the normal jump diffusion process where the jumps follow a normal distribution (see Merton (1976)) and the double exponential jump diffusion process where the jumps follow a double exponential distribution (see Kou (2002)). Meanwhile, Perry et al. (2002) studies the one-sided and two-sided first-exit problems for a compound Poisson process with both positive and negative jumps of hyperexponential, Erlang and Coxian types and with linear deterministic negative drift between jumps. Kou and Wang (2003) use a doubly exponential jump diffusion process to model asset prices in finance. The process is of the form

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad X_0 \equiv 0.$$

Here $\{W_t : t \geq 0\}$ is a standard Brownian motion with $W_0 = 0$, while $\{N_t : t \geq 0\}$ is a Poisson process with rate λ, constants μ and $\sigma > 0$ are the drift and volatility of the diffusion part respectively, and the jump sizes $\{Y_1, Y_2, ...\}$ are independent and identically distributed random variables. The common density of Y is given by

$$f_Y(y) = p\eta_1 e^{-\eta_1 y} I_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y} I_{\{y<0\}},$$

where $p, q \geq 0$ are constants, $p + q = 1$ and $\eta_1, \eta_2 > 0$.

Define $\tau_b := \inf\{t \geq 0; X_t \geq b\}, b > 0$.

Theorem 2.1. *(Kou and Wang (2003)) For any $\alpha \in (0, \infty)$, let β_1 and β_2 be the only two positive roots of the equation*

$$x\mu + \frac{1}{2}x^2\mu^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1\right) = \alpha,$$

where $0 < \beta_1 < \eta_1 < \beta_2 < \infty$. Then we have the following results concerning the Laplace transforms of τ_b and X_{τ_b}.

$$E\left(e^{-\alpha\tau_b}\right) = \frac{\eta_1 - \beta_1}{\eta_1}\frac{\beta_2}{\beta_2 - \beta_1}e^{-b\beta_1} + \frac{\beta_2 - \eta_1}{\eta_1}\frac{\beta_1}{\beta_2 - \beta_1}e^{-b\beta_2},$$

$$E\left(e^{-\alpha\tau_b}; X_{\tau_b} - b > y\right) = e^{-\eta_1 y}\frac{(\eta_1 - \beta_1)(\beta_2 - \eta_1)}{\eta_1(\beta_2 - \beta_1)}[e^{-b\beta_1} - e^{-b\beta_2}], \quad \textit{for any } y \geq 0,$$

$$E\left(e^{-\alpha\tau_b}; X_{\tau_b} = b\right) = \frac{\eta_1 - \beta_1}{\beta_2 - \beta_1}e^{-b\beta_1} + \frac{\beta_2 - \eta_1}{\beta_2 - \beta_1}e^{-b\beta_2}.$$

The Laplace transforms of the joint distribution of the first passage time it gives is consistent with the results in Chapter 3 of this thesis.

The probability

$$P\left(X_t \geq a, \max_{0\leq s\leq t} X_s \geq b\right) = P(X_t \geq a, \tau_b \leq t),$$

for some fixed numbers $a \leq b$ and $b > 0$ is useful, for example, in pricing barrier options while the logarithm of the underlying asset price is modeled by a jump diffusion process. The following theorem gives the joint distribution of the jump diffusion and its running maxima.

Theorem 2.2. *(Kou and Wang (2003)) The Laplace transform of the joint distribution is given by*

$$\begin{aligned}
&\int_0^\infty e^{-\alpha t} P(X_t \geq a, \tau_b \leq t)dt \\
=&E\left(e^{-\alpha\tau}; X_{\tau_b} = b\right)\int_0^\infty e^{-\alpha t} P(X_t \geq a-b)dt \\
&+ E\left(e^{-\alpha\tau}; X_{\tau_b} > b\right)\int_0^\infty e^{\alpha t} P(X_t + \xi \geq a-b)dt.
\end{aligned}$$

Here ξ is an independent exponential random variable with rate η_1 and $E\left(e^{-\alpha\tau}; X_{\tau_b} > b\right)$ is obtain by letting $y = 0$ in Theorem 2.1.

Wang and Zhang (2009) discuss the two-sided first-exit problems of a diffusion process with both positive and negative Erlang jumps using a martingale approach. They introduce the surplus process

$$X_t = u + ct + \sigma W_t - \sum_{i=1}^{N_t} Y_i, \quad t \geq 0,$$

where $u \geq 0$ is the initial capital of the company, and c is the positive constant premium income rate, $\{W_t : t \geq 0\}$ is a standard Brownian motion with dispersion parameter $\sigma > 0$. Let $\{N_t : t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$, counting the total number of claims from an insurance portfolio and $\{Y_i : i \in N\}$ be a sequence of independent and identically distributed random variables with density

$$f_Y(y) = p\frac{\eta_1^n y^{n-1}}{(n-1)!}e^{\eta_1 y}I_{\{y\geq 0\}} + q\frac{\eta_2^m(-y)^{m-1}}{(m-1)!}e^{\eta_2 y}I_{\{y<0\}},$$

where $p, q \geq 0$ are constants with $p+q=1, \eta_1, \eta_2 > 0$. In particular, explicit expressions for the Laplace transform of first exit times, as well as the distribution of the time of ruin are obtained. Define

$$\tau_a := \inf\{t \geq 0 : X_t \leq a\}, \quad 0 < a < u,$$

$$\tau_0 := \inf\{t \geq 0 : X_t \leq 0\},$$

$$\tau_b := \inf\{t \geq 0 : X_t \geq b\}, \quad b > u,$$

$$\tau := \inf\{t \geq 0 : X_t \notin (0, b)\}, \quad b > u.$$

Theorem 2.3. *(Wang and Zhang (2009)) For any $\theta > 0$, we have the Laplace transforms of τ_a and X_{τ_a}*

$$E^u\left(e^{-\theta\tau_a}\right) = \sum_{i=1}^{n+1} E^u\left(e^{-\theta\tau_a} 1_{G_i}\right) = \sum_{i=1}^{n+1} \pi_i,$$

$$E^u\left(e^{-\theta\tau_a}; X_{\tau_a} < a\right) = \sum_{i=1}^{n} E^u\left(e^{-\theta\tau_a} 1_{G_i}\right) = \sum_{i=1}^{n} \pi_i,$$

$$E^u\left(e^{-\theta\tau_a}; X_{\tau_a} = a\right) = E^u\left(e^{-\theta\tau_a} 1_{G_{n+1}}\right) = \pi_{n+1},$$

where

$G_i = \{X.$ *crosses the level a at time τ_a by the ith phase of a downward jump*$\}, i = 1, 2, ..., n,$

$G_n = \{X_{\tau_a} = a\}$

and the vector π satisfies the following system

$$\pi\widehat{f}[\beta_i] = e^{\beta_i(u-a)}, \quad i = 1, 2, ..., n+1, \tag{2.1}$$

π is the unique solution of (2.1).

Theorem 2.4. *(Wang and Zhang (2009)) The Laplace transforms in connection with the first-exit time τ are obtained as follows:*

$$E^u\left(e^{-\theta\tau}\right) = \sum_{i=1}^{n+3} E^u\left(e^{-\theta\tau} 1_{K_i}\right),$$

$$E^u\left(e^{-\theta\tau}; X_\tau \geq b\right) = E^u\left(e^{-\theta\tau_b} 1_{\{\tau_b < \tau_0\}}\right) = \sum_{i=1}^{n+1} E^u\left(e^{-\theta\tau} 1_{K_i}\right),$$

$$E^u\left(e^{-\theta\tau}; X_\tau \leq 0\right) = E^u\left(e^{-\theta T} 1_{\{\tau_0 < \tau_b\}}\right) = \sum_{i=n+2}^{n+3} E^u\left(e^{-\theta\tau} 1_{K_i}\right),$$

where the events $K_1, K_2, ..., K_{m+n+2}$ *are defined as*

$$K_i = \{X.\ \textit{crosses b at time } \tau \textit{ by the ith phase of a positive jump}\},\ i = 1, 2, ..., m,$$

$$K_{m+1} = \{X_\tau = b\},$$

$$K_{m+1+j} = \{X.\ \textit{crosses 0 at time } \tau \textit{ by the jth phase of a negative jump}\},\ j = 1, 2, ..., n,$$

$$K_{m+n+2} = \{X_\tau = 0\}.$$

Chapter 3

First passage times of a jump diffusion process

Let

$$X_t = \sigma W_t + \mu t + \sum_{i=1}^{N_t} Y_i, \tag{3.1}$$

where W is a Brownian motion with $W_0 = 0$, N is a rate λ Poisson process and (Y_i) are i.i.d. random variables with density function

$$f_Y(y) = p\eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q\eta_2 e^{-\eta_2 y} 1_{\{y<0\}}, \quad p, q > 0, p + q = 1.$$

In addition, processes $W_\cdot$, $N_\cdot$ and (Y_i) are independent. Note that the jump part is a special case of the so-called marked point processes; further background on marked point processes can be found in Brémaud (1981) and Jacod and Shiryaev (1987). The process in (3.1) is indeed special cases of Lévy processes, processes with stationary and independent, see Bertoin (1996) and Sato (1999).

Normally, the two-sided exit problem is difficult for general Lévy processes. It can only be solved for certain kinds of jump distributions, such as the exponential-type, thanks to the memoryless property of the exponential distribution.

