# A New Three-parameter Lifetime Distribution with Bathtub Shape or Increasing Failure Rate Function and A Flood Data Application

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# Abstract


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In this thesis, a new three-parameter lifetime distribution with bathtub shaped or increasing failure rate function is given by adding a shape parameter in the distribution which is suggested by Chen(2000). It can be used for modeling lifetime data from reliability, survival analysis and various extreme value data. The detailed analysis of this distribution includes density shapes, tail classification and character, hazard function shapes and the extremes and extreme spacings distributions. The confidence intervals for the parameters are discussed by using bootstrap method. Its application in modeling extreme value data is illustrated by the floods data of the Floyd River at James, Iowa.

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# Chapter 1

# Introduction

## 1.1 Introduction

As a human attribute, reliability has been used for a long time. For technical systems, however, the reliability concept has been applied for not more than 60 years. It is observed that the lifetime distribution of many electronic, mechanical and electro-mechanical products often has non-monotone failure rate functions. In many reliability analyses, especially over the life-cycle of the product, it usually involves high initial failure rates (infant mortality) and eventual high failure rates due to aging and wearout, indicating a bathtub shape failure rate.

In fact, from the 60's and 70's, researchers got interested in the distributions with non-monotone hazard function, such as bathtub shape and unimodal hazard functions and noticed that distributions with one or two parameters like the Weibull distribution have very strong restrictions on the data. Smith and Bain (1975) gave the exponential power distribution whose hazard function has a bathtub shape. Mudholkar and Srivastava (1993) provided an exponentiated-Weibull distribution. This distribution has monotone increasing, monotone decreasing, bathtub or unimodal failure rate depending on the different parameter ranges. Chen (2000) proposed a two-parameter lifetime distribution with bathtub shape or increasing hazard function. Its distribution function is:

$$F_C(x) = 1 - e^{\lambda(1-e^{x^\beta})}, \qquad (x > 0) \tag{1.1}$$

In this thesis, a new three-parameter distribution is given, which is obtained by adding a shape parameter to the Chen's two-parameter lifetime distribution (2000). Its cumulative distribution function is given by:

$$F(x) = \left(1 - e^{\lambda(1-e^{x^\beta})}\right)^\alpha, \qquad (x > 0) \tag{1.2}$$

where $\lambda > 0$, $\beta > 0$ and $\alpha > 0$ are the parameters. The purpose of this thesis is to provide a structural analysis of this distribution in a method similar to that of the exponentiated-Weibull distribution by Mudholkar and Hutson(1996) and the two-parameter distribution by Chen (2000).

Chapter 1 includes an introduction. A brief review of basic concepts and theorems of survival analysis are given in Chapter 2. Especially, the concepts of extremes, extreme spacing and tail classification are discussed in detail, along with the basic bootstrap method for a confidence interval for a parameter.

Chapter 3 contains the basic facts about the new distribution and some properties of its density function, along with a discussion of the shape and properties of the corresponding hazard function. Chapter 4 presents an analysis of the extremes and extreme spacings of this distribution. It also provides the classification according to their tail lengths. In Chapter 5, we provide an application of this distribution in the extreme-value analysis using flood data for the Floyd River at James, Iowa. We use maximum loglikelihood method to estimate the parameters and give the confidence intervals for each parameter by using the bootstrap method and the likelihood ratio test to test some hypothesis about the distribution. The empirical TTT transform is used to justify the appropriateness of this distribution. Here we also provide a comparison of this distribution with well known exponentiated-Weibull distribution by using the same data set.

The final part contains some remarks and conclusions.

# Chapter 2

# Review of The Definitions and Theorems

## 2.1 Introduction to Basic Concepts

Until the 1960s, reliability was defined as: " the probability that an item will perform a required function under stated conditions for a stated period of time." In fact, reliability analysis includes a variety of statistical techniques for analyzing positive-valued random variables. These techniques were primarily developed in the medical and biological sciences, and they were also widely used in the social and econnomic sciences, as well as in engineering (reliability and failure time analysis).

### 2.1.1 Reliability Function and Failure Rate

Let random variable $T$ be the lifetime or time to failure of a component, having probability density function (p.d.f) $f(t)$ and distribution function (d.f) $F(t)$. The probability that the component survives beyond some time $t$ is called the reliability (survival) $R(t)$ of the component. Thus,

$$R(t) = 1 - F(t) = P\{T > t\}, \qquad (t > 0) \tag{2.1}$$

The component is assumed to be working properly at time $t = 0\,(R(0) = 1)$ and no component can work forever without failure $(\lim_{t \to +\infty} R(t) = 0)$. $R(t)$ is a monotone non-increasing function of t. Reliability has no meaning for $t < 0$. $F(t)$ is called the unreliability.

The probability that a component will fail in the interval $(t, t + \Delta t]$ given that the component is working at time t is:

$$\begin{aligned} P(t < T \leq t + \Delta t | T > t) &= \frac{P(t < T \leq t + \Delta t)}{P(T > t)} \\ &= \frac{F(t + \Delta t) - F(t)}{R(t)} \end{aligned} \tag{2.2}$$

By dividing this probability by the length of the time interval $\Delta t$ and letting $\Delta t \to 0$, we can get the failure rate (hazard) function $h(t)$ at time t :

$$\begin{aligned} h(t) &= \lim_{\Delta t \to 0} \frac{P(t < T \leq t + \Delta t | T > t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t} \frac{1}{R(t)} \\ &= \frac{f(t)}{R(t)} \end{aligned} \tag{2.3}$$

A failure rate function may be classfied as an IFR (increasing failure rate) , or DFR (decr-

easing failure rate).

Since

$$f(t) = \frac{d}{dt} F(t) = -R'(t) \tag{2.4}$$

then

$$h(t) = -\frac{R'(t)}{R(t)} = -\frac{d}{dt} \log R(t) \tag{2.5}$$

then using $R(0) = 1$, we have

$$\int_0^t h(t) dt = -\log R(t)$$

and

$$R(t) = e^{-\int_0^t h(u)du} \tag{2.6}$$

**Remark**: From the above concepts and formulae, the reliability function $R(t)$ and the distribution function $F(t) = 1 - R(t)$ are uniquely determined by the failure rate function $h(t)$. We can also find the relationships between the functions $F(t), f(t), R(t)$ and $h(t)$ as given below:

Table 2.1: relationships between the $F(t), f(t), R(t)$ and $h(t)$

| Expressed by | $F(t)$ | $f(t)$ | $R(t)$ | $h(t)$ |
|---|---|---|---|---|
| $F(t) =$ | - | $\int_0^t f(u)du$ | $1 - R(t)$ | $1 - e^{-\int_0^t h(u)du}$ |
| $f(t) =$ | $\frac{d}{dt}F(t)$ | - | $-\frac{d}{dt}R(t)$ | $h(t)e^{-\int_0^t h(u)du}$ |
| $R(t) =$ | $1 - F(t)$ | $\int_t^\infty f(u)du$ | - | $e^{-\int_0^t h(u)du}$ |
| $h(t) =$ | $\frac{dF(t)/dt}{1-F(t)}$ | $\frac{f(t)}{\int_t^\infty f(u)du}$ | $-\frac{d}{dt}\log R(t)$ | - |

In order to understand the relationships between them, the graph is drawn for example $f(t) = \frac{0.4}{(0.2t+1)^3}$

### 2.1.2 Bathtub Hazard Function

In many industrial applications, the hazard function is observed to have the so-called bathtub shape as shown in figure 2.2. The failure rate with bathtub shape is often high at the beginning phase. This can explain that there must be undiscovery defects in the items, these will show up when these items are activated. When the item has survived in an undiscovery defect phase, the failure rate often stabilizes at a level

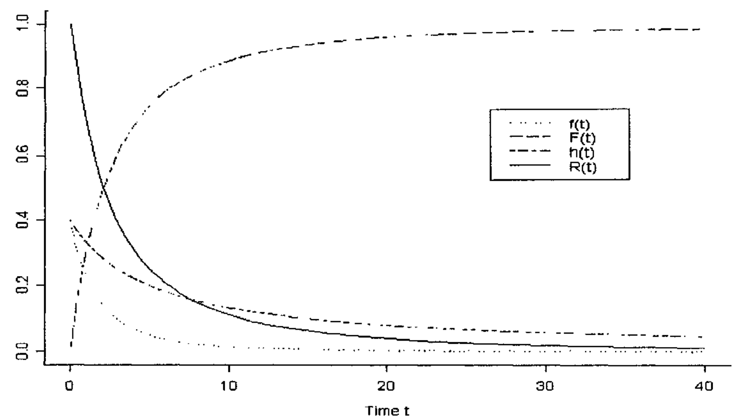


Figure 2.1: The relationships between the $F(t)$, $f(t)$, $h(t)$ and $R(t)$

where it remains for a certain time until it starts to increase as the items start to wear out. From the shape of bathtub hazard curve, the lifetime of an item can be divided into three parts: the *burn-in period*, the *useful period* and the *wear-out period.* A bathtub curve is called degenerate if either the decreasing or increasing part is not present (i.e., it is either always increasing or always decreasing).

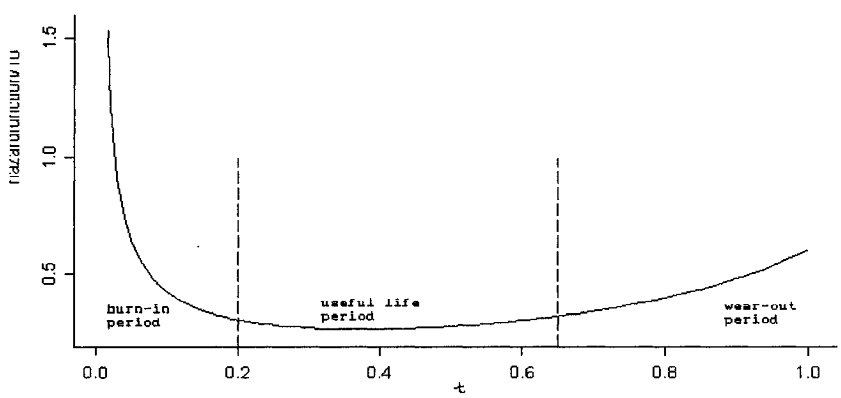


Figure 2.2: The bathtub curve

During the burn-in period, the failure rate is expected to drop with age. In the second period, failure rate is approximately constant and exponential model is usually acceptable, whereas components begin to reach the *wear-out phase* with increasing of the failure rate. The wear-out failure is the outcome of a depleted process due to abrasion, fatigue, and so on.

## 2.2 Classification of Probability Laws by Tail Behavior

The probability laws can be classified as one of three types of tail behavior: short, medium, or long. Parzen (1979) used the limiting behavior of the density-quantile function $f(Q(U))$ (as $U$ goes to 0 or 1, where $Q(U)$ is the inverse function of distribution function $F(t)$) to classify the probability laws. The right-tail behavior of a probability law with density function $f(t)$ is classified according to the value of the right-tail exponent $\alpha_0$, defined by

$$f(Q(U)) \sim (1-U)^{\alpha_0}, \qquad (U \to 1)$$

When the density function is differentiable, the definition of the tail exponent $\alpha_0$ can be defined as:

$$\alpha_0 = \lim_{U \to 1^-} \frac{(1-U)J(U)}{f(Q(U))}$$

where

$$J(U) = \frac{-f'(Q(U))}{f(Q(U))}, \tag{2.7}$$

The above formula is also equivalent to:

$$\alpha_0 = \lim_{t \to \infty} \frac{-(1-F(t))f'(t)}{f(t)^2}$$

Then right-tail classification would have the following three categories:

$$\begin{cases} \alpha_0 < 1: & \text{short tail} \\ \alpha_0 = 1: & \text{medium tail} \\ \alpha_0 > 1: & \text{long tail} \end{cases}$$

A similar classification for a left tail holds in terms of the left-tail exponent $\alpha_1$, defined as:

$$\alpha_1 = \lim_{U \to 0^+} \frac{-UJ(U)}{f(Q(U))}$$

equivalently,

$$\alpha_1 = \lim_{t \to 0^+} \frac{f'(t)F(t)}{f(t)^2}$$

with the following rules similar to the left tail classification:

$$\begin{cases} \alpha_1 < 1: & \text{short tail} \\ \alpha_1 = 1: & \text{medium tail} \\ \alpha_1 > 1: & \text{long tail} \end{cases}$$

When $\alpha_0 = 1$, Parzen's (1979) medium-tailed distribution may further be classified by limiting value of the hazard hunction $h(t)$. Suppose:

$$h_1 = \lim_{U \to 1^-} \frac{1-U}{f(Q(U))} = \lim_{U \to 1^-} \frac{1}{h(Q(U))} = \lim_{t \to \infty} \frac{1}{h(t)}$$

So a right-tail density-quantile classification has five categories:

$$\begin{cases} \alpha_0 < 1: & \text{short tail} \\ \alpha_0 = 1, h_1 = 0: & \text{medium} - \text{short} \\ \alpha_0 = 1, 0 < h_1 < \infty: & \text{medium} - \text{medium} \\ \alpha_0 = 1, h_1 = \infty: & \text{medium} - \text{long} \\ \alpha_0 > 1: & \text{long tail} \end{cases}$$

A similar classification for a left tail holds with $\alpha_1$ replacing $\alpha_0$ and $h_0 = \lim_{U \to 0^+} \frac{U}{f(Q(U))}$ $= \lim_{t \to 0^+} \frac{F(t)}{f(t)}$ replacing $h_1$. In the following chapters, we will give the quantile, density-quantile function, classification of the tails and their relations to extreme value theory for this new distribution.

