Absolutely Continuous Invariant Measures For Random Maps

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Abstract

Absolutely Continuous Invariant Measures For Random Maps

Wael Bahsoun

A random map is a discrete time dynamical system consisting of a collection of transformations which are selected randomly by means of probabilities at each iteration. We prove the existence of absolutely continuous invariant measures for different classes of position dependent random maps under mild conditions. Moreover, we prove that these measures are stable under small stochastic perturbations. We also apply these results to forecasting in financial markets.

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Contents

Introduction

Ergodic theory of dynamical systems is concerned with the qualitative analysis of iterations of a single transformation. In general, the long time behavior of trajectories of a chaotic dynamical system is unpredictable. Therefore, it is natural to describe the behavior of the system as a whole by statistical means. In this approach, one attempts to describe the dynamics by proving the existence of an invariant measure and determining its ergodic properties. In particular, the existence of invariant measures which are absolutely continuous with respect to Lebesgue measure is very important from a physical point of view, because computer simulations of orbits of the system reveal only invariant measures which are absolutely continuous with respect to Lebesgue measure.

Ulam and von Neumann [42] suggested the study of more general systems where, at each iteration, a transformation is selected randomly from a collection of transformations. Such dynamical systems have recently found application in the study of fractals [8] and in modeling interference effects in quantum mechanics [15].

Recently, random dynamical systems have been shown to provide a useful framework for modeling and analysis of economic phenomena with stochastic components [30]. In fact, the theory and applications of random dynamical systems are at the frontier of research in both mathematics and economics [30].

We now describe briefly what is meant by a random map. Let $\tau_1, \tau_2, ..., \tau_K$ be a collection of transformations. Usually, a random map T is defined by choosing τ_k with constant probability p_k, $p_k > 0$, $\sum_{k=1}^{K} p_k = 1$. The ergodic theory of such dynamical systems was studied in [37, 34, 29]. Random maps with position dependent probabilities with $\tau_1, \tau_2, ..., \tau_K$ continuous contracting transformations were studied in [41].

In this thesis we deal with piecewise continuous transformations $\tau_1, \tau_2, ..., \tau_K$ and position dependent probabilities $p_k(x)$, $k = 1, ..., K$, $p_k(x) \geq 0$, $\sum_{k=1}^{K} p_k(x) = 1$, i.e., the p_k's are functions of position. We point out that studying such dynamical systems was first introduced in [23] where sufficient conditions for the existence of an absolutely continuous invariant measure were given. The conditions in [23] are applicable only when $\tau_1, \tau_2, ..., \tau_K$ are piecewise C^2 with a relatively big expanding factor.

We prove the existence of absolutely continuous invariant measures for different classes of position dependent random maps under fairly mild conditions. Moreover, we prove that these measures are stable under small stochastic perturbations. We also study absolutely continuous conditionally invariant measures of random maps with holes. Finally, we apply our results to forecasting in financial markets.

In Chapter 1, we present preliminaries that we need. Chapter 2 includes the derivation of the Frobenius-Perron operator of a position dependent random map and a proof of a random ergodic theorem. In Chapter 3 we prove the existence of absolutely continuous invariant measures for position dependent random maps of the interval. Chapter 4 contains existence results in higher dimensions and on the real line. It also includes the existence of absolutely continuous conditionally invariant measures for random maps of an interval with holes. In Chapter 5, we prove that absolutely continuous invariant measures for position dependent random maps on the interval are stable under small stochastic perturbations. Different models of perturbations are considered. Finally, in Chapter 6, we apply our results to forecasting in financial markets. We represent the binomial model by a position dependent random map and answer the following question: With what frequency do the future prices of a risky security occur in an interval? Markets which are arbitrage free and markets with arbitrage opportunities are considered.

Chapter 1

Preliminaries

After a brief review of measure theory, this chapter presents various results about functions of bounded variation, which will play an important role throughout this text.

1.1 Review of Measure Theory

We recall some fundamental ideas from measure theory. Let X be a set. In most cases we will assume that X is a compact metric space.

Definition 1.1.1 *A family $\mathfrak{B}$ of subsets of X is called a σ-algebra if and only if:*

1) $X \in \mathfrak{B}$;

2) *for any* $B \in \mathfrak{B}$, $X \backslash B \in \mathfrak{B}$;

3) *if* $B_n \in \mathfrak{B}$, *for* $n = 1, 2, ...$, *then* $\bigcup_{n=1}^{\infty} B_n \in \mathfrak{B}$.

Elements of $\mathfrak{B}$ are usually referred to as *measurable sets.*

Definition 1.1.2 *A function $\mu : \mathfrak{B} \to \mathbb{R}^+$ is called a measure on $\mathfrak{B}$ if and only if:*

1) $\mu(\emptyset) = 0$*;*

2) for any sequence $\{B_n\}$ of disjoint measurable sets, $B_n \in \mathfrak{B}$, $n = 1, 2, \ldots$,

$$\mu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n).$$

The triplet $(X, \mathfrak{B}, \mu)$ is called a *measure space.* If $\mu(X) = 1$, we say it is a *normalized measure space* or *probability space.* If X is a countable union of sets of finite μ-measure, then we say that μ is a σ-finite measure. Later on we shall work with probability spaces.

Definition 1.1.3 *A family $\mathfrak{A}$ of subsets of X is called an algebra if:*

1) $X \in \mathfrak{A}$*;*

2) for any $A \in \mathfrak{A}$, $X \backslash A \in \mathfrak{A}$;

3) for any $A_1, A_2 \in \mathfrak{A}$, $A_1 \cup A_2 \in \mathfrak{A}$.

For any family $\mathfrak{J}$ of subsets of X there exists a smallest σ-algebra, $\mathfrak{B}$, containing $\mathfrak{J}$. We say that $\mathfrak{J}$ generates $\mathfrak{B}$ and write $\mathfrak{B} = \sigma(\mathfrak{J})$.

In practice, when defining a measure μ on a space $(X, \mathfrak{B})$, μ is known only on an algebra $\mathfrak{A}$ generating $\mathfrak{B}$. We would like to know if μ can be extended to a measure on $\mathfrak{B}$. The answer is contained in

Theorem 1.1.1 *Given a set X and an algebra $\mathfrak{A}$ of subsets of X, let $\mu_1 \colon \mathfrak{A} \to \mathbb{R}^+$ be a function satisfying $\mu_1(X) = 1$ and*

$$\mu_1(\bigcup_n A_n) = \sum_n \mu_1(A_n)$$

whenever $A_n \in \mathfrak{A}$, for $n = 1, 2, ..., \cup_{n=1}^{\infty} A_n \in \mathfrak{A}$ and $\{A_n\}$ disjoint. Then there exists a unique normalized measure μ on $\mathfrak{B} = \sigma(\mathfrak{A})$ such that $\mu(A) = \mu_1(A)$ whenever $A \in \mathfrak{A}$.

Definition 1.1.4 *Let X be a topological space. Let $\mathfrak{O}$ denote the family of all open subsets of X. Then the σ-algebra $\mathfrak{B} = \sigma(\mathfrak{O})$ is called the Borel σ-algebra of X and its elements, Borel subsets of X.*

Definition 1.1.5 *Let $(X, \mathfrak{B}, \mu)$ be a measure space. The function $f : X \to \mathbb{R}$ is said to be measurable if for all $c \in \mathbb{R}$, $f^{-1}(c, \infty) \in \mathfrak{B}$, or, equivalently, if $f^{-1}(A) \in \mathfrak{B}$ for any Borel set $A \subset \mathbb{R}$.*

If X is a topological space and $\mathfrak{B}$ is the σ-algebra of Borel subsets of X, then each continuous function $f : X \to \mathbb{R}$ is measurable.

Definition 1.1.6 *Let $\mathfrak{B}_n$ be a σ-algebra of Borel subsets of X, $n = 1, 2 \ldots.$ Let $n_1 < n_2 < ... < n_r$ be integers and $A_{n_i} \in \mathfrak{B}_{n_i}$, $i = 1, ..., r$. We define a cylinder set to be a set of the form*

$$C(A_{n_1}, ..., A_{n_r}) = \{\{x_1, x_2, \ldots\} \in X^{\mathbb{N}} : x_{n_i} \in A_{n_i}, \quad 1 \leq i \leq r\}.$$

Definition 1.1.7 *(Direct Product of Measure Spaces)*

Let $(X_i, \mathfrak{B}_i, \mu_i)$, $i \in \mathbb{N}$ be normalized measure spaces. The direct product measure *space* $(X, \mathfrak{B}, \mu) = \Pi_{i=1}^{\infty}(X_i, B_i, \mu_i)$ *is defined by*

$$X = \Pi_{i=1}^{\infty} X_i \quad and \quad \mu(C(A_{n_1}, ..., A_{n_r})) = \Pi_{i=1}^{r} \mu_{ni}(A_{n_i}).$$

It is easy to see that finite unions of cylinders form an algebra of subsets of X. By Theorem 1.1.1 it can be uniquely extended to a measure on $\mathfrak{B}$, the smallest σ-algebra containing all cylinders.

1.2 Spaces of Functions and Measures

Let $\mathfrak{F}$ be a linear space. A function $|| \cdot || : \mathfrak{F} \to \mathbb{R}^+$ is called a *norm* if it has the following properties:

$$||f|| = 0 \Leftrightarrow f \equiv 0$$

$$||\alpha f|| = \mid \alpha \mid ||f||$$

$$||f + g|| \leq ||f|| + ||g||,$$

for $f, g \in \mathfrak{F}$ and $\alpha \in \mathbb{R}$. The space $\mathfrak{F}$ endowed with a norm $|| \cdot ||$ is called a *normed linear space.*

Definition 1.2.1 *A sequence* $\{f_n\}$ *in a normed linear space is a Cauchy sequence if, for any* $\varepsilon > 0$, *there exists an* $N \geq 1$ *such that for any* $n, m \geq N$,

$$||f_n - f_m|| < \varepsilon.$$

Every convergent sequence is a Cauchy sequence.

Definition 1.2.2 *A normed linear space $\mathfrak{F}$ is complete if every Cauchy sequence converges, i.e., if for each Cauchy sequence $\{f_n\}$ there exists $f \in \mathfrak{F}$ such that $f_n \to f$. A complete normed space is called a Banach space.*

Let $(X, \mathfrak{B}, \mu)$ be a normalized measure space.

Definition 1.2.3 *Let $1 \leq p < \infty$. The family of real valued measurable functions (or rather a.e.-equivalence classes of them) $f : X \to \mathbb{R}$ satisfying*

$$\int_X |f(x)|^p d\mu < \infty \tag{1.1}$$

is called the $\mathfrak{L}^p(X, \mathfrak{B}, \mu)$ space and is denoted by $\mathfrak{L}^p(\mu)$ when the underlying space is clearly known, and by $\mathfrak{L}^p$ where both the space and the measure are known.

The integral in (1.1) is assigned a special notation

$$||f||_p = \left(\int_X |f(x)|^p d\mu \right)^{\frac{1}{p}},$$

and is called the $\mathfrak{L}^p$ *norm* of f. $\mathfrak{L}^p$ with the norm $|| \cdot ||_p$ is a complete normed space, i.e., a Banach space.

The space of almost everywhere bounded measurable functions on $(X, \mathfrak{B}, \mu)$ is denoted by $\mathfrak{L}^\infty$. Functions that differ only on a set of μ-measure 0 are considered to represent the same element of $\mathfrak{L}^\infty$. The $\mathfrak{L}^\infty$ norm is given by

$$||f||_\infty = \text{ess sup}|f(x)| = \inf \{M : \mu\{x : f(x) > M\} = 0\}.$$

The space $\mathfrak{L}^\infty$ with the norm $|| \cdot ||_\infty$ is a Banach space.

Definition 1.2.4 *The space of bounded linear functionals on a normed space $\mathfrak{F}$ is called the adjoint space to $\mathfrak{F}$ and is denoted by $\mathfrak{F}^*$. The weak convergence in $\mathfrak{F}$ is defined as follows: A sequence $\{f_n\}_1^\infty \subset \mathfrak{F}$ converges weakly to an $f \in \mathfrak{F}$ if and only if for any $F \in \mathfrak{F}^*$, $F(f_n) \to F(f)$ as $n \to +\infty$. Similarly, a sequence of functionals $\{F_n\}_1^\infty \subset \mathfrak{F}^*$ converges in the weak-$*$ topology to a functional $F \in \mathfrak{F}^*$ if and only if for any $f \in \mathfrak{F}$, $F_n(f) \to F(f)$ as $n \to +\infty$.*

Theorem 1.2.1 *Let $1 \leq p < \infty$ and let q satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1\,,\ (\frac{1}{\infty} = 0).$$

Then $\mathfrak{L}^q$ is the adjoint space of $\mathfrak{L}^p$.

If $f \in \mathfrak{L}^p$, $g \in \mathfrak{L}^q$, then fg is integrable and the Hölder inequality holds:

$$\int_X |fg|\, d\mu \leq \|f\|_p \|g\|_q.$$

Let $g \in \mathfrak{L}^q$. We define a functional F_g on $\mathfrak{L}^p$ by setting

$$F_g(f) = \int_X fg d\mu$$

$$\|F_g\| = \sup_{f \neq 0} \{\frac{| F_g(f) |}{\|f\|}\}.$$

Clearly F_g is linear.

Proposition 1.2.1 *Each function $g \in \mathfrak{L}^q$ defines a bounded linear functional F_g on $\mathfrak{L}^P$ with $F_g(f) = \int_X fg d\mu$ and $\|F_g\| = \|g\|_q$.*

Theorem 1.2.2 *(Riesz Representation Theorem) [18] Let F be a bounded linear functional on $\mathfrak{L}^p, 1 \leq p < \infty$. Then there exists a function g in $\mathfrak{L}^q$ such that*

$$F(f) = \int_X fg d\mu.$$

Furthermore, $\|F\| = \|g\|_q$.

We will use the following kinds of convergence in $\mathfrak{L}^p$ spaces.

(1) Norm (or strong) convergence:

$$f_n \to f \text{ in } \mathfrak{L}^p - \text{norm} \iff \|f_n - f\|_p \to 0, \; n \to +\infty.$$

(2) Weak convergence: $f_n \to f$ weakly in $\mathfrak{L}^p$, $1 \leq p < +\infty$, $\iff$

$$\forall g \in \mathfrak{L}^q, \quad \int f_n g d\mu \to \int fg d\mu, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

(3) Pointwise convergence:

$$f_n \to f \quad \text{almost everywhere (a.e.)} \iff f_n(x) \to f(x),$$

for almost every $x \in X$.

The following results give several characterizations of these types of convergence and connections between them:

Theorem 1.2.3 *Let a sequence $\{f_n\}_{n=1}^{\infty}$, $f_n \in \mathfrak{L}^1$, $n = 1, 2, ...$ satisfy*

(1) $\|f_n\|_1 \leq M$ *for some* M;

(2) $\forall \varepsilon > 0 \; \exists \; \delta > 0$ *such that for any* $A \in \mathfrak{B}$, *if* $\mu(A) < \delta$ *then for all* n,

$$|\int_A f_n d\mu| < \varepsilon.$$

Then $\{f_n\}$ contains a weakly convergent subsequence, i.e., $\{f_n\}$ is weakly compact.

Corollary 1.2.1 *If there exists $g \in \mathfrak{L}^1$ such that $f_n \leq g$ for $n = 1, 2, ...,$ then $\{f_n\}$ is weakly compact.*

Theorem 1.2.4 *(Scheffé's Theorem) [11] If $f_n \geq 0$, $\int f_n d\mu = 1$, $n = 1, 2, ...$ and $f_n \to f$ a.e. with $\int f d\mu = 1$, then $f_n \to f$ in $\mathfrak{L}^1$-norm.*

Theorem 1.2.5 *If $f_n \to f$ weakly in $\mathfrak{L}^1$ and almost everywhere, then $f_n \to f$ in $\mathfrak{L}^1$-norm.*

We now consider spaces of continuous and differentiable functions. Let X be a compact metric space.

Definition 1.2.5 *$C^0(X) = C(X)$ is the space of all continuous real functions $f : X \to \mathbb{R}$, with the norm*

$$\|f\|_{C^0} = \sup_{x \in X} | f(x) | .$$

Definition 1.2.6 *$\mathfrak{M}(X)$ denotes the spaces of all measures μ on $\mathfrak{B}(X)$. The norm, called the total variation norm on $\mathfrak{M}(X)$, is defined by*

$$\|\mu\| = \sup_{A_1 \cup \ldots \cup A_N = X} \{| \mu(A_1) | + \ldots + | \mu(A_N) |\},$$

where the supremum is taken over all finite partitions of X.

A more frequently used topology on $\mathfrak{M}(X)$ is the *weak topology of measures*, which we can define with the aid of the following result [18]:

Theorem 1.2.6 *Let X be a compact metric space. Then the adjoint space of $C(X)$, $C^*(X)$, is equal to $\mathfrak{M}(X)$.*

Definition 1.2.7 *The weak topology of measures is a topology of weak convergence on $\mathfrak{M}(X)$, i.e.,*

$$\mu_n \to \mu \Leftrightarrow \int_X g d\mu_n \to \int_X g d\mu, \quad \text{for any } g \in C(X).$$

In view of Theorem 1.2.6 this is sometimes referred to as the topology of weak-$*$ convergence.

Theorem 1.2.7 *The weak topology of measures is metrizable and any bounded (in norm) subset of $\mathfrak{M}(X)$ is compact in the weak topology of measures.*

We now present two important corollaries of Theorem 1.2.6.

Corollary 1.2.2 *Two measures μ_1 and μ_2 are identical if and only if*

$$\int_X g d\mu_1 = \int_X g d\mu_2$$

for all $g \in C(X)$.

Corollary 1.2.3 *The set of probability measures is compact in the weak topology of measures.*

For excellent accounts on the weak topology of measures, the reader is referred to [10] and [36]. We now collect a number of results which will be needed in the sequel.

Theorem 1.2.8 *[18] Let $\mathfrak{F}, \mathfrak{G}$ be Banach spaces and let $\{T_n\}$ be a sequence of bounded linear operators on $\mathfrak{F}$ into $\mathfrak{G}$. Then the limit $Tf = \lim_{n\to+\infty} T_n f$ exists for every f in $\mathfrak{F}$ if and only if* *(i)* *the limit* Tf *exists for every* f *in a set dense in* $\mathfrak{F}$

and

(ii) *for each* f *in* $\mathfrak{F}$, $sup_n\ |T_n f| < \infty$.

When the limit Tf *exists for each* f *in* $\mathfrak{F}$, *the operator* T *is bounded and*

$$\|T\| \leq \ lim\ inf_{n \to +\infty}\ \|T_n\| \leq \ sup_n\ \|T_n\| < +\infty.$$

Definition 1.2.8 *Let* ν *and* μ *be two measures on the same measure space* $(X, \mathfrak{B})$. *We say that* ν *is absolutely continuous with respect to* μ *if for any* $A \in \mathfrak{B}$, *such that* $\mu(A) = 0$, *it follows that* $\nu(A) = 0$. *We write* $\nu << \mu$.

A useful condition for testing absolute continuity is given by

Theorem 1.2.9 $\nu << \mu$ *if and only if given* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $\mu(A) < \delta$ *implies* $\nu(A) < \varepsilon$.

The proof of this theorem can be found in [18].

If $\nu << \mu$, then it is possible to represent ν in terms of μ. This is the essence of the Radon–Nikodym Theorem.

Theorem 1.2.10 *(Radon-Nikodym) Let* $(X, \mathfrak{B})$ *be a space and let* ν *and* μ *be two normalized measures on* $(X, \mathfrak{B})$. *If* $\nu << \mu$, *then there exists a unique* $f \in \mathfrak{L}^1(X, \mathfrak{B}, \mu)$ *such that for every* $A \in \mathfrak{B}$,

$$\nu(A) = \int_A f d\mu.$$

f *is called the Radon–Nikodym derivative and is denoted by* $\frac{d\nu}{d\mu}$.

Definition 1.2.9 *Let X be a compact metric space and let μ be a measure on $(X, \mathfrak{B})$, where $\mathfrak{B}$ is the Borel σ-algebra of subsets of X. We define the support of μ as the smallest closed set of full μ measure, i.e.,*

$$supp(\mu) = X \setminus \bigcup_{\substack{\mathcal{O}-open \\ \mu(\mathcal{O})=0}} \mathcal{O}.$$

It is worth noting that two mutually singular measures may have the same support.

Let $\mathfrak{M}(X)$ denote the space of measures on $(X, \mathfrak{B})$. Let $\tau : X \to X$ be a measurable transformation (i.e., $\tau^{-1}(A) \in \mathfrak{B}$ for $A \in \mathfrak{B}$). τ induces a transformation τ_* on $\mathfrak{M}(X)$ by means of the definition: $(\tau_*\mu)(A) = \mu(\tau^{-1}A)$. Since τ is measurable, it is easy to see that $\tau_*\mu \in \mathfrak{M}(X)$. Hence, τ_* is well defined.

Definition 1.2.10 *Let $(X, \mathfrak{B}, \mu)$ be a normalized measure space. Then $\tau : X \to X$ is said to be nonsingular if and only if $\tau_*\mu << \mu$, i.e., if for any $A \in \mathfrak{B}$ such that $\mu(A) = 0$, we have $\tau_*\mu(A) = \mu(\tau^{-1}A) = 0$.*

Proposition 1.2.2 *Let $(X, \mathfrak{B}, \mu)$ be a normalized measure space, and let $\tau : X \to X$ be nonsingular. Then, if $\nu << \mu$, $\tau_*\nu << \tau_*\mu << \mu$.*

Proof. Since $\nu << \mu$, $\mu(A) = 0 \Rightarrow \nu(A) = 0$. Since τ is non-singular, $\mu(A) = 0 \Rightarrow \mu(\tau^{-1}A) = 0 \Rightarrow \nu(\tau^{-1}A) = 0$. Thus, $\tau_*\nu << \tau_*\mu$. Since τ is nonsingular, we obtain $\tau_*\mu << \mu$. ■

Definition 1.2.11 *Let $(X, \mathfrak{B}, \mu)$ be a normalized measure space. Let*

$$\mathfrak{D} = \mathfrak{D}(X, \mathfrak{B}, \mu) = \{f \in \mathfrak{L}^1(X, \mathfrak{B}, \mu) : \ f \ \geq 0 \ and \ \|f\|_1 = 1\}$$

denote the space of probability density functions. A function $f \in \mathfrak{D}$ is called a density function or simply a density.

If $f \in \mathfrak{D}$, then

$$\mu_f(A) = \int_A f d\mu << \mu$$

is a measure and f is called the *density of* μ_f and is written as $d\mu_f/d\mu$.

Theorem 1.2.11 *(Kakutani–Yosida Theorem) [18] Let $\mathfrak{F}$ be a Banach space and let $T : \mathfrak{F} \to \mathfrak{F}$ be a bounded linear operator. Assume there exists $c > 0$ such that $\|T^n\| \leq c$, $n = 1, 2, \ldots$. Furthermore, if for any $f \in A \subset \mathfrak{F}$, the sequence $\{f_n\}$, where*

$$f_n = \frac{1}{n}\sum_{k=1}^{n} T^k f \ ,$$

contains a subsequence $\{f_{n_k}\}$ which converges weakly in $\mathfrak{F}$, then for any $f \in \overline{A}$,

$$\frac{1}{n}\sum_{k=1}^{n} T^k f \to f^* \in \mathfrak{F}$$

(norm convergence) and $T(f^*) = f^*$.

Theorem 1.2.12 *(Schauder–Tykhonov Theorem) [18] Let Λ be a convex subset of a Banach space. Let T be continuous such that $T(\Lambda) \subset \Lambda$. Then T has a fixed point in Λ.*

1.3 Functions of Bounded Variation

Let $[a, b] \subset \mathbb{R}$ be a bounded interval and let λ denote Lebesgue measure on $[a, b]$. For any sequence of points $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \, , n \geq 1$, we define a partition $\mathcal{P} = \{I_i = [x_{i-1}, x_i) : i = 1, ..., n\}$ of $[a, b]$. The points $\{x_0, x_1, ..., x_n\}$ are called *end-points of the partition* $\mathcal{P}$. Sometimes we will write $\mathcal{P} = \mathcal{P}\{x_0, x_1, ..., x_n\}$.

Definition 1.3.1 *Let $f : [a, b] \to \mathbb{R}$ and let $\mathcal{P} = \mathcal{P}\{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$. If there exists a positive number M such that*

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq M$$

for all partitions $\mathcal{P}$, then f is said to be of bounded variation on $[a, b]$.

If f is increasing or if it satisfies the Lipschitz condition

$$|f(x) - f(y)| < K|x - y|,$$

then it is of bounded variation.

Note that the Hölder condition

$$|f(x) - f(y)| < H|x - y|^{\varepsilon},\ 0 < \varepsilon < 1\ ,$$

is not enough to guarantee that f is of bounded variation. This can be seen by considering the function

$$f(x) = \begin{cases} x \sin(\frac{1}{x}), & 0 < x \leq 2\pi \\ 0, & x = 0 \end{cases},$$

which is Hölder continuous, but not of bounded variation.

Definition 1.3.2 *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation. The number*

$$V_{[a,b]} f = \sup_{\mathcal{P}} \{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|\}$$

is called the total variation or, simply, the variation of f on $[a, b]$.

Many of the following results are well known and can be found in the excellent book [35].

Theorem 1.3.1 *If f is of bounded variation on $[a, b]$, then f is bounded on $[a, b]$. In fact,*

$$|f(x)| \leq |f(a)| + V_{[a,b]}f$$

for all $x \in [a, b]$.

Lemma 1.3.1 *Let f be a function of bounded variation such that $\|f\|_1 < \infty$. Then $|f(x)| \leq V_{[a,b]}f + \frac{\|f\|_1}{b-a}$ for all $x \in [a, b]$, where $\|\cdot\|_1$ is the $\mathfrak{L}^1$ norm on $[a, b]$.*

Proof. We claim there exists $y \in [a, b]$ such that $|f(y)| \leq \frac{\|f\|_1}{b-a}$. If not, then for any $x \in [a, b]$

$$(b-a)|f(x)| > \|f\|_1.$$

Hence, $\|f\|_1 = \int_a^b |f(x)| d\lambda(x) > \int_a^b \frac{\|f\|_1}{b-a} d\lambda(x) = \|f\|_1$ and we have a contradiction. Since

$$| f(x) | \leq | f(x) - f(y) | + | f(y) |$$

we have

$$|f(x)| \leq V_{[a,b]}f + \frac{\|f\|_1}{b-a}.$$

■

Theorem 1.3.2 *Let f and g be of bounded variation on $[a,b]$. Then so are their sum, difference and product. Also, we have*

$$V_{[a,b]}(f \pm g) \leq V_{[a,b]}f + V_{[a,b]}g$$

and

$$V_{[a,b]}(f \cdot g) \leq AV_{[a,b]}f + BV_{[a,b]}g,$$

where $A = \sup\{|g(x)| : x \in [a,b]\}$, $B = \sup\{|f(x)| : x \in [a,b]\}$.

Quotients are not included in Theorem 1.3.2 because the reciprocal of a function of bounded variation need not be of bounded variation. For example, if $f(x) \to 0$ as $x \to x_0$, then $1/f$ will not be bounded on any interval containing x_0 and therefore $1/f$ cannot be of bounded variation on such an interval. To extend Theorem 1.3.2 to quotients, we must exclude functions whose values can be arbitrarily close to zero.

Theorem 1.3.3 *Let $f : [a,b] \to \mathbb{R}$ be of bounded variation and assume f is bounded away from 0; i.e., there exists a positive number $\alpha > 0$ such that $|f(x)| \geq \alpha$ for all $x \in [a,b]$. Then $g = 1/f$ is of bounded variation on $[a,b]$ and*

$$V_{[a,b]}g \leq \frac{1}{\alpha^2}V_{[a,b]}f.$$

Proof. Let $\{x_0, ..., x_n\}$ be a partition of $[a,b]$. Since $f \in BV[a,b]$, we have

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| < M_1$$

Then,

$$\sum_{k=1}^{n} \left| \frac{1}{f(x_k)} - \frac{1}{f(x_{k-1})} \right| = \sum_{k=1}^{n} \frac{|f(x_k) - f(x_{k-1})|}{|f(x_k)||f(x_{k-1})|}$$
$$\leq \frac{1}{\alpha^2} M_1.$$

Therefore, $\frac{1}{f} \in BV[a,b]$ and $V_{[a,b]}\frac{1}{f} \leq \frac{1}{\alpha^2} V_{[a,b]} f$. ■

If we keep f fixed and study the total variation as a function of the interval $[a,b]$, we have the following property:

Theorem 1.3.4 *Let $f : [a,b] \to \mathbb{R}$ be of bounded variation and assume $c \in (a,b)$. Then f is of bounded variation on $[a,c]$ and on $[c,b]$ and we have*

$$V_{[a,b]} f = V_{[a,c]} f + V_{[c,b]} f.$$

The following result characterizes functions of bounded variation.

Theorem 1.3.5 *Let f be defined on $[a,b]$. Then f is of bounded variation if and only if f can be expressed as the difference of two increasing functions.*

Theorem 1.3.6 *Let f be of bounded variation on $[a,b]$. If $x \in [a,b]$, let $V(x) = V_{[a,x]} f$ and let $V(a) = 0$. Then every point of continuity of f is also a point of continuity of V. The converse is also true.*

Combining the two foregoing theorems, we have

Theorem 1.3.7 *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is of bounded variation on $[a,b]$ if and only if f can be represented as the difference of two increasing continuous functions.*

We now distinguish an important subspace of functions of bounded variation.

Definition 1.3.3 *Let $f : [a,b] \to \mathbb{R}$. f is called an absolutely continuous function if and only if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $\{s_i, t_i\}_{i=1}^N$*

$$\sum_{i=1}^{N} |t_i - s_i| < \delta \implies \sum_{i=1}^{N} |f(t_i) - f(s_i)| < \varepsilon.$$

If f has a continuous derivative (or more generally, if f is absolutely continuous), there is a very useful representation for its variation.

Theorem 1.3.8 *Let $f : [a,b] \to \mathbb{R}$ have a continuous derivative f' on $[a,b]$. Then*

$$V_{[a,b]} f = \int_a^b |f'(x)| d\lambda(x).$$

We now briefly discuss the interesting relation between absolute continuity (nonsingularity) of a function and nonsingularity of a transformation defined by this function.

Recall, that $f : [0,1] \to [0,1]$ is called nonsingular (as a transformation) $\Leftrightarrow$ for any $A \in \mathfrak{B}([0,1])$ $\lambda(A) = 0 \Rightarrow \lambda(f^{-1}(A)) = 0$ (i.e. $\Leftrightarrow$ $f_*\lambda << \lambda$ for $f_*\lambda(A) = \lambda(f^{-1}(A))$). Then, by the Radon–Nikodym Theorem, there exists a function $g(x) \geq 0$ such that

$$\lambda(f^{-1}(A)) = \int_A g(t) d\lambda(t), \tag{1.2}$$

for all $A \in \mathfrak{B}([0,1])$. Note that the function g may vanish on some set of positive measure. If f is increasing and $f(0) = 0$, then applying the formula (1.2) to $A = [0,x]$, we obtain

$$f^{-1}(x) = \int_0^x g(t) d\lambda(t) \quad \text{for} \quad x \in [0,1].$$

On the other hand, the function $\varphi : [0,1] \to [0,1]$ is called nonsingular or absolutely continuous (as a function) $\Leftrightarrow \varphi$ is differentiable a.e., and

$$\varphi(x) = \int_0^x \varphi'(t)d\lambda(t) \text{ for } x \in [0,1].$$

(This characterization is equivalent to Definition 1.3.3).

