Weak Convergence Approach to Compound Poisson Risk Processes Perturbed by Diffusion

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Abstract

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The ruin probability, the joint density function of the surplus immediately prior to ruin, the deficit at ruin, and the time to ruin and the expected discounted penalty function for the classical as well as for the diffusion risk model have been studied by many authors. We consider a sequence of risk processes, which converges weakly to the standard Wiener process when, for instance, the number of policies in a large insurance portfolio goes to infinity, and is added to the classical risk process. The resultant process is a diffusion perturbed classical risk model. We study this model and obtained the ruin probabilities, the joint density function of the surplus immediately prior to ruin, the deficit at ruin, the time to ruin and the expected discounted penalty function for the diffusion risk model by the weak convergence. In other words, we show that these quantities converge weekly to the corresponding quantities of the diffusion risk model for large number of policies. Numerical illustrations of the expected discounted penalty function and ruin probabilities are also presented.

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Contents

Introduction

The claim size distribution, of course, represents one major aspect of a risk model. The specification of the structure of the claim counting process is an another important aspect. In the classical model of risk theory, the number of claims occurred in an insurance business is assumed to follow a Poisson process. The surplus immediately prior to ruin, the deficit at ruin and the time to ruin play important roles in the classical risk theory. As usual these quantities are denoted by $U(T^-)$, $|U(T)|$ and T respectively. Extensive literature on the topics of ruin probability, the joint and marginal distributions of $U(T^-)$, $|U(T)|$ and T for the classical risk model (CRM) as well as for the diffusion risk model (DRM) is available. For instance, see, Gerber et al. (1987), Dufresne and Gerber (1988, 1991), Dickson (1992), Dos Reis (1993), Gerber and Shiu (1997), Gerber and Landry (1998), Lin and Willmot (1999), Tsai (2001), Tsai and Willmot (2002), Wang (2001) or Zhang and Wang (2003). In particular, Gerber and Shiu (1997) obtained the joint density function of the random variables $U(T^-)$, $|U(T)|$ and T for the CRM. To find the joint density function they use a duality argument. They also introduce an expected discounted penalty function with the force of interest $\delta \geq 0$ and show that its explicit expression converges to the traditional ruin probability when the force of interest δ is zero. Gerber (1970) extended the CRM by adding an independent diffusion process to this model which is called the diffusion model.

Based on this model Dufresne and Gerber (1991) derive explicit expressions of ruin probabilities caused by either oscillations in the diffusion or a claim. In the DRM, the diffusion term is a Weiner process with zero infinitesimal drift and the infinitesimal variance of $2D > 0$. It is assumed that the aggregate claims and the Wiener process are independent. The physical interpretation of the diffusion term is an additional uncertainty of the aggregate claims or alternately an uncertainty to the premium income of the insurance company.

Zhang and Wang (2003) obtain the explicit formula of the joint density function of $U(T^-)$, $|U(T)|$ and T for the DRM by using the similar argument to Gerber and Shiu (1997) and the strong Markov property. For $D = 0$, the DRM turns to be the CRM and it is shown that the ruin probability and the joint density function of the DRM coincide with the corresponding quantities of the CRM respectively.

Gerber and Shiu (1997) derive a defective renewal equation for the expected discounted penalty function in the CRM and obtain an explicit expression for this function by using the Laplace transform technique. If the discounted factor is set to zero this function then coincides with the ruin probability.

Dufresne and Gerber (1991) consider the DRM and derive a defective renewal equation for the survival probability and similarly for the ruin probability caused by oscillations. By applying standard technique of renewal theory argument, they solve the renewal equation and obtain explicit expressions. From these, it easy to get the ruin probability either due to oscillations or due to claims, and the ruin probability caused by a claim for the DRM. Gerber and Landry (1998) consider a more general expected discounted penalty function (EDPF) and show that it satisfies a defective renewal equation.

Goals of the Thesis

In Chapter 1 of the thesis, a brief review of the compound Poisson process and the ruin probability is presented. The joint and marginal distributions of $U(T^-)$, $|U(T)|$ and T are studied. The expected discounted penalty function and its defective renewal equation is discussed in Chapter 1. Though the CRM model is mathematically simple to analyze but it also admits physical interpretations (e.g., a large portfolio of insurance policy holders, each having a (time homogeneous) small rate of experiencing a claim, gives rise to a claim counting process that is very close to a Poisson process).

The DRM is introduced in Chapter 2. The ruin probability, the joint and marginal densities, and distributions of the random variables $U(T^-)$, $|U(T)|$ and T are discussed for this process. Also, the defective renewal equation of an EDPF is studied.

Finally, Chapter 3 is devoted to the main work of the thesis. First all basic and necessary theorems of weak convergence are discussed. A sequence of risk processes which converges weakly to the standard Wiener process when the number of policies is infinite, is added to the CRM. This new model is again a CRM, i.e., compound Poisson process. We show that the ruin probability, the joint density function of

$U(T^{-})$, $|U(T)|$ and T, and the EDPF for this new model converges weakly to the corresponding quantities of the DRM when, for instance, the number of policies in insurance business goes to infinity.

Our approach provides a simpler way of obtaining the results of , for example, Dufresne and Gerber (1991) and Tsai and Willmot (2002). In particular, we show that the expected discounted penalty function obeys a defective renewal equation in the DRM, under an easily verifiable condition on the penalty function, compared to the condition of Tsai and Willmot (2002). The appendix contains some numerical results for our model.

Chapter 1

Classical Model

1.1 Classical Risk Model

Here the claim counting process is considered to be a homogeneous Poisson process (simply called Poisson process). We review here the theory of Poisson processes and compound Poisson processes (CPP).

1.1.1 The Poisson Process

Consider a claim counting process $N = \{N(t) : t \geq 0\}$ with a claim arrival intensity parameter $\lambda > 0$ such that:

1. $N(0) = 0$;
2. N has stationary and independent increments;

then $N(t)$ is Poisson distributed with mean λt. That is, for all $s, t > 0$,

$$Pr\{N(s+t) - N(s) = n\} = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \qquad n = 0, 1, 2, 3, \ldots$$

Note that if the claim inter-occurrence times are i.i.d. random variables with a common distribution function, then $\{N(t) : t \geq 0\}$ is called a renewal process. The Poisson process is a special case of renewal process, i.e, when the claim inter-occurrence times are exponentially distributed.

1.1.2 Compound Poisson Process

Suppose that individual claim sizes X_1, X_2, X_3, $\ldots$ are nonnegative, i.i.d. random variables with a common distribution function $P(x) = Pr\{X \leq x\}$, where $P(0) = 0$, p is the corresponding density function. Claim sizes are assumed independent of the claim counting process $\{N(t) : t \geq 0\}$. Let μ be the mean claim sizes. The aggregate claim process $\{S(t) : t \geq 0\}$, where $S(t) = \sum_{i=1}^{N(t)} X_i$ (with $S(t) = 0$ if $N(t) = 0$) denotes the aggregate claims in the time interval $[0, t)$, is called the CPP with parameter $\lambda > 0$. Thus, for any fixed $t \geq 0$, $S(t)$ has a compound Poisson distribution which is denoted by $S(t) \sim C.P.[\lambda t; P]$. Therefore the insurer's surplus process for the CPP is defined by

$$U(t) = u + ct - S(t), \qquad t \geq 0, \tag{1.1}$$

where $U(0) = u \geq 0$ is the initial reserve of the insurance company, $c = \lambda\mu(1 + \theta)$ is the premium received continuously at a constant rate per unit time, $\theta > 0$ is the relative security loading. The surplus process (1.1) is known as the classical risk model (CRM).

Definition 1.1 *Let $T = inf\{t : U(t) < 0\}$ be the first time that the surplus becomes negative with initial reserve u, then T is called the time to ruin ($T = \infty$ if ruin does* *not occur). The probability $\psi(u) = Pr\{T < \infty | U(0) = u\}$ is called the ultimate ruin probability and the distribution function $\psi(u,t) = Pr\{T \le t | U(0) = u\}$ is called the finite time ruin probability.*

It is note that $\psi(u)$ is not necessarily 1 as T is a defective random variable. Also, we have $\psi(u,t) \le \psi(u)$ for all $u \ge 0$, while $\psi(\infty) = 0$ and $\psi(u) = 1$ if $u < 0$ are two useful boundary conditions.

1.2 Ruin Probability

1.2.1 An Integro-differential Equation

Theorem 1.1 *The ruin probability $\psi(u,t)$, within time $t \ge 0$, satisfies the partial integro-differential equation*

$$\frac{\partial}{\partial t}\psi(u,t) = c\frac{\partial}{\partial u}\psi(u,t) + \lambda[1 - P(u)] - \lambda\psi(u,t) + \lambda \int_0^u \psi(u-x,t)dP(x), \qquad u \ge 0.$$

Proof. Panjer and Willmot (1992), p. 389. □

An explicit expression for the ruin probability $\psi(0,t)$ is given by the following theorem.

Theorem 1.2 *The ruin probability, within the time $t \ge 0$ with zero initial reserve, is given by*

$$\psi(0,t) = \frac{\lambda}{c}\int_0^t [1 - P(x)]dx.$$

Proof. Klugman et al. (1998), p. 534. □

Corollary 1.1 *The ultimate ruin probability $\psi(0)$, with the initial reserve $u = 0$, is given by*

$$\psi(0) = \frac{1}{1+\theta}.$$

Proof. By the definition of the ultimate ruin probability, we have $\psi(0) = \lim_{t\to\infty}\psi(0,t)$. From Theorem 1.2, when $t \to \infty$,

$$\begin{aligned}
\psi(0) &= \frac{\lambda}{c}\int_0^\infty [1-P(x)]dx \\
&= \frac{\lambda\mu}{c} \\
&= \frac{\lambda\mu}{\lambda\mu(1+\theta)} \\
&= \frac{1}{1+\theta}.
\end{aligned}$$

□

It is remarkable that the ruin probability with initial reserve zero is independent of the claim size distribution and is completely determined by the relative security loading $\theta > 0$ only.

1.2.2 Volterra Integro-differential Equation

An important equation for the ultimate ruin probability is the Volterra integro-differential equation.

Theorem 1.3 *The probability of ultimate ruin $\psi(u)$ satisfies the following Volterra integro-differential equation:*

$$\frac{\partial}{\partial u}\psi(u) = \frac{\lambda}{c}\psi(u) - \frac{\lambda}{c}\int_0^u \psi(u-x)dP(x) - \frac{\lambda}{c}[1-P(u)], \qquad u \geq 0.$$

Proof. Rolski et al. (1999), p. 162. □

The analytical or numerical solution of the equation in Theorem 1.3 for the ruin probability $\psi(u)$ is rather complicated for a general claim size distribution P. For some special choices of P, analytical or numerical solutions are possible, subject to the boundary conditions $\psi(\infty) = 0$ and $\psi(-u) = 1$ for $u > 0$. For example, if P is exponential or a mixture of exponential claim size distributions, then an exact analytical expression for $\psi(u)$ can be obtained.

Now consider another equation for the ultimate ruin probability by using Theorem 1.3 which gives a different approach to calculate $\psi(u)$. Let

$$P_e(x) = \frac{1}{\mu}\int_0^x \bar{P}(y)dy, \qquad x \geq 0, \tag{1.2}$$

be an equilibrium distribution function of P and $\bar{P}(x) = 1 - P(x)$ with $\bar{P}(0) = 1$.

Theorem 1.4 *The probability of ultimate ruin $\psi(u)$ satisfies the following defective renewal equation*

$$\psi(u) = \frac{1}{1+\theta}\int_0^u \psi(u-x)dP_e(x) + \frac{\bar{P}_e(u)}{1+\theta}, \qquad u \geq 0, \tag{1.3}$$

where $\bar{P}_e(u) = 1 - P_e(u)$ with $P_e(0) = 0$.

Proof. Klugman et al. (1998), p. 544. □

The Theorem 1.4 may be solved for $\psi(u)$ by using Laplace transform technique. Though an analytical solution of (1.3) is possible for some choices of claim severity distributions, it is not for general claim severity distributions, but can be solved numerically. This will be discussed in detail in Section 1.5. For the time being we just state that the ruin probability $\psi(u)$, that solves (1.3), can be expressed as the convolution of the tail of a compound geometric distribution, i.e.,

$$\psi(u) = \sum_{j=1}^{\infty} \frac{\theta}{1+\theta}(\frac{1}{1+\theta})^j \bar{P}_e^{*j}(u), \qquad u \geq 0, \tag{1.4}$$

where $\bar{P}_e^{*j}(u) = 1 - P_e^{*j}(u)$ is the tail of j-fold convolution of $\bar{P}_e$ with itself and $\theta > 0$ is the relative security loading.

Therefore we can calculate the ruin probability numerically by computing the tail of the j-fold convolution of a compound geometric distribution, using some known technique, e.g., recursive method, inverse transform method or direct evaluation of convolution. In some cases analytical forms of the j-fold convolution exist, that is, when the claim size distribution is closed under convolution (e.g., gamma, inverse Gaussian).

1.3 Asymptotic Ruin Probability

We start this section by defining the adjustment coefficient which will be used to obtain a bound of the ultimate ruin probability $\psi(u)$.

The adjustment coefficient r is the unique positive solution, if it exists, to the equation

$$1 + (1+\theta)\mu r = E(e^{rX}) \tag{1.5}$$

or equivalently

$$(1+\theta)\mu = \int_0^{\infty} e^{rx}\bar{P}(x)dx, \qquad x > 0.$$

Remark: The positive solution of equation (1.5) may not always exist, i.e., if the claim size distribution has no moment generating function (e.g., Pareto, lognormal or

Weibull).

Theorem 1.5 *(Cramer-Lundberg bound): Suppose the adjustment coefficient* $r > 0$ *satisfies that (1.5), then the ruin probability* $\psi(u)$ *satisfies*

$$\psi(u) \leq e^{-ru}, \qquad u \geq 0.$$

Proof. Klugman et al. (1998), p. 544. □

This bound is very simple and gives accurate results for some choices of claim severity distributions. But in practical situations the adjustment coefficient does not always exist, then the above inequality is not suitable to calculate the upper bound of ruin probability in insurance business. Many authors have derived lower and upper bounds for the ruin probability $\psi(u)$ based on the fact that $\psi(u)$ can be expressed as convolution of the tail of compound geometric distribution (e.g., De Vylder and Goovaerts (1984) or Cai and Garrido (1999)).

We now turn our discussion to evaluate the probability of ruin $\psi(u)$ with a different approach. The following Cramer's formula gives the asymptotic ruin probability. We use the notation $a(u) \sim b(u)$ as $u \to \infty$, to mean $\lim_{u \to \infty} \frac{a(u)}{b(u)} = 1$

Theorem 1.6 *Assume that* $r > 0$ *satisfies (1.5), then the probability of ruin is given by*

$$\psi(u) \sim \frac{\theta\mu}{\{E(Xe^{rX}) - \mu(1+\theta)\}} e^{-ru}, \qquad u \to \infty,$$

or equivalently

$$\psi(u) \sim \frac{\theta\mu}{\{M'_X(r) - \mu(1+\theta)\}} e^{-ru}, \qquad u \to \infty.$$

$M'_X(r)$ *is the derivative of the moment generating function of* X.

Proof. Panjer and Willmot (1992). □

Since the adjustment coefficient $r > 0$ satisfying (1.5) always exists for the light tailed claim severity distribution (e.g., exponential, gamma, or inverse Gaussian), so the ultimate ruin probability for the large initial reserve can be evaluated by the above Cramer's formula (Theorem 1.6. But for the heavy tailed subexponential distributions (e.g., Pareto, lognormal or Weibull) $r > 0$, satisfying (1.5), does not always exist. In these situations, there are some generalized asymptotic formula available such as Embrechts and Veraverbeke (1982) obtained $\psi(u) \sim \frac{\int_u^\infty [1-P(y)]dy}{\theta\mu} = \frac{\bar{P}(u)}{\theta}$ as $u \to \infty$ and for the medium tailed distributions they show that $\psi(u) \sim \frac{\theta\mu\xi\bar{P}(u)}{1+(1+\theta)\mu\xi - M_X(\xi)}$ as $u \to \infty$, where $M_X(\xi) < 1 + (1+\theta)\mu\xi$ if $M_X(\xi)$ exists for any $0 < \xi < t$ and $M_X(t) = \infty$.

In the next section, the joint and marginal density functions of the surplus immediately before ruin, the deficit at ruin, and the time to ruin is obtained. Also, the joint density function of the surplus immediately before ruin and the deficit at ruin are shown with initial reserve zero.

1.4 Joint and Marginal Density Functions

Let $U(T^-) = x$ be the surplus immediately before ruin, $|U(T)| = y$, the deficit at ruin and $T = t$, the time to ruin. For $U(0) = u$, let $f(x, y, t)$ be the joint density function of x, y and t. Then we have

$$f(x, y, t|u)dxdydt = Pr\{U(T^-) \in [x, x+dx], |U(T)| \in [y, y+dy], T \in [t, t+dt] \mid u\}$$

Gerber and Shiu (1997) prove the following theorem:

Theorem 1.7 *For $u \geq 0$,*

$$f(x, y, t|u) = \frac{\lambda}{c} p(x+y)\tilde{\pi}(t; u, x), \qquad x, y, t \geq 0,$$

where $\tilde{\pi}(t; u, x)dt$ is the probability that ruin does not occur by the time t and that there is an upcrossing of the surplus process at level x between times t and $t + dt$.

Proof. Gerber and Shiu (1997). $\square$

Let $T_x = \inf\{t : U(t) = x\}$ be the first time that the surplus reaches the level x. Suppose $\pi(t; u, x)$ and $\tilde{\pi}(t; u, x)dt$ denote the probability density function of T_x and the probability that ruin does not occur by time t, respectively and that there is an upcrossing of the surplus process at level x between times t and $t + dt$. Then by the duality argument Gerber and Shiu (1997) obtained the following result.

Corollary 1.2 *For $x, t > 0$,*

$$\pi(t; 0, x) = \tilde{\pi}(t; 0, x).$$

Proof. Gerber and Shiu (1997). $\square$

The process $\{e^{-\delta t + \rho U(t)} : t \geq 0\}$ is a martingale, where $\delta > 0$ is the force of interest, ρ is the solution of the equation $-\delta + c\xi + \lambda[\hat{p}(\xi) - 1] = 0$ and $\hat{p}(\xi)$ is the Laplace transform of $p(x)$. By the optional sampling theorem, Gerber and Shiu (1997) show the following result.