### 2.2.1 Extremes and Extreme Spacings

Extreme value and extreme spacings distributions are elegant important parts of statistical theory and practice. The asymptotic distributions of the smallest and the largest observations in a random sample have interested many prominent statisticians including Fisher, Fréchet, Gumbel and Gnedenko, from the earliest days of modern statistics. The large sample distributions of the extreme value have been traditionally used to classify the tail-behaviors of populations.

**Definition 1:** Let $X_1, X_2, \ldots X_n$ be a random sample from a random variable X having continuous distribution function $F(y)$ and density function $f(y)$, and let $X_{(1)} = X_{1:n} \leq \ldots \leq X_{n:n} = X_{(n)}$ be the corresponding order statistics. $S_{1:n} = X_{(2)} - X_{(1)}$ and $S_{n:n} = X_{(n)} - X_{(n-1)}$ are the extreme spacings (ES).

**Definition 2:** If, as $n \to \infty$, $S_{n:n}$ converges to 0 in probability, then the right tail is called ES-short. If $S_{n:n}$ diverges in probability, then the tail is called ES-long. It is called ES-medium if $S_{n:n}$ remains bounded but non-zero in probability. $S_{1:n} \equiv X_{2:n} - X_{1:n}$ can be used similarly to describe the left tails.

The classical extreme value classification is a little crude. And exponential and normal distributions have medium right tails. To refine it, Schuster (1984) proposed using limiting distributions of the extreme spacings to divide the medium tail distributions into three subclasses.

**Theorem 1:** [Fisher and Tippett (1928), Gnedenko (1943)] Let $\{X_n\}$ be a sequence of $i.i.d$ random variables. If there exist costant $c_n > 0$, $d_n \in R$ and some non-degenerate distribution function $H$, such that, for $M_n = max\{X_1, X_2, \ldots, X_n\}$,

$$\frac{M_n - d_n}{c_n} \xrightarrow{d} H$$

then $H$ belongs to one of the three standard extreme value distribution.

$$\text{Fréchet:} \qquad \Phi_\alpha(x) = \begin{cases} 0, & \text{if} x \leq 0, \\ e^{-x^{-\alpha}}, & \text{if} x > 0. \end{cases}$$

$$\text{Weilbull:} \qquad \Psi_\alpha(x) = \begin{cases} e^{-(-x)^\alpha}, & \text{if} x \leq 0, \\ 1, & \text{if} x > 0. \end{cases}$$

$$\text{Gumbel:} \qquad \Lambda(x) = e^{-e^{-x}}, x \in R.$$

This theorem is equivalent to saying that as $n \to \infty$, the limiting distribution of $a_n X_{n:n} + b_n$, for suitably chosen $a_n$ and $b_n$, must be that of $Y^{-1/\alpha}$ or $-Y^{1/\alpha}$ or $-\log Y$ for some $\alpha > 0$. Where $Y$ is a standard exponential random variable. The corresponding population is then said to have a long or short or medium tail, respectively.

**Theorem 2:** [Schuster (1984)] Suppose $h_0$ exists and is possibly infinite. Then

1. $h_0 = 0$ if and only if $S_{1:n} = o_p(1)$.
2. $h_0 = a, 0 < a < \infty$, if and only if $S_{1:n} = O_p(1)$ but $S_{1:n} \neq o_p(1)$.
3. $h_0 = \infty$ if and only if $S_{1:n} \xrightarrow{P} \infty$.

It implies that the distribution can be classified as an ES short, ES medium, or ES long left tail by statement 1, 2, 3 in this theorem respectively.

**Lemma:** [Schuster (1984)] If $\beta(F) = inf\{x : F(x) > 0\} > -\infty$, then F has an ES short left tail.

**Lemma:** [Schuster (1984)] Suppose the distribution $F$ has no mgf, then $F$ has an ES long right tail.

**Lemma:** The ES short- (right-) tailed distributions are composed of the refined Parzen(RP) short- and the RP medium-short tailed distributions. The RP medium-medium distributions are the ES medium tailed, and the RP long and the RP medium-long are the ES long tailed.

These theorems give a physical interpretation of the short-, medium- and long-tail distributions in terms of the limiting size of the extreme spacings in a random sample. Now divide the five RP classes into three extreme spacing(ES) groups:

(i) ES short =RP short, RP medium-short

(ii) ES medium =RP medium-medium

(iii) ES long =RP medium-long, RP long

Generally speaking, an ES short-tailed distribution will rarely have outliers, an ES medium-tailed distribution will occasionally have some outliers, and an ES long-tailed distribution will often have extreme outliers.

**Theorem 3:** When $n$ goes to infinity,both $nU_{1:n}$ and $n(1-U_{n:n})$ converge in law to the standard exponential r.v.Z.

Proof:

Let $U_1, U_2, \ldots, U_n$ be a random sample from a uniform(0,1) population, and let $U_{1:n}, U_{2:n}, \ldots, U_{n:n}$ denote the sample order statistics. By the formula, we can get the density function of $f_{U_{1:n}}(u_1)$:

$$\begin{aligned}
f_{U_{1:n}}(u_1) &= \frac{n!}{(n-1)!}[F(u_1)]^{1-1}[1-F(u_1)]^{n-1}f(u_1), \qquad (0 < u_1 < 1) \\
&= n[1-\frac{u_1}{1}]^{n-1} \\
&= n(1-u_1)^{n-1} \\
\text{Then: } P\{nU_{1:n} \le t\} &= P\{U_{1:n} \le \frac{t}{n}\} \\
&= \int_0^{\frac{t}{n}} f_{U_{1:n}}(u_1)du_1 \\
&= \int_0^{\frac{t}{n}} n(1-u_1)^{n-1}du_1 \\
&= 1-(1-\frac{t}{n})^n
\end{aligned}$$

The limit value of $P\{nU_{1:n} \le t\}$ is :

$$\lim_{n\to\infty} P\{nU_{1:n} \le t\} = \lim_{n\to\infty} 1-(1-\frac{t}{n})^n = 1-e^{-t}$$

So $nU_{1:n}$ converges in law to the standard exponential r.v.Z.

we also can get that $n(1-U_{n:n})$ converges in law to the standard exponential r.v. $Z$ by using the similar proof. □

**Lemma** :(cf.[Mudholkar and Hutson(1996)]) For any sequence $Y_n$ of random variables such that $Y_n \xrightarrow{L} Y$ in law as $n \to \infty$, and $g_n(y)$ converges uniformly to g(y) over all compact sets, then $g_n(Y_n) \xrightarrow{L} g(Y)$.

## 2.3 Bootstrap Method

Bootstrap method [see Efron and Tibshirani (1985)] is a recently developed technique for making certain kinds of statistical inferences. It is only recently developed because it requires computer to simplify the intricate calculation of traditional statistical theory. Now researchers need no longer rely on asymptotic theory to estimate the distribution of a statistic. Instead, they may use resampling method which return inferential results for either normal or non-normal distributions. Here we mainly use this method to get the confidence interval (empirical percentiles and $BC_a$ (Bartlett-Correction adjusted) percentiles [see Davison amd Hinkley (1997)].

Suppose original sample $x = (x_1, x_2, \ldots, x_n)$, $\hat{\theta}$ is the estimator of $\theta$ and B is the number of resamples. In order to give the distribution " bootstrap" of the estimator, at first:

(1) Generate a large number $B$ of bootstrap samples of size n $x_b^* = (x_{1b}^*, x_{2b}^*, \ldots, x_{nb}^*)$, $b = 1, 2, ..., B$.

(2) Calculate an estimator $\hat{\theta}^*_{(b)}$ using $x_b^*$

Then

$$\hat{a} = \frac{\sum_{i=1}^{n}(\hat{\theta}_{(.)} - \hat{\theta}_{(i)})^3}{6(\sum_{i=1}^{n}(\hat{\theta}_{(.)} - \hat{\theta}_{(i)})^2)^{\frac{3}{2}}}$$

where:

$\hat{\theta}_{(.)} = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_{(i)}$;

$\hat{\theta}_{(i)}$: an estimator of $\theta$ gotten from the original sample without i-th;

$n$: the size of the original sample;

$$\hat{Z}_0 = \Phi^{-1}\left(\frac{\#\{\hat{\theta}^*_{(b)} < \hat{\theta}\}}{B}\right)$$

where:

$\hat{\theta}$: estimator of $\theta$ of the original sample;

$\hat{\theta}^*_{(b)}$: an estimator of $\theta$ for $b$-th " bootstrap";

B:" bootstrap" number(recommended value B=1000);

$\Phi^{-1}(.)$: the inverse function of the normal distribution accumulated standard.

In this form, the $BC_a$ confidence interval is given by:

$$BC_a = (\hat{\theta}^{*(\alpha_1)}, \hat{\theta}^{*(\alpha_2)})$$

where:

$$\alpha_1 = \Phi\left(\hat{Z}_0 + \frac{\hat{Z}_0 + Z^{(\alpha)}}{1 - \hat{a}(\hat{Z}_0 + Z^{(\alpha)})}\right)$$

$$\alpha_2 = \Phi\left(\hat{Z}_0 + \frac{\hat{Z}_0 + Z^{(1-\alpha)}}{1 - \hat{a}(\hat{Z}_0 + Z^{(1-\alpha)})}\right)$$

and

$Z^{()}$: the point percentile of the standardized normal distribution;

$\Phi(.)$: indicating the cumulative normal distribution accumulated standard.

# Chapter 3

# Density Function and Tail Shape Classification

## 3.1 Density Function and Its Properties

In this Chapter, we consider the classification of distributions given by Eq.(3.2) and analyze its nature with respect to the parameters $\alpha, \beta,$ and $\lambda$. To simplify the discussions, we let $\lambda = 1$. The general case can be dealt similarly. Then the distribution function becomes:

$$F(t) = \left(1 - e^{1-e^{t^\beta}}\right)^\alpha, \qquad (t > 0, \alpha > 0, \beta > 0), \tag{3.1}$$

The corresponding p.d.f. is:

$$f(t) = \alpha\beta \left(1 - e^{1-e^{t^\beta}}\right)^{\alpha-1} e^{1+t^\beta-e^{t^\beta}} t^{\beta-1}, \qquad (t > 0, \alpha > 0, \beta > 0), \tag{3.2}$$

The quantile function obtained by inverting the distribution function in (3.1) is given by:

$$Q(U) = (\log(1 - \log(1 - U^{\frac{1}{\alpha}})))^{\frac{1}{\beta}}, \qquad (\alpha > 0, \beta > 0), \tag{3.3}$$

In order to discuss the properties of density function, we derive the derivative of $f(t)$. Note that we are dealing with the case $\lambda = 1$. The general case$(\lambda \neq 1)$ is very cumbersome. Even the case of $\lambda = 1$ is quite involved, but a simple transformation simplifies the analysis. Let us write:

$$z \equiv z(t) = e^{t^\beta}.$$

Then writing $f(t) = g(z)$; we have:

$$\begin{aligned} f(t) &= g(z) \\ &= \alpha\beta(1-e^{1-z})^{\alpha-1}e^{1-z}z(\log z)^{\frac{\beta-1}{\beta}}, \end{aligned}$$

and the derivative of $g(z)$ is given by :

$$\begin{aligned} \frac{1}{\alpha\beta}g'(z) &= ((1-e^{1-z})^{\alpha-1}e^{1-z})'z(\log z)^{\frac{\beta-1}{\beta}} + (1-e^{1-z})^{\alpha-1}e^{1-z} \\ &\quad (z(\log z)^{\frac{\beta-1}{\beta}})' \\ &= z(\log z)^{\frac{\beta-1}{\beta}}\{(\alpha-1)(1-e^{1-z})^{\alpha-2}e^{2(1-z)} - (1-e^{1-z})^{\alpha-1}e^{1-z}\} \\ &\quad +(1-e^{1-z})^{\alpha-1}e^{1-z}\{(\log z)^{\frac{\beta-1}{\beta}} + \frac{\beta-1}{\beta}(\log z)^{-\frac{1}{\beta}}\} \\ &= (1-e^{1-z})^{\alpha-2}(\log z)^{\frac{\beta-1}{\beta}-1}e^{1-z}[(\alpha-1)z\log ze^{1-z} \\ &\quad +\log z(1-e^{1-z})(1-z) + (1-e^{1-z})\frac{\beta-1}{\beta}] \qquad (3.4) \\ &= (1-e^{1-z})^{\alpha-2}(\log z)^{\frac{\beta-1}{\beta}-1}e^{1-z}[T_1(z)+T_2(z)+T_3(z)] \qquad (3.5) \end{aligned}$$

Where $T_1(z) = (\alpha-1)z\log ze^{1-z}$, $T_2(z) = \log z(1-e^{1-z})(1-z)$ and $T_3(z) = (1-e^{1-z})\frac{\beta-1}{\beta} = \frac{\beta-1}{\beta}$. Now we need judge the symbol of each part of $g'(z)$ equation. At first, consider the case of $z \to \infty$ (that is $t \to \infty$). The sign of $g'(z)$ depends on the three terms in square brackets. Since,

$$
\begin{aligned}
\lim_{z\to\infty} T_1(z) &= \lim_{z\to\infty} (\alpha-1) z \log z e^{1-z} \\
&= \lim_{z\to\infty} \frac{(\alpha-1)(\log z + 1)}{e^{z-1}} \\
&= \lim_{z\to\infty} \frac{(\alpha-1)\frac{1}{z}}{e^{z-1}} \\
&= 0 \\
\lim_{z\to\infty} T_2(z) &= \lim_{z\to\infty} \log z(1-e^{1-z})(1-z) \\
&= \lim_{z\to\infty} \log z(1-z) \\
&= -\infty
\end{aligned}
$$

$$\lim_{z\to\infty} T_3(z) = \lim_{z\to\infty} (1-e^{1-z})\frac{\beta-1}{\beta} = \frac{\beta-1}{\beta}$$

We find that for large $z$, $g'(z) < 0$; hence as $t \to \infty$, $f(t)$ is decreasing. Now we consider four cases where we can explicitely discuss the nature of $f(t)$.

