STATIC AND BUCKLING ANALYSES OF CURVED METALLIC AND COMPOSITE BEAMS USING HIERARCHICAL FEM

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ABSTRACT

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The conventional finite element formulation has limitations in performing the static and buckling analyses of composite curved beams. The hierarchical finite element formulation provides us with the advantages of using fewer elements and obtaining better accuracy in the calculation of displacements, stresses and critical buckling loads. The hierarchical finite element formulation for uniform curved beams made of isotropic and composite materials is developed in the present work. Two sub-formulations of hierarchical finite element method viz. polynomial and trigonometric sub-formulations have been developed. The efficiency and accuracy of the developed formulation are established in comparison with the closed form solutions for uniform isotropic and composite curved beams. The central deflection values of uniform isotropic and composite curved beams are evaluated using the hierarchical finite element method. The critical buckling loads of composite curved beams are calculated based on the developed formulation and the results are validated with the approximate solution by the Ritz method. A detailed parametric study encompassing the influences of boundary conditions, laminate configuration, and the internal degrees of freedom is performed to see their effect on the central deflection and the critical buckling load. The NCT-301 graphite-epoxy composite material is considered in the analysis and in the parametric study.

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Nomenclature

E	Young's modulus
I	mass moment of inertia
EA	constant axial rigidity
EI	constant bending rigidity
h	thickness of the curved beam
A	cross sectional area of the curved beam
s	distance variable of 8 d.o.f. curved beam element
θ	angular variable of 8 d.o.f. curved beam element
β	subtending angle along the curvilinear distance s
R	constant radius of curvature
L	length of the curved beam element that is equal to $R\theta$
v	tangential displacement in the s direction as a function of s
v_s	derivative of the tangential displacement $\partial v / \partial s$
w	radial displacement in the z direction as a function of s
w_s	derivative of the radial displacement $\partial w / \partial s$ or slope θ
U	total strain energy of the curved beam
ε	axial strain in the s direction
κ	curvature of the middle surface
P	transverse load applied to the circular arch
$\{d\}$	displacement matrix
[T]	transformation matrix

[N]	interpolation function matrix
N_i'	derivative of the interpolation function N_i
ξ	non-dimensional co-ordinate
v_n	number of hierarchical terms used with tangential displacement
w_n	number of hierarchical terms used with radial displacement
A_r	coefficients of the polynomial hierarchical terms for tangential displacement
B_r	coefficients of the polynomial hierarchical terms for radial displacement
u	displacement in the x direction
ε_s	in-plane strain in the s direction
ε_x	in-plane strain in the x direction
ε_{sx}	transverse shear strain
ε_s^0	strain component of the middle plane in the s direction
κ_s	curvature in the s direction
ε_x^0	strain component of the middle plane in the x direction
κ_x	curvature in the x direction
N_s	in-plane force resultant in the s direction
M_s	bending moment resultant in the s direction
$[Q_{ij}]$	reduced stiffness matrix for the plane stress
[A]	laminate axial stiffness matrix (relating normal and shear forces per unit width to mid-plane strain components)

[B] bending-stretching coupling matrix

[D] laminate bending or flexural stiffness matrix (relating bending and twisting moments per unit width to curvatures)

A_{ij} stiffnesses defined for the A matrix

B_{ij} stiffnesses defined for the B matrix

D_{ij} stiffnesses defined for the D matrix

W_f potential energy owed to the transverse point load P

S_m polynomials and trigonometric functions defined to find the approximate radial displacement for the Ritz method

S_n polynomials and trigonometric functions defined to find the approximate tangential displacement for the Ritz method

A_m coefficients of the polynomials and trigonometric functions for the radial displacement solution

B_n coefficients of the polynomials and trigonometric functions for the tangential displacement solution

[F] force matrix

m variable defining the coefficients of the terms for radial displacement function for Ritz solution

n variable defining the coefficients of the terms for tangential displacement function for Ritz solution

a radius of the undeformed centroidal surface of the circular ring

b width of the circular ring

ds	circumferential element of the length before deformation
ds^*	circumferential element of the length after deformation
Δ	displacement of the free end of the curved beam
W_n	work done by the axial force at the free end of the curved beam
$[k]$	stiffness matrix of the finite element
$[n]$	geometric stiffness matrix associated with the bending deflection
G_{12}	in-plane shear modulus
G_{23}	out-of-plane shear modulus
e_k	ply thickness
z_k	centroidal distance of the ply from the reference plane
ν_{12}	Poisson's ratio between the fiber direction (1) and the transverse direction (2)

Chapter 1

Introduction

1.1 Buckling Analysis in Mechanical Design

Buckling is the finite bowing, warping, wrinkling, or twisting deformation that accompanies the development of excessive compressive stresses in isotropic and composite structures, particularly in thin walled structures. These structures include rods, Euler columns, plates and shells. The problem itself is a difficult one, for instability can be affected by various factors such as initial geometrical and material imperfections, non uniformity in load distribution, and the pattern of loading. Most studies are concerned with isotropic and orthotropic structures, the latter including beams, finite strips, plates and shells. Arbitrary geometries are much more difficult to analyze. In view of these factors, it is imperative that we develop efficient techniques and algorithms for the study of buckling.

1.2 Composite Materials and Structures

Basically, a composite material consists of two or more constituent materials or phases that have significantly different macroscopic behavior and a distinct interface between each constituent (on the microscopic level). This includes the continuous fiber-reinforced laminated composites that are of primary concern herein, as well as a variety of composites not specifically addressed.

The term composite material is usually referred to materials that are combinations of two or more organic or inorganic components, of which one serves as the matrix and the other as fiber. Individual fiber is usually stiffer and stronger than the matrix. The central concept behind composites is that the fibers and the matrix can blend into a new material with properties that are better than those of the constituent parts. In addition, by changing the orientation of the fibers, the composites can be optimized for strength, stiffness, fatigue, heat and moisture resistance, etc. It is therefore feasible to tailor the material to meet specific needs. Composite materials also have much higher strength to weight and stiffness to weight ratios than the conventional materials. The intrinsic mechanical behavior of composite laminates offers tremendous possibilities. Two composite laminates $[0/90]_s$ and $[90/0]_s$ consist of the same geometry, four layers of the same thickness, and are subjected to the same axial force and moments. The results show [1] that when the same axial force is applied, both stretch the same amount. However, when the same moment is applied, the laminates behave differently, with $[0/90]_s$ exhibiting a stiffer response than the $[90/0]_s$ plate. This illustrative example shows the great advantages of the structures made from the composite materials; by varying the fiber orientation we can alter and optimize the mechanical response of the structure under certain loadings [1].

A shell is a thin walled body, just as a beam or plate is, whose middle surface is curved in at least one direction. For instance, a cylindrical shell has only one direction in which the middle surface is curved. In many practical applications, such as in aircraft structures, we encounter plates having curvature in at least one direction. The strain energy expressions for a curved beam in a special reduced form of that for a thin shell are given by Novozhilov [2]. These expressions will be used in the present work for the static and buckling analyses of the composite curved beams.

1.3 Finite Element Method in Mechanical Design

The analysis of laminated composite beams is usually based on four approaches, classical theory of elasticity, theory of mechanics of materials, variational methods and strain energy statements. The governing equations of motion are generally nonlinear partial differential equations, which are extremely difficult to solve in the closed form. The availability and sophistication of modern digital computers have made possible the extensive use of the finite element method for analyzing complex structures. Finite Element Method (FEM) is one of the most powerful numerical analysis tools in the engineering and physical sciences. In spite of its tremendous potential, the FEM has some drawbacks too. The Hierarchical Finite Element Method (HFEM) provides us with critical advantages of using fewer elements and obtaining better accuracy in the calculation of displacements, stresses and for the buckling analysis of composite curved beams.

1.4 Literature Survey

In the following sub-sections a comprehensive and up-to-date literature survey on relevant topics is presented. Important works done on the finite element methodologies and buckling analysis of the composite shell structures have been chronicled. After a brief history of the hierarchical finite element method, seminal works on the HFEM analysis of beams are given.