Corollary 1.3 *For $x > u \geq 0$, $t > 0$,*

$$\int_0^\infty e^{-\delta t}\pi(t; u, x)\, dt = e^{-\rho(x-u)}.$$

Proof. Gerber and Shiu (1997). □

For an initial reserve of $u = 0$, from Theorem 1.7 and using Corollary 1.2, we obtain

$$f(x, y, t|0) = \frac{\lambda}{c} p(x+y)\bar{\pi}(t; 0, x) = \frac{\lambda}{c} p(x+y)\pi(t; 0, x), \qquad x, y, t \geq 0. \tag{1.6}$$

Multiply (1.6) by $e^{-\delta t}$ and integrate it from 0 to ∞ with respect to t, then by Corollary 1.3, we get

$$f(x, y|0) = \frac{\lambda}{c} p(x+y)\, e^{-\rho x}, \qquad x, y > 0. \tag{1.7}$$

Integrating (1.7) with respect to y and x respectively, gives

$$f(x|0) = \frac{\lambda}{c}[1 - P(x)]\, e^{-\rho x}, \qquad x > 0. \tag{1.8}$$

and

$$f(y|0) = \frac{\lambda}{c} \int_0^\infty p(x+y)\, e^{-\rho x}\, dx, \qquad x, y > 0. \tag{1.9}$$

1.5 Defective Renewal Equation

Gerber and Shiu (1997) introduce an EDPF for $\delta \geq 0$,

$$\phi(u) = E\{e^{-\delta T + \rho U(T)} I(T < \infty) | U(0) = u\}, \qquad u \geq 0, \tag{1.10}$$

where

$$I(T < \infty) = \begin{cases} 1 & if \quad T < \infty \\ 0 & otherwise \end{cases}.$$

In particular, when $\delta = 0$, then

$$\phi(u) = E\{I(T < \infty)|U(0) = u\} = Pr\{T < \infty|U(0) = u\},$$

which is just the ultimate ruin probability. They also derive the following defective renewal equation by conditioning on whether or not ruin occurs at the first time that the surplus falls below the initial level u, and applying the law of iterated expectations.

$$\phi(u) = \int_0^u \phi(u-x)\, f(y|0)\, dy + \int_u^\infty e^{-\rho(y-u)}\, f(y|0)\, dy. \tag{1.11}$$

The general solution of (1.11) can be obtained by Laplace transform technique, and it is

$$\phi(u) = e^{\rho u} - [1 - \phi(0)][e^{\rho u} + \sum_{j=1}^{\infty} \int_0^u e^{\rho y} f^*(u-y|0)\, dy], \tag{1.12}$$

where $\phi(0) = \int_0^\infty e^{-\rho(y)}\, f(y|0)\, dy = \hat{f}(\xi)$.

For $\delta = 0$, (1.12) becomes $\phi(u) = \sum_{j=1}^{\infty} \frac{\theta}{1+\theta}(\frac{1}{1+\theta})^j \bar{P}_e^{*j}(u)$, which is exactly the ruin probability with initial reserve u.

In addition, Gerber and Shiu (1998) generalized (1.10) by introducing a non-negative penalty function $w(x_1, x_2)$, $0 \leq x_1,\ x_2 < \infty$, such that

$$\phi_w(u) = E\{e^{-\delta T} w(U(T^-), |U(T)|) I(T < \infty)|U(0) = u\},$$

where I is the same as defined in (1.10). If $\delta = 0$ and $w(x_1, x_2) = 1$, obviously, $\phi_w(u)$ coincides with the ruin probability, i.e., $\phi_w(u) = \psi(u)$.

For the evaluation of $\phi_w(u)$, Gerber and Shiu (1998) used an approach where $\phi_w(u)$ can be expressed as a function of the tail of a related compound geometric distribution. This is useful as there are many results available regarding the tail of compound geometric distributions, e.g., recursive formula [Panjer and Willmot (1992)], upper and lower bounds [Willmot (1994), Lin (1996), Willmot and Lin (1998)]. Also, Tijms (1994) and Dufresne and Gerber (1988) found the exact solutions for the tail of compound geometric distributions. Cramer-Lundberg's well-known asymptotic formula [e.g., Gerber (1979)] and Tijms approximation [e.g., Tijms (1986), Willmot (1997)] are also available. It is also possible to evaluate $\phi_w(u)$ for various choices of $w(x_1, x_2)$ under this approach.

Gerber and Shiu (1998) stated that $\phi_w(u)$ satisfies the defective renewal equation

$$\begin{aligned}\phi_w(u) &= \frac{\lambda}{c}\int_0^u \phi_w(u-x)\int_x^\infty e^{-\rho(y-x)}\,dP(y)\,dx \\ &\quad +\frac{\lambda}{c}e^{\rho u}\int_u^\infty e^{-\rho x}\int_x^\infty w(x,y-x)\,dP(y)\,dx, \qquad u \geq 0, \qquad (1.13)\end{aligned}$$

where $\rho(\delta) = \rho$ is the unique nonnegative solution of the equation

$$c\rho - \delta = \lambda - \lambda\int_0^\infty e^{-\rho y}dP(y),$$

with $\rho(0) = 0$. For the analytical representation of ρ, see De Vylder and Goovaerts (1998). To obtain the solution of (1.12) another defective renewal equation is considered as follows:

$$\phi_w(u) = \frac{1}{1+\beta}\int_0^u \phi_w(u-x)\,dG(x) + \frac{1}{1+\beta}H(u), \qquad u \geq 0 \qquad (1.14)$$

where $\beta > 0$, $G(x) = 1 - \bar{G}(x)$ is a distribution function with $G(0) = 0$, $H(u)$ is continuous for $u \geq 0$, and

$$K(u) = 1 - \bar{K}(u) = 1 - \sum_{j=1}^{\infty}\frac{\beta}{1+\beta}\,(\frac{1}{1+\beta})^j\,\bar{G}^{*j}(u), \qquad u \geq 0, \qquad (1.15)$$

is the associated compound geometric distribution and $\bar{G}^{*j}(u)$ is the j-fold convolution of $\bar{G}(u)$. The solution $\phi_w(u)$ to (1.13) in terms of $\bar{K}(u)$ is stated by the following theorem.

Theorem 1.8 *The solution $\phi_w(u)$ to (1.14) can be expressed as*

$$\begin{aligned}\phi_w(u) &= \frac{1}{\beta}\int_0^u H(u-x)\,dK(x) + \frac{1}{1+\beta}H(u) \\ &= -\frac{1}{\beta}\int_0^u \bar{K}(u-x)\,dH(x) - \frac{1}{\beta}H(0)\bar{K}(u) + \frac{1}{\beta}H(u).\end{aligned}$$

If $H(u)$ is differentiable, then

$$\phi_w(u) = -\frac{1}{\beta}\int_0^u \bar{K}(u-x)H'(x)\,dx - \frac{1}{\beta}H(0)\bar{K}(u) + \frac{1}{\beta}H(u), \qquad u \geq 0.$$

Proof. Lin and Willmot (1999). □

Now the solution of (1.13) can be obtained via the solution of (1.14). For this purpose, in particular, we have to show that (1.13) and (1.14) have the same form and by comparing them identify β, $G(x)$ and $H(u)$ in (1.13). The rearrangement of (1.13) gives

$$\begin{aligned}\phi_w(u) &= \frac{\lambda}{c}\int_0^\infty e^{-\rho y}\bar{P}(y)dy \int_0^u \phi_w(u-x)\{\frac{\int_x^\infty e^{-(y-x)}dP(y)}{\int_0^\infty e^{-\rho y}\bar{P}(y)dy}\}dx + \frac{\lambda}{c}\int_0^\infty e^{-\rho y}\bar{P}(y)dy \\ &\quad \{\frac{e^{\rho u}\int_u^\infty e^{-\rho x}\int_x^\infty w(x,y-x)dP(y)dx}{\int_0^\infty e^{-\rho y}\bar{P}(y)dy}\}, \qquad u \geq 0. \qquad (1.16)\end{aligned}$$

The equation (1.16) has the same form as (1.14), and so by comparing them we have

$$\beta = \frac{1+\theta}{\int_0^\infty e^{-\rho y}\,dP_e(y)} - 1, \qquad (1.17)$$

$$G'(x) = \frac{e^{\rho x}\int_x^\infty e^{-\rho y}\,dP(y)}{\int_0^\infty e^{-\rho y}\bar{P}(y)\,dy}, \qquad x \geq 0, \qquad (1.18)$$

$$\bar{G}(x) = \frac{e^{\rho x}\int_x^\infty e^{-\rho y}\bar{P}(y)\,dy}{\int_0^\infty e^{-\rho y}\bar{P}(y)\,dy}, \qquad x \geq 0, \tag{1.19}$$

and

$$H(u) = \frac{e^{\rho u}\int_u^\infty e^{-\rho x}\int_x^\infty w(x, y-x)dP(y)dx}{\int_0^\infty e^{-\rho y}\bar{P}(y)dy}, \qquad u \geq 0. \tag{1.20}$$

Since the solution of (1.14) is stated by Theorem 1.8 and hence the general solution of (1.16) (i.e., of (1.13)) is also stated by Theorem 1.8 with $\bar{K}(u)$, β, $\bar{G}(x)$, and $H(u)$ given by (1.15), (1.17), (1.19), and (1.20), respectively.

Let us investigate the special cases of the EDPF $\phi_w(u)$, i.e., the solution of the defective renewal equation (1.13).

Case (i): Consider the discounted factor $\delta = 0$ (i.e., $\rho(0) = 0$). Then (1.17), (1.19), and (1.20) imply that $\beta = \theta$, $\bar{G}(x) = \bar{P}_e(x)$ and $H(u) = \frac{1}{\mu}\int_u^\infty\int_x^\infty w(x, y-x)\,dP(y)\,dx$, respectively. Therefore, in this case the solution of (1.13) is given by Theorem 1.8 with these β, $\bar{G}(x)$ and $H(u)$. Another important fact is that the tail of the compound geometric distribution $\bar{K}(u) = \sum_{j=1}^\infty \frac{\theta}{1+\theta}(\frac{1}{1+\theta})^j\,\bar{P}_e^{*j}(u) = \psi(u)$. Hence when the discounted factor or the force of interest is zero, the EDPF $\phi_w(u)$ can be expressed in terms of the ultimate ruin probability $\psi(u)$. Also $\phi_w(u)$ reduces to $E\{w(U(T^-)), |U(T)|)I(T < \infty)|U(0) = u\}$, i.e., the expected value of the penalty function at ruin.

Case (ii): Consider $w(x_1, x_2) = 1$. Therefore (1.20) becomes $H(u) = \bar{G}(u)$ while (1.17) and (1.19) the remain same. Now the solution of (1.13) is stated by Theorem 1.8, where β, $\bar{G}(x)$ are given by (1.17), (1.19), respectively and $H(u) = \bar{G}(u)$. In this case, $\phi_w(u)$ reduces to $E\{e^{-\delta t}I(T < \infty)|U(0) = u\}$, i.e., the expectation of the present value of the ruin time or the Laplace transform of ruin time. Let us define the Laplace transforms of $G(u)$, $K(u)$ and $\bar{K}(u)$ as follows:

$$
\begin{aligned}
\hat{g}(\xi) &= \int_0^\infty e^{-\xi u}\, dG(u), \\
\hat{K}(\xi) &= K(0) + \int_0^\infty e^{-\xi u} K(u)\, du = \frac{\beta}{1+\beta-\hat{g}(\xi)}, \\
\hat{\bar{K}}(\xi) &= \int_0^\infty e^{-\xi u} \bar{K}(u) du.
\end{aligned}
$$

We also have that

$$
\begin{aligned}
\hat{\bar{K}}(\xi) &= \frac{1}{\xi}\{1 - \int_0^\infty e^{-\xi u} dK(u)\} \\
&= \frac{1}{1+\beta}\hat{g}(\xi)\hat{\bar{K}}(\xi) + \frac{1}{1+\beta}\frac{1-\hat{g}(\xi)}{\xi}.
\end{aligned}
$$

Taking the inverse Laplace transform, we get

$$
\bar{K}(u) = \frac{1}{1+\beta}\int_0^u \bar{K}(u-x)\, dG(x) + \frac{1}{1+\beta}\bar{G}(u), \qquad u \geq 0. \tag{1.21}
$$

Note that equation (1.21) has the same form as the defective renewal equation (1.14) and hence we can say that (1.21) is a defective renewal equation for the function $\bar{K}(u)$. By comparing (1.14) and (1.21), one immediately obtains that $\phi_w(u) = \bar{K}(u)$, where β and $\bar{G}(x)$ are given by (1.17) and (1.19), respectively.

Case *(iii)*: Suppose that the discounted factor $\delta = 0$ (i.e., $\rho(0) = 0$) and $w(x_1, x_2) = 1$. In this case, (1.17), (1.19) and (1.20) reduce to the simple forms $\beta = \theta$, $\bar{G}(x) = \bar{P}_e(x)$ and $H(u) = \bar{P}_e(x)$, respectively. Thus the solution $\phi_w(u)$ to (1.13) is stated by Theorem 1.8 with $\beta = \theta$, $\bar{G}(x) = \bar{P}_e(x)$ and $H(u) = \bar{P}_e(x)$. It can be easily shown that $\phi_w(u) = \sum_{j=1}^\infty \frac{\theta}{1+\theta}(\frac{1}{1+\theta})^j\, \bar{P}_e^{*j}(u) = \psi(u)$, i.e., the ultimate ruin probability.

1.6 Distribution and Density Functions with Penalty Function

The joint distribution function of $U(T^-)$ and $|U(T)|$ can be obtained by an appropriate choice of the non-negative penalty function $w(U(T^-), |U(T)|)$ in $\phi_w(u)$ in of Section 1.5. Then the joint density function from the joint distribution, and marginal distribution and density functions are also presented in this section.

1.6.1 Joint Distribution and Density Functions

To obtain the explicit expression of the joint distribution function, we choose the penalty function as follows.

For any fixed x and y,

$$w(x_1, x_2) = \begin{cases} 1 & if \quad x_1 \le x, \quad x_2 \le y \\ 0 & otherwise \end{cases}.$$

For $0 \le u < x$, by (1.20), we have

$$\begin{aligned} H(u) &= \frac{\lambda(1+\beta)}{c} e^{\rho u} \int_u^\infty e^{-\rho x} \int_x^\infty w(x, y-x)\, dP(y)\, dx \\ &= \frac{\lambda(1+\beta)}{c} e^{\rho u} \int_u^\infty e^{-\rho x} \bar{P}(x)\, dP(y)\, dx \\ &= \bar{G}(u) - \bar{G}(u+y) - e^{\rho(u-x)}[\bar{G}(x) - \bar{G}(x+y)]. \end{aligned}$$

Again, for $0 < x \le u$ and by using (1.19), we have $H(u) = 0$. By Theorem 1.8, for $0 \le u < x$ and since $x_1 \le x$, the explicit expression of the EDPF $\phi_w(u)$ is

$$\phi_w(u) = -\frac{1}{\beta} \int_0^u \bar{K}(u - x_1)\, dH(x_1) - \frac{1}{\beta} H(0)\bar{K}(u) + \frac{1}{\beta} H(u).$$

Substitute $H(u)$ in this equation, then it follows that

$$\begin{aligned}\phi_w(u) &= \frac{(1+\beta)}{\beta}\bar{K}(u) - \frac{1}{\beta}\int_0^u \bar{K}(u-x_1)dG(x_1+y) - \frac{1}{\beta}\bar{K}(u)G(y) + \\ &\frac{1}{\beta}e^{-\rho x}[\bar{G}(x) - \bar{G}(x+y)][\bar{K}(u) + \rho\int_0^u e^{\rho x_1}\bar{K}(u-x_1)dx_1 - e^{\rho u} - \frac{1}{\beta}\bar{G}(u+y)].\end{aligned}$$

If $0 < x \le u$, then $H(u) = 0$ and $\phi_w(u)$ becomes

$$\phi_w(u) = -\frac{1}{\beta}\int_0^x \bar{K}(u-x_1)\, dH(x_1) - \frac{1}{\beta}H(0)\bar{K}(u).$$

As previously, we substitute $H(u)$, to obtain

$$\begin{aligned}\phi_w(u) &= \frac{1}{\beta}\int_0^x \bar{K}(u-x_1)d[G(x_1) - G(x_1+y)] - \frac{1}{\beta}\bar{K}(u)G(y) \\ &+\frac{1}{\beta}e^{-\rho x}[\bar{G}(x) - \bar{G}(x+y)][\bar{K}(u) + \rho\int_0^u e^{\rho x_1}\bar{K}(u-x_1)dx_1 \\ &-e^{\rho x}\{\bar{K}(u-x) + \rho\int_0^{u-x} e^{\rho x_1}\bar{K}(u-x-x_1)dx_1 - \bar{K}(u-x)\}], \qquad u \ge 0.\end{aligned}$$

Now for $\delta = 0$ and $w(x_1, x_2) = 1$, if $x_1 \le x$, $x_2 \le y$, the EDPF $\phi_w(u)$ turns to have the following form

$$\begin{aligned}\phi_w(u) &= E\{w(x_1, x_2)I(T < \infty)|U(0) = u\} \\ &= Pr\{x_1 \le x,\ x_2 \le y,\ T < \infty|U(0) = u\} \\ &= F(x,\ y|U(0) = u) \\ &= F(x,\ y|u), \qquad u \ge 0,\end{aligned}$$

i.e., the joint distribution function of $U(T^-) = x$ and $|U(T)| = y$. Lin and Willmot (1999) derived an explicit expression for $F(x,\ y|u)$, which is stated in the following theorem.

Theorem 1.9 *The joint distribution function of $U(T^-)$ and $|U(T)|$ is as follows: If $0 < x \leq u$, then*

$$\begin{aligned} F(x,y|u) &= \frac{1}{\theta\mu}\int_0^x \psi(u-x_1)\{\bar{P}(x_1)-\bar{P}(x_1+y)\}\,dx_1 \\ &\quad +\frac{\psi(u)}{\theta}[P_e(x+y)-P_e(x)-P_e(y)]. \end{aligned}$$

If $0 \leq u < x$, then

$$\begin{aligned} F(x,y|u) &= \frac{1}{\theta\mu}\int_0^y \psi(u+y-x_1)\bar{P}(x_1)\,dx_1 + \frac{\psi(u)}{\theta}[P_e(x+y)-P_e(x)-P_e(y)] \\ &\quad +\frac{1+\theta}{\theta}[\psi(u)-\psi(u+y)] - \frac{1}{\theta}[P_e(x+y)-P_e(x)]. \end{aligned}$$

Proof. Lin and Willmot (1999). □

We obtain an explicit expression for the corresponding joint density function $f(x,y|u)$, simply by differentiating $F(x,y|u)$ with respect to x and y.

Corollary 1.4 *The joint density function of $U(T^-)$ and $|U(T)|$ is given by*

$$f(x,y|u) = \begin{cases} \frac{\lambda}{c}p(x+y)\frac{1-\psi(u)}{1-\psi(0)}, & if \quad 0 \leq u < x \\ \frac{\lambda}{c}p(x+y)\frac{\psi(u-x)-\psi(u)}{1-\psi(0)}, & if \quad 0 < x \leq u \end{cases}.$$

1.6.2 The Marginal Distribution and Density Functions

The marginal distribution and density functions can be derived from the joint distribution and density functions respectively.

Taking $y \to \infty$ and $x \to \infty$ in Theorem 1.9, then the marginal distributions of x and y are described by the following corollary.

Corollary 1.5

$$F(x|u) = \begin{cases} \frac{1}{\theta\mu}\int_0^x \psi(u-x_1)\bar{P}(x_1)dx_1 - \frac{\psi(u)}{\theta}P_e(x), & if \quad 0 < x \leq u \\ \{1+\frac{\bar{P}_e(x)}{\theta}\}\psi(u) - \frac{\bar{P}_e(x)}{\theta}, & if \quad 0 \leq u < x \end{cases}.$$

$$
F(y|u) = \begin{cases} \frac{1}{\theta\mu}\int_0^y \psi(u+y-x_1)\bar{P}(x_1)\,dx_1 + \frac{1+\theta}{\theta}[\psi(u)-\psi(u+y)] \\ -\frac{1}{\theta}P_e(y)\psi(u), \qquad if \qquad 0 \le u < x \\ 0, \qquad if \qquad 0 < x \le u \end{cases}.
$$

By differentiating the marginal distribution functions in Corollary 1.5, the marginal density functions are obtained.