Case I: $(\alpha < 1, \beta < 1)$

If $\alpha < 1$ and $\beta < 1$, then: Since,

$T_1(z) < 0$ for all $z > 1$ or $t > 0$,

$T_2(z) < 0$ for all $z > 1$ and

$T_3(z) < 0$, it follows from Eq.(3.4), that $g'(z) < 0$ for all $z > 1$. That means $g(z)$ is decreasing or $f(t)$ is strictly decreasing.

Case II: $(\alpha > 1, \beta > 1)$

For this case, we show that $f(t)$ is unimodal. Let:

$$\Psi(z) = (\alpha-1) z \log z e^{1-z} + \log z(1-e^{1-z})(1-z) + (1-e^{1-z})\frac{\beta-1}{\beta}$$

then

$$
\begin{aligned}
\Psi(z) &= (\alpha-1) z \log z e^{1-z} + (1-e^{1-z})(\log z - z\log z + \frac{\beta-1}{\beta}) \\
&= (\alpha-1) z \log z e^{1-z} + (1-e^{1-z})\psi(z)
\end{aligned}
$$

Where $\psi(z) = \log z - z\log z + \frac{\beta-1}{\beta}$. We find that,

$$\psi'(z) = \frac{1}{z} - \log z - z\frac{1}{z}$$
$$\psi''(z) = -\frac{1}{z^2} - \frac{1}{z}$$

It is obvious that $\psi''(z) < 0$ for $z > 1$. This implies that $\psi'(z)$ is decreasing function. Hence; $\psi'(z) < \psi'(1) = 0 \Rightarrow \psi(z)$ is a decreasing function. Since $\lim_{z\to\infty} \psi(z) = -\infty$, and $\psi(1) = \frac{\beta-1}{\beta} > 0$, there exists a $z^*$ such that $\psi(z^*) = 0$ and, $0 < \psi(z) < \psi(1) = \frac{\beta-1}{\beta}$ for $1 < z < z^*$. This implies $\Psi(z) > 0$ for $1 < z < z^*$. Futher since, $\lim_{z\to\infty} g'(z) = -\infty$, using the same argument, we find that there exists $z^{**} \geq z^*$, such that $\Psi(z^{**}) = 0$. This provides that $g(z)$ is unimodal or equivalently $f(t)$ is unimodal.

Case III: $(\alpha < 1, \beta > 1)$

Note that $g'(z)$ has the same sign as $\Psi(z)$. Since

$$\Psi(z) = (\alpha - 1)z\log ze^{1-z} + (1 - e^{1-z})\psi(z)$$

For $\alpha < 1$; $(\alpha - 1)z\log ze^{1-z} < 0$ for all $z > 1$.

Also, $\psi(z)$ is decreasing function and $\psi(z) < \psi(1) = \frac{\beta-1}{\beta}$.

Let $z^*$ be such that $\psi(z^*) = 0$; then

For $z > z^*$; $\Psi(z) < 0$, hence;

$g(z) \searrow$ for $z > z^*$.

For $z \leq z^*$,

$$\begin{aligned}
\Psi(z) \geq 0 &\Leftrightarrow (\alpha - 1)z\log ze^{1-z} + (1 - e^{1-z})\psi(z) \geq 0 \\
&\Leftrightarrow \frac{(1 - e^{1-z})\psi(z)}{z\log ze^{1-z}} \geq 1 - \alpha \\
&\Leftrightarrow \alpha \geq 1 - \frac{(1 - e^{1-z})\psi(z)}{z\log ze^{1-z}} \ \forall z^* \geq z \\
&\Leftrightarrow \alpha \geq 1 - sup_{z\leq z^*}\frac{(1 - e^{1-z})\psi(z)}{z\log ze^{1-z}} = 1 - u_{z^*}
\end{aligned}$$

If $\alpha$ satisfies the above condition, then $f(t)$ is unimodal, otherwise $f(t)$ is decreasing with $t$.

Case IV: $(\alpha > 1, \beta < 1)$

This case is very similar to the case III. Maybe $\Psi(z)$ is always non-positive, or at the begining, it is non-negative and eventually becomes non-positive. That means $g(z)$ is decreasing or unimodal. It is equivalent to saying $f(t)$ may be decreasing or unimodal.

By the above analysis, it is clear that when $\alpha$ and $\beta$ both are larger than 1, the density function is unimodal, whereas, when both are smaller than 1, the density function is decreasing. Another case, we may get unimodal or decreasing density function. This is summarized in Table 3.1

Table 3.1: illustrates the four types of density shapes

| $\alpha$ | $\beta$ | density behavior |
|---|---|---|
| 1 | 1 | decreasing |
| $< 1$ | $< 1$ | decreasing |
| $> 1$ | $> 1$ | unimodal |
| $< 1$ | $> 1$ | decreasing or unimodal |
| $> 1$ | $< 1$ | decreasing or unimodal |

Some graphs showing the density function plot for various values of $\alpha$, $\beta$ are given in Fig 3.1-3.6:

In the above graphs, they are respectively the cases: i) $\alpha > 1,\ \beta > 1$, ii) $\alpha < 1,\ \beta < 1$, iii) $\alpha > 1,\ \beta < 1$ and $\alpha\beta < 1$, iv) $\alpha < 1,\ \beta > 1$ and $\alpha\beta < 1$, v) $\alpha < 1,\ \beta > 1$ and $\alpha\beta > 1$ and vi) $\alpha > 1,\ \beta < 1$ and $\alpha\beta > 1$. We find that when at least one of $\alpha, \beta$ is less than 1 and the other one is larger than 1, the corresponding density function is decreasing or unimodal depending on whether $\alpha\beta > 1$ or not. Of course, it is only a conjecture.