1.4.1 Hierarchical Finite Element Method (HFEM)

The finite element method has been serving as a powerful tool for the analysis of structures. The finite element method in general, is a special case of the Rayleigh-Ritz method, with the main difference between the two lying in the choice of admissible functions used in the series representation of the solution. The standard Finite Element Method consists of dividing the domain of interest into a number of smaller – although not necessarily identical–convex sub-domains called Finite Elements. The solution is then approximated by locally admissible polynomial functions, which are piecewise smooth only over each individual sub-domain [3].

Most of the literature is devoted to the development of isotropic beam elements. Laminated beams have received less attention. Various straight and curved laminated beam finite elements were developed by Venkatesh and Rao [4], Yuan and Miller [5], Chan and Yang [6]. Most isotropic and composite beam finite elements are based on classical finite element methods in conjunction with classical, first or higher order lamination theories. The first finite elements were developed by Gallagher to model thin plates in bending and shells based on the Kirchhoff plate theory. The difficulties in these approaches are that the elements must satisfy the convergence requirements and be relatively effective in their applications. To arrive at a 3-D curved beam element formulation, we interpolate the curved geometry and corresponding beam displacements. With these interpolations a pure displacement-based element is derived. For curved elements spurious membrane strains are also obtained. Hence, a curved element also displays membrane locking.

There are various procedures that exist for the refinement of the finite element solutions. Broadly these fall into two categories: The first, and the most common, involves refining the mesh while keeping the degree of the elements fixed. This is termed as the *h*-version of the finite element method, or simply the finite element method. The second method involves keeping the mesh size constant and letting the degree of the approximating polynomial to tend to infinity [7,8]. This approach is better known as the *p*-version of the finite element method or the Hierarchical Finite Element Method (HFEM). Clearly, the HFEM has much in common with the classical Rayleigh–Ritz method; however, greater versatility and improved rates of convergence always result, since local (as opposed to global) admissible displacement functions are used [9].

Hierarchical functions were initially introduced by Zienkiewicz et al. [10] with the objective of introduction of p-graded meshes in an *a priori* chosen manner. Initial applications included the analysis of the nuclear reactor vessels [11]. Subsequently, new and useful families of p-type elements were introduced by Peano et al. [4, 12-13]. Explicit discussion of hierarchical functions has been done by Zienkiewicz et al. [14]. The use of non-uniform p-refinement in finite element method done hierarchically was initiated by Kelly et al. [15] and Gago et al. [16]. These papers as parts I and II respectively, deal with error analysis and adaptive processes applied to finite element calculations. In part I, they derive the basic theory and methods of deriving error estimates for second order problems. In part II, they provide in detail a strategy for adaptive refinement and concentrate on the p-convergent methods. It is shown that an extremely high rate of convergence is reached in practical problems using such procedures. They also present applications to realistic stress analysis and potential problems. Babuska et al. [8] describe the mathematical aspect of the convergence of the finite element solution for p-refinement. Szabo [17] showed that uniform p-refinement also allows the global energy norm error to be approximately extrapolated by three consecutive solutions.

The transition of the hierarchical finite element method from the developmental stages to the application stages has been rather arduous. In general, it offers superior performance to the h-version, but it took a long time for its merits to be recognized at the commercial level [18]. Polynomial functions are more common in the finite element analysis. With regards to the HFEM, Legendre polynomials in the Rodrigues form are quite popular. They have, for example been applied to linear analysis of plates in references [19,20]. In these references, it has been shown that convergence is achieved with far fewer degrees of freedom in the HFEM than that in the h-version of the FEM. Bardell et al. [21] applied the h-p method to study linear vibrations of shells.

Beam eigenfunctions, exact solutions of the linear problems, are hyperbolic-trigonometric or only trigonometric, depending on the boundary conditions. Another advantage of these functions is that the linear stiffness matrices and the mass matrices are diagonal, and therefore they are well-conditioned and they have several computational benefits. Also, since higher order polynomials are ill-conditioned [22], some researchers advised the use of trigonometric displacement shape functions [22-27].

The idea of using trigonometric terms in the finite element method is not new. Pian [28] described the concept of using more co-ordinates than the element nodal displacements in deriving element stiffness matrices. Krahula and Polhemus [27] used the Fourier series for a rectangular plane stress element.

1.4.2 Buckling Analysis of Composite Curved Beams and Shells

The objective of a nonlinear analysis is in many cases to estimate the maximum load that a structure can support prior to structural instability or collapse. In the analysis, the load distribution on the structure is known, but the load magnitude that the structure can sustain is unknown. Different structures respond differently to collapse or buckling. A thin plate under a transverse load does not have a collapse point; indeed because of membrane action, the plate increases its stiffness as the displacements grow. An arch, however, for specific geometric parameters, will collapse if load increases [29].

Wilkins and Love [30] examined the combined compression and shear buckling behavior of laminated composite cylindrical shells characteristic of the fuselage structure. Boron-epoxy and graphite epoxy shells of both $[\pm 45]_s$ and $[0/\pm 45]_s$ laminates were tested. Specimen sizes were $15''$ diameter and $15''$ length, with wall thickness of $0.0212'' - 0.0336''$.Compression-shear interaction curves were obtained for all the above. Compared to classical buckling theory, the actual compression buckling values were 65% of the theoretical value. The disparity was attributed to imperfections. Good agreement between theory and experiment were realized for shear buckling. It was observed that the compression-shear interaction was essentially linear.

Waltz and Vinson [31] presented methods of analysis for the determination of interlaminar stresses in laminated cylindrical shells of composite materials. El Naschie [32] investigated the large deflection behavior of composite material shells in determining the lower limit of the asymmetric buckling load. Ecord [33] wrote on a very practical shell structure, namely pressure vessels for the space shuttle orbiter. Here, a Kevlar 49 overwrap is used over a titanium and Inconel spherical pressure vessel structure. The Kevlar overwrap was designed to retain the internal pressure without metallic liner.

Johnson, Reck, and Davis [34] published a paper dealing with the design, fabrication and testing of a 10 foot long, 10 foot diameter ring stiffened corrugated graphite-epoxy cylindrical shell, typical of a large space structure, capable of resisting buckling. The results of the project established the feasibility of efficiently utilizing composites in structural shell applications. Compared with an aluminum design for the same application, the use of composites resulted in a 23% weight reduction. Fujczak [35] studied the torsional fatigue behavior of graphite-epoxy cylindrical shells. New important information resulted from that study. Booton [36] investigated the buckling of imperfect composite material cylinders under combined loadings, both theoretically and experimentally. The combined loadings involved axial compressions, external pressure and torsion. Donnell-Mushtari theory was used. Imperfections were more critical in axial compression than in external pressure or torsion, as expected.

Raju, Chandra, and Rao [37] studied the determination of transient temperatures in laminated composite conical shells caused by aerodynamic heating. Varadan [38] studied the snap-through buckling of composite shallow spherical shells. He calculated the critical buckling external pressure as a function of both the shell geometry and material properties. Also, Rhodes and Marshall [39] studied the asymmetric buckling of laterally loaded composite material shells. Montague [40] experimentally studied the behavior of double-skinned composite, circular cylindrical shells subjected to external pressure.

Humphrey [41] experimentally investigated hygrothermal effects on composite material pressure vessels, to be used as rocket motor cases. The tests show that Kevlar composites suffer far less degradation than fiberglass.

More recently, Bert [42, 43] and his colleagues have been very prolific in the area of shell theory of composite materials. He concentrated in the behavior of composites which have different properties in tension and compression, which he termed bi-modulus composites. These are typical of some composites such as fibre reinforced tires, and some biological materials. Yuceogle and Updike [44] have studied the stress concentrations in bonded, multilayer cylindrical shells.

1.4 Scope and Objectives of the Thesis

The objectives of the present thesis are, (1) to develop and evaluate the hierarchical finite element formulation for the static analysis of the curved beams made of isotropic and composite materials; (2) to conduct the buckling analysis of composite curved beams using the developed hierarchical finite element formulation; and, (3) to conduct a detailed parametric study of isotropic and composite curved beams.

Hierarchical finite element formulations are developed, viz. the trigonometric and polynomial formulations. Both the formulations are analyzed for their performance in the analysis of uniform curved beams made of isotropic and composite materials. All possible combinations of both symmetric and non-symmetric polynomial and trigonometric hierarchical terms are studied. The best combination is figured out to calculate the central deflection and the critical buckling load for the composite curved beams. The developed methodology not only gives more accurate and faster convergence, but also uses less number of elements, which is extremely advantageous in the analysis of composite structures. Finally, a detailed parametric study of composite curved beams is conducted for the buckling load and central deflection.