Corollary 1.6

$$
f(x|u) = \begin{cases} \frac{\lambda}{c}\,\bar{p}(x)\frac{1-\psi(u)}{1-\psi(0)}, \qquad if \qquad 0 \le u < x \\ \frac{\lambda}{c}\,\bar{p}(x)\frac{\psi(u-x)-\psi(u)}{1-\psi(0)}, \qquad if \qquad 0 < x \le u \end{cases}.
$$

$$
f(y|u) = \begin{cases} \frac{1}{\theta\mu}\int_0^y \psi'(u+y-x_1)\bar{P}(x_1)\,dx_1 + \frac{1+\theta}{\theta} - \psi'(u+y), \qquad if \qquad 0 \le u < x \\ 0, \qquad if \qquad 0 < x \le u \end{cases}.
$$

Chapter 2

Diffusion Model

In this Chapter, we define the diffusion process (or Wiener process) and then introduce the classical model that is perturbed by diffusion, called diffusion model (or perturbed model).

For the CRM in Chapter 1, the income of an insurer is a linear function of time. But this is not realistic. Actually, the income of an insurer is a non-linear function of time because of some sources of uncertainty. For example, there are fluctuations in the number of customers, the claim arrival intensity may depend on time, the insurer investment of surplus, claims as well as premiums increase with inflation and also the difference of interest and inflation is not constant always in time. To add these additional uncertainties to the insurer surplus process, a diffusion process is added to the CRM.

We study the defective renewal equation to obtain the explicit expression of the ruin probability for this model. The joint density function of the surplus immediately prior to ruin, the deficit at ruin, and the time to ruin is presented. Also joint and marginal distributions, and density functions of the surplus immediately prior to ruin, and the deficit at ruin will be discussed briefly.

2.1 Diffusion Risk Model

Definition 2.1 *Diffusion process (or Wiener process or Brownian motion):*

Let $\{W(t) : t \geq 0\}$ be a continuous time stochastic process. Then, it is called diffusion process if

1. $W(0) = 0$,
2. $\{W(t) : t \geq 0\}$ *has stationary and independent increments, and*
3. *for all* $t > s \geq 0$, $W(t) - W(s)$ *is normally distributed with mean zero and variance* $2D(t-s) > 0$.

Remark: The process $\{W(t) : t \geq 0\}$ is called a Brownian motion with drift if it satisfies properties $(1) - (2)$, above, and additionally for all $t > 0$, $W(t)$ is normally distributed with mean $\mu_B t$ and variance $2Dt > 0$. That is, the diffusion process (or Wiener process) is a special case of Brownian motion with drift (i.e, when $\mu_B = 0$).

First Gerber (1970) extended the CRM by adding a diffusion process which is known as the CRM perturbed by diffusion, in short, the Diffusion Risk Model (DRM). For this model the surplus of the insurance company is given by

$$U_D(t) = u + ct - S(t) + W(t), \qquad t \geq 0, \tag{2.1}$$

where u, c and $S(t)$ are defined as in the CRM of Section 1.1 with $c - \lambda\mu > 0$. W is a diffusion process with infinitesimal drift zero, and infinitesimal variance

$2D > 0$. So, for any $t > 0$, the random variable $W(t)$ is normally distributed with mean zero and variance $2Dt$, i.e., $W(t) \sim N(0, 2Dt)$. Furthermore, it is supposed that $\{N(t) : t \geq 0\}$, the claim counting process; $\{S(t) : t \geq 0\}$, the aggregate claim process and $\{W(t) : t \geq 0\}$, the diffusion process are mutually independent. It is noted that the physical interpretation of the diffusion process is that an additional uncertainty of the aggregate claims; or an alternative interpretation is that it adds an uncertainty to the premium income of the insurance company.

2.2 Joint Density Function

Many authors study joint and marginal density functions of the characteristics: the surplus immediately prior to ruin, the deficit at ruin and the time to ruin for the CRM, for example, Gerber et al. (1987), Dufresne and Gerber (1988), Dickson and Waters (1992), Dickson (1992, 1993), Gerber and Shiu (1997), Willmot and Lin (1998, 2000), Lin and Willmot (1999), Wu et al. (2003) and references therein. In particular, Gerber and Shiu (1997) obtained an expression of the joint density of $U(T^-)$, $|U(T)|$ and T, for the CRM given in Section 1.4 by Theorem 1.7. Wu et al. (2003) give an explicit expression for Theorem 1.7 based on the CRM. Thereafter Zhang and Wang (2003) studied the DRM and obtained an explicit expression of the joint density of the random variables $U(T^-)$, $|U(T)|$ and T, which coincides with the one in Wu et al. (2003) and thus coincides with Theorem 1.7 when the diffusion process is removed from the DRM.

Before going to the joint density function, denote by $R(t) = ct - S(t) + W(t)$ and assume that g_{tD} is the density function of $R(t)$. Then an explicit expression for g_{tD} is presented by the following lemma.

Lemma 2.1 *The density function of $R(t)$ is given by*

$$g_{tD}(x) = \sum_{r=0}^{\infty} \frac{(\lambda t)^r}{r!} e^{-\lambda t} \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi D t}} e^{-\frac{y^2}{4Dt}} p^{*r}(ct + y - x)dy, \qquad x \in \Re.$$

Proof. Zhang and Wang (2003). □

Let us assume that $\pi_D(t; u, x)dx$ is the probability of non-ruin before time t and that the surplus then lies between x and $x + dx$, i.e.,

$$\pi_D(t; u, x)dx = Pr\{t < T,\ U_D(t) \in [x,\ x + dx]|U(0) = u\}$$

and $\tilde{\pi}_D(t; u, x)dx$ is the probability that the surplus does not reach the level $x + u$ by the time t and still lies between x and $x + dx$, i.e.,

$$\tilde{\pi}_D(t; u, x)dx = Pr\{t < T_{x+u},\ U_D(t) \in [x,\ x + dx]|U(0) = u\}.$$

Now by the duality argument it can be shown that

$$\pi_D(t; u, x) = \tilde{\pi}_D(t; u, x).$$

Zhang and Wang (2003) derive an explicit expression for $\pi_D(t; u, x)$ which is stated below.

Proposition 2.1 *The density $\pi_D(t; u, x)$ is as follows:*

$$\pi_D(t; u, x) = g_{tD}(x - u) - \int_0^t \frac{x}{s}\, g_{sD}(x)\, g_{(t-s)D}(-u)\, ds, \qquad u, x > 0, \qquad t \geq 0,$$

where $g_{tD}(x)$ is given by Lemma 2.1.

Proof. Zhang and Wang (2003). $\square$

Let $\psi_D(u,t)$ be the finite time ruin probability. Then ruin probability $\psi_D(u,t)$ can be decomposed as follows (based on the idea of Dufresne and Gerber (1991) for the ultimate ruin probability $\psi_D(u)$):

$$\psi_D(u,t) = \psi_d(u,t) + \psi_c(u,t), \qquad u,t \geq 0,$$

where $\psi_d(u,t)$ is the finite time ruin probability caused by oscillation and $\psi_c(u,t)$ is the finite time ruin probability caused by a claim. So we can write

$$\psi_d(u,t) = Pr\{T \leq t,\ |U(T)| = 0 \mid U(0) = u\}, \qquad u,t \geq 0,$$

and

$$\psi_c(u,t) = Pr\{T \leq t,\ |U(T)| > 0 \mid U(0) = u\}, \qquad u,t \geq 0.$$

Let $f_D(x,y,t|u)$ be the density function of $U(T^-)$, $|U(T)|$ and T for the DRM. Also, suppose that $f_d(x,y,t|u)$ and $f_c(x,y,t|u)$ are the corresponding density functions due to oscillations and due to claims respectively. Then, obviously, we have

$$f_D(x,y,t|u) = f_d(x,y,t|u) + f_c(x,y,t|u), \qquad u,t \geq 0,$$

Which is derived explicitely by Zhang and Wang (2003) and is stated below.

Theorem 2.1 *For $u \geq 0$, the joint density function $f_D(x,y,t|u)$ is*

$$f_D(x,y,t|u) = \begin{cases} \lambda p(x+y)[g_{tD}(x-u) - \int_0^t \frac{x}{s}\, g_{sD}(x)\, g_{(t-s)D}(-u)\, ds], & if \quad x > 0, \\ 0, \quad if \quad x = 0, \quad y \neq 0, \end{cases}$$

and also

$$f_D(0,0,t|u) = \frac{\partial \psi_d(u,t)}{\partial t},$$

where $g_{tD}(x)$ is given by Lemma 2.1.

Proof. Zhang and Wang (2003). $\square$

2.2.1 An Explicit Expression for $f(x, y, t|u)$

Here we try to derive a similar explicit expression of $f(x, y, t|u)$ for the CRM, from $f_D(x, y, t|u)$. If we remove the diffusion term from the density function $g_{tD}(x)$ (i.e., if $D = 0$), then $g_{tD}(x)$ becomes

$$g_t(x) = \sum_{r=0}^{\infty} \frac{(\lambda t)^r}{r!} e^{-\lambda t} p^{*r}(ct - x). \tag{2.2}$$

That is, $g_t(x)$ is the density function of $ct - \sum_{i=1}^{N(t)} X_i$. For the CRM, the probability of ruin due to oscillation is zero, i.e., $\psi_d(u, t) = 0$, which implies that $f_D(0, 0, t) = 0$.

Now we are in a position to write an explicit expression for the joint density function $f(x, y, t|u)$ in terms of $g_t(x)$ by Theorem 2.1. As we have, for $D = 0$, $g_{tD}(x) = g_t(x)$. Thus the joint density function $f(x, y, t|u)$ in terms of $g_t(x)$ is as follows.

Corollary 2.1

$$f(x, y, t|u) = \lambda p(x + y)[g_t(x - u) - \int_0^t \frac{x}{s}\, g_s(x)\, g_{(t-s)}(-u)\, ds], \qquad t \geq 0,$$

where $g_t(x)$ is given by (2.2).

Proof. Proof follows from Theorem 2.1 by letting $D = 0$. $\square$

It is difficult to get analytical expressions for the joint density function $f(x, y, t|u)$, even for simple severity distributions. A numerical evaluation is sometimes possible.

If the severity distribution is closed under convolutions, then an analytical expression for $f(x, y, t|u)$ may be found (i.e., exponential severities).

2.3 Ruin Probability

Let $\bar{\psi}_D(u) = Pr\{U_D(t) \geq 0 : \forall\, t \geq 0\}$ be the ultimate survival probability for the DRM. Then $\psi_D(u) = 1 - \bar{\psi}_D(u)$ is defined as the ultimate probability of ruin. It is well known that the ruin probability $\psi_D(u)$ can be decomposed as the sum of the probability of ruin caused by oscillations, $\psi_d(u)$ and that caused by a claim, $\psi_c(u)$, i.e.

$$\psi_D(u) = \psi_d(u) + \psi_c(u), \qquad u \geq 0. \tag{2.3}$$

We also have that $\bar{\psi}_D(0) = \psi_c(0) = 0$ which implies that $\psi_D(0) = \psi_d(0) = 1$, from the oscillating nature of sample paths. As in the CRM, here we are interested in ruin probability for the DRM. Dufresne and Gerber (1991) give an analytical expression of the ruin probability. Specially, they derive a defective renewal equation for the survival probability $\bar{\psi}_D(u)$ and solve it analytically.

2.3.1 Defective Renewal Equation for $\bar{\psi}_D(u)$

Dufresne and Gerber (1991) show that the survival probability $\bar{\psi}_D(u)$ satisfies the defective renewal equation

$$\bar{\psi}_D(u) = \frac{c - \lambda\mu}{c} P_1(u) + \frac{\lambda\mu}{c} \int_0^u \bar{\psi}_D(z) p_1 * p_e(u - z)\, dz, \qquad u \geq 0, \tag{2.4}$$

where $P_1(x) = 1 - e^{-\frac{c}{D}x}$ and $P_e(x) = \int_0^x \frac{1-P(y)}{\mu} dy$, $x \geq 0$, while p_1 and p_e are the corresponding density functions.

Theorem 2.2 *The solution $\bar{\psi}_D(u)$ to (2.4) is given by*

$$\bar{\psi}_D(u) = \sum_{j=0}^{\infty} (\frac{c-\lambda\mu}{c})(\frac{\lambda\mu}{c})^j P_1^{*(j+1)} * P_e^{*j}(u), \qquad u \geq 0.$$

Proof. Dufresne and Gerber (1991). □

Hence, from Theorem 2.2, it follows that the ultimate ruin probability $\psi_D(u)$ is

$$\psi_D(u) = \begin{cases} 1 - \sum_{j=0}^{\infty} (\frac{c-\lambda\mu}{c})(\frac{\lambda\mu}{c})^j P_1^{*(j+1)} * P_e^{*j}(u) \\ 1 - \sum_{j=0}^{\infty} (\frac{\theta}{1+\theta})(\frac{1}{1+\theta})^j P_1^{*(j+1)} * P_e^{*j}(u), \qquad u \geq 0 \end{cases}. \tag{2.5}$$

Remark: For $D = 0$, the ultimate ruin probability for the DRM coincides to the ultimate ruin probability for the CRM, i.e., $\psi_D(u) = \psi(u)$, for $u \geq 0$.

By a similar approach, used to obtain the survival probability $\bar{\psi}_D(u)$, the ruin probabilities $\psi_d(u)$ and $\psi_c(u)$ can be calculated. For example, we write down the defective renewal equation and the solution for $\psi_d(u)$. The defective renewal equation of $\psi_d(u)$ is

$$\psi_d(u) = 1 - P_1(u) + \frac{\lambda\mu}{c} \int_0^u \psi_d(u-z) p_1 * p_e(z)\, dz, \qquad u \geq 0. \tag{2.6}$$

Corollary 2.2 *For $u \geq 0$, the solution $\psi_d(u)$ to (2.6) is given by*

$$\psi_d(u) = \frac{D\bar{\psi}_D'(u)}{c-\lambda\mu} = \frac{D(1-\psi_D(u))'}{c-\lambda\mu}, \qquad u \geq 0.$$

Proof. Dufresne and Gerber (1991). □

Thus according to Corollary 2.2, the ultimate survival probability or ruin probability is determined, the ruin probability $\psi_d(u)$ can then be obtained numerically or if possible analytically, and hence $\psi_c(u)$ can be calculated immediately as we know that $\psi_D(u) = \psi_d(u) + \psi_c(u)$. If p is continuous, Wang (2001) show that $\psi_d(u)$ and $\psi_c(u)$ are twice continuously differentiable, and also gives explicit expressions for $\psi_d(u)$ and $\psi_c(u)$ if the severity distribution is exponential.

2.4 An Explicit Expression of a Generalized EDPF

In Chapter 1, we present the expected discounted penalty function $\phi_w(u)$ for the compound Poisson process. The defective renewal equations of $\bar{\psi}_D(u)$ and $\psi_d(u)$ are studied for the DRM in Section 2.3 of this chapter. Based on the idea of Dufresne and Gerber (1991), Gerber and Landry (1998) introduced the EDPF in the DRM associated with a non-negative constant penalty w_0 at ruin if ruin is due to oscillations, and a penalty function $w(U(T))$ at ruin, if ruin is due to claims. That is, for $\delta \geq 0$:

$$\phi_{Dy}(u) = w_0\ \phi_d(u) + E[e^{-\delta T} w(|U(T)|)\ I(T < \infty, U(T) < 0) \mid U(0) = u], \qquad u \geq 0,$$

where w_0 is a nonnegative constant, I is an indicator function, i.e., $I(T < \infty, U(T) < 0) = 1$ if $T < \infty$ and $U(T) < 0$, but 0 otherwise. Similarly, $\phi_d(u) = E[e^{-\delta T} I(T < \infty, U(T) = 0) \mid U(0) = u]$, also $I(T < \infty, U(T) = 0) = 1$ if $T < \infty$ and $U(T) = 0$, but 0 otherwise. We see that $\phi_d(u)$ is the Laplace transform or the expectation of the present value of the time of ruin T due to oscillation and the other term on the right hand side of $\phi_{Dy}(u)$ is the expected discounted penalty if the ruin occurs by a claim. When $\delta = 0$, then $\phi_d(u) = E[I(T < \infty, U(T) = 0) \mid U(0) = u]$ is the probability of ruin due to oscillations, i.e., $\phi_d(u) = \psi_d(u)$, and if $\delta = 0$ and $w(U(T)) = 1$, then $E[e^{-\delta T} w(U(T))\ I(T < \infty, U(T) < 0) \mid U(0) = u] = E[I(T < \infty, U(T) <$

$0) \mid U(0) = u]$, i.e., the probability of ruin due to a claim. Thus $\phi_{Dy}(u)$ becomes the ruin probability $\psi_D(u)$ in this case, given by (2.5) for the DRM.

Gerber and Landry (1998) also derive a defective renewal equation for the function $\phi_{Dy}(u)$:

$$\begin{aligned}\phi_{Dy}(u) &= \int_0^u \phi_{Dy}(u-y)g_D(y)dy + w_0 e^{-bu} + \int_u^\infty w(u-y)g_D(y)dy \\ &\quad -e^{-bu}\int_0^\infty w(-y)g_D(y)dy, \qquad u \geq 0,\end{aligned}$$

where $D = \frac{1}{2}\sigma^2$, $b = \rho_D + \frac{c}{D}$, $\rho_D = \rho_D(\delta, D)$ is the unique nonnegative solution of the generalized Lundberg's equation

$$c\xi + D\xi^2 + \lambda \int_0^\infty e^{-\xi y} dP(y) = \lambda + \delta$$

with $\rho_D(0, D) = 0$ and

$$g_D(y) = \frac{\lambda}{c}\int_0^y e^{-b(y-s)} \int_s^\infty e^{-\rho_D(x-s)} dP(x)ds. \tag{2.7}$$

But there are some restrictions for the applications of $\phi_{Dy}(u)$ as the given penalty $w(|U(T)|)$ is a function of $|U(T)|$ alone. To remove the restrictions Tsai and Willmot (2002) generalize the penalty function in such a way that w is a function of both $U(T^-)$ and $U(T)$, i.e., $w(U(T^-), |U(T)|)$. Then the corresponding generalized EDPF $\phi_D(u)$ involving $w(U(T^-), |U(T)|)$ is defined by

$$\phi_D(u) = w_0\phi_d(u) + \phi_c(u), \qquad u \geq 0, \tag{2.8}$$

where $\phi_c(u) = E[e^{-\delta T} w(U(T^-), |U(T)|) I(T < \infty, U(T) < 0) \mid U(0) = u]$. Note that $\phi_D(u)$ also coincides with $\psi_D(u)$ when $\delta = 0$ and $w(U(T^-), |U(T)|) = 1$.

Tsai and Willmot (2002) prove that the generalize EDPF $\phi_D(u)$ satisfies a defective renewal equation given by the following theorem.

Theorem 2.3 *For $D > 0$, if $\lim_{u\to\infty} e^{-\rho_D u}\phi_c(u) = 0$ and $\lim_{u\to\infty} e^{-\rho_D u}\phi_c'(u) = 0$, then $\phi_D(u)$ satisfies the defective renewal equation*

$$\phi_D(u) = \int_0^u \phi_D(u-y)g_D(y)dy + w_0 e^{-bu} + H_{\omega D}(u), \qquad u \geq 0,$$

where $g_D(y)$ is given by (2.8),

$$\omega(x) = \int_x^\infty w(x, y-x)dP(y), \qquad x \geq 0,$$

and

$$H_{\omega D}(u) = \frac{\lambda}{D}\int_0^u e^{-b(u-s)}\int_s^\infty e^{-\rho_D(x-s)}\omega(x)dxds.$$

Proof. Tsai and Willmot (2002). □

Lemma 2.2 *For $D > 0$, the Laplace transforms of $g_D(y)$, and $H_{\omega D}(u)$, respectively are given by*

$$\hat{g}_D(\xi) = \frac{\lambda[\hat{p}(\xi) - \hat{p}(\rho)]}{D(\rho-\xi)(b+\xi)}, \qquad \xi \geq 0,$$

$$\hat{H}_{\omega D}(\xi) = \frac{\lambda[\hat{\omega}(\xi) - \hat{\omega}(\rho)]}{D(\rho-\xi)(b+\xi)}, \qquad \xi \geq 0,$$

where $\hat{\omega}(\xi) = \int_0^\infty e^{-\xi x}[\int_x^\infty w(x, y-x)dP(y)]dx$ and $b = \frac{c}{D} + \rho$.