We present an inequality that will play an important role in the sequel:

Theorem 1.3.9 *Let $f : [a,b] \to \mathbb{R}$ be of bounded variation. Let $x, y \in [a,b]$ and $x < y$. Then*

$$|f(x)| + |f(y)| \leq V_{[x,y]}f + \frac{2}{|y-x|}\int_x^y |f(t)|dt.$$

Proof. We have

$$|f(x)| + |f(y)| \leq 2 \inf_{x \leq t \leq y} |f(t)| + |f(x) - f(t)| + |f(t) - f(y)|.$$

By the Mean Value Theorem for integrals, we obtain

$$| f(x) | + | f(y) | \leq \frac{2}{| y - x |}\int_x^y | f(t) | d\lambda(t) + V_{[x,y]}f.$$

■

We define a norm on $BV([a,b])$ as follows: For $f \in BV([a,b])$,

$$\|f\|_{BV} = \|f\|_1 + \inf_{f_1 = f a.e.} V_{[a,b]}f_1.$$

Without the $\mathfrak{L}^1$-norm, $\|\cdot\|_{BV}$ would not be a norm, since a function that is not 0 could have 0 variation.

We now collect some miscellaneous properties of $BV([a,b])$.

Proposition 1.3.1 *$BV([a,b])$ is dense in $\mathfrak{L}^1([a,b])$.*

Proof. Since $C^1([a,b])$ is dense in $\mathfrak{L}^1([a,b])$ and $BV([a,b])$ contains $C^1([a,b])$, the result is true. ∎

Below we present two results of [28], which we will use later in the text. Let us define the indefinite integral $\int(\Phi)$ of a function $\Phi \in \mathfrak{L}^1$ by

$$\int(\Phi)(z) = \int_{x \le z} \Phi(x) d\lambda(x).$$

Lemma 1.3.2 *Let $f \in BV$ and $\Phi \in \mathfrak{L}^1$. Then,*

$$|\int f\Phi d\lambda| \le V(f) \cdot \|\int(\Phi)\|_\infty + |\int \Phi d\lambda| \cdot \|f\|_\infty \le 2\|f\|_{BV}\|\int(\Phi)\|_\infty.$$

Theorem 1.3.10 *For $f \in \mathfrak{L}^1$,*

$$V(f) = \sup_{\Phi} \left| \int f\Phi d\lambda \right|,$$

where the supremum extends over all $\Phi \in \mathfrak{L}^1$ with $\|\int(\Phi)\|_\infty \le 1$ and $\int \Phi d\lambda = 0$.

1.4 Conditional Expectations

Definition 1.4.1 *Let $(X, \mathfrak{B}, \mu)$ be a normalized measure space. For f μ-measurable, we define the expectation of f, will be denoted by $E(f)$, by*

$$E(f) = \int_X f d\mu,$$

if the integral exists.

Definition 1.4.2 *Let* $(X, \mathfrak{B}, \mu)$ *be a normalized measure space and let* $\mathfrak{C} \subset \mathfrak{B}$ *be a* σ*-algebra. For* $f \in \mathfrak{L}^1(X, \mathfrak{B}, \mu)$, *we define the conditional expectation of* f *with respect to* $\mathfrak{C}$ *as the Radon–Nikodym derivative of the measure* $f\mu_{|\mathfrak{C}}$ *with respect to the measure* $\mu_{|\mathfrak{C}}$ *and denote it by* $E(f|\mathfrak{C})$*:*

$$E(f|\mathfrak{C}) = \frac{d(f\mu_{|\mathfrak{C}})}{d(\mu_{|\mathfrak{C}})}.$$

Theorem 1.4.1 *For a function* $g \in \mathfrak{L}^1(X, \mathfrak{C}, \mu)$, *we have* $g = E(f|\mathfrak{C})$ *if and only if*

$$\int_A g d\mu = \int_A f d\mu$$

for any $A \in \mathfrak{C}$.

1.5 Stochastic Processes

In this section, we briefly present definitions and theorems of the theory of stochastic processes, mainly Markov processes. For a deep treatment of the subject, we refer the reader to [17].

Let $(\Omega, \mathfrak{B}, \mathbf{P})$ be a probability space. A function $\mathcal{X}$, defined on Ω, is called *random variable* if it is $\mathfrak{B}$-measurable.

Definition 1.5.1 *A stochastic process is a family of random variables* $\{\mathcal{X}_t, t \in T\}$. *If* T *is an infinite sequence, the process is called a discrete stochastic process.*

An important class of stochastic processes are martingales. Martingales are useful tools in game theory, in particular in financial mathematics. One would like to differentiate between an unbiased game and an unfair game. Martingales are the mathematical tool which detect whether a game is fair or not.

Definition 1.5.2 *A stochastic process* $\{\mathcal{X}_t, t \in T\}$ *is called a martingale if* $E(|\mathcal{X}_t|) < \infty$ *for all* t*, and if for any times* $t_1 < \cdots < t_{n+1}$*,*

$$E(\mathcal{X}_{t_{n+1}}|\mathcal{X}_{t_1}, \ldots, \mathcal{X}_{t_n}) = \mathcal{X}_{t_n},$$

with probability 1.

Another important class of stochastic processes, which plays crucial role in later chapters of this book, are Markov processes:

Definition 1.5.3 *A discrete Markov process is a stochastic process* $\{\mathcal{X}_t, t \in T\}$ *satisfying the following condition: for any integer* $n \geq 1$*, if* $t_1 < \cdots < t_n$ *are any parameter values, the conditional expectation of* $\mathcal{X}_{t_n}$ *relative to* $\mathcal{X}_{t_1}, \ldots, \mathcal{X}_{t_{n-1}}$ *is the same as the one relative to* $\mathcal{X}_{t_{n-1}}$*, i.e.,*

$$E(\mathcal{X}_{t_{n+1}}|\mathcal{X}_{t_1}, \ldots, \mathcal{X}_{t_n}) = E(\mathcal{X}_{t_{n+1}}|\mathcal{X}_{t_n}),$$

with probability 1.

In general the dependence of the $\mathcal{X}_{t_{n+1}}$ on $\mathcal{X}_{t_n}$ may depend on time t_n. We will consider only the case where this dependence does not change, i.e., stationary Markov processes.

A special case of Markov process is called a Markov chain. In a Markov chain, the phase space, i.e., the space of possible states of the process is finite or at most countable. Let the consecutive states of the chain be $\mathcal{X}_1, \mathcal{X}_2, \ldots$. If $\mathbf{P}\{\mathcal{X}_m(\omega) = i\} > 0$, define p_{ij} by

$$p_{ij} = \mathbf{P}\{\mathcal{X}_{m+1}(\omega) = j | \mathcal{X}_m(\omega) = i\} = \frac{\mathbf{P}\{\mathcal{X}_m(\omega) = i, \mathcal{X}_{m+1}(\omega) = j\}}{\mathbf{P}\{\mathcal{X}_m(\omega) = i\}},$$

and let $\mathbb{P}$ be the matrix (p_{ij}). Note, we assume that these probabilities are independent of m. Then , the matrix $\mathbb{P}$ called the transition matrix, has the following properties:

$$p_{ij} \geq 0, \qquad \sum_j p_{ij} = 1. \tag{1.3}$$

Any matrix satisfying the above conditions ia called a stochastic matrix. Since

$$\begin{aligned}
&\mathbf{P}\{\mathcal{X}_{m+2}(\omega) = j | \mathcal{X}_m(\omega) = i\} \\
&= \sum_k \mathbf{P}\{\mathcal{X}_{m+2}(\omega) = j | \mathcal{X}_{m+1}(\omega) = k\} \cdot \mathbf{P}\{\mathcal{X}_{m+1}(\omega) = k | \mathcal{X}_m(\omega) = i\} \qquad (1.4) \\
&= \sum_k p_{ik} p_{kj}.
\end{aligned}$$

the matrix of transition probabilities from i at time m to j at time $m+2$ is the product $\mathbb{P} \cdot \mathbb{P}$ of the matrices of one step transition probabilities. In general the matrix of transition probabilities in n steps is

$$(p_{ij}^{(n)}) = \mathbb{P}^n = \left(\sum_k p_{ik}^{(n-1)} p_{kj}^{(1)} \right) = \left(\sum_k p_{ik}^{(n-s)} p_{kj}^{(s)} \right),$$

for any $0 \leq s \leq n$.

If there exists a vector of initial probabilities $\{p_i\}$ satisfying

$$p_i \geq 0, \qquad \sum_i p_i = 1, \qquad \sum_i p_i p_{ij} = p_j, \text{ for any } j, \tag{1.5}$$

they are called invariant or stationary probabilities of the Markov chain. The following theorem can be proved:

Theorem 1.5.1 *If the matrix $\mathbb{P}^n$ is positive for some $n \geq 1$ then the limit $\lim_{n\to\infty} \mathbb{P}^n = \mathbb{Q}$ exists. The matrix $\mathbb{Q} = (q_{ij})$ has constant columns and the probabilities $q_{1,j}$ form a vector of stationary probabilities. We have $q_{1j} > 0$, for all j.*

For a nondiscrete phase space we define Markov process by a transition function:

Definition 1.5.4 *A function $\mathbb{P} : \Omega \times \mathfrak{B} \to [0,1]$ is called a stochastic transition function if it has the following properties:*

(i) for any $A \in \mathfrak{B}$, $\mathbb{P}(\cdot, A) : \Omega \to [0,1]$ is a $\mathfrak{B}$-measurable function;

(ii) for any $x \in \Omega$, $\mathbb{P}(x, \cdot) : \mathfrak{B} \to [0,1]$ is a measure.

Then, if p is a probabilistic measure on $\mathfrak{B}$ called initial probability, we can define all probabilities related to the Markov process using p and $\mathbb{P}$:

$$\begin{aligned} \mathbf{P}(\mathcal{X}_0 \in A) &= p(A); \\ \mathbf{P}(\mathcal{X}_1 \in A | \mathcal{X}_0 = x) &= \mathbb{P}(x, A); \\ \mathbf{P}(\mathcal{X}_1 \in A) &= \int_\Omega \mathbb{P}(x, A) dp(x); \end{aligned} \tag{1.6}$$

and in general:

$$\begin{aligned} &\mathbf{P}(\mathcal{X}_{n+1} \in A | \mathcal{X}_n = x) = \mathbb{P}(x, A); \\ &\mathbf{P}(\mathcal{X}_{n+1} \in A) = \underbrace{\int_\Omega \cdots \int_\Omega}_{(n+1)-\text{times}} dp(x_0)\mathbb{P}(x_0, dx_1)\mathbb{P}(x_1, dx_2)\ldots\mathbb{P}(x_{n-1}, dx_n)\mathbb{P}(x_n, A). \end{aligned} \tag{1.7}$$

Equivalently, Markov process can be understood as a measure on the product space $\Omega_+ = \Omega^{\mathbb{N}\cup\{0\}}$ given by:

$$\mathbf{P}(A_0 \times A_1 \times \cdots \times A_n) = \mathbf{P}(\mathcal{X}_0 \in A_0, \mathcal{X}_1 \in A_1, \ldots, \mathcal{X}_n \in A_n),$$

for all $n \geq 1$ and $A_0, A_1, \ldots, A_n \in \mathfrak{B}$.

Now, we state a generalization of Theorem 1.5.1 and an ergodic theorem for Markov processes. We define

$$\begin{aligned} \mathbb{P}^n(x, A) &= \mathbf{P}\{\mathcal{X}_n \in A | \mathcal{X}_0 = x\} \\ &= \underbrace{\int_\Omega \cdots \int_\Omega}_{(n-1)-\text{times}} \mathbb{P}(x_0, dx_1)\mathbb{P}(x_1, dx_2) \ldots \mathbb{P}(x_{n-2}, dx_{n-1})\mathbb{P}(x_{n-1}, A). \end{aligned} \tag{1.8}$$

The measure m on $\mathfrak{B}$ is called a stationary probability of the Markov process if

$$m(A) = \int_\Omega dm(x)\mathbb{P}(x, A), \tag{1.9}$$

for all $A \in \mathfrak{B}$. Then, obviously $m(A) = \int_\Omega dm(x)\mathbb{P}^n(x, A)$.

Hypothesis: There is a finite measure ϕ on $\mathfrak{B}$ with $\phi(\Omega) > 0$, an integer $\nu \geq 1$, and a positive ε such that

$$\phi(A) \leq \varepsilon \quad \Longrightarrow \mathbb{P}^{(\nu)}(x, A) \leq 1 - \varepsilon,$$

for all $x \in \Omega$. An ergodic Markov process means: if $\mathbb{P}(x, A) = 1$ for all $x \in A$ then $m(A) = 0$ of $m(A^c) = 0$. The following theorem can be found in [17].

Theorem 1.5.2 *If the hypothesis holds, then the limit*

$$m(A) = \lim_{n\to\infty} \mathbb{P}^n(x, A),$$

exists for any $A \in \mathfrak{B}$ *and is independent of the initial point* x. *The measure* m *is a stationary measure for the Markov process. Moreover, for any* $\mathfrak{B}$*-measurable function* f *with*

$$\boldsymbol{E}\{|f(\mathcal{X}_1)|\} = \int_\Omega |f(\xi)| m(d\xi) < \infty,$$

the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f(\mathcal{X}_m)$$

exists with probability one. In particular, under the above hypothesis, if there is only one ergodic set,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} f(\mathcal{X}_m) = \int_\Omega f(\xi) m(d\xi)$$

with probability one.

Remark 1.5.1 *Theorem 1.5.2 will be used to prove an ergodic theorem for position dependent random maps.*

The Markov process induces two operators:

$$\mathcal{M} : L^\infty(\Omega, \mathfrak{B}) \to L^\infty(\Omega, \mathfrak{B}),$$

defined by

$$(\mathcal{M}f)(y) = \int_\Omega f(x) \mathbb{P}(y, dx),$$

and

$$\mathcal{M}_* : M(\Omega, \mathfrak{B}) \to M(\Omega, \mathfrak{B}),$$

defined by

$$(\mathcal{M}_*\mu)(A) = \int_\Omega d\mu(x)\mathbb{P}(x, A), \quad A \in \mathfrak{B}.$$

Note that a measure m is stationary for the Markov process if and only if m is a fixed point of $\mathcal{M}_*$. The operators $\mathcal{M}$ and $\mathcal{M}_*$ are conjugate to each other, i.e.,

$$\int_\Omega (\mathcal{M}f)d\mu = \int_\Omega f d(\mathcal{M}_*\mu),$$

for any bounded function f.

1.6 Review of Ergodic Theory

Now, we present a brief review of ergodic theory. Many of the results will be used in the sequel. For a more complete study of ergodic theory the reader is referred to the excellent text [38]. We start the chapter with a brief review for one map dynamical systems. Then we study random maps with constant probabilities. For detailed study on the existence of absolutely continuous measures and their properties for one map dynamical systems the reader is referred to [14]. Let $(X, \mathfrak{B}, \mu)$ be a normalized measure space. Let $\tau : X \to X$ be transformation. The nth iterate of τ is denoted by τ^n, i.e.,

$$\tau^n(x) = \tau \circ \ldots \circ \tau(x)$$

n times.

Definition 1.6.1 *The transformation $\tau : X \to X$ is measurable if $\tau^{-1}(\mathfrak{B}) \subset \mathfrak{B}$, i.e., $B \in \mathfrak{B} \Rightarrow \tau^{-1}(B) \in \mathfrak{B}$, where $\tau^{-1}(B) \equiv \{x \in X : \tau(x) \in B\}$.*

Definition 1.6.2 *We say the measurable transformation $\tau : X \to X$ preserves measure μ or that μ is τ-invariant if $\mu(\tau^{-1}(B)) = \mu(B)$ for all $B \in \mathfrak{B}$.*

Remark 1.6.1 *We interpret the Definition 1.6.2 as follows: we have $\mu(X) = 1$. Think of X as a metal bar with mass 1 and non-homogenous mass density determined by μ. A measurable transformation $\tau : X \to X$ preserves μ means that if one folds X via τ one obtains again the original distribution on X.*

Definition 1.6.3 *Let $(X, \mathfrak{B}, \mu)$ be a normalized measure space and let $\tau : X \to X$ preserve μ. The quadruple $(X, \mathfrak{B}, \mu, \tau)$ is called a dynamical system.*

The following theorem gives a necessary and sufficient condition for τ-invariance of μ.

Theorem 1.6.1 *Let $\tau : X \to X$ be a measurable transformation of $(X, \mathfrak{B}, \mu)$. Then τ is μ-preserving if and only if*

$$\int_X f(x) d\mu = \int_X f(\tau(x)) d\mu \tag{1.10}$$

for any $f \in \mathfrak{L}^\infty$. If X is compact and (1.10) holds for any continuous function f, then τ is μ-preserving.

Let $(X, \mathfrak{B}, \mu)$ be a normalized measure space and let $\tau : X \to X$ be a measure-preserving transformation on $(X, \mathfrak{B}, \mu)$. If $\tau^{-1}B = B$ for some $B \in \mathfrak{B}$, then $\tau^{-1}(X \backslash B) = X \backslash B$ and the study of τ splits into two parts: $\tau\mid_B$ and $\tau\mid_{X \backslash B}$. It is useful to have a concept of *indecomposability* for measure-preserving transformations, so that if τ has this indecomposability property then the study of τ cannot be split into separate parts. This property is called *ergodicity.*

Definition 1.6.4 *We call a measure-preserving transformation $\tau : X \to X$ ergodic if for any $B \in \mathfrak{B}$, such that $\tau^{-1}B = B$, $\mu(B) = 0$ or $\mu(X \setminus B) = 0$.*

Since ergodicity is a property of the pair (τ, μ), we often say that (τ, μ) is ergodic. Now, we define the notion of *random map* with constant probabilities.

Definition 1.6.5 *Let $(X, \mathfrak{B}, \mu)$ be a measure space. Let $\tau_1, \tau_{2,}, \ldots, \tau_K$ be a collection of point transformations from X into X and define the random map T by choosing τ_k with probability p_k, $p_k > 0$, $\sum_{k=1}^{K} p_k = 1$, where p_k's are constants. The collection $T = \{\tau_1, \tau_{2,}, \ldots, \tau_K; p_1, \ldots, p_K\}$ is called a random map with constant probabilities.*

Definition 1.6.6 *A measure μ on X is called invariant under T if*

$$\mu(A) = \sum_{k=1}^{K} p_k \mu(\tau_k^{-1} A)$$

for all $A \in \mathfrak{B}$.

Example 1.6.1 *Let T be a random map which is given by $\{\tau_1, \tau_2; p_1, p_2\}$ where*

$$\tau_1(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} < x \le 1 \end{cases}, \tag{1.11}$$

$$\tau_2(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2x - 1, & \frac{1}{2} < x \le 1 \end{cases}. \tag{1.12}$$

Then, for any p_1, p_2, obviously T preserves Lebesgue measure on $[0, 1]$.

In [37] the following sufficient condition is used ensure the existence of absolutely continuous invariant measures for these random maps:

$$\sum_{k=1}^{K} \frac{p_k}{|\tau'_k|} \leq \gamma < 1$$

for some constant γ. In [34] a spectral decomposition theorem is proved.

There is a rich literature on random maps with constant probabilities which are often treated as random perturbations of transformations [29]. Usually, random maps with constant probabilities are represented by skew products [37].

Definition 1.6.7 *Let* $(\Omega, \mathfrak{A}, \sigma, \nu)$ *be a dynamical system and let* $(S, \mathfrak{B}, \tau_\omega, \mu_\omega)_{\omega\in\Omega}$ *be a family of dynamical systems such that the function* $\tau_\omega(x)$ *is* $\mathfrak{A} \times \mathfrak{B}$ *measurable. A skew product of* σ *and* $\{\tau_\omega\}_{\omega\in\Omega}$ *is a transformation* $T : \ \Omega \times X \to \Omega \times X$ *defined by*

$$T(\omega, x) = (\sigma(\omega), \tau_\omega(x)),$$

$\omega \in \Omega, \ x \in X.$

In fact an important application of a skew product construction is the *random map*. Let $\Omega = \Sigma^+ = Y^{\{\mathbb{N}\cup 0\}}$, where Y is a compact space with Borel probability measure p. Let $\nu = p^{\{\mathbb{N}\cup(0)\}}$ be the product measure and $\sigma : \Omega \to \Omega$ be the shift to the left. Let $\{X, \mathfrak{B}, \lambda\}$ be a measure space and $\{\tau_y\}_{y\in Y}$ a family of transformations $\tau_y : X \to X$, such that $\tau_y(x)$ is a $\mathfrak{A} \times \mathfrak{B}$ measurable function. A skew product T of σ and $\{\tau_y\}_{y\in Y}$ can be interpreted as a "random map" $\{\tau_y, \eta\}$, where the transformation τ_y is chosen according to the probability p. If Y is a finite space, this model (introduced in

Definition 1.6.5) is especially simple: we have a finite number of transformations $\{\tau_i\}_{i=1}^k$ that act with probabilities $\{p_i\}_{i=1}^k$.

Chapter 2

Position Dependent Random Maps

2.1 Introduction

There is a rich literature on random maps with position dependent probabilities with $\tau_1, \tau_2, \ldots, \tau_K$ being continuous contracting transformations [41]. In this chapter, we deal with piecewise monotonic nonsingular transformations $\tau_1, \tau_2, \ldots, \tau_K$ and position dependent probabilities $p_k(x)$, $k = 1, \ldots, K$, $p_k(x) \geq 0$, $\sum_{k=1}^{K} p_k(x) = 1$, i.e., the p_k's are functions of position. We point out that the study of such dynamical systems was introduced in [23].

2.2 Invariant Measures for Random Maps

In this section we formulate the definition of a random map T with position dependent probabilities. Let $(X, \mathfrak{B}, \lambda)$ be a measure space, where λ is an underlying measure.

Let $\tau_k : X \to X$, $k = 1, \dots, K$ be piecewise one-to-one, non-singular transformations on a common partition $\mathcal{P}$ of X : $\mathcal{P} = \{I_1, \dots, I_q\}$ and $\tau_{k,i} = \tau_k \mid_{I_i}$, $i = 1, \dots, q$, $k = 1, \dots, K$ ($\mathcal{P}$ can be found by considering finer partitions).

Definition 2.2.1 *Let $T = \{\tau_1, \dots \tau_K; p_1(x), \dots p_K(x)\}$. We call T a random map with position dependent probabilities or simply a position dependent random map.*

Definition 2.2.2 *Let $T = \{\tau_1, \dots \tau_K; p_1(x), \dots p_K(x)\}$ be a position dependent random map. Then T preserves a measure μ if and only if*

$$\mu(A) = \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(x) d\mu,$$

for all $A \in \mathfrak{B}$.

Example 2.2.1 *Let T be a random map which is given by $\{\tau_1, \tau_2; p_1(x), p_2(x)\}$*

$$\tau_1(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2x - 1, & \frac{1}{2} < x \le 1 \end{cases}, \tag{2.1}$$

$$\tau_2(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} < x \le 1 \end{cases}; \tag{2.2}$$

$$p_1(x) = \begin{cases} \frac{1}{3}, & 0 \le x \le \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} < x \le 1 \end{cases}, \tag{2.3}$$

and $p_2(x) = 1 - p_1(x)$. Observe that

$$\begin{aligned} \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(x) d\lambda &= 1/6 \cdot \lambda(A) + 2/6 \cdot \lambda(A) + 1/4 \cdot \lambda(A) + 1/4 \cdot \lambda(A) \\ &= \lambda(A). \end{aligned} \tag{2.4}$$

Thus, T preserves Lebesgue measure on $[0, 1]$.

Lemma 2.2.1 *μ is T-invariant if and only if $\int_I g d\mu = \sum_{k=1}^K \int_I p_k(x) \cdot (g \circ \tau_k) d\mu(x)$ for $g \in C(I)$.*

Proof. μ is T-invariant if and only if $\mu(A) = \sum_{k=1}^K \int_{\tau_k^{-1}(A)} p_k(x) d\mu(x)$ for any $A \in \mathfrak{B}$; i.e., if and only if

$$\int_I \chi_A(x) d\mu(x) = \sum_{k=1}^K \int_I p_k(x) \cdot (\chi_A \circ \tau_k) d\mu(x).$$

For any $g \in C(I)$, we can find a simple function which is arbitrarily close to g. Thus, the lemma is true for any $g \in C(I)$. ∎

2.3 The Frobenius-Perron Operator of a Position Dependent Random Map

In this section, following the ideas of [23], we introduce the Frobenius-Perron operator of a position dependent random map and study its properties.

We define the transition function for the random map

$$T = \{\tau_1, \dots \tau_k; p_1(x), \dots p_k(x)\}$$

as follows:

$$\mathbb{P}(x, A) = \sum_{k=1}^K p_k(x) \chi_A(\tau_k(x)), \tag{2.5}$$

where A is any measurable set and $\{p_k(x)\}_{k=1}^K$ is a set of position dependent measurable probabilities, i.e., $\sum_{k=1}^K p_k(x) = 1$, $p_k(x) \geq 0$, for any $x \in X$ and χ_A denotes the characteristic function of the set A. We define

$$T(x) = \tau_k(x) \text{ with probability } p_k(x)$$

and

$$T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)$$

with probability

$$p_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k,1}(x)).p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots p_{k_1}(x).$$

The transition function $\mathbb{P}$ induces an operator $\mathbb{P}_*$ on measures on $(X, \mathfrak{B})$ defined by

$$\begin{aligned} \mathbb{P}_*\mu(A) &= \int \mathbb{P}(x, A) d\mu(x) \\ &= \sum_{k=1}^{K} \int p_k(x)\chi_A(\tau_k(x))d\mu(x) \\ &= \sum_{k=1}^{K}\sum_{i=1}^{q} \int_{\tau_{k,i}^{-1}(A)} p_k(x)d\mu(x) \end{aligned} \tag{2.6}$$

If μ has density f with respect to λ, then $\mathbb{P}_*\mu$ has also a density which we denote by $P_T f$. By a change of variables, we obtain

$$\begin{aligned} \int_A P_T f(x) d\lambda(x) &= \sum_{k=1}^{K}\sum_{i=1}^{q} \int_{\tau_{k,i}^{-1}(A)} p_k(x) f(x) d\lambda(x) \\ &= \sum_{k=1}^{K}\sum_{i=1}^{q} \int_A p_k(\tau_{k,i}^{-1}x) f(\tau_{k,i}^{-1}x) \frac{1}{J_{k,i}(\tau_{k,i}^{-1})} d\lambda(x) \end{aligned} \tag{2.7}$$

where $J_{k,i}$ is the Jacobian of $\tau_{k,i}$ with respect to λ, $J(\tau) = \frac{d\tau_*(\lambda)}{d\lambda}$ the Radon-Nikodym derivative. Since this holds for any measurable set A, we obtain an a.e. equality:

$$(P_T f)(x) = \sum_{k=1}^{K}\sum_{i=1}^{q} p_k(\tau_{k,i}^{-1}x) f(\tau_{k,i}^{-1}x) \frac{1}{J_{k,i}(\tau_{k,i}^{-1})} \chi_{\tau_k(I_i)}(x), \tag{2.8}$$

or

$$(P_T f)(x) = \sum_{k=1}^{K} P_{\tau_k}\left(p_k f\right)(x), \tag{2.9}$$

where P_{τ_k} is the Frobenius-Perron operator corresponding to the transformation τ_k [14]. We call P_T the Frobenius-Perron operator of the random map T. This operator has very useful properties which will be developed in the following Lemma. These properties resemble the properties of the traditional Frobenius-Perron operator for a single map.

Lemma 2.3.1 *P_T satisfies the following properties:*

(i) P_T is linear;

(ii) P_T is non-negative; i.e., $f \geq 0 \implies P_T f \geq 0$;

(iii) $P_T f = f \Leftrightarrow d\mu = f d\lambda(x)$ is T-invariant;

(iv) $||P_T f||_1 \leq ||f||_1$, where $||\cdot||_1$ denotes the $\mathfrak{L}^1$ norm;

(v) $P_{T\circ R} = P_T \circ P_R$. In particular, $(P_T^N f)(x) = (P_{T^N} f)(x)$.

Proof. The proofs of (i)-(iv) follow from the properties of P_τ (See [14], Chapter 4). For (v), let T and R be two random maps corresponding to $\{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\}$ and $\{\zeta_1, \zeta_2, \ldots, \zeta_L; r_1, r_2, \ldots, r_L\}$ respectively. We define $\{\tau_k\}_{k=1}^K$ and $\{\zeta_l\}_{l=1}^L$ on a common partition $\mathcal{P}$. We have

$$\begin{aligned}
(P_T \circ P_R f)(x) &= \left(P_R \sum_{k=1}^{K} P_{\tau_k}(p_k f) \right)(x) \\
&= \sum_{l=1}^{L} \sum_{k=1}^{K} \left(P_{\zeta_l} r_l P_{\tau_k}(p_k f) \right)(x) \\
&= \sum_{l=1}^{L} \sum_{k=1}^{K} \sum_{i=1}^{q} \left(r_l(\zeta_{l,i}^{-1}) P_{\tau_k}(p_k f)(\zeta_{l,i}^{-1}) \frac{1}{J_{l,i}(\zeta_{l,i}^{-1})} \chi_{\zeta_{l,i}(I_i)} \right)(x) \\
&= \sum_{k=1}^{K} \sum_{l=1}^{L} \sum_{j=1}^{q} \sum_{i=1}^{q} (r_l(\zeta_{l,i}^{-1}) p_k(\tau_{k,j}^{-1} \circ \zeta_{l,i}^{-1}) f(\tau_{k,j}^{-1} \circ \zeta_{l,i}^{-1}) \qquad (2.10) \\
&\qquad \times \frac{1}{J_{k,j}(\tau_{k,j}^{-1} \circ \zeta_{l,i}^{-1})} \frac{1}{J_{l,i}(\zeta_{l,i}^{-1})} \chi_{\tau_k(I_j)}(\zeta_{l,i}^{-1}) \chi_{\zeta_{l,i}(I_i)})(x) \\
&= \sum_{k=1}^{K} \sum_{l=1}^{L} P_{\tau_k \circ \zeta_l} \left(p_k(\zeta_l) r_l f \right)(x) \\
&= (P_{T \circ R} f)(x).
\end{aligned}$$

In particular, $(P_T^N f)(x) = (P_{T^N} f)(x)$. ■

Remark 2.3.1 *In the case where $X = [a, b]$, to prove that T admits an absolutely continuous invariant measure on $[a, b]$, it is enough to prove that for any $f \in BV(X)$ there exist an $N \in \mathbb{N}$, and real numbers A, B such that*

$$||P_T^N f||_{BV} \leq A||f||_{BV} + B||f||_1, \tag{2.11}$$

where $0 < A < 1$ and $0 < B < \infty$ (See [14] for details).