1.5 Layout of the Thesis

The present chapter provided a brief introduction and literature survey regarding the hierarchical finite element method and the static and buckling analyses of composite curved beams.

In chapter 2 the hierarchical finite element method is developed and applied to calculate the central deflection of the isotropic curved beams. Both the hierarchical sub-formulations, viz. trigonometric and polynomial formulations are developed and validated using closed form solutions. Finally a detailed comparison is made between the conventional and the hierarchical finite element formulations.

Chapter 3 gives the application of hierarchical finite element method to composite curved beams for calculating the central deflection. The results obtained by applying both trigonometric and polynomial hierarchical formulations are validated using the approximate solution obtained by using Ritz method.

In chapter 4 buckling analysis of the composite curved beam is performed by using hierarchical finite element formulation. Both trigonometric and polynomial hierarchical formulations are applied and the results are again validated using the approximate solution obtained using Ritz method.

Chapter 5 is devoted to the parametric study, which includes the effects of the internal degrees of freedom, boundary conditions, and the laminate configurations.

Chapter 6 brings the thesis to its end by providing the conclusions of the present work and some recommendations for future work.

Chapter 2

Hierarchical Finite Element Formulation for Curved Beams Made of Isotropic Material

2.1 Introduction

Beams with curvatures are another form of structures that we often encounter in practical structures. In the following, we limit our discussion to the beams curved and bent only in the plane of curvature so that no torsion is involved. Examples of application of curved beams can be found in fuselage rings, reinforcement rings for cylindrical and conical shells, arches, curved bridge girders, hooks, and so on. It is common for curved beams to have non-uniform cross section.

In this chapter the stiffness matrix is derived for a circularly curved beam element. Both the tangential and radial displacement functions are based on polynomials. The curved element is used to analyze a circular arch problem. Curved beam finite elements developed using hierarchical finite element method are studied as well.

2.2 Mathematical Model

Beams are actually three-dimensional solids. One-dimensional mathematical models of plane beams are constructed on the basis of beam theories. All such theories involve some form of approximation that describes the behavior of the cross-sections in terms of quantities evaluated at the longitudinal axis. More precisely, the kinematics of a plane straight beam is completely defined if the following functions are given: the transverse displacement $W(X)$ and the cross-section rotation $\theta_Z(X) = \theta\ (X)$, where X denotes the longitudinal co-ordinate in the reference configuration (Figure 2.1). The following beam model is in common use in structural mechanics.

2.2.1 Euler-Bernoulli (EB) Model

This is also called as *classical beam theory* or the *engineering beam theory model.* This model accounts for bending moment effects on stresses and deformations. Transverse shear forces are recovered from equilibrium but their effect on beam deformations is neglected. Its fundamental assumption is that cross-sections remain plane and normal to the deformed longitudinal axis. The rotation occurs about a neutral axis that passes through the centroid of the cross-section [45]. The rotation, θ (X) and the displacement, W(X) are related as indicated in Figure 2.1. In the present work, the Euler-Bernoulli (EB) Model is used for curved beam.

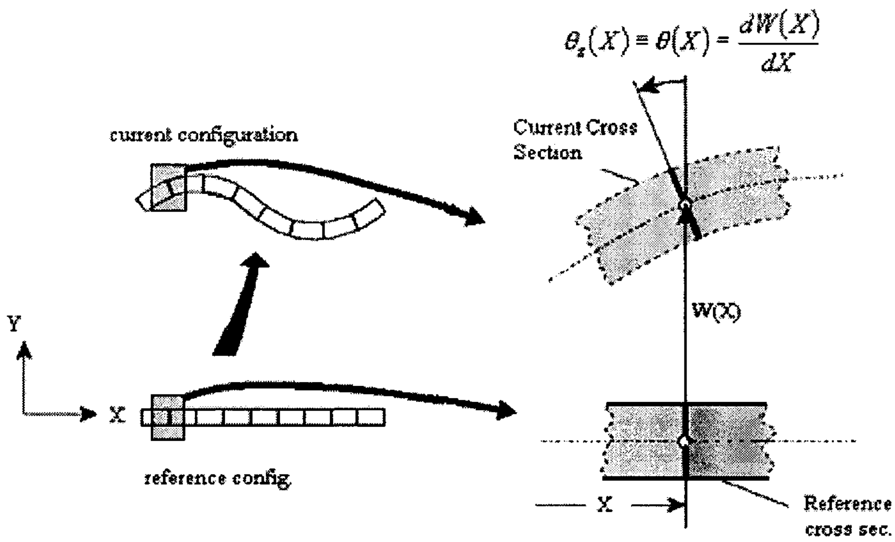


Figure 2.1 Beam Kinematics for EB Model

2.3 The Conventional Finite Element Formulation

2.3.1 Circularly Curved Beam Finite Elements

Circularly curved beams are a special form of curved shells. A study of curved beams is an important first step toward gaining insight into more complex shells.

The basic difference between a curved and a straight beam is that, in the small deflection theory, axial and flexural behaviors are coupled in the curved beam but not in the straight beam. Furthermore, in the finite element formulation, the displacement functions for curved beam elements must be capable of representing three rigid-body displacements: two orthogonal displacements and a rotation, all in the plane of curvature of the element.

There are many circularly curved beam finite elements available. But for the present case only one element is used by describing the assumptions of displacement functions for tangential (v) and radial (w) displacements and the assumed degrees of freedom at each nodal point. To explain this in detail, this element is formulated and evaluated in detail. Finally, an arch example [46] will be used for comparing results.

Element: Cubic functions are used for v and w

$$\begin{aligned} v &= a_1 + a_2 s + a_3 s^2 + a_4 s^3 \\ w &= a_5 + a_6 s + a_7 s^2 + a_8 s^3 \end{aligned} \tag{2.1}$$

where s is the tangential distance variable shown in Figure 2.2.

In order to demonstrate in depth the formulation and application of curved beam elements, we choose cubic-cubic element for which the stiffness matrix can be formulated explicitly and accurate results can be obtained.

2.3.2 Interpolation Functions

A circularly curved beam finite element is shown in Figure 2.2. The element has constant bending rigidity EI, axial rigidity EA, constant radius of curvature R, and subtending angle β, and length L that is equal to $R\,\beta$. The angular variable θ and distance variable s that is equal to $R\,\theta$ are measured from nodal point 1. The element possesses four degrees of freedom at each nodal point: a tangential displacement v, a derivative of tangential displacement $(\partial v/\partial s)$ or v_s, a radial displacement w, and a derivative of radial displacement $(\partial w/\partial s)$ or w_s, or slope θ.

The interpolation functions for both the tangential (v) and radial (w) displacements for this cubic element are of cubic polynomials in s as given in Equation (2.1). The eight constants are obtained by using the conditions of eight nodal degrees of freedom at both ends.

$$
\begin{array}{lll}
 & \text{at} & s = 0 \\
v = v_1 & \textit{and} & \dfrac{\partial v}{\partial s} = v_{s1} \\
w = w_1 & \textit{and} & \dfrac{\partial w}{\partial s} = w_{s1} = \theta_1 \\
 & \text{at} & s = L \\
v = v_2 & \textit{and} & \dfrac{\partial v}{\partial s} = v_{s2} \\
w = w_2 & \textit{and} & \dfrac{\partial w}{\partial s} = w_{s2} = \theta_2
\end{array}
\tag{2.2}
$$

Application of boundary conditions (2.2) yields

$$
\begin{Bmatrix} v_1 \\ v_{s1} \\ v_2 \\ v_{s2} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \tag{2.3a}
$$

Similarly

$$
\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{Bmatrix} a_5 \\ a_6 \\ a_7 \\ a_8 \end{Bmatrix} \tag{2.3b}
$$