Proof. Tsai and Willmot (2002). □

Theorem 2.4 *For $D > 0$, the Laplace transform of $\phi_D(u)$ is given by*

$$\hat{\phi}_D(\xi) = \frac{\hat{H}_{\omega D}(\xi)}{1-\hat{g}_D(\xi)} + \frac{w_0}{(b+\xi)[1-\hat{g}_D(\xi)]}, \qquad \xi \geq 0.$$

Proof. For $D > 0$, $\phi_D(u)$ satisfies the defective renewal equation [Theorem 2.3]

$$\phi_D(u) = \int_0^u \phi_D(u-y)g_D(y)dy + w_0 e^{-bu} + H_{\omega D}(u), \qquad u \geq 0.$$

Taking Laplace transforms, then

$$\begin{aligned}
\hat{\phi}_D(\xi) &= \int_0^\infty e^{-\xi u} \int_0^u \phi_D(u-y)g_D(y)dy\, du \\
&\quad + w_0 \int_0^\infty e^{-\xi u} e^{-bu} du + \int_0^\infty e^{-\xi u} H_{\omega D}(u) du \\
&= \hat{\phi}_D(\xi)\hat{g}_D(\xi) + \frac{w_0}{b+\xi} + \hat{H}_{\omega D}(\xi) \\
&= \frac{\hat{H}_{\omega D}(\xi)}{1-\hat{g}_D(\xi)} + \frac{w_0}{(b+\xi)[1-\hat{g}_D(\xi)]}, \qquad \xi \geq 0.
\end{aligned}$$

□

Note that the explicit expression of the function $\phi_D(u)$, i.e., the solution of the defective renewal equation in Theorem 2.3 can be easily obtained, given by the following corollary.

Corollary 2.3 *For $D > 0$,*

$$\phi_D(u) = \int_0^u [H_{\omega D}(y) + w_0 e^{-by}] \sum_{r=1}^\infty g_D^{*r}(u-y)dy + w_0 e^{-bu} + H_{\omega D}(u), \qquad u \geq 0.$$

Proof. From Theorem 2.4

$$\begin{aligned}
\hat{\phi}_D(\xi) &= \frac{\hat{H}_{\omega D}(\xi)}{1-\hat{g}_D(\xi)} + \frac{w_0}{(b+\xi)[1-\hat{g}_D(\xi)]} \\
&= \hat{H}_{\omega D}(\xi) \sum_{r=0}^\infty g_D^{*r}(\xi) + \frac{w_0}{b+\xi} \sum_{r=0}^\infty g_D^{*r}(\xi) \\
&= \hat{H}_{\omega D}(\xi) + \hat{H}_{\omega D}(\xi) \sum_{r=1}^\infty g_D^{*r}(\xi) + \frac{w_0}{b+\xi} + \frac{w_0}{b+\xi} \sum_{r=1}^\infty g_D^{*r}(\xi), \qquad \xi \geq 0.
\end{aligned}$$

Taking inverse Laplace transforms, we have

$$\begin{aligned}\phi_D(u) &= H_{\omega D}(u) + \int_0^u H_{\omega D}(y) \sum_{r=1}^{\infty} g_D^{*r}(u-y)dy + w_0 e^{-bu} \\ &\quad + \int_0^u w_0 e^{-by} \sum_{r=1}^{\infty} g_D^{*r}(u-y)dy \\ &= \int_0^u [H_{\omega D}(y) + w_0 e^{-by}] \sum_{r=1}^{\infty} g_D^{*r}(u-y)dy + w_0 e^{-bu} + H_{\omega D}(u), \qquad u \geq 0.\end{aligned}$$

Which is an explicit expression for $\phi_D(u)$. □

The defective renewal equation [Gerber and Landry (1998)] for the function $\phi_d(u)$ is

$$\phi_d(u) = \int_0^u \phi_d(u-y) g_D(y)dy + e^{-bu}, \qquad u \geq 0. \tag{2.9}$$

The function $\phi_c(u)$ satisfies the defective renewal equation [Tsai and Willmot (2002)]

$$\phi_c(u) = \int_0^u \phi_c(u-y) g_D(y)dy + H_{\omega D}(u), \qquad u \geq 0, \tag{2.10}$$

where $g_D(y)$ and $H_{\omega D}(u)$ are defined in Theorem 2.3.

We can also find the explicit expression for $\phi_D(u)$ by solving the defective renewal equations (2.9) and (2.10).

2.5 Discounted Distribution and Density Functions with Penalty Function

Section 1.6 of Chapter 1 discusses the joint, marginal distributions, and density functions of $U(T^-)$ and $|U(T)|$, when the discounted factor $\delta = 0$ for the CRM. In this section, we describe briefly the joint, marginal distributions, and density functions of $U(T^-)$ and $|U(T)|$ with the discounted factor $\delta \geq 0$ for the DRM.

2.5.1 Discounted Joint Distribution and Density Functions

We rewrite equation (2.10) as follows:

$$\phi_c(u) = \frac{1}{1+\beta}\int_0^u \phi_c(u-y)dG_D(y) + \frac{1}{1+\beta}H_D(u), \qquad u \geq 0, \tag{2.11}$$

where $H_D(u) = (1+\beta)H_{\omega D}(u)$, while $G_D(y) = \Gamma_1 * \Gamma_2(y)$ with $\bar{\Gamma}_1(x) = e^{-bx}$, $\bar{\Gamma}_1(x) = 1 - \Gamma_1(x)$ and

$$\bar{\Gamma}_2(x) = \frac{e^{\rho_D x}\int_x^\infty e^{-\rho_D y}\bar{P}(y)dy}{\int_0^\infty e^{-\rho_D y}\bar{P}(y)dy}, \qquad x \geq 0.$$

The solution of (2.11) can be expressed as (based on Theorem 1.8 in Chapter 1)

$$\phi_c(u) = -\frac{1}{\beta}\int_0^u \bar{K}_D(u-x)dH_D(x) + \frac{1}{\beta}H_D(u) - \frac{1}{\beta}H_D(0)\bar{K}_D(u), \qquad u \geq 0, \tag{2.12}$$

where $\bar{K}_D(u) = \sum_{r=1}^{\infty}(\frac{\beta}{1+\beta})(\frac{1}{1+\beta})^r\bar{G}_D^{*r}(u)$, which satisfies the following defective renewal equation:

$$\bar{K}_D(u) = \frac{1}{1+\beta}\int_0^u \bar{K}_D(u-x)dG_D(x) + \frac{1}{1+\beta}\bar{G}_D(u), \qquad u \geq 0. \tag{2.13}$$

Here at first we study the joint distribution of $U(T^-)$ and $|U(T)|$ by appropriately choosing the penalty function $w(U(T^-), |U(T)|)$. For instance choose $w(U(T^-), |U(T)|)$ such that

$$w(x_1, x_2) = \begin{cases} 1 & if \quad x_1 \leq x, \quad x_2 \leq y \\ 0 & otherwise \end{cases}.$$

Then $H_D(u)$ becomes

$$\begin{aligned} H_D(u) &= \bar{G}_D(u) - \bar{G}_D(u+y) - \bar{\Gamma}_1(u)G_D(y) - \frac{b}{b+\rho_D}e^{-\rho_D x}(e^{\rho_D u} - e^{-bu}) \\ &\quad [\bar{\Gamma}_2(x) - \bar{\Gamma}_2(x+y)], \qquad u \geq 0. \end{aligned} \tag{2.14}$$

With this choice of penalty function w, $\phi_c(u)$ to be of the form

$$\phi_c(u) = \int_0^x \int_0^y \int_0^\infty e^{-\delta t} f_D(x_1, x_2, t|u)\, dt\, dx_2\, dx_1 = F_D(x, y; \delta|u),$$

i.e., the discounted joint distribution of $U(T^-)$ and $|U(T)|$, where $f_D(x_1, x_2, t|u)$ is the joint density function of $U(T^-)$, $|U(T)|$ and T.

Now for the explicit expressions of $F_D(x, y; \delta|u)$ and $f_D(x, y; \delta|u)$, the discounted joint density function of $U(T^-)$ and $|U(T)|$, we refer to Tsai (2001). It also gives explicit expressions for the discounted marginal distributions and density functions of $U(T^-)$ and $|U(T)|$. When the diffusion term is removed (i.e. $D = 0$) all the discounted joint, marginal distributions and density functions converge to corresponding quantities for the CRM. If both $D = 0$ and $\delta = 0$, then all the discounted joint, marginal distributions and density functions coincide with those presented in Section 1.6.

Chapter 3

Results Under the DRM via Weak Convergence

At the beginning of this chapter, we would like to discuss the convergence of the surplus process U_n by using weak convergence theorems, our main mathematical tools. As far as we know that the first application of weak convergence in risk theory is due to Iglehart (1969), which shows its usefulness. In Section 3.2, we consider a sequence of aggregate claim processes, which converges weakly to a Wiener process when, for instance, the number of policies of a large insurance portfolio goes to infinity. Then by adding this sequence of risk processes to the CRM, we derive the ultimate ruin probability, the joint density function of the surplus immediately before ruin, the deficit at ruin and time to ruin as well as the expected discounted penalty function.

3.1 Weak Convergence of the Surplus Process U_n

In this section, first we consider a large portfolio of insurance business. A sequence of aggregate claims processes is introduced as follows (see Iglehart 1969):

$$S_n(t) = \frac{n^{-\frac{1}{2}}\sqrt{2D}}{\sqrt{\alpha(\mu' + \sigma'^2)}}[\sum_{j=1}^{M(nt)} Y_j' - \alpha n \mu' t], \qquad t \geq 0, \tag{3.1}$$

where, the claim sizes Y_j' are i.i.d. random variables for $j = 1, 2, ...$ with mean $E(Y_j') = \mu'$ and variance $V(Y_j') = \sigma'^2$. The claim counting process, $M(nt) \sim$ Poisson(αnt) for $(\alpha > 0)$, while the infinitesimal variance $2D > 0$, of the Wiener process W, is defined in Section 2.1 and n is a large quantity, e. g., the number of policies in a large insurance portfolio.

We add the sequence $S_n(t)$ in (3.1) to $S(t)$ defined in the surplus process (1.1), then the resulting surplus process, $U_n(t)$, can be written as

$$U_n(t) = U(t) - S_n(t), \qquad t \geq 0.$$

In order to establish the weak convergence of $U_n(t)$ to $U(t) - W(t)$, we quote the following result from Grandell (1977):

Theorem 3.1 *(Grandell 1977, Theorem 9): Let $X_1, X_2, ...$ be either summation processes (i.e., $X_n(t) = \sum_{i=1}^{[nt]} Y_i$, where Y_i are i.i.d., $E(Y_i) = 0$) or stochastic processes with stationary and independent increments. Define ξ_n by $\xi_n(t) = X_n(nt)$ and let ξ be a stochastic process with stationary and independent increments. If $\xi_n(t) \xrightarrow{d} \xi(t)$ for all $t \in [0, \infty)$ then $\xi_n \xrightarrow{d} \xi$.*

By Theorem 3.1, clearly $S_n \xrightarrow{d} W$ as $S_n(t) \xrightarrow{d} W(t)$. It follows that $(U, S_n) \xrightarrow{d} (U, W)$ by independence and consequently our surplus process U_n converges weakly to the process $U-W$ in distribution as $n \to \infty$. Thus the process $U_n(t)$ approximates the DRM.

3.2 Convergence of Ruin Probability

For simplicity of calculations we suppose that $\alpha = 1$, $\mu' = 1$ and $\sigma' = 1$. Then (3.1) takes the following simple form:

$$S_n(t) = \sum_{j=1}^{M(nt)} \sqrt{\frac{D}{n}} Y_j' - \sqrt{nD}t, \qquad t \geq 0. \tag{3.2}$$

Let us rewrite the surplus process U_n as follows:

$$\begin{aligned} U_n(t) &= u + ct - [S(t) + S_n(t)], \qquad t \geq 0, \\ &= u + ct - [\sum_{i=1}^{N(t)} X_i + \sum_{j=1}^{M(nt)} \sqrt{\frac{D}{n}} Y_j' - \sqrt{nD}\,t] \\ &= u + (c + \sqrt{nD})t - [\sum_{i=1}^{N(t)} X_i + \sum_{j=1}^{M(nt)} \sqrt{\frac{D}{n}} Y_j']. \end{aligned} \tag{3.3}$$

Here assume that $P_{Y'}(y)$ is the distribution function of i.i.d. random variables Y_j'. The claim counting process, $M(nt)$ ~Poisson(nt). Thus $\bar{S}_n(t) = \sum_{j=1}^{M(nt)} \sqrt{\frac{D}{n}} Y_j'$ is a compound Poisson process (CPP) with parameter nt and severity distribution $P_{Y'}(\sqrt{\frac{n}{D}}y)$, i.e.,

$$\bar{S}_n(t) \sim C.P.[nt;\ P_{Y'}(\sqrt{\frac{n}{D}}y)], \qquad t \geq 0,$$

and

$$S(t) \sim C.P.[\lambda t;\ P(x)], \qquad t \geq 0,$$

as defined in Subsection 1.1.2. Also, S and $\bar{S}_n$ are independent aggregate claim processes.

Theorem 3.2 *Suppose $S_i(t)$ are independent compound Poisson processes with parameter $\lambda_i t$ and distribution function $F_i(x)$, for $i = 1, 2, \ldots.., r$, i.e.,*

$$S_i(t) \sim C.P.[\lambda_i t;\ F_i], \qquad t \geq 0.$$

Then

$$S(t) = \sum_{i=1}^{r} S_i(t) \sim C.P.[\lambda t;\ F], \qquad t \geq 0,$$

where $F(x) = \sum_{i=1}^{r} \frac{\lambda_i}{\lambda} F_i(x)$ and $\lambda = \sum_{i=1}^{r} \lambda_i$.

Denote

$$c + \sqrt{nD} = c_n,$$

$$\lambda + n = \lambda_n,$$

and

$$S(t) + \bar{S}_n(t) = \sum_{k=1}^{L_n(t)} Z_k,$$

where $L_n(t)$ ~Poisson($\lambda_n t$) and Z_k are i.i.d. random variables for $k = 1, 2, \ldots..$ Theorem 3.2 asserts that the aggregate claim process $\sum_{k=1}^{L_n(t)} Z_k$ is another CPP with parameter $\lambda_n t$ and distribution function $P_n(x) = \frac{\lambda}{\lambda_n} P(x) + \frac{n}{\lambda_n} P_{Y'}(\sqrt{\frac{n}{D}} x)$.

Now (3.3) takes the following form

$$U_n(t) = u + c_n t - \sum_{k=1}^{L_n(t)} Z_k, \qquad t \geq 0. \tag{3.4}$$

The surplus process in (3.4) is similar to the surplus process in (1.1). Assume that θ_n is the relative security loading, μ_n is the mean of Z_k and $\psi_n(u)$ is the ultimate ruin probability for the surplus process in (3.4). We have

$$\begin{aligned}\mu_n &= E(Z_k)\\ &= \frac{\lambda}{\lambda_n}E(X_i) + \frac{n}{\lambda_n}E(\sqrt{\frac{D}{n}}Y_j')\\ &= \frac{\lambda\mu}{\lambda_n} + \frac{\sqrt{nD}}{\lambda_n}\\ &= \frac{\lambda\mu + \sqrt{nD}}{\lambda_n}\end{aligned}$$

and

$$\begin{aligned}c_n &= (1+\theta_n)\lambda_n\mu_n,\\ \Rightarrow \frac{1}{1+\theta_n} &= \frac{\lambda\mu + \sqrt{nD}}{c+\sqrt{nD}}.\end{aligned} \tag{3.5}$$

The ultimate ruin probability for the CRM can be expressed as the convolution of the tail of a compound geometric distribution, given by (1.4) in Section 1.2. Therefore, for the surplus process (3.4), the ultimate ruin probability $\psi_n(u)$ can be written as

$$\psi_n(u) = 1 - \sum_{j=0}^{\infty}(\frac{\theta_n}{1+\theta_n})(\frac{1}{1+\theta_n})^j P_{en}^{*j}(u), \qquad u \geq 0, \tag{3.6}$$

where

$$P_{en}(x) = \frac{1}{\mu_n}\int_0^x [1 - P_n(y)]\, dy, \qquad x \geq 0, \tag{3.7}$$

and $\bar{P}_{en}^{*j}(u) = 1 - P_{en}^{*j}(u)$ is the tail of j-fold convolution of P_{en} with $(P_{en}(0) = 0)$. (3.5) gives

$$\theta_n = \frac{c - \lambda\mu}{\lambda\mu + \sqrt{nD}}$$

and

$$\frac{\theta_n}{1+\theta_n} = \frac{c-\lambda\mu}{c_n}. \tag{3.8}$$

By applying Corollary 1.1 for the surplus process (3.4) and equation (3.5), it follows that:

$$\psi_n(0) = \frac{1}{1+\theta_n} = \frac{\lambda\mu + \sqrt{nD}}{c + \sqrt{nD}} \to 1 \; as \; n \to \infty.$$

Based on the oscillating nature of the (surplus process) sample paths and the definition of the ultimate ruin probability, ψ_D in Section 2.3, with an initial reserve of $u = 0$ gives $\psi_d(0) = 1$, while $\psi_c(0) = 0$, and hence $\psi_n(0) \to \psi_d(0) + \psi_c(0)$ as $n \to \infty$, which implies that $\psi_n(0) \to \psi_D(0)$ as $n \to \infty$, i.e., the ultimate ruin probability for the CRM converges to that for the DRM when the initial reserve $u = 0$ and the number of policies goes to infinity.

Now we are going to prove the main result of this section. Here we will show that the ultimate ruin probability, given by (3.6), converges to the ultimate ruin probability, given by (2.5), for $u \geq 0$, i.e., $\psi_n(u) \to \psi_D(u)$ as $n \to \infty$. To prove the main result, we first need the following lemmas.