The process X in (3.1) has a generator

$$Lu(x) = \frac{1}{2}\sigma^2 u''(x) + \mu u'(x) + \lambda \int_{-\infty}^{\infty} [u(x+y) - u(x)] f_Y(y) dy.$$

For any $\theta \in (-\eta_2, \eta_1)$, $E[e^{\theta X_t}]$ exists and

$$E[e^{\theta X_t}] = e^{\psi(\theta) t},$$

where

$$\psi(\theta) = \theta\mu + \frac{1}{2}\theta^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - \theta} + \frac{q\eta_2}{\eta_2 + \theta} - 1\right).$$

Lemma 3.1. *For all $\alpha > 0$, the equation $\psi(x) = \alpha$ has exactly four roots $\beta_1, \beta_2, -\beta_3, -\beta_4$, such that*

$$0 < \beta_1 < \eta_1 < \beta_2 < \infty, \quad 0 < \beta_3 < \eta_2 < \beta_4 < \infty.$$

In addition, let the overall drift of the jump diffusion process be

$$\overline{u} := \mu + \lambda\left(\frac{p}{\eta_1} - \frac{q}{\eta_2}\right).$$

Then, as $\alpha \to 0+$,

$$\beta_1 \to \begin{cases} 0, & \text{if } \overline{u} \geq 0; \\ \widetilde{\beta}_1, & \text{if } \overline{u} < 0; \end{cases}$$

and

$$\beta_2 \to \widetilde{\beta}_2, \beta_3 \to \widetilde{\beta}_3, \beta_4 \to \widetilde{\beta}_4,$$

where $\widetilde{\beta}_1$, $\widetilde{\beta}_2$, $\widetilde{\beta}_3$ and $\widetilde{\beta}_4$ are defined as the unique roots of equation

$$\psi(x) = 0, \quad \text{such that} \quad 0 < \widetilde{\beta}_1 < \eta_1 < \widetilde{\beta}_2 < \infty; 0 < \widetilde{\beta}_3 < \eta_2 < \widetilde{\beta}_4 < \infty.$$

Proof. Function ψ is convex in $(-\eta_2, \eta_1)$ with $\psi(0) = 0$. Since

$$\lim_{x \to \eta_1^-} \psi(x) = +\infty \text{ and } \lim_{x \to -\eta_2^+} \psi(x) = +\infty,$$

there is exactly one root β_1 for equation $\psi(x) = \alpha$ in $(-\eta_2, 0)$ and another root in $(0, \eta_1)$. Furthermore, since

$$\lim_{x \to \eta_1^+} \psi(x) = -\infty \text{ and } \lim_{x \to +\infty} \psi(x) = +\infty,$$

there is at least one root in $(\eta_1, +\infty)$. Similarly, since

$$\lim_{x \to -\eta_2^-} \psi(x) = -\infty \text{ and } \lim_{x \to -\infty} \psi(x) = +\infty,$$

so there is at least one root in $(-\infty, -\eta_2)$. The equation $\psi(x) = \alpha$ is a polynomial equation of degree four, so it has at most four roots. It follows that, there is exactly one root in $(-\infty, -\eta_2)$ and (η_1, ∞), respectively.

The limiting results when $\alpha \to 0$ follow easily once we note that $\psi'(0) = \overline{u}$. $\square$

For any $a, b > 0$ let

$$\tau := \inf\{t \geq 0 : X_t \leq -a \text{ or } X_t \geq b\}.$$

So τ is the first time when process $X_t \notin [-a, b]$. We want to find $E\left(e^{-\alpha\tau}; X_\tau \geq b\right)$, $E\left(e^{-\alpha\tau}; X_\tau \leq -a\right)$, $E\left(e^{-\alpha\tau}; X_\tau < -a - x\right)$ and $E\left(e^{-\alpha\tau}; X_\tau - b > x\right)$ for $x > 0$.

We first define some constants. Put

$$\begin{aligned}
A :=& \frac{(\beta_2 - \beta_1)(\beta_4 - \beta_3)}{(\beta_1 - \eta_1)(\beta_2 - \eta_1)(\beta_3 - \eta_2)(\beta_4 - \eta_2)} \\
&+ \frac{e^{-(a+b)\beta_2}}{(\beta_1 - \eta_1)(\beta_2 + \eta_2)} \left(\frac{(\beta_1 + \beta_3)(\beta_2 + \beta_4)}{(\beta_3 + \eta_1)(\beta_4 - \eta_2)} e^{-(a+b)\beta_3} - \frac{(\beta_1 + \beta_4)(\beta_2 + \beta_3)}{(\beta_4 + \eta_1)(\beta_3 - \eta_2)} e^{-(a+b)\beta_4} \right) \\
&+ \frac{(\beta_1 + \beta_3)(\beta_2 + \beta_4)}{(\beta_2 - \eta_1)(\beta_4 + \eta_1)(\beta_1 + \eta_2)(\beta_3 - \eta_2)} e^{-(a+b)(\beta_1 + \beta_4)} \\
&+ \frac{e^{-(a+b)(\beta_1 + \beta_3)}}{(\beta_3 + \eta_1)(\beta_1 + \eta_2)} \left(\frac{(\beta_1 + \beta_4)(\beta_2 + \beta_3)}{(\beta_2 - \eta_1)(-\beta_4 + \eta_2)} + \frac{(\beta_2 - \beta_1)(\beta_4 - \beta_3)}{(\beta_4 + \eta_1)(\beta_2 + \eta_2)} e^{-(a+b)(\beta_2 + \beta_4)} \right),
\end{aligned}$$

$$
\begin{aligned}
A_{11} :=& \frac{e^{-(a+b)\beta_2}}{\beta_2+\eta_2}\left(\frac{e^{-(a+b)\beta_4}(\beta_2+\beta_3)\beta_4}{(\beta_4+\eta_1)(\beta_3-\eta_2)} - \frac{e^{-(a+b)\beta_3}(\beta_2+\beta_4)\beta_3}{(\beta_3+\eta_1)(\beta_4-\eta_2)}\right) \\
&- \frac{\beta_2(\beta_3-\beta_4)}{(\beta_2-\eta_1)(\beta_4-\eta_2)(-\beta_3+\eta_2)},
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
A_{31} :=& \frac{e^{-(a+b)\beta_1}}{\beta_1+\eta_2}\left(\frac{e^{-(a+b)(\beta_4+\beta_2)}(\beta_2-\beta_1)\beta_4}{(\beta_4+\eta_1)(\beta_2+\eta_2)} + \frac{(\beta_1+\beta_4)\beta_2}{(\beta_2-\eta_1)(-\beta_4+\eta_2)}\right) \\
&+ \frac{e^{-(a+b)\beta_2}\beta_1(\beta_2+\beta_4)}{(\beta_1-\eta_1)(\beta_4-\eta_2)(\beta_2+\eta_2)}.
\end{aligned} \tag{3.3}
$$

Define A_{21} and A_{41} such that $-A_{21}$ is obtained from A_{11} by changing β_2 to β_1 in (3.2), and $-A_{41}$ is obtained from A_{31} by changing β_4 to β_3 in (3.3).

$$
B_{11} := \frac{-\beta_3+\beta_4}{(\beta_4-\eta_2)(-\beta_3+\eta_2)} + \frac{e^{-(a+b)(\beta_4+\beta_2)}(\beta_2+\beta_3)}{(\beta_2+\eta_2)(\beta_3-\eta_2)} - \frac{e^{-(a+b)(\beta_3+\beta_2)}(\beta_2+\beta_4)}{(\beta_2+\eta_2)(\beta_4-\eta_2)}, \tag{3.4}
$$

$$
B_{31} := \frac{e^{-(a+b)\beta_2}(\beta_2+\beta_4)}{(\beta_4-\eta_2)(\beta_2+\eta_2)} + \frac{e^{-(a+b)\beta_1}(\beta_1+\beta_4)}{(\beta_1+\eta_2)(-\beta_4+\eta_2)} + \frac{(-\beta_1+\beta_2)e^{-(a+b)(\beta_2+\beta_4+\beta_1)}}{(\beta_1+\eta_2)(\beta_2+\eta_2)}. \tag{3.5}
$$

Define B_{21} and B_{41} such that $-B_{21}$ is obtained from B_{11} by changing β_2 to β_1 in (3.4), and $-B_{41}$ is obtained from B_{31} by changing β_4 to β_3 in (3.5).

Theorem 3.1. *For any $\alpha_1 > 0$, $b > 0$,*

$$
E\left(e^{-\alpha_1\tau}; X_\tau \geq b\right) = A_1 e^{-\beta_1 b} + A_2 e^{-\beta_2 b} + A_3 e^{-\beta_3 a} + A_4 e^{-\beta_4 a},
$$

where

$$
A_1 = \frac{A_{11}}{\eta_1 A}, A_2 = \frac{A_{21}}{\eta_1 A}, A_3 = \frac{A_{31}}{\eta_1 A} \quad \text{and} \quad A_4 = \frac{A_{41}}{\eta_1 A}.
$$

Proof. We consider the two-sided exit problem under the same settings of Kou and Wang (2003). To this end, we modify the approach of Kou and Wang (2003) for the one-sided exit problem. Let

$$u(x) = \begin{cases} 1, & x > b; \\ A_1 e^{-\beta_1(b-x)} + A_2 e^{-\beta_2(b-x)} + A_3 e^{-\beta_3(x+a)} + A_4 e^{-\beta_4(x+a)}, & -a \le x \le b; \\ 0, & x < -a. \end{cases}$$