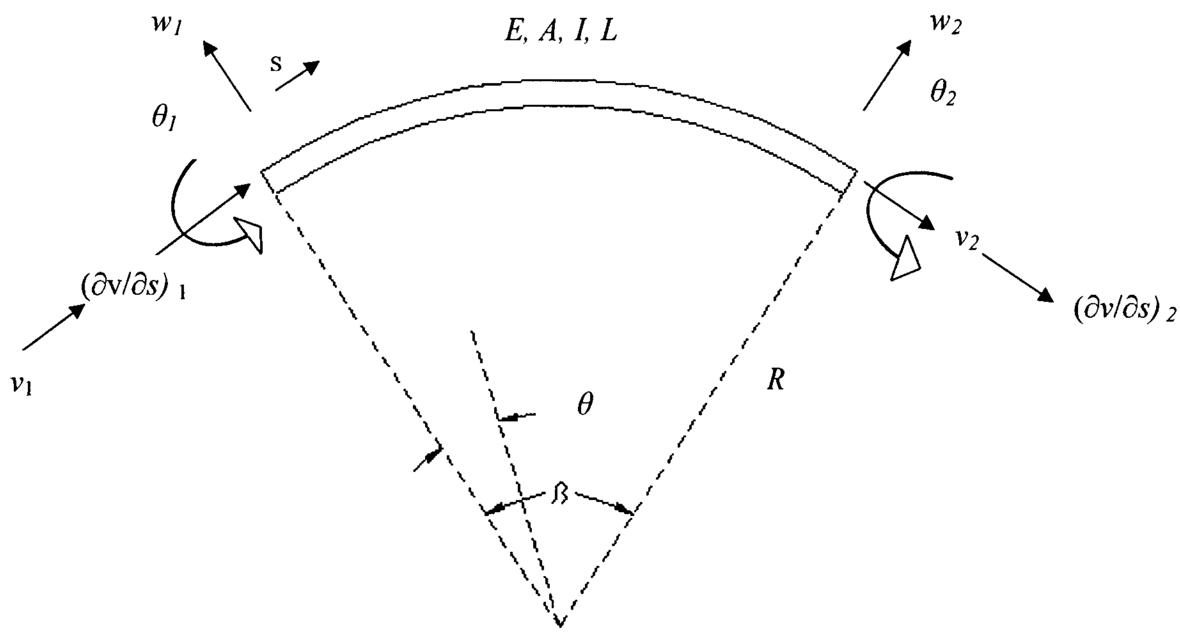


Figure 2.2 Eight Degrees-of- Freedom Circular Beam Element

In inverse form Equations (2.3a) and (2.3b) become

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \frac{1}{L^3}\begin{bmatrix} L^3 & 0 & 0 & 0 \\ 0 & L^3 & 0 & 0 \\ -3L & -2L & 3L & -L^2 \\ 2 & L & -2 & L \end{bmatrix}\begin{Bmatrix} v_1 \\ v_{s1} \\ v_2 \\ v_{s2} \end{Bmatrix} \tag{2.4a}$$

and

$$\begin{Bmatrix} a_5 \\ a_6 \\ a_7 \\ a_8 \end{Bmatrix} = \frac{1}{L^3}\begin{bmatrix} L^3 & 0 & 0 & 0 \\ 0 & L^3 & 0 & 0 \\ -3L & -2L & 3L & -L^2 \\ 2 & L & -2 & L \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \tag{2.4b}$$

or symbolically

$$\{a\} = [T]\,\{d\} \tag{2.5}$$

After substituting the a's back into the displacement functions and factoring out each degree of freedom, we obtain the displacement functions in the form of interpolation functions.

$$v\,(s) = N_1 v_1 + N_2 v_{s1} + N_3 v_2 + N_4 v_{s2} \tag{2.6}$$

$$w\,(s) = N_1 w_1 + N_2 w_{s1} + N_3 w_2 + N_4 w_{s2} \tag{2.7}$$

where the interpolation functions are

$$\begin{aligned} N_1 &= 1 - 3\left(\frac{s}{L}\right)^2 + 2\left(\frac{s}{L}\right)^3 \\ N_2 &= x - 2\left(\frac{s^2}{L}\right) + \left(\frac{s^3}{L^2}\right) \\ N_3 &= 3\left(\frac{s}{L}\right)^2 - 2\left(\frac{s}{L}\right)^3 \\ N_4 &= -\left(\frac{s^2}{L}\right) + \left(\frac{s^3}{L^2}\right) \end{aligned} \tag{2.8}$$

In normalized form

$$\xi = \frac{s}{L} = \frac{R\theta}{R\beta} = \frac{\theta}{\beta} \tag{2.9}$$

$$\begin{aligned} N_1 &= 1 - 3\xi^2 + 2\xi^3 \\ N_2 &= L\left(\xi - 2\xi^2 + \xi^3\right) \\ N_3 &= 3\xi^2 - 2\xi^3 \\ N_4 &= L\left(-\xi^2 + \xi^3\right) \end{aligned} \tag{2.10}$$

2.3.3 Strain Energy expression

The strain energy expressions for general thin shells are well known. The strain energy expression for a curved beam is in a special reduced form of that for a thin shell [46].

$$U = \frac{EA}{2}\int \varepsilon^2\, ds \;+\; \frac{EI}{2}\int \kappa^2\, ds \tag{2.11}$$

where ε and κ are the axial strain and curvature of the middle surface, respectively, with

$$\varepsilon = \frac{\partial v}{\partial s} + \frac{w}{R} = v' + \frac{w}{R} \tag{2.12}$$

$$\kappa = \frac{1}{R}\frac{\partial v}{\partial s} - \frac{\partial^2 w}{\partial s^2} = \frac{1}{R}v' - w'' \tag{2.13}$$

Substituting Equations (2.12) and (2.13) into Equation (2.11) gives

$$U = U_{vv} + U_{vw} + U_{ww} \tag{2.14a}$$

where

$$U_{vv} = \frac{EA}{2}\int_0^L (v')^2\, ds \;+\; \frac{EI}{2R^2}\int_0^L (v')^2\, ds$$

$$U_{vw} = \frac{EA}{R}\int_0^L v'\, w\, ds \;-\; \frac{EI}{R}\int_0^L v'\, w''\, ds \tag{2.14b}$$

$$U_{ww} = \frac{EA}{2R^2}\int_0^L (w)^2\, ds + \frac{EI}{2}\int_0^L (w'')^2\, ds$$

The energy expressions U_{vv}, U_{vw}, and U_{ww} are associated with axial, axial-flexural coupling, and flexural behaviors, respectively.

2.3.4 Stiffness Equations

Substituting the displacement functions for v and w given by Equation (2.1) into the energy expressions (2.14a) and (2.14b), and then performing partial differentiations of the strain energy with respect to each of the eight degrees of freedom, the 8×8 stiffness matrix for the element is obtained.

$$\begin{Bmatrix} X_1 \\ X_1' \\ X_2 \\ X_2' \\ \cdots \\ Y_1 \\ M_1 \\ Y_2 \\ M_2 \end{Bmatrix} = \left[\begin{array}{c|c} \left[k_{vv\,4\times4}\right] & \left[k_{vw\,4\times4}\right] \\ \hline \left[k_{wv\,4\times4}\right] & \left[k_{ww\,4\times4}\right] \end{array} \right] \begin{Bmatrix} v_1 \\ v_{s1} \\ v_2 \\ v_{s2} \\ \cdots \\ w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \tag{2.15}$$

where X_1' and X_2' are the counterpart generalized forces in inch-pounds associated with the degrees of freedom v_{s1} and v_{s2}, respectively. The coefficients in the 4×4 sub-matrices are obtained as

$$\begin{aligned} k_{vv_{ij}} &= \int_0^L EA\left(1+\frac{\alpha}{R^2}\right) N_i\, N_j'\, ds \\ k_{vw_{ij}} &= k_{wv_{ji}} = \int_0^L \frac{EA}{R}\left(N_i\, N_j' - \alpha\, N_i'\, N_j''\right) ds \\ k_{ww_{ij}} &= \int_0^L EA\left(\frac{N_i\, N_j}{R^2} + \alpha\, N_i''\, N_j''\right) ds \end{aligned} \tag{2.16}$$

where the primes indicate differentiation with respect to s and $\alpha = EI/EA$.