Lemma 3.1 *The Laplace transform of P_{en} is given by*

$$\hat{p}_{en}(\xi) = \frac{\lambda\mu}{\lambda\mu + \sqrt{nD}}\hat{p}_e(\xi) + \frac{n}{\xi(\lambda\mu + \sqrt{nD})}[1 - \hat{p}_{Y'}(\sqrt{\frac{D}{n}}\xi)], \qquad \xi \geq 0.$$

Proof. Let $\hat{p}_{en}$, $\hat{p}_e$ and $\hat{p}_{Y'}$ be the Laplace transforms of p_{en}, p_e and $p_{Y'}$, respectively. From (3.7) we get

$$\begin{aligned}
P_{en}(x) &= \frac{1}{\mu_n}\int_0^x [1 - \frac{\lambda}{\lambda_n}P(y) - \frac{n}{\lambda_n}P_{Y'}(\sqrt{\frac{n}{D}}y)]\,dy, \qquad x \geq 0, \\
&= \frac{\lambda\mu}{\lambda\mu + \sqrt{nD}}\int_0^x \{\frac{1 - P(y)}{\mu}\}dy + \frac{n}{\lambda\mu + \sqrt{nD}}\int_0^x [1 - P_{Y'}(\sqrt{\frac{n}{D}}y)]\,dy \\
&= \frac{\lambda\mu}{\lambda\mu + \sqrt{nD}}P_e(x) + \frac{n}{\lambda\mu + \sqrt{nD}}\int_0^x [1 - P_{Y'}(\sqrt{\frac{n}{D}}y)]\,dy.
\end{aligned}$$

Multiplying both sides of the above equation by $e^{-\xi x}$ and integrating from 0 to ∞, then

$$\begin{aligned}
\hat{p}_{en}(\xi) &= \frac{\lambda\mu}{\lambda\mu+\sqrt{nD}}\hat{p}_e(\xi) + \frac{n}{\lambda\mu+\sqrt{nD}}\int_0^\infty e^{-\xi x} d\{\int_0^x [1-P_{Y'}(\sqrt{\frac{n}{D}}y)]\, dy\} \\
&= \frac{\lambda\mu}{\lambda\mu+\sqrt{nD}}\hat{p}_e(\xi) + \frac{n}{\lambda\mu+\sqrt{nD}}\int_0^\infty e^{-\xi x}[1-P_{Y'}(\sqrt{\frac{n}{D}}x)]\, dx \\
&= \frac{\lambda\mu}{\lambda\mu+\sqrt{nD}}\hat{p}_e(\xi) + \frac{n}{\lambda\mu+\sqrt{nD}}[\frac{1}{\xi} - \int_0^\infty e^{-\xi x} P_{Y'}(\sqrt{\frac{n}{D}}x)\, dx] \\
&= \frac{\lambda\mu}{\lambda\mu+\sqrt{nD}}\hat{p}_e(\xi) + \frac{n}{\lambda\mu+\sqrt{nD}}[1 - \int_0^\infty e^{-\xi x} dP_{Y'}(\sqrt{\frac{n}{D}}x)] \\
&= \frac{\lambda\mu}{\lambda\mu+\sqrt{nD}}\hat{p}_e(\xi) + \frac{n}{\lambda\mu+\sqrt{nD}}[1-\hat{p}_{Y'}(\sqrt{\frac{D}{n}}\xi)], \qquad \xi \geq 0.
\end{aligned}$$

□

Using (2.5), Corollary 2.2 gives

$$\psi_d(u) = \frac{D}{c-\lambda\mu}\sum_{j=0}^{\infty}(\frac{\theta}{1+\theta})(\frac{1}{1+\theta})^j p_1^{*(j+1)} * p_e^{*j}(u), \qquad u \geq 0. \tag{3.9}$$

Lemma 3.2 *The Laplace transform of ψ_d is given by*

$$\hat{\psi}_d(\xi) = \frac{D}{c+\xi D - \lambda\mu\hat{p}_e(\xi)}, \qquad \xi \geq 0\,.$$

Proof. The Laplace transform of $\psi_d(u)$ is defined as

$$\begin{aligned}
\hat{\psi}_d(\xi) &= \int_0^\infty e^{-\xi u}\psi_d(u)du, \qquad \xi \geq 0, \\
&= \frac{D}{c-\lambda\mu}\sum_{j=0}^{\infty}(\frac{\theta}{1+\theta})(\frac{1}{1+\theta})^j \int_0^\infty e^{-\xi u} p_1^{*(j+1)} * p_e^{*j}(u)du \\
&= \frac{D}{c-\lambda\mu}\sum_{j=0}^{\infty}(\frac{\theta}{1+\theta})(\frac{1}{1+\theta})^j \hat{p}_1^{(j+1)}(\xi)\hat{p}_e^j(\xi) \\
&= \frac{D}{c-\lambda\mu}(\frac{\theta}{1+\theta})\hat{p}_1(\xi)\sum_{j=0}^{\infty}[(\frac{1}{1+\theta})\hat{p}_1(\xi)\hat{p}_e(\xi)]^j.
\end{aligned}$$

Since $\frac{1}{1+\theta} = \frac{\lambda\mu}{c}$, $\frac{\theta}{1+\theta} = \frac{c-\lambda\mu}{c}$ and $\hat{p}_1(\xi) = \frac{c}{c+\xi D}$, so

$$\begin{aligned}\hat{\psi}_d(\xi) &= \frac{D}{c-\lambda\mu}\frac{c-\lambda\mu}{c}\frac{c}{c+\xi D}\sum_{j=0}^{\infty}[\frac{\lambda\mu}{c}\frac{c}{c+\xi D}\hat{p}_e(\xi)]^j, \qquad \xi \geq 0,\\ &= \frac{D}{c+\xi D}[\frac{1}{1-\frac{\lambda\mu}{c+\xi D}\hat{p}_e(\xi)}]\\ &= \frac{D}{c+\xi D-\lambda\mu\hat{p}_e(\xi)}.\end{aligned}$$

□

The following propositions show how $\psi_D(u)$ can be derived from $\psi_n(u)$.

Proposition 3.1 *When $n \to \infty$, the limit of the Laplace transform of ψ_n is given by*

$$lim_{n\to\infty}\hat{\psi}_n(\xi) = \frac{1}{\xi} - \frac{c-\lambda\mu+D}{c+\xi D-\lambda\mu\hat{p}_e(\xi)} + \frac{D}{c+\xi D-\lambda\mu\hat{p}_e(\xi)}, \qquad \xi \geq 0.$$

Proof. Multiplying (3.6) by $e^{-\xi u}$ and integrating from 0 to ∞ with respect to u, we have

$$\begin{aligned}\int_0^\infty e^{-\xi u}d\psi_n(u) &= \int_0^\infty e^{-\xi u}du - \sum_{j=0}^{\infty}(\frac{\theta_n}{1+\theta_n})(\frac{1}{1+\theta_n})^j\int_0^\infty e^{-\xi u}dP_{en}^{*j}(u)\\ \Rightarrow \hat{\psi}_n(\xi) &= \frac{1}{\xi} - \frac{\theta_n}{1+\theta_n}\sum_{j=0}^{\infty}[\frac{1}{1+\theta_n}\hat{p}_{en}(\xi)]^j, \qquad \xi \geq 0,\\ &= \frac{1}{\xi} - \frac{\theta_n}{1+\theta_n-\hat{p}_{en}(\xi)}.\end{aligned}$$

Using (3.5) and Lemma 3.1, $\hat{\psi}_n$ becomes

$$\begin{aligned}\hat{\psi}_n(\xi) &= \frac{1}{\xi} - \frac{c-\lambda\mu}{c-\lambda\mu\hat{p}_e(\xi)+\sqrt{nD}-\frac{n}{\xi}[1-\hat{p}_{Y'}(\sqrt{\frac{D}{n}}\xi)]}, \qquad \xi \geq 0,\\ &= \frac{1}{\xi} - \frac{c-\lambda\mu+D}{c-\lambda\mu\hat{p}_e(\xi)+\sqrt{nD}-\frac{n}{\xi}[1-\hat{p}_{Y'}(\sqrt{\frac{D}{n}}\xi)]}\\ &\quad + \frac{D}{c-\lambda\mu\hat{p}_e(\xi)+\sqrt{nD}-\frac{n}{\xi}[1-\hat{p}_{Y'}(\sqrt{\frac{D}{n}}\xi)]}.\end{aligned}$$

Since $\hat{p}_e(\xi) = 1 - \xi E(X) + \frac{\xi^2}{2!}E(X^2) + O(\xi^3)$ when $\xi \to 0$, we can write $\hat{p}_{Y'}(\xi\sqrt{\frac{D}{n}}) = 1 - \xi\sqrt{\frac{D}{n}}E(Y') + \frac{\xi^2}{2!}\frac{D}{n}E(Y'^2) + O((\xi\sqrt{\frac{D}{n}})^3)$ as $n \to \infty$, i.e., $\xi\sqrt{\frac{D}{n}} \to 0$. Since $E(Y') = 1$ and $E(Y'^2) = 2$, therefore $\hat{p}_{Y'}(\xi\sqrt{\frac{D}{n}}) = 1 - \xi\sqrt{\frac{D}{n}} + \xi^2\frac{D}{n} + O((\xi\sqrt{\frac{D}{n}})^3)$ as $n \to \infty$. Hence, $\lim_{n\to\infty}\hat{\psi}_n(\xi) = \frac{1}{\xi} - \frac{c-\lambda\mu+D}{c+\xi D-\lambda\mu\hat{p}_e(\xi)} + \frac{D}{c+\xi D-\lambda\mu\hat{p}_e(\xi)}, \qquad \xi \geq 0.$ □

Proposition 3.2 *The Laplace transform of ψ_D is given by*

$$\hat{\psi}_D(\xi) = \frac{1}{\xi} - \frac{c-\lambda\mu+D}{c+\xi D-\lambda\mu\hat{p}_e(\xi)} + \frac{D}{c+\xi D-\lambda\mu\hat{p}_e(\xi)}, \qquad \xi \geq 0.$$

Proof. Multiplying (2.5) by $e^{-\xi u}$ and integrating from 0 to ∞ with respect to u, we have

$$\begin{aligned}
\int_0^\infty e^{-\xi u}d\psi_D(u) &= \int_0^\infty e^{-\xi u}du - \sum_{j=0}^\infty(\frac{\theta}{1+\theta})(\frac{1}{1+\theta})^j\int_0^\infty e^{-\xi u}dP_1^{*(j+1)} * P_e^{*j}(u) \\
\Rightarrow \hat{\psi}_D(\xi) &= \frac{1}{\xi} - \sum_{j=0}^\infty(\frac{\theta}{1+\theta})(\frac{1}{1+\theta})^j\hat{p}_1^{(j+1)}(\xi)\hat{p}_e^j(\xi), \qquad \xi \geq 0, \\
&= \frac{1}{\xi} - \frac{\theta}{1+\theta}\hat{p}_1(\xi)\sum_{j=0}^\infty[(\frac{1}{1+\theta})\hat{p}_1(\xi)\hat{p}_e(\xi)]^j.
\end{aligned}$$

Since $\frac{1}{1+\theta} = \frac{\lambda\mu}{c}$, $\frac{\theta}{1+\theta} = \frac{c-\lambda\mu}{c}$ and $\hat{p}_1(\xi) = \frac{c}{c+\xi D}$, hence

$$\begin{aligned}
\hat{\psi}_D(\xi) &= \frac{1}{\xi} - \frac{c-\lambda\mu}{c}\frac{c}{c+\xi D}\sum_{j=0}^\infty[\frac{\lambda\mu}{c}\frac{c}{c+\xi D}\hat{p}_e(\xi)]^j, \qquad \xi \geq 0, \\
&= \frac{1}{\xi} - \frac{c-\lambda\mu}{c+\xi D}\frac{1}{1-\frac{\lambda\mu}{c+\xi D}\hat{p}_e(\xi)} \\
&= \frac{1}{\xi} - \frac{c-\lambda\mu}{c+\xi D-\lambda\mu\hat{p}_e(\xi)} \\
&= \frac{1}{\xi} - \frac{c-\lambda\mu+D}{c+\xi D-\lambda\mu\hat{p}_e(\xi)} + \frac{D}{c+\xi D-\lambda\mu\hat{p}_e(\xi)}.
\end{aligned}$$

□

Lemma 3.3 *The Laplace transform of ψ_c is as follows:*

$$\hat{\psi}_c(\xi) = \frac{1}{\xi} - \frac{c-\lambda\mu+D}{c+\xi D-\lambda\mu\hat{p}_e(\xi)}, \qquad \xi \geq 0.$$

Proof. We have

$$\hat{\psi}_c(\xi) = \hat{\psi}_D(\xi) - \hat{\psi}_d(\xi) = \frac{1}{\xi} - \frac{c - \lambda\mu + D}{c + \xi D - \lambda\mu\hat{p}_e(\xi)}, \qquad \xi \geq 0,$$

from Lemma 3.2 and Proposition 3.2. □

The following theorem establishes the main result of this section.

Theorem 3.3 *For $u \geq 0$, $\psi_n(u)$ converges to $\psi_D(u)$ as $n \to \infty$.*

Proof. The proof follows from Propositions 3.1 and 3.2. □

3.3 Convergence of the Joint Density Function

We are interested in that the joint density function of $U(T^-)$, $|U(T)|$ and T for the CRM converges to that for the DRM as $n \to \infty$. We give an explicit expression for this joint density function for the CRM in Corollary 2.1 in terms of density function g_t, given by (2.2). For the DRM, the explicit expression for this joint density function is presented in Theorem 2.1 in terms of the density function g_{tD}, given in Lemma 2.1.

To establish the main result of the present section, we first present some needed propositions. Recall that the surplus process U_n is given by (3.4) and $g_t(x)$ is the density function of $ct - \sum_{i=1}^{N(t)} X_i$ for $U(t)$, given by (1.1). Let $g_{t,n}(x)$ be the density function corresponding to $g_t(x)$ for $U_n(t)$. Then we have

$$g_{t,n}(x) = \sum_{r=0}^{\infty} \frac{(\lambda_n t)^r}{r!} e^{-\lambda_n t} p_n^{*r}(c_n t - x), \qquad x \geq 0, \tag{3.10}$$

where

$$p_n(c_n t - x) = \frac{\lambda}{\lambda_n} p(c_n t - x) + \frac{n}{\lambda_n}\sqrt{\frac{n}{D}} p_{Y'}(\sqrt{\frac{n}{D}}(c_n t - x)), \qquad x, t \geq 0.$$

Proposition 3.3 *When $n \to \infty$, the limit of the characteristic function of $g_{t,n}$ is given as follows:*

$$\lim_{n\to\infty} \varphi_{g_{t,n}}(\xi) = e^{\lambda t[1-\varphi_p(\xi)]+ict\xi-Dt\xi^2}, \qquad \xi \geq 0.$$

Proof. By the definition of the characteristic function of $g_{t,n}$:

$$\begin{aligned}
\varphi_{g_{t,n}}(\xi) &= \int_0^\infty e^{i\xi x} g_{t,n}(x)dx, \qquad \xi \geq 0, \\
&= \sum_{r=0}^{\infty} \frac{(\lambda_n t)^r}{r!} e^{-\lambda_n t} \int_0^\infty e^{i\xi x} p_n^{*r}(c_n t - x)dx \\
&= \sum_{r=0}^{\infty} \frac{(\lambda_n t)^r}{r!} e^{-\lambda_n t + ic_n t\xi} \int_0^\infty e^{i(-\xi)(c_n t - x)} p_n^{*r}(c_n t - x)dx \\
&= \sum_{r=0}^{\infty} \frac{(\lambda_n t)^r}{r!} e^{-\lambda_n t + ic_n t\xi} \varphi_{p_n}^r(-\xi),
\end{aligned}$$

where

$$\begin{aligned}
\varphi_{p_n}(-\xi) &= \frac{\lambda}{\lambda_n} \int_0^\infty e^{i(-\xi)(c_n t - x)} p(c_n t - x)dx \\
&\quad + \frac{n}{\lambda_n}\sqrt{\frac{n}{D}} \int_0^\infty e^{i(-\xi)(c_n t - x)} p_{Y'}(\sqrt{\frac{n}{D}}(c_n t - x))dx \\
&= \frac{\lambda}{\lambda_n}\varphi_p(-\xi) + \frac{n}{\lambda_n}\varphi_{p_{Y'}}(-\xi\sqrt{\frac{D}{n}}).
\end{aligned}$$

$\varphi_{g_{t,n}}(\xi)$ becomes

$$\begin{aligned}
\varphi_{g_{t,n}}(\xi) &= \sum_{r=0}^{\infty} \frac{(\lambda_n t)^r}{r!} e^{-\lambda_n t + ic_n t\xi} [\frac{\lambda}{\lambda_n}\varphi_p(-\xi) + \frac{n}{\lambda_n}\varphi_{p_{Y'}}(-\xi\sqrt{\frac{D}{n}})]^r, \qquad \xi \geq 0, \\
&= e^{-\lambda_n t + ic_n t\xi} \sum_{r=0}^{\infty} \frac{[\lambda t\,\varphi_p(-\xi) + nt\,\varphi_{p_{Y'}}(-\xi\sqrt{\frac{D}{n}})]^r}{r!} \\
&= e^{-\lambda t[1-\varphi_p(-\xi)]}\, e^{-nt[1-\varphi_{p_{Y'}}(-\xi\sqrt{\frac{D}{n}})]+it(c+\sqrt{nD})\xi}\,.
\end{aligned}$$

By using the fact $\varphi_p(\xi) = 1 + i\xi\mu - \frac{\xi^2}{2!}E(X^2) + O(\xi^3)$ as $\xi \to 0$. We have $\varphi_{p_{Y'}}(-\xi\sqrt{\frac{D}{n}}) =$

$1 - i\xi\sqrt{\frac{D}{n}} - \xi^2\frac{D}{n} + O((\xi\sqrt{\frac{D}{n}})^3)$, since $\mu' = 1$ and $\sigma' = 1$. Again, $\varphi_{g_{t,n}}(\xi)$ follows:

$$\begin{aligned}\varphi_{g_{t,n}}(\xi) &= e^{-\lambda t[1-\varphi_p(-\xi)]}\, e^{-nt[i\xi\sqrt{\frac{D}{n}}+\xi^2\frac{D}{n}+O((\xi\sqrt{\frac{D}{n}})^3)]+it(c+\sqrt{nD})\xi}, \qquad \xi \geq 0,\\ &= e^{-\lambda t[1-\varphi_p(-\xi)]-\xi^2 Dt+ict\xi+O((\xi\sqrt{\frac{D}{n}})^3)},\end{aligned}$$

which in turn implies $\lim_{n\to\infty}\varphi_{g_{t,n}}(\xi) = e^{-\lambda t[1-\varphi_p(-\xi)]-\xi^2 Dt+ict\xi}$. □

Proposition 3.4 *The characteristic function of g_{tD} is given by:*

$$\varphi_{g_{tD}}(\xi) = e^{-\lambda t[1-\varphi_p(-\xi)]-\xi^2 Dt+ict\xi}, \qquad \xi \geq 0.$$

Proof. The characteristic function of g_{tD} is given by:

$$\begin{aligned}\varphi_{g_{tD}}(\xi) &= \sum_{r=0}^{\infty}\frac{(\lambda t)^r}{r!}e^{-\lambda t}\int_0^{\infty}e^{i\xi x}\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi Dt}}e^{-\frac{y^2}{4Dt}}p^{*r}(ct+y-x)dydx, \qquad \xi \geq 0,\\ &= e^{-\lambda t+ict\xi}\sum_{r=0}^{\infty}\frac{(\lambda t)^r}{r!}\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi Dt}}e^{-\frac{y^2}{4Dt}+iy\xi}\\ &\quad [\int_0^{\infty}e^{i(-\xi)(ct+y-x)}p^{*r}(ct+y-x)dx]dy\\ &= e^{-\lambda t+ict\xi}\sum_{r=0}^{\infty}\frac{(\lambda t)^r}{r!}\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi Dt}}e^{-\frac{y^2}{4Dt}+iy\xi}\varphi_p^r(-\xi)dy\\ &= e^{-\lambda t+ict\xi}\sum_{r=0}^{\infty}\frac{[\lambda t\varphi_p(-\xi)]^r}{r!}\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi Dt}}e^{-\frac{y^2}{4Dt}+iy\xi}dy\\ &= e^{-\lambda t+ict\xi+\lambda t\varphi_p(-\xi)-\xi^2 Dt}\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi Dt}}e^{-\frac{(y-2iDt\xi)^2}{4Dt}}dy\\ &= e^{-\lambda t[1-\varphi_p(-\xi)]-\xi^2 Dt+ict\xi},\end{aligned}$$

which is exactly the characteristic function of $g_{t,n}$ if $n \to \infty$. □

From Propositions 3.3 and 3.4, we can make a conclusion given by the following theorem.