2.4 Random Ergodic Theorem

The main goal of this section is to prove that position dependent random maps satisfy an ergodic theorem. In the case of a one map dynamical system, the Birkhoff Ergodic

Theorem establishes the dynamical importance of invariant measures in general, and of the absolutely continuous invariant measures in particular [14]. This motivates us to prove that position dependent random maps satisfy a Birkhoff "type" Ergodic Theorem. The following theorem was proved in [7] and the proof is based on ideas from [19].

Let $\Omega = K^\infty = \{(k_1, k_2, \dots) : 1 \le k_j \le K$ and k_j is an integer for each $j\}$. Let $\mathfrak{A}$ be the σ-algebra generated by the cylinders in Ω. For each $x \in X$, let P_x be the probability measure on $\mathfrak{A}$ defined on cylinders by $P_x((k_1, k_2, \dots, k_N)) = p_{k_N}(\tau_{k_{N-1}} \circ \dots \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ \dots \circ \tau_{k_1}(x)) \cdots p_{k_1}(x)$. This is the probability measure for realizations of the Markov process starting at x. For instance, if we consider a Markov process $\{Z_n, n = 0, 1, \dots\}$ with state space X and transition probability $\mathbb{P}$ as defined in (2.5), then

$$P(Z_0, Z_1, \dots) \in B | Z_0 = x) = P_x\{(k_1, k_2, \dots) : (x, \tau_{k_1}(x), \tau_{k_2}(\tau_{k_1}(x)), \dots) \in B\}$$

for any measurable $B \in (\Omega \times X)^\infty$.

Theorem 2.4.1 *If μ is T-invariant, μ is absolutely continuous and unique among absolutely continuous invariant measures, P_T satisfies (2.11), then for almost every μ point x with probability 1:*

$$\frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) \to \mu(f)$$

for any $f \in \mathfrak{L}^1(X, \mu)$.

Proof. Let $\{Z_n\}$ be the Markov process with transition probability $\mathbb{P}$ such that Z_0 has distribution μ. Then the process is stationary since μ is an invariant measure and it is ergodic and P_T satisfies (2.11). Let $f \in \mathfrak{L}^1(X, \lambda)$. Define

$$\Lambda = \{(x_0, x_1, \dots) \in X^\infty : \frac{1}{n}\sum_{j=0}^{n-1} f(x_j) \to \int f d\mu\}.$$

By Theorem 1.5.2,

$$P((Z_0, Z_1, \dots) \in \Lambda) = 1. \tag{2.12}$$

Observe that

$$\begin{aligned} P((Z_0, Z_1, \dots) \in \Lambda) &= \int P((Z_0, Z_1, \dots) \in \Lambda | Z_0 = x) d\mu(x) \\ &= \int P_x((k_1, k_2, \dots) : (x, \tau_{k_1}(x), \tau_{k_2}(\tau_{k_1}(x)), \dots) \in \Lambda) d\mu(x). \end{aligned} \tag{2.13}$$

Then by (2.12) and (2.13) we have

$$P_{x_0}((k_1, k_2, \dots) : (x_0, \tau_{k_1}(x_0), \tau_{k_2}(\tau_{k_1}(x_0)), \dots) \in \Lambda)) = 1$$

for some $x_0 \in X$. Let $H = \{(k_1, k_2, \dots) : x_0, \tau_{k_2}(\tau_{k_1}(x_0)), \dots) \in \Lambda)\}$. Thus, $P_{x_0}(H) = 1$ and for $(k_1, k_2, \dots) \in H$,

$$\frac{1}{n}\sum_{j=0}^{n-1} f(\tau_{k_j} \circ \dots \circ \tau_{k_1}(x_0)) \to \int f d\mu.$$

Thus, for almost every μ point x with probability 1:

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x)) \to \mu(f)$$

for any $f \in \mathfrak{L}^1(X, \mu)$. ■

Chapter 3

Absolutely Continuous Invariant Measures on the Interval

In this chapter we present results on the existence of absolutely continuous invariant measures for position dependent random maps of an interval [1, 2].

3.1 The Expanding on Average Case

Let $([a,b], \mathfrak{B}, \lambda)$ be a measure space, where λ is Lebesgue measure on $[a,b]$; $\tau_k : [a,b] \to [a,b]$, $k = 1, \ldots, K$ be piecewise one-to-one and differentiable, non-singular transformations on a partition $\mathcal{P}$ of $[a,b]$: $\mathcal{P} = \{I_1, \ldots, I_q\}$ and $\tau_{k,i} = \tau_k \mid_{I_i}$, $i = 1, \ldots, q$, $k = 1, \ldots, K$. Let $p_k(x)$ be the probability of choosing τ_k. Let $g_k(x) = \frac{p_k(x)}{|\tau_k'(x)|}$. We assume that

Condition (A): $\sum_{k=1}^{K} \sup_x g_k(x) \leq \alpha < 1$, and

Condition (B): $g_k \in BV(I); k = 1, ..K$.

Under the above conditions our goal is to prove:

$$V_I P_T^n f \leq A V_I f + B ||f||_1 \tag{3.1}$$

where $A < 1$ and $B > 0$. The inequality in equation (3.1) guarantees the existence of a T-invariant measure absolutely continuous with respect to Lebesgue measure.

Lemma 3.1.1 *Let T be a random map which satisfies condition (A). Then*

$$\sum_{w \in \{1,2,\ldots,K\}^N} \sup_x \frac{p_w(x)}{|T_w'(x)|} \leq \alpha^N. \tag{3.2}$$

Proof. We write

$$T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)$$

with probability

$$p_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k,1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots p_{k_1}(x).$$

The maps defining T^N may be indexed by $\{1, 2, \ldots, K\}^N$. Set

$$T_w(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)$$

where $w = (k_1, \ldots, K_N)$,

$$p_w(x) = p_{k_N}(\tau_{k_N-1} \circ \cdots \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots p_{k_1}(x)$$

and

$$T_w'(x) = \tau_{k_N}'(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x))\tau_{k_{N-1}}'(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots \tau_{k_1}'(x).$$

Suppose that T satisfies condition (A). Using induction on N, we will show that

$$\sum_{w\in\{1,2,\ldots,K\}^N} \sup_x \frac{p_w(x)}{|T_w'(x)|} \leq \alpha^N. \tag{3.3}$$

For $N = 1$, we have

$$\sum_{w\in\{1,2,\ldots,K\}} \sup_x \frac{p_w(x)}{|T_w'(x)|} \leq \alpha \tag{3.4}$$

by condition (A). Assume (3.3) is true for $N-1$. Then,

$$\begin{aligned}\sum_{w\in\{1,2,\ldots,K\}^N} \sup_x \frac{p_w(x)}{|T_w'(x)|} &\leq \sum_{\overline{w}\in\{1,2,\ldots,K\}^{N-1}} \sum_{k=1}^{K} \sup_x \frac{p_k(x)p_{\overline{w}}(\tau_k(x))}{|\tau_k'(x)||T_{\overline{w}}'(\tau_k(x))|} \\ &\leq \sum_{k=1}^{K} \sup_x \frac{p_k(x)}{|\tau_k'(x)|} \sum_{\overline{w}\in\{1,2,\ldots,K\}^{N-1}} \sup_x \frac{p_{\overline{w}}(\tau_k(x))}{|T_{\overline{w}}'(\tau_k(x))|} \\ &\leq \alpha^{N-1} \sum_{k=1}^{K} \sup_x \frac{p_k(x)}{|\tau_k'(x)|} \leq \alpha^N.\end{aligned} \tag{3.5}$$

■

Lemma 3.1.2 *Let T satisfy conditions (A) and (B) and let $\delta = \min_{i=1,\ldots,q} \lambda(I_i)$. Then for any $f \in BV(I)$,*

$$V_I P_T f \leq A V_I f + B||f||_1, \tag{3.6}$$

where $A = \sum_{k=1}^K \max_{1\leq i\leq q}(V_{I_i} g_k) + 2\sum_{k=1}^K \sup_x g_k(x)$ and $B = \frac{2}{\delta}\sum_{k=1}^K \sup_x g_k(x) + \frac{1}{\delta}\sum_{k=1}^K \max_{1\leq i\leq q} V_{I_i} g$.

Proof. Let $a = x_0 < x_1 < \cdots < x_r = b$ be an arbitrary partition of I. Define $\phi_i = \tau_i^{-1}$. We have,

$$V_I P_T f \leq \sum_{k=1}^{K} V_I P_{\tau_k}(p_k f)(x). \tag{3.7}$$

We will estimate $V_I P_{\tau_k}(p_k f)$.

$$\begin{aligned}
V_I P_{\tau_k}(p_k f)(x) &= \sum_{j=1}^{r} |P_{\tau_k}(p_k f)(x_j) - P_{\tau_k}(p_k f)(x_{j-1})| \\
&= \sum_{j=1}^{r} |(\sum_{i=1}^{q} g_k(\phi_{k,i}(x_j)) f(\phi_{k,i}(x_j)) \chi_{\tau_k(I_i)}(x_j) - \\
&\qquad \sum_{i=1}^{q} g_k(\phi_{k,i}(x_{j-1})) f(\phi_{k,i}(x_{j-1})) \chi_{\tau_k(I_i)}(x_{j-1}))| \qquad (3.8) \\
&\le \sum_{j=1}^{r} \sum_{i=1}^{q} |g_k(\phi_{k,i}(x_j)) f(\phi_{k,i}(x_j)) \chi_{\tau_k(I_i)}(x_j) \\
&\qquad - g_k(\phi_{k,i}(x_{j-1})) f(\phi_{k,i}(x_{j-1})) \chi_{\tau_k(I_i)}(x_{j-1}))|
\end{aligned}$$

We divide the sum on the right hand side into three parts:

(I) the summands for which $\chi_{\tau_k(I_i)}(x_j) = \chi_{\tau_k(I_i)}(x_{j-1}) = 1$,

(II) the summands for which $\chi_{\tau_k(I_i)}(x_j) = 1$ and $\chi_{\tau_k(I_i)}(x_{j-1}) = 0$,

(III) the summands for which $\chi_{\tau_k(I_i)}(x_j) = 0$ and$\chi_{\tau_k(I_i)}(x_{j-1}) = 1$.

First, we will estimate (I).

$$\begin{aligned}
&\sum_{j=1}^{r} \sum_{i=1}^{q} |g_k(\phi_{k,i}(x_j)) f(\phi_{k,i}(x_j)) - g_k(\phi_{k,i}(x_{j-1})) f(\phi_{k,i}(x_{j-1}))| \\
&\le \sum_{i=1}^{q} \sum_{j=1}^{r} |f(\phi_{k,i}(x_j))[g_k(\phi_{k,i}(x_j)) - g_k(\phi_{k,i}(x_{j-1}))]| \\
&\qquad + \sum_{i=1}^{q} \sum_{j=1}^{r} |g_k(\phi_{k,i}(x_{j-1}))[f(\phi_{k,i}(x_j)) - f(\phi_{k,i}(x_{j-1}))]| \qquad (3.9) \\
&\le \sum_{i=1}^{q} \left(\sup_{I_i} |f| V_{I_i} g_k \right) + (\sup_x g_k(x)) \sum_{i=1}^{q} V_{I_i} f \\
&\le \max_{1 \le i \le q} (V_{I_i} g_k) \sum_{i=1}^{q} \left(V_{I_i} |f| + \frac{1}{\lambda(I_i)} \int_{I_i} |f| d\lambda \right) + (\sup_x g_k(x)) \sum_{i=1}^{q} V_{I_i} f \\
&\le \max_{1 \le i \le q} (V_{I_i} g_k)(V_I f + \frac{1}{\delta} \int_I |f| \lambda(dx)) + (\sup_x g_k(x)) V_I f
\end{aligned}$$

We now consider the subsums (II) and (III) together. For $1 \leq k \leq K$ we notice that $\chi_{\tau_k(I_i)}(x_j) = 1$ and $\chi_{\tau_k(I_i)}(x_{j-1}) = 0$ occurs only if $x_j \in \tau_k(I_i)$ and $x_{j-1} \notin \tau_k(I_i)$, i.e., if x_j and x_{j-1} are on opposite sides of an end point of $\tau_k(I_i)$, we can have at most one pair x_j, x_{j-1} like this and another pair $x'_j \notin \tau_k(I_i)$ and $x_{j'-1} \in \tau_k(I_i)$.

$$\begin{aligned}\sum_{i=1}^{q} (|g_k(\phi_{k,i}(x_j))f(\phi_{k,i}(x_j))| + |g_k(\phi_{k,i}(x_{j'-1}))f(\phi_{k,i}(x_{j'-1}))|) \\ \leq \sup_x g_k(x) \sum_{i=1}^{q}(|f(\phi_{k,i}(x_j))| + f(\phi_{k,i}(x_{j'-1}))|)\end{aligned} \tag{3.10}$$

Since $s_i = \phi_{k,i}(x_j)$ and $r_i = \phi_{k,i}(x_{j'-1})$ are both points in I_i, we can write

$$\sum_{i=1}^{q}(|f(s_i)| + |f(r_i)|) \leq \sum_{i=1}^{q}(2|f(v_i)| + |f(v_i) - f(r_i)| + |f(v_i) - f(s_i)|),$$

where $v_i \in I_i$ is such a point that $|f(v_i)| \leq \frac{1}{\lambda(I_i)} \int_{I_i} |f|\lambda(dx)$. Thus, (3.10) is bounded by

$$\sup_x g_k(x) \sum_{i=1}^{q} \left(V_{I_i} f + \frac{2}{\lambda(I_i)} \int_{I_i} |f|\lambda(dx) \right) \leq \sup_x g_k(x) V_I f + \frac{2 \sup_x g_k(x)}{\delta} \int_I |f|\lambda(dx) \tag{3.11}$$

Therefore,

$$\begin{aligned}V_I P_{\tau_k}(p_k f)(x) \leq \left(\max_{1\leq i\leq q}(V_{I_i} g_k) + 2 \sup_x g_k(x) \right) V_I f \\ + \left(2\frac{\sup_x g_k(x)}{\delta} + \frac{1}{\delta} \max_{1\leq i\leq q} V_{I_i} g_k \right) \int_I |f|\lambda(dx)\end{aligned} \tag{3.12}$$

It follows that

$$\begin{aligned}V_I P_T f \leq \left(\sum_{k=1}^{K} \max_{1\leq i\leq q}(V_{I_i} g_k) + 2 \sum_{k=1}^{K} \sup_x g_k(x) \right) V_I f \\ + \left(\frac{2}{\delta} \sum_{k=1}^{K} \sup_x g_k(x) + \frac{1}{\delta} \sum_{k=1}^{K} \max_{1\leq i\leq q} V_{I_i} g_k \right) \int_I |f|\lambda(dx)\end{aligned} \tag{3.13}$$

■

Lemma 3.1.3 *Let* $g_w = \frac{p_w}{|T'_w|}$, *where* T_w *is defined as in Lemma 3.1.1,* $w \in \{1, \ldots, K\}^n$.

Define

$$W_n \equiv \sum_{w \in \{1,\ldots,K\}^n} \max_{J \in \mathcal{P}^{(n)}} V_J g_w \quad \text{and} \quad W_1 \equiv \sum_{k=1}^{K} \max_{J \in \mathcal{P}} V_J g_k,$$

where $\mathcal{P}^{(n)}$ *is the partition for all* T_w. *Then for all* $n \geq 1$

$$W_n \leq n\alpha^{n-1} W_1,$$

where α *is defined as in condition (A).*

Proof. We prove the lemma by induction on n. For $n = 1$ the lemma is true by definition of W_n. Assume that the lemma is true for n; i.e.,

$$W_n \leq n\alpha^{n-1} W_1. \tag{3.14}$$

Let $J \in \mathcal{P}^{(n+1)}$ and $x_0 < x_1 < \cdots < x_l$ be a sequence of points in J. Then

$$\begin{aligned}
\sum_w \sum_{j=0}^{l-1} |g_w(x_{j+1}) - g_w(x_j)| &= \sum_{j=0}^{l-1} \sum_{w \in \{1,\dots,K\}^{n+1}} |g_w(x_{j+1}) - g_w(x_j)| \\
&\leq \sum_{j=0}^{l-1} \sum_{\overline{w} \in \{1,\dots,K\}^n} \sum_{k=1}^{K} |g_{\overline{w}}(\tau_k(x_{j+1}))g_k(x_{j+1}) - g_{\overline{w}}(\tau_k(x_j))g_k(x_j)| \\
&\leq \sum_{j=0}^{l-1} \sum_{\overline{w} \in \{1,\dots,K\}^n} \sum_{k=1}^{K} |g_{\overline{w}}(\tau_k(x_{j+1}))g_k(x_{j+1}) - g_{\overline{w}}(\tau_k(x_{j+1}))g_k(x_j)| \\
&+ \sum_{j=0}^{l-1} \sum_{\overline{w} \in \{1,\dots,K\}^n} \sum_{k=1}^{K} |g_{\overline{w}}(\tau_k(x_{j+1}))g_k(x_j) - g_{\overline{w}}(\tau_k(x_j))g_k(x_j)| \\
&\leq \sum_{j=0}^{l-1} \sum_{k=1}^{K} |g_k(x_{j+1}) - g_k(x_j)| \sum_{\overline{w} \in \{1,\dots,K\}^n} g_{\overline{w}}(\tau_k(x_{j+1})) \\
&+ \sum_{j=0}^{l-1} \sum_{k=1}^{K} g_k(x_j) \sum_{\overline{w} \in \{1,\dots,K\}^n} |g_{\overline{w}}(\tau_k(x_{j+1})) - g_{\overline{w}}(\tau_k(x_j))| \\
&\leq \alpha^n \sum_{j=0}^{l-1} \sum_{k=1}^{K} |g_k(x_{j+1}) - g_k(x_j)| \\
&+ \alpha \sum_{j=0}^{l-1} \sum_{\overline{w} \in \{1,\dots,K\}^n} |g_{\overline{w}}(\tau_k(x_{j+1})) - g_{\overline{w}}(\tau_k(x_j))| \\
&\leq \alpha^n W_1 + \alpha W_n \\
&\leq \alpha W_1 + n\alpha^n W_1 = (n+1)\alpha^n W_1.
\end{aligned} \tag{3.15}$$

We have used condition (A), Lemma 3.1.1 and the fact that $\{\tau(x_j)\}_{j=0}^{l}$ form a partition of some element of $\mathcal{P}^{(n)}$. ∎

Lemma 3.1.4 *Let* $\gamma_n \equiv n\alpha^{n-1}W_1 + 2\alpha^n$. *Then, there exists an* N *such that* $\gamma_N < 1$.

Proof. Since $\alpha < 1$, we can find an N such that $\gamma_N < 1$. ∎

Theorem 3.1.1 *Let T be a random map which satisfies conditions (A) and (B). Then T preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator P_T is quasi-compact on $BV(I)$ and has a number of useful properties [14].*

Proof. Let N satisfy Lemma 3.1.4. Let $\gamma \equiv \gamma_N$. Then, by Lemma 3.1.2, we get

$$||P_T^{n \cdot N} f||_{BV} \leq \gamma^n ||f||_{BV} + R||f||_1, \tag{3.16}$$

and the theorem follows by (2.11) (See [26] and [14] for details). ■

Remark 3.1.1 *It is enough to assume that condition (A) is satisfied for some iterate $T^m, m \geq 1$.*

Remark 3.1.2 *The number of absolutely continuous invariant measures for random maps has been studied in [21]. The proof of [21], which uses graph theoretic methods, goes through analogously in our case. Thus, if T is a random map with position dependent probabilities built from piecewise expanding maps τ_k, $k = 1, \ldots K$, then the number of T-acim is not greater than the minimum of the τ_k-acim.*

We now present two examples of random maps which are expanding on average.

Example 3.1.1 *Let T be a random map which is given by $\{\tau_1, \tau_2; p_1(x), p_2(x)\}$ where*

$$\tau_1(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ x, & \frac{1}{2} < x \leq 1 \end{cases}, \tag{3.17}$$

$$\tau_2(x) = \begin{cases} x + \frac{1}{2}, & 0 \le x \le \frac{1}{2} \\ 2x - 1, & \frac{1}{2} < x \le 1 \end{cases}; \tag{3.18}$$

and

$$p_1(x) = \begin{cases} \frac{2}{3}, & 0 \le x \le \frac{1}{2} \\ \frac{1}{3}, & \frac{1}{2} < x \le 1 \end{cases}, \tag{3.19}$$

$$p_2(x) = \begin{cases} \frac{1}{3}, & 0 \le x \le \frac{1}{2} \\ \frac{2}{3}, & \frac{1}{2} < x \le 1 \end{cases}. \tag{3.20}$$

Then, $\sup_x g_1(x) = \frac{1}{3}$, $\sup_x g_2(x) = \frac{1}{3}$ *and* $\sum_{k=1}^{2} \sup_x g_k(x) = \frac{2}{3} < 1$. *Therefore, T satisfies conditions (A) and (B). Consequently, by Theorem 3.1.1, T preserves an invariant measure absolutely continuous with respect to Lebesgue measure. Notice that τ_1, τ_2 are piecewise linear Markov maps defined on the same Markov partition* $\mathcal{P} : \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$. *For such maps the Frobenius-Perron operator reduces to a matrix [14]. The corresponding matrices are:*

$$P_{\tau_1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \qquad P_{\tau_2} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{3.21}$$

Their invariant densities are $f_{\tau_1} = [0, 2]$ *and* $f_{\tau_2} = [2, 0]$. *The Frobenius-Perron operator of the random map T is given by [23]:*

$$P_T = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}. \tag{3.22}$$

If the invariant density of T *is* $f = [f_1, f_2]$*, normalized by* $f_1 + f_2 = 2$ *and satisfying equation* $fP_T = f$*, then* $f_1 = \frac{2}{3}$ *and* $f_2 = \frac{4}{3}$*.*

Example 3.1.2 *Let* $T = \{\tau_1, \tau_2; p_1, p_2\}$ *be a random map with position dependent probabilities. We give an example of a family of random maps which admits an absolutely continuous invariant measure for a set of parameters of positive measure. The family of the random maps consists of two families, the first family is the tent map family and the second family is the logistic map family. Let* $\kappa \in (0, 1]$*. Let*

$$\tau_1(x) = \begin{cases} 2\kappa x, & 0 \le x \le \frac{1}{2} \\ 2\kappa(1-x), & \frac{1}{2} < x \le 1 \end{cases}, \tag{3.23}$$

and

$$\tau_2(x) = 4\kappa x(1-x); \tag{3.24}$$

with

$$p_1(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2-2x, & \frac{1}{2} < x \le 1 \end{cases}, \tag{3.25}$$

$$p_2(x) = \frac{|4-8x|}{4}. \tag{3.26}$$

Let $T_\kappa = \{\tau_1, \tau_2; p_1(x), p_2(x)\}$*. Using the convention* $\frac{0}{0} = 0$*, we see that* $\sup_x \frac{p_1(x)}{|\tau_1'(x)|} = \frac{1}{2\kappa}$ *and* $\sup_x \frac{p_2(x)}{|\tau_2'(x)|} = \frac{1}{4\kappa}$*. Therefore,* $\sum_{k=1}^{2} \sup_x \frac{p_k(x)}{|\tau_k'(x)|} = \frac{3}{4\kappa} < 1$ *for* $\kappa \in (\frac{3}{4}, 1]$*. Thus, for* $\kappa \in (\frac{3}{4}, 1]$*,* T *admits an absolutely continuous invariant measure. It is worth mentioning that* τ_2 *does not admit an acim for certain values in* $(\frac{3}{4}, 1]$*.*

3.2 Convex and Concave Random Maps

In this section we consider position dependent random maps which are not necessarily expanding on average, i.e., $\sum_{k=1}^{K} \sup_x \frac{p_k(x)}{|\tau_k'(x)|}$ may be greater than 1. However, we assume a convexity (or concavity) condition on the transformations τ_k. The main result of this section proves the existence of absolutely continuous invariant measures for weakly convex and weakly concave random maps with position dependent probabilities.

3.3 Absolutely Continuous Invariant Measures for Weakly Convex Random Maps

Let $([0,1], \mathfrak{B}, \lambda)$ be a measure space, where λ denotes Lebesgue measure on $[0,1]$. Let $\tau_k : [0,1] \to [0,1]$, $k = 1, \ldots, K$ be piecewise continuous, increasing transformations on a common partition $\mathcal{P}$ of $[0,1]$: $\mathcal{P} = \{I_1, \ldots, I_q\}$, $I_i = [a_{i-1}, a_i]$ and $\tau_{k,i} = \tau_k |_{I_i}$, $i = 1, \ldots, q$, $k = 1, \ldots, K$. Define $F_{k,i} = \frac{p_k(\tau_{k,i}^{-1}(x))}{\tau_k'(\tau_{k,i}^{-1}(x))}$. We assume that

(D) $\tau_{k,i}(a_{i-1}) = 0$, $1 \le i \le q$, for all $1 \le k \le K$.

(E) $\sum_{i=1}^{l} F_{k,i}$, $1 \le l \le q$, is decreasing, for all $1 \le k \le K$.

(F) $\sum_{k=1}^{K} \frac{p_k(0)}{\tau_k'(0)} \le \alpha < 1$.

We call a random map T weakly convex if it satisfies the above conditions. Our goal is to prove the existence of absolutely continuous invariant measures for weakly convex random maps.

Remark 3.3.1 *In the previous section, we could consider maps defined on different partitions and redefine them on a common one. Here we assume that the transformations are already defined on a common partition.*

The following lemma is a slight modification of Lemma 2.2 of [12].

Lemma 3.3.1 *Let T be a weakly convex random map and let f be a nonincreasing positive function. Then $P_T f$ is also nonincreasing.*

Proof. Let f be a nonincreasing positive function. Define $\tau_{k,i}^{-1}(x) = x_{k,i}$. Let $x < y$. Since $\tau_{k,i}$ is increasing and $\tau_k(a_{i-1}) = 0$, if $\chi_{\tau_k(I_i)}(x) = 0$, then $\chi_{\tau_k(I_i)}(y) = 0$. Thus, we consider the case when they are both nonzero and we have

$$
\begin{aligned}
(P_T f)(x) - (P_T f)(y) &= \sum_{k=1}^{K} (P_{\tau_k} p_k f)(x) - (P_{\tau_k} p_k f)(y) \\
&= \sum_{k=1}^{K} \sum_{i=1}^{q} (F_{k,i}(x) f(x_{k,i}) - F_{k,i}(y) f(y_{k,i})) \\
&= \sum_{k=1}^{K} \sum_{i=1}^{q} (F_{k,i}(x) f(x_{k,i}) - F_{k,i}(y) f(x_{k,i}) \\
&\qquad + F_{k,i}(y) f(x_{k,i}) - F_{k,i}(y) f(y_{k,i})) \qquad (3.27) \\
&= \sum_{k=1}^{K} \sum_{i=1}^{q} (F_{k,i}(x) - F_{k,i}(y)) f(x_{k,i}) \\
&\qquad + \sum_{k=1}^{K} \sum_{i=1}^{q} (f(x_{k,i}) - f(y_{k,i})) F_{k,i}(y) \\
&\geq \sum_{k=1}^{K} \sum_{i=1}^{q} (F_{k,i}(x) - F_{k,i}(y)) f(x_{k,i})
\end{aligned}
$$

since f is nonincreasing and $F_{k,i} > 0$. Equation (3.27) implies that $P_T f$ is nonincreasing, if $\sum_{i=1}^{q} (F_{k,i}(x) - F_{k,i}(y)) f(x_{k,i}) \geq 0$ for all k. Define the following q-dimensional vectors

$$\begin{aligned}\overrightarrow{F_k} &= \langle F_{k,i}(x) - F_{k,i}(y)\rangle \\ \overrightarrow{f_k} &= \langle f(x_{k,i})\rangle\end{aligned},$$

and

$$\overrightarrow{b_j} = \langle \underbrace{1, \ldots, 1}_{j \text{ times}}, 0, \ldots, 0\rangle;\ j = 1, 2, \ldots, q.$$

Using this notation, we can rewrite $\sum_{i=1}^{q}(F_{k,i}(x) - F_{k,i}(y))f(x_{k,i})$ as

$$\overrightarrow{F_k} \cdot \overrightarrow{f_k}. \tag{3.28}$$

Condition (E) implies

$$\overrightarrow{F_k} \cdot \overrightarrow{b_j} \geq 0;\ j = 1, 2, \ldots, q. \tag{3.29}$$

Moreover,

$$\overrightarrow{f_k} = \sum_{j=1}^{q-1}(f(x_{k_j}) - f(x_{k_{j+1}}))\overrightarrow{b_j} + f(x_{k_q})\overrightarrow{b_q}. \tag{3.30}$$

Then

$$\overrightarrow{F_k} \cdot \overrightarrow{f_k} = \sum_{j=1}^{q-1}(f(x_{k_j}) - f(x_{k_{j+1}}))\overrightarrow{F_k} \cdot \overrightarrow{b_j} + f(x_{k_q})\overrightarrow{F_k} \cdot \overrightarrow{b_q} \geq 0 \tag{3.31}$$

by (3.29) and the fact that f is positive and nonincreasing. ■

Lemma 3.3.2 *If f is positive and nonincreasing, then $f(x) \leq \frac{1}{x}\lambda(f)$, for $x \in [0,1]$, where*

$$\lambda(f) = \int_I f d\lambda.$$

Proof. For any $0 < x \leq 1$, we have

$$\lambda(f) \geq \int_0^x f(t)d\lambda(t) \geq x \cdot f(x).$$

■

Lemma 3.3.3 *Let T be a weakly convex random map. If $f : [0,1] \to \mathbb{R}^+$ is nonincreasing, then*

$$||P_T f||_\infty \leq \alpha ||f||_\infty + \beta ||f||_1, \tag{3.32}$$

where $\beta = \sum_{k=1}^{K} \sum_{i=2}^{q} \frac{p_k(a_{i-1})}{a_{i-1} \cdot \tau_k'(a_{i-1})}$.

Proof. By Lemma 3.3.1, since f is nonincreasing, $P_T f$ is nonincreasing. It follows that

$$\begin{aligned} ||P_T f||_\infty &\leq (P_T f)(0) \\ &= \sum_{k=1}^{K} \frac{p_k(0)}{\tau_k'(0)} f(0) + \sum_{k=1}^{K} \sum_{i=2}^{q} \frac{p_k(\tau_{k,i}^{-1}(0))}{\tau_k'(\tau_{k,i}^{-1}(0))} f(\tau_{k,i}^{-1}(0)) \\ &\leq \alpha f(0) + \sum_{k=1}^{K} \sum_{i=2}^{q} \frac{p_k(a_{i-1})}{\tau_k'(a_{i-1})} f(a_{i-1}) \\ &\leq \alpha f(0) + \sum_{k=1}^{K} \sum_{i=2}^{q} \frac{p_k(a_{i-1})}{\tau_k'(a_{i-1})} \frac{\lambda(f)}{a_{i-1}} \\ &\leq \alpha ||f||_\infty + \beta ||f||_1. \end{aligned} \tag{3.33}$$

■

Theorem 3.3.1 *Let T be a weakly convex random map. Then T admits an absolutely continuous invariant measure, $\mu = f^* \lambda$, and the density f^* is nonincreasing.*

Proof. Let $f \equiv 1$. f is nonincreasing. Then by Lemma 3.3.3, we apply inequality (3.33) iteratively. We obtain

$$||P_T^n f||_\infty \leq \alpha^n ||f||_\infty + \beta(1 + \alpha + \cdots + \alpha^{n-1}||f||_1 \leq 1 + \beta\frac{1}{1-\alpha}.$$

Thus, the sequence $\{P_T^n f\}_{n=1}^\infty$ is uniformly bounded and thus weakly compact in $\mathfrak{L}^1$. By the Yosida-Kakutani Theorem (Chapter 1), the sequence $\frac{1}{n}\sum_{i=1}^{n-1} P_T^i f$ converges in $\mathfrak{L}^1$ to a P_T-invariant function f^*. It is nonincreasing since it is the a.e. limit of a subsequence of nonincreasing functions. ∎

Now, we present an example of a weakly convex random map T which satisfies conditions (D), (E) and (F) and thus preserves an absolutely continuous invariant measure.