For $-a < x < b$,

$$\begin{aligned}
&\int_{-\infty}^{\infty} u(x+y) f_Y(y) dy \\
&= \int_{b-x}^{\infty} p\eta_1 e^{-\eta_1 y} dy + \int_{-\infty}^{-a-x} 0 \times q\eta_2 e^{\eta_2 y} dy \\
&+ \int_0^{b-x} \left(A_1 e^{-\beta_1(b-x-y)} + A_2 e^{-\beta_2(b-x-y)} + A_3 e^{-\beta_3(x+y+a)} + A_4 e^{-\beta_4(x+y+a)}\right) p\eta_1 e^{-\eta_1 y} dy \\
&+ \int_{-a-x}^{0} \left(A_1 e^{-\beta_1(b-x-y)} + A_2 e^{-\beta_2(b-x-y)} + A_3 e^{-\beta_3(x+y+a)} + A_4 e^{-\beta_4(x+y+a)}\right) q\eta_2 e^{\eta_2 y} dy,
\end{aligned}$$

where

$$\int_{b-x}^{\infty} p\eta_1 e^{-\eta_1 y} dy = p e^{-\eta_1(b-x)},$$

$$\begin{aligned}
&\int_0^{b-x} e^{-\beta_1(b-x-y)} p\eta_1 e^{-\eta_1 y} dy + \int_{-a-x}^{0} e^{-\beta_1(b-x-y)} q\eta_2 e^{\eta_2 y} dy \\
&= \left(\frac{p\eta_1}{\eta_1 - \beta_1} + \frac{q\eta_2}{\eta_2 + \beta_1}\right) e^{-\beta_1(b-x)} - \frac{p\eta_1}{\eta_1 - \beta_1} e^{-\eta_1(b-x)} - \frac{q\eta_2}{\eta_2 + \beta_1} e^{-\beta_1(a+b) - \eta_2(x+a)},
\end{aligned}$$

similarly for β_2,

$$\begin{aligned}
&\int_0^{b-x} e^{-\beta_2(b-x-y)} p\eta_1 e^{-\eta_1 y} dy + \int_{-a-x}^{0} e^{-\beta_2(b-x-y)} q\eta_2 e^{\eta_2 y} dy \\
&= \left(\frac{p\eta_1}{\eta_1 - \beta_2} + \frac{q\eta_2}{\eta_2 + \beta_2}\right) e^{-\beta_2(b-x)} - \frac{p\eta_1}{\eta_1 - \beta_2} e^{-\eta_1(b-x)} - \frac{q\eta_2}{\eta_2 + \beta_2} e^{-\beta_2(a+b) - \eta_2(x+a)},
\end{aligned}$$

for β_3,

$$\begin{aligned}
&\int_{b-x}^{\infty} e^{-\beta_3(x+y+a)} p\eta_1 e^{-\eta_1 y} dy + \int_{-a-x}^{0} e^{-\beta_3(x+y+a)} q\eta_2 e^{\eta_2 y} dy \\
&= \left(\frac{p\eta_1}{\eta_1 + \beta_3} + \frac{q\eta_2}{\eta_2 - \beta_3}\right) e^{-\beta_3(x+a)} - \frac{p\eta_1}{\eta_1 + \beta_3} e^{-\beta_3(a+b) - \eta_1(b-x)} - \frac{q\eta_2}{\eta_2 - \beta_3} e^{-\eta_2(x+a)},
\end{aligned}$$

and finally for β_4,

$$\int_{b-x}^{\infty} e^{-\beta_4(x+y+a)} p\eta_1 e^{-\eta_1 y} dy + \int_{-a-x}^{0} e^{-\beta_4(x+y+a)} q\eta_2 e^{\eta_2 y} dy$$
$$= \left(\frac{p\eta_1}{\eta_1+\beta_4} + \frac{q\eta_2}{\eta_2-\beta_4}\right) e^{-\beta_4(x+a)} - \frac{p\eta_1}{\eta_1+\beta_4} e^{-\beta_4(a+b)-\eta_1(b-x)} - \frac{q\eta_2}{\eta_2-\beta_4} e^{-\eta_2(x+a)}.$$

Put

$$F_1(A_1,A_2,A_3,A_4) := A_1 + A_2 + e^{-\beta_3(a+b)}A_3 + e^{-\beta_4(a+b)}A_4 - 1,$$

$$F_2(A_1,A_2,A_3,A_4) := e^{-\beta_1(a+b)}A_1 + e^{-\beta_2(a+b)}A_2 + A_3 + A_4,$$

$$F_3(A_1,A_2,A_3,A_4) := \frac{\eta_1}{\eta_1-\beta_1}A_1 + \frac{\eta_1}{\eta_1-\beta_2}A_2 + \frac{\eta_1}{\eta_1+\beta_3}e^{-\beta_3(a+b)}A_3 + \frac{\eta_1}{\eta_1+\beta_4}e^{-\beta_4(a+b)}A_4 - 1,$$

$$F_4(A_1,A_2,A_3,A_4) := \frac{\eta_2}{\eta_2+\beta_1}e^{-\beta_1(a+b)}A_1 + \frac{\eta_2}{\eta_2+\beta_2}e^{-\beta_2(a+b)}A_2 + \frac{\eta_2}{\eta_2-\beta_3}A_3 + \frac{\eta_2}{\eta_2-\beta_4}A_4.$$

Since (A_1,A_2,A_3,A_4) solves the following system of equations

$$F_i(A_1,A_2,A_3,A_4) = 0, \quad i = 1,2,3,4, \tag{3.6}$$

we can verify that

$$u(b) = F_1(A_1,A_2,A_3,A_4) + 1 = 1,$$

$$u(-a) = F_2(A_1,A_2,A_3,A_4) = 0,$$

and for $-a < x < b$,

$$\begin{aligned} Lu(x) - \alpha_1 u(x) =& A_1 e^{-\beta_1(b-x)}(\psi(\beta_1)-\alpha_1) + A_2 e^{-\beta_2(b-x)}(\psi(\beta_2)-\alpha_1) \\ & + A_3 e^{-\beta_3(x+a)}(\psi(-\beta_3)-\alpha_1) + A_4 e^{-\beta_4(x+a)}(\psi(-\beta_4)-\alpha_1) \\ & - F_3(A_1,A_2,A_3,A_4)pe^{-\eta_1(b-x)} - F_4(A_1,A_2,A_3,A_4)qe^{-\eta_2(x+a)} \\ =& 0. \end{aligned}$$

Similar to Kou and Wang (2003), we can select a sequence of uniformly bounded functions (u_n) such that $u_n \in C^2(\mathbb{R})$, where $C^2(\mathbb{R})$ refers the twice-differentiable function whose second derivative is continuous. $u_n(x) = u(x)$ for $-a \le x \le b$, $u_n(x) = 0$

for $x \leq -a - 1/n$, $u_n(x) = 1$ for $x \geq b + 1/n$, and both (u_n') and (u_n'') are uniformly bounded. Then

$$\|Lu_n - \alpha_1 u_n\|_\infty \to 0.$$

Applying Itô's formula, (see Rao (1995) and Protter (1990)), we see that the process

$$M_\cdot^n := e^{-\alpha_1(\cdot\wedge\tau)} u_n(X_{\cdot\wedge\tau}) - \int_0^{\cdot\wedge\tau} (Lu_n(X_s) - \alpha_1 u_n(X_s)) ds$$

is a martingale. Taking a limit in n we have that

$$\begin{aligned} E\left(e^{-\alpha_1(t\wedge\tau)} u(X_{t\wedge\tau})\right) &= \lim_{n\to\infty} E\left(e^{-\alpha_1(t\wedge\tau)} u_n(X_{t\wedge\tau})\right) \\ &= \lim_{n\to\infty} u_n(0) \\ &= u(0) \\ &= A_1 e^{-\beta_1 b} + A_2 e^{-\beta_2 b} + A_3 e^{-\beta_3 a} + A_4 e^{-\beta_4 a}. \end{aligned}$$

Then

$$\begin{aligned} E\left(e^{-\alpha_1 \tau}; X_\tau \geq b\right) &= \lim_{t\to\infty} E\left(e^{-\alpha_1(t\wedge\tau)} u(X_{t\wedge\tau})\right) \\ &= A_1 e^{-\beta_1 b} + A_2 e^{-\beta_2 b} + A_3 e^{-\beta_3 a} + A_4 e^{-\beta_4 a}. \end{aligned}$$

□

Remark 3.1. *Note that in the above proof we show that there is a unique function* $u \in C((-\infty,\infty)) \cap C^2((-a,b))$ *satisfying*

$$\begin{cases} u(x) = 0, & x \leq -a; \\ Lu(x) - \alpha_1 u(x) = 0, & -a < x < b; \\ u(x) = 1, & x \geq b. \end{cases}$$

Remark 3.2. *It seems to be very difficult to give the distribution of the two-sided exit time analytically. Because when we apply the inversion of Laplace transforms on* $E\left(e^{-\alpha_1\tau}; X_\tau \geq b\right)$*, we need to do the integration for* $e^{\alpha_1\tau}E\left(e^{-\alpha_1\tau}; X_\tau \geq b\right)$ *on* α_1*. From the theorem 3.1, we can see that the result for* $E\left(e^{-\alpha_1\tau}; X_\tau \geq b\right)$ *is not directly expressed by* α_1*. Instead, it is expressed by the solutions of the equation* $\psi(x) = \alpha_1$*, which have very complicated expressions. But we can give the distribution of the two-sided exit time numerically.*

Theorem 3.2. *For* $\alpha_1 > 0$,$y > 0$,

$$E\left(e^{-\alpha_1\tau}; X_\tau - b > y\right) = B_1 e^{-\beta_1 b} + B_2 e^{-\beta_2 b} + B_3 e^{-\beta_3 a} + B_4 e^{-\beta_4 a},$$

where

$$B_1 = \frac{e^{-y\eta_1}B_{11}}{\eta_1 A}, B_2 = \frac{e^{-y\eta_1}B_{21}}{\eta_1 A}, B_3 = \frac{e^{-y\eta_1}B_{31}}{\eta_1 A} \quad \text{and} \quad B_4 = \frac{e^{-y\eta_1}B_{41}}{\eta_1 A}.$$

Proof. Similar to the proof for Theorem 3.1, for $y > 0$ define

$$v(x) = \begin{cases} 1, & x > b+y; \\ 0, & b < x \leq b+y; \\ B_1 e^{-\beta_1(b-x)} + B_2 e^{-\beta_2(b-x)} + B_3 e^{-\beta_3(x+a)} + B_4 e^{-\beta_4(x+a)}, & -a \leq x \leq b; \\ 0, & x < -a, \end{cases}$$

where (B_1, B_2, B_3, B_4) solves the equations

$$\begin{cases} B_1 + B_2 + e^{-\beta_3(a+b)}B_3 + e^{-\beta_4(a+b)}B_4 = 0, \\ e^{-\beta_1(a+b)}B_1 + e^{-\beta_2(a+b)}B_2 + B_3 + B_4 = 0, \\ \frac{\eta_1}{\eta_1-\beta_1}B_1 + \frac{\eta_1}{\eta_1-\beta_2}B_2 + \frac{\eta_1}{\eta_1+\beta_3}e^{-\beta_3(a+b)}B_3 + \frac{\eta_1}{\eta_1+\beta_4}e^{-\beta_4(a+b)}B_4 = e^{-\eta_1 y}, \\ \frac{\eta_2}{\eta_2+\beta_1}e^{-\beta_1(a+b)}B_1 + \frac{\eta_2}{\eta_2+\beta_2}e^{-\beta_2(a+b)}B_2 + \frac{\eta_2}{\eta_2-\beta_3}B_3 + \frac{\eta_2}{\eta_2-\beta_4}B_4 = 0. \end{cases}$$