For the convenience of assemblage, it is desirable to number all the degrees of freedom at each nodal point in a certain sequence. For this purpose the above matrix is rearranged as:

$$\begin{Bmatrix} X_1 \\ X_1' \\ Y_1 \\ M_1 \\ \cdots\cdots \\ X_2 \\ X_2' \\ Y_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \\ \\ \\ & & [k_{8\times 8}] & & \\ \\ \\ \end{bmatrix} \begin{Bmatrix} v_1 \\ v_{s1} \\ w_1 \\ \theta_1 \\ \cdots\cdots \\ v_2 \\ v_{s2} \\ w_2 \\ \theta_2 \end{Bmatrix} \tag{2.17}$$

2.3.5 Curved Beam Example: Analytical Solution

In order to evaluate the performance of the 8 degrees-of-freedom element, an example of a semicircular arch as shown in Figure 2.3(a) and Figure 2.3(b) is analyzed. The parameters are defined as $A = 1\times 1$ in^2, $R = 17$ in, $P = 2000$ lb, and $E = 10^7$ psi. Three different approaches are used to analyze this problem: 1. Analytical solution using Castigliano's theorem, 2. Conventional FEM solution using 8 degrees-of-freedom curved element, 3. Conventional FEM solution using 6 degrees-of-freedom frame element [46].

First, the analytical solution is obtained. Due to symmetry, only half of the arch as shown in Figure 2.3(b) needs to be analyzed. At an arbitrary point B, the bending moment and axial force are respectively given by

$$
\begin{aligned}
M &= PR\sin\theta - QR(1-\cos\theta) - M_o \\
S &= P\sin\theta + Q\cos\theta
\end{aligned}
\tag{2.18}
$$

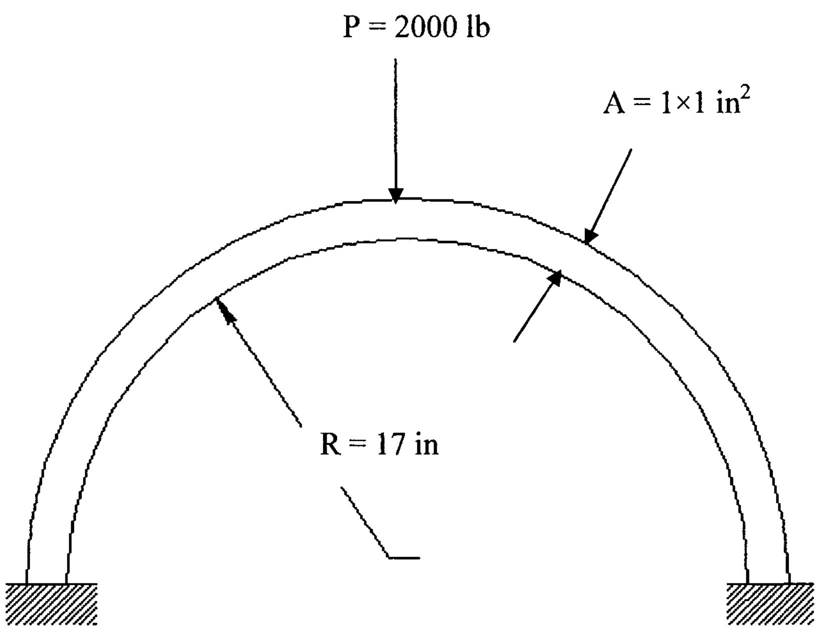


Figure 2.3 (a) Circular Arch under a Central Load

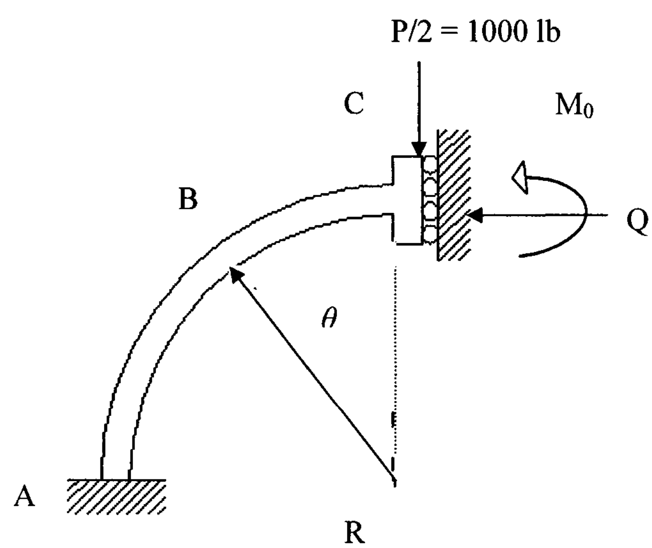


Figure 2.3(b) Half of the Arch Being Analyzed

The strain energy expressions are

$$U = \int_0^{\frac{\pi}{2}} \frac{M^2 R}{2EI} d\theta + \int_0^{\frac{\pi}{2}} \frac{S^2 R}{2EA} d\theta \tag{2.19}$$

Because of symmetry, the tangential displacement v_c and rotation θ_c at point C are both zero.

$$\begin{aligned} v_c &= \frac{\partial U}{\partial Q} = \frac{R^2}{EI}\left[-\frac{PR}{2} + QR\left(\frac{3\pi}{4} - 2\right) + M_0\left(\frac{\pi}{2} - 1\right)\right] + \frac{R}{EA}\left(\frac{Q\pi}{4} + \frac{P}{2}\right) = 0 \\ \theta_c &= \frac{\partial U}{\partial M_o} = \frac{R}{EI}\left[M_0 \frac{\pi}{2} - PR + QR\left(\frac{\pi}{2} - 1\right)\right] = 0 \end{aligned} \tag{2.20}$$

For $R = 17$ in , $A =$ $1{\times}1$ in^2 , $I = 1/12$ in^4 , and $P/2 =$ 1000 lb ,

Solving the two foregoing equations simultaneously gives,

Q = $0.9159P$ lb

$M_0 =$ $0.3037PR$ = $5.1645P$ in-lb

$w_c =$ 0 .14152 in

2.3.6 Solution using Eight Degrees of Freedom (D.O.F.) Curved Beam Element

If one element is used to model half of the arch, the boundary conditions are

$$v_1 = w_1 = \left(\frac{\partial w}{\partial s}\right)_1 = v_2 = \left(\frac{\partial w}{\partial s}\right)_2 = 0 \tag{2.21}$$

From Equation (2.17), the stiffness equations can be obtained as

$$\begin{Bmatrix} X_1' \\ X_2' \\ Y_2 \end{Bmatrix} = 10^5 \begin{bmatrix} 356.150 & -89.0375 & -15.7263 \\ -89.0375 & 356.150 & 15.7263 \\ -15.7263 & 15.7263 & 3.43724 \end{bmatrix} \begin{Bmatrix} v_{s1} \\ v_{s2} \\ w_2 \end{Bmatrix} \tag{2.22}$$

Inverting the matrix gives

$$\begin{Bmatrix} v_{s1} \\ v_{s2} \\ w_2 \end{Bmatrix} = 10^{-9} \begin{bmatrix} 35.3144 & 2.12302 & 151.860 \\ 2.12302 & 35.3144 & -151.860 \\ 151.860 & -151.860 & 4298.91 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1000 \end{Bmatrix} \tag{2.23}$$

which gives $w_2 = .004299 \ in$, with -97 % error compared to the analytical solution of w_c = 0 .14152 in.

The solution for the central deflection using 2, 3, 4…12 equal-length elements is given in Table 2.1. The solution converges rapidly and monotonically as the number of elements increases as shown in Figure 2.7. The error reduces to less than 1 % with the four- elements mesh (15 degrees-of-freedom). It is noted that better accuracy may be obtained if elements of unequal length are used i.e., using smaller elements near the central load.

2.3.7 Solution using Six Degrees-of-Freedom (D.O.F) Frame Element

A typical frame finite element with field variables defined at each node is drawn in Figure 2.4. Stiffness matrix for the frame element can be obtained by superposing augumented stiffness matrices of bar and beam elements [47].