Theorem 3.4 *The density function $g_{t,n}$ converges to the density function g_{tD} if $n \to \infty$.*

Proof. Propositions 3.3, 3.4 and the following theorem guarantees the proof. $\square$

Theorem 3.5 *We have $S_n(t) = \sqrt{\frac{D}{n}}\sum_{j=1}^{M(nt)} Y_j' - \sqrt{nD}t$ and assume that $f_{Y'}$, the common density function of $Y_j'(j = 1, 2, ...)$, is bounded. Then the density function $f_{S_n(t)}$ of $S_n(t)$ satisfies $|f_{S_n(t)}(x) - f_{tD}(x)| \to 0$ as $n \to \infty$, where f_{tD} is the density function of $N(0, 2tD)$.*

Proof. We can write

$$\begin{aligned} S_n(t) &= \sqrt{\frac{D}{n}}\sum_{j=1}^{M(nt)} Y_j' - \sqrt{nD}t, \qquad t \geq 0, \\ &= \sqrt{\frac{D}{n}}\sum_{j=1}^{M(nt)} (Y_j' - 1) + \sqrt{\frac{D}{n}}[M(nt) - nt]. \end{aligned}$$

Let us suppose that $D = 1$. Then $S_n(t)$ becomes

$$S_n(t) = \frac{1}{\sqrt{n}}\sum_{j=1}^{M(nt)} (Y_j' - 1) + \frac{M(nt) - nt}{\sqrt{n}}, \qquad t \geq 0.$$

The density function of $S_n(t)$ is given by

$$f_{S_n(t)}(x) = \sum_{k=1}^{\infty} e^{-nt}\frac{(nt)^k}{k!}\sqrt{n} f^{*k}_{Y'-1}(\sqrt{n}x - (k - nt)), \qquad x \geq 0.$$

The characteristic function of $S_n(t)$ is given by:

$$\begin{aligned} \phi_{S_n(t)}(\xi) &= \sum_{k=1}^{\infty} e^{-nt}\frac{(nt)^k}{k!}\sqrt{n}\int_{-\infty}^{\infty} e^{i\xi x} f^{*k}_{Y'-1}(\sqrt{n}x - (k - nt))dx, \qquad \xi \geq 0, \\ &= \sum_{k=1}^{\infty} e^{-nt}\frac{(nt)^k}{k!} e^{i\xi\frac{(k-nt)}{\sqrt{n}}}[\phi_{Y'-1}(\frac{\xi}{\sqrt{n}})]^k. \end{aligned}$$

The boundedness of $f_{Y'-1}(.)$ implies, as in the proof of Theorem 1, p. 224, of Gnedenko and Kolmogorov (1954, Ch. 8), that $\int |\phi_{Y'-1}(\xi)|^k d\xi < \infty$ for $k \geq 2$.

Hence $\int |e^{i\xi\frac{(k-nt)}{\sqrt{n}}}[\phi_{Y'-1}(\frac{\xi}{\sqrt{n}})]^k| d\xi \leq \int |\phi_{Y'-1}(\frac{\xi}{\sqrt{n}})|^k d\xi < \infty$. So, $\sup_x f^{*k}_{Y'-1}(\sqrt{n}x - (k - nt)) < \infty$ for $k \geq 2$.

Now we have

$$
\begin{aligned}
&|f_{S_n(t)}(x) - f_{tD}(x)| \\
&= |\sum_{k=1}^{\infty} e^{-nt}\frac{(nt)^k}{k!}\sqrt{n}f_{Y'-1}^{*k}(\sqrt{n}x-(k-nt)) - f_{tD}(x)|, \qquad x \geq 0, \\
&\leq e^{-nt}f_t(x) + \sum_{|k-nt|\leq Mn^{\frac{3}{4}}} e^{-nt}\frac{(nt)^k}{k!}|\sqrt{n}f_{Y'-1}^{*k}(\sqrt{n}x-(k-nt)) - f_{tD}(x)| \\
&+B\sqrt{n}\sum_{|k-nt|>Mn^{\frac{3}{4}}} e^{-nt}\frac{(nt)^k}{k!}, \qquad (3.11)
\end{aligned}
$$

where $M > 0$ is any number and $B > 0$ is an upper bound for $|f_{Y'-1}^{*k}(\sqrt{n}x - (k - nt)) - f_{tD}(x)|$.

By the Fourier inversion formula, the 2nd term on the right-hand side of (3.11) can be written as:

$$
\begin{aligned}
&\sum_{|k-nt|\leq Mn^{\frac{3}{4}}} e^{-nt}\frac{(nt)^k}{k!}|\sqrt{n}f_{Y'-1}^{*k}(\sqrt{n}x-(k-nt)) - f_{tD}(x)| \\
&\leq \frac{1}{2\pi}\sum_{|k-nt|\leq Mn^{\frac{3}{4}}} e^{-nt}\frac{(nt)^k}{k!}\int |e^{i\xi\frac{(k-nt)}{\sqrt{n}}}[\phi_{Y'-1}(\frac{\xi}{\sqrt{n}})]^k - e^{-\xi^2 t}|d\xi \\
&\leq \frac{1}{2\pi}\sum_{|k-nt|\leq Mn^{\frac{3}{4}}} e^{-nt}\frac{(nt)^k}{k!}[\int |[\phi_{Y'-1}(\frac{\xi}{\sqrt{n}})]^k - e^{-\frac{1}{2}\xi^2 t}|d\xi \\
&+\int e^{-\frac{1}{2}\xi^2 t}|e^{i\xi\frac{(k-nt)}{\sqrt{n}}} - e^{-\frac{1}{2}\xi^2 t}|d\xi] \\
&\leq \frac{1}{2\pi}\sum_{|k-nt|\leq Mn^{\frac{3}{4}}} e^{-nt}\frac{(nt)^k}{k!}\int |[\phi_{Y'-1}(\frac{\xi}{\sqrt{n}})]^k - e^{-\frac{1}{2}\xi^2 t}|d\xi \\
&+\frac{1}{2\pi}\int |\sum_{k=0}^{\infty} e^{-nt}\frac{(nt)^k}{k!}e^{i\xi\frac{(k-nt)}{\sqrt{n}}} - e^{-\frac{1}{2}\xi^2 t}|e^{-\frac{1}{2}\xi^2 t}|d\xi. \qquad (3.12)
\end{aligned}
$$

For every $M > 0$ and k with $|k - nt| \leq Mn^{\frac{3}{4}}$, we have

$$
\begin{aligned}
[\phi_{Y'-1}(\frac{\xi}{\sqrt{n}})]^k &= [1 - \frac{\xi^2}{2n} + O(\frac{1}{n^{\frac{3}{2}}})]^k, \qquad \xi \geq 0, \\
&= [1 - \frac{\xi^2}{2n} + O(\frac{1}{n^{\frac{3}{2}}})]^{nt(1+\frac{k-nt}{nt})} \to e^{-\frac{1}{2}\xi^2 t} \qquad as \qquad n \to \infty.
\end{aligned}
$$

Hence the first term on the right-hand side of (3.12) can be made arbitrarily small by letting $n \to \infty$ for any $M > 0$ as in the proof of the Theorem 1 of Gnedenko and Kolmogorov (1954). The second term on the right-hand side of (3.12) can be written as:

$$\frac{1}{2\pi}\int |E\{e^{i\xi\frac{M(nt)-nt}{\sqrt{n}}}\} - e^{-\frac{1}{2}\xi^2 t}|e^{-\frac{1}{2}\xi^2 t}d\xi$$

and hence converges to zero as $n \to \infty$ by the dominated convergence theorem since $\frac{M(nt)-nt}{\sqrt{n}} \to N(0,t)$. Thus the second term of (3.11) tends to zero for every $M > 0$ as $n \to \infty$. The first term of (3.11), $e^{-nt}f_{tD}(x) \to 0$ as $n \to \infty$ uniformly in x and the third term of (3.11), by the Chebyshev's inequality,

$$B\sqrt{n}\sum_{|k-nt|>Mn^{\frac{3}{4}}} e^{-nt}\frac{(nt)^k}{k!} \le B\sqrt{n}\frac{V[M(nt)]}{M^2 n^{\frac{3}{2}}} = \frac{Bt}{M^2},$$

which can be made arbitrarily small by letting $M > 0$ be sufficiently large. Hence , finally we obtain from (3.11), $|f_{S_n(t)}(x) - f_{tD}(x)| \to 0$ as $n \to \infty$ uniformly in x $(-\infty < x < \infty)$. □

Lemma 3.4 *Let us assume that* $p_{Y'}(z) = O(\frac{1}{z^k})$ *for some* $k > 3$ *as* $z \to \infty$. *Then for* $D > 0$, $\lambda_n p_n(x+y) \to \lambda p(x+y)$ *as* $n \to \infty$, *where* $\lambda_n = n + \lambda$.

Proof. We have

$$\begin{aligned}\lambda_n p_n(x+y) &= \lambda_n[\frac{\lambda}{\lambda_n}p(x+y) + \frac{n}{\lambda_n}\sqrt{\frac{n}{D}}p_{Y'}(\sqrt{\frac{n}{D}}(x+y))], \qquad x,y \ge 0,\\ &= \lambda p(x+y) + n\sqrt{\frac{n}{D}}\, p_{Y'}(\sqrt{\frac{n}{D}}(x+y)).\end{aligned}$$

Then, for some $k > 3$,

$$
\begin{aligned}
n\sqrt{\frac{n}{D}}\, p_{Y'}(\sqrt{\frac{n}{D}}(x+y)) &= n\sqrt{\frac{n}{D}}\, O(\frac{1}{(\sqrt{\frac{n}{D}}(x+y))^k}) \\
&= \frac{n^{\frac{3}{2}}}{\sqrt{D}} O(\frac{1}{(\frac{n}{D})^{\frac{k}{2}}(x+y)^k}) \\
&= O(\frac{D^{\frac{(k-1)}{2}}}{n^{\frac{(k-3)}{2}}(x+y)^k}) \to 0 \qquad as \qquad n \to \infty\,.
\end{aligned}
$$

Which implies that

$$
\frac{n^{\frac{3}{2}}}{\sqrt{D}}\, p_{Y'}(\sqrt{\frac{n}{D}}(x+y)) \to 0 \qquad as \qquad n \to \infty,
$$

under the assumption, made in the statement of the lemma. □

Example 3.1 *Consider Y' has exponential claims, i.e., $Y' \sim Exp(1)$.*

Then $p_{Y'}(z) = e^{-z}$, $z > 0$. Now we have

$$
\begin{aligned}
&\frac{n^{\frac{3}{2}}}{\sqrt{D}} p_{Y'}(\sqrt{\frac{n}{D}}(x+y)) \\
&= \frac{n^{\frac{3}{2}}}{\sqrt{D}}\sqrt{\frac{n}{D}} e^{-\sqrt{\frac{n}{D}}(x+y)} \\
&= \frac{n^2}{D}\frac{1}{e^{\sqrt{\frac{n}{D}}(x+y)}} \\
&= \frac{\frac{n^2}{D}}{1+\sqrt{\frac{n}{D}}(x+y)+\frac{n}{D}\frac{(x+y)^2}{2!}+(\frac{n}{D})^{\frac{3}{2}}\frac{(x+y)^3}{3!}+(\frac{n}{D})^2\frac{(x+y)^4}{4!}+O((\frac{n}{D})^{\frac{5}{2}}\frac{(x+y)^5}{5!})} \\
&= \frac{1}{\frac{D}{n^2}+\frac{\sqrt{D}}{n^{\frac{3}{2}}}(x+y)+\frac{1}{n}\frac{(x+y)^2}{2!}+\frac{1}{\sqrt{nD}}\frac{(x+y)^3}{3!}+\frac{1}{D}\frac{(x+y)^4}{4!}+\frac{D}{n^2}O((\frac{n}{D})^{\frac{5}{2}}\frac{(x+y)^5}{5!})}
\end{aligned}
$$

Which in turn implies $\frac{n^{\frac{3}{2}}}{\sqrt{D}} p_{Y'}(\sqrt{\frac{n}{D}}(x+y)) \to 0$ as $n \to \infty$. Hence, in case of exponential claims, by Lemma 3.5, for $D > 0$, $\lambda_n p_n(x+y) \to \lambda p(x+y)$ as $n \to \infty$.

□

Let us now write down the joint density function of $U(T^-)$, $|U(T)|$ and T for $U_n(t)$, given by (3.4), from the Corollary 2.1 as follows:

$$f_n(x,y,t|u) = \lambda_n p_n(x+y)[g_{t,n}(x-u) - \int_0^t \frac{x}{s} g_{s,n}(x) g_{(t-s),n}(-u) ds], \qquad t \geq 0,$$

where $g_{t,n}(x)$ is given by (3.10). Also

$$f_D(x,y,t|u) = \begin{cases} \lambda p(x+y)[g_{tD}(x-u) - \int_0^t \frac{x}{s} g_{sD}(x) g_{(t-s)D}(-u) ds], & if \quad x > 0, \\ 0, \quad if \quad x = 0,\ y \neq 0, \end{cases}$$

and

$$f_D(0,0,t|u) = \frac{\partial \psi_d(u,t)}{\partial t},$$

where $g_{tD}(x)$ is given by Lemma 2.1.

The main result of this section is stated by the following theorem:

Theorem 3.6 *For $x > 0, u \geq 0$, the joint density function $f_n(x,y,t|u)$ converges to the joint density function $f_D(x,y,t|u)$, pointwise as $n \to \infty$.*

Proof. Theorem 3.4 and Lemma 3.4 illustrate the convergence of $g_{t,n}$ and $\lambda_n p_n$ to g_{tD} and λp, respectively as $n \to \infty$. Consequently, the joint density function $f_n(x,y,t|u)$ converges to the joint density function $f_D(x,y,t|u)$, pointwise as $n \to \infty$. □

3.4 Convergence of EDPF

Here we prove that a generalized EDPF under the DRM can be obtained from the EDPF under the CRM using a weak convergence approach. Defective renewal equations with their solutions are given in Sections 1.5 and 2.4 for the CRM and DRM, respectively. Since (1.14) is the defective renewal equation for the CRM, then the corresponding defective equation for the CRM, defined by (3.4), can be written as

$$\phi_{w,n}(u) = \frac{1}{1+\beta_n}\int_0^u \phi_{w,n}(u-x)dG_n(x) + \frac{1}{1+\beta_n}H_n(u), \qquad u \geq 0, \tag{3.13}$$

where $\rho = \rho(\delta)$ is the unique nonnegative solution of Lundberg's equation

$$\lambda_n + \delta = c_n\xi + \lambda_n \int_0^\infty e^{-\xi y} dP_n(y)$$

with $\rho(0) = 0$.

Lemma 3.5 *For $\delta \geq 0$, when $n \to \infty, \lambda_n + \delta = c_n\xi + \lambda_n \hat{P}_n(\xi)$ converges to the generalized Lundberg's equation $\lambda\hat{p}(\xi) = \lambda + \delta - c\xi - D\xi^2$, where $D > 0$.*

Proof. We have

$$\begin{aligned} \lambda_n + \delta &= c_n\xi + \lambda_n \int_0^\infty e^{-\xi y} dP_n(y) \\ &= c_n\xi + \lambda\hat{p}(\xi) + n\hat{p}_{Y'}(\xi\sqrt{\frac{D}{n}}). \end{aligned}$$

Now by the same fact used in Proposition 3.1, i.e., $\hat{p}_{Y'}(\xi\sqrt{\frac{D}{n}}) = 1 - \xi\sqrt{\frac{D}{n}} + \xi^2\frac{D}{n} + O(\frac{1}{n^{\frac{3}{2}}})$ as $n \to \infty$, we have

$$\begin{aligned} \lambda + n + \delta &= (c + \sqrt{nD})\xi + \lambda\hat{p}(\xi) + n[1 - \xi\sqrt{\frac{D}{n}} + \xi^2\frac{D}{n} + O(\frac{1}{n^{\frac{3}{2}}})] \\ \Rightarrow \lambda\hat{p}(\xi) &= \lambda + \delta - c\xi - D\xi^2, \ \textit{as } n \to \infty. \end{aligned}$$

□

The following theorem illustrates how the Laplace transform of the EDPF under the CRM converges as $n \to \infty$ to the Laplace transform of the EDPF under the DRM.

Theorem 3.7 *Let $w_0 = w(0,0)$ and $w_e(x,y) = w(x,y) - w_0$. Assume that $|w_e(x,y)| \leq \alpha(x+y)^r$ for $x, y \geq 0$, $\alpha > 0$ and $r > 1$. Then for $D > 0$, when $n \to \infty$, the limit of the Laplace transform of $\phi_{w,n}(u)$ is given by*

$$\lim_{n\to\infty} \hat{\phi}_{w,n}(\xi) = \frac{\hat{H}_{\omega D}(\xi)}{1-\hat{g}_D(\xi)} + \frac{w_0}{(b+\xi)[1-\hat{g}_D(\xi)]}, \qquad \xi \geq 0,$$

where $\omega(x) = \int_x^\infty w(x, y-x)dP(y)$, $\hat{\omega}(\xi) = \int_0^\infty e^{-\xi x}\omega(x)dx$, $w_0 = w(0,0)$ and $\hat{g}_D(\xi)$ and $\hat{H}_{\omega D}(\xi)$ are given by Lemma 2.2.