Example 3.3.1 *Let T be a random map which is given by $\{\tau_1, \tau_2; p_1(x), p_2(x)\}$ where*

$$\tau_1(x) = \begin{cases} \frac{x}{1-x}, & 0 \leq x < \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \leq x \leq 1 \end{cases}, \tag{3.34}$$

$$\tau_2(x) = \begin{cases} \frac{3x}{2-x}, & 0 \leq x < \frac{1}{2} \\ x - \frac{1}{2}, & \frac{1}{2} \leq x \leq 1 \end{cases}; \tag{3.35}$$

and

$$p_1(x) = \begin{cases} \frac{1+x}{2}, & 0 \leq x < \frac{1}{2} \\ \frac{1}{3}, & \frac{1}{2} \leq x \leq 1 \end{cases}, \tag{3.36}$$

$$p_2(x) = \begin{cases} \frac{1-x}{2}, & 0 \leq x < \frac{1}{2} \\ \frac{2}{3}, & \frac{1}{2} \leq x \leq 1 \end{cases}. \tag{3.37}$$

Observe that,

$$\tau_{k,1}(0) = \tau_{k,2}(\frac{1}{2}) = 0$$

for $k = 1, 2$;

$$F_{1,1} = \frac{1+2x}{2(1+x)}\frac{1}{(x+1)^2},\ F_{1,2} = (\frac{1}{3})(\frac{1}{2})$$

are nonincreasing;

$$F_{2,1} = \frac{3-2x}{2(3+x)}\frac{6}{(3+x)^2},\ F_{2,2} = (\frac{2}{3})(1)$$

are nonincreasing;

$$\frac{p_1(0)}{\tau_1'(0)} + \frac{p_2(0)}{\tau_2'(0)} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} < 1.$$

Therefore, T satisfies conditions (D), (E) and (F) and thus, preserves an absolutely continuous invariant measure.

3.4 Absolutely Continuous Invariant Measures for Weakly Concave Random maps

Let $([0,1], \mathfrak{B}, \lambda)$ be a measure space, where λ denotes Lebesgue measure on $[0,1]$. Let $\tau_k : [0,1] \to [0,1]$, $k = 1, \dots, K$ be piecewise continuous, increasing transformations on a common partition $\mathcal{P}$ of $[0,1]$: $\mathcal{P} = \{I_1, \dots, I_q\}$, $I_i = [a_{i-1}, a_i]$ and $\tau_{k,i} = \tau_k \mid_{I_i}$, $i = 1, \dots, q$, $k = 1, \dots, K$. Define $F_{k,i} = \frac{p_k(\tau_{k,i}^{-1}(x))}{\tau_k'(\tau_{k,i}^{-1}(x))}$. We assume that

(D*) $\tau_{k,i}(a_i) = 1$, $1 \le i \le q$, for all $1 \le k \le K$.

(E*) $\sum_{i=1}^{l} F_{k,i}$, $1 \le l \le q$, is increasing, for all $1 \le k \le K$.

(F*) $\sum_{k=1}^{K} \frac{p_k(1)}{\tau_k'(1)} \le \alpha < 1$.

We call a random map T weakly concave if it satisfies the above conditions. Our goal is to prove the existence of absolutely continuous invariant measures for weakly concave random maps. In this section we only state the results since the proofs are analogous to those in the weakly convex case.

Lemma 3.4.1 *Let T be a weakly concave random map and let f be an increasing positive function. Then $P_T f$ is also increasing.*

Lemma 3.4.2 *If f is positive and increasing, then $f(x) \leq \frac{1}{1-x}\lambda(f)$, for $x \in [0,1]$, where*

$$\lambda(f) = \int_I f d\lambda.$$

Lemma 3.4.3 *Let T be a weakly concave random map. If $f : [0,1] \to \mathbb{R}^+$ is increasing, then*

$$||P_T f||_\infty \leq \alpha ||f||_\infty + \beta ||f||_1, \tag{3.38}$$

where $\beta = \sum_{k=1}^{K} \sum_{i=1}^{q-1} \frac{p_k(a_i)}{(1-a_i)\cdot \tau_k'(a_i)}$.

Theorem 3.4.1 *Let T be a weakly concave random map. Then T admits an absolutely continuous invariant measure, $\mu = f^*\lambda$, and the density f^* is increasing.*

Remark 3.4.1 *The existence of an absolutely continuous invariant measure for piecewise concave maps of $[0,1]$ was studied in [16]. Condition (E^*) is a weaker concavity assumption. Thus, the generalization of the result of [16] is a corollary of Theorem 3.4.1.*

Now, we present an example of a weakly concave random map T which satisfies conditions (D*), (E*) and (F*) and thus, preserves an absolutely continuous invariant measure.

Example 3.4.1 *Let T be a random map which is given by $\{\tau_1, \tau_2; p_1(x), p_2(x)\}$*

$$\tau_1(x) = \begin{cases} -x^2 + 2x + \frac{1}{4}, & 0 \le x \le \frac{1}{2} \\ 2x - 1, & \frac{1}{2} < x \le 1 \end{cases}, \tag{3.39}$$

$$\tau_2(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2x - 1, & \frac{1}{2} < x \le 1 \end{cases}; \tag{3.40}$$

and

$$p_1(x) = \begin{cases} \frac{3}{4}, & 0 \le x \le \frac{1}{2} \\ \frac{1}{3}, & \frac{1}{2} < x \le 1 \end{cases}, \tag{3.41}$$

$$p_2(x) = \begin{cases} \frac{1}{4}, & 0 \le x \le \frac{1}{2} \\ \frac{2}{3}, & \frac{1}{2} \le x \le 1 \end{cases}. \tag{3.42}$$

Observe that,

$$\tau_{k,2}(1) = \tau_{k,1}(\frac{1}{2}) = 1$$

for $k = 1, 2$;

$$F_{1,1} = (\frac{1}{2})\left(\frac{1}{2}(\frac{5}{4} - x)^{-\frac{1}{2}}\right), \ F_{1,2} = (\frac{1}{3})(\frac{1}{2})$$

are increasing;

$$F_{2,1} = (\frac{1}{2})(\frac{1}{2}), \ F_{2,2} = (\frac{2}{3})(1)$$

are increasing;

$$\frac{p_1(1)}{\tau_1'(1)} + \frac{p_2(1)}{\tau_2'(1)} = (\frac{1}{2})(\frac{1}{3}) + (\frac{1}{2})(\frac{2}{3}) = \frac{1}{2} < 1.$$

Therefore, T satisfies conditions (D^*), (E^*) and (F^*) and thus, preserves an absolutely continuous invariant measure.

Chapter 4

Other Existence Results

4.1 Absolutely Continuous Invariant Measures in Higher Dimensions

In this section we prove the existence of absolutely continuous invariant measures for position dependent random maps in higher dimensions [1]. Let S be a bounded region in $\mathbb{R}^n$ and let λ_n be Lebesgue measure on S. Let $\tau_k : S \to S$, $k = 1, \dots, K$ be piecewise, one-to-one, and C^2 non-singular transformations on a partition $\mathcal{P}$ of S, where $\mathcal{P} = \{S_1, \dots, S_q\}$ and $\tau_{k,i} = \tau_k \mid_{S_i}$, $i = 1, \dots, q$, $k = 1, \dots, K$ and each S_i is a bounded closed domain having a piecewise C^2 boundary of finite $(n-1)$-dimensional measure. We will also assume that the probabilities $p_k(x)$ are piecewise C^1 functions on the partition $\mathcal{P}$. Set $\sup_x ||D\tau_{k,i}^{-1}|| := \sigma_k$, where $D\tau_{k,i}^{-1}$ is the derivative matrix of $\tau_{k,i}^{-1}$ and $\sup_x p_k(x) := \pi_k$. We assume:

Condition (C): $\sum_{k=1}^{K} \sigma_k \pi_k \leq \sigma < 1$.

Under this condition, our goal is to prove the existence of an acim for the random map $T = \{\tau_1, \ldots, \tau_K; p_1, \ldots, p_K\}$. The main tool of this section is multidimensional variation defined using derivatives in the distributional sense [20]:

$$V(f) = \int_{\mathbb{R}^n} ||Df|| = \sup\{\int_{\mathbb{R}^n} f \mathrm{div}(g) d\lambda_n : g = (g_1, \ldots, g_n) \in C_0^1(\mathbb{R}^n, \mathbb{R}^n)\},$$

where $f \in \mathfrak{L}_1(\mathbb{R}^n)$ has bounded support, Df denotes the gradient of f in the distributional sense, and $C_0^1(\mathbb{R}^n, \mathbb{R}^n)$ is the space of continuously differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}^n$ having a compact support. We will use the following property of variation which is derived from [20], Remark 2.14: If $f = 0$ outside a closed domain A whose boundary is Lipschitz continuous, $f_{|A}$ is continuous, $f_{|\mathrm{int}(A)}$ is C^1, then

$$V(f) = \int_{\mathrm{int}(A)} ||Df|| d\lambda_n + \int_{\partial A} |f| d\lambda_{n-1},$$

where λ_{n-1} is the $(n-1)$-dimensional measure on the boundary of A. In this section we shall consider the Banach space ([20], Remark 1.12),

$$BV(S) = \{f \in \mathfrak{L}_1(S) : V(f) < +\infty\},$$

with the norm $||f||_{BV} = V(f) + ||f||_1$. We adapt the following two lemmas from [24]. The proofs of the following two lemmas are exactly the same as in [24].

Lemma 4.1.1 *Consider $S_i \in \mathcal{P}$. Let x be a point in ∂S_i and $y = \tau_k(x)$ a point in $\partial(\tau_k(S_i))$. Let J_k be the Jacobian of $\tau_{k|S_i}$ at x and J_k^0 be the Jacobian of $\tau_{k|\partial S_i}$ at x. Then*

$$\frac{J_k^0}{J_k} \leq \sigma_k.$$

Let Z denote the set of singular points of ∂S. Let us construct at any $x \in Z$ the largest cone having a vertex at x and which lies completely in S. Let $\theta(x)$ denote the angle subtended at the vertex of this cone. Then define

$$\beta(S) = \min_{x \in Z} \theta(x).$$

Since the faces of ∂S meet at angles away from 0, $\beta(S) > 0$. Let $\alpha(S) = \pi/2 + \beta(S)$ and

$$a(S) = |cos\alpha(S)|.$$

Now we will construct a C^1 field of segments L_y, $y \in \partial S$, every L_y being a central ray of a regular cone contained in S, with angle subtended at the vertex y greater than or equal to $\beta(S)$.

We start at points $y \in Z$, where the minimal angle $\beta(S)$ is attained, defining L_y to be central rays of the largest regular cones contained in S. Then we extend this field of segments to the C^1 field we want, making L_y short enough to avoid overlapping. Let $\delta(y)$ be the length of L_y, $y \in \partial S$. By the compactness of ∂S we have

$$\delta(S) = \inf_{y \in \partial S} \delta(y) > 0.$$

Now, we shorten L_y of our field, making them of length $\delta(S)$.

Lemma 4.1.2 *If S is some closed domain with piecewise C^2 boundary of finite $(n-1)$-dimensional measure, whose smooth surfaces meet at angles bounded away from zero, and f is a C^1 function on S, then*

$$\int_{\partial S} f(y) d\lambda_{n-1}(y) \leq \frac{1}{a(S)} \left(\frac{1}{\delta(S)} \int_S f d\lambda_n + V(f, int(S)) \right).$$

We prove the following :

Theorem 4.1.1 *If T is a random map which satisfies condition (C), then*

$$V(P_T f) \leq \sigma(1+1/a)V(f) + (M + \frac{\sigma}{a\delta})||f||_1,$$

where $a = \min\{a(S_i) : i = 1, \ldots q\} > 0$, $\delta = \min\{\delta S_i, : i = 1, \ldots q\} > 0$, $M = \sum_{k=1}^{K} M_k$ *and* $M_k = \sup_x (Dp_k(x) - \frac{DJ_k}{J_k} p_k(x))$.

Proof. We have $V(P_T f) \leq \sum_{k=1}^{K} V(P_{\tau_k} p_k f)$. We first estimate $V(P_{\tau_k} p_k f)$. Let $G_{k,i} = \frac{f(\tau_{k,i}^{-1})}{J_k(\tau_{k,i}^{-1})}$, $i = 1, \ldots, q$. Then,

$$\begin{aligned} \int_{\mathbb{R}^n} ||DP_{\tau_k} p_k f|| d\lambda_n &\leq \sum_{i=1}^{q} \int_{\mathbb{R}^n} ||D(G_{k,i} p_k \chi_{R_i})|| d\lambda_n \\ &\leq \sum_{i=1}^{q} \left(\int_{\mathbb{R}^n} ||D(G_{k,i} p_k)\chi_{R_i}|| d\lambda_n + \int_{\mathbb{R}^n} ||G_{k,i} p_k (D\chi_{R_i})|| d\lambda_n \right). \end{aligned} \tag{4.1}$$

Now, for the first integral we have,

$$\begin{aligned} &\int_{\mathbb{R}^n} ||D(G_{k,i} p_k)\chi_{R_i}|| d\lambda_n = \int_{R_i} ||D(G_{k,i} p_k)|| d\lambda_n \\ &\leq \int_{R_i} ||D(f(\tau_{k,i}^{-1})) \frac{p_k(\tau_{k,i}^{-1})}{J(\tau_{k,i}^{-1})}|| d\lambda_n + \int_{R_i} ||f(\tau_{k,i}^{-1}) D\left(\frac{p_k(\tau_{k,i}^{-1})}{J(\tau_{k,i}^{-1})}\right)|| d\lambda_n \\ &\leq \int_{R_i} ||Df(\tau_{k,i}^{-1})|| ||D\tau_{k,i}^{-1}|| \frac{p_k(\tau_{k,i}^{-1})}{J(\tau_{k,i}^{-1})} d\lambda_n + \int_{R_i} ||f(\tau_{k,i}^{-1})|| \frac{M_k}{J(\tau_{k,i}^{-1})} d\lambda_n \\ &\leq \sigma_k \pi_k \int_{S_i} ||Df|| d\lambda_n + M_k \int_{S_i} ||f|| d\lambda_n. \end{aligned} \tag{4.2}$$

For the second integral, we have

$$\begin{aligned} \int_{\mathbb{R}^n} ||G_{k,i} p_k (D\chi_{R_i})|| d\lambda_n &= \int_{\partial R_i} |f(\tau_{k,i}^{-1}))| \frac{p_k(\tau_{k,i}^{-1})}{J(\tau_{k,i}^{-1})} d\lambda_{n-1} \\ &= \int_{\partial S_i} |f| p_k \frac{J_k^0}{J_k} d\lambda_{n-1}. \end{aligned} \tag{4.3}$$

By Lemma 4.1.1, $\frac{J_k^0}{J_k} \le \sigma_k$. Using Lemma 4.1.2, we obtain,

$$\begin{aligned}\int_{\mathbb{R}^n} ||G_{k,i} p_k(D\chi_{R_i})|| d\lambda_n &\le \sigma_k \pi_k \int_{\partial S_i} |f| d\lambda_{n-1} \\ &\le \frac{\sigma_k \pi_k}{a} V(f, S_i) + \frac{\sigma_k \pi_k}{a\delta} \int_{S_i} |f| d\lambda n.\end{aligned} \tag{4.4}$$

Using (4.2) and (4.4) and summing over i, we get

$$V(P_{\tau_k} p_k f) \le \sigma_k \pi_k (1 + 1/a) V(f) + (M_k + \frac{\sigma_k \pi_k}{a\delta}) ||f||_1.$$

Summing over k, we get

$$V(P_T f) \le \sigma(1 + 1/a) V(f) + (M + \frac{\sigma}{a\delta}) ||f||_1.$$

■

Theorem 4.1.2 *Let T be a random map which satisfies condition (C). If $\sigma(1+1/a) < 1$, then T preserves a measure which is absolutely continuous with respect to Lebesgue measure.*

Proof. The proof of the theorem follows by the standard technique [14]. ■ The operator P_T is quasi-compact on $BV(S)$ and has a number of useful properties [14]. Now, We present an example of a random map which satisfies condition (C) of Theorem 4.1.1. It preserves an absolutely continuous invariant measure by satisfying the condition of Theorem 4.1.2.

Example 4.1.1 *Let T be a random map which is given by $\{\tau_1, \tau_2; p_1(x), p_2(x)\}$ where*

$\tau_1, \tau_2 : I^2 \longrightarrow I^2$ *defined by:*

$$\tau_1(x_1,x_2) = \begin{cases} (3x_1, 2x_2), & (x_1,x_2) \in S_1 = \{0 \leq x_1, x_2 \leq \frac{1}{3}\} \\ (3x_1 - 1, 2x_2), & (x_1,x_2) \in S_2 = \{\frac{1}{3} < x_1 \leq \frac{2}{3}; 0 \leq x_2 \leq \frac{1}{3}\} \\ (3x_1 - 2, 2x_2), & (x_1,x_2) \in S_3 = \{\frac{2}{3} < x_1 \leq 1; 0 \leq x_2 \leq \frac{1}{3}\} \\ (3x_1, 3x_2 - 1), & (x_1,x_2) \in S_4 = \{0 < x_1 \leq \frac{1}{3}; \frac{1}{3} < x_2 \leq \frac{2}{3}\} \\ (3x_1 - 1, 3x_2 - 1), & (x_1,x_2) \in S_5 = \{\frac{1}{3} < x_1, x_2 \leq \frac{2}{3}\} \\ (3x_1 - 2, 3x_2 - 1), & (x_1,x_2) \in S_6 = \{\frac{2}{3} < x_1 \leq 1; \frac{1}{3} < x_2 \leq \frac{2}{3}\} \\ (3x_1, 3x_2 - 2), & (x_1,x_2) \in S_7 = \{0 \leq x_1 \leq \frac{1}{3}; \frac{2}{3} < x_2 \leq 1\} \\ (3x_1 - 1, 3x_2 - 2), & (x_1,x_2) \in S_8 = \{\frac{1}{3} < x_1 \leq \frac{2}{3}; \frac{2}{3} < x_2 \leq 1\} \\ (3x_1 - 2, 3x_2 - 2), & (x_1,x_2) \in S_9 = \{\frac{2}{3} < x_1 \leq 1; \frac{2}{3} < x_2 \leq 1\} \end{cases},$$

$$\tau_2(x_1,x_2) = \begin{cases} (3x_1, 3x_2), & (x_1,x_2) \in S_1 \\ (2 - 3x_1, 3x_2), & (x_1,x_2) \in S_2 \\ (3x_1 - 2, 3x_2), & (x_1,x_2) \in S_3 \\ (3x_1, 3x_2 - 1), & (x_1,x_2) \in S_4 \\ (2 - 3x_1, 3x_2 - 1), & (x_1,x_2) \in S_5 \\ (3x_1 - 2, 3x_2 - 1), & (x_1,x_2) \in S_6 \\ (3x_1, 3x_2 - 2), & (x_1,x_2) \in S_7 \\ (2 - 3x_1, 3x_2 - 2), & (x_1,x_2) \in S_8 \\ (3x_1 - 2, 3x_2 - 2), & (x_1,x_2) \in S_9 \end{cases},$$

and

$$p_1(x) = \begin{cases} 0.215, & (x_1, x_2) \in S_1 \\ 0.216, & (x_1, x_2) \in S_2 \\ 0.216, & (x_1, x_2) \in S_3 \\ 0.216, & (x_1, x_2) \in S_4 \\ 0.215, & (x_1, x_2) \in S_5 \\ 0.216, & (x_1, x_2) \in S_6 \\ 0.216, & (x_1, x_2) \in S_7 \\ 0.216, & (x_1, x_2) \in S_8 \\ 0.215, & (x_1, x_2) \in S_9 \end{cases}, \quad p_2(x) = \begin{cases} 0.785, & (x_1, x_2) \in S_1 \\ 0.784, & (x_1, x_2) \in S_2 \\ 0.784, & (x_1, x_2) \in S_3 \\ 0.784, & (x_1, x_2) \in S_4 \\ 0.785, & (x_1, x_2) \in S_5 \\ 0.784, & (x_1, x_2) \in S_6 \\ 0.784, & (x_1, x_2) \in S_7 \\ 0.784, & (x_1, x_2) \in S_8 \\ 0.785, & (x_1, x_2) \in S_9 \end{cases}$$

The derivative matrix of $(\tau_{1,i})^{-1}$, *is*

$$\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \quad or \quad \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

and the derivative matrix of $(\tau_{2,i})^{-1}$, *is*

$$\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \quad or \quad \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}.$$

Therefore, the Euclidean matrix norm, $||D(\tau_{1,i})^{-1}||$ *is* $\frac{\sqrt{2}}{3}$, *or* $\frac{\sqrt{13}}{6}$ *and the Euclidean matrix norm,* $||D(\tau_{2,i})^{-1}||$ *is* $\frac{\sqrt{2}}{3}$. *Then*

$$\sigma_1\pi_1 + \sigma_2\pi_2 = 0.216\frac{\sqrt{13}}{6} + 0.785\frac{\sqrt{2}}{3}.$$

For this partition $\mathcal{P}$, *we have* $a = 1$, *which implies*

$$\sigma(1 + 1/a) = 2(0.216\frac{\sqrt{13}}{6} + 0.785\frac{\sqrt{2}}{3}) \approx 0.9998 < 1.$$

Therefore, by Theorem 4.1.2, the random map T admits an absolutely continuous invariant measure. Notice that τ_1, τ_2 are piecewise linear Markov maps defined on the same Markov partition $\mathcal{P} = \{S_1, S_2, \ldots, S_9\}$. For such maps the Frobenius-Perron operator reduces to a matrix and the invariant density is constant on the elements of the partition. The Frobenius-Perron operator of the random map T is represented by the following matrix

$$M = \Pi_1 M_1 + \Pi_2 M_2,$$

where M_1, M_2 are the matrices of P_{τ_1} and P_{τ_2} respectively, and Π_1, Π_2 are the diagonal *matrices of* $p_1(x)$ *and* $p_2(x)$ *respectively. Then,* M *is given by*

$$M = p_1\mathbf{Id}_9 \times \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix}$$

$$+ p_2\mathbf{Id}_9 \times \begin{pmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix} = \begin{pmatrix} a & a & a & a & a & a & b & b & b \\ c & c & c & c & c & c & d & d & d \\ c & c & c & c & c & c & d & d & d \\ e & e & e & e & e & e & e & e & e \\ e & e & e & e & e & e & e & e & e \\ e & e & e & e & e & e & e & e & e \\ e & e & e & e & e & e & e & e & e \\ e & e & e & e & e & e & e & e & e \\ e & e & e & e & e & e & e & e & e \end{pmatrix},$$

where $p_1 = (0.215, 0.216, 0.216, 0.216, 0.215, 0.216, 0.216, 0.216, 0.215)$,

$p_2 = (0.785, 0.784, 0.784, 0.784, 0.785, 0.784, 0.784, 0.784, 0.785)$, $\mathbf{Id}_9$ *is* 9×9 *identity* *matrix and*

$$a = 0.12306$$

$$b = 0.087222$$

$$c = 0.12311$$

$$d = 0.087111$$

$$e = 0.11111.$$

The invariant density of T *is*

$$f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9), \quad f_i = f_{|S_i}, \quad i = 1, 2, \ldots, 9,$$

normalized by

$$f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8 + f_9 = 9,$$

and satisfying equation $fM = f$. *Then,* $f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = \frac{9}{6.29739}$ *and* $f_7 = f_8 = f_9 = \frac{0.29739}{3} f_1$.

4.2 Position Dependent Random Maps On The Real Line

In [27], Góra and Jabłoński proved a general result on the existence of absolutely continuous invariant measures for piecewise nonsingular transformations on the real line. In this section, we deal with piecewise monotonic nonsingular transformations $\tau_1, \tau_2, \ldots, \tau_K$, where τ_k is a transformation from $\mathbb{A}$ to itself, $\mathbb{A}$ is an interval of $\mathbb{R}$

not necessarily bounded, with countably many branches, $\mathbb{A} \subset \mathbb{R}$ not necessarily bounded, and position dependent probabilities $p_k(x)$, $k = 1, \ldots, K$, satisfy $p_k(x) \geq 0$, $\sum_{k=1}^{K} p_k(x) = 1$. Sufficient conditions for the existence of absolutely continuous invariant measures for position dependent random maps on the real line is the main result of this section [3].

Notation and Conditions: Let m denote the Lebesgue measure on $\mathbb{R}$. Let us recall the definition of $BV(\mathbb{A})$, which will be used below:

$$BV(\mathbb{A}) = \{f \in \mathfrak{L}^1(\mathbb{A}) : \exists g \in \mathfrak{L}^1(\mathbb{A}), g = f \text{ a.e.}, V_{\mathbb{A}} g < \infty\}.$$

Let $T = \{\tau_1, \ldots, \tau_K; p_1(x), \ldots, p_K(x)\}$. For $k = 1, \ldots, K$ we assume:

(I) $\tau_k : \cup_{i=1}^{\infty} I_i^k \to \mathbb{A}$, where I_i^k, $i = 1, \ldots,$ are open intervals, $I_i^k \subset \mathbb{A}$ and $\mathbb{A}$ is an interval not necessarily bounded,

(II) $I_i^k \cap I_j^k = \emptyset$ for $i \neq j$,

(III) $\sup_{i \geq 1} m(I_i^k) = L < \infty$, and $m(\mathbb{A} \setminus \cup_{i=1}^{\infty} I_i^k) = 0$,

(IV) $\tau_{k,i} = \tau_k|_{I_i}$ is of class C^1, $i = 1, 2, \ldots$.

(V) $\sup_{x \in \mathbb{A}} \frac{p_k(x)}{|\tau_k'(x)|} \leq \lambda_k$,

(VI) There exists constants $M_k, \delta_k, \gamma_1^k, \gamma_2^k, \gamma_3^k$ such that

$$\sum_{k=1}^{K} (2\lambda_k + \gamma_1^k + \gamma_2^k + \gamma_3^k) \leq \alpha < 1,$$

and

(a) for any $i \in J_1^k$, where $J_1^k = \{i \in \mathbb{N} : V_{I_i^k}\left(\frac{p_k}{|\tau_k'|}\right) \leq M_k m(I_i^k)\}$, and for any points

$x, x' \in I_i^k$ with $|x - x'| < \delta_k$, we have

$$V_{[x,x']} \frac{p_k}{|\tau_k'|} < \gamma_1^k;$$

(b) $2\sum_{i \in J_2^k} V_{I_i^k}\left(\frac{p_k}{|\tau_k'|}\right) < \gamma_2^k$, where $J_2^k = \mathbb{N} \setminus J_1^k$,

(c) $\sum_{i \in J_3^k} \left(|\psi_{k,i}'(\tau_k(a_i))| p_k(a_i) + |\psi_{k,i}'(\tau_k(b_i))| p_k(b_i)\right) < \gamma_3^k$, where

$$J_3^k = \{i \in \mathbb{N} : |\psi_{k,i}'(\tau_k(a_i))| p_k(a_i) > M_k m(I_i^k) \text{ or } |\psi_{k,i}'(\tau_k(b_i))| p_k(b_i) > M_k m(I_i^k)\},$$

$\psi_{k,i} = \tau_{k,i}^{-1}$ and $I_i^k = (a_i, b_i)$,

(VII) There exists $W_1^k, W_2^k \subset \mathbb{N}$, $W_1^k \cap W_2^k = \emptyset$ and $W_1^k \cup W_2^k = \mathbb{N}$, such that the functions $\sup_{l \in W_1^k} \frac{|\psi_{k,l}'(x)| p_k(\psi_{k,l}(x))}{m(I_l^k)}$ and $\sum_{l \in W_2^k} |\psi_{k,l}'(x)| p_k(\psi_{k,l}(x))$ are integrable.