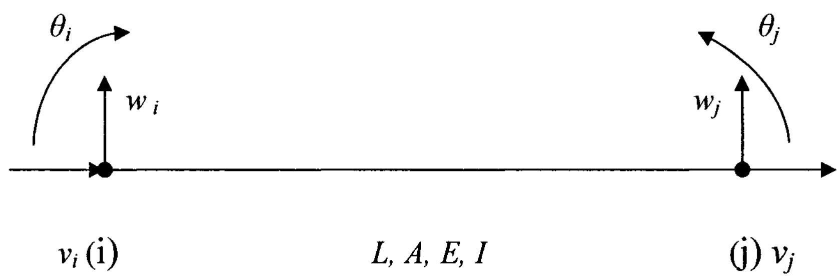


Figure 2.4 Frame Element

Frame element stiffness matrix in the local coordinate system is given by

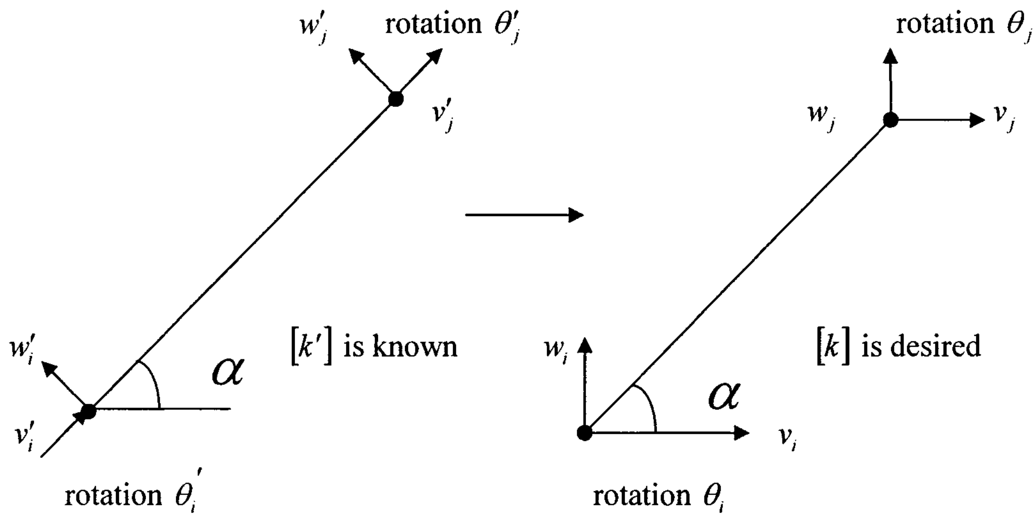
$$[k'] = \frac{AE}{L}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{EI}{L^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \tag{2.24}$$

The nodal degrees of freedom are shown in Figure 2.5. Note that for clarity and brevity, rotations at nodal points, that are θ_i' , θ_j' , θ_i and θ_j were not shown in Figure 2.5. Also note that $\theta_i' = \theta_i$; $\theta_j' = \theta_j$.

$$\begin{Bmatrix} v_i' \\ w_i' \\ \theta_i' \\ v_j' \\ w_j' \\ \theta_j' \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_i \\ w_i \\ \theta_i \\ v_j \\ w_j \\ \theta_j \end{Bmatrix} \tag{2.25}$$

where $c = \cos\alpha$ and $s = \sin\alpha$

The transformation is given by



Note: $\theta_i' = \theta_i$; $\theta_j' = \theta_j$

Figure 2.5 Transformation of Displacements for a Frame Element

$$\{d'\} = [T]\{d\} \tag{2.26}$$

where $\{d'\}^T = \{v_i' \;\; w_i' \;\; \theta_i' \;\; v_j' \;\; w_j' \;\; \theta_j'\}$ and $\{d\}^T = \{v_i \;\; w_i \;\; \theta_i \;\; v_j \;\; w_j \;\; \theta_j\}$

Since there is no change in the θ values, that is $\theta_i' = \theta_i \;\; \theta_j' = \theta_j$, the value of 1 appears in [T].

$$[k]\{d\} = \{r\} \tag{2.27}$$

$$\{r\} = [T]^T\{r'\} \tag{2.28}$$

$$[k']\{d'\} = \{r'\} \tag{2.29}$$

$$\{r\} = [T]^T[k']\{d'\} = [T]^T[k'][T]\{d\} = [k]\{d\} \tag{2.30}$$

where

$$[T]^{T}[k'][T]\{d\}=[k] \tag{2.31}$$

Transformed stiffness matrix [k] is given by

$$[k]=\frac{AE}{L}\begin{bmatrix} c^2 & cs & 0 & -c^2 & -cs & 0 \\ cs & s^2 & 0 & -cs & -s^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -c^2 & -cs & 0 & c^2 & cs & 0 \\ -cs & -s^2 & 0 & cs & s^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}+\frac{EI}{L^3}\begin{bmatrix} 12.s^2 & & & & & \\ -12.c.s & 12.c^2 & & & sym & \\ -6.L.s & 6.L.c & 4.L^2 & & & \\ -12.s^2 & 12.c.s & 6.L.S & 12.s^2 & & \\ 12.c.s & -12.c^2 & -6.L.c & -12.c.s & 12.c^2 & \\ -6.L.s & 6.L.c & 2L^2 & 6.L.s & -6.L.c & 4.L^2 \end{bmatrix} \tag{2.32}$$

where as before, the abbreviations $c=\cos\alpha$, $s=\sin\alpha$ are used.

To evaluate the performance of curved beam element, a straight frame element with 6 degrees of freedom was used to analyze the same semi-circular arch problem instead of using curved beam 8 degrees-of-freedom element. A mesh of 3 straight 6 d.o.f. frame elements is shown in Figure 2.6. The results for the central deflection obtained by using up to 16 equal-length straight frame elements are given in Table 2.1 for comparison with the curved beam element solution.

It is seen that for nearly the same numbers of degrees-of-freedom (except for the first two elements) the solution obtained using curved beam elements are better than that of the straight frame elements. The results obtained by using the curved beam elements get even better when we increase the number of degrees-of-freedom (D.O.F.). The curve (Figure 2.7) for the straight frame elements starts with quite good accuracy

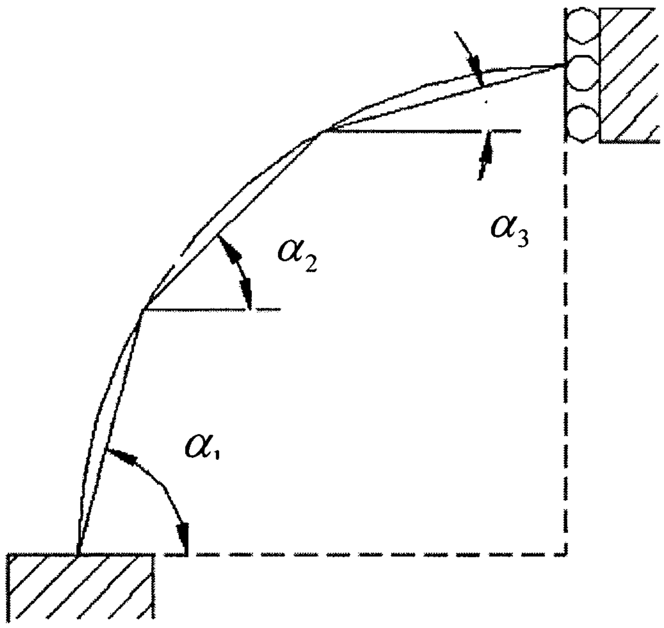


Figure 2.6 A Mesh of 3 Straight 6 D.O.F. Frame Elements

Table 2.1 Convergence Study of the Two Types of Finite Elements for The Arch Problem

8 D.O.F. Curved Beam Elements				**6 D.O.F. Frame Elements**			
Number of Elements	*Number of D.O.F.*	*Centre Deflection(in)*	*Error (%)*	*Number of Elements*	*Number of D.O.F.*	*Centre Deflection(in)*	*Error (%)*
1	3	0.0042	96.9	1	1	0.00480	97
2	7	0.1146	18.9	2	4	0.12780	96.9
3	11	0.1371	3.08	3	7	0.13474	4.79
4	15	0.1403	0.808	4	10	0.13748	2.86
5	19	0.1411	0.282	6	16	0.13964	1.33
6	23	0.1413	0.119	7	19	0.14012	0.99
7	27	0.1414	0.06	8	22	0.14044	0.76
8	31	0.1414	0.031	10	28	0.14082	0.49
9	35	0.1414	0.02	11	31	0.14094	0.41
10	39	0.1415	0.0117	12	34	0.14103	0.34
11	43	0.14151	0.00712	14	40	0.14116	0.25
12	47	0.14151	0.00677	15	43	0.14121	0.22
				16	46	0.14125	0.19