Proof. We have

$$\begin{aligned}
\int_0^\infty e^{-\rho y}\bar{P}_n(y)dy &= \int_0^\infty e^{-\rho y}[1-P_n(y)]dy \\
&= \int_0^\infty e^{-\rho y}\int_y^\infty dP_n(z)\, dy \\
&= \frac{1}{\rho}\int_0^\infty [1-e^{-\rho z}]dP_n(z) \\
&= \frac{1}{\rho}[1-\hat{p}_n(\rho)],
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty e^{-\xi x}[e^{\rho x}\int_x^\infty e^{-\rho y}dP_n(y)]dx &= \int_0^\infty [\int_0^y e^{-x(\xi-\rho)}dx]e^{-\rho y}dP_n(y), \qquad \xi \geq 0, \\
&= \frac{1}{\xi-\rho}[\int_0^\infty e^{-\rho y}dP_n(y) - \int_0^\infty e^{-\xi y}dP_n(y)] \\
&= \frac{1}{\rho-\xi}[\hat{p}_n(\xi)-\hat{p}_n(\rho)].
\end{aligned}$$

So

$$\hat{g}_n(\xi) = \frac{\rho[\hat{p}_n(\xi)-\hat{p}_n(\rho)]}{(\rho-\xi)[1-\hat{p}_n(\rho)]}, \qquad \xi \geq 0.$$

Thus

$$\begin{aligned}
1+\beta_n-\hat{g}_n(\xi) &= \frac{1+\theta_n}{\int_0^\infty e^{-\rho y}dP_{en}(y)} - \frac{\rho[\hat{p}_n(\xi)-\hat{p}_n(\rho)]}{(\rho-\xi)[1-\hat{p}_n(\rho)]} \\
&= \frac{c_n}{\lambda_n\int_0^\infty e^{-\rho y}d\bar{P}_n(y)} - \frac{\rho[\hat{p}_n(\xi)-\hat{p}_n(\rho)]}{(\rho-\xi)[1-\hat{p}_n(\rho)]} \\
&= \frac{\rho c_n}{\lambda_n[1-\hat{p}_n(\rho)]} - \frac{\rho[\hat{p}_n(\xi)-\hat{p}_n(\rho)]}{(\rho-\xi)[1-\hat{p}_n(\rho)]} \\
&= \frac{\rho}{\lambda_n(\rho-\xi)[1-\hat{p}_n(\rho)]}[(\rho-\xi)c_n-\lambda_n\{\hat{p}_n(\xi)-\hat{p}_n(\rho)\}].
\end{aligned}$$

Let $\frac{\rho}{\lambda_n(\rho-\xi)[1-\hat{p}_n(\rho)]} = Q_n(\rho)$ and using $\hat{p}_n(\xi) = \frac{\lambda}{\lambda_n}\hat{p}(\xi) + \frac{n}{\lambda_n}\hat{p}_{Y'}(\xi\sqrt{\frac{D}{n}})$, $\lambda_n = \lambda + n$, $c_n = c+n$, we get

$$\begin{aligned}
1+\beta_n-\hat{g}_n(\xi) &= Q_n(\rho)[(\rho-\xi)(c+\sqrt{nD})-\lambda\{\hat{p}(\xi)-\hat{p}(\rho)\} \\
&\quad -n\{\hat{p}_{Y'}(\xi\sqrt{\frac{D}{n}})-\hat{p}_{Y'}(\rho\sqrt{\frac{D}{n}})\}].
\end{aligned}$$

By the fact used in Proposition 3.1, i.e., $\hat{p}_{Y'}(\xi\sqrt{\frac{D}{n}}) = 1-\xi\sqrt{\frac{D}{n}}+\xi^2\frac{D}{n}+O(\frac{1}{n^{\frac{3}{2}}})$ as $n\to\infty$, we get

$$\begin{aligned}
&1+\beta_n-\hat{g}_n(\xi) \\
&= Q_n(\rho)[(\rho-\xi)(c+\sqrt{nD})-\lambda\{\hat{p}(\xi)-\hat{p}(\rho)\}+(\xi-\rho)\sqrt{nD}-(\xi^2-\rho^2)D+O(\frac{1}{\sqrt{n}}) \\
&= Q_n(\rho)[(\rho-\xi)(c+\rho D+\xi D)-\lambda\{\hat{p}(\xi)-\hat{p}(\rho)\}+O(\frac{1}{\sqrt{n}})].
\end{aligned}$$

Using Corollary 2.2 and $b=\frac{c}{D}+\rho$, then

$$1+\beta_n-\hat{g}_n(\xi) = Q_n(\rho)(\rho-\xi)(b+\xi)D[1-\hat{g}_D(\xi)+O(\frac{1}{\sqrt{n}})]. \tag{3.14}$$

Further suppose that

$$\omega_n(x) = \int_x^\infty w(x,y-x)dP_n(y)$$

$$
\begin{aligned}
\hat{\omega}_n(\xi) &= \int_0^\infty e^{-\xi x}\omega_n(x)dx \\
&= \int_0^\infty e^{-\xi x}[\frac{\lambda}{\lambda_n}\int_x^\infty w(x,y-x)dP(y) + \frac{n}{\lambda_n}\int_x^\infty w(x,y-x)dP_{Y'}(y\sqrt{\frac{n}{D}})]dx \\
&= \frac{\lambda}{\lambda_n}\hat{\omega}(\xi) + \frac{n}{\lambda_n}\int_0^\infty[\int_0^{y\sqrt{\frac{D}{n}}} e^{-\xi x}w(x,y\sqrt{\frac{D}{n}}-x)dx]dP_{Y'}(y)
\end{aligned}
$$

with $\omega(x) = \int_x^\infty w(x,y-x)dP(y)$ and $\hat{\omega}(\xi) = \int_0^\infty e^{-\xi x}\omega(x)dx$.

Similarly, $\hat{\omega}_n(\rho) = \frac{\lambda}{\lambda_n}\hat{\omega}(\rho) + \frac{n}{\lambda_n}\int_0^\infty[\int_0^{y\sqrt{\frac{D}{n}}} e^{-\rho x}w(x,y\sqrt{\frac{D}{n}}-x)dx]dP_{Y'}(y)$, then

$$
\begin{aligned}
\hat{H}_n(\xi) &= \int_0^\infty e^{-\xi u}H_n(u)du \\
&= \frac{\int_0^\infty e^{-\xi u}[e^{\rho u}\int_u^\infty e^{-\rho x}\int_x^\infty w(x,y-x)dP_n(y)\,dx]du}{\int_0^\infty e^{-\rho y}\bar{P}_n(y)dy} \\
&= \frac{\int_0^\infty e^{-(\xi-\rho)u}[\int_u^\infty e^{-\rho x}\omega_n(x)dx]du}{\frac{1-\hat{p}_n(\rho)}{\rho}} \\
&= Q_n(\rho)\lambda_n[\hat{\omega}_n(\xi) - \hat{\omega}_n(\rho)] \\
&= Q_n(\rho)[\lambda\{\hat{\omega}(\xi) - \hat{\omega}(\rho)\} \\
&\quad +n\int_0^\infty\{\int_0^{y\sqrt{\frac{D}{n}}}(e^{-\xi x} - e^{-\rho x})w(x,y\sqrt{\frac{D}{n}}-x)dx\}dP_{Y'}(y)].
\end{aligned}
$$

Therefore using $w(x,y) = w_e(x,y) + w_0$ and the assumption made in the statement of the theorem, we have

$$
\begin{aligned}
\hat{H}_n(\xi) &= Q_n(\rho)[\lambda\{\hat{\omega}(\xi) - \hat{\omega}(\rho)\} + nw_0\int_0^\infty\{\int_0^{y\sqrt{\frac{D}{n}}}(e^{-\xi x} - e^{-\rho x})dx\}dP_{Y'}(y), \qquad \xi \geq 0, \\
&\quad +n\int_0^\infty\{\int_0^{y\sqrt{\frac{D}{n}}}(e^{-\xi x} - e^{-\rho x})w_e(x,y\sqrt{\frac{D}{n}}-x)dx\}dP_{Y'}(y)] \\
&= Q_n(\rho)[\lambda\{\hat{\omega}(\xi) - \hat{\omega}(\rho)\} + nw_0\int_0^\infty\{\frac{1-e^{-\xi\sqrt{\frac{D}{n}}y}}{\xi} - \frac{1-e^{-\rho\sqrt{\frac{D}{n}}y}}{\rho}\}dP_{Y'}(y) \\
&\quad +n\int_0^\infty\{\int_0^{y\sqrt{\frac{D}{n}}}(e^{-\xi x} - e^{-\rho x})w_e(x,y\sqrt{\frac{D}{n}}-x)dx\}dP_{Y'}(y)] \\
&= Q_n(\rho)[\lambda\{\hat{\omega}(\xi) - \hat{\omega}(\rho)\} + nw_0\{\frac{1-\hat{p}_{Y'}(\xi\sqrt{\frac{D}{n}})}{\xi} - \frac{1-\hat{p}_{Y'}(\rho\sqrt{\frac{D}{n}})}{\rho}\} \\
&\quad +n\int_0^\infty\{\int_0^{y\sqrt{\frac{D}{n}}}(e^{-\xi x} - e^{-\rho x})w_e(x,y\sqrt{\frac{D}{n}}-x)dx\}dP_{Y'}(y)].
\end{aligned}
$$

Now the middle term of the above equation is

$$nw_0\{\frac{1-\hat{p}_{Y'}(\xi\sqrt{\frac{D}{n}})}{\xi}-\frac{1-\hat{p}_{Y'}(\rho\sqrt{\frac{D}{n}})}{\rho}\} = w_0 D(\rho-\xi)+O(\frac{1}{\sqrt{n}}),$$

since $\hat{p}_{Y'}(\xi\sqrt{\frac{D}{n}}) = 1-\xi\sqrt{\frac{D}{n}}+\xi^2\frac{D}{n}+O(\frac{1}{n^{\frac{3}{2}}})$ as $n\to\infty$.

The last term of the above equation is

$$\begin{aligned}
&|n\int_0^\infty \{\int_0^{y\sqrt{\frac{D}{n}}}(e^{-\xi x}-e^{-\rho x})w_e(x, y\sqrt{\frac{D}{n}}-x)dx\}dP_{Y'}(y)| \\
&\le n\alpha\int_0^\infty \{\int_0^{y\sqrt{\frac{D}{n}}}(y\sqrt{\frac{D}{n}})^r|e^{-\xi x}-e^{-\rho x}|dx\} \\
&\le n\alpha\int_0^\infty (y\sqrt{\frac{D}{n}})^{r+1}dP_{Y'}(y) \qquad \textit{since} \qquad |e^{-\xi x}-e^{-\rho x}|\le 1, \\
&= \alpha D^{\frac{r+1}{2}}n^{\frac{1-r}{2}}\int_0^\infty y^{r+1}dP_{Y'}(y).
\end{aligned}$$

for some $r>1$ and $\int_0^\infty y^{r+1}dP_{Y'}(y)<\infty$ as $n\to\infty$. Finally $\hat{H}_n(\xi)$ takes the following nice form

$$\hat{H}_n(\xi) = Q_n(\rho)[\lambda\{\hat{\omega}(\xi)-\hat{\omega}(\rho)\}+w_0D(\rho-\xi)+O(\frac{1}{\sqrt{n}})]. \tag{3.15}$$

Multiply (3.13) by $e^{-\xi u}$ and integrate from 0 to ∞, then

$$\begin{aligned}
\int_0^\infty e^{-\xi u}\phi_{w,n}(u)du &= \frac{1}{1+\beta_n}\int_0^\infty e^{-\xi u}\int_0^u \phi_{w,n}(u-x)dG_n(x)du \\
&\quad +\frac{1}{1+\beta_n}\int_0^\infty e^{-\xi u}H_n(u)du, \qquad \xi\ge 0, \\
&= \frac{1}{1+\beta_n}\hat{\phi}_{w,n}(\xi)\hat{g}_n(\xi)+\frac{1}{1+\beta_n}\hat{H}_n(\xi) \\
&= \frac{\hat{H}_n(\xi)}{1+\beta_n-\hat{g}_n(\xi)}.
\end{aligned}$$

Using equations (3.14) and (3.15) in the above equation, we get

$$\begin{aligned}
\hat{\phi}_{w,n}(\xi) &= \frac{Q_n(\rho)[\lambda\{\hat{\omega}(\xi)-\hat{\omega}(\rho)\}+w_0D(\rho-\xi)+O(\frac{1}{\sqrt{n}})]}{Q_n(\rho)(\rho-\xi)(b+\xi)D[1-\hat{g}_D(\xi)+O(\frac{1}{\sqrt{n}})]} \\
\Rightarrow \lim_{n\to\infty}\hat{\phi}_{w,n}(\xi) &= \frac{\lambda\{\hat{\omega}(\xi)-\hat{\omega}(\rho)\}}{D(\rho-\xi)(b+\xi)[1-\hat{g}_D(\xi)]}+\frac{w_0}{(b+\xi)[1-\hat{g}_D(\xi)]}.
\end{aligned}$$

By Lemma 2.2

$$\lim_{n\to\infty} \hat{\phi}_{w,n}(\xi) = \frac{\hat{H}_{\omega D}(\xi)}{1-\hat{g}_D(\xi)} + \frac{w_0}{(b+\xi)[1-\hat{g}_D(\xi)]}, \qquad \xi \geq 0.$$

Which is exactly the Laplace transform of the expected discounted penalty function $\phi_D(u) = \phi_c(u) + w_0\phi_d(u)$, shown by Theorem 2.4. □

The following examples illustrate the function $|w_e(x,y)| \leq \alpha(x+y)^r$.

Example 3.2 *For any fixed $x_1, y_1 > 0$, let*

$$w(x,y) = \begin{cases} 1 & if \quad x \leq x_1,\ y \leq y_1 \\ 0 & otherwise \end{cases},$$

i.e, $w(x,y)$ gives the distribution function of $U(T^-)$ and $|U(T)|$, when $\delta = 0$ and the discounted distribution function, while $\delta > 0$. Then $w(0,0) = w_0 = 1$, and hence

$$|w_e(x,y)| = \begin{cases} 0 & if \quad x \leq x_1,\ y \leq y_1 \\ 1 & if \quad x > x_1 \text{ or } y > y_1 \end{cases}.$$

So, it is clear that $|w_e(x,y)| \leq (\frac{x+y}{min\{x_1,y_1\}})^2$.

□

Example 3.3 *Again, for any fixed $v > 0$, let*

$$w(x,y) = \begin{cases} 1 & if \quad x+y \leq v \\ 0 & otherwise \end{cases},$$

i.e., $w(x,y)$ gives the distribution function of $U(T^-) + |U(T)|$, when $\delta = 0$ and the discounted distribution function, while $\delta > 0$. This kind of distribution is interesting because $U(T^-) + |U(T)|$ is a claim causing ruin. Then $w(0,0) = w_0 = 1$, and hence

$$|w_e(x,y)| = \begin{cases} 0 & if \quad x+y \leq v \\ 1 & if \quad x+y > v \end{cases}.$$

Thus, clearly $|w_e(x,y)| \leq (\frac{x+y}{v})^2$.

$\square$

Example 3.4 *Finally, for any* $k, l \geq 1$, *let*

$$w(x,y) = x^k y^l,$$

i.e., the product moments of $U(T^-)$ *and* $|U(T)|$. *Then* $w(0,0) = w_0 = 0$, *therefore*

$$|w_e(x,y)| \leq (x+y)^{k+l}.$$

$\square$

Appendix

This section illustrates some numerical results for our model. We consider claims are exponentially distributed, i.e., $p(x) = \beta e^{-\beta x},\ x > 0$ with a positive security loading condition $c > \lambda\mu$. Also, let Y' have the exponential distribution with mean $\mu' = 1,\ i.e.,\ p_{Y'}(y) = e^{-y},\ y > 0$. To keep calculations simple let $w(x, y) = 1$. Then

$$\hat{p}(\xi) = \frac{\beta}{\beta + \xi}, \tag{3.16}$$

$$\hat{p}_{Y'}(\xi\sqrt{\frac{D}{n}}) = \frac{1}{1 + \xi\sqrt{\frac{D}{n}}} \to 1 - \xi\sqrt{\frac{D}{n}} + \xi^2\frac{D}{n} \quad as \ \ n \to \infty. \tag{3.17}$$

Also we have

$$\begin{aligned} \hat{p}_n(\xi) &= \frac{\lambda}{\lambda_n}\hat{p}(\xi) + \frac{n}{\lambda_n}\hat{p}_{Y'}(\xi\sqrt{\frac{D}{n}}) \\ &= \frac{\lambda}{\lambda_n}\frac{\beta}{\beta + \xi} + \frac{n - \xi\sqrt{nD} + \xi^2 D}{\lambda_n}. \end{aligned}$$

Now, for $\delta \geq 0$, we see that the exponential claim size distribution satisfies Lemma 3.4, when $n \to \infty$.

$$\begin{aligned} & c_n\xi + \lambda_n\hat{p}_n(\xi) = \lambda_n + \delta, \\ \Rightarrow \quad & (c + \sqrt{nD})\xi + \lambda_n[\frac{\lambda}{\lambda_n}\frac{\beta}{\beta + \xi} + \frac{n - \xi\sqrt{nD} + \xi^2 D}{\lambda_n}] = \lambda + n + \delta, \\ \Rightarrow \quad & c\xi + \frac{\lambda\beta}{\beta + \xi} + \xi^2 D = \lambda + \delta, \end{aligned} \tag{3.18}$$

which is the generalized Lundberg's equation for exponential claims.

By Lemma 2.2, we have

$$\hat{g}_D(\xi) = \frac{\lambda\beta}{D(b+\xi)(\beta+\xi)(\beta+\rho)}, \qquad \xi \geq 0, \tag{3.19}$$

$$\hat{H}_{\omega D}(\xi) = \frac{\lambda}{D(b+\xi)(\beta+\xi)(\beta+\rho)}. \tag{3.20}$$

Let us now obtain the explicit expressions for $\phi_c(u), \phi_d(u)$ and $\phi_D(u)$. Moreover we derive the ultimate ruin probabilities for the DRM through our model. We have

$$\begin{aligned} 1+\beta_n - \hat{g}_n(\xi) &= Q_n(\rho)[(\rho-\xi)(c+\sqrt{nD}) - \lambda\{\hat{p}(\xi)-\hat{p}(\rho)\} \\ &\quad -n\{\hat{p}_{Y'}(\xi\sqrt{\frac{D}{n}}) - \hat{p}_{Y'}(\rho\sqrt{\frac{D}{n}})\}]. \end{aligned}$$

Using (3.16) and (3.17) in the above equation

$$1+\beta_n - \hat{g}_n(\xi) = DQ_n(\rho)(b+\xi)(\rho-\xi)[1-\hat{g}_D(\xi)] \tag{3.21}$$

and

$$\begin{aligned} \hat{H}_n(\xi) &= Q_n(\rho)[\lambda\{\hat{\omega}(\xi)-\hat{\omega}(\rho)\} + nw_0\{\frac{1-\hat{p}_{Y'}(\xi\sqrt{\frac{D}{n}})}{\xi} - \frac{1-\hat{p}_{Y'}(\rho\sqrt{\frac{D}{n}})}{\rho}\}, \qquad \xi \geq 0, \\ &\quad +n\int_0^\infty \{\int_0^{y\sqrt{\frac{D}{n}}} (e^{-\xi x} - e^{-\rho x}) w_e(x, y\sqrt{\frac{D}{n}} - x)dx\} dP_{Y'}(y)]. \end{aligned}$$

Since $w(x,y) = 1$ and using (3.16) and (3.17), the first term in the above equation becomes

$$\lambda\{\hat{\omega}(\xi) - \hat{\omega}(\rho)\} = \frac{\lambda(\rho-\xi)}{(\beta+\rho)(\beta+\xi)}.$$

The second term in $\hat{H}_n(\xi)$ as $n \to \infty$:

$$nw_0\{\frac{1-\hat{p}_{Y'}(\xi\sqrt{\frac{D}{n}})}{\xi} - \frac{1-\hat{p}_{Y'}(\rho\sqrt{\frac{D}{n}})}{\rho}\} \to w_0 D(\rho-\xi).$$

By the assumption $|w_e(x,y)| \leq \alpha(x+y)^r$ made in the statement of Theorem 3.7 and $|e^{-\xi x} - e^{-\rho x}| \leq 1$. The last term in $\hat{H}_n(\xi)$ as $n \to \infty$:

$$\begin{aligned}
&|n \int_0^\infty \{\int_0^{y\sqrt{\frac{D}{n}}} (e^{-\xi x} - e^{-\rho x}) w_e(x, y\sqrt{\frac{D}{n}} - x) dx\} dP_{Y'}(y)| \\
&\leq n\alpha \int_0^\infty (y\sqrt{\frac{D}{n}})^{r+1} dP_{Y'}(y) \\
&= \alpha D^{\frac{r+1}{2}} n^{\frac{1-r}{2}} \int_0^\infty y^{r+1} e^{-y} dy \to 0,
\end{aligned}$$

since for some $r > 1$, $\int_0^\infty y^{r+1} e^{-y} dy < \infty$. So

$$\hat{H}_n(\xi) = Q_n(\rho)[\frac{\lambda(\rho-\xi)}{(\beta+\rho)(\beta+\xi)} + w_0 D(\rho - \xi)], \qquad \xi \geq 0\,. \tag{3.22}$$

Note that $w_0 = 1$ as $|w_e(0,0)| \leq 0$. Hence by Theorem 3.7:

$$\begin{aligned}
\lim_{n\to\infty} \hat{\phi}_{w,n}(\xi) &= \frac{\lambda}{D(b+\xi)(\beta+\xi)(\beta+\rho)[1-\hat{g}_D(\xi)]} + \frac{1}{(b+\xi)[1-\hat{g}_D(\xi)]} \\
&= \hat{\phi}_c(\xi) + \hat{\phi}_d(\xi),
\end{aligned}$$

by the defective renewal equations (2.9) and (2.10).