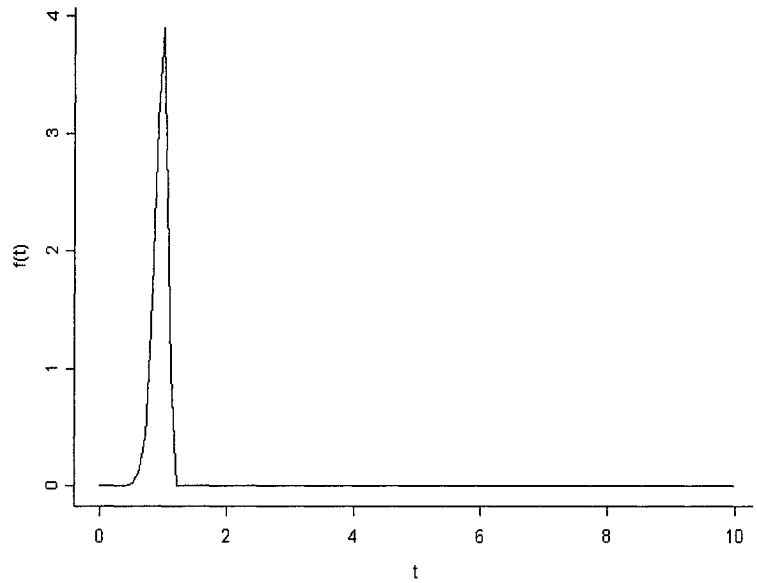


Figure 3.1: The density function curve with $\alpha = 2$, $\beta = 5$

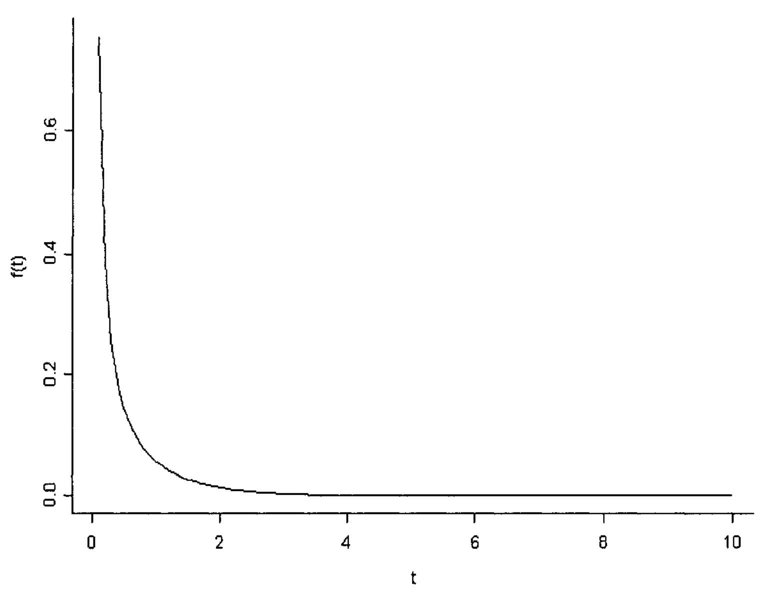


Figure 3.2: The density function curve with $\alpha = 0.2$, $\beta = 0.5$

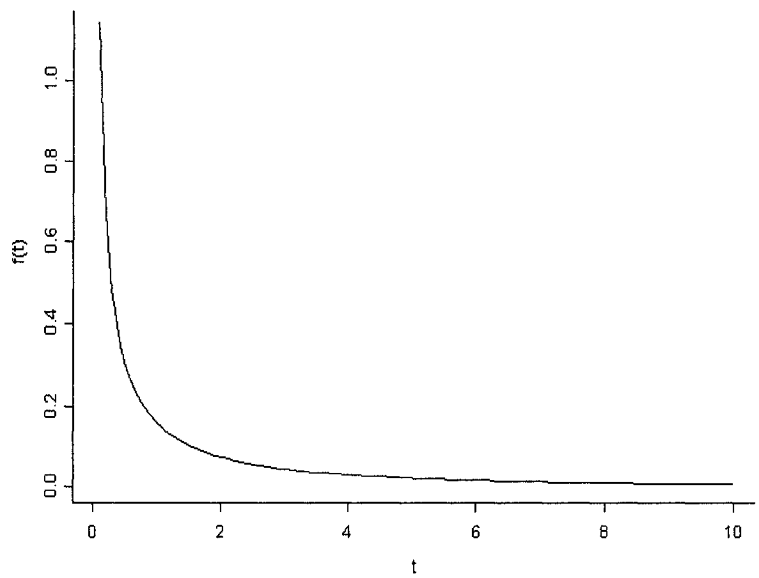


Figure 3.3: The density function curve with $\alpha = 2$, $\beta = 0.2$

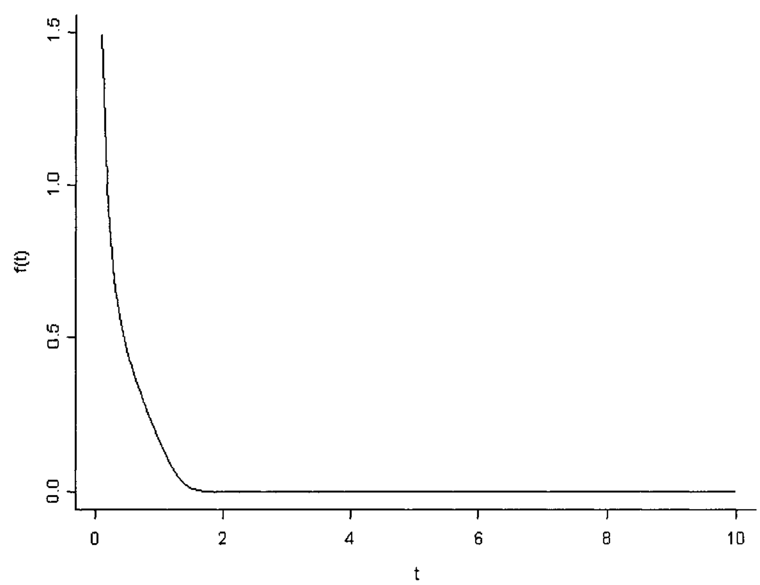


Figure 3.4: The density function curve with $\alpha = 0.2$, $\beta = 1.5$

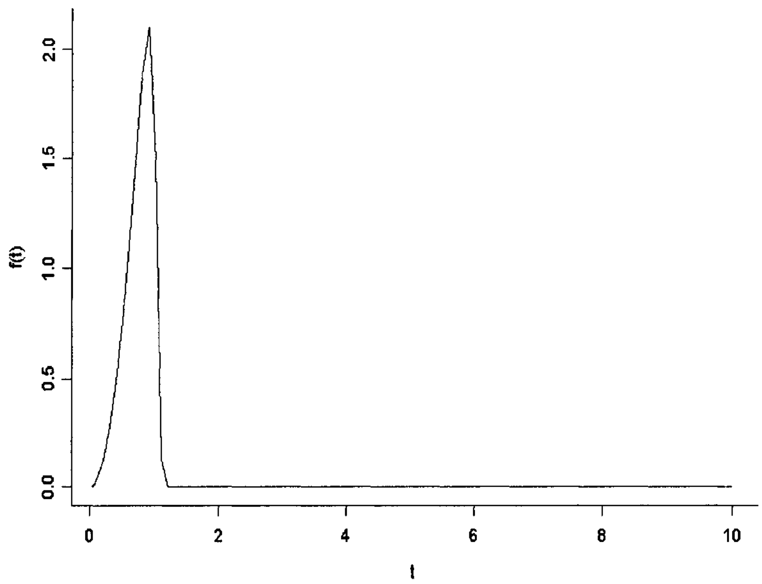


Figure 3.5: The density function curve with $\alpha = 0.5$, $\beta = 6$

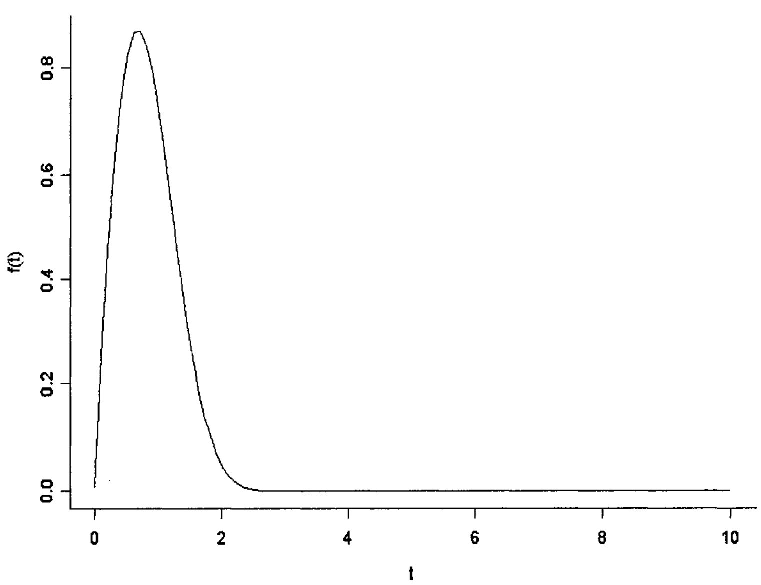


Figure 3.6: The density function curve with $\alpha = 2$, $\beta = 0.9$

## 3.2 Tail Shape Classification

According to the definition in the Chapter 2, we need to compute $\alpha_0$ and $\alpha_1$ for classifying the tail shapes.

$$\begin{aligned}
\alpha_0 &= \lim_{t\to\infty} \frac{-(1-F(t))f'(t)}{f(t)^2} \\
&= \lim_{t\to\infty} \frac{\log f(t)}{\log(1-F(t))} \\
&= \lim_{z\to\infty} \frac{\log\alpha + \log\beta + (\alpha-1)\log(1-e^{1-z}) + 1 - z + \log z + \frac{\beta-1}{\beta}\log\log z}{\log(1-(1-e^{1-z})^\alpha)} \\
&= \lim_{z\to\infty} \frac{[(\alpha-1)\frac{e^{1-z}}{1-e^{1-z}} - 1 + \frac{1}{z} + \frac{\beta-1}{\beta}\frac{1}{z\log z}][1-(1-e^{1-z})^\alpha]}{-\alpha e^{1-z}(1-e^{1-z})^{\alpha-1}} \qquad (3.6)
\end{aligned}$$

Now let $k(z) = (\alpha-1)\frac{e^{1-z}}{1-e^{1-z}} - 1 + \frac{1}{z} + \frac{\beta-1}{\beta}\frac{1}{z\log z}$, then

$$\lim_{z\to\infty} k(z) = -1$$

and

$$\begin{aligned}
\lim_{z\to\infty} \frac{1-(1-e^{1-z})^\alpha}{-\alpha(1-e^{1-z})^{\alpha-1}e^{1-z}} &= \lim_{z\to\infty} \frac{(1-e^{1-z})^{\alpha-1}}{(\alpha-1)(1-e^{1-z})^{\alpha-2}e^{1-z} - (1-e^{1-z})^{\alpha-1}} \\
&= -1
\end{aligned}$$

Hence, from Eq.(3.6),

$$\alpha_0 = \lim_{z\to\infty} k(z)\frac{1-(1-e^{1-z})^\alpha}{-\alpha(1-e^{1-z})^{\alpha-1}e^{1-z}} = 1$$

Hence, by Parzen's classification scheme, the density function has a right medium tail. A refined classification is needed by considering $h_1$.

$$\begin{aligned}
h_1 &= \lim_{t\to\infty} \frac{1}{h(t)} \\
&= \lim_{z\to\infty} \frac{1-(1-e^{1-z})^\alpha}{\alpha\beta(1-e^{1-z})^{\alpha-1}e^{1-z}z(\log z)^{\frac{\beta-1}{\beta}}} \\
&= \lim_{z\to\infty} \frac{-\alpha(1-e^{1-z})^{\alpha-1}}{\alpha\beta(-z(\log z)^{\frac{\beta-1}{\beta}} + (\log z)^{\frac{\beta-1}{\beta}} + \frac{\beta-1}{\beta}(\log z)^{\frac{\beta-1}{\beta}-1})} \\
&= \lim_{z\to\infty} \frac{-(1-e^{1-z})^{\alpha-1}}{\beta(-z(\log z)^{\frac{\beta-1}{\beta}} + (\log z)^{\frac{\beta-1}{\beta}})} \\
&= 0
\end{aligned}$$

Since $h_1 = 0$, so it has a medium-short tail.(see the definition in Chapter 2)

For the left tail case, we find

$$\begin{aligned}
\alpha_1 &= \lim_{t\to 0^+} \frac{f'(t)F(t)}{f(t)^2} \\
&= \lim_{z\to 1} \frac{\log\alpha + \log\beta + (\alpha-1)\log(1-e^{1-z}) + 1 - z + \log z + \frac{\beta-1}{\beta}\log\log z}{\alpha\log(1-e^{1-z})} \\
&= \lim_{z\to 1} \frac{(\alpha-1)e^{1-z} - 1 + e^{1-z} + \frac{1-e^{1-z}}{z} + \frac{\beta-1}{\beta}\frac{1}{\log z}\frac{1-e^{1-z}}{z}}{\alpha e^{1-z}} \\
&= \frac{\alpha-1}{\alpha} + \frac{\beta-1}{\alpha\beta} \\
&= 1 - \frac{1}{\alpha\beta} < 1 \ (\text{since}\ \alpha > 0, \beta > 0)
\end{aligned}$$

Hence, it follows that the density function has a short left tail.

In addition, the shapes of the density function of this distribution can be understood by considering the limits of $f(Q(U))$ as $U \to 0 (z \to 1)$ and $U \to 1 (z \to \infty)$. From the Eq.(3.3), we know that $Q(U) = (\log(1-\log(1-U^{1/\alpha})))^{1/\beta}$, $(\alpha > 0, \beta > 0)$, then

$$f(Q(U)) = \frac{dU}{dQ(U)} = \frac{1}{\frac{dQ(U)}{dU}} \tag{3.7}$$

Since,

$$\frac{dQ(U)}{dU} = \frac{1}{\alpha\beta}(\log(1-\log(1-U^{\frac{1}{\alpha}})))^{\frac{1}{\beta}-1}\frac{1}{1-\log(1-U^{\frac{1}{\alpha}})}\frac{1}{1-U^{\frac{1}{\alpha}}}U^{\frac{1}{\alpha}-1},$$

Eq.(3.7) becomes,

$$f(Q(U)) = \alpha\beta(\log(1-\log(1-U^{\frac{1}{\alpha}})))^{1-\frac{1}{\beta}}(1-\log(1-U^{\frac{1}{\alpha}}))(1-U^{\frac{1}{\alpha}})U^{1-\frac{1}{\alpha}} \tag{3.8}$$

Now we examine the limits of $f(Q(U))$ $(U \to 0\ ,\ U \to 1)$:

I) Consider $U \to 1$:

$$\lim_{U\to 1} f(Q(U)) = \lim_{U\to 1} \alpha\beta(\log(1-\log(1-U^{\frac{1}{\alpha}})))^{1-\frac{1}{\beta}}(1-\log(1-U^{\frac{1}{\alpha}}))(1-U^{\frac{1}{\alpha}}) \tag{3.9}$$

Using transformation, $t \equiv 1 - \log(1 - U^{\frac{1}{\alpha}})$, we have

$$\lim_{U \to 1} f(Q(U)) = \lim_{t \to \infty} \alpha\beta(\log t)^{1-\frac{1}{\beta}} t e^{1-t}.$$

Since

$$\lim_{t \to \infty} t e^{1-t} = 0$$

and when $\beta < 1$, $\lim_{t \to \infty} (\log t)^{1-\frac{1}{\beta}} = 0$,

so

$$\lim_{t \to \infty} \alpha\beta(\log t)^{1-\frac{1}{\beta}} t e^{1-t} = 0 \text{ for } \beta < 1$$

That is

$$\lim_{U \to 1} f(Q(U)) = 0 \text{ when } \beta < 1.$$

The same result holds for the case $\beta > 1$,

Since,

$$\begin{aligned}
\lim_{U \to 1} f(Q(U)) &= \lim_{t \to \infty} \alpha\beta(\log t)^{1-\frac{1}{\beta}} t e^{1-t} \\
&= \lim_{t \to \infty} \alpha\beta \frac{(\log t)^{1-\frac{1}{\beta}} t}{e^{t-1}} \\
&= \lim_{t \to \infty} \alpha\beta \frac{(\log t)^{1-\frac{1}{\beta}} + (1 - \frac{1}{\beta})(\log t)^{-\frac{1}{\beta}}}{e^{t-1}} \\
&= 0
\end{aligned}$$

II) Consider $U \to 0$:

$$\lim_{U \to 0} f(Q(U)) = \lim_{U \to 0} \alpha\beta(\log(1 - \log(1 - U^{\frac{1}{\alpha}})))^{1-\frac{1}{\beta}} U^{1-\frac{1}{\alpha}}$$

i) $\alpha > 1, \beta > 1$:

$$\lim_{U \to 0} f(Q(U)) = 0$$

ii) $\alpha < 1, \beta < 1$:

$$\lim_{U \to 0} f(Q(U)) = \infty$$

iii) $\alpha > 1, \beta < 1$:

then

$$
\begin{aligned}
\lim_{U\to 0} f(Q(U)) &= \lim_{U\to 0} \alpha\beta \frac{U^{1-\frac{1}{\alpha}}}{(\log(1-\log(1-U^{\frac{1}{\alpha}})))^{\frac{1}{\beta}-1}} \\
&= \lim_{U\to 0} \alpha\beta \frac{(1-\frac{1}{\alpha})U^{-\frac{1}{\alpha}}(1-\log(1-U^{\frac{1}{\alpha}}))(1-U^{\frac{1}{\alpha}})}{(\frac{1}{\beta}-1)(\log(1-\log(1-U^{\frac{1}{\alpha}})))^{\frac{1}{\beta}-2}\frac{1}{\alpha}U^{\frac{1}{\alpha}-1}} \\
&= \lim_{U\to 0} \alpha\beta \frac{1-\frac{1}{\alpha}}{\frac{1}{\alpha}(\frac{1}{\beta}-1)(\log(1-\log(1-U^{\frac{1}{\alpha}})))^{\frac{1}{\beta}-2}U^{\frac{2}{\alpha}-1}} \qquad (3.10)
\end{aligned}
$$

From the Eq.(3.10), the limiting value of $f(Q(U))$ $(U \to 0)$ may be 0 or $\infty$, depending on the parameters $\alpha$, $\beta$.

iv) $\alpha < 1, \beta > 1$:

$$
\begin{aligned}
\lim_{U\to 0} f(Q(U)) &= \lim_{U\to 0} \alpha\beta \frac{(\log(1-\log(1-U^{\frac{1}{\alpha}})))^{1-\frac{1}{\beta}}}{U^{\frac{1}{\alpha}-1}} \\
&= \lim_{U\to 0} \alpha\beta \frac{(1-\frac{1}{\beta})U}{(\frac{1}{\alpha}-1)(\log(1-\log(1-U^{\frac{1}{\alpha}})))^{\frac{1}{\beta}}} \\
&= \lim_{U\to 0} \alpha\beta \frac{(1-\frac{1}{\beta})U^{1-\frac{1}{\alpha}}}{(\frac{1}{\alpha}-1)\frac{1}{\beta}\frac{1}{\alpha}((\log(1-\log(1-U^{\frac{1}{\alpha}})))^{\frac{1}{\beta}-1})}
\end{aligned}
$$

There are similar results in this case as in case iii). The limiting value of $f(Q(U))$ $(U \to 0)$ may be 0 or $\infty$, depending on the value of $\alpha$, $\beta$.