Remark 4.2.1 *The above assumptions allow the transformation τ_k to have critical points; i.e., $\tau_k'(c) = 0$. This does not violate (V) since we can compensate by taking $p_k(c) = 0$. Similarly, we do not assume that $\psi_{k,i}'(a_i) = 0$ if $\tau_k(a_i) = \pm\infty$. This does not violate (VI) since we can compensate by taking $p_k(a_i) = 0$.*

Absolutely Continuous Invariant Measures on $\mathbb{R}$:

Lemma 4.2.1 *Let T be a random map satisfying conditions (I)-(VII). Then there exist constants $0 < \alpha < 1$ and $C > 0$ such that*

$$V_{\mathbb{A}} P_T f \leq \alpha V_{\mathbb{A}} f + C||f||_1.$$

Proof. Let $f \in BV(\mathbb{A})$. We have

$$V_{\mathbb{A}} P_T f \leq \sum_{k=1}^{K} V_{\mathbb{A}} P_{\tau_k} p_k f. \tag{4.5}$$

First, we estimate $V_{\mathbb{A}} P_{\tau_k} p_k f$ for $k \in \{1, \ldots, K\}$ which is given by:

$$\begin{aligned} V_{\mathbb{A}} P_{\tau_k} p_k f &= V_{\mathbb{A}} \sum_{i=1}^{\infty} f(\psi_{k,i}(x)) p_k(\psi_{k,i}(x)) |\psi'_{k,i}(x)| \chi_{\tau_k(I_i^k)}(x) \\ &\leq \sum_{i=1}^{\infty} V_{\tau_k(I_i^k)} (f \circ \psi_{k,i}(x)) p_k(\psi_{k,i}(x)) |(\psi'_{k,i}(x))| \\ &\quad + \sum_{i=1}^{\infty} \left(|f(a_i)| p_k(a_i) |\psi'_{k,i}(\tau_k(a_i)) + |f(b_i)| p_k(b_i) |\psi'_{k,i}(\tau_k(b_i)) \right) \\ &= S_1 + S_2, \end{aligned} \tag{4.6}$$

where $I_i^k = (a_i, b_i)$. Since $V_{\tau(I_i^k)} \left(p_k(\psi_{k,i}) |(\psi'_{k,i})| \right) = V_{I_i^k}(\frac{p_k}{|\tau'_k|}) < \infty$, we can assume that either $\psi'_{k,i}(\tau_k(a_i)) = 0$ or $p_k(\psi(a_i)) = 0$ (either $\psi'_{k,i}(\tau_k(b_i)) = 0$ or $p_k(\psi(b_i)) = 0$) if $\tau_k(a_i) = \pm\infty$ ($\tau_k(b_i) = \pm\infty$). For every $h : [a, b] \to \mathbb{R}$ with $V_{[a,b]} h < \infty$ there is $c \in [a, b]$ such that

$$h(c) \leq \frac{1}{m([a, b])} \int_a^b |h| dm.$$

For such c, $|h(a)| \leq |h(c)| + V_{[a,c]} h$ and $|h(b)| \leq |h(c)| + V_{[c,b]} h$. Therefore, for each $i \geq 1$ and appropriate $c_i \in [a_i, b_i]$ we have

$$\begin{aligned} &|f(a_i)| p_k(a_i) |\psi'_{k,i}(\tau_k(a_i))| + |f(b_i)| p_k(b_i) |\psi'_{k,i}(\tau_k(b_i))| \\ &\quad \leq \left(|f(c_i)| + V_{[a_i, c_i]} f \right) p_k(a_i) |\psi'_{k,i}(\tau_k(a_i))| + \left(|f(c_i)| + V_{[c_i, b_i]} f \right) p_k(b_i) |\psi'_{k,i}(\tau_k(b_i))| \\ &\quad \leq \frac{p_k(a_i) |\psi'_{k,i}(\tau_k(a_i))|}{m([a_i, b_i])} \int_{a_i}^{b_i} |f| dm + \frac{p_k(b_i) |\psi'_{k,i}(\tau_k(b_i))|}{m([a_i, b_i])} \int_{a_i}^{b_i} |f| dm + \lambda_k V_{[a_i, b_i]} f. \end{aligned} \tag{4.7}$$

Now, we estimate the second summand S_2 of (4.6). Let $J_4^k = \mathbb{N} \setminus J_3^k$. By (4.7) and

(VI)(c) we have

$$
\begin{aligned}
S_2 &= \sum_{i=1}^{\infty} \left(|f(a_i)|p_k(a_i)|\psi'_{k,i}(\tau(b_i)) + |f(b_i)|p_k(b_i)|\psi'_{k,i}(\tau(b_i))\right) \\
&\leq \sum_{i\in J_4^k} \left(\frac{p_k(a_i)|\psi'_{k,i}(\tau_k(a_i))|}{m([a_i,b_i])} \int_{a_i}^{b_i} |f|dm + \frac{p_k(b_i)|\psi'_{k,i}(\tau_k(b_i))|}{m([a_i,b_i])} \int_{a_i}^{b_i} |f|dm + \lambda_k V_{I_i} f \right) \\
&\quad + \sum_{i\in J_3^k} \left(\sup_{\mathbb{A}} |f| \left(p_k(a_i)|\psi'_{k,i}(\tau_k(a_i))| + p_k(b_i)|\psi'_{k,i}(\tau_k(b_i))|\right) \right) \\
&\leq 2M_k ||f||_1 + \lambda_k V_{\mathbb{A}} f + \left(\frac{||f||_1}{m(\mathbb{A})} + V_{\mathbb{A}} f \right) \gamma_3^k \\
&= \left(\lambda_k + \gamma_3^k\right) V_{\mathbb{A}} f + \left(2M_k + \frac{\gamma_3^k}{m(\mathbb{A})} \right) ||f||_1.
\end{aligned}
\tag{4.8}
$$

In (4.8), we have used the inequality $\sup_{\mathbb{A}} |f| \leq \frac{||f||_1}{m(\mathbb{A})} + V_{\mathbb{A}} f$, with $\frac{||f||_1}{m(\mathbb{A})} = 0$ if $m(\mathbb{A}) = \infty$, which holds for functions in $BV(\mathbb{A})$. Now, we estimate the first summand S_1 in

(4.6). Let δ_k be as in (VI) and $y_r = \tau_{k,i}(x_r)$, where $x_r \in I_i^k$. If $m(I_i^k) \leq \delta_k$, we have

$$
\begin{aligned}
&V_{\tau_k(I_i^k)}(f \circ \psi_{k,i})p_k(\psi_{k,i})|\psi'_{k,i}| \\
&= \overline{\sup_{\tau_k(I_i^k)}} \sum_{r=1}^{q} \big|(f \circ \psi_{k,i})(y_r)p_k(\psi_{k,i}(y_r))|(\psi'_{k,i}(y_r))| \\
&\qquad\qquad - (f \circ \psi_{k,i})(y_{r-1})p_k(\psi_{k,i}(y_{r-1}))|(\psi'_{k,i}(y_{r-1}))|\big| \\
&\leq \overline{\sup_{\tau_k(I_i^k)}} \sum_{r=1}^{q} \big|(f \circ \psi_{k,i})(y_r)p_k(\psi_{k,i}(y_r))|(\psi'_{k,i}(y_r))| \\
&\qquad\qquad - (f \circ \psi_{k,i})(y_{r-1})p_k(\psi_{k,i}(y_r))|(\psi'_{k,i}(y_r))|\big| \qquad (4.9) \\
&\quad + \overline{\sup_{\tau_k(I_i^k)}} \sum_{r=1}^{q} \big|(f \circ \psi_{k,i})(y_{r-1})p_k(\psi_{k,i}(y_r))|(\psi'_{k,i}(y_r))| \\
&\qquad\qquad - (f \circ \psi_{k,i})(y_{r-1})p_k(\psi_{k,i}(y_{r-1}))|(\psi'_{k,i}(y_{r-1}))|\big| \\
&\leq \lambda_k V_{I_i^k} f + \overline{\sup_{I_i^k}} \sum_{r=1}^{q} |f(x_{r-1})| \left| \frac{p_k(x_r)}{|\tau'_k(x_r)|} - \frac{p_k(x_{r-1})}{|\tau'_k(x_{r-1})|} \right| \\
&\leq \lambda_k V_{I_i^k} f + (\sup_{I_i^k} |f|) \overline{\sup_{I_i^k}} \sum_{r=1}^{q} \left| \frac{p_k(x_r)}{|\tau'_k(x_r)|} - \frac{p_k(x_{r-1})}{|\tau'_k(x_{r-1})|} \right|,
\end{aligned}
$$

where $\overline{\sup}_{\tau_k(I_i^k)}$ and $\overline{\sup}_{I_i^k}$ indicate the suprema over all finite partitions of $\tau_k(I_i^k)$ and I_i^k respectively. To estimate the second summand on the right hand side of (4.9), we consider the cases of $i \in J_1^k$ and $i \in J_2^k$ separately. For $i \in J_1^k$ we estimate (4.9) as follows:

$$
\begin{aligned}
V_{\tau_k(I_i^k)}(f \circ \psi_{k,i})p_k(\psi_{k,i})|\psi'_{k,i}| &\leq \lambda_k V_{I_i^k} f + (\inf_{I_i^k} |f| + V_{I_i^k} f) V_{I_i^k} \frac{p_k}{|\tau'_k|} \\
&\leq \lambda_k V_{I_i^k} f + \left(\frac{1}{m(I_i^k)} \int_{I_i^k} |f| \right) V_{I_i^k} \frac{p_k}{|\tau'_k|} + (V_{I_i^k} f) V_{I_i^k} \frac{p_k}{|\tau'_k|} \\
&\leq \left(\lambda_k + \gamma_1^k\right) V_{I_i^k} f + M_k \int_{I_i^k} |f|.
\end{aligned}
\tag{4.10}
$$

For $i \in J_2^k$ we estimate (4.9) as follows:

$$V_{\tau(I_i^k)}(f \circ \psi_{k,i})p_k(\psi_{k,i})|\psi_{k,i}'| \le \lambda_k V_{I_i^k} f + \sup_{\mathbb{A}} |f| V_{I_i^k} \frac{p_k}{|\tau_k'|}. \tag{4.11}$$

This finishes the estimate on S_1 when $m(I_i^k) \le \delta_k$. If $m(I_i^k) > \delta_k$, then there is a partition $a_i = c_0 < c_1 < \cdots < c_{n_i} = b_i$ such that

$$\frac{\delta_k}{2} \le |c_j - c_{j-1}| < \delta_k \quad \text{for } j = 1, \dots, n_i. \tag{4.12}$$

We have

$$V_{\tau_k(I_i^k)}(f \circ \psi_{k,i})p_k(\psi_{k,i})|\psi_{k,i}'| = \sum_{j=1}^{n_i} V_{\tau_k(c_{j-1},c_j)}(f \circ \psi_{k,i})p_k(\psi_{k,i})|\psi_{k,i}'|. \tag{4.13}$$

Using the same estimate of (4.9), we get

$$V_{\tau_k(c_{j-1},c_j)}(f \circ \psi_{k,i})p_k(\psi_{k,i})|\psi_{k,i}'| \le \lambda_k V_{[c_{j-1},c_j]} f + (\sup_{[c_{j-1},c_j]} |f|) V_{[c_{j-1},c_j]} \frac{p_k}{|\tau_k'|} = R_{ij}. \tag{4.14}$$

We consider the cases $i \in J_1^k$ and $i \in J_2^k$ separately. If $i \in J_1^k$, then using the estimates of (4.11) and the inequality of (4.12) we get

$$\begin{aligned} R_{ij} &\le \lambda_k V_{[c_{j-1},c_j]} f + \left(\frac{1}{m([c_{j-1},c_j]} \int_{c_{j-1}}^{c_j} |f| \right) V_{[c_{j-1},c_j]} \frac{p_k}{|\tau_k'|} + V_{[c_{j-1},c_j]} f V_{[c_{j-1},c_j]} \frac{p_k}{|\tau_k'|} \\ &\le \lambda_k V_{[c_{j-1},c_j]} f + \frac{2V_{I_i^k}}{\delta_k} \int_{c_{j-1}}^{c_j} |f| + \gamma_1^k V_{[c_{j-1},c_j]} f \\ &\le (\lambda_k + \gamma_1^k) V_{[c_{j-1},c_j]} f + \frac{2M_k L}{\delta_k} \int_{c_{j-1}}^{c_j} |f|. \end{aligned} \tag{4.15}$$

Summing over all $j's$ we get

$$V_{\tau_k(I_i^k)}(f \circ \psi_{k,i})p_k(\psi_{k,i})|\psi_{k,i}'| \le (\lambda_k + \gamma_1^k) V_{\tau_k(I_i^k)} f + \frac{2M_k L}{\delta_k} \int_{I_i^k} |f|. \tag{4.16}$$

If $i \in J_2^k$, then we have

$$R_{ij} \leq \lambda_k V_{[c_{j-1},c_j]} f + \sup_{\mathbb{A}} V_{[c_{j-1},c_j]} \frac{p_k}{|\tau_k'|}. \tag{4.17}$$

Summing over all $j's$ we get

$$V_{\tau_k(I_i^k)}(f \circ \psi_{k,i}) p_k(\psi_{k,i}) |\psi_{k,i}'| \leq \lambda_k V_{I_i^k} f + \sup_{\mathbb{A}} V_{I_i^k} \frac{p_k}{|\tau_k'|}. \tag{4.18}$$

Using the estimates of equations (4.18),(4.15), (4.10), (4.11) and using assumption (VI)(b) we obtain the following estimate on S_1:

$$\begin{aligned} S_1 &\leq \left(\lambda_k + \gamma_1^k\right) V_{\mathbb{A}} f + \gamma_2^k \sup_{\mathbb{A}} |f| + \max\left(\frac{2M_k L}{\delta_k}, M_k\right) ||f||_1 \\ &\leq \left(\lambda_k + \gamma_1^k + \gamma_2^k\right) V_{\mathbb{A}} f + \left(\frac{\gamma_2^k}{m(\mathbb{A})} + \max\left(\frac{2M_k L}{\delta_k}, M_k\right)\right) ||f||_1. \end{aligned} \tag{4.19}$$

Now, using (4.19) and (4.8) we obtain

$$\begin{aligned} V_{\mathbb{A}} P_{\tau_k p_k} f \leq& \left(2\lambda_k + \gamma_1^k + \gamma_2^k + \gamma_3^k\right) V_{\mathbb{A}} f \\ &+ \left(2M_k + \frac{\gamma_3^k}{m(\mathbb{A})} + \frac{\gamma_2^k}{m(\mathbb{A})} + \max\left(\frac{2M_k L}{\delta_k}, M_k\right)\right) ||f||_1. \end{aligned} \tag{4.20}$$

Thus, using (4.20) and (4.5) we have

$$\begin{aligned} V_{\mathbb{A}} P_T f \leq& \sum_{k=1}^{K} \left(2\lambda_k + \gamma_1^k + \gamma_2^k + \gamma_3^k\right) V_{\mathbb{A}} f \\ &+ \sum_{k=1}^{K} \left(2M_k + \frac{\gamma_3^k}{m(\mathbb{A})} + \frac{\gamma_2^k}{m(\mathbb{A})} + \max\left(\frac{2M_k L}{\delta_k}, M_k\right)\right) ||f||_1 \\ \leq& \alpha V_{\mathbb{A}} f + C||f||_1, \end{aligned} \tag{4.21}$$

where

$$\sum_{k=1}^{K} \left(2\lambda_k + \gamma_1^k + \gamma_2^k + \gamma_3^k\right) \leq \alpha < 1,$$

and

$$C = \sum_{k=1}^{K} \left(2M_k + \frac{\gamma_3^k}{m(\mathbb{A})} + \frac{\gamma_2^k}{m(\mathbb{A})} + \max \left(\frac{2M_k L}{\delta_k}, M_k \right) \right) > 0.$$

■

Lemma 4.2.2 *If T satisfies (I)-(VII) and $B \subset \mathfrak{L}^1(\mathbb{A})$ is such that*

$$V_{\mathbb{A}} f + ||f||_1 \leq D$$

for some D and any $f \in B$, then $P_T B$ is weakly compact.

Proof. Recall that $P_T B = \sum_k^K P_{\tau_k} p_k B$. Therefore, it suffices to prove that the lemma is true for $P_{\tau_k} p_k B$ for all k. For $f \in B$ we choose points $z_l \in I_l^k$ such that

$$\sum_l m(I_l^k)|f(z_l)| \leq \int_{\mathbb{A}} |f(x)| dx.$$

We have

$$\begin{aligned}
|P_{\tau_k} p_k f| &\leq P_{\tau_k} p_k |f| \\
&= \sum_l |f(\psi_{k,l}(x))| p_k(\psi_{k,l}(x)) || \psi'_{k,l}(x)| \\
&= \sum_l \frac{p_k(\psi_{k,l}(x))||\psi'_{k,l}(x)|}{m(I_l^k)} m(I_l^k)|f(\psi_{k,l}(x))| \\
&= \sum_{l \in W_1^k} \frac{p_k(\psi_{k,l}(x))||\psi'_{k,l}(x)|}{m(I_l^k)} m(I_l^k)|f(\psi_{k,l}(x))| \\
&\quad + \sum_{l \in W_2^k} |f(\psi_{k,l}(x))| p_k(\psi_{k,l}(x)) || \psi'_{k,l}(x)| \\
&\leq \sup_{l \in W_1^k} \frac{p_k(\psi_{k,l}(x))||\psi'_{k,l}(x)|}{m(I_l^k)} \sum_{l \in W_1^k} m(I_l^k)|f(\psi_{k,l}(x))| \\
&\quad + \sup_{\mathbb{A}} |f| \sum_{l \in W_2^k} p_k(\psi_{k,l}(x)) || \psi'_{k,l}(x)|.
\end{aligned}$$

By (III), we get

$$\sum_{l\in W_1^k} m(I_l^k)|f(\psi_{k,l}(x))| \le \sum_{l\in W_1^k} m(I_l^k)(|f(\psi_{k,l}(x)) - f(z_l)| + |f(z_l))|$$

$$\le LV_{\mathbb{A}}f + ||f||_1 \le (L+1)D,$$

and

$$\sup_{\mathbb{A}} |f| \le \frac{||f||_1}{m(\mathbb{A})} + V_{\mathbb{A}}f \le D\left(\frac{1}{m(\mathbb{A})} + 1\right).$$

Hence, (VII) implies uniform integrability of the set $P_{\tau_k}B$ for $k = 1, \ldots, K$. ■

Theorem 4.2.1 *Let T be a random map satisfying conditions (I)-(VII). Then T admits an absolutely continuous invariant measure on $\mathbb{A}$.*

Proof. Let $f \in BV(\mathbb{A})$. Then, by Lemma 4.2.1 we have

$$V_{\mathbb{A}}P_T^n f \le \alpha^n V_{\mathbb{A}}f + (\alpha^{n-1} + \cdots + \alpha + 1)||f||_1. \tag{4.22}$$

Thus, for every n,

$$V_{\mathbb{A}}P_T^n f \le \alpha^n V_{\mathbb{A}}f + \frac{C}{1-\alpha}||f||_1,$$

and $||P_T^n f||_1 \le ||f||_1$. Therefore, by Lemma 4.2.2, the set $P_T\{P_T^n f\}_{n=0}^{\infty} = \{P_T^n f\}_1^{\infty}$ is weakly compact in $\mathfrak{L}^1(\mathbb{A})$. By the Yosida-Kakutani Theorem (see Chapter 1), the sequence $\frac{1}{n}\sum_{i=1}^{n-1} P_T^i f$ converges in $\mathfrak{L}^1$ to a P_T-invariant function f^*. ■

Now, we present an example of a random map which satisfies conditions (I)-(VII) and thus admits an absolutely continuous invariant measure.

Example 4.2.1 *Let $T = \{\tau_1, \tau_2; p_1(x), p_2(x)\}$ be position dependent random map where $\tau_1(x) = 10\tan(x)$, $\tau_2(x) = \sin(x)$, $p_1(x) = \frac{10-|\cos(x)|}{10}$ and $p_2(x) = \frac{|\cos(x)|}{10}$.*

Obviously, conditions (I)-(V) are satisfied. First, we check condition (VII). For τ_1 *and* τ_2 *we consider* $W_1 = \mathbb{N}$. *For* $k = 1$, *we have*

$$\int_{-\infty}^{+\infty} \sup_{l \in W_1^1} \frac{|\psi_{1,l}'(x)| p_1(\psi_{1,l}(x))}{m(I_l^1)} \le \frac{10}{4\pi} \int_{-\infty}^{+\infty} (\tan^{-1}(x))' dx = \frac{5}{2}.$$

For $k = 2$, *we have*

$$\int_{-1}^{1} \sup_{l \in W_1^2} \frac{|\psi_{2,l}'(x)| p_2(\psi_{1,2}(x))}{m(I_l^2)} \le \frac{1}{\pi} \int_{-1}^{1} \frac{1}{10} dx = \frac{1}{10\pi}.$$

It remains to prove that T *satisfies condition (VI). Observe that* $J_2^k = J_3^k = \emptyset$ *for* $k = 1, 2$. *Therefore, we only prove that*

$$\sum_{k=1}^{2} \left(2\lambda_k + \gamma_1^k\right) \le \alpha < 1.$$

For $k = 2$, *we have* $\gamma_1^2 = 0$ *and* $\lambda_2 = \frac{1}{10}$. *For* $k = 1$, *we have* $\lambda_1 \le \frac{1}{10}$ *and calculate, with out loss of generality,* γ_1^1 *on* $[-\frac{\pi}{2}, \frac{\pi}{2}]$. *Observe that* $\left(\frac{10-|\cos(x)|}{10}\right) \frac{\cos^2(x)}{10}$ *is monotonic on* $[-\frac{\pi}{2}, 0]$ *and* $[0, \frac{\pi}{2}]$. *Then*

$$\begin{aligned} \gamma_1^1 &= \left| \left(\frac{10 - |\cos(-\frac{\pi}{2})|}{10}\right) \frac{\cos^2(-\frac{\pi}{2})}{10} - \left(\frac{10 - |\cos(0)|}{10}\right) \frac{\cos^2(0)}{10} \right| \\ &+ \left| \left(\frac{10 - |\cos(0)|}{10}\right) \frac{\cos^2(0)}{10} - \left(\frac{10 - |\cos(\frac{\pi}{2})|}{10}\right) \frac{\cos^2(\frac{\pi}{2})}{10} \right| \\ &= \frac{18}{100}. \end{aligned}$$

Therefore,

$$\sum_{k=1}^{2} \left(2\lambda_k + \gamma_1^k\right) \le 2\left(\frac{1}{10} + \frac{1}{10}\right) + \frac{18}{100} = 0.58 < 1.$$

Thus, T *satisfies conditions (I)-(VII) and, by Theorem 4.2.1, admits an absolutely continuous invariant measure.*

4.3 Random Maps of an Interval With Holes

The theory of expanding maps with "holes" has been studied recently in [9] and [33]. The word "holes" means that some open subintervals are removed from the domain. In the case of expanding maps with holes, almost every point of the domain eventually escapes through the holes. Thus, such transformations have no absolutely continuous invariant measure. In this case, conditionally invariant measures which are absolutely continuous with respect to Lebesgue measures are constructed. Expanding maps with holes are believed to have interesting applications in chaos and statistical mechanics [9]. This motivates us to consider the problem in a more general setting.

In the following sections, we study random maps with holes and we allow the probabilities to be functions of position. We prove the existence of absolutely continuous conditionally invariant measures for position dependent random maps on the unit interval with holes [4].

4.4 Notation and the Frobenius-Perron Operator

In this section, we formulate the definition of position dependent random maps with holes and introduce the "normalized" Frobenius-Perron operator. Let

$$\hat{T} = \{\hat{\tau}_1, \dots, \hat{\tau}_K; p_1(x), \dots, p_K(x)\}$$

be a position dependent random map, where $\hat{\tau}_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \dots, K$, are piecewise one-to-one, nonsingular differentiable transformations and $p_k(x) \geq 0$, $k = 1, \dots, K$, are position dependent probabilities, $\sum_{k=1}^{K} p_k(x) = 1$; i.e., $\hat{T}(x) = \hat{\tau}_k(x)$ with probability $p_k(x)$. Let H be a finite union of disjoint open intervals, $H = \bigcup_{j=1}^{L} H_j$, $H \subset [0,1]$, $I = [0,1] \setminus H$ and $\hat{\tau}_k(I) \subset [0,1]$. Let $\mathcal{P}_k$ be a partition of $[0,1]$, $\mathcal{P}_k = \{\hat{I}_i^k, \ldots, \hat{I}_{q_k}^k\}$ and $\hat{\tau}_{k,i} = \hat{\tau}_k \mid_{\hat{I}_i^k}$, $i = 1, \ldots, q_k$, $k = 1, \ldots, K$. Define $\tau_k : I \to [0,1]$ to be the restriction of $\hat{\tau}_k$ to I, $k = 1, \ldots, K$. We define the position dependent random map with holes by $T = \{\tau_1, \ldots, \tau_K; p_1(x), \ldots, p_K(x)\}$. Let $I_i^k = \hat{I}_i^k \cap I$ and $\tau_{k,i} = \tau_k \mid_{I_i^k}$, $i = 1, \ldots, q_k$. We denote by L the number of holes and by $h = \lambda(H)$ the Lebesgue measure of H.

Recall that the Frobenius-Perron operator of a position dependent random map is given by:

$$(P_T f)(x) = \sum_{k=1}^{K} P_{\tau_k} (p_k f)(x),$$

where P_{τ_k} is the Frobenius-Perron operator corresponding to the transformation τ_k. In the case of random maps with holes, the Frobenius-Perron operator decreases the norm of measures; i.e., for $f \geq 0$, we have

$$\begin{aligned} ||P_T f||_1 = \int_I P_T f dx &= \sum_{k=1}^{K} \int_I (P_{\tau_k} p_k f)(x) dx \\ = \sum_{k=1}^{K} \int_{\tau_k^{-1}(I)} p_k(x) f(x) dx &\leq \sum_{k=1}^{K} \int_I p_k(x) f(x) dx = ||f||_1. \end{aligned}$$

Therefore, we consider the normalized Frobenius-Perron operator

$$\overline{P}_T f = \frac{P_T f}{||P_T f||_1},$$

which is defined only when $||P_T f||_1 \neq 0$. A probability measure μ on $[0,1]$, $\mu = f^* \lambda$, is said to be an absolutely continuous conditionally invariant measure if there exists

$\rho > 0$ such that $P_T f^* = \rho f^*$. Let $\Lambda_b = \{f \in \mathfrak{L}^1(I) : f \geq 0, ||f||_1 = 1, V_I f \leq b\}$. Observe that Λ_b is compact convex subset of $\mathfrak{L}^1(I)$.

4.5 Absolutely Continuous Conditionally Invariant Measures. Result Independent of the Number of Holes

In this section we assume:

(i) No hole covers the interior of the domain of $\tau_{k,i}$ for $1 \leq i \leq q_k; 1 \leq k \leq K$.

(ii) $\frac{p_k}{|\tau_k'|} \in BV(I)$, $k = 1, \dots, K$.

(iii) $\left(2\sum_{k=1}^K \sup_x \frac{p_k(x)}{|\tau_k'(x)|} + \sum_{k=1}^K V_I \frac{p_k}{|\tau_k'|}\right) \leq \alpha < 1$.

Lemma 4.5.1 *Let T be a random map satisfying conditions (i)-(iii). Then there exist constants $0 < \alpha < 1$ and $\beta > 0$ such that*

$$V_I P_T f \leq \alpha V_I f + \beta ||f||_1,$$

for $f \in BV(I)$.

Proof. Let $f \in BV(I)$. Define $\psi_{k,i} = \tau_{k,i}^{-1}$. We have

$$V_I P_T f \leq \sum_{k=1}^{K} V_I P_{\tau_k} p_k f. \tag{4.23}$$

First, we estimate $V_I P_{\tau_k} p_k f$, $k \in \{1, \ldots, K\}$, as follows:

$$
\begin{aligned}
V_I P_{\tau_k} p_k f \le & \sum_{i=1}^{q_k} V_{\tau_k(I_i^k)} \left[(f \circ \psi_{k,i}(x)) p_k(\psi_{k,i}(x)) |(\psi'_{k,i}(x))| \right] \\
& + \sum_{i=1}^{q_k} \sum_j \left(|f(a_{i,j})| p_k(a_{i,j}) |\psi'_{k,i}(\tau(a_{i,j}))| + |f(b_{i,j})| p_k(b_{i,j}) |\psi'_{k,i}(\tau(b_{i,j})|\right) \\
& = S_1 + S_2,
\end{aligned} \tag{4.24}
$$

where $I_i^k = [a_i, b_i] = \bigcup_j [a_{i,j}, b_{i,j}]$. Define $[a_{i,j}, b_{i,j}] = I_{i,j}^k$ and $\delta_k = \min \lambda(I_{i,j}^k)$. Let us estimate S_2 of (4.24). We have

$$
S_2 \le \sup_x \frac{p_k(x)}{|\tau'_k(x)|} \sum_{i=1}^{q_k} \sum_j \left(|f(a_{i,j})| + |f(b_{i,j})| \right). \tag{4.25}
$$

Now, observe that

$$
|f(a_{i,j})| + |f(b_{i,j})| \le 2|f(v_{i,j})| + |f(v_{i,j}) - f(a_{i,j})| + |f(v_{i,j}) - f(b_{i,j})|,
$$

where $v_{i,j} \in [a_{i,j}, b_{i,j}]$ such that $|f(v_{i,j})| \le \frac{1}{\lambda(I_{i,j}^k)} \int_{I_{i,j}^k} |f| dx$. Therefore,

$$
\begin{aligned}
\sum_j |f(a_{i,j})| + |f(b_{i,j})| &\le \sum_j \frac{2}{\delta_k} \int_{I_{i,j}^k} |f(x)| dx + V_{I_{i,j}^k} f \\
&\le \frac{2}{\delta_k} \int_{I_i^k} |f(x)| dx + V_{I_i^k} f.
\end{aligned} \tag{4.26}
$$

Summing over i in (4.26) and using (4.25) we get

$$
S_2 \le \sup_x \frac{p_k(x)}{|\tau'_k(x)|} \left(\frac{2}{\delta_k} ||f||_1 + V_I f \right). \tag{4.27}
$$

Now, we estimate S_1 of (4.24). Let $y_r = \tau_{k,i}(x_r)$, where $x_r \in I_i^k$. We have

$$
\begin{aligned}
&V_{\tau(I_i^k)}\left[(f\circ\psi_{k,i})p_k(\psi_{k,i})|\psi'_{k,i}|\right] \\
&= \overline{\sup_{\tau(I_i^k)}}\sum_{r=1}^{l}\big|(f\circ\psi_{k,i})(y_r)p_k(\psi_{k,i}(y_r))|(\psi'_{k,i}(y_r))| \\
&\qquad\qquad - (f\circ\psi_{k,i})(y_{r-1})p_k(\psi_{k,i}(y_{r-1}))|(\psi'_{k,i}(y_{r-1}))|\big| \\
&\le \overline{\sup_{\tau(I_i^k)}}\sum_{r=1}^{l}\big|(f\circ\psi_{k,i})(y_r)p_k(\psi_{k,i}(y_r))|(\psi'_{k,i}(y_r))| \\
&\qquad\qquad - (f\circ\psi_{k,i})(y_{r-1})p_k(\psi_{k,i}(y_r))|(\psi'_{k,i}(y_r))|\big| \\
&\quad + \overline{\sup_{\tau(I_i^k)}}\sum_{r=1}^{l}\big|(f\circ\psi_{k,i})(y_{r-1})p_k(\psi_{k,i}(y_r))|(\psi'_{k,i}(y_r))| \qquad\qquad (4.28)\\
&\qquad\qquad - (f\circ\psi_{k,i})(y_{r-1})p_k(\psi_{k,i}(y_{r-1}))|(\psi'_{k,i}(y_{r-1}))|\big| \\
&\le \sup_x \frac{p_k(x)}{|\tau'_k(x)|}V_{I_i^k}f + \overline{\sup_{I_i^k}}\sum_{r=1}^{l}|f(x_{r-1})|\left|\frac{p_k(x_r)}{|\tau'_k(x_r)|}-\frac{p_k(x_{r-1})}{|\tau'_k(x_{r-1})|}\right| \\
&\le \sup_x \frac{p_k(x)}{|\tau'_k(x)|}V_{I_i^k}f + (\sup_{I_i^k}|f|)\overline{\sup_{I_i^k}}\sum_{r=1}^{l}\left|\frac{p_k(x_r)}{|\tau'_k(x_r)|}-\frac{p_k(x_{r-1})}{|\tau'_k(x_{r-1})|}\right| \\
&\le \sup_x \frac{p_k(x)}{|\tau'_k(x)|}V_{I_i^k}f + (\sup_{I_i^k}|f|)V_{I_i^k}\frac{p_k}{|\tau'_k|} \\
&\le \sup_x \frac{p_k(x)}{|\tau'_k(x)|}V_{I_i^k}f + \left(\frac{2}{\delta_k}\int_{I_i^k}|f(x)|dx + V_{I_i^k}f\right)V_{I_i^k}\frac{p_k}{|\tau'_k|},
\end{aligned}
$$

where $\overline{\sup}_{\tau(I_i^k)}$ and $\overline{\sup}_{I_i^k}$ indicate the suprema over all finite partitions of $\tau(I_i^k)$ and I_i^k respectively. Summing over i in (4.28), we get

$$
S_1 \le \left(\sup_x \frac{p_k(x)}{|\tau'_k(x)|} + V_I\frac{p_k}{|\tau'_k|}\right)V_I f + \left(V_I\frac{p_k}{|\tau'_k|}+\frac{2}{\delta_k}\right)||f||_1. \qquad (4.29)
$$

Therefore,

$$\begin{aligned} S_1 + S_2 \le & \left(2 \sup_x \frac{p_k(x)}{|\tau_k'(x)|} + V_I \frac{p_k}{|\tau_k'|} \right) V_I f \\ & + \left(V_I \frac{p_k}{|\tau_k'|} + \sup_x \frac{p_k(x)}{|\tau_k'(x)|} \right) \frac{2}{\delta_k} ||f||_1. \end{aligned} \tag{4.30}$$

It follows that

$$\begin{aligned} V_I P_T f \le & \left(2 \sum_{k=1}^{K} \sup_x \frac{p_k(x)}{|\tau_k'(x)|} + \sum_{k=1}^{K} V_I \frac{p_k}{|\tau_k'|} \right) V_I f \\ & + \sum_{k=1}^{K} \frac{2}{\delta_k} \left(V_I \frac{p_k}{|\tau_k'|} + \sup_x \frac{p_k(x)}{|\tau_k'(x)|} \right) ||f||_1. \end{aligned} \tag{4.31}$$

Let $\delta = \min\{\delta_k\}_{k=1}^{K}$. Thus $V_I P_T f \le \alpha V_I f + \beta ||f||_1$ with $\beta = \frac{2\alpha}{\delta}$. ∎

Lemma 4.5.2 *Let T be a random map satisfying conditions (i)-(iii). There exists a $b_{\min} > 0$ such that for every $b_{\max} \ge b_{\min}$ there is an $h_0 > 0$, h_0 depends on $b_{\max}$, $b_{\min}$, such that whenever $h < h_0$, then the normalized Frobenius-Perron operator $\overline{P}_T f$ is well defined on Λ_b for $b \in (b_{\min}, b_{\max})$ and $\overline{P}_T f \in \Lambda_b$ for $f \in \Lambda_b$.*

Proof. By Lemma 4.5.1, we have

$$V_I P_T f \le \alpha V_I f + \beta ||f||_1,$$

with $\beta = \frac{2\alpha}{\delta}$. Put $b_{\min} = \frac{2\beta}{1-\alpha}$. Let $b_{\max} > b_{\min}$ be given and observe that $||f||_\infty < b_{\max} + 2$.

$$\begin{aligned} ||P_T f||_1 &= \int_I P_T f dx = \sum_{k=1}^{K} \int_I (P_{\tau_k} p_k f)(x) dx = \sum_{k=1}^{K} \int_{\tau_k^{-1}(I)} p_k(x) f(x) dx \\ &= \sum_{k=1}^{K} \int_I p_k(x) f(x) dx - \sum_{k=1}^{K} \int_{I \setminus \tau_k^{-1}(I)} p_k(x) f(x) dx \\ &= 1 - \sum_{k=1}^{K} \int_{I \setminus \tau_k^{-1}(I)} p_k(x) f(x) dx \end{aligned} \tag{4.32}$$

Observe that $I \backslash \tau_k^{-1}(I) = \tau_k^{-1}(H)$. Let $q = \max\{q_1, \ldots, q_K\}$ and $\sum_{k=1}^{K} \sup_x \frac{p_k(x)}{|\tau'(x)|} \leq \gamma$.