Then one can show that

$$v(-a) = 0 = v(b)$$

and for $-a < x < b$,

$$Lv(x) - \alpha_1 v(x) = 0.$$

We first approximate v by continuous functions (v_n) such that $0 \le v_n(x) \le 1$ for $x \in (b+y, b+y+1/n)$ and $v_n(x) = v(x)$ for $x \in (b+y, b+y+1/n)^c$. Then approximate v_n using a smooth function and finally apply Ito's formula. We can show that

$$\begin{aligned}
E\left(e^{-\alpha_1 \tau}; X_\tau - b > y\right) &= E\left(e^{-\alpha_1 \tau} v(X_\tau)\right) \\
&= \lim_{n\to\infty} E\left(e^{-\alpha_1 \tau} v_n(X_\tau)\right) \\
&= \lim_{n\to\infty} v_n(0) \\
&= v(0) \\
&= B_1 e^{-\beta_1 b} + B_2 e^{-\beta_2 b} + B_3 e^{-\beta_3 a} + B_4 e^{-\beta_4 a}.
\end{aligned}$$

□

Combining Theorems 3.1 and 3.2, we have

Proposition 3.1.

$$E\left(e^{-\alpha\tau}; X_\tau = b\right) = C_1 e^{-\beta_1 b} + C_2 e^{-\beta_2 b} + C_3 e^{-\beta_3 a} + C_4 e^{-\beta_4 a}.$$

where

$$C_1 = A_1 - B_1,\ C_2 = A_2 - B_2,\ C_3 = A_3 - B_3 \quad \textit{and} \quad C_4 = A_4 - B_4.$$

Letting $a \to \infty$ we can recover Theorem 2.1.

Corollary 3.1. *For any $\alpha \in (0,\infty)$, let β_1 and β_2 be the only two positive roots of the equation*

$$\psi(x) = \alpha,$$

where $0 < \beta_1 < \eta_1 < \beta_2 < \infty$. Then we have the following results concerning the Laplace transforms of τ and X_τ:

$$E\left(e^{-\alpha\tau}\right) = \frac{\eta_1 - \beta_1}{\eta_1}\frac{\beta_2}{\beta_2 - \beta_1}e^{-b\beta_1} + \frac{\beta_2 - \eta_1}{\eta_1}\frac{\beta_1}{\beta_2 - \beta_1}e^{-b\beta_2},$$

$$E\left(e^{-\alpha\tau}; X_\tau - b > y\right) = e^{-\eta_1 y}\frac{(\eta_1 - \beta_1)(\beta_2 - \eta_1)}{\eta_1(\beta_2 - \beta_1)}\left(e^{-b\beta_1} - e^{-b\beta_2}\right), \quad for\ any\ y \geq 0,$$

$$E\left(e^{-\alpha\tau}; X_\tau = b\right) = \frac{\eta_1 - \beta_1}{\beta_2 - \beta_1}e^{-b\beta_1} + \frac{\beta_2 - \eta_1}{\beta_2 - \beta_1}e^{-b\beta_2}.$$

Proof. In Theorems 3.1, 3.2 and Proposition 3.1, let $a \to \infty$, then

$$\begin{aligned}
A_1 &= \frac{A_{11}}{\eta_1 A} \\
&= -\frac{\beta_2(\beta_3 - \beta_4)}{(\beta_2 - \eta_1)(\beta_4 - \eta_2)(-\beta_3 + \eta_2)}\Bigg/\frac{\eta_1(\beta_2 - \beta_1)(\beta_4 - \beta_3)}{(\beta_1 - \eta_1)(\beta_2 - \eta_1)(\beta_3 - \eta_2)(\beta_4 - \eta_2)} \\
&= \frac{\beta_2(\eta_1 - \beta_1)}{\eta_1(\beta_2 - \beta_1)},
\end{aligned}$$

$$\begin{aligned}
A_2 &= \frac{A_{21}}{\eta_1 A} \\
&= \frac{\beta_1(\beta_3 - \beta_4)}{(\beta_1 - \eta_1)(\beta_4 - \eta_2)(-\beta_3 + \eta_2)}\Bigg/\frac{\eta_1(\beta_2 - \beta_1)(\beta_4 - \beta_3)}{(\beta_1 - \eta_1)(\beta_2 - \eta_1)(\beta_3 - \eta_2)(\beta_4 - \eta_2)} \\
&= \frac{\beta_1(\beta_2 - \eta_1)}{\eta_1(\beta_2 - \beta_1)},
\end{aligned}$$

$$\begin{aligned}
B_1 &= \frac{e^{-y\eta_1}B_{11}}{\eta_1 A} \\
&= \frac{e^{-y\eta_1}(\beta_4 - \beta_3)}{(\beta_4 - \eta_2)(-\beta_3 + \eta_2)}\Bigg/\frac{\eta_1(\beta_2 - \beta_1)(\beta_4 - \beta_3)}{(\beta_1 - \eta_1)(\beta_2 - \eta_1)(\beta_3 - \eta_2)(\beta_4 - \eta_2)} \\
&= e^{-\eta_1 y}\frac{(\eta_1 - \beta_1)(\beta_2 - \eta_1)}{\eta_1(\beta_2 - \beta_1)},
\end{aligned}$$

and

$$\begin{aligned}
B_2 &= \frac{e^{-y\eta_1}B_{21}}{\eta_1 A} \\
&= -\frac{e^{-y\eta_1}(\beta_4-\beta_3)}{(\beta_4-\eta_2)(-\beta_3+\eta_2)} \Bigg/ \frac{\eta_1(\beta_2-\beta_1)(\beta_4-\beta_3)}{(\beta_1-\eta_1)(\beta_2-\eta_1)(\beta_3-\eta_2)(\beta_4-\eta_2)} \\
&= e^{-\eta_1 y}\frac{(\beta_1-\eta_1)(\beta_2-\eta_1)}{\eta_1(\beta_2-\beta_1)}.
\end{aligned}$$

Let $y=0$, we then get

$$\begin{aligned}
C_1 &= A_1 - B_1 \\
&= \frac{\beta_2(\eta_1-\beta_1)}{\eta_1(\beta_2-\beta_1)} - \frac{(\eta_1-\beta_1)(\beta_2-\eta_1)}{\eta_1(\beta_2-\beta_1)} \\
&= \frac{\eta_1-\beta_1}{\beta_2-\beta_1},
\end{aligned}$$

and

$$\begin{aligned}
C_2 &= A_2 - B_2 \\
&= \frac{\beta_1(\beta_2-\eta_1)}{\eta_1(\beta_2-\beta_1)} - \frac{(\beta_1-\eta_1)(\beta_2-\eta_1)}{\eta_1(\beta_2-\beta_1)} \\
&= \frac{\beta_2-\eta_1}{\beta_2-\beta_1}.
\end{aligned}$$

So these are all consistent with Theorem 2.1 of Kou and Wang (2003) for the one-sided exit problem of the doubly exponential jump diffusion process. $\square$

Remark 3.3. *To find $E\left(e^{-\alpha\tau}; X_\tau \le -a\right)$, we can consider the time when $-X$ first exits the interval $[-b, a]$ at level a and apply the previous results.*

Define

$$\hat{X}_t = -X_t = -\sigma W_t - \mu t - \sum_{i=1}^{N_t} Y_i,$$

where W is a Brownian motion with $W_0 = 0$, N is a Poisson process with rate λ and (Y_i) are i.i.d. random variables with density function

$$\hat{f}_{-Y}(y) = p\eta_1 e^{\eta_1 y}1_{\{y<0\}} + q\eta_2 e^{-\eta_2 y}1_{\{y>0\}}, \;\; p,q>0, \;\; p+q=1.$$

In addition, W, N and (G_i) are independent. Process $\hat{X}_t$ has a generator

$$\hat{L}u(x) = \frac{1}{2}\sigma^2 u''(x) - \mu u'(x) + \lambda \int_{-\infty}^{\infty} [u(x+y) - u(x)] f_{-Y}(y) dy.$$

For any $\theta \in (-\eta_2, \eta_1)$,

$$E\left(e^{\theta \hat{X}_t}\right) = e^{\hat{\psi}(\theta) t},$$

where

$$\hat{\psi}(\theta) = -\theta\mu + \frac{1}{2}\theta^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 + \theta} + \frac{q\eta_2}{\eta_2 - \theta} - 1\right).$$

For $\alpha > 0$, equation

$$\hat{\psi}(x) = \alpha$$

has exactly four solutions $\widehat{\beta}_1, \widehat{\beta}_2, -\widehat{\beta}_3, -\widehat{\beta}_4$ such that

$$0 < \widehat{\beta}_1 < \eta_1 < \widehat{\beta}_2, \text{ and } 0 < \widehat{\beta}_3 < \eta_2 < \widehat{\beta}_4;$$

We can get similar results to the above theorems for $\hat{X}_t$ with $\widehat{A}_i$, $\widehat{B}_i$ and $\widehat{C}_i$ following the similar notation as for A_i, B_i and C_i, where $i = 1, 2, 3, 4$.

Chapter 4

Range time for jump diffusion

Next, we will discuss the range time of the jump diffusion X in (3.1). As we defined in Chapter 1, T is the first time when the range of the process reaches r, the process should be necessarily at an extremum at time T. Correspondingly, S is the first time when the process is at the other extremum up to time T. The joint Laplace transform for S and T is given in the following theorem and it is the main result of my thesis.