Analytical Solution: Central Deflection = 0.1415238 in

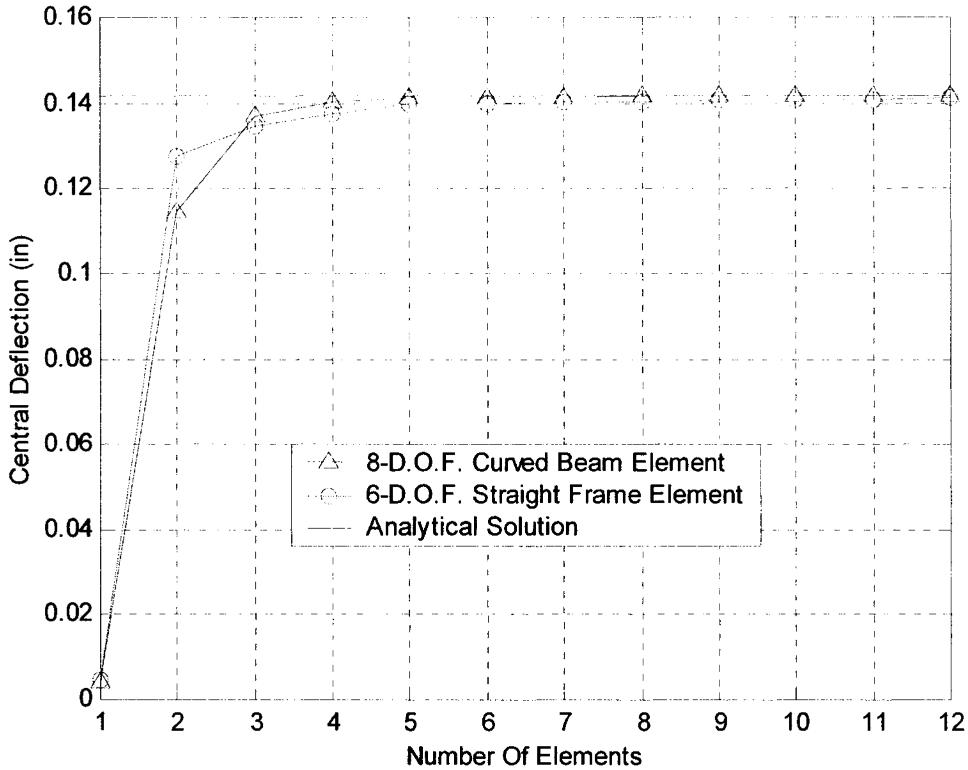


Figure 2.7 Comparison between Analytical Solution, Curved Beam Element Solution and Straight Frame Element Solution

at a low number of degrees of freedom but converges very slowly as the number of degrees of freedom increases.

In general, the axial-flexural behaviors are intricately coupled in the arch structures. It is recommended that we not be prejudiced against the axial displacement function *v*; that is, the degree of accuracy or order of polynomials assumed for the axial displacement function *v* should be comparable to that for the flexural displacement function *w*.

2.4 The Hierarchical Finite Element Method (HFEM)

In the formulation of the finite element model using the conventional formulation, we assumed a cubic polynomial function for both the tangential displacement (v) and radial displacement (w) (Equation 2.1). In the hierarchical formulation, we modify the approximating functions (i) by adding trigonometric functions and (ii) by adding polynomial functions. We shall study both these cases simultaneously and ascertain the pros and cons of them as we proceed in our analysis.

2.4.1 Formulation Based on Euler – Bernoulli Theory

The co-ordinate system used to define the geometry of the two-node curved beam element is shown in Figure 2.2.

2.4.2 Trigonometric Hierarchical Formulation

The tangential displacement (v) and radial displacement (w) functions of the above mentioned element are given as:

$$
\begin{aligned}
v(s) &= a_1 + a_2 s + a_3 s^2 + a_4 s^3 + \sum_{r=1}^{N} a_{r+4} \sin[\delta_r s] \\
w(s) &= c_1 + c_2 s + c_3 s^2 + c_4 s^3 + \sum_{r=1}^{N} c_{r+4} \sin[\delta_r s]
\end{aligned}
\tag{2.33}
$$

where $\delta_r = \dfrac{r\pi}{L}$, $\qquad r = 1, 2, 3 \ldots N$

and a_i and c_i are coefficients to be determined.

The element degrees of freedom in this case are the same as in the conventional case, viz. tangential displacement (v) and its derivative $\frac{\partial v}{\partial s}$, and radial displacement (w) and slope ($\theta = \frac{\partial w}{\partial s}$). The polynomial terms in the assumed displacement field are used as before to define the element nodal degrees of freedom and the trigonometric terms are used to provide additional degrees-of-freedom that are not physical to the interior of the element. The above equation can be written in the matrix form as

$$
\begin{aligned}
v(s) &= [g][a] \\
w(s) &= [g][c]
\end{aligned}
\tag{2.34}
$$

where

$$
\begin{aligned}
g &= \left[1, s, s^2, s^3, \{\sin[\delta_r s]\}\right] \\
a &= \left[a_1, a_2, a_3, a_4, \{a_{r+4}\}\right]^T \\
c &= \left[c_1, c_2, c_3, c_4, \{c_{r+4}\}\right]^T
\end{aligned}
\tag{2.35}
$$

where $\{a_{r+4}\}$ contains terms such as $a_5, a_{6,} a_7, a_8$ and so on. In a similar manner $\{\sin[\delta_r s]\}$ and $\{c_{r+4}\}$ are defined.

Now, upon evaluating $v, v_s, w, and\ \theta$ at node 1 (i.e. at $s = 0$) and at node 2 (at $s = L$) and evaluating the hierarchical term when r = 1, we get the following matrices.

$$\begin{Bmatrix} v_1 \\ v_{s1} \\ v_2 \\ v_{s2} \\ v_{v5} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \delta_1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & -\delta_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} \tag{2.36}$$

Similarly

$$\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_{w5} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \delta_1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & -\delta_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{Bmatrix} \tag{2.37}$$

When r = 2,

$$\begin{Bmatrix} v_1 \\ v_{s1} \\ v_2 \\ v_{s2} \\ v_{v5} \\ v_{v6} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \delta_1 & \delta_2 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -\delta_1 & \delta_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} \tag{2.38}$$

Similarly

$$\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_{w5} \\ w_{w6} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \delta_1 & \delta_2 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -\delta_1 & \delta_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix} \tag{2.39}$$

The first matrix given (Equation 2.36) can be written in the following form:

$$[p] = [h][a] \tag{2.40}$$

Therefore,

$$[a] = [h]^{-1}[p] \tag{2.41}$$

Upon substitution of [a] in Equation (2.34) we get,

$$v = [g][h]^{-1}[p] \tag{2.42}$$

or,

$$v = [N][p] \tag{2.43}$$

where

[N] = Interpolation function matrix

$$\begin{aligned} [N] &= [g][h]^{-1} \\ &= \left[1,\, s,\, s^2,\, s^3,\, \{\sin(\delta_r s)\}\right][h]^{-1} \end{aligned} \tag{2.44}$$