Notice that condition (iii) implies $0 < \gamma < \alpha < 1$. Using this and (4.32), we obtain

$$
\begin{aligned}
||P_T f||_1 &= 1 - \sum_{k=1}^{K} \sum_{i=1}^{q_k} \int_{\tau_{k_i}^{-1}(H)} p_k(x) f(x) d\lambda(x) \\
&= 1 - \sum_{k=1}^{K} \sum_{i=1}^{q_k} \int_H p_k(\tau_{k_i}^{-1}x) f(\tau_{k_i}^{-1}x) \frac{1}{|\tau'(\tau_{k_i}^{-1}x)|} d\lambda(x) \\
&\geq 1 - \sum_{k=1}^{K} \sup_x \frac{p_k(x)}{|\tau'(x)|} \sum_{i=1}^{q_k} \int_H f(\tau_{k_i}^{-1}x) d\lambda(x) \\
&\geq 1 - \sum_{k=1}^{K} \sup_x \frac{p_k(x)}{|\tau'(x)|} \lambda(H) \cdot q \cdot ||f||_\infty \\
&\geq 1 - h \cdot \gamma \cdot q(b_{\max} + 2).
\end{aligned}
$$

Let

$$
h_0 = \frac{1-\alpha}{2} \left((b_{\max} + 2)\gamma \cdot q\right)^{-1}.
$$

We have

$$
||P_T f||_1 > 1 - \frac{1-\alpha}{2} = \alpha + \frac{1-\alpha}{2} > 0.
$$

Thus, $\overline{P}_T f$ is well defined on Λ_b, $b \leq b_{\max}$, and

$$
\begin{aligned}
V_I \overline{P}_T f &\leq ||P_T f||_1^{-1} (\alpha V_I f + \beta) \\
&< (\alpha + \frac{1-\alpha}{2})^{-1} (\alpha b + \frac{1-\alpha}{2} b) = b.
\end{aligned}
$$

Thus, $\overline{P}_T(\Lambda_b) \subset \Lambda_b$. ∎

Theorem 4.5.1 *Let T satisfy conditions (i)-(iii) and*

$$
h < \frac{(1-\alpha)^2 \delta}{8\alpha + 4\delta(1-\alpha)} \cdot \frac{1}{\gamma} \cdot \frac{1}{q}. \tag{4.33}
$$

Then T admits an absolutely continuous conditionally invariant measure.

Proof. By Lemma 4.5.2, the Schauder-Tykhonov Theorem (Chapter 1) implies that $\overline{P}_T$ has a fixed point f^*, $f^* \in \Lambda_b$. Thus, $P_T f^* = \rho f^*$ with $\rho = ||P_T f^*||_1$. ■

4.6 Absolutely Continuous Conditionally Invariant Measure. Result Depending on the Number of Holes but Allowing Larger Holes

In this section we assume:

(I*) $\Delta_k > h$ for $k = 1, \ldots, K$, where $\Delta_k = \min_i \lambda(\hat{I}_i^k)$.

(II*) $\frac{p_k}{|\tau_k'|} \in BV(I)$, $k = 1, \ldots, K$.

(III*) $\left((L+2)\sum_{k=1}^K \sup_x \frac{p_k(x)}{|\tau_k'(x)|} + \sum_{k=1}^K V_I \frac{p_k}{|\tau_k'|}\right) \leq \eta < 1$.

Lemma 4.6.1 *Let T be a random map satisfying conditions (I*)-(III*). Then there exist constants $0 < \eta < 1$ and $C > 0$ such that*

$$V_I P_T f \leq \eta V_I f + C||f||_1,$$

for $f \in BV(I)$.

Proof. Let $f \in BV(I)$. Define $\psi_{k,i} = \tau_{k,i}^{-1}$. We have

$$V_I P_T f \leq \sum_{k=1}^K V_I P_{\tau_k} p_k f. \tag{4.34}$$

First, we estimate $V_I P_{\tau_k} p_k f$, $k \in \{1, \dots, K\}$, which is given by:

$$
\begin{aligned}
V_I P_{\tau_k} p_k f &\le \sum_{i=1}^{q_k} V_{\tau_k(I_i^k)} \left[(f \circ \psi_{k,i}(x)) p_k(\psi_{k,i}(x)) |(\psi'_{k,i}(x))| \right] \\
&+ \sum_{i=1}^{q_k} \sum_j \left(|f(a_{i,j})| p_k(a_{i,j}) |\psi'_{k,i}(\tau(a_{i,j}))| + |f(b_{i,j})| p_k(b_{i,j}) |\psi'_{k,i}(\tau(b_{i,j}))| \right) \\
&= S_1 + S_2,
\end{aligned} \tag{4.35}
$$

where $I_i^k = [a_i, b_i] = \bigcup_j [a_{i,j}, b_{i,j}]$. Let us estimate S_2 of (4.35). We have

$$
S_2 \le \sup_x \frac{p_k(x)}{|\tau'_k(x)|} \sum_{i=1}^{q_k} \sum_j \left(|f(a_{i,j})| + |f(b_{i,j})| \right). \tag{4.36}
$$

Now, observe that

$$
\begin{aligned}
|f(a_{i,j})| + |f(b_{i,j})| &\le \frac{2}{\lambda(\hat{I}_i^k \setminus H)} \int_{\hat{I}_i^k \setminus H} |f(x)| dx + V_{\hat{I}_i^k \setminus H} f \\
&\le \frac{2}{\Delta_k - h} \int_{\hat{I}_i^k \setminus H} |f(x)| dx + V_{\hat{I}_i^k \setminus H} f,
\end{aligned} \tag{4.37}
$$

where we used fact that $\lambda(\hat{I}_i^k \setminus H) \ge \Delta_k - h$. Summing over i in (4.37) and using (4.36) we get

$$
S_2 \le (L+1) \sup_x \frac{p_k(x)}{|\tau'_k(x)|} \left(\frac{2}{\Delta_k - h} ||f||_1 + V_I f \right). \tag{4.38}
$$

The estimate on S_1 of (4.36) is similar to that in Lemma 4.5.1.

$$
S_1 \le \left(\sup_x \frac{p_k(x)}{|\tau'_k(x)|} + V_I \frac{p_k}{|\tau'_k|} \right) V_I f + \left(V_I \frac{p_k}{|\tau'_k|} + \frac{1}{\Delta_k - h} \right) ||f||_1. \tag{4.39}
$$

Therefore,

$$
\begin{aligned}
S_1 + S_2 &\le \left((L+2) \sup_x \frac{p_k(x)}{|\tau'_k(x)|} + V_I \frac{p_k}{|\tau'_k|} \right) V_I f \\
&+ \left(V_I \frac{p_k}{|\tau'_k|} + \sup_x \frac{p_k(x)}{|\tau'_k(x)|} \right) \frac{2}{\Delta_k - h} ||f||_1.
\end{aligned} \tag{4.40}
$$

It follows that

$$V_I P_T f \leq \left((L+2)\sum_{k=1}^{K} \sup_x \frac{p_k(x)}{|\tau_k'(x)|} + \sum_{k=1}^{K} V_I \frac{p_k}{|\tau_k'|} \right) V_I f$$
$$+ \sum_{k=1}^{K} \frac{2}{\Delta_k - h} \left(V_I \frac{p_k}{|\tau_k'|} + (L+1) \sup_x \frac{p_k(x)}{|\tau_k'(x)|} \right) ||f||_1.$$

Let $\Delta = \min\{\Delta_k\}_{k=1}^{K}$. Thus $V_I P_T f \leq \eta V_I f + C||f||_1$ with $C = \frac{2\eta}{\Delta - h}$. ■

Lemma 4.6.2 *Let T be a random map satisfying conditions (I*)-(III*). There exists a $b_{\min} > 0$ such that for every $b_{\max} \geq b_{\min}$ there is an $h_1 > 0$, h_1 depends on $b_{\max}$, $b_{\min}$, such that whenever $h < h_1$, then the normalized Frobenius-Perron operator $\overline{P}_T f$ is well defined on Λ_b for $b \in (b_{\min}, b_{\max})$ and $\overline{P}_T f \in \Lambda_b$ for $f \in \Lambda_b$.*

Proof. Let $0 < h_1 < \frac{\Delta}{2}$. Then by Lemma 4.6.1

$$V_I P_T f \leq \eta V_I f + C||f||_1,$$

with $C = \frac{4\eta}{\Delta}$. Put $b_{\min} = \frac{2C}{1-\eta}$. Let $b_{\max} > b_{\min}$ be given and observe that $||f||_\infty < b_{\max} + 2$. Using (4.32), we obtain

$$||P_T f||_1 = 1 - \sum_{k=1}^{K} \int_{I \setminus \tau_k^{-1}(I)} p_k(x) f(x) dx. \tag{4.41}$$

Observe that $I \setminus \tau_k^{-1}(I) = \tau_k^{-1}(H)$. Let $q = \max\{q_1, \ldots, q_K\}$ and $\sum_{k=1}^{K} \sup_x \frac{p_k(x)}{|\tau'(x)|} \leq \gamma$. Notice that condition (III) implies $0 < \gamma < \eta < 1$. Using this and (4.41), we obtain

$$||P_T f||_1 = 1 - \sum_{k=1}^{K} \sum_{i=1}^{q_k} \int_{\tau_{k_i}^{-1}(H)} p_k(x) f(x) d\lambda(x)$$
$$\geq 1 - h \cdot \gamma \cdot q(b_{\max} + 2).$$

Let

$$h_1 = \min\{\frac{1-\eta}{2}\left((b_{\max}+2)\gamma \cdot q\right)^{-1}, \frac{\Delta}{2}\}.$$

We have

$$||P_T f||_1 > 1 - \frac{1-\eta}{2} = \eta + \frac{1-\eta}{2} > 0.$$

Thus, $\overline{P}_T f$ is well defined on Λ_b, $b \leq b_{\max}$, and

$$V_I \overline{P}_T f \leq ||P_T f||_1^{-1} (\eta V_I f + \beta) < b.$$

Thus, $\overline{P}_T(\Lambda_b) \subset \Lambda_b$. ■

Theorem 4.6.1 *Let T satisfy conditions (I*)-(III*) and*

$$h < \frac{(1-\eta)^2\Delta}{16\eta + 4\Delta(1-\eta)} \cdot \frac{1}{\gamma} \cdot \frac{1}{q}.$$

Then T admits an absolutely continuous conditionally invariant measure.

Proof. By Lemma 4.6.2, the Schauder-Tykhonov Theorem implies that $\overline{P}_T$ has a fixed point f^*, $f^* \in \Lambda_b$. Thus, $P_T f^* = \rho f^*$ with $\rho = ||P_T f^*||_1$. ■

4.7 Absolutely Continuous Invariant Measures and Conditionally Invariant Measures

In this section we give an example showing that the conditional invariant measure is not necessarily a restriction of an invariant measure to a smaller domain. More precisely, let $\hat{\tau} : \hat{I} \to \hat{I}$ be a transformation preserving a measure $\hat{\mu}$ and let $\tau : I \to I$ be a map with holes obtained by restricting $\hat{\tau}$ to $I \subset \hat{I}$ preserving a conditionally invariant measure μ. Then, it may happen that $\mu \neq \hat{\mu}_{|I}$, properly normalized. (Example, where this equality holds is easy to construct.)

Example 4.7.1 *Let* $\hat{I} = [0, 2]$ *with partition* $\mathcal{P} = \{\hat{I}_1, \ldots, \hat{I}_8\}$, *where* $\hat{I}_i = [(i-1)/4, i/4]$, $i = 1, \ldots, 8$. *Let us define*

$$\hat{\tau}(x) = \begin{cases} 4x, & \text{for } x \in \hat{I}_1 \cup \hat{I}_2; \\ 4 - 4x, & \text{for } x \in \hat{I}_3 \cup \hat{I}_4; \\ 2(x-1), & \text{for } x \in \hat{I}_5; \\ 2(x - 5/4), & \text{for } x \in \hat{I}_6; \\ 8(x - 6/4), & \text{for } x \in \hat{I}_7; \\ 8(x - 7/4), & \text{for } x \in \hat{I}_8. \end{cases}$$

Let $I = \hat{I}_1 \cup \hat{I}_4$ *and* $\tau = \hat{\tau}_{|I}$. *It is easy to see that* τ*-conditionally invariant absolutely continuous measure is normalized Lebesgue measure on* I. $\hat{\tau}$*-invariant density is piecewise constant on* $\mathcal{P}$ *and can be found from the equation* $f = f\mathbb{M}$, *where* $f =$

$[f_1, f_2, \ldots, f_8]$ *and*

$$\mathbb{M} = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 \\ 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 \end{pmatrix}.$$

Then, we have $f_1 = 2f_4$. *Thus,* $\mu \neq \hat{\mu}_{|I}$.

Chapter 5

Stochastic Perturbations of Random Maps

Physical systems are usually subjected to small perturbations from external noise or round-off errors. There are well known results [14] that study the stability of absolutely continuous invariant measures for piecewise expanding transformations. In this chapter, we prove that absolutely continuous invariant measures for position dependent random maps are stable under small stochastic perturbations [5, 6].

5.1 Notation and Set Up

Let $T = \{\tau_1, \dots, \tau_K; p_1(x), \dots, p_K(x)\}$ be a position dependent random map where τ_k are piecewise expanding and piecewise C^2, $\tau_k : [0,1] \to [0,1]$, defined on the same partition $\mathcal{P}$. $\mathcal{P}$ can be found by refining the defining partitions of all the transformations. We define

$$\alpha = \min_k \inf_x |\tau_k'(x)| > 1,$$

and assume that $\{p_k(x)\}_{k=1}^K$ are piecewise C^1 with respect to the partition $\mathcal{P}$.

We consider a perturbation of the random map T. Let G be a family of functions g that are piecewise C^2 on the partition $\mathcal{P}$, $g(x) : [0,1] \to [0,1]$ such that $\sup_x |g'(x)| \leq \frac{\alpha-1}{2}$ and $|g''(x)| \leq M$, where M is independent of $g \in G$. We further assume that G is endowed with a regular probability measure η. Usually G will be a family of functions with parameter in a bounded region of $\mathbb{R}^n$ having normalized Lebesgue measure. At each iteration, to the map τ_k we add a function g from G chosen at random. Thus, at each iteration, with probability $p_k(x)$, the next map is

$$\tau_{k,g}(x) = \tau_k(x) + g(x) \pmod 1,$$

where each g is chosen from G according to the probability η. The perturbed random map is denoted by T_g if the perturbing maps $\{g\}$ are fixed and by T_G if $\{g\}$'s are chosen at random from G. For the case of one map similar models were considered in [34].

The iteration of the random map T_G is performed as follows:

$$T_G^N(x) = \tau_{k_N,g} \circ \tau_{k_{N-1},g} \circ \cdots \circ \tau_{k_1,g}(x)$$

with probability

$$p_{k_N}(\tau_{k_{N-1},g} \circ \cdots \circ \tau_{k_1,g}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2},g} \circ \cdots \circ \tau_{k_1,g}(x)) \ldots p_{k_1}(x),$$

where the perturbations are chosen independently at each step.

T_G can be viewed as a Markov process with the transition function

$$\mathbb{P}(x, A) = \sum_{k=1}^{K} p_k(x) \int_G \chi_A(\tau_{k,g}(x)) d\eta(g),$$

where A is a measurable set and χ_A denotes the characteristic function of the set A. We say that a measure μ is T_G-invariant if it is invariant for the above Markov process. We are interested in absolutely continuous invariant measures with respect to Lebesgue measure. Our main tool is the Frobenius-Perron operator which describes the transformation of densities under the action of the process T_G. For fixed perturbations $\{g\}$, the Frobenius-Perron operator of T_g is given by:

$$(P_{T_g} f)(x) = \sum_{k=1}^{K} P_{\tau_{k,g}} (p_k f)(x), \tag{5.1}$$

where $P_{\tau_{k,g}}$ is the Frobenius-Perron operator corresponding to the transformation $\tau_{k,g}$. The transition function $\mathbb{P}$ induces an operator $\mathbb{P}_*$ on measures on $([0,1], \mathfrak{B})$ defined by

$$\begin{aligned}
\mathbb{P}_*\mu(A) = \int \mathbb{P}(x, A) d\mu(x) &= \int \int_G \sum_{k=1}^{K} p_k(x) \chi_A(\tau_{k,g}(x)) d\eta(g) d\mu(x) \\
&= \int_G \left(\sum_{k=1}^{K} \int p_k(x) \chi_A(\tau_{k,g}(x)) d\mu(x) \right) d\eta(g) \\
&= \int_G \left(\sum_{k=1}^{K} \sum_{p=1}^{t_g} \int_{(\tau_{k,g,p})^{-1} A} p_k(x) d\mu(x) \right) d\eta(g),
\end{aligned}$$

where t_g is the number of branches of $\tau_{k,g}$. We used Fubini's Theorem in the above argument. If μ has a density f with respect to λ, then $\mathbb{P}_*\mu(A)$ also has a density which we denote by $P_{T_G} f$, where P_{T_G} denotes the Frobenius-Perron operator associated with

T_G. By a change of variables, we obtain,

$$\int_A (P_{T_G} f)(x) d\lambda(x) = \int_G \left(\sum_{k=1}^{K} \sum_{p=1}^{t_g} \int_{(\tau_{k,g,p})^{-1}A} p_k(x) f(x) d\lambda(x) \right) d\eta(g)$$

$$= \int_G \left(\sum_{k=1}^{K} \sum_{p=1}^{t_g} \int_A p_k((\tau_{k,g,p})^{-1}x) f((\tau_{k,g,p})^{-1}x) \frac{1}{J_{(\tau_{k,g,p})}((\tau_{k,g,p})^{-1}(x))} d\lambda(x) \right) d\eta(g)$$

$$= \int_A \left(\int_G \sum_{k=1}^{K} \sum_{p=1}^{t_g} p_k((\tau_{k,g,p})^{-1}x) f((\tau_{k,g,p})^{-1}x) \frac{1}{J_{(\tau_{k,g,p})}((\tau_{k,g,p})^{-1}(x))} d\eta(g) \right) d\lambda(x), \tag{5.2}$$

where $J_{(\tau_{k,g,p})}$ denotes the Jacobian of $\tau_{k,g,p}$ with respect to λ. We have again used Fubini's Theorem in the above argument. Since (5.2) holds for any measurable set A, we obtain an a.e. equality:

$$(P_{T_G} f)(x) = \int_G \sum_{k=1}^{K} \sum_{p=1}^{t_g} p_k((\tau_{k,g,p})^{-1}x) f((\tau_{k,g,p})^{-1}x) \frac{1}{J_{(\tau_{k,g,p})}((\tau_{k,g,p})^{-1}(x)} d\eta(g). \tag{5.3}$$

Therefore, by (5.1), the Frobenius-Perron operator of T_G is given by:

$$(P_{T_G} f)(x) = \int_G (P_{T_g} f)(x) d\eta(g). \tag{5.4}$$

5.2 Stability of Invariant Measures

The following theorem was proved in [23].

Theorem 5.2.1 *Let T be as above. If $\frac{2\sum_{k=1}^{K} \sup_x p_k(x)}{\alpha} < 1$, then T admits an absolutely continuous invariant measure.*

In this section, we assume:

$$2 \sum_{k=1}^{K} \sup_x p_k(x) (1 + \frac{\alpha - 1}{2})^{-1} < 1. \tag{5.5}$$

If (5.5) is satisfied then by Theorem 5.2.1, T admits an absolutely continuous invariant measure. We prove:

Lemma 5.2.1 *For any $g \in G$, T_g admits an absolutely continuous invariant measure. In particular, for $f \in BV_I$ and, for any $g \in G$, we have:*

$$V_I P_{T_g} f \leq A V_I f + B||f||_1, \tag{5.6}$$

where $A = 2\Theta(1 + \frac{\alpha-1}{2})^{-1}$,

$$B = (\frac{2}{\delta}\Theta(1 + \frac{\alpha - 1}{2})^{-1} + K(1 + \frac{\alpha - 1}{2})^{-2}(\max_k \sup_x(|\tau_k''(x)| + M)(p_k(x) + p_k'(x)(\tau_k'(x) + \frac{\alpha - 1}{2})),$$

$\delta = \min_{i=1,\dots,q} \lambda(I_i)$, *and* $\Theta = \sum_{k=1}^K \sup_x p_k(x)$.

The proof of Lemma 5.2.1 follows from the following lemma. Now, define

$$\tau_g(x) = \tau(x) + g(x) \pmod 1, \;\; x \in [0,1].$$

Lemma 5.2.2 *Let $\tau : [0,1] \to [0,1]$ be piecewise expanding and piecewise C^2 on $\mathcal{P} = \{I_i\}_{i=1}^q$. Let $g \in G$ and τ_g be as above. Then, for any $f \in BV_I$, and for any g we have:*

$$V_I P_{\tau_g} f \leq A V_I f + B||f||_1, \tag{5.7}$$

where $A = 2(1 + \frac{\alpha-1}{2})^{-1}$, $B = \left(\frac{2}{\delta}(1 + \frac{\alpha-1}{2})^{-1} + (1 + \frac{\alpha-1}{2})^{-2}(\sup_x |\tau''(x)| + M)\right)$, *and* $\delta = \min_{i=1,\dots,q} \lambda(I_i)$.

The goal of this lemma is to obtain an uniform bound for the variation of the invariant densities over all perturbed maps and thus for random maps.

$\mathcal{P} = \{I_i\}_{i=1}^q$ is the defining partition of the unperturbed map τ. $\{I_p\}_{p=1}^t$ will denote the defining partition of the perturbed map τ_g. Note that $\{I_p\}_{p=1}^t$ is a refinement of $\mathcal{P}$, see Figures 1 and 2.

Proof. Since f is Riemann integrable, for any arbitrary $\varepsilon > 0$, we can find a number θ such that for any $I_i \in \mathcal{P}$ and any partition finer than: $I_i = \cup_{l=1}^{L_i}[s_{l-1}, s_l]$ with $s_l - s_{l-1} < \theta$, we have

$$\sum_{l=1}^{L_i} |f(s_{l-1})||s_l - s_{l-1}| \leq \int_{I_i} |f| d\lambda + \varepsilon.$$

Let $0 = x_0 < x_1 < \cdots < x_r = 1$ be such a fine partition of I. When τ is perturbed by g, $\tau_{g,i}$ is not necessarily injective on I_i. The condition $|g_k'| \leq \frac{\alpha-1}{2}$ ensures the monotonicity of $\tau_k + g_k$, but taking the values modulo 1 may introduce small extra intervals of monotonicity, see Figures 1 and 2. The dotted lines in the figures are the pieces of the perturbed map branches which are moved up or down by (mod 1) operation. Thus, a refinement of the original partition is required. We refine the partition, $I = \cup_{p=1}^t I_p$, such that $\tau_{g,p} = \tau_{g|_{I_p}}$ is injective.

Define $\phi_p = \tau_{g,p}^{-1}$. Let $h(x) = \frac{1}{|\tau'(x)+g'(x)|}$ and observe that $\sup_x h(x) \leq (1 + \frac{\alpha-1}{2})^{-1} < 1$. We have,

$$\begin{aligned} V_I P_{\tau_g}(f)(x) \leq \sum_{j=1}^{r} |(\sum_{p=1}^{t} h(\phi_p(x_j))f(\phi_p(x_j))\chi_{\tau_g(I_p)}(x_j) - \\ \sum_{p=1}^{t} h(\phi_p(x_{j-1}))f(\phi_p(x_{j-1}))\chi_{\tau_g(I_p)}(x_{j-1}))| \\ \leq \sum_{j=1}^{r}\sum_{p=1}^{t} |h(\phi_p(x_j))f(\phi_p(x_j))\chi_{\tau_g(I_p)}(x_j) \\ - h(\phi_p(x_{j-1}))f(\phi_p(x_{j-1}))\chi_{\tau_g(I_p)}(x_{j-1}))| \end{aligned} \tag{5.8}$$

We divide the sum on the right hand side into three parts:

(I) the summands for which $\chi_{\tau_g(I_p)}(x_j) = \chi_{\tau_g(I_p)}(x_{j-1}) = 1$,

(II) the summands for which $\chi_{\tau_g(I_p)}(x_j) = 1$ and $\chi_{\tau_g(I_p)}(x_{j-1}) = 0$,

(III) the summands for which $\chi_{\tau_g(I_p)}(x_j) = 0$ and$\chi_{\tau_g(I_p)}(x_{j-1}) = 1$.

First, we will estimate (I).

$$
\begin{aligned}
&\sum_{j=1}^{r}\sum_{p=1}^{t}|h(\phi_p(x_j))f(\phi_p(x_j)) - h(\phi_p(x_{j-1}))f(\phi_p(x_{j-1}))| \\
&\leq \sum_{p=1}^{t}\sum_{j=1}^{r}|f(\phi_p(x_j))[h(\phi_p(x_j)) - h(\phi_p(x_{j-1}))]| \\
&\qquad\qquad + \sum_{p=1}^{t}\sum_{j=1}^{r}|h(\phi_p(x_{j-1}))[f(\phi_p(x_j)) - f(\phi_p(x_{j-1}))]| \\
&\leq \sup_x |h'(x)| \sum_{p=1}^{t}\sum_{j=1}^{r}|f(\phi_p(x_j))[\phi_p(x_j) - \phi_p(x_{j-1})]| + (\sup_x h(x))\sum_{p=1}^{t} V_{I_p} f \\
&\leq \sup_x |h'(x)| \sum_{p=1}^{t}\left(\int_{I_p}|f|d\lambda(x) + \varepsilon\right) + (\sup_x h(x))\sum_{p=1}^{t} V_{I_p} f \\
&\leq (1+\frac{\alpha-1}{2})^{-2}(\sup_x|\tau''(x)| + M)\int_I |f|d\lambda(x)) + (1+\frac{\alpha-1}{2})^{-1}V_I f + t(\sup_x h(x))\varepsilon.
\end{aligned}
\tag{5.9}
$$

We now consider (II) and (III) together. Here, we make use of the old partition $\{I_i\}_{i=1}^{q}$. Notice that $\chi_{\tau_g(I_p)}(x_j) = 1$ and $\chi_{\tau_g(I_p)}(x_{j-1}) = 0$ occurs only if $x_j \in \tau_g(I_p)$ and $x_{j-1} \notin \tau_g(I_p)$. For this situation we have four cases. This is illustrated in Figures 1 and 2: Recall that $\mathcal{P} = \{I_i\}_{i=1}^{q}$ is the defining partition of the unperturbed map τ. $\{I_p\}_{p=1}^{t}$ will denote the defining partition of the perturbed map τ_g.

(a) τ_g is monotonic on I_p and $I_p \in \{I_i\}_{i=1}^{q}$, then it happens if x_j and x_{j-1} are on opposite sides of an end point of $\tau_g(I_p)$. We can have at most one pair x_j, x_{j-1} like this and another pair $x_{j'} \notin \tau_g(I_p)$ and $x_{j'-1} \in \tau_g(I_p)$.

(b) If $\tau_g(I_{p+1}) \cap \tau_g(I_p) = \emptyset$, then $\tau_{g|_{I_{p+1}\cup I_p}} = \tau_{g|_{I_i}}$ for some I_i. The above situation happens if x_j and x_{j-1} are on opposite sides of an end point of $\tau_g(I_p)$. We can have at most one pair x_j, x_{j-1} like this and another pair $x_{j'} \notin \tau_g(I_{p+1})$ and $x_{j'-1} \in \tau_g(I_{p+1})$.

(c) If $\tau_{g|_{I_p \cup I_{p+1}}}$ is not injective on some I_i, then either $\tau_g(I_{p+1}) \subset \tau_g(I_p)$ or $\tau_g(I_p) \subset \tau_g(I_{p+1})$ or $\tau_g(I_{p+1}) \cup \tau_g(I_{p-1}) \subset \tau_g(I_p)$ with $\tau_g(I_{p-1}) \cap \tau_g(I_{p+1}) = \emptyset$ and $I_{p-1} \cup I_p \cup I_{p+1} = I_i$ for some I_i. Moreover, $\tau_g(I_p) = [0, 1]$, $1 \in \tau_g(I_{p+1})$ and $0 \in \tau_g(I_{p-1})$ (or vice versa). The last statement is true because $g(x)$ is continuous on I_i. We consider the last case because it is more general. Then the above situation happens if x_j and x_{j-1} are on opposite sides of an end point of $\tau_g(I_{p+1})$ or on opposite sides of $\tau_g(I_{p-1})$. We can have at most one pair $x_j \in \tau_g(I_{p+1})$, $x_{j-1} \notin \tau_g(I_{p+1})$

and another pair $x_{j'} \notin \tau_g(I_{p-1})$ and $x_{j'-1} \in \tau_g(I_{p-1})$.