Theorem 4.1. *For any $\alpha_1, \alpha_2 > 0$ and $r > 0$,*

$$E(e^{-\alpha_1 S - \alpha_2 (T-S)}; X_S - X_T = r) = \sum_{i=1}^{4} \frac{C_i}{\beta_i}(1 - e^{-\beta_i r}) \left(\sum_{i=1}^{2} \widehat{C}_i \widehat{\beta}_i e^{-\widehat{\beta}_i r} - \sum_{i=3}^{4} \widehat{C}_i \widehat{\beta}_i \right).$$

Proof. Our idea is to divide the interval $[0, r]$ into 2^n parts and consider the problem on each small interval. Define

$$A_{k,n} = \left\{ \frac{kr}{2^n} \le \overline{X}_T < \frac{(k+1)r}{2^n}, k = 0, 1, 2 \cdots, 2^n - 1 \right\},$$

and

$$A_n = \bigcup_{k=0}^{2^n - 1} A_{k,n}.$$

For $0 \le k \le 2^n - 1$, define

$$S_{k,n}^{-} = \inf \left\{ 0 \le t \le T : X_t \ge \frac{kr}{2^n} \right\},$$

and

$$T_{k,n}^- = \inf\left\{0 \le t \le T, X_t \le -\frac{r(2^n-k-1)}{2^n}\right\},$$

with the convention that $\inf\emptyset = \infty$. Denote

$$S_n^- = \sum_{k=0}^{2^n-1} S_{k,n}^- I_{A_{k,n}} \text{ and } T_n^- = \sum_{k=0}^{2^n-1} T_{k,n}^- I_{A_{k,n}}.$$

We can see that

$$S_n^- \le S \text{ and } T_n^- \le T.$$

Since with probability one, S can not be a jump time for X, then $S_n^- \uparrow S$ and $X_{S_n^-} \to X_S$ $a.s.$ In addition, $T_n^- \uparrow T$; $X_{T_n^-} \to X_T$ $a.s$ when $X_T - X_S = r$ and $X_{T_n^-} = X_T$ for n large enough when $X_T - X_S > r$.

In view of two-sided jumps, the process can reach S in two ways: the process X may either reach $kr/2^n$, $k = 0, 1, 2, ..., 2^n - 1$, continuously or by an upward jump. Define

$$\lfloor x \rfloor := \begin{cases} \frac{ri}{2^n} I_{\{\frac{ri}{2^n} \le x < \frac{r(i+1)}{2^n}\}}, & i = 0, ..., 2^n-1; \\ \frac{r(i+1)}{2^n} I_{\{\frac{ri}{2^n} \le x < \frac{r(i+1)}{2^n}\}}, & i = -2^n+1, ..., 0. \end{cases}$$

$$\begin{aligned}
&E(e^{-\alpha_1 S - \alpha_2(T-S)}; X_S - X_T = r) \\
=&\lim_{n\to\infty} E\left(e^{-\alpha_1 S_n^- - \alpha_2(T_n^- - S_n^-)}; \lfloor X_{S_n^-} \rfloor - \lfloor X_{T_n^-} \rfloor = r - \frac{r}{2^n}\right) \\
=&\lim_{n\to\infty} \sum_{k=0}^{2^n-1} E\left(e^{-\alpha_1 S_{k,n}^- - \alpha_2(T_{k,n}^- - S_{k,n}^-)}; X_{S_{k,n}^-} > \frac{kr}{2^n}, \left\lfloor X_{T_{k,n}^-} \right\rfloor = -\frac{(2^n-k-1)r}{2^n}\right) \\
&+ \lim_{n\to\infty} \sum_{k=0}^{2^n-1} E\left(e^{-\alpha_1 S_{k,n}^- - \alpha_2(T_{k,n}^- - S_{k,n}^-)}; X_{S_{k,n}^-} = \frac{kr}{2^n}, \left\lfloor X_{T_{k,n}^-} \right\rfloor = -\frac{(2^n-k-1)r}{2^n}\right) \\
:=& I + II. \qquad (4.1)
\end{aligned}$$

For term I, let $y^* = X_{S_{k,n}^-}$, $y = y^* - \frac{kc}{2^n}$. Then X_t first crosses $kr/2^n$, for $k =$

$0, 1, 2, ..., 2^n - 1$, because of the jumps.

Then by Theorem 3.2 with $b = \frac{kr}{2^n}, a = \frac{r(2^n-k)}{2^n}$ and Proposition 3.1 with $\widehat{a} = \frac{r}{2^n} - (y^* - \frac{kr}{2^n})$ and $\widehat{b} = r + y^* - \frac{r(k+1)}{2^n}$, we have

$$\begin{aligned}
&\sum_{k=0}^{2^n-1} E\left(e^{-\alpha_1 S_{k,n}^- - \alpha_2 (T_{k,n}^- - S_{k,n}^-)}; X_{S_{k,n}^-} > \frac{kr}{2^n}, \left\lfloor X_{T_{k,n}^-} \right\rfloor = -\frac{(2^n - k - 1)r}{2^n}\right) \qquad (4.2)\\
=&\sum_{k=0}^{2^n-1} \int_0^{\frac{r}{2^n}} E_0(e^{-\alpha_1 S_{k,n}^-}, S_{k,n}^- < T_{k,n}^-, X_{S_{k,n}^-} - \frac{kr}{2^n} \in dy^*) E_y^*(e^{-\alpha_2 T}; S_{k+1,n}^- > T_{k,n}^-)\\
=&\sum_{k=0}^{2^n-1} \int_0^{\frac{r}{2^n}} \left(-\frac{e^{-y\eta_1} B_{11}}{A} e^{-\frac{\beta_1 kr}{2^n}} - \frac{e^{-y\eta_1} B_{21}}{A} e^{-\frac{\beta_2 kr}{2^n}} - \frac{e^{-y\eta_1} B_{31}}{A} e^{-\frac{\beta_3 r(2^n-k)}{2^n}} - \frac{e^{-y\eta_1} B_{41}}{A} e^{-\frac{\beta_4 r(2^n-k)}{2^n}}\right)\\
&\times \left(\widehat{C}_1 e^{-\widehat{\beta}_1 (r+y^*-\frac{(k+1)r}{2^n})} + \widehat{C}_2 e^{-\widehat{\beta}_2 (r+y^*-\frac{(k+1)r}{2^n})} + \widehat{C}_3 e^{-\widehat{\beta}_3 (\frac{(k+1)r}{2^n}-y^*)} + \widehat{C}_4 e^{-\widehat{\beta}_4 (\frac{(k+1)r}{2^n}-y^*)}\right) dy^*\\
:=&\mathbf{E}_n.
\end{aligned}$$

By (3.6) with $i = 2$, we have

$$\widehat{C}_1 e^{-\widehat{\beta}_1 r} + \widehat{C}_2 e^{-\widehat{\beta}_2 r} + \widehat{C}_3 + \widehat{C}_4 = 0.$$

Hence, using Taylor expansions for the exponential functions at 0,

$$\begin{aligned}
&\widehat{C}_1 e^{-\widehat{\beta}_1 (r+y^*-\frac{(k+1)r}{2^n})} + \widehat{C}_2 e^{-\widehat{\beta}_2 (r+y^*-\frac{(k+1)r}{2^n})} + \widehat{C}_3 e^{-\widehat{\beta}_3 (\frac{(k+1)r}{2^n}-y^*)} + \widehat{C}_4 e^{-\widehat{\beta}_4 (\frac{(k+1)r}{2^n}-y^*)} - 0\\
=&\widehat{C}_1 e^{-\widehat{\beta}_1 r} (e^{-\widehat{\beta}_1 (y^*-\frac{(k+1)r}{2^n})} - 1) + \widehat{C}_2 e^{-\widehat{\beta}_2 r} (e^{-\widehat{\beta}_2 (y^*-\frac{(k+1)r}{2^n})} - 1)\\
&+ \widehat{C}_3 (e^{-\widehat{\beta}_3 (\frac{(k+1)r}{2^n}-y^*)} - 1) + \widehat{C}_4 (e^{-\widehat{\beta}_4 (\frac{(k+1)r}{2^n}-y^*)} - 1)\\
=&\left(\widehat{C}_1 \widehat{\beta}_1 e^{-\widehat{\beta}_1 r} + \widehat{C}_2 \widehat{\beta}_2 e^{-\widehat{\beta}_2 r} - \widehat{C}_3 \widehat{\beta}_3 - \widehat{C}_4 \widehat{\beta}_4\right) \times \left(\frac{(k+1)r}{2^n} - y\right) + o(2^{-2n})\\
\leq&\left(\widehat{C}_1 \widehat{\beta}_1 e^{-\widehat{\beta}_1 r} + \widehat{C}_2 \widehat{\beta}_2 e^{-\widehat{\beta}_2 r} - \widehat{C}_3 \widehat{\beta}_3 - \widehat{C}_4 \widehat{\beta}_4\right) \times \frac{r}{2^n} + o(2^{-2n})
\end{aligned}$$

On the other hand,

$$\int_0^{\frac{r}{2^n}} e^{-y\eta_1} dy = \frac{1}{\eta_1}(1 - e^{-\frac{r\eta_1}{2^n}}).$$

Calculating the following sums over k, we have

$$\sum_{k=0}^{2^n-1} e^{\frac{-\beta_i rk}{2^n}} = \frac{1-e^{-\beta_i r}}{1-e^{\frac{-\beta_i r}{2^n}}}, \quad i=1,2,$$

and

$$\sum_{k=0}^{2^n-1} e^{\frac{-\beta_j r(2^n-k)}{2^n}} = \frac{e^{-\beta_j r}-1}{1-e^{\frac{\beta_j r}{2^n}}}, \quad j=3,4.$$