Therefore

$$\begin{aligned}
\hat{\phi}_c(\xi) &= \frac{\lambda}{D(b+\xi)(\beta+\xi)(\beta+\rho)[1-\hat{g}_D(\xi)]}, \qquad \xi \geq 0, \\
&= \frac{\frac{\lambda}{D(\beta+\rho)}}{(b+\xi)(\beta+\xi) - \frac{\lambda\beta}{D(\beta+\rho)}}.
\end{aligned}$$

Suppose that ξ_1 and ξ_2 are two roots of the following equation

$$(b+\xi)(\beta+\xi) - \frac{\lambda\beta}{D(\beta+\rho)} = 0. \tag{3.23}$$

Then

$$\hat{\phi}_c(\xi) = \frac{\lambda}{D(\beta+\rho)(\xi_1-\xi_2)}[\frac{1}{\xi-\xi_1} - \frac{1}{\xi-\xi_2}].$$

Taking inverse Laplace transforms, we get

$$\phi_c(u) = \frac{\lambda}{D(\beta+\rho)(\xi_1-\xi_2)}[e^{\xi_1 u} - e^{\xi_2 u}], \qquad u \geq 0. \tag{3.24}$$

Similarly

$$\hat{\phi}_d(\xi) = \frac{1}{\xi_1-\xi_2}[\frac{\beta+\xi_1}{\xi-\xi_1} - \frac{\beta+\xi_2}{\xi-\xi_2}], \qquad u \geq 0 \, .$$

By inverse Laplace transforms

$$\phi_d(u) = \frac{1}{\xi_1-\xi_2}[(\beta+\xi_1)e^{\xi_1 u} - (\beta+\xi_2)e^{\xi_2 u}], \qquad u \geq 0. \tag{3.25}$$

Numerical Results

Suppose that $c = 2, \lambda = \beta = D = 1$ and $\delta = 0.1$. Then the nonnegative solution of (3.18) is $\rho = 0.0858441545$. Equation (3.23) gives $\xi_1 = -0.4403311035$, $\xi_2 = -2.645513051$.

So, (3.24) and (3.25) take the following form:

$$\phi_c(u) = 0.417626524[e^{-0.4403311035u} + e^{-2.645513051u}], \qquad u \geq 0,$$

$$\phi_d(u) = 0.253797151\, e^{-0.4403311035u} + 0.746202848\, e^{-2.645513051u}.$$

Hence the total expected discounted penalty function:

$$\phi_D(u) = 0.671423675\, e^{-0.4403311035u} + 0.328576324\, e^{-2.645513051u}, \qquad u \geq 0.$$

Now assume that $c = 2,\ \lambda = \beta = D = 1$ and $\delta = 0$. By the same procedure we have $\rho = 0$, $\xi_1 = -0.381966011$ and $\xi_2 = -2.618033989$.

In this case, $\phi_c(u) = \psi_c(u)$ and $\phi_d(u) = \psi_d(u)$. Thus the corresponding explicit expressions are given by

$$\psi_c(u) = 0.447213595[e^{-0.381966011u} - e^{-2.618033989u}], \qquad u \geq 0,$$

$$\psi_d(u) = 0.276393202\, e^{-0.381966011u} + 0.723606797\, e^{-2.618033989u},$$

$$\psi_D(u) = 0.723606797\, e^{-0.381966011u} + 0.276393202\, e^{-2.618033989u}.$$

Let us now investigate the effect of different values of D.

Case(i): when $D = 0.25$. Then $\rho = 0.09045174916$, $\xi_1 = -0.5157307462$ and $\xi_2 = -8.574721003$.

$$\phi_c(u) = 0.455169242[e^{-0.5157307462u} - e^{-8.574721003u}], \qquad u \geq 0,$$

$$\phi_d(u) = 0.060090562\, e^{-0.5157307462u} + 0.939909438\, e^{-8.574721003u},$$

$$\phi_D(u) = 0.515259804\, e^{-0.5157307462u} + 0.484740169\, e^{-8.574721003u},$$

with $\delta = 0$: $\rho = 0$, $\xi_1 = -0.468871125$ and $\xi_2 = -8.531128874$.

$$\psi_c(u) = 0.4961389384[e^{-0.468871125u} - e^{-8.53112887u}], \qquad u \geq 0,$$

$$\psi_d(u) = 0.06587842904\, e^{-0.468871125u} + 0.934121571\, e^{-8.53112887u},$$

$$\psi_D(u) = 0.5620173674\, e^{-0.468871125u} + 0.4379826326\, e^{-8.53112887u}.$$

Case(ii): when $D = 0.50$. Then $\rho = 0.08881201191$, $\xi_1 = -0.4896422020$ and $\xi_2 = -4.59916981$.

$$\phi_c(u) = 0.446977014[e^{-0.4896422020u} - e^{-4.59916981u}], \qquad u \geq 0,$$

$$\phi_d(u) = 0.12418892\, e^{-0.4896422020u} + 0.875811079\, e^{-4.59916981u},$$

$$\phi_D(u) = 0.571165934\, e^{-0.4896422020u} + 0.428834065\, e^{-4.59916981u},$$

with $\delta = 0$: $\rho = 0, \xi_1 = -0.438447187$ and $\xi_2 = -4.561552813$.

$$\psi_c(u) = 0.48507125[e^{-0.438447187u} - e^{-4.561552813u}], \qquad u \geq 0,$$

$$\psi_d(u) = 0.1361965625\, e^{-0.438447187u} + 0.8638034375\, e^{-4.561552813u},$$

$$\psi_D(u) = 0.6212678125\, e^{-0.438447187u} + 0.3787321875\, e^{-4.561552813u}.$$

Case(iii): when $D = 0.75$. Then $\rho = 0.08728028151$, $\xi_1 = -0.4643929012$ and $\xi_2 = -3.289554048$.

$$\phi_c(u) = 0.4340642313[e^{-0.4643929012u} - e^{-3.289554048u}], \qquad u \geq 0,$$

$$\phi_d(u) = 0.1895846187\, e^{-0.4643929012u} + 0.8104153812\, e^{-3.289554048u},$$

$$\phi_D(u) = 0.62364885\, e^{-0.4643929012u} + 0.3763511499\, e^{-3.289554048u},$$

with $\delta = 0$: $\rho = 0$, $\xi_1 = -0.4093327093$ and $\xi_2 = -3.257333958$.

$$\psi_c(u) = 0.4681645886[e^{-0.4093327093u} - e^{-3.257333958u}], \qquad u \geq 0,$$

$$\psi_d(u) = 0.2073971319\, e^{-0.4093327093u} + 0.792602868\, e^{-3.257333958u},$$

$$\psi_D(u) = 0.6755617205\, e^{-0.4093327093u} + 0.3244382794\, e^{-3.257333958u}.$$

Graphical Illustrations of EDPF

Figure 3.1 explains the effect of different values of D in EDPF. It is clear that for small values of u, the graphs increase very sharply (approximately for $0.4 < u < 1$) and then also decrease sharply. For large values of u (i.e., $u \geq 10$) the rate of decreasing is very slow. We also see that for greater values of D, the rate of increasing (for smaller

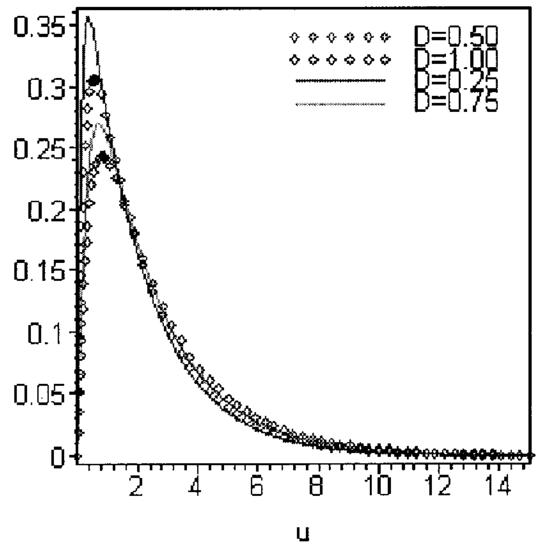

Figure 3.1: EDPF due to claims for different D.

values of u) of graphs is comparatively less than that for the smaller values of D. On the other hand, for larger D graphs decrease faster than that for smaller D.

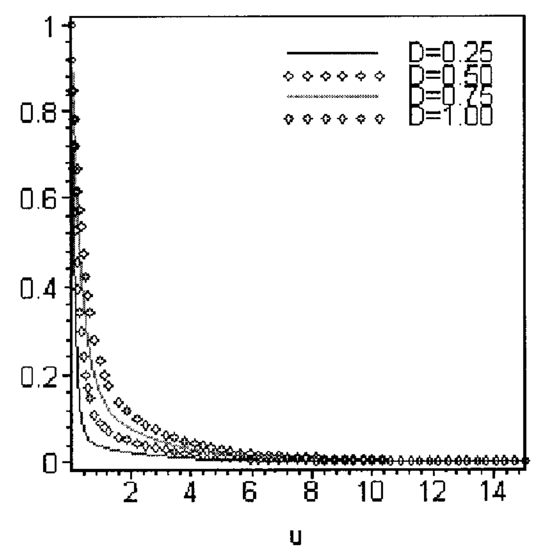

Figure 3.2: EDPF due to oscillations for different D.

In Figure 3.2, we see that the graphs of EDPF due to oscillations are totally different from the graphs due to claims. All the graphs due to oscillations are decreasing for $u \geq 0$, because claims dominate for large u and oscillations dominate for small u. Approximately, for $0 < u < 1.2$, graphs decrease sharply and then decrease very slowly. Also, graphs decrease faster with smaller values of D.

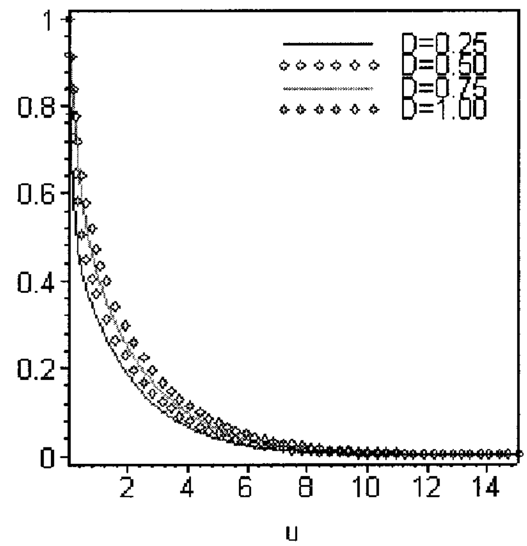

Figure 3.3: EDPF due to either claims or oscillations for different D.

In Figure 3.3, let us see the complete graph of EDPF, i.e., due to either claim or oscillations. These graphs behave like the graphs due to oscillations. But these have smoothness rather than sharpness. These also have the property that the smaller D, the faster decreasing.

Figure (3.4) illustrates the graphs of EDPF due to claims, due to oscillations and due to claims or oscillations for $D = 1.00$. We can easily distinguish and compare the behavior of all the graphs. Clearly, for $0 < u < 0.8$ (approximately), the graphs due to oscillations and either claims or oscillations decrease. On the contrary, the graph due to claims increases. As for small initial reserve oscillations are more dominating than claims. Then for $u > 1$ (approximately), all graphs decrease in different rate.

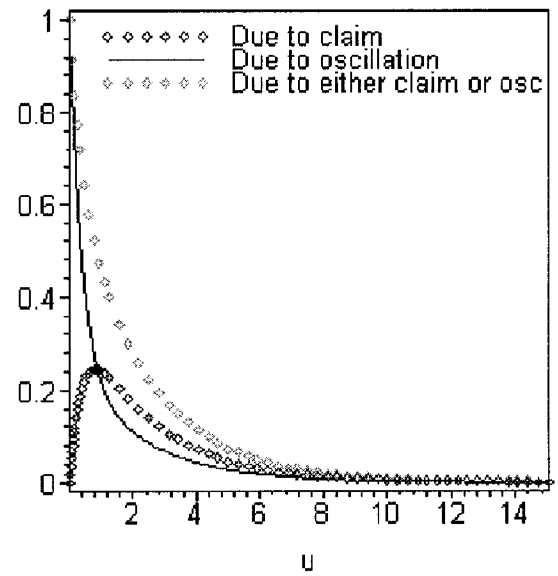

Figure 3.4: Comparison of the graphs of EDPF for D=1.00.

Graphical Illustration of Ruin Probabilities

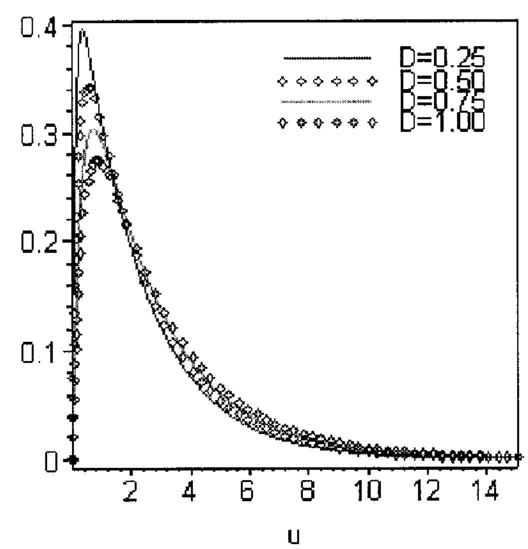

Figure 3.5: Ruin probability due to claims for different D.

Figure 3.5 describes the graphs of the ruin probability due to claims for different D. The ruin probability due to claims for $u = 0$, is 0 and for large u tends to 0. The ruin probability due to claims increases sharply with the increasing values of

u (approximately when $0 < u < 1$) but decreases for $u > 1$ (approximately), since the larger the initial reserve, the smaller the ruin probability due to claims. Another aspect is that for $0 < u < 1.2$ (approximately), the ruin probability due to claim is bigger with smaller D but for $u > 1.8$ (approximately), the ruin probability due to a claim is smaller with smaller D.

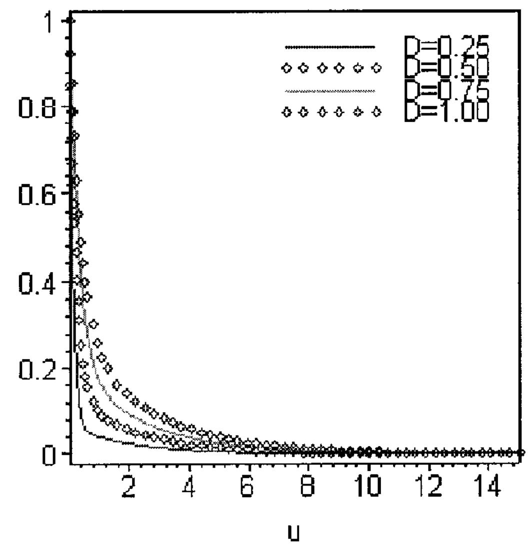

Figure 3.6: Ruin probability due to oscillations for different D.

Figure 3.6 tells us that the ruin probability due to oscillations always decreases. For $u = 0$, the ruin probability due to oscillations is 1. But for large u, the ruin probability due to oscillations tends to 0 because then ruin is possible by big jumps. The graphs show that the ruin probability due to oscillations varies with different values of D, specifically, the smaller D the smaller ruin probability due to oscillations.

We can say from Figure 3.7 that the ruin probability due to either claims or oscillations decreases if the initial reserve increases. If $u = 0$, this is just the ruin

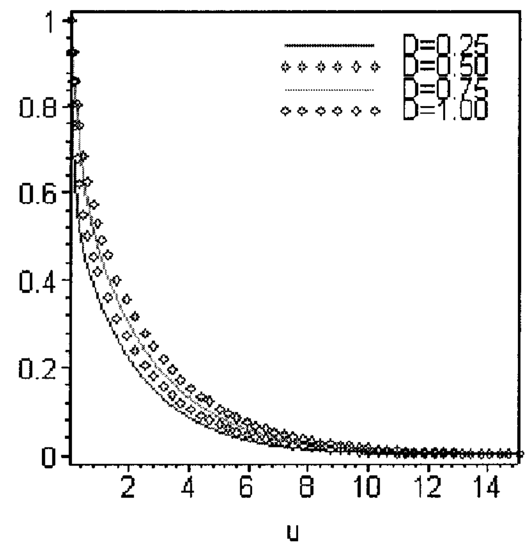

Figure 3.7: Ruin probability due to either claim or oscillations for different D.

probability due to oscillations. There is no possibility of ruin by claims. And for large u, it tends to 0. Further the smaller D, the smaller the ruin probability due to either claims or oscillations.

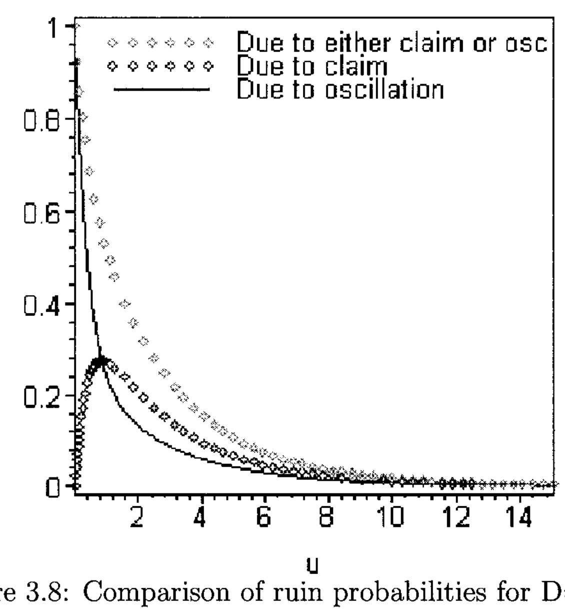

Figure 3.8: Comparison of ruin probabilities for D=1.00.

In Figure 3.8, the ruin probability due to claims, due to oscillations and due to either claims or oscillations are plotted. Graphs give an important conception that for small initial reserve the ruin probability due to oscillations and either claims or oscillations decreases but that due to claims increases as the probability of ruin due to oscillations is more than that due to claims for small u. For large initial reserve all the ruin probabilities decrease but at different rates.

Conclusion

In classical risk analysis, the most popular risk process, easy to analyze and suitable for practical applications is the compound Poisson. This explains the rigorous research done on the compound Poisson process by many authors for a long time. But due to practical reasons, now a days the CPP is no longer the fundamental risk model in modern risk theory.

People are interested in incorporating the phenomena arising in practical situations (e.g., the effect of the fluctuations in the number of customers, the investment of surplus, the inflation of claims and premiums, etc.) in the risk model to make it more practical. That is why Gerber (1970) established the diffusion risk model. The ruin probabilities due to claims, due to oscillations/diffusion, and due to either to a claim or oscillation/diffusion, are obtained for the DRM by Dufresne and Gerber (1991). We derive the above quantities for the same model by using the weak convergence approach from the CPP, given by (3.4) which is made by adding the well known and established sequence of risk processes converges weakly to the standard Wiener process when, for instance, the number of policies in a large insurance portfolio goes to infinity. It is very interesting that not only the ruin probabilities but also we showed that the more general EDPF and the joint density function of the random variables, the surplus immediately before ruin, the deficit at ruin, and the time to ruin are easily obtained for the DRM from the process (3.4). These results are derived by Tsai and Willmot (2002), and Zhang and Wang (2003) respectively with different approaches. The EDPF and the ruin probabilities are illustrated in the Appendix by a numerical example for the exponential claim size distribution .

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