(d) If $\tau_{g|_{I_p \cup I_{p+1}}}$ is not injective on some I_i with $\tau_g(I_{p+1}) \not\subset \tau_g(I_p)$, $\tau_g(I_p) \not\subset \tau_g(I_{p+1})$ and $I_p \cup I_{p+1} = I_i$ for some I_i. Then, $1 \in \tau_g(I_p)$ and $0 \in \tau_g(I_{p+1})$ (or vice versa). The last statement is true because $g(x)$ is continuous on I_i. Therefore, the above situation happens if x_j and x_{j-1} are on opposite sides of an end point of $\tau_g(I_{p+1})$. We can have at most one pair x_j, x_{j-1} like this and another pair $x_{j'} \notin \tau_g(I_p)$ and $x_{j'-1} \in \tau_g(I_p)$. Thus, (II) and (III) can be estimated by:

$$\begin{aligned} &\sum_{p=1}^{t} (|h(\phi_p(x_j))f(\phi_p(x_j))| + |h(\phi_p(x_{j'-1}))f(\phi_p(x_{j'-1}))|) \\ &\qquad \leq \sup_x h(x) \sum_{p=1}^{t} (|f(\phi_p(x_j))| + f(\phi_p(x_{j'-1}))|) \end{aligned} \tag{5.10}$$

Since $s_i = \phi_p(x_j)$ and $r_i = \phi_p(x_{j'-1})$ are both points in I_i, we can write (and switch the sum to i)

$$\sum_{i=1}^{q} (|f(s_i)| + |f(r_i)|) \leq \sum_{i=1}^{q} (2|f(v_i)| + |f(v_i) - f(r_i)| + |f(v_i) - f(s_i)|),$$

where $v_i \in I_i$ is such a point that $|f(v_i)| \leq \frac{1}{\lambda(I_i)} \int_{I_i} |f|\lambda(dx)$. We get

$$(II) + (III) \leq$$

$$\sup_x h(x) \sum_{i=1}^{q} \left(V_{I_i} f + \frac{2}{\lambda(I_i)} \int_{I_i} |f|\lambda(dx) \right) \leq (1 + \frac{\alpha - 1}{2})^{-1} V_I f + \frac{2}{\delta}(1 + \frac{\alpha - 1}{2})^{-1} ||f||_1. \tag{5.11}$$

Thus,

$$V_I P_{\tau_g} f \leq 2(1 + \frac{\alpha - 1}{2})^{-1} V_I f + \left(\frac{2}{\delta}(1 + \frac{\alpha - 1}{2})^{-1} + (1 + \frac{\alpha - 1}{2})^{-2} (\sup_x |\tau''(x)| + M) \right) ||f||_1 + t(\sup_x h'(x))\varepsilon. \tag{5.12}$$

Since ε is arbitrarily small this proves the lemma. ■

Lemma 5.2.3 *Let g, G and η be as above . Let F_g be a family of functions such that $f_g \in BV(I)$ for $f_g \in F_g$. If $V(f_g) \leq A$, $A > 0$, for all $f_g \in F_g$, then $V(\int_G f_g(x) d\eta(g)) \leq A$.*

Proof. Let $0 = x_0 < x_1 < \cdots < x_r = 1$ be an arbitrary partition of I. We have

$$\begin{aligned} V(\int_G f_g(x) d\eta(g)) &= \sum_{j=1}^{r} | \int_G f_g(x_j) d\eta(g)) - \int_G f_g(x_{j-1}) d\eta(g))| \\ &\leq \sum_{j=1}^{r} \int_G |f_g(x_j) - f_g(x_{j-1})| d\eta(g) \\ &= \int_G \sum_{j=1}^{r} |f_g(x_j) - f_g(x_{j-1})| d\eta(g) \\ &= \int_G V(f_g) d\eta(g) < A. \end{aligned}$$

■

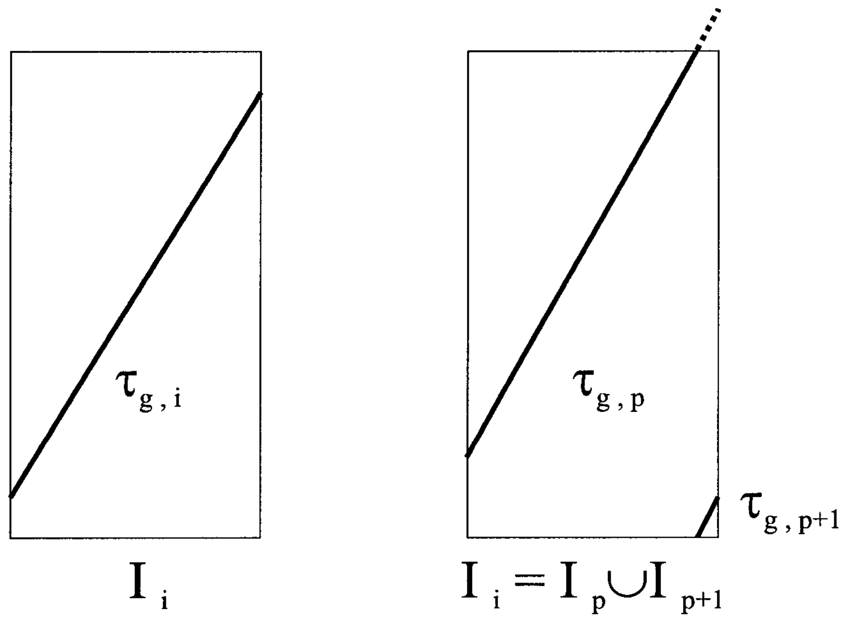

Figure 5.1: Cases (a) and (b)

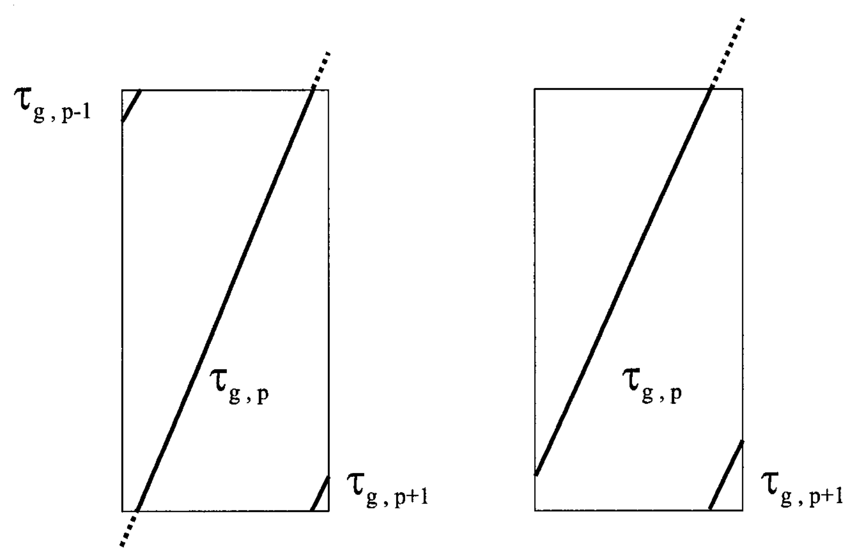

Figure 5.2: Cases (c) and (d)

Corollary 5.2.1 *Under the assumptions of Section 5.1, for any* $f \in BV([0,1])$,

$$V_I(P_{T_G} f) \leq A \cdot V_I(f) + B \cdot \|f\|_1,$$

where the constants A, B *are the same as in Lemma 5.2.1. This implies that* P_{T_G} *admits an invariant density.*

Proof. We have

$$(P_{T_G} f)(x) = \int_G (P_{T_g} f)(x) d\eta(g). \tag{5.13}$$

Thus, by Lemma 5.2.1 and Lemma 5.2.3, we obtain the inequality

$$V_I(P_{T_G} f) \leq A \cdot V_I(f) + B \cdot \|f\|_1,$$

with $0 < A < 1$. From here, the proof may follow, for example, the classical proof of [32]. Applying the inequality iteratively to $P^n_{T_G} f$ we prove that the sequence $\{P^n_{T_G} f\}_{n=1}^{\infty}$ is bounded in the space $BV(I)$. Thus, the sequence of the averages $\{(1/n)\sum_{k=1}^{n} P^k_{T_G} f\}_{n=1}^{\infty}$ is also bounded in $BV(I)$ and thus weakly precompact in $\mathfrak{L}^1([0,1],\lambda)$. By the Yosida-Kakutani theorem the operator P_{T_G} has a fixed point f^*. It can be proved that $f^* \in BV(I)$. ■

Theorem 5.2.2 *Let* g, G, η, T_G *and* T *be as above. Let us consider a family of sets* $G : \{G_\varepsilon\}_{\varepsilon>0}$ *such that* $\sup_{g\in G_\varepsilon} \sup_x |g(x)| \leq \varepsilon$. *Let* f_ε *be an invariant density of* $P_{T_{G_\varepsilon}}$. *Then, the densities* $\{f_\varepsilon\}$ *form a precompact set in* $\mathfrak{L}^1([0,1],\lambda)$. *Any weak limit point* f^* *of invariant densities* f_ε *as* $\varepsilon \to 0$ *is an invariant density of* T.

Proof. The density of f_ε, where index "ε" denotes dependence on the perturbation family G_ε, is a fixed point of the operator $P_{T_{G_\varepsilon}}$. By Corollary 5.2.1 all $P_{T_{G_\varepsilon}}$ satisfy the same estimates independently of the functions G_ε. Thus, the densities f_ε have uniformly bounded variation and form a precompact set in $\mathfrak{L}^1([0,1],\lambda)$. Let G_n be a sequence of perturbations such that $\sup_{g\in G_n}\sup_x |g(x)| \to 0$, as $n \to +\infty$. The corresponding sequence of invariant densities f_n contains a convergent subsequence. We can assume it converges: $f_n \to f^*$ in $\mathfrak{L}^1([0,1],\lambda)$. We claim that f^* is an invariant density of T. It is enough to show that $P_T f^* = f^*$. Since T_{G_n} converges almost uniformly to T, as $n \to +\infty$, this can be proved in the same way as in Lemma 11.2.2 of [14]. ■

5.3 Expanding on Average

In the previous section, we assumed that all the transformations are piecewise expanding. This condition can be weakened to expanding on average case (see Section 3.1) if we modify our conditions. We need three assumptions:

1. $\sum_{k=1}^K \sup_x \frac{p_k(x)}{|\tau_k'(x)|} \le \rho < 1$. (expanding on average).
2. $\sum_{k=1}^K \sup_x \frac{p_k(x)}{|\tau_k'(x)+g_k'(x)|} \le \beta < 1$. (expanding on average of the random map T_g).
3. $\sup_x |g'(x)| < \min_k \inf_x |\tau_k'(x)|$ (mild noise).

The following theorem is a weaker version of the existence Theorem of 3.1.1:

Theorem 5.3.1 *Let T be as above. If $\sum_{k=1}^K \sup_x \frac{p_k(x)}{|\tau_k'(x)|} \le \rho < 1$, then T admits an* *absolutely continuous invariant measure.*

The following lemma is analogous to Lemma 5.2.1.

Lemma 5.3.1 *For any $g \in G$, T_g admits an absolutely continuous invariant measure. In particular, for $f \in BV_I$ and, for any g, we have:*

$$V_I P_{T_g} f \leq A V_I f + B||f||_1, \tag{5.14}$$

where $A = 2\beta$, $B = (\frac{2}{\delta}\beta + K \max_k \sup_x \left(\frac{p_k(x)}{|\tau_k' + g_k'|}\right)')$, $\delta = \min_{i=1,\dots,q} \lambda(I_i)$.

Remark 5.3.1 *All the results of Section 5.2 remain valid in the expanding on average case if we assume that conditions 1, 2 and 3 hold.*

5.4 Estimates in Case of Perturbations with Small Derivative

In this section we assume that T, T_g satisfy conditions of Section 5.3. We estimate the error in the invariant density in the case when derivatives of perturbing maps g converge to 0. Our considerations are based on the ideas of Keller ([28] or [14]).

We introduce the *Skorokhod metric* on the space of all piecewise monotonic maps on I:

$$d_S(\tau_1, \tau_2) = \inf \Big\{ \varepsilon > 0 : \ \exists A \subseteq I \text{ and } \exists \sigma : I \to I \text{ such that}$$
$$\lambda(A) > 1 - \varepsilon, \sigma \text{ is a diffeomorphism}, \tau_{1|_A} = \tau_2 \circ \sigma_{|_A}$$
$$\text{and } \forall x \in A, \ \ |\sigma(x) - x| < \varepsilon, \ \ \left|\frac{1}{\sigma'(x)} - 1\right| < \varepsilon \Big\}.$$

Let $G_{\varepsilon_1,\varepsilon_2,\alpha}$ be a family of piecewise C^2 functions such that

$$\sup_{x\in I}|g(x)| \le \varepsilon_1 \quad , \quad \sup_{x\in I}|g'(x)| \le \varepsilon_2 < \alpha = \inf|\tau'(x)| > 0,$$

and $\sup_{x\in I}|g''(x)| \le M$.

Proposition 5.4.1 *Let τ be a piecewise C^2, piecewise monotonic map with $\alpha = \inf|\tau'(x)| > 0$. Let $g \in G_{\varepsilon_1,\varepsilon_2,\alpha}$ and $\tau_g = \tau + g$ (mod 1). Then,*

$$d_S(\tau, \tau_g) \le \max\left\{\frac{4q\varepsilon_1}{\alpha}, \frac{\max|\tau''|\varepsilon_1/\alpha + \varepsilon_2}{\alpha}\right\},$$

where q is the number of inverse branches of τ.

Proof. We will construct the diffeomorphism σ. Let $I_i = [a_i, a_{i+1}]$ be an element of the defining partition. We will show how to construct σ on I_i. We assume that τ_i (and thus also $\tau_i + g$) is increasing.

First, we find small subintervals next to a_i and a_{i+1} which will belong to the complement of the set A from the definition. Let $y_1 = \max\{\tau_i(a_i), \tau_i(a_i) + g(a_i)\} + \varepsilon_1$, where we assume that $\varepsilon_1 > 0$ is sufficiently small. Let $b_1 = \tau_i^{-1}(y_1)$. Since $|y_1 - \tau_i(a_i)| < 2\varepsilon_1$ we have $b_1 - a_i < 2\varepsilon_1/\alpha$. Let $b_2 = (\tau_i + g)^{-1}(y_1)$.

Similarly, let $y_2 = \min\{\tau_i(a_{i+1}), \tau_i(a_{i+1}) + g(a_{i+1})\} - \varepsilon_1$. Let $b_3 = \tau_i^{-1}(y_2)$. Since $|y_2 - \tau_i(a_{i+1})| < 2\varepsilon_1$ we have $a_{i+1} - b_3 < 2\varepsilon_1/\alpha$. Let $b_4 = (\tau_i + g)^{-1}(y_2)$. The intervals $[a_i, b_1]$, $[b_3, a_{i+1}]$ are the intersection of A with the interval I_i. Thus, our construction gives $\lambda(A) \le q \cdot 4\varepsilon_1/\alpha$. We define $\sigma : [a_i, b_1] \to [a_i, b_2]$ and $\sigma : [b_3, a_{i+1}] \to [b_4, a_{i+1}]$ as linear with necessary smoothing at the end points to make σ a diffeomorphism.

On the remaining interval we define $\sigma : [b_1, b_3] \to [b_2, b_4]$ as $\sigma(x) = \tau_g^{-1}(\tau(x))$. Obviously, we have $\tau = \tau_g \circ \sigma$ on $[b_1, b_3]$. We have to estimate $|\sigma(x) - x|$ and $\left|\frac{1}{\sigma'(x)} - 1\right|$. Since $|\tau(x) - \tau(\sigma(x))| = |\tau_g(\sigma(x)) - \tau(\sigma(x)| < \varepsilon_1$, we have

$$|\sigma(x) - x| < \varepsilon_1/\alpha.$$

Now, $\sigma'(x) = \tau_g'(\tau(x)) \cdot \tau'(x) = \dfrac{\tau'(x)}{(\tau' + g')(\sigma(x))}$, and

$$\begin{aligned}\left|\frac{1}{\sigma'(x)} - 1\right| &= \left|\frac{\tau'(\sigma(x)) - \tau'(x) + g'(\sigma(x))}{\tau'(x)}\right| \\ &= \left|\frac{\tau''(\xi)(\sigma(x) - x) + g'(\sigma(x))}{\tau'(x)}\right| \leq \frac{\max|\tau''|(\varepsilon_1/\alpha) + \max|g'|}{\alpha}.\end{aligned} \tag{5.15}$$

■ For operators $P : BV \to \mathfrak{L}^1$, we introduce the norm

$$|||P||| = \sup\{||Pf||_1 : f \in BV, ||f||_{BV} \leq 1\}.$$

Lemma 5.4.1 *Let P_T and P_{T_g} be the Frobenius-Perron operators of T and T_g respectively. Then $|||P_T - P_{T_g}||| \leq 14 \sum_{k=1}^{K} \sup_x p_k(x) \cdot d_{\mathcal{S}}(\tau_k, \tau_{k,g})$.*

Proof. The proof follows that of Lemma 11.2.1 of [14]. Let $f \in BV$, $h = \frac{|P_T f - P_{T_g} f|}{(P_T f - P_{T_g} f)}$. Then

$$\begin{aligned}\int |P_T f - P_{T_g} f| d\lambda &= \int h \cdot (P_T f - P_{T_g} f) d\lambda \\ &= \sum_{k=1}^{K} \int h \cdot (P_{\tau_k} p_k f - P_{\tau_{k,g}} p_k f) d\lambda \\ &= \sum_{k=1}^{K} \int (h \circ \tau_k - h \circ \tau_{k,g}) \cdot p_k f d\lambda \\ &\leq \sum_{k=1}^{K} \sup_x p_k(x) \int (h \circ \tau_k - h \circ \tau_{k,g}) \cdot f d\lambda \\ &\leq 2||f||_{BV} \sum_{k=1}^{K} \sup_x p_k(x) \cdot \sup_z \left| \int_0^z (h \circ \tau_k - h \circ \tau_{k,g}) d\lambda \right|,\end{aligned}$$

by Lemma 1.3.2. The estimate of $\sup_z |\int_0^z (h\circ\tau_k - h\circ\tau_{k,g})d\lambda|$ is the same as in Lemma 11.2.1 of [14]. ■

Theorem 5.4.1 *Let T satisfy assumptions of Section 5.3. We also assume that T has a unique invariant density. Let $g_{\varepsilon_1,\varepsilon_2} \in G_{\varepsilon_1,\varepsilon_2,\alpha}$, where $\alpha = \min_k \inf|\tau_k'|$. Let $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$, then*

$$||f - f_{g_{\varepsilon_1,\varepsilon_2}}||_1 \leq O(\varepsilon \ln \varepsilon) \ \text{as}\ \varepsilon \to 0^+, \tag{5.16}$$

where f and $f_{g_{\varepsilon_1,\varepsilon_2}}$ are the invariant densities of T and $T_{g_{\varepsilon_1,\varepsilon_2}}$ respectively.

Proof. By Lemma 5.3.1 the family $\{T_{g_{\varepsilon_1,\varepsilon_2}}\}$, $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$ is small enough, is $\mathcal{S}$-bounded, i.e., satisfies inequality (5.14) with common constants A and B (see Chapter 11 of [14] for details). Then by Lemma 5.4.1 and Proposition 5.4.1 and Theorem 11.1.1 of [14] inequality (5.16) holds. ■

The following is a consequence

Corollary 5.4.1 *Let T satisfy assumptions of Section 5.3. We also assume that T has a unique invariant density. Let $G_{\varepsilon_1,\varepsilon_2,\alpha}$ be as in Section 5.1, where $\alpha = \min_k \inf|\tau_k'|$. Then*

$$||f - f_{G_{\varepsilon_1,\varepsilon_2}}||_1 \leq O(\varepsilon \ln \varepsilon) \ \text{as}\ \varepsilon \to 0^+,$$

where f and $f_{g_{\varepsilon_1,\varepsilon_2}}$ are the invariant densities of T and $T_{G_{\varepsilon_1,\varepsilon_2}}$ respectively.

5.5 Perturbations and Stochastic Operators

In the remaining part of this chapter, we deal with a different model of stochastic perturbations. Let $I = [0, 1]$, and let $\mathfrak{B}$ be Borel σ-algebra, λ Lebesgue measure on $(I, \mathfrak{B})$. Let $\mathfrak{L}^1 = \mathfrak{L}^1(I, \mathfrak{B}, \lambda)$. Let $T = \{\tau_1, \dots, \tau_K; p_1(x), \dots, p_K(x)\}$ be a position dependent random map where τ_k are piecewise differentiable nonsingular transformations, $\tau_k : [0, 1] \to [0, 1]$, defined on the same partition $\mathcal{P}$. We assume expanding on average conditions (see Chapter 3):

Condition (A): $\sum_{k=1}^{K} \sup_x \frac{p_k(x)}{|\tau_k'(x)|} \le \alpha < 1$,

Condition (B): $\frac{p_k(x)}{|\tau_k'(x)|} \in BV(I); k = 1, ..K$,

which imply

$$||P_T^N||_{BV} \le \alpha ||f||_{BV} + C||f||_1,$$

where $C > 0$, and T admits an absolutely continuous invariant measure. Following [28], we present,

Definition 5.5.1 *We say that an operator $P : \mathfrak{L}^1 \to \mathfrak{L}^1$ is a linear stochastic operator if it satisfies the following:*

(a) $P(BV) \subset BV$;

(b) there exist constants $\alpha < 1$, $C > 0$ and a positive integer N such that, $||P||_{BV} < \infty$, and $||P^N||_{BV} \le \alpha ||f||_{BV} + C||f||_1$, $f \in BV$; and

(c) P is stochastic, i.e., $P \ge 0$ and $\int_I Pf d\mu = \int_I f d\mu$, $f \in L^1$. Hence $||P||_1 = 1$.

Let $\mathcal{S}$ denote the class of all linear stochastic operators and let $\mathcal{S}(\alpha, C)$ be a subclass of $\mathcal{S}$ that satisfies Definition 5.5.1 for a fixed α and C.

5.6 Stochastic Perturbations

Let $\mathbb{Q}_\varepsilon(x, A)$, $\varepsilon > 0$ be a family of probability transition functions. The family of Markov processes $\mathfrak{T}_\varepsilon$ with transition functions

$$\sum_{k=1}^{K} p_k(x)\mathbb{Q}_\varepsilon(\tau_k(x), A),$$

is called a small stochastic perturbation of T if, for any $\theta > 0$,

$$E_\varepsilon(\theta) = \inf_{x \in I} \mathbb{Q}_\varepsilon(x, [x - \theta, x + \theta] \cap I) \underset{\varepsilon \to 0}{\longrightarrow} 1, \tag{5.17}$$

i.e., the measures $\mathbb{Q}_\varepsilon(x, \cdot)$ are concentrated around x uniformly as $\varepsilon \to 0$. Denote by μ^ε the invariant measure associated with $\mathfrak{T}_\varepsilon$ (Section 1.5). (We prove the existence of μ^ε later). Our goal is to prove that $\mu^\varepsilon \to \mu$ weakly, where μ is the invariant measure of the position random map T.

Proposition 5.6.1 *Let the family $\mathfrak{T}_\varepsilon$, $\varepsilon > 0$, be a small stochastic perturbation of a position dependent random map T which admits an invariant measure μ. If μ^ε is an invariant measure of $\mathfrak{T}_\varepsilon$, then any weak limit point of the family $\{\mu^\varepsilon\}_{\varepsilon > 0}$ is a T-invariant measure.*

Proof. Let us assume that $\mu^\varepsilon \to \mu$ in the weak topology of measures. We will prove that μ is T-invariant. By Lemma 2.2.1, it is enough to prove that for any $g \in C(I)$,

$$\int_I g d\mu = \sum_{k=1}^{K} \int_I p_k(x) \cdot (g \circ \tau_k) d\mu(x).$$

We have

$$\begin{aligned}
|\int_I g d\mu(x) - \sum_{k=1}^{K} \int_I p_k(x) \cdot (g \circ \tau_k) d\mu(x)| &\leq |\int_I g d\mu(x) - \sum_{k=1}^{K} \int_I p_k(x) \cdot g d\mu^\varepsilon| \\
&+ |\sum_{k=1}^{K} \int_I p_k(x) \cdot g d\mu^\varepsilon - \sum_{k=1}^{K} \int_I p_k(x) \cdot (g \circ \tau_k) d\mu^\varepsilon| \\
&+ |\sum_{k=1}^{K} \int_I p_k(x) \cdot (g \circ \tau_k) d\mu^\varepsilon - \sum_{k=1}^{K} \int_I p_k(x) \cdot (g \circ \tau_k) d\mu| \\
&\leq |\int_I g d\mu(x) - \int_I g d\mu^\varepsilon| + \max_k \sup_x p_k(x) |\sum_{k=1}^{K} \int_I g d\mu^\varepsilon - \sum_{k=1}^{K} \int_I g \circ \tau_k d\mu^\varepsilon| \\
&+ |\sum_{k=1}^{K} \int_I p_k(x) \cdot (g \circ \tau_k) d\mu^\varepsilon - \sum_{k=1}^{K} \int_I p_k(x) \cdot (g \circ \tau_k) d\mu|.
\end{aligned}$$

The first and the third summands on the right-hand side converge to 0 as $\varepsilon \to 0$ since $\mu^\varepsilon \to \mu$ weakly. To estimate the second summand, it is enough to show that $|\int_I g d\mu^\varepsilon - \sum_{k=1}^{K} \int_I g \circ \tau_k d\mu^\varepsilon|$ is arbitrarily small for all k. The proof of this fact is exactly the same as in Proposition 11.3.1 of [14]. ■

5.7 Stability of Invariant Measures

In this section we consider the class of probability transition functions which is generated by doubly stochastic kernels. We define $q(x, y)$ to be a doubly stochastic kernel if

(1) $q(\cdot,\cdot) : I \times I \to \mathbb{R}^+$,

(2) $\int_I q(x,y)d\lambda(y) = 1$ for any $x \in I$,

(3) $\int_I q(x,y)d\lambda(x) = 1$ for any $y \in I$.

Let

$$s_k(x,y) = q(\tau_k(x), y), k = 1, \dots, K.$$

Define the transition function

$$\mathbb{P}_q(x,A) = \sum_{k=1}^{K} \int_A p_k(x) s_k(x,y) d\lambda(y).$$

The Markov process $\mathfrak{T}$ defined by the transition function $\mathbb{P}_q(x,A)$ is called a non-singular perturbation of the random map T. The following corresponds to (5.17):

If

$$\inf_{x\in I} \int_{x-\theta}^{x+\theta} q_\varepsilon(x,y)d\lambda(y) \to 1 \text{ as } \varepsilon \to 0, \theta > 0 \tag{5.18}$$

is satisfied for the family $\mathfrak{T}_\varepsilon$, then we call them small stochastic perturbations of T.

Let Q be the stochastic operator induced by the kernel $q(\cdot,\cdot)$, $Q : \mathfrak{L}^1 \to \mathfrak{L}^1$, defined by

$$(Qf)(x) = \int_I f(u) q(u,x) d\lambda(u).$$

The time evolution of the densities of the process $\mathfrak{T}$ is given by $P_{\mathfrak{T}} : \mathfrak{L}^1 \to \mathfrak{L}^1$, where

$$\begin{aligned}(P_{\mathfrak{T}}f)(x) &= \sum_{k=1}^{K} \int_I p_k(u) f(u) s_k(u,x) d\lambda(u) = \sum_{k=1}^{K} \int_I p_k(u) f(u) q(\tau_k(u), x) d\lambda(u) \\ &= \sum_{k=1}^{K} \int_I (P_{\tau_k} p_k f)(u) q(u,x) d\lambda(u) = \int_I (P_T f)(u) q(u,x) d\lambda(u) \qquad (5.19) \\ &= ((Q \circ P_T) f)(x).\end{aligned}$$

For operators $P : BV \to \mathfrak{L}^1$, we introduce the norm

$$|||P||| = \sup\{||Pf||_1 : f \in BV, ||f||_{BV} \leq 1\}.$$

Theorem 5.7.1 *Let P_T be the Frobenius-Perron operator induced by the random map T. For $z \in I$, let us define*

$$q_z(y) = \int_{x \leq z} q(x,y) d\lambda(x),$$
$$b_z = \iint\limits_{B_z} q(x,y) d\lambda(x) d\lambda(y),$$

where $B(z) = \{(x,y) : x \leq z < y \text{ or } y \leq z < x\}$, and let $c(q) = \sup_{z \in I} b_z$. Then,

(a) $||P_{\mathfrak{T}}||_1 \leq ||P_T||_1 = 1$;

(b) $V(P_{\mathfrak{T}}) \leq \sup_{z \in I} V(q_z) \cdot V(P_T f)$;

(c) $|||P_T - P_{\mathfrak{T}}||| \leq c(q) \cdot ||P_T||_{BV}$.

Proof. (a) We have

$$\begin{aligned} ||P_{\mathfrak{T}} f||_1 &= \int_I |P_{\mathfrak{T}} f| d\lambda(x) \leq \sum_{k=1}^{K} \int_I \left| \int_I p_k(u) f(u) s_k(u,x) d\lambda(u) \right| d\lambda(x) \\ &\leq \sum_{k=1}^{K} \int_I p_k(u) |f(u)| \left(\int_I s_k(u,x) d\lambda(x) \right) d\lambda(u) \\ &= \int_I \sum_{k=1}^{K} p_k(u) |f(u)| d\lambda(u) = \int_I |f(u)| d\lambda(u) = ||f||_1. \end{aligned} \tag{5.20}$$

We have used Fubini's Theorem and property (2) of $q(\cdot,\cdot)$.

(b) to estimate $V(P_{\mathfrak{T}} f)$, we will use Theorem 1.3.10. Take $\Phi \in \mathfrak{L}^1$ with $|| \int \Phi||_\infty \leq 1$ and $\int \Phi d\lambda = 0$. Then,

$$\begin{aligned} \int_I P_{\mathfrak{T}} f \Phi d\lambda(x) &= \int_I \left(\int_I P_T f(u) q(u,x) d\lambda(u) \right) \Phi(x) d\lambda(x) \\ &= \int_I P_T f \Psi d\lambda(u), \end{aligned} \tag{5.21}$$

where $\Psi(x) = \int_I q(x,y)\Phi(y)d\lambda(y)$. We have

$$\int_I \Psi d\lambda = \int_I \Phi d\lambda = 0,$$

and

$$|\int_I (\Psi)(z)| = |\int_I q_z \cdot \Phi d\lambda \leq V(q_z).$$

Using Theorem 1.3.10, we obtain $V(P_{\mathfrak{T}} f) \leq \sup_{z\in I} V(q_z) \cdot V(P_T f)$.