So,

$$\begin{aligned}\mathbf{E}_n \leq & \frac{r}{A\eta_1 2^n}(1-e^{-\frac{r\eta_1}{2^n}}) \times \left(\widehat{C}_1\widehat{\beta}_1 e^{-\widehat{\beta}_1 r} + \widehat{C}_2\widehat{\beta}_2 e^{-\widehat{\beta}_2 r} - \widehat{C}_3\widehat{\beta}_3 - \widehat{C}_4\widehat{\beta}_4\right) \\ & \times \left[\sum_{i=1}^{2} \frac{B_{i1}(1-e^{-\beta_i r})}{1-e^{-\frac{\beta_i r}{2^n}}} + \sum_{j=3}^{4} \frac{B_{j1}(e^{-\beta_j r}-1)}{1-e^{\frac{\beta_j r}{2^n}}}\right]. \end{aligned} \tag{4.3}$$

Let $n \to \infty$, by (4.3) we can get

$$\begin{aligned} & \lim_{n\to\infty} \sum_{k=0}^{2^n-1} E\left(e^{-\alpha_1 S_{k,n}^- - \alpha_2(T_{k,n}^- - S_{k,n}^-)}; X_{S_{k,n}^-} > \frac{kr}{2^n}, \left\lfloor X_{T_{k,n}^-}\right\rfloor = -\frac{(2^n-k-1)r}{2^n}\right) \\ \leq & \lim_{n\to\infty} \frac{r}{A2^n}\left(\widehat{C}_1\widehat{\beta}_1 e^{-\widehat{\beta}_1 r} + \widehat{C}_2\widehat{\beta}_2 e^{-\widehat{\beta}_2 r} - \widehat{C}_3\widehat{\beta}_3 - \widehat{C}_4\widehat{\beta}_4\right) \sum_{i=1}^{4} \frac{e^{-\beta_i r}-1}{\beta_i} \\ = & \ 0 \end{aligned}$$

For the term II in (4.1), X reaches the running maximum up to time T due to the Brownian motion, we want to apply Proposition 3.1 with $b = \frac{kr}{2^n}, a = \frac{r(2^n-k)}{2^n}$ for X and with $\widehat{a} = \frac{r}{2^n}, \widehat{b} = r$ for $\widehat{X}$. We can take the limit on n directly to get the exact results. By Proposition 3.1,

$$\begin{aligned} & \lim_{n\to\infty} \sum_{k=0}^{2^n-1} E\left(e^{-\alpha_1 S_{k,n}^- - \alpha_2(T_{k,n}^- - S_{k,n}^-)}; X_{S_{k,n}^-} = \frac{kr}{2^n}, \left\lfloor X_{T_{k,n}^-}\right\rfloor = -\frac{(2^n-k-1)r}{2^n}\right) \\ = & \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \left(C_1 e^{\frac{-\beta_1 rk}{2^n}} + C_2 e^{\frac{-\beta_2 rk}{2^n}} + C_3 e^{\frac{-\beta_3 r(2^n-k)}{2^n}} + C_4 e^{\frac{-\beta_4 r(2^n-k)}{2^n}}\right) \\ & \times (\widehat{C}_1 e^{-\widehat{\beta}_1 r} + \widehat{C}_2 e^{-\widehat{\beta}_2 r} + \widehat{C}_3 e^{\frac{-\widehat{\beta}_3 r}{2^n}} + \widehat{C}_4 e^{\frac{-\widehat{\beta}_4 r}{2^n}}). \end{aligned} \tag{4.4}$$

Taking sums over k, we have

$$\sum_{k=0}^{2^n-1} e^{\frac{-\beta_i rk}{2^n}} = \frac{1-e^{-\beta_i r}}{1-e^{\frac{-\beta_i r}{2^n}}}, \quad i=1,2,$$

and

$$\sum_{k=0}^{2^n-1} e^{\frac{-\beta_j r(2^n-k)}{2^n}} = \frac{e^{-\beta_j r}-1}{1-e^{\frac{\beta_j r}{2^n}}}, \quad j=3,4.$$

By (3.6) with $i=2$, we also have

$$\widehat{C}_1 e^{-\widehat{\beta}_1(r+\frac{r}{2^n})} + \widehat{C}_2 e^{-\widehat{\beta}_2(r+\frac{r}{2^n})} + \widehat{C}_3 + \widehat{C}_4 = 0.$$

Hence, using Taylor expansions for the exponential functions at 0,

$$\begin{aligned}
&\widehat{C}_1 e^{-\widehat{\beta}_1 r} + \widehat{C}_2 e^{-\widehat{\beta}_2 r} + \widehat{C}_3 e^{\frac{-\widehat{\beta}_3 r}{2^n}} + \widehat{C}_4 e^{\frac{-\widehat{\beta}_4 r}{2^n}} - 0 \\
=&\widehat{C}_1(e^{-\widehat{\beta}_1 r} - e^{-\widehat{\beta}_1(r+\frac{r}{2^n})}) + \widehat{C}_2(e^{-\widehat{\beta}_2 r} - e^{-\widehat{\beta}_2(r+\frac{r}{2^n})}) \\
&+ \widehat{C}_3(e^{\frac{-\widehat{\beta}_3 r}{2^n}} - 1) + \widehat{C}_4(e^{\frac{-\widehat{\beta}_4 r}{2^n}} - 1) \\
=&\widehat{C}_1 e^{-\widehat{\beta}_1 r}\frac{\widehat{\beta}_1 r}{2^n} + \widehat{C}_2 e^{-\widehat{\beta}_2 r}\frac{\widehat{\beta}_2 r}{2^n} - \widehat{C}_3\frac{\widehat{\beta}_3 r}{2^n} - \widehat{C}_4\frac{\widehat{\beta}_4 r}{2^n} + o(2^{-2n})
\end{aligned}$$

So we have

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{k=0}^{2^n-1} E\left(e^{-\alpha_1 S_{k,n}^- - \alpha_2(T_{k,n}^- - S_{k,n}^-)}; X_{S_{k,n}^-} = \frac{kr}{2^n}, \left\lfloor X_{T_{k,n}^-}\right\rfloor = -\frac{(2^n-k-1)r}{2^n}\right) \\
=&\sum_{i=1}^{4}\frac{C_i}{\beta_i}(1-e^{-\beta_i r})(\widehat{C}_1\widehat{\beta}_1 e^{-\widehat{\beta}_1 r} + \widehat{C}_2\widehat{\beta}_2 e^{-\widehat{\beta}_2 r} - \widehat{C}_3\widehat{\beta}_3 - \widehat{C}_4\widehat{\beta}_4).
\end{aligned}$$

Combining these two cases, we can get the desired results for this theorem. □

Theorem 4.2. *For any $\alpha_1, \alpha_2 > 0$ and $r > 0$,*

$$\begin{aligned}
&E(e^{-\alpha_1 S - \alpha_2(T-S)}; X_S - X_T > r) \\
=&\sum_{i=1}^{4}\frac{C_i}{\beta_i}(1-e^{-\beta_i r})(\widehat{B}_1\widehat{\beta}_1 e^{-\widehat{\beta}_1 r} + \widehat{B}_2\widehat{\beta}_2 e^{-\widehat{\beta}_2 r} - \widehat{B}_3\widehat{\beta}_3 - \widehat{B}_4\widehat{\beta}_4).
\end{aligned}$$

Proof. In this theorem, we consider the case that the process jumps over the previous running minimum by a downward jump. We just need to follow the proof of Theorem 4.1 by changing $\widehat{C}_i$ to $\widehat{B}_i$. □

The joint Laplace transform of (S, T, X_T) for two-sided jump diffusion processes is given in the following Theorem.

Theorem 4.3. *Given any $x < 0, \alpha_1, \alpha_2 > 0$, we have for $-r < x < 0$,*

$$\begin{aligned}
&E(e^{-\alpha_1 S - \alpha_2 (T-S)}; X_S - X_T = r, X_T < x) \\
&= \left(\sum_{i=1}^{2} \frac{C_i}{\beta_i}(1 - e^{-\beta_i(r+x)}) + \sum_{i=3}^{4} \frac{C_i}{\beta_i}(e^{\beta_i x} - e^{-\beta_i r}) \right) \\
&\quad \times \left(\sum_{i=1}^{2} \widehat{C}_i \widehat{\beta}_i e^{-\widehat{\beta}_i r} - \sum_{j=3}^{4} \widehat{C}_j \widehat{\beta}_j \right),
\end{aligned}$$

and for $x < -r$,

$$\begin{aligned}
&E(e^{-\alpha_1 S - \alpha_2 (T-S)}; X_S - X_T > r, X_T < x) \\
&= e^{\eta_1 (x+r)} \left(\sum_{i=1}^{2} \frac{C_i}{\eta_1 + \beta_i}(1 - e^{-(\beta_i + \eta_1) r}) - \sum_{i=3}^{4} \frac{C_i}{\beta_i - \eta_1}(e^{-\beta_i r} - e^{-\eta_1 r}) \right) \\
&\quad \times \left(\sum_{i=1}^{2} \widehat{B}_{i1} \widehat{\beta}_i e^{-\widehat{\beta}_i r} - \sum_{i=3}^{4} \widehat{B}_{i1} \widehat{\beta}_i \right),
\end{aligned}$$

while for $-r < x < 0$,

$$\begin{aligned}
&E(e^{-\alpha_1 S - \alpha_2 (T-S)}; X_S - X_T > r, X_T < x) \\
&= \Bigg\{ \sum_{i=1}^{2} \frac{C_i}{\beta_i}(1 - e^{-\beta_i(r+x)}) + \sum_{i=3}^{4} \frac{C_i}{\beta_i}(e^{\beta_i x} - e^{-\beta_i r}) \\
&\quad + e^{\eta_1 (x+r)} \left(\sum_{i=1}^{2} \frac{C_i e^{-(\eta_1 + \beta_i) r}}{\eta_1 + \beta_i}(e^{-(\eta_1 + \beta_i) x} - 1) + \sum_{i=3}^{4} \frac{C_i e^{-\eta_1 r}}{\eta_1 - \beta_i}(e^{-(\eta_1 - \beta_i) x} - 1) \right) \Bigg\} \\
&\quad \times \frac{1}{\eta_1 A} \left(\sum_{i=1}^{2} \widehat{B}_{i1} \widehat{\beta}_i e^{-\widehat{\beta}_i r} - \sum_{i=3}^{4} \widehat{B}_{i1} \widehat{\beta}_i \right).
\end{aligned}$$