These results are summarized in the following table.

Table 3.2: illustrates the limits of four cases

| | $U \to 0$ | $U \to 1$ |
|---|---|---|
| $\alpha > 1$, $\beta > 1$ | 0 | 0 |
| $\alpha > 1$, $\beta < 1$ | 0 or $\infty$ | 0 |
| $\alpha < 1$, $\beta > 1$ | 0 or $\infty$ | 0 |
| $\alpha < 1$, $\beta < 1$ | $\infty$ | 0 |

From the above classification and limit results, we may conclude that when the density function has monotone decreasing density function, it becomes unbounded in the left-tail. Otherwise, it always has a short left tail. About its right tail, it has a medium short right tail which doesn't depend on $\alpha$, $\beta$. The density function is either unimodal or monotone decreasing.

## 3.3 The Hazard Function Analysis

The hazard function (also known as the failure rate, hazard rate, or force of mortality) $h(t)$ is the ratio of the density function $f(t)$ to the survival function $R(t)$, given by:

$$h(t) = \frac{f(t)}{R(t)} \tag{3.11}$$

For the distribution, introduced in Eq.(3.11), the hazard function is given by:

$$h(t) = \frac{\alpha\beta(1-e^{1-e^{t^\beta}})^{\alpha-1}e^{1+t^\beta-e^{t^\beta}}t^{\beta-1}}{1-(1-e^{1-e^{t^\beta}})^\alpha} \tag{3.12}$$

As in section 3.1, we may analyze $h(t)$ in term of $z = e^{t^\beta}$, and consider

$$\begin{aligned} h(t = (\log z)^{\frac{1}{\beta}}) = r(z) &= \frac{\alpha\beta(1-e^{1-z})^{\alpha-1}e^{1-z+\log z}(\log z)^{\frac{\beta-1}{\beta}}}{1-(1-e^{1-z})^\alpha} \\ &= \frac{\alpha\beta(1-e^{1-z})^{\alpha-1}e^{1-z}z(\log z)^{\frac{\beta-1}{\beta}}}{1-(1-e^{1-z})^\alpha} \end{aligned} \tag{3.13}$$

$$\begin{aligned} \text{Now} \qquad r'(z) &= \frac{\alpha\beta((1-e^{1-z})^{\alpha-1}e^{1-z}z(\log z)^{\frac{\beta-1}{\beta}})'(1-(1-e^{1-z})^\alpha)}{(1-(1-e^{1-z})^\alpha)^2} \\ &\quad -\frac{\alpha\beta(1-e^{1-z})^{\alpha-1}e^{1-z}z(\log z)^{\frac{\beta-1}{\beta}}(1-(1-e^{1-z})^\alpha)'}{(1-(1-e^{1-z})^\alpha)^2} \\ &= \alpha\beta\frac{\phi_1(z)(1-(1-e^{1-z}))-(1-e^{1-z})^{\alpha-1}e^{1-z}z(\log z)^{\frac{\beta-1}{\beta}}\phi_2(z)}{(1-(1-e^{1-z})^\alpha)^2} \end{aligned}$$

where

$$\phi_1(z) = \frac{d((1-e^{1-z})^{\alpha-1}e^{1-z}z(\log z)^{\frac{\beta-1}{\beta}})}{dz}$$

and

$$\phi_2(z) = \frac{d}{dz}(1-(1-e^{1-z})^{\alpha})$$

It can be seen that

$$\begin{aligned}
\phi_1(z) &= ((1-e^{1-z})^{\alpha-1}e^{1-z})'z(\log z)^{\frac{\beta-1}{\beta}} + (1-e^{1-z})^{\alpha-1}e^{1-z}(z(\log z)^{\frac{\beta-1}{\beta}})' \\
&= z(\log z)^{\frac{\beta-1}{\beta}}((\alpha-1)(1-e^{1-z})^{\alpha-2}e^{2-2z} - (1-e^{1-z})^{\alpha-1}e^{1-z}) \\
&\quad +(1-e^{1-z})^{\alpha-1}e^{1-z}((\log z)^{\frac{\beta-1}{\beta}} + z\frac{\beta-1}{\beta}(\log z)^{\frac{\beta-1}{\beta}-1}\frac{1}{z}) \\
&= (1-e^{1-z})^{\alpha-2}(\log z)^{\frac{\beta-1}{\beta}-1}e^{1-z}z\log z((\alpha-1)e^{1-z} - (1-e^{1-z})) \\
&\quad +(1-e^{1-z})^{\alpha-2}(\log z)^{\frac{\beta-1}{\beta}-1}e^{1-z}(1-e^{1-z})(\log z + \frac{\beta-1}{\beta}) \qquad (3.14)
\end{aligned}$$

and $$\phi_2(z) = -\alpha(1-e^{1-z})^{\alpha-1}e^{1-z} \qquad (3.15)$$

Then $r'(z)$ can be written as:

$$r'(z) = \phi_3(z)\phi_4(z)$$

Where $\phi_3(z) = \alpha\beta\frac{(1-e^{1-z})^{\alpha-2}(\log z)^{\frac{\beta-1}{\beta}-1}e^{1-z}}{(1-(1-e^{1-z})^{\alpha})^2}$ and

$$\begin{aligned}
\phi_4(z) &= (z\log z((\alpha-1)e^{1-z} - (1-e^{1-z})) + (1-e^{1-z}(\log z + \frac{\beta-1}{\beta}))(1-(1-e^{1-z})^{\alpha}) \\
&\quad +(1-e^{1-z})^{\alpha}\alpha z e^{1-z}\log z
\end{aligned}$$

It is very easy to see that $\phi_3(z)$ is always larger than 0, so we need to consider $\phi_4(z)$ in detail.

$$\begin{aligned}
\phi_4(z) &= z\log z(\alpha-1)e^{1-z} - (1-e^{1-z})z\log z + (1-e^{1-z})(\log z + \frac{\beta-1}{\beta}) \\
&\quad -z\log z(\alpha-1)e^{1-z}(1-e^{1-z})^{\alpha} + (1-e^{1-z})^{\alpha+1}z\log z \\
&\quad -(1-e^{1-z})^{\alpha+1}(\log z + \frac{\beta-1}{\beta}) \\
&\quad +(1-e^{1-z})^{\alpha}\alpha z e^{1-z}\log z \\
&= (1-e^{1-z})(\log z + \frac{\beta-1}{\beta})(1-(1-e^{1-z})^{\alpha}) + z\log z(\alpha e^{1-z} - 1 + (1-e^{1-z})^{\alpha}) \\
&= (1-e^{1-z})G_1(z) + G_2(z)
\end{aligned}$$

Where

$$G_1(z) = (\log z + \frac{\beta-1}{\beta})(1-(1-e^{1-z})^\alpha)$$

and

$$G_2(z) = z\log z(\alpha e^{1-z} - 1 + (1-e^{1-z})^\alpha)$$

Now we can find some limits when $t \to \infty (z \to \infty)$:

$$\begin{aligned}\lim_{z\to\infty} G_1(z) &= \lim_{z\to\infty} \frac{1-(1-e^{1-z})^\alpha}{\frac{1}{\log z}} \\ &= \lim_{z\to\infty} \alpha z e^{1-z}(\log z)^2 \\ &= 0 \end{aligned} \tag{3.16}$$

$$\lim_{z\to\infty} \alpha e^{1-z} - 1 + (1-e^{1-z})^\alpha = 0 \tag{3.17}$$

$$\begin{aligned}\lim_{z\to\infty} G_2(z) &= \lim_{z\to\infty} \frac{2z\log z}{\alpha(\alpha-1)e^{2z-2}} \\ &= \lim_{z\to\infty} \frac{\frac{1}{z}}{\alpha(\alpha-1)2e^{2z-2}} \\ &= 0 \end{aligned} \tag{3.18}$$

Since

$$G_1(z) \overset{z\to\infty}{\approx} \log z(\alpha e^{1-z} - \frac{\alpha(\alpha-1)}{2}(e^{1-z})^2 + o((e^{1-z})^2))$$
$$G_2(z) \overset{z\to\infty}{\approx} z\log z(\frac{\alpha(\alpha-1)}{2}(e^{1-z})^2 + o((e^{1-z})^2))$$

and

$$\lim_{z\to\infty} ze^{1-z} = \lim_{z\to\infty} \frac{z}{e^{z-1}} = 0$$

$G_2(z)$ converges to zero at a faster rate than $G_1(z)$.

Suppose $k(\alpha) = \alpha e^{1-z} - 1 + (1-e^{1-z})^\alpha$ and $e^{1-z} = t, \quad (0 < t < 1)$

so that

$$k(\alpha) = t\alpha - 1 + (1-t)^\alpha,$$

$$k'(\alpha) = t + (1-t)^\alpha \log(1-t),$$

$$k''(\alpha) = (1-t)^\alpha \log(1-t)\log(1-t) = (1-t)^\alpha(\log(1-t))^2 > 0.$$

And

$$\alpha = 0, k(0) = 0,$$

$$\alpha = 1, k(1) = 0.$$

Let $k'(\alpha) = 0$, then $\alpha^* = \log_{1-t}^{-\frac{t}{\log(1-t)}}$, $\alpha = \alpha^*$ is a minimum value point.

Now we need to prove that:

$$-\frac{t}{\log(1-t)} > 1-t \qquad (3.19)$$

$$-\frac{t}{\log(1-t)} < 1 \qquad (3.20)$$

Proof:

Let $T(t) = (1-t)\log(1-t) + t$

$$T(0) = 0$$

$$T'(t) = -\log(1-t) + \frac{1-t}{1-t}(-1) + 1 = -\log(1-t) > 0$$

So $T(t) \nearrow$, and $T(t) > T(0) = 0$, that is $-\frac{t}{\log(1-t)} > 1-t$. This proves Eq(3.19).

To prove Eq.(3.20), let $g(t) = \log(1-t) + t$

$$g(0) = 0$$

$$g'(t) = \frac{-1}{1-t} + 1 < 0$$

Which is equivalent to $g(t) \searrow$, and $g(t) < g(0) = 0$, that is $-\frac{t}{\log(1-t)} < 1$.

Now, we go back to consider the behavior of $k(\alpha)$. Since $\alpha^*$ is the minimum value of $k(\alpha)$

$$k'(\alpha) < 0, \qquad \text{for}(\alpha < \alpha^*)$$

and

$$k'(\alpha) > 0, \qquad \text{for}(\alpha > \alpha^*)$$

Also when $\alpha > 1,\ k'(\alpha) > 0$ and $k(1) = 0$; *i.e*, for $\alpha > 1,\ k(\alpha) \nearrow$, hence

$$k(\alpha) > 0, \qquad \text{for}(\alpha > 1)$$

$$k(\alpha) < 0, \qquad \text{for}(0 < \alpha < 1).$$

There are four cases:

Case I: $(\alpha \geq 1,\ \beta > 1)$

When $\alpha \geq 1,\ \beta > 1,\ \phi_4(z) > 0$,

so $r(z)$ is increasing, that is $h(t)$ is increasing.

Case II: $(\alpha > 1,\ \beta < 1)$

When $\alpha > 1,\ \beta < 1,\ \phi_4(z) > 0$, for all $z$ or at the beginning, $\phi_4(z) < 0$, then it becomes positive. In this case, therefore $r(z)$ is increasing or bathtub, that is $h(t)$ is increasing or bathtub.

Case III: $(\alpha < 1,\ \beta > 1)$

When $\alpha < 1,\ \beta > 1,\ \phi_4(z) > 0$, for all $z$ or at the beginning, $\phi_4(z) < 0$, then it becomes positive. Hence $r(z)$ is increasing or bathtub, that is $h(t)$ is increasing or bathtub.

Case IV: $(\alpha \leq 1,\ \beta < 1)$

When $\alpha \leq 1, \beta < 1$, at the begining, if $z < e^{\frac{1-\beta}{\beta}}$, $\phi_4(z) < 0$, then $\phi_4(z) > 0$, hence $r(z)$ is bathtub, that is $h(t)$ is bathtub.