(c) See [14]. ■

Our considerations prove the following theorem:

Theorem 5.7.2 *Let T be a position dependent random map which satisfy conditions (A) and (B), and condition (b) of Definition 5.5.1 for $N = 1$. If $q(\cdot,\cdot)$ is a doubly stochastic kernel and $q_z(y)$ is nonincreasing, i.e., $q_z(y_1) \geq q_z(y_2)$, for $y_1 \leq y_2$, then P_T and $P_{\mathfrak{T}}$ are in the same class $\mathcal{S}(\alpha, C)$. In particular, $\mathfrak{T}$ admits an absolutely continuous invariant measure.*

Theorem 5.7.3 *Let T be a position dependent random map which satisfy conditions (A) and (B), and condition (b) of Definition 5.5.1 for $N = 1$. Let $q_\varepsilon(\cdot,\cdot)$ be a family of doubly stochastic kernels satisfying condition (5.18) and such that $q_{\varepsilon,z}$ is nonincreasing for any ε and $z \in I$. Let f^ε denote the invariant density of the Markov process $\mathfrak{T}_\varepsilon$. Then any limit point of $\{f^\varepsilon\}_{\varepsilon>0}$ in the weak topology of $\mathfrak{L}^1$ is a T-invariant density.*

Proof. *The operators $P_{\mathfrak{T}_\varepsilon}$ are all in the same class $\mathcal{S}(\alpha, C)$. Thus $\{f^\varepsilon\}_{\varepsilon>0}$ is a precompact set in $\mathfrak{L}^1$. Let us assume that $f^\varepsilon \to f^*$ as $\varepsilon \to 0$ weakly in $\mathfrak{L}^1$. Then the measures $f^\varepsilon\lambda$ converge to $f^*\lambda$ in the weak topology of measures. By Proposition*

5.6.1, since $q_\varepsilon(\cdot,\cdot)$ satisfy (5.18), $f^*\lambda$ is an invariant measure of T. Thus, f is an invariant density of T. ■

5.8 Additive Noise

A common model of a dynamical system contaminated by noise is given in [13], namely:

$$x_{n+1} = \tau(x_n) + \xi,$$

where τ is an expanding transformation of the interval to itself and ξ is a random variable having a probability density function ϕ. This model is often used in physical applications such as filtering and stochastic control.

Let us suppose that X is a random variable with probability density function $f(x)$, then the probability density function of the random variable $\tau(X)$ is given by the Frobenius-Perron operator $P_\tau f$. Consider the random variable $\tau(X_n) + \xi$ (mod 1), where X_n is a random variable with density function $f(x)$ and ξ has probability density function $\phi(x)$. If we assume that ξ is independent of X_n, the density of $\tau(X_n) + \xi$ is given by $(P_\tau f) \star \phi$, where $\star$ denotes convolution. For the stochastic equation

$$X_{n+1} = \tau(X_n) + \xi, \tag{5.22}$$

it is also assumed that the density of X_{n+1} is $f(x)$. The solution of

$$(P_\tau f) \star \phi = f$$

is the invariant density of the Markov chain (5.22).

Similarly, if we consider instead of τ a random map T we get the equation

$$(P_T f) \star \phi = f$$

for the invariant density of the perturbed process.

In our model, any point x is first moved by T to $T(x)$ and then dispersed around $T(x)$ according to a probability distribution $q(T(x), y)$. This generalizes adding a random variable ξ at each iteration of the process. Moreover, the operator $P_{\mathfrak{T}}$ which was constructed in Section 5.7 generalizes the convolution $(P_T f) \star \phi$. Observe that

$$\left((P_T f) \star \phi\right)(x) = \int (P_T f)(x - y)\phi(y) d\lambda(y).$$

Let $u = x - y$. Then

$$\left((P_T f) \star \phi\right)(x) = \int (P_T f)(u)\phi(x - u) d\lambda(y),$$

which is the same as $P_{\mathfrak{T}}$ when $q(u, x) = \phi(x - u)$.

Chapter 6

Financial Application

In this chapter we find an application for position dependent random maps in finance [7].

6.1 Financial Markets Driven by Random Maps

In this section, we suggest a discrete-time model. We define a multiperiod model as follows:

1. $N+1$ trading dates: $n = 0, 1, \ldots, N$, $\mathbb{T} = \{0, 1, \ldots, N\}$, where the trading Horizon N is treated as the terminal date of the economy activity being modeled.

2. A finite probability space Ω with $K < \infty$ elements:

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_K\},$$

for instance, when dealing with binomial model, the cardinality of Ω is 2.

3. A probability measure $\overline{P}$ on Ω with a $\overline{P}(\omega) > 0$ for all $\omega \in \Omega$.

4. A bank account process $B = \{B_n; n = 0, \dots, N\}$, where B is a stochastic process with $B_0 = 1$, $B_n(\omega) > 0$ for all n and B_n is the value of the bank account at time n. The quantity $r_n \equiv \frac{B_n - B_{n-1}}{B_{n-1}} \geq 0$, $n = 1, \dots, N$ is the interest rate in the interval $(n-1, n)$. We suppose that the interest rate is constant over time and in some situations, with out loss of generality, we suppose it is equal to 1.

5. L risky security processes $s_l = (s_1(n), \dots, s_L(n))$, $n = 0, 1, \dots, N$, where s_l is a Markov process for $l = 1, \dots, L$. $s_l(n)$ is the price of the risky security l at time n. For example, s_l is the price of one share of common stock of a particular corporation. In our discussion, we deal with $L = 1$.

6. Let $\mathbb{F} = \{\mathcal{F}_n; n = 0, \dots, N\}$ be a filtration defined on $[0, 1]$, with the Lebesgue measurable sets $\mathfrak{B}$, where $\mathcal{F}_n$ is the smallest sub-σ-algebra generated by

$$(s_1(0), \dots, s_1(n)).$$

The measure we consider is the invariant measure for the transition function of Markov process s_1.

We assume that the price of the s_l risky security is an adapted, i.e. $s_1(n)$ is $\mathcal{F}_n$ measurable, stochastic process. Thus, the investors will have full knowledge of the past and present prices. For instance , at time n $s_l(n)$ will be known.

The prices of the securities are assumed to be smaller than a finite number (see Remark 6.1.1); i.e., the prices have an upper bound $M \in \mathbb{R}$, $0 < M < \infty$, such that $0 < s_l(n) < M$. We normalize the prices over M so that

$$0 < s_l(n) < 1$$

for $1 \leq l \leq L$ and $n = 0, \ldots, N$.

Remark 6.1.1 *This assumption can be removed if we work on the real line instead of the unit interval. Our goal is to build a random map which drives the risky prices and at the same time has an absolutely continuous invariant measure. The conditions for the existence of absolutely continuous invariant measure of a random map on the unit interval are weaker than those of a random map on the real line (compare the conditions in [1] and [3]). Thus, we can remove the condition that the prices has an upper bound M; however, we have to compensate by imposing stronger conditions which assure the existence of the absolutely continuous invariant measure [3].*

In this chapter, we suggest a model where we give the investor some knowledge about the stochastic process. Here is a description of the process that drives the prices:

Let $T = \{\tau_1, \ldots, \tau_K; p_1(x), \ldots, p_K(x)\}$ be a position dependent random map where τ_k are piecewise monotonic nonsingular transformations, $\tau_k : [0, 1] \to [0, 1]$, defined on the same partition $\mathcal{P}$.

With out loss of generality, we focus our attention on the binomial model. The binomial model is a simple yet very important model for the price of a single risky security. It is commonly used by practitioners, for example, to determine the price of various kinds of stock options. In this model we study one risky security price s_1. At each period there are two possibilities: the security price may go up by the factor u or it may go down by a factor d; i.e., $s_1(n) = u \cdot s_1(n-1)$ or $s_1(n) = d \cdot s_1(n-1)$. The probability of an up move during a period is equal to the parameter $\overline{p}_u$, and the probability of going down is $\overline{p}_d = 1 - \overline{p}_u$. The moves over time are independent of each other.

We introduce a generalization of the binomial model. We assume that the factors u and d are functions of the prices, $u(x) : (0,1) \rightarrow (1, \infty)$ and $d(x) : (0,1) \rightarrow (0,1)$; i.e., at time n, u and d depend on the price of the risky security s_1 at time $n-1$. The examples of u and d are: u and d are constant over subsets of $(0,1)$; u and d are piecewise linear or piecewise non-linear over $(0,1)$. Similarly, the probabilities $\overline{p}_u$ and $\overline{p}_d$ can be constant or price dependent. Price dependent probabilities are more realistic because in practice the probability for the market to go up or down is not constant in time and may depend on current price.

We believe that this generalization of the binomial model is more realistic since the real market does not go up always with the same factor. In all other models, for instance Black Scholes, or models based on time series, the change in the price is not constant.

At time $n = 0$, we estimate the functions u, d, and the probabilities $\overline{p}_u$ and $\overline{p}_d$. For example , we can estimate u and d by the historical data. The only assumption that we need is that u tends to 1 as the price tends to 1. This assumption is natural because as the price approaches our bound M, the price may still go up but by a very small amount. We need this assumption because our transformation τ is defined as a map from $[0, 1]$ to itself. For this reason, the estimation of M is very important in the model.

Once we are given the functions $u(x)$, $d(x)$ and the probabilities $\overline{p}_u$ and $\overline{p}_d$ at time $n = 0$, we can construct the random map T which consists of the transformations τ_u, τ_d and the position dependent probabilities p_u and p_d. The subscript u for τ_u illustrates that the transformation τ_u is the law which moves the price up and the subscript d for τ_d illustrates that the transformation τ_d is the law which moves the price down. The construction of the random map T is straight forward. At time $n + 1$, consider the up price to be $\tau_u(s_n)$ and the down price to be $\tau_d(s_n)$. Also $s_{n+1} = u(s_n) \cdot s_n$ or $s_{n+1} = d(s_n) \cdot s_n$. Therefore, the transformations τ_u and τ_d are given by the following formulas:

$$\tau_u(x) = u(x) \cdot x \qquad \text{and} \qquad \tau_d(x) = d(x) \cdot x. \tag{6.1}$$

Moreover, we extend τ_u and τ_d from (0,1) to the closed interval $[0, 1]$ continuously. For the probabilities, we assume $\overline{p}_u = p_u$ and $\overline{p}_d = p_d$.

We give an example to illustrate our model:

Example 6.1.1 *Suppose that* $u(x), d(x), \overline{p}_u$ *and* $\overline{p}_d$ *are given (for instance they can be calculated from historical data) by:*

$$u(x) = \begin{cases} 2, & 0 < x < \frac{1}{2} \\ \frac{3}{2} - \frac{1}{10x}, & \frac{1}{2} \leq x \leq \frac{2}{3} \\ \frac{3}{4} + \frac{1}{4x}, & \frac{2}{3} < x < 1 \end{cases}, \tag{6.2}$$

$$d(x) = \begin{cases} \frac{4}{5}, & 0 < x < \frac{1}{2} \\ \frac{3}{4} + \frac{1}{10x}, & \frac{1}{2} \leq x \leq \frac{2}{3} \\ \frac{2}{x} - 2, & \frac{2}{3} < x < 1 \end{cases}, \tag{6.3}$$

and

$$\overline{p}_u(x) = \begin{cases} \frac{3}{4}, & 0 \leq x < \frac{1}{2} \\ \frac{3}{4}, & \frac{1}{2} \leq x \leq \frac{2}{3} \\ \frac{1}{4}, & \frac{2}{3} < x \leq 1 \end{cases}, \tag{6.4}$$

$$\overline{p}_d(x) = 1 - \overline{p}_u(x).$$

By the above discussion, we construct a random map $T = \{\tau_u(x), \tau_d(x); p_u(x), p_d(x)\}$, *where*

$$\tau_u(x) = \begin{cases} 2x, & 0 \leq x < \frac{1}{2} \\ \frac{3}{2}x - \frac{1}{10}, & \frac{1}{2} \leq x \leq \frac{2}{3} \\ \frac{3}{4}x + \frac{1}{4}, & \frac{2}{3} < x \leq 1 \end{cases}, \tag{6.5}$$

$$\tau_d(x) = \begin{cases} \frac{4}{5}x, & 0 \leq x < \frac{1}{2} \\ \frac{3}{4}x + \frac{1}{10}, & \frac{1}{2} \leq x \leq \frac{2}{3} \\ 2 - 2x, & \frac{2}{3} < x \leq 1 \end{cases}, \tag{6.6}$$

$p_u(x) = \overline{p}_u(x)$ *and* $p_d(x) = \overline{p}_d(x)$*. For example, if the price of the risky security at time* $n = 0$ *is* 0.25*, then the orbit of the price at times* $n = 1, 2$ *is given by:*

$$
\begin{array}{cccccc}
 & & & & \nearrow & \xrightarrow{p_u=0.75,u=1.3} \tau_u(0.5) = 0.65 \\
 & & \xrightarrow{p_u=0.75,u=2} \tau_u(0.25) = 0.5 & & & \\
 & \nearrow & & & \searrow & \xrightarrow{p_d=0.25,d=0.75} \tau_d(0.5) = 0.475 \\
s_1(0) = 0.25 & & & & & \\
 & \searrow & & & \nearrow & \xrightarrow{p_u=0.75,u=2} \tau_u(0.2) = 0.4 \\
 & & \xrightarrow{p_d=0.25,d=0.8} \tau_d(0.25) = 0.2 & & & \\
 & & & & \searrow & \xrightarrow{p_d=0.25,d=0.8} \tau_d(0.2) = 0.16
\end{array}
\tag{6.7}
$$

Remark 6.1.2 *The random map* T *of Example 6.1.1 satisfies the assumptions of Theorem 3.1.1. Thus, it admits an absolutely continuous invariant measure* μ*. In Figure 6.1, the histogram approximating the invariant density of* T *is shown after 2,000,000 iterations of random map* T*. The invariant density allows us to find the following probability:* $\mu\{x : T(x) \in (\delta_1, \delta_2)\} = \mu(\delta_1, \delta_2)$*, where* $\mu = f^*\lambda$*,* f^* *is the invariant density.*

Under some conditions on $u(x)$ and $d(x)$, we prove that $\tau_u = u(x) \cdot x$ and $\tau_d = d(x) \cdot x$ are arbitrage free prices. First, we need some definitions.

We define a trading strategy $H = (H_0, H_1 \dots H_L)$ as a vector of stochastic processes $H_l = \{H_l(n); n = 0, \dots N\}$, $l = 0, 1, \dots, L$. $H_l(n)$, $l = 1, \dots L$, is the number of units

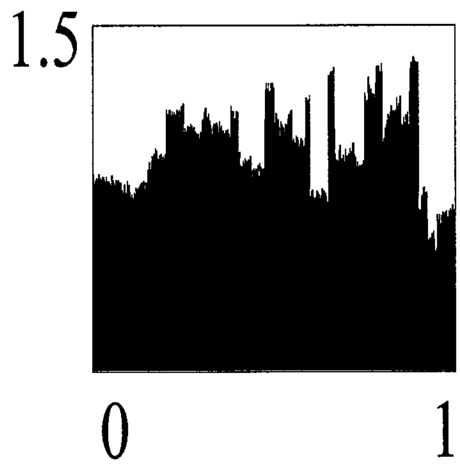

Figure 6.1:

The invariant density of T in Example 6.1.1, histogram after 2 000 000 iterations.

of security with price s_l that the investors owns from time $n-1$ to time n, whereas $H_0(n)B_{n-1}$ is the amount of money invested in the bank account at time $n-1$.

We define a value process by $V = \{V_n; n = 0, \dots, N\}$:

$$V_0 = H_0(1)B_0 + \sum_{l=1}^{L} H_l(1)s_l(0) \tag{6.8}$$

$$V_n = H_0(n)B_n + \sum_{l=1}^{L} H_l(n)s_l(n) \quad n \geq 1. \tag{6.9}$$

A trading strategy H is said to be self-financing if

$$V_n = H_0(n+1)B_n + \sum_{l=1}^{L} H_l(n+1)s_l(n), \quad n = 1, \dots, N-1,$$

i.e., the time n values of the portfolio just before and just after any time n transactions are equal. Intuitively, if no money is added to or withdraw from the portfolio between times $n = 0$ and time N, then any change in the portfolio's value must be due to a gain or loss in the investments.

Definition 6.1.1 *An arbitrage opportunity in the case of a multiperiod securities market is some trading self-financing strategy H such that:*

1. $V_0 = 0$,

2. $V_N \geq 0$,

3. $E[V_N] > 0$.

Definition 6.1.2 *A risk neutral probability measure (also called a martingale measure) is a probability measure Q such that*

1. $Q(J) > 0$ *for all* $J \in [0, 1]$,

2. *the discounted price process.*

$$s_l^*(n) \equiv \frac{s_l(n)}{B_n} \quad n = 0, \ldots, N \quad l = 0, \ldots, L$$

   is a martingale under Q for every $l = 0, \ldots, L$.

In other words, a risk neutral probability measure Q satisfies

$$E_Q[s_l^*(n+t)|\mathcal{F}_n] = s_l^*(n), \quad n \geq 0, t \geq 1. \tag{6.10}$$

One of the principal results in finance is the following :

Theorem 6.1.1 *There are no arbitrage opportunities if and only if there exists a martingale measure Q.*

In our case, there are no arbitrage opportunities if and only if the process s_1 satisfy

$$E_Q[\frac{s_1^*(n+1)}{B_n}|\mathcal{F}_n] = \frac{s_1^*}{B_n}, \quad n \geq 0. \tag{6.11}$$

In the binomial model, if we suppose that the interest rate, r, is constant over the time, then by using (6.1) and (6.11) we obtain for $q(x) = p_u(x)$:

$$q(x)\left[\frac{u(x)-1-r}{1+r}\right] + (1-q(x))\left[\frac{d(x)-1-r}{1+r}\right] = 0 \tag{6.12}$$

for all x.

Observe that the martingale measure Q depends on s_0 because the functions u and d depends on the initial price. Let $q(s_0)$ be the conditional probability that the next move is an up move given the information $\mathcal{F}_n$. Hence:

$$q(s_0) = \frac{1+r-d(s_0)}{u(s_0)-d(s_0)}$$

for all $\mathcal{F}_n$ and n. Since Q is a probability, it is easy to see that:

$$u(x) > 1+r > d(x) \quad \text{for all } x. \tag{6.13}$$

Thus, if we assume that $u(x)$ and $d(x)$ satisfy (6.13), which is satisfied by the binomial model built in this section, then s_1 is arbitrage free price.

If the interest rate changes with the time, the we require the functions u and d to satisfy a more general condition:

$$u(x) > 1+r_n > d(x) \quad \text{for all } x \text{ and } n = 0, \ldots, N. \tag{6.14}$$

6.2 Arbitrage Opportunities

In this section, we suggest a discrete-time model of a market which is not arbitrage free. Discrete time models which are not arbitrage free are believed to be realistic and may lead to a deeper understanding of the nature of markets.

We define our multiperiod model as in Section 6.1 and assume that (6.13) is satisfied. The prices will be driven by the random map T_G which is a perturbation of the random map T(See Section 5.1 and assumptions 6.15). Thus, the price of stock l at time n will be given by

$$s_l(n) = T_G(s_l(n-1)),$$

where $n = 1, \ldots, N$.

In this model, the g's create the arbitrage opportunities. Hence, the prices will be really driven by the random map T_G, associated with the random map $T = \{\tau_u, \tau_p; p_u(x), p_d(x)\}$. The g's are considered small perturbations which are unknown to the investor. Moreover, since in our model we assume $\tau_{u,g}(x) > x$ and $\tau_{d,g}(x) < x$ for all x, and we want the same properties of the perturbed maps, the perturbation T_G is slightly modified. Assume τ_u is increasing. For τ_u we define $\tau_{u,g}(x)$ as follows. Let $x \in (a_i, a_{i+1}]$

$$\tau_{u,g}(x) = \begin{cases} \tau_u(x) + g(x), & x < \tau_u(x) + g(x) \leq 1 \\ \tau_u(x) + g(x) - (\tau_u(a_{i+1}) + g(a_{i+1})), & \tau_u(x) + g(x) > 1 \\ \tau_u(x) + g(x) - (\tau_u(a_i) + g(a_i)), & \tau_u(x) + g(x) < x \end{cases} . \quad (6.15)$$

The definitions for decreasing τ_u and for τ_d are similar. All perturbation results of Chapter 5 hold for this model as well.

When the market is very sensitive, the g's may cause considerable changes in the outcomes of the risky security price. Therefore, in the following section, instead of viewing the long term behavior of the outcomes themselves, which we do not really know because the g's are unknowns, we will view the probabilities of these outcomes.

6.3 Frequency of Future Prices

In this section we answer the following question: starting with price $s_l(0)$, With what frequency do the points of the orbit $\{s_l(0), T_G(s_l(0)), T_G^2(s_l(0)), \ldots, T_G^{n-1}(s_l(0))\}$ occur in the set E? We answer this question when T, T_G, G satisfy the assumptions of Section 5.3.

Example 6.3.1 *In this example, we suppose that in the market the changes (d and u) in the price are constants when the price is smaller than half of the maximum price (M). Then, when the price becomes bigger than half of the maximum price, the change in the price to go up (u) is decreasing and the change in the price to go down (d) is increasing. For example, this situation may occur if we suppose that some of the investors sell their shares when they see an increase in the price.*

We model $u(x), d(x), \overline{p}_u$ and $\overline{p}_d$ by (for instance they can be calculated from his-

torical data):

$$u(x) = \begin{cases} 2, & 0 < x < \frac{1}{2} \\ \frac{5}{4} + \frac{1}{10x}, & \frac{1}{2} \le x \le \frac{2}{3} \\ \frac{3}{4} + \frac{1}{4x}, & \frac{2}{3} < x < 1 \end{cases}, \tag{6.16}$$

$$d(x) = \begin{cases} \frac{1}{2}, & 0 < x < \frac{1}{2} \\ \frac{3}{4} - \frac{1}{8x}, & \frac{1}{2} \le x \le \frac{2}{3} \\ \frac{3}{2} - \frac{1}{2x}, & \frac{2}{3} < x < 1 \end{cases}, \tag{6.17}$$

and

$$\overline{p}_u(x) = \begin{cases} 0.8, & 0 \le x < \frac{1}{2} \\ 0.725, & \frac{1}{2} \le x \le \frac{2}{3} \\ 0.4, & \frac{2}{3} < x \le 1 \end{cases}, \tag{6.18}$$

$$\overline{p}_d(x) = 1 - p_1(x).$$

Observe that $u(x)$ *is decreasing and* $d(x)$ *is increasing. From* $u(x), d(x), \overline{p}_u(x)$ *and* $\overline{p}_d(x)$, *we construct a random map* $T = \{\tau_u(x), \tau_d(x); p_u(x), p_d(x)\}$,

$$\tau_u(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ \frac{5}{4}x + \frac{1}{10}, & \frac{1}{2} \le x \le \frac{2}{3} \\ \frac{3}{4}x + \frac{1}{4}, & \frac{2}{3} < x \le 1 \end{cases}, \tag{6.19}$$

$$\tau_d(x) = \begin{cases} \frac{1}{2}x, & 0 \le x < \frac{1}{2} \\ \frac{3}{4}x - \frac{1}{8}, & \frac{1}{2} \le x \le \frac{2}{3} \\ \frac{3}{2}x - \frac{1}{2}, & \frac{2}{3} < x \le 1 \end{cases}, \tag{6.20}$$

$p_u(x) = \overline{p}_u(x)$ *and* $p_d(x) = \overline{p}_d(x)$. *Now, observe that*

$$\sup_x \frac{p_u(x)}{|\tau_u'(x)|} + \sup_x \frac{p_d(x)}{|\tau_d'(x)|} = 0.58 + 0.4 = 0.98 < 1.$$

Remark 6.3.1 *The random map T of Example 6.3.1 satisfies the assumptions of Theorem 3.1.1. Thus, it admits an absolutely continuous invariant measure μ. In the left hand side part of Figure 2, the histogram approximating the invariant density of T is shown after 2,000,000 iterations. The invariant density allows us to find the following probability: $\mu\{x : T(x) \in (\delta_1, \delta_2)\} = \mu(\delta_1, \delta_2)$, where $\mu = f^*\lambda$, f^* is the invariant density.*

Remark 6.3.2 *In most cases, condition 2 is satisfied. In Example 6.1.1, $u(x)$ is not decreasing between $1/2$ and $2/3$ and $d(x)$ is not increasing between $1/2$ and $2/3$. Moreover, some extreme situations were allowed in the same example. For instance, in the interval $(0, 1/2)$, we have a big gain with a high probability and in the interval $(2/3, 1)$ we have a big loss with a high probability.*

The results of Chapter 5 say that the invariant densities of T_G and T are "very close". Thus, to find the relative frequency that points of the orbit

$$\{s_l(0), T_G(s_l(0)), T_G^2(s_l(0)), \dots, T_G^{n-1}(s_l(0))\}$$

occur in the set E, it is enough to find f^* and use the ergodic Theorem 2.4.1. The method of approximating f^* will be discussed in the following section. For now, let us

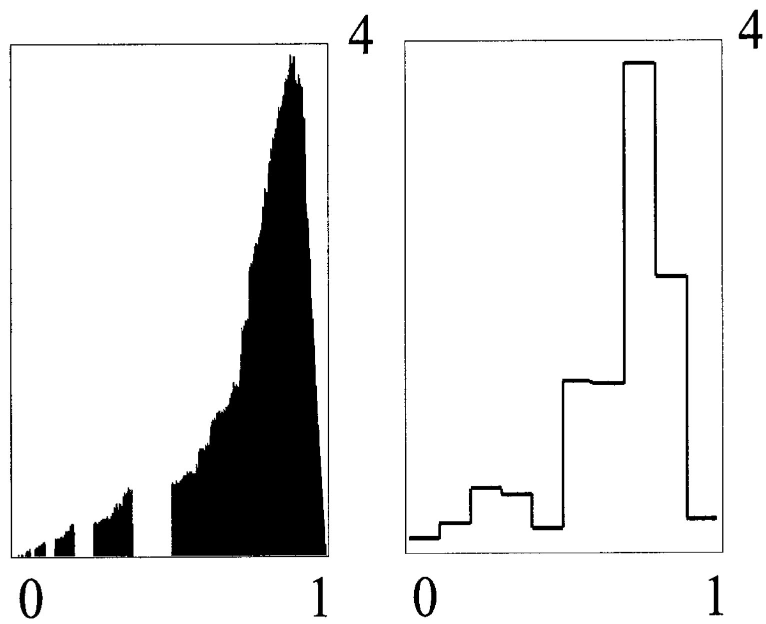

Figure 6.2:

On the left: The histogram of invariant density of T of Example 6.3.1.

On the right: The density of Markov random map T_M.

assume that we have already approximated f^* and illustrate the meaning of Theorem 2.4.1:

For a price $s_l(0)$ we are interested in the following question: With what frequency do the points of the orbit $\{s_l(0), T_G(s_l(0)), T_G^2(s_l(0)), \ldots, T_G^{n-1}(s_l(0))\}$ occur in the set E. Clearly, $T_G^i(s_l(0)) \in E$ if and only if $\chi_E(T_G^i(s_l(0))) = 1$. Thus, the number of points of the orbit $\{s_l(0), \ldots, T_G^{n-1}(s_l(0))\}$ in E is equal to $\sum_{i=0}^{n-1} \chi_E(T_G^i(s_l(0)))$, and the relative frequency of the elements $\{s_l(0), T_G(s_l(0)), T_G^2(s_l(0)), \ldots, T_G^{n-1}(s_l(0))\}$ in E equals to $\frac{1}{n}\sum_{i=0}^{n-1} \chi_E(T_G^i(s_l(0)))$.

6.4 Approximation of the Invariant Density

In this section, we approximate the invariant density f^*. Note that, if the transformations τ_u and τ_d are Markov [14], and the probabilities p_u and p_d are piecewise constant, we can find the exact unique invariant density f^* using the methods of [23]. When the transformations are not Markov, the invariant density can be approximated by matrix operators [23]. This method is known as Ulam's method. Now, we are going to approximate the invariant density of T in Example 6.3.1. First, we find two Markov transformations τ_{u_m} and τ_{d_m} which approximate τ_u and τ_d respectively.

Let

$$\tau_{u_m}(x) = \begin{cases} 2x, & 0 \le x < 0.5 \\ 2x - 0.3, & 0.5 \le x < 0.6 \\ x + 0.1, & 0.6 \le x < 0.7 \\ x, & 0.7 \le x \le 1 \end{cases}, \tag{6.21}$$

and

$$\tau_{d_m}(x) = \begin{cases} x, & 0 \le x < 0.1 \\ x - 0.1, & 0.1 \le x < 0.3 \\ x - 0.2, & 0.3 \le x < 0.5 \\ 2x - 0.9, & 0.7 \le x < 0.9 \\ 2x - 0.1, & 0.9 \le x \le 1 \end{cases}. \tag{6.22}$$

Observe that τ_{u_m} and τ_{d_m} are Markov transformations on the common partition $[i/10, (i+1)/10)_{i=0}^{9}$. The Frobenius-Perron operator of a Markov transformation can be represented by a matrix [14]. Also, the Frobenius-Perron operator of the random map T_M, $T_M = \{\tau_{u_m}, \tau_{d_m}; p_u, p_d\}$, [23] is represented by the following matrix

$$M = \Pi_u M_u + \Pi_d M_d, \tag{6.23}$$

where M_u, M_d are the matrices of $P_{\tau_{um}}$ and $P_{\tau_{dm}}$ respectively, and Π_u, Π_d are the diagonal matrices of $p_u(x)$ and $p_d(x)$ respectively. We have

$$M_u = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_d = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

and

$$M = \begin{pmatrix} 0.6 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0.2 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0.2 & 0.275 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0.275 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0. & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0.3625 & 0.725 & 0.4 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0.3625 & 0 & 0 & 0.7 & 0.3 \\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0.7 \end{pmatrix},$$

where $p_u = (0.8, 0.8, 0.8, 0.8, 0.8, 0.725, 0.725, 0.4, 0.4, 0.4);$, $p_d = 1 - p_u$, $\mathbf{Id}_{10}$ is 10×10 identity matrix. The invariant density of T_M is

$$f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}), \quad f_i = f_{|I_i}, \quad i = 1, 2, \ldots, 9, \tag{6.24}$$

normalized by

$$f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8 + f_9 + f_{10} = 10, \tag{6.25}$$

and satisfying equation $fM = f$. Then,

$$f_1 = 0.11591, \quad f_2 = 0.23183, \quad f_3 = 0.48548, \quad f_4 = 0.44184, \quad f_5 = 0.19419,$$

$$f_6 = 1.28694, \quad f_7 = 1.26949, \quad f_8 = 3.64250, \quad f_9 = 2.07290, \quad f_{10} = 0.25892 \ .$$

The T_M-invariant density is shown on the right hand side of Figure 6.3. Comparing left and right parts of Figure 6.3, we see that the invariant density of T_M approximates the invariant density of T in Example 6.3.1. Notice that, had we used Markov transformations on a finer partition than that in the above construction, we would have obtained a better approximation for the invariant density of T in Example 6.3.1 [23].

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