Proof. Similarly to Theorems 4.1 and 4.2, we already know that (4.2) is 0 when n goes to infinity,

$$
\begin{aligned}
&E(e^{-\alpha_1 S-\alpha_2(T-S)}; X_S - X_T = r, X_T < x)\\
&= \lim_{n\to\infty} \sum_{k=0}^{\left\lfloor \frac{2^n(x+r)}{r} - 1 \right\rfloor} \left(C_1 e^{-\frac{\beta_1 rk}{2^n}} + C_2 e^{-\frac{\beta_2 rk}{2^n}} + C_3 e^{-\frac{\beta_3 r(2^n-k)}{2^n}} + C_4 e^{-\frac{\beta_4 r(2^n-k)}{2^n}} \right)\\
&\quad \times \left(\widehat{C}_1 e^{-\widehat{\beta}_1 r} + \widehat{C}_2 e^{-\widehat{\beta}_2 r} + \widehat{C}_3 e^{-\frac{\widehat{\beta}_3 r}{2^n}} + \widehat{C}_4 e^{-\frac{\widehat{\beta}_4 r}{2^n}} \right)\\
&= \left(\sum_{i=1}^{2} \frac{C_i}{\beta_i} (1 - e^{-\beta_i(r+x)}) + \sum_{i=3}^{4} \frac{C_i}{\beta_i} (e^{\beta_i x} - e^{-\beta_i r}) \right) \times \left(\sum_{i=1}^{2} \widehat{C}_i \widehat{\beta}_i e^{-\widehat{\beta}_i r} - \sum_{j=3}^{4} \widehat{C}_j \widehat{\beta}_j \right).
\end{aligned}
$$

Then

$$
\begin{aligned}
&E(e^{-\alpha_1 S-\alpha_2(T-S)}; X_S - X_T > r, X_T < x)\\
=&E(e^{-\alpha_1 S-\alpha_2(T-S)}; X_S - X_T > r, X_T < x, 1_{\{x<-r\}})\\
&+ E(e^{-\alpha_1 S-\alpha_2(T-S)}; X_S - X_T > r, X_T < x, 1_{\{-r<x<0\}}).
\end{aligned}
$$

When $x < -r$, by Theorem 3.2 with $b = \frac{kr}{2^n}$, $a = \frac{r(2^n-k)}{2^n}$ and Proposition 3.1 with $\widehat{a} = \frac{r}{2^n}$, $\widehat{b} = r$, $\widehat{y} = -x - \frac{(2^n-k)r}{2^n}$, we get

$$
\begin{aligned}
&E(e^{-\alpha_1 S-\alpha_2(T-S)}; X_S - X_T > r, X_T < x)\\
&= \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \left(C_1 e^{-\frac{\beta_1 rk}{2^n}} + C_2 e^{-\frac{\beta_2 rk}{2^n}} + C_3 e^{-\frac{\beta_3 r(2^n-k)}{2^n}} + C_4 e^{-\frac{\beta_4 r(2^n-k)}{2^n}} \right)\\
&\quad \times \left(\widehat{B}_1 e^{-\widehat{\beta}_1 r} + \widehat{B}_2 e^{-\widehat{\beta}_2 r} + \widehat{B}_3 e^{-\frac{\widehat{\beta}_3 r}{2^n}} + \widehat{B}_4 e^{-\frac{\widehat{\beta}_4 r}{2^n}} \right)\\
&= \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \left(C_1 e^{-\frac{\beta_1 rk}{2^n}} + C_2 e^{-\frac{\beta_2 rk}{2^n}} + C_3 e^{-\frac{\beta_3 r(2^n-k)}{2^n}} + C_4 e^{-\frac{\beta_4 r(2^n-k)}{2^n}} \right)\\
&\quad \times \frac{1}{\eta_1 A} e^{\eta_1 (x + \frac{(2^n-k)r}{2^n})} \left(\widehat{B}_{11} e^{-\widehat{\beta}_1 r} + \widehat{B}_{21} e^{-\widehat{\beta}_2 r} + \widehat{B}_{31} e^{-\frac{\widehat{\beta}_3 r}{2^n}} + \widehat{B}_{41} e^{-\frac{\widehat{\beta}_4 r}{2^n}} \right)\\
&=e^{\eta_1(x+r)} \left(\sum_{i=1}^{2} \frac{C_i}{\eta_1 + \beta_i} (1 - e^{-(\beta_i+\eta_1)r}) - \sum_{i=3}^{4} \frac{C_i}{\beta_i - \eta_1} (e^{-\beta_i r} - e^{-\eta_1 r}) \right)\\
&\quad \times \left(\sum_{i=1}^{2} \widehat{B}_{i1} \widehat{\beta}_i e^{-\widehat{\beta}_i r} - \sum_{i=3}^{4} \widehat{B}_{i1} \widehat{\beta}_i \right).
\end{aligned}
$$

When $-r < x < 0$, by Theorem 3.2 with $b = \frac{kr}{2^n}$, $a = \frac{r(2^n-k)}{2^n}$ and Theorem 3.1 with $\widehat{a} = \frac{r}{2^n}$, $\widehat{b} = r$, $\widehat{y} = 0$ for the first part and $\widehat{a} = \frac{r}{2^n}$, $\widehat{b} = r$, $\widehat{y} = -x - \frac{(2^n-k)r}{2^n}$ for the second part, we have

$$
\begin{aligned}
& E(e^{-\alpha_1 S - \alpha_2 (T-S)}; X_S - X_T > r, X_T < x) \\
&= \lim_{n\to\infty} \sum_{k=0}^{\left\lfloor \frac{2^n(x+r)}{r} - 1 \right\rfloor} \left(C_1 e^{-\frac{\beta_1 rk}{2^n}} + C_2 e^{-\frac{\beta_2 rk}{2^n}} + C_3 e^{-\frac{\beta_3 r(2^n-k)}{2^n}} + C_4 e^{-\frac{\beta_4 r(2^n-k)}{2^n}} \right) \\
&\quad \times \left(\widehat{B}_1 e^{-\widehat{\beta}_1 r} + \widehat{B}_2 e^{-\widehat{\beta}_2 r} + \widehat{B}_3 e^{-\frac{\widehat{\beta}_3 r}{2^n}} + \widehat{B}_4 e^{-\frac{\widehat{\beta}_4 r}{2^n}} \right) \\
&\quad + \sum_{k=\left\lfloor \frac{2^n(x+r)}{r} \right\rfloor}^{2^n-1} \left(C_1 e^{-\frac{\beta_1 rk}{2^n}} + C_2 e^{-\frac{\beta_2 rk}{2^n}} + C_3 e^{-\frac{\beta_3 r(2^n-k)}{2^n}} + C_4 e^{-\frac{\beta_4 r(2^n-k)}{2^n}} \right) \\
&\quad \times \left(\widehat{B}_1 e^{-\widehat{\beta}_1 r} + \widehat{B}_2 e^{-\widehat{\beta}_2 r} + \widehat{B}_3 e^{-\frac{\widehat{\beta}_3 r}{2^n}} + \widehat{B}_4 e^{-\frac{\widehat{\beta}_4 r}{2^n}} \right) \\
&= \lim_{n\to\infty} \sum_{k=0}^{\left\lfloor \frac{2^n(x+r)}{r} - 1 \right\rfloor} \left(C_1 e^{-\frac{\beta_1 rk}{2^n}} + C_2 e^{-\frac{\beta_2 rk}{2^n}} + C_3 e^{-\frac{\beta_3 r(2^n-k)}{2^n}} + C_4 e^{-\frac{\beta_4 r(2^n-k)}{2^n}} \right) \\
&\quad \times \frac{1}{\eta_1 A} \left(\widehat{B}_{11} e^{-\widehat{\beta}_1 r} + \widehat{B}_{21} e^{-\widehat{\beta}_2 r} + \widehat{B}_{31} e^{-\frac{\widehat{\beta}_3 r}{2^n}} + \widehat{B}_{41} e^{-\frac{\widehat{\beta}_4 r}{2^n}} \right) \\
&\quad + \sum_{k=\left\lfloor \frac{2^n(x+r)}{r} \right\rfloor}^{2^n-1} \left(C_1 e^{-\frac{\beta_1 rk}{2^n}} + C_2 e^{-\frac{\beta_2 rk}{2^n}} + C_3 e^{-\frac{\beta_3 r(2^n-k)}{2^n}} + C_4 e^{-\frac{\beta_4 r(2^n-k)}{2^n}} \right) \\
&\quad \times \frac{1}{\eta_1 A} e^{\eta_1 (x + \frac{(2^n-k)r}{2^n})} \left(\widehat{B}_{11} e^{-\widehat{\beta}_1 r} + \widehat{B}_{21} e^{-\widehat{\beta}_2 r} + \widehat{B}_{31} e^{-\frac{\widehat{\beta}_3 r}{2^n}} + \widehat{B}_{41} e^{-\frac{\widehat{\beta}_4 r}{2^n}} \right) \\
&= \Bigg\{ \sum_{i=1}^{2} \frac{C_i}{\beta_i} (1 - e^{-\beta_i (r+x)}) + \sum_{i=3}^{4} \frac{C_i}{\beta_i} (e^{\beta_i x} - e^{-\beta_i r}) \\
&\quad + e^{\eta_1 (x+r)} \left(\sum_{i=1}^{2} \frac{C_i e^{-(\eta_1+\beta_i) r}}{\eta_1 + \beta_i} (e^{-(\eta_1+\beta_i)x} - 1) + \sum_{i=3}^{4} \frac{C_i e^{-\eta_1 r}}{\eta_1 - \beta_i} (e^{-(\eta_1 - \beta_i)x} - 1) \right) \Bigg\} \\
&\quad \times \frac{1}{\eta_1 A} \left(\sum_{i=1}^{2} \widehat{B}_{i1} \widehat{\beta}_i e^{-\widehat{\beta}_i r} - \sum_{i=3}^{4} \widehat{B}_{i1} \widehat{\beta}_i \right).
\end{aligned}
$$

□

With the above theorem, we can find the defective distribution for X_T.