Figures 3.7-3.12 depict different shapes of the hazard function for four cases:

Table 3.3: Four types of hazard shapes

| $\alpha$ | $\beta$ | failure behavior |
|---|---|---|
| 1 | 1 | constant |
| $<1$ | $<1$ | bathtub |
| $>1$ | $>1$ | increasing |
| $<1$ | $>1$ | increasing or bathtub |
| $>1$ | $<1$ | increasing or bathtub |

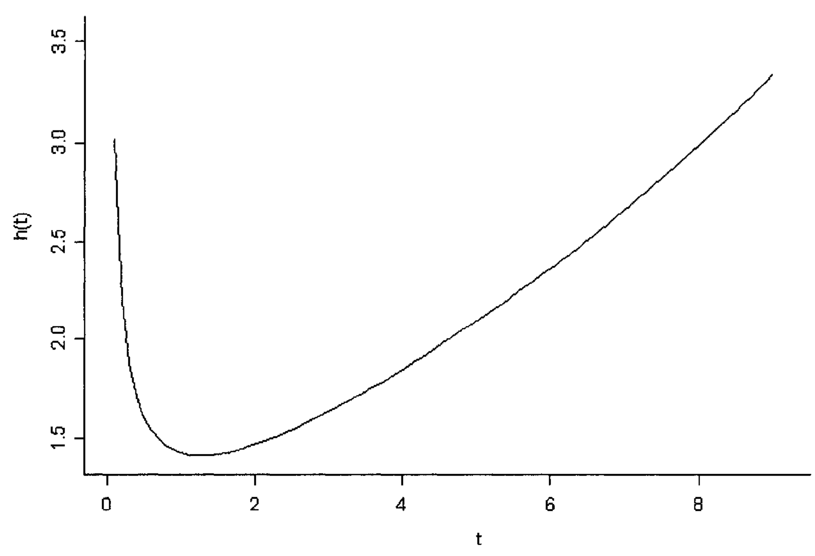


Figure 3.7: The hazard function curve with $\alpha = 0.5$, $\beta = 0.5$

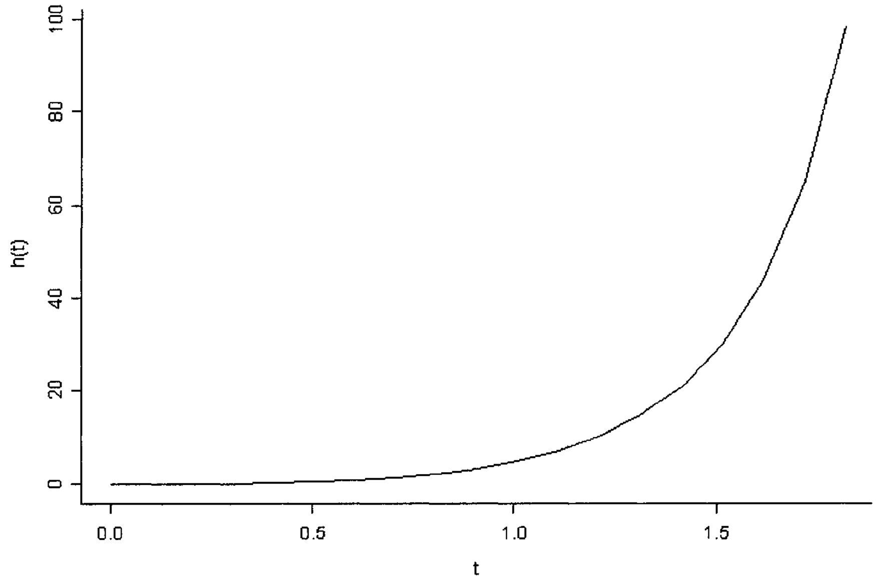


Figure 3.8: The hazard function curve with $\alpha = 2$, $\beta = 2$

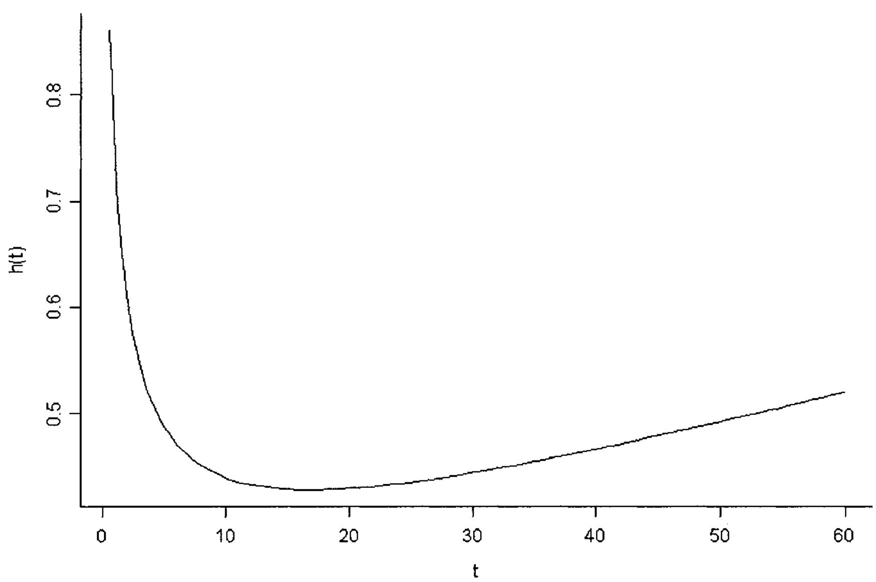


Figure 3.9: The hazard function curve with $\alpha = 2$, $\beta = 0.3$

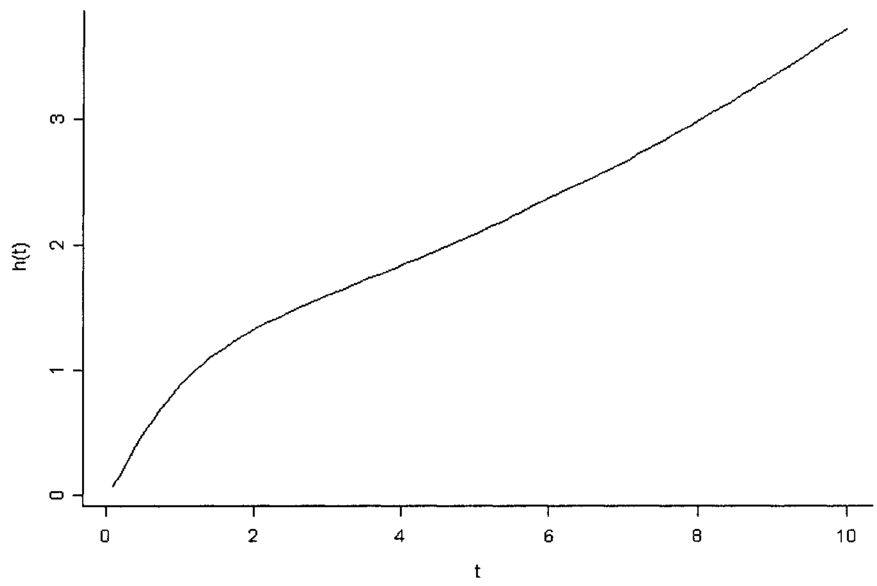


Figure 3.10: The hazard function curve with $\alpha = 5$, $\beta = 0.5$

In the above the graphs, they are respectively the cases: i) $\alpha > 1$, $\beta > 1$, ii) $\alpha < 1$, $\beta < 1$, iii) $\alpha > 1$, $\beta < 1$ and $\alpha\beta < 1$, iv) $\alpha < 1$, $\beta > 1$ and $\alpha\beta < 1$, v) $\alpha < 1$, $\beta > 1$ and $\alpha\beta > 1$ and vi) $\alpha > 1$, $\beta < 1$ and $\alpha\beta > 1$. we see that when at least one of the $\alpha, \beta$ is less than 1 and the other one is larger than 1, the corresponding hazard function is decreasing or bathtub which seems related to $\alpha\beta > 1$ or not. Of course, it is only a conjecture which is similar to the cases of the density function.

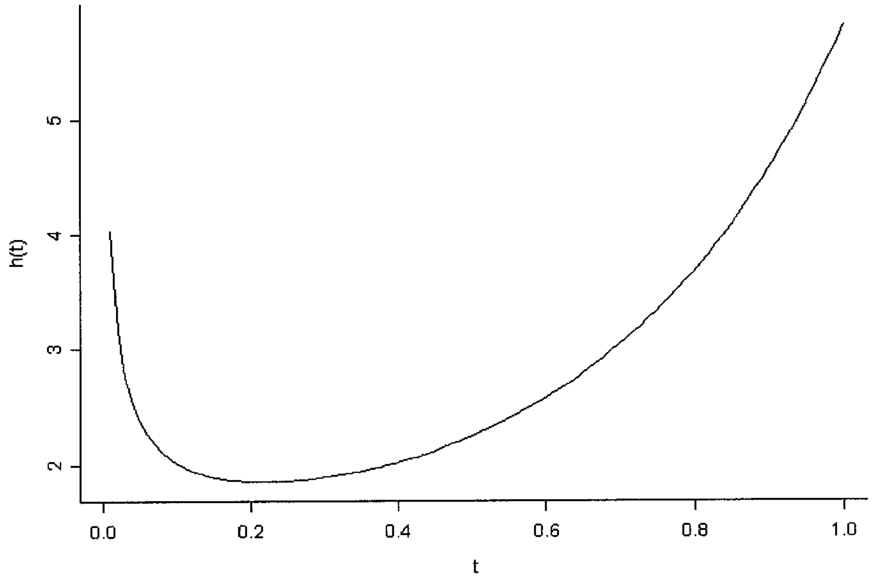


Figure 3.11: The hazard function curve with $\alpha = 0.3$, $\beta = 2$

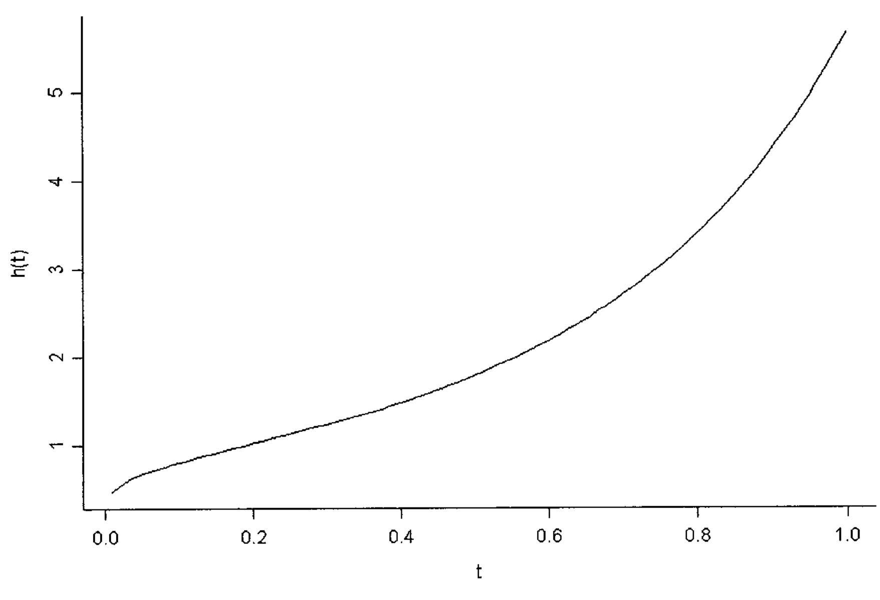


Figure 3.12: The hazard function curve with $\alpha = 0.6$, $\beta = 2$

# Chapter 4

# Extremes and Extreme Spacings

The tail-character of a population is closely related to the asymptotic distributions of the sample extremes. Freimer *et al.*(1989) discuss the relationship between the extreme value distribution and the character of the tails, and demonstrate, using the generalized Tukey lambda and the Weibull families as examples, how simple expansions of quantile functions may be used to derive the extreme value distribution. They also prove that expansions may be used for obtaining the limiting distributions of the extreme spacings.

## 4.1 Extreme Value Distribution

Extreme value distributions are the limiting distributions for the minimum or the maximum of a very large collection of random observations from the same arbitrary distribution. Gumbel (1958) showed that for any well-behaved initial distribution (i.e. $F(x)$ is continuous and has an inverse), only a few models are needed, depending on whether you are interested in the maximum or the minimum, and also if the observations are bounded above or below. As demonstrated in Freimer *et al.* (1989), the asymptotic distributions of the extremes and extreme spacings of random samples, and the related theory can be derived and developed by applying elementary methods to the population quantile functions.

Let $U_1, U_2, \ldots, U_n$ be a random sample from a uniform (0,1) population, and let $U_{1:n}$, $U_{2:n}$, $\ldots$, $U_{n:n}$ denote the sample order statistics. Then the order statistics $Y_{i:n}$ of random samples of size $n$ from this new distribution is gotten by using the quantile function:

$$Y_{i:n} = \left(\log(1 - \log(1 - U_{i:n}^{\frac{1}{\alpha}}))\right)^{\frac{1}{\beta}}, \qquad \alpha,\, \beta > 0,\ i = 1, 2, \ldots, n.$$

Hence the limiting distribution of $Y_{1:n}$ and $Y_{n:n}$ can be obtained from the above equation for $i = 1$ and $i = n$ respectively given by:

$$Y_{1:n} = \left(\log(1 - \log(1 - U_{1:n}^{\frac{1}{\alpha}}))\right)^{\frac{1}{\beta}}, \qquad \alpha,\, \beta > 0.$$

and

$$Y_{n:n} = \left(\log(1 - \log(1 - U_{n:n}^{\frac{1}{\alpha}}))\right)^{\frac{1}{\beta}}, \qquad \alpha,\, \beta > 0.$$

We can use the Theorem 2 (see Chapter 2) that as $n \to \infty$, both $nU_{1:n}$ and $n(1-U_{n:n})$ converge in law to the standard exponential r.v. Z. In particular, we get the following theorem.