Corollary 4.1. *Given any $x < 0, \alpha_1, \alpha_2 \to 0+$, we have for $-r < x < 0$,*

$$
\begin{aligned}
&P(X_T < x, X_S - X_T = r) \\
=&\left(C_1(r+x) + \frac{C_2}{\widetilde{\beta}_2}(1 - e^{-\widetilde{\beta_2}(r+x)}) + \sum_{i=3}^{4} \frac{C_i}{\widetilde{\beta}_i} e^{\widetilde{\beta}_i x}(1 - e^{-\widetilde{\beta}_i (r+x)}) \right) \\
&\times \left(\widehat{C}_2 \widetilde{\widehat{\beta}}_2 e^{-\widetilde{\widehat{\beta}}_2 r} - \widehat{C}_3 \widetilde{\widehat{\beta}}_3 - \widehat{C}_4 \widetilde{\widehat{\beta}}_4 \right),
\end{aligned}
$$

while for $x < -r$,

$$
\begin{aligned}
&P(X_T < x, X_S - X_T > r) \\
=&e^{\eta_1(x+r)} \left(\frac{C_1(1 - e^{-\eta_1 r})}{\eta_1} + \frac{C_2}{\eta_1 + \beta_2}(1 - e^{-(\beta_2+\eta_1) r}) - \sum_{i=3}^{4} \frac{C_i}{\beta_i - \eta_1}(e^{-\beta_i r} - e^{-\eta_1 r}) \right) \\
&\times \left(\widehat{B}_{21} \widehat{\beta}_2 e^{-\widehat{\beta}_2 r} - \sum_{i=3}^{4} \widehat{B}_{i1} \widehat{\beta}_i \right),
\end{aligned}
$$

and for $-r < x < 0$, we have

$$
\begin{aligned}
&P(X_T < x, X_S - X_T > r) \\
=&\Bigg\{ C_1(r+x) + \frac{C_2}{\beta_2}(1 - e^{-\beta_2(r+x)}) + \sum_{i=3}^{4} \frac{C_i}{\beta_i}(e^{\beta_i x} - e^{-\beta_i r}) \\
&+ e^{\eta_1(x+r)} \left(\frac{C_1 e^{-\eta_1 r}}{\eta_1}(e^{-\eta_1 x} - 1) + \frac{C_2 e^{-(\eta_1+\beta_2) r}}{\eta_1 + \beta_2}(e^{-(\eta_1+\beta_2)x} - 1) + \sum_{i=3}^{4} \frac{C_i e^{-\eta_1 r}}{\eta_1 - \beta_i}(e^{-(\eta_1 - \beta_i)x} - 1) \right) \Bigg\} \\
&\times \frac{1}{\eta_1 A} \left(\widehat{B}_{21} \widehat{\beta}_2 e^{-\widehat{\beta}_2 r} - \sum_{i=3}^{4} \widehat{B}_{i1} \widehat{\beta}_i \right),
\end{aligned}
$$

where $\widetilde{\beta}_i$ is the solution of $\psi(x) = 0$ and $\widetilde{\widehat{\beta}}_i$ is the solution of $\hat{\psi}(x) = 0$, for $i = 1, ..., 4$, while C_i, $\widehat{C}_i$ and $\widehat{B}_{i1}$ are redefined by letting $\beta_1, \widehat{\beta}_1 = 0$ (see Lemma 3.1).

Proof. Let $\alpha_1, \alpha_2 \to 0+$ in Theorem 4.3, then the distribution for X_T is given by

$$
\begin{aligned}
&P(X_T < x, X_S - X_T = r)\\
=& \lim_{\substack{\alpha_1 \to 0^+ \\ \alpha_2 \to 0^+}} \left(\sum_{i=1}^{2} \frac{C_i}{\beta_i}(1 - e^{-\beta_i(r+x)}) + \sum_{i=3}^{4} \frac{C_i}{\beta_i}(e^{\beta_i x} - e^{-\beta_i r}) \right) \times \left(\sum_{i=1}^{2} \widehat{C}_i \widehat{\beta}_i e^{-\widehat{\beta}_i r} - \sum_{j=3}^{4} \widehat{C}_j \widehat{\beta}_j \right)\\
=& \left(C_1(r+x) + \frac{C_2}{\widetilde{\beta}_2}(1 - e^{-\widetilde{\beta}_2(r+x)}) + \sum_{i=3}^{4} \frac{C_i}{\widetilde{\beta}_i} e^{\widetilde{\beta}_i x}(1 - e^{-\widetilde{\beta}_i(r+x)}) \right)\\
&\times \left(\widehat{C}_2 \widetilde{\widehat{\beta}}_2 e^{-\widetilde{\widehat{\beta}}_2 r} - \widehat{C}_3 \widetilde{\widehat{\beta}}_3 - \widehat{C}_4 \widetilde{\widehat{\beta}}_4 \right),
\end{aligned}
$$

where we need Lemma 3.1 for the last two equations.

When $x < -r$,

$$
\begin{aligned}
&P(X_T < x, X_S - X_T > r)\\
=& \lim_{\substack{\alpha_1 \to 0^+ \\ \alpha_2 \to 0^+}} e^{\eta_1(x+r)} \left(\sum_{i=1}^{2} \frac{C_i}{\eta_1 + \beta_i}(1 - e^{-(\beta_i + \eta_1)r}) - \sum_{i=3}^{4} \frac{C_i}{\beta_i - \eta_1}(e^{-\beta_i r} - e^{-\eta_1 r}) \right)\\
&\times \left(\sum_{i=1}^{2} \widehat{B}_{i1} \widehat{\beta}_i e^{-\widehat{\beta}_i r} - \sum_{i=3}^{4} \widehat{B}_{i1} \widehat{\beta}_i \right)\\
=& e^{\eta_1(x+r)} \left(\frac{C_1(1 - e^{-\eta_1 r})}{\eta_1} + \frac{C_2}{\eta_1 + \beta_2}(1 - e^{-(\beta_2 + \eta_1)r}) - \sum_{i=3}^{4} \frac{C_i}{\beta_i - \eta_1}(e^{-\beta_i r} - e^{-\eta_1 r}) \right)\\
&\times \left(\widehat{B}_{21} \widehat{\beta}_2 e^{-\widehat{\beta}_2 r} - \sum_{i=3}^{4} \widehat{B}_{i1} \widehat{\beta}_i \right),
\end{aligned}
$$

where we need Lemma 3.1 for the last two equations.

When $-r < x < 0$,

$$
\begin{aligned}
&P(X_T < x, X_S - X_T > r) \\
=&\lim_{\substack{\alpha_1\to 0^+\\ \alpha_2\to 0^+}} \Bigg\{ \sum_{i=1}^{2} \frac{C_i}{\beta_i}(1 - e^{-\beta_i(r+x)}) + \sum_{i=3}^{4} \frac{C_i}{\beta_i}(e^{\beta_i x} - e^{-\beta_i r}) \\
&+ e^{\eta_1(x+r)} \left(\sum_{i=1}^{2} \frac{C_i e^{-(\eta_1+\beta_i)r}}{\eta_1 + \beta_i}(e^{-(\eta_1+\beta_i)x} - 1) + \sum_{i=3}^{4} \frac{C_i e^{-\eta_1 r}}{\eta_1 - \beta_i}(e^{-(\eta_1-\beta_i)x} - 1) \right) \Bigg\} \\
&\times \frac{1}{\eta_1 A} \left(\sum_{i=1}^{2} \widehat{B}_{i1} \widehat{\beta}_i e^{-\widehat{\beta}_i r} - \sum_{i=3}^{4} \widehat{B}_{i1} \widehat{\beta}_i \right) \\
=&\Bigg\{ C_1(r+x) + \frac{C_2}{\beta_2}(1 - e^{-\beta_2(r+x)}) + \sum_{i=3}^{4} \frac{C_i}{\beta_i}(e^{\beta_i x} - e^{-\beta_i r}) \\
&+ e^{\eta_1(x+r)} \left(\frac{C_1 e^{-\eta_1 r}}{\eta_1}(e^{-\eta_1 x} - 1) + \frac{C_2 e^{-(\eta_1+\beta_2)r}}{\eta_1 + \beta_2}(e^{-(\eta_1+\beta_2)x} - 1) + \sum_{i=3}^{4} \frac{C_i e^{-\eta_1 r}}{\eta_1 - \beta_i}(e^{-(\eta_1-\beta_i)x} - 1) \right) \Bigg\} \\
&\times \frac{1}{\eta_1 A} \left(\widehat{B}_{21} \widehat{\beta}_2 e^{-\widehat{\beta}_2 r} - \sum_{i=3}^{4} \widehat{B}_{i1} \widehat{\beta}_i \right),
\end{aligned}
$$

where we need Lemma 3.1 for the last two equations. □

Chapter 5

Conclusion and future work

Previous results on the range process, the first range time and the first passage time can be found in several papers. In this thesis, we first obtain the Laplace transforms of the first passage times for two-sided exit problem. It is difficult to study the first passage times for general jump diffusion processes with arbitrary jumps, due to a possible overshoot, $X_{\tau_b} - b$, over the boundary. The double exponential jump diffusion process offers a rare case in which analytical solutions for the first passage times are feasible. The results are consistent with those for one-sided exit problem in Kou and Wang (2003). We also derive explicit expressions for the joint distribution of the times when the process reaches the running maximum and minimum, up to range r, using the idea of considering each interval with length $r/2^n$ and applying the two-sided exit time's results. The distribution for the process at the range time is also given.

In the future, we can work on the range time for jump diffusion process where the jumps follow a mixture of exponential distribution. We can also consider the range time for Lévy process with one-sided jumps. For Lévy process with general two-sided jumps, it is still an open problem to characterize its range time.

Following Kou and Wang (2004), we would like to apply the results we obtained on the range time to price the options related to the fluctuation of an asset value process. Also, by using the Gaver-Stehfest algorithm, we can try to fine the numerical values for the distribution of the two-sided exit time through the inversion of Laplace transforms.

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