**Theorem 1:** Let $Y_{1:n}$ and $Y_{n:n}$ be the minimum and maximum of a random sample of size n from this population respectively, and let Z denote the standard exponential r.v.with c.d.f. $F_Z(z) = 1 - e^{-z}, z \geq 0$, then as $n \to \infty$,

$$n^{\frac{1}{\alpha\beta}} Y_{1:n} \xrightarrow{L} Z^{\frac{1}{\alpha\beta}}$$

and

$$\beta(\log(1 + \log n))^{1-\frac{1}{\beta}}(1 + \log n)Y_{n:n} - \beta \log(1 + logn)(1 + \log n) - \log \alpha \xrightarrow{L} -\log Z$$

Proof:

Since $Y_{1:n} = (\log(1-\log(1-U_{1:n}^{\frac{1}{\alpha}})))^{\frac{1}{\beta}}$, Taylor expansion of $\log(1-\log(1-u^{\frac{1}{\alpha}}))$ about $u=0$, gives:

$$\log(1-\log(1-u^{\frac{1}{\alpha}})) = u^{\frac{1}{\alpha}} + o(u^{\frac{1}{\alpha}})$$

that is:

$$\begin{aligned}
\log(1-\log(1-U_{1:n}^{\frac{1}{\alpha}})) &= U_{1:n}^{\frac{1}{\alpha}} + o_P\left((\frac{1}{n})^{\frac{1}{\alpha}}\right) \\
n^{\frac{1}{\alpha}}\log(1-\log(1-U_{1:n}^{\frac{1}{\alpha}})) &= n^{\frac{1}{\alpha}}U_{1:n}^{\frac{1}{\alpha}} + n^{\frac{1}{\alpha}}o_P\left((\frac{1}{n})^{\frac{1}{\alpha}}\right) \\
\text{therefore} \qquad n^{\frac{1}{\alpha\beta}}Y_{1:n} \approx (nU_{1:n})^{\frac{1}{\alpha\beta}} &\xrightarrow{L} Z^{\frac{1}{\alpha\beta}}
\end{aligned}$$

Now we show the second part of the theorem.

Expanding $\log(1-u^{\frac{1}{\alpha}})$ around $u=1$ gives,

$$\begin{aligned}
\log(1-u^{\frac{1}{\alpha}}) &= \log(1-(u-1+1)^{\frac{1}{\alpha}}) \\
&= \log(1-[1+\frac{1}{\alpha}(u-1)+o((u-1))]) \\
&= \log(\frac{1-u}{\alpha} - o((u-1))) \\
&= \log(\frac{1-u}{\alpha}(1-\alpha\frac{o((u-1))}{1-u})) \\
\text{then: } \log(1-\log(1-U_{n:n}^{\frac{1}{\alpha}})) &= \log\left(1-\log\frac{1-U_{n:n}}{\alpha} - \log(1-\alpha\frac{o((U_{n:n}-1))}{1-U_{n:n}})\right)
\end{aligned}$$

$$(\log(1-\log(1-U_{n:n}^{\frac{1}{\alpha}})))^{\frac{1}{\beta}} \approx \left(\log(1+\log n - \log\frac{n(1-U_{n:n})}{\alpha})\right)^{\frac{1}{\beta}}$$

That is:

$$Y_{n:n} \approx \left(\log(1+\log n - \log\frac{n(1-U_{n:n})}{\alpha})\right)^{\frac{1}{\beta}}$$

and from the lemma (see Chapter 2), we can show that for any sequence $Y_n$ of random variables such that $Y_n \xrightarrow{L} Y$ in law as $n \to \infty$, and $g_n(y)$ converges uniformly to $g(y)$ over all compact sets, then $g_n(Y_n) \xrightarrow{L} g(Y)$. Hence, we let

$$g_n(y) = \frac{(\log(1+\log n - y))^{\frac{1}{\beta}} - (\log(1+\log n))^{\frac{1}{\beta}}}{\frac{1}{\beta}(\log(1+\log n))^{\frac{1}{\beta}-1}\frac{1}{1+\log n}}$$

Here we show that

$$g_n(y) \to -y = g(y) \text{ on compact sets.}$$

We can write

$$\begin{aligned}
g_n(y) &= \frac{(\log(1+\log n - y))^{\frac{1}{\beta}} - (\log(1+\log n))^{\frac{1}{\beta}}}{\frac{1}{\beta}(\log(1+\log n))^{\frac{1}{\beta}-1}\frac{1}{1+\log n}} \\
&= \frac{(\log(1+\log n) + \log(1-\frac{y}{1+\log n}))^{\frac{1}{\beta}}}{\frac{1}{\beta}(\log(1+\log n))^{\frac{1}{\beta}-1}\frac{1}{1+\log n}} \\
&\quad - \frac{(\log(1+\log n))^{\frac{1}{\beta}}}{\frac{1}{\beta}(\log(1+\log n))^{\frac{1}{\beta}-1}\frac{1}{1+\log n}} \\
&= \frac{\left(1+\frac{\log(1-\frac{y}{1+\log n})}{\log(1+\log n)}\right)^{\frac{1}{\beta}} - 1}{\frac{1}{\beta}\frac{1}{\log(1+\log n)}\frac{1}{1+\log n}}
\end{aligned}$$

For $y$ in a compact set, there must exist a positive number M which makes $|y| < M$, we can choose $n$, such that $\frac{|y|}{1+\log n} < 1$, that means $|y| - 1 < \log n, \text{or}\, n > e^{|y|-1}$, we choose $N = max(3, \lfloor e^{M-1} \rfloor)$, when $n \geq N$, then:

$$g_n(y) = \frac{1 + \frac{1}{\beta}\frac{\log(1-\frac{y}{1+\log n})}{\log(1+\log n)} + o\left(\frac{\log(1-\frac{y}{1+\log n})}{\log(1+\log n)}\right) - 1}{\frac{1}{\beta}\frac{1}{\log(1+\log n)}\frac{1}{1+\log n}}$$

So

$$g_n(y) \approx \frac{\log(1-\frac{y}{1+\log n})}{\frac{1}{1+\log n}} \to -y \ \ (n \geq N) \text{ uniformly}$$

Let $X_n = \log\frac{n(1-U_{n:n})}{\alpha}$, it converges to $\log\frac{Z}{\alpha} = \log Z - \log\alpha$ in law. And since:

$$Y_{n:n} \approx (\log(1+\log n - \log\frac{n(1-U_{n:n})}{\alpha}))^{\frac{1}{\beta}}$$

$$\frac{(\log(1+\log n - \log\frac{n(1-U_{n:n})}{\alpha}))^{\frac{1}{\beta}} - (\log(1+\log n))^{\frac{1}{\beta}}}{\frac{1}{\beta}(\log(1+\log n))^{\frac{1}{\beta}-1}\frac{1}{1+\log n}} \xrightarrow{L} \log\alpha - \log Z$$

Then:

$$\beta(\log(1+\log n))^{\frac{1}{\beta}-1}(1+\log n)Y_{n:n} - \beta\log(1+\log n)(1+\log n) - \log\alpha \xrightarrow{L} -\log Z.$$

$\square$

## 4.2 Extreme Spacings

Schuster (1984) used the convergence in probability of the the extreme spacings $Y_{n:n} - Y_{n-1:n}$ to refine the classical classifications of the extreme value distributions into subclasses named medium-medium, medium-long and medium-short, as explained in Chapter 2. Freimer *et al* (1989) used the convergence in distribution of the extreme spacings to clarify the refinement by caculating the magnitudes in probability of the extreme spacings in large samples and interpreting them as tail lengths. Mudholkar and Kollia (1994)examined the tail lengths of the generalized Weibull family. Mudholkar and Hutson(1996) examined the tail lengths of the exponential Weibull family using the similar method. Now we consider the family introduced in Chapter 1 for a similar treatment.

Let $S_{n:n} = Y_{n:n} - Y_{n-1:n}$ and $S_{1:n} = Y_{2:n} - Y_{1:n}$, then convergence results of the extreme spacings of this distribution are given in the following theorem.

**Theorem 2:** For a random sample of size $n$ from the family and random variable $(Z, X)$ with joint p.d.f.

$$f_{Z,X} = \begin{cases} e^{-z}, & \text{if } 0 \leq x \leq z, \\ 0, & \text{otherelse} \end{cases}$$

as $n \longrightarrow \infty$

$$n^{\frac{1}{\alpha\beta}} S_{1:n} \xrightarrow{L} Z^{\frac{1}{\alpha\beta}} - X^{\frac{1}{\alpha\beta}}$$

and

$$(\log(1+\log n))^{1-\frac{1}{\beta}}(1+\log n)S_{n:n} \xrightarrow{L} \frac{1}{\beta}(\log Z - \log X)$$

Proof:

Here we use Lemma 4.1 from Freimer *et al.* (1989) as given below.

**Lemma:** [Freimer *et al.* (1989)] If $U_{1:n} \leq U_{2:n} \leq ... \leq U_{n:n}$ are the ordered statistics from a random sample from $U(0,1)$ distribution, then as $n \to \infty$, $(n(1-U_{n-1:n}), n(1-U_{n:n}))$ converges in law to the random variable $(Z, X)$ having the joint distribution as given in the theorem.

Since $S_{n:n} = Y_{n:n} - Y_{(n-1):n}$ and $S_{1:n} = Y_{2:n} - Y_{1:n}$, using Theorem 1, we get the following results:

$$n^{\frac{1}{\alpha\beta}} S_{1:n} \xrightarrow{L} Z^{\frac{1}{\alpha\beta}} - X^{\frac{1}{\alpha\beta}}$$

$$(\log(1+\log n))^{1-\frac{1}{\beta}}(1+\log n)S_{n:n} \xrightarrow{L} \frac{1}{\beta}(\log Z - \log X)$$

□

**Corollary:** The left and right extreme spacings of a sample of size $n$ from the distribution in Eq(1.2) satisfy:

$$S_{1:n} = O_p(n^{-\frac{1}{\alpha\beta}})$$

and

$$S_{n:n} = O_p\left(\frac{(\log(1+\log n))^{\frac{1}{\beta}-1}}{1+\log n}\right)$$

Note: From the Corollary, it an be seen that in Schuster's terminology, classically medium right tail of this distribution is always medium-short. and the convergence rate of the extreme spacings is in terms of powers of $\log(1+\log n)$. So the medium-short tail is not very short.

# Chapter 5

# An Application

The flood rate of rivers have important economic, social, political and engineering implications. The modeling of flood data and analyses involving indications constitute an important application of the extreme value theory. Mudholkar and Hutson(1996) use the empirical TTT transform to demonstrate that exponential Weibull family provides a pratical model for the analysis of the flood data. Here we use the similar method to examine the model introduced in this thesis for the flood data and compare the model used by Mudholkar and Hutson(1996).

Table 5.1: The Consecutive Annual Flood Discharge Rates of the Floyd River at James, Iowa

| Year | Flood Discharge in($ft^3/s$) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1935-1944 | 1460 | 4050 | 3570 | 2060 | 1300 | 1390 | 1720 | 6280 | 1360 | 7440 |
| 1945-1954 | 5320 | 1400 | 3240 | 2710 | 4520 | 4840 | 8320 | 13900 | 71500 | 6250 |
| 1955-1964 | 2260 | 318 | 1330 | 970 | 1920 | 15100 | 2870 | 20600 | 3810 | 726 |
| 1965-1973 | 7500 | 7170 | 2000 | 829 | 17300 | 4740 | 13400 | 2940 | 5660 | |

## 5.1 Model Fits

The Floyd River flood rate data for the years 1935-1973 are given in Table 5.1. An exponentiated Weilbull model for the these data was constructed by estimating the parameters $\alpha, \theta$ and $\sigma$ by using the method of maximum likelihood as in Mudholkar *et al.*(1995): $\hat{\alpha} = 0.232$, $\hat{\theta} = 77.958$ and $\hat{\sigma} = 4.241$. Now we consider the model given in Eq(1.2).

### 5.1.1 Parameter Estimation

We use Maximum likelihood method to estimate the parameters and consider three cases: i) $\lambda = 1$ in Eq.(1.2), ii)$\alpha = 1$ in Eq.(1.2) and iii)full model in Eq.(1.2). In each case, the density and likelihood functions are given. For maximizalim , we use Excel.

i) Suppose $\lambda = 1$, then the model becomes:

$$F(x) = (1 - e^{1-e^{x^\beta}})^\alpha$$

and density function becomes:

$$f(x) = \alpha\beta(1 - e^{1-e^{x^\beta}})^{\alpha-1} e^{1+x^\beta-e^{x^\beta}} x^{\beta-1}$$

Maximum likelihood function is:

$$\begin{aligned} L(x,\alpha,\beta) &= \Pi_{i=1}^{n} \alpha\beta(1 - e^{1-e^{x_i^\beta}})^{\alpha-1} e^{1+x_i^\beta-e^{x_i^\beta}} x_i^{\beta-1} \\ &= \alpha^n \beta^n \Pi_{i=1}^{n} (1 - e^{1-e^{x_i^\beta}})^{\alpha-1} e^{n+\sum_{i=1}^{n} x_i^\beta - \sum_{i=1}^{n} e^{x_i^\beta}} \Pi_{i=1} n x_i^{\beta-1} \end{aligned}$$

log-likelihood function is:

$$\begin{aligned} \log L(x,\alpha,\beta) &= n\log\alpha + n\log\beta + (\alpha-1)\sum_{i=1}^{n}\log(1 - e^{1-e^{x_i^\beta}}) + n + \sum_{i=1}^{n} x_i^\beta - \sum_{i=1}^{n} e^{x_i^\beta} \\ &\quad +(\beta-1)\sum_{i=1}^{n}\log x_i \end{aligned}$$

Using Excel, we can get $\hat{\alpha} = 266.7722$, $\hat{\beta} = 0.081388$ and the maximum value of the log-likelihood function is -376.369.

ii)Suppose $\alpha = 1$, then:

$$F(x) = 1 - e^{\lambda(1-e^{x^\beta})}$$

density function is:

$$\begin{aligned} f(x) &= -e^{\lambda(1-e^{x^\beta})}(-\lambda)e^{x^\beta}\beta x^{\beta-1} \\ &= \lambda\beta e^{\lambda(1-e^{x^\beta})+x^\beta}x^{\beta-1} \end{aligned}$$

Maximum likelihood function is:

$$\begin{aligned} L(x,\lambda,\beta) &= \Pi_{i=1}^{n}\lambda\beta e^{\lambda(1-e^{x_i^\beta})+x_i^\beta}x_i^{\beta-1} \\ &= \lambda^n\beta^n e^{n\lambda-\lambda\sum_{i=1}^{n}e^{x_i^\beta}+\sum_{i=1}^{n}x_i^\beta}\Pi_{i=1}^{n}x_i^{\beta-1} \end{aligned}$$

log-likelihood function is:

$$\log L(x,\lambda,\beta) = n\log\lambda + n\log\beta + n\lambda - \lambda\sum_{i=1}^{n}e^{x_i^\beta} + \sum_{i=1}^{n}x_i^\beta + (\beta-1)\sum_{i=1}^{n}\log x_i$$

Using Excel, we can get $\hat{\beta} = 0.17023$ and $\hat{\lambda} = 0.011379$ and the maximum value of the log-likelihood function is -387.784.

iii)Consider the regular case:

$$F(x) = (1 - e^{\lambda(1-e^{x^\beta})})^\alpha$$

$$\begin{aligned} f(x) &= \alpha(1-e^{\lambda(1-e^{x^\beta})})^{\alpha-1}(-e^{\lambda(1-e^{x^\beta})})(-\lambda e^{x^\beta})\beta x^{\beta-1} \\ &= \alpha\lambda\beta(1-e^{\lambda(1-e^{x^\beta})})^{\alpha-1}e^{\lambda(1-e^{x^\beta})+x^\beta}x^{\beta-1} \end{aligned}$$

Maximum likelihood function is:

$$\begin{aligned} L(x,\alpha,\beta,\lambda) &= \Pi_{i=1}^{n}\alpha\lambda\beta(1-e^{\lambda(1-e^{x_i^\beta})})^{\alpha-1}e^{\lambda(1-e^{x_i^\beta})+x_i^\beta}x_i^{\beta-1} \\ &= \alpha^n\beta^n\lambda^n\Pi_{i=1}^{n}(1-e^{\lambda(1-e^{x_i^\beta})})^{\alpha-1}e^{n\lambda-\lambda\sum_{i=1}^{n}e^{x_i^\beta}+\sum_{i=1}^{n}x_i^\beta}\Pi_{i=1}^{n}x_i^{\beta-1} \end{aligned}$$

log-likelihood function is:

$$\log L(x,\alpha,\beta,\lambda) = n\log\alpha + n\log\beta + n\log\lambda + (\alpha-1)\sum_{i=1}^{n}\log(1-e^{\lambda(1-e^{x_i^\beta})}) + n\lambda$$
$$-\lambda\sum_{i=1}^{n}e^{x_i^\beta} + \sum_{i=1}^{n}x_i^\beta + (\beta-1)\sum_{i=1}^{n}\log x_i$$

Using Excel, we can get $\hat{\alpha} = 387.0979, \hat{\beta} = 0.077969$ and $\hat{\lambda} = 1.132394$ and the maximum value of log-likelihood function is -376.362.

### 5.1.2 Confidence Intervals for Parameters

A confidence interval gives an estimated range of values which is likely to include an unknown population parameter, the estimated range being calculated from a given set of sample data. If independent samples are taken repeatedly from the same population, and a confidence interval calculated for each sample, then a certain percentage (confidence level) of the intervals will include the unknown population parameter. Confidence intervals are usually calculated so that this percentage is 95%. The width of the confidence interval gives us some idea about how uncertain we are about the unknown parameter. A very wide interval may indicate that more data should be collected before anything very definite can be said about the parameter.

Traditionally, the central limit theorem and normal approximations are used to obtain standard errors and confidence intervals. But when the samples are not very large, the normal approximation may provide poor results. We may not be able to rely on normal-theory methods in the present case. Hence, we use resampling methods which provide inferential results for either normal or nonormal distribitions. Resampling techniques include bootstrap and jackknife. We mainly use bootstrap method. In the bootstrap, B new samples, each of the same size as the observed data, are drawn

with replacement from the observed data. The statistic is calculated for each new set of data, yeilding a bootstrap distribution for the statistic. We suppose $B = 1000$, confidence level 95%, and use the empirial percentiles and BCa (bias-corrected and adjusted )percentiles to get the confidence intervals.
Confidence intervals for parameters in three cases are provided belows:

i)Suppose $\lambda = 1$:
CI for $\alpha$:(157.3348, 640.1938)
CI for $\beta$:(0.07494897, 0.08966388)

ii)Suppose $\alpha = 1$:
CI for $\lambda$:(0.007851, 0.014907)
CI for $\beta$:(0.17020197, 0.1702328)

iii)The regular case:
CI for$\alpha$:(4.223883, 1414.707)
CI for$\lambda$:(0.04455415, 1.680493)
CI for$\beta$:(0.05785784, 0.1090082)

By comparison, the empirical percentiles are very easy to calculate, but may not be very accurate unless the sample size is very large. The BCa percentiles require more computation but they are more accurate and the length of confidence interval is shorter.

### 5.1.3 Model Suitability

The appropriateness of this model can be checked by using likelihood ratio test. The likelihood ratio test (LRT) is a statistical test of the goodness-of-fit between two models. A relatively more complex model is compared to a simpler model to see if it fits a particular dataset significantly better. If so, the additional parameters of the more complex model are often used in subsequent analyses. The LRT is only valid if used to compare hierarchically nested models. That is, the more complex model must differ from the simple model only by the addition of one or more parameters. Adding additional parameters will always result in a higher likelihood score. However, there comes a point when adding additional parameters is no longer justified in terms of significant improvement in fit of a model to a particular dataset. The LRT provides one objective criterion for selecting among possible models. The LRT begins with a comparison of the likelihood scores of the two models: $LR = 2 * (\log L1 - \log L2)$ This LRT statistic approximately follows a chi-square distribution. To determine if the difference in likelihood scores among the two models is statistically significant, we next must consider the degrees of freedom. In the LRT, degrees of freedom is equal to the number of additional parameters in the more complex model. Using this information we can then determine the critical value of the test statistic from standard statistical tables. The LRT is explained in more detail by Felsenstein (1981), Huelsenbeck and Crandall (1997), Huelsenbeck and Rannala (1997), and Swofford *et al.* (1996). While the focus of this part is using the LRT to compare two competing models, under some circumstances one can compare two competing trees estimated using the same likelihood model. There are many additional considerations as discussed in Kishino and Hasegawa (1989), Shimodaira and Hasegawa (1999) and Swofford *et al.* (1996).

Now consider our models, the maximum value of the loglikelihood for three cases are -376.369, -387.784 and -376.362 respectively.

Suppose $H_0 : \alpha = 1$, $H_1 : \alpha > 1$

log(likelihood ratio)$= 2(387.784 - 376.362) = 22.844$

degree of freedom$= 1$

critical value$(P = 0.05) = 3.84$

We reject the $H_0$. In this case, case II does not fit the data significantly better than case III, and it infers that the additional rate parameter is meaningful.

Suppose $H_0 : \lambda = 1$, $H_1 : \lambda > 1$

log(likelihood ratio)$= 2(376.369 - 376.362) = 0.014$

degree of freedom$= 1$

critical value$(P = 0.05) = 3.84$

We accept $H_0$, reject $H_1$.

Because this distribution has increasing or bathtub shaped hazard function, a graphical method based on the TTT (total time on test) transform introduced by Barlow and Campo(1975). Aarset(1987)proposes and illustrates the use of empirical TTT-tranform for identifying bathtub failure rates and offers a goodness of fittest of exponentiality. The TTT transform is a convenient tool for checking the nature of hazard rate. One of the principle use of the TTT concept has been in obtaining approximate optimal solutions for age replacement and also in obtaining approximate optimal burn-in times. The scaled TTT transform of a probability distribution with d.f. F(.) and quantile function Q(.) is:

$$\phi(U) = \frac{1}{\mu}\int_0^{Q(U)} (1 - F(t))dt.$$

Where $\mu$ is mean of this distribution. Then the scaled TTT transform $\phi(U) = U$ for the exponential distribution which has a constant hazard function. If $\phi(U)$ is convex then the hazard function $h(U)$ is decreasing, and $h(U)$ is increasing if $\phi(U)$ is concave. And If $\phi(U)$ is concave-convex then $h(U)$ is unimodal; and it is convex-concave if $h(U)$ is bathtub shaped. Given a sample from a population which is $x_{(1)} \leq x_{(2)} \cdots \leq x_{(n)}$, the hazard function of the population can be obtained by using the empirical scaled TTT transform:

$$\phi_n(i/n) = \left(\sum_{j=1}^{i}(n-j+1)(x_{(j)} - x_{(i)})\right)/\hat{\mu},$$

where $x_{(0)} = 0$ and $\hat{\mu} = \sum_{j=1}^{n}(n-j+1)(x_{(j)} - x_{(i)})$

For the above three cases, the quantile function are:

$$Q(U) = (\log(1 - \log(1 - U^{\frac{1}{\alpha}})))^{\frac{1}{\beta}}$$

,

$$Q(U) = (\log(1 - \frac{\log(1-U)}{\lambda}))^{\frac{1}{\beta}}$$

and

$$Q(U) = (\log(1 - \frac{\log(1-U^{\frac{1}{\alpha}})}{\lambda}))^{\frac{1}{\beta}}$$

respectively. Here gives the graph of the empirical scaled TTT transform for the Floyd River flood data. And the corresponding scaled TTT trandforms for that three cases, exponentiated-Weibull and exponential fits.

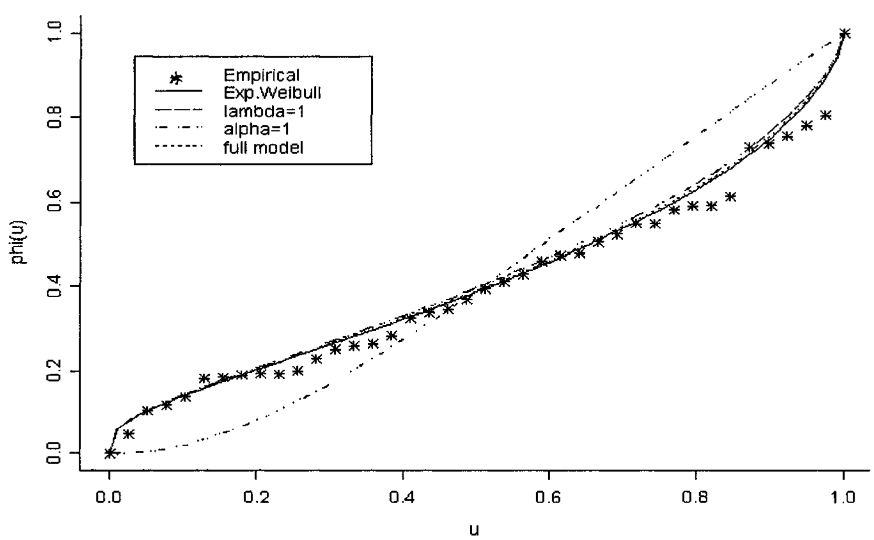


Figure 5.1: The scaled TTT Transforms for the Floyd river at James, Iowa

We can notice that this distribution almost has the same shape as the exponentiated-Weibull's. It is an ideal model to fit the flood data.

## 5.2 Conclusions and Remarks

A new three-parameter lifetime distribution with bathtub shape or increasing failure rate function is introduced in this thesis. We mainly studied the properties of the density function, tail shapes, hazard function and extremes and extreme spacings of this distribution in the similar method as the structural analysis of the Tukey lambda family in Freimer *et al.* (1988), of the Weibull family by Mudholkar and Kollia (1994) and Exponentiated-Weibull family by Mudholkar and Hutson (1996). The principal applications are in survival, reliability and the extreme-value analysis. For all the analysis given in this thesis, we consider $\lambda = 1$; for other values similar properties are postulated. We also checked it for the flood data by testing $\lambda = 1$. Another use of this distribution is to test the composite good-of-fit hypothesis of the distribution given by Chen(2000) by testing $\alpha = 1$. And by adding $\alpha$ shape parameter, we can find that the convergence of the sample extremes to their limiting distribution becomes faster and it fits the flood data for the Floyd River located at James, Iowa very well.

This distribution, just like exponentiated-Weibull distribution, is very useful in the lifetime, reliability and extreme-value data analysis. They are regular and amenable to simpler methods of analysis and inference.

For this model, the further work is still needed. Its focus can be on confidence interval, exact joint confidence regions for the parameters, the developments in accelerated life testing or optimal burn-in time or control. In other words, we can make use of the bathtub property of this distribution and let it be used in wider field.

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