Hence the individual interpolation functions will be

$$\begin{aligned} N_1 &= 1 - 3\left(\frac{s}{L}\right)^2 + 2\left(\frac{s}{L}\right)^3 \\ N_2 &= x - 2\left(\frac{s^2}{L}\right) + \left(\frac{s^3}{L^2}\right) \\ N_3 &= 3\left(\frac{s}{L}\right)^2 - 2\left(\frac{s}{L}\right)^3 \\ N_4 &= -\left(\frac{s^2}{L}\right) + \left(\frac{s^3}{L^2}\right) \end{aligned} \tag{2.45}$$

or

$$
\begin{aligned}
N_1 &= 1 - 3\xi^2 + 2\xi^3 \\
N_2 &= L\left(\xi - 2\xi^2 + \xi^3\right) \\
N_3 &= 3\xi^2 - 2\xi^3 \\
N_4 &= L\left(-\xi^2 + \xi^3\right)
\end{aligned}
\tag{2.46}
$$

where

$$
\xi = \frac{s}{L} = \frac{R\theta}{R\beta} = \frac{\theta}{\beta}
\tag{2.47}
$$

and the trigonometric hierarchical shape functions are

$$
N_{r+4} = -\delta_r s + \left(2\delta_r + (-1)^r \delta_r\right)s^2 + \left(-\delta_r - (-1)^r \delta_r\right)s^3 + \sin[\delta_r s]
\tag{2.48}
$$

where

$$
\delta_r = \frac{r\pi}{L}, \qquad \text{r} = 1, 2, 3, \ldots..\text{N}
$$

Hence, the displacement field for the element, in terms of the nodal degrees of freedom and the hierarchical degrees of freedom, can now be written as,

for tangential displacement (v)

$$
v = N_1 v_1 + N_2 v_{s1} + N_3 v_2 + N_4 v_{s2} + \sum_{r=1}^{N} N_{r+4} v_{vr+4}
\tag{2.49}
$$

and similarly for radial displacement (w)

$$
w = N_1 w_1 + N_2 \theta_1 + N_3 w_2 + N_4 \theta_2 + \sum_{r=1}^{N} N_{r+4} w_{wr+4}
\tag{2.50}
$$

The values of N_1, N_2, N_3 and N_4 at s = *0* and s = L are the same as given in Equations (2.8) and (2.10).

The hierarchical term(s) N_{r+4} have the values as follows at each end.

$$N_{r+4} = 0 \qquad \text{at} \qquad s = 0 \quad \text{and} \quad s = L \tag{2.51}$$

$$N'_{r+4} = 0 \qquad \text{at} \qquad s = 0 \quad \text{and} \quad s = L \tag{2.52}$$

Figures 2.8, 2.9, 2.10, and 2.11 show the values of the hierarchical shape functions N_5, N_6, N_7, N_8 and their derivatives at various locations within the element. These functions provide zero displacement and zero slope at each end. This feature is highly significant, since these functions only provide additional freedom to the interior of the element and do not affect the element's nodal degrees of freedom.

2.4.3 Generation of the Finite Element Model

To generate the finite element model using the HFEM, different combinations of trigonometric hierarchical terms were tried to get the most accurate results. Combinations involve from one to four trigonometric terms for each of the tangential (v) and radial (w) displacements. Firstly for symmetric combinations of hierarchical terms, same number of hierarchical terms are used with tangential (v) and radial (w) displacement functions e.g. $v_1 - w_1, v_2 - w_2, v_3 - w_3, v_4 - w_4$, where $v_1 - w_1$ means that one hierarchical term is used with tangential displacement (v) function and one with radial displacement (w) function. For non-symmetric combinations hierarchical terms are used in different numbers with both tangential (v) and radial (w) displacement functions e.g. $v_0 - w_n, v_1 - w_n, v_2 - w_n, v_3 - w_n$ and $v_4 - w_n$ where n = 1, 2, 3, 4.

For instance, for the case of two symmetric terms (Equation 2.53), two trigonometric hierarchical terms were added to both tangential (v) and radial (w) displacement functions. Initially the order of the matrix was 8×8, which changed to 12×12 for this particular matrix.

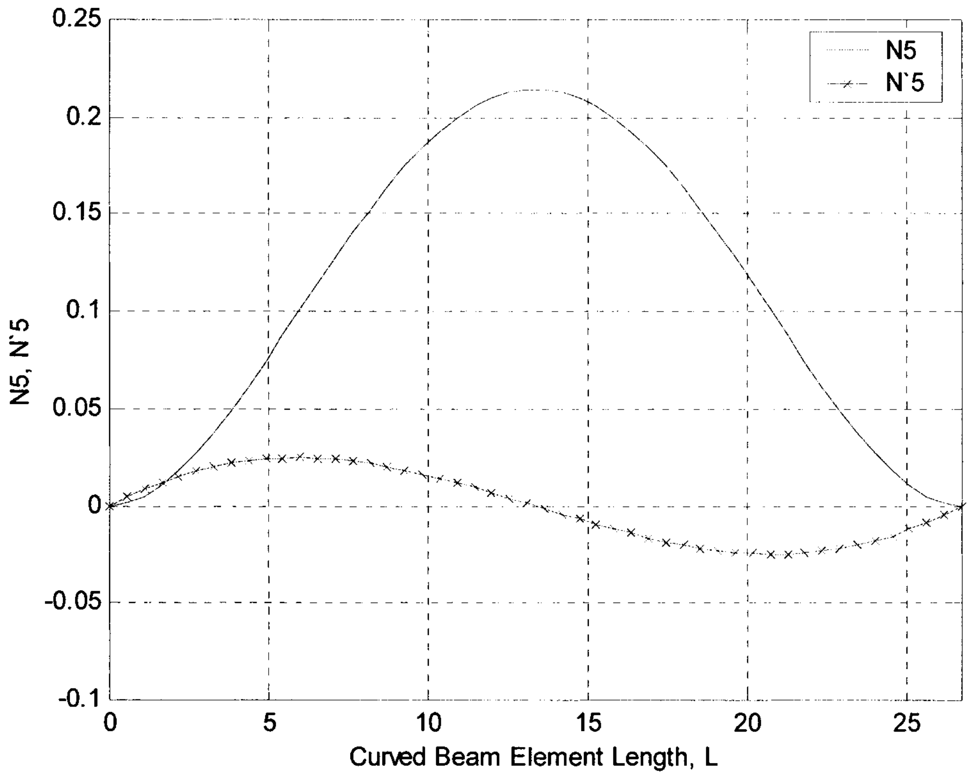


Figure 2.8 The First Trigonometric Hierarchical Shape Function (N_5) and its Derivative (N_5')

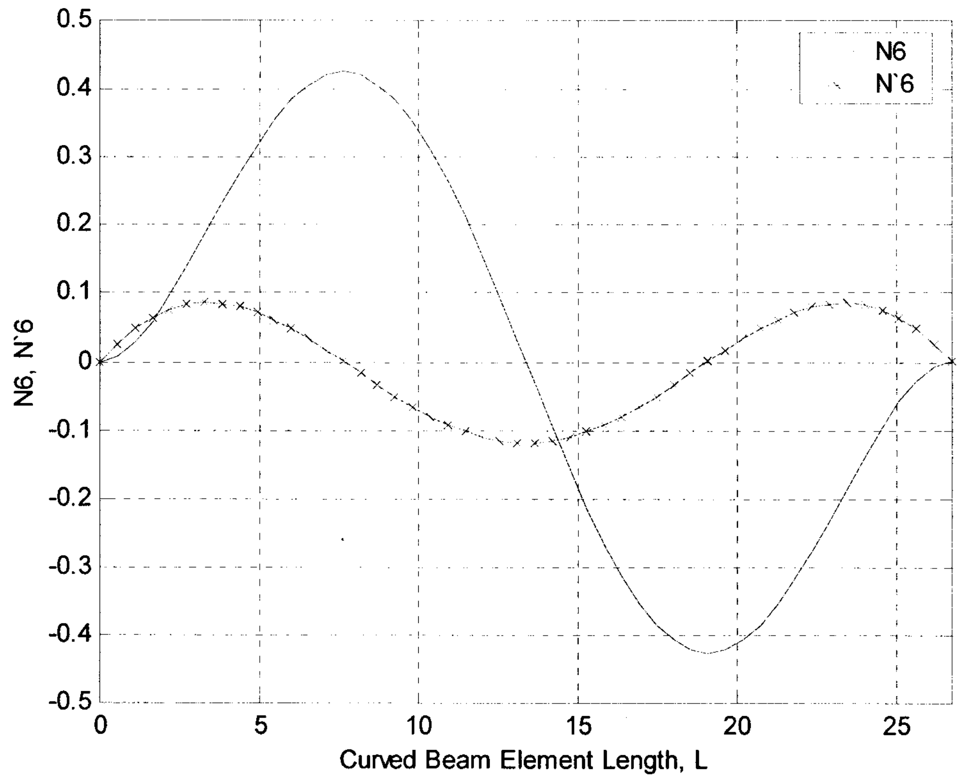


Figure 2.9 The Second Trigonometric Hierarchical Shape Function (N_6) and its Derivative (N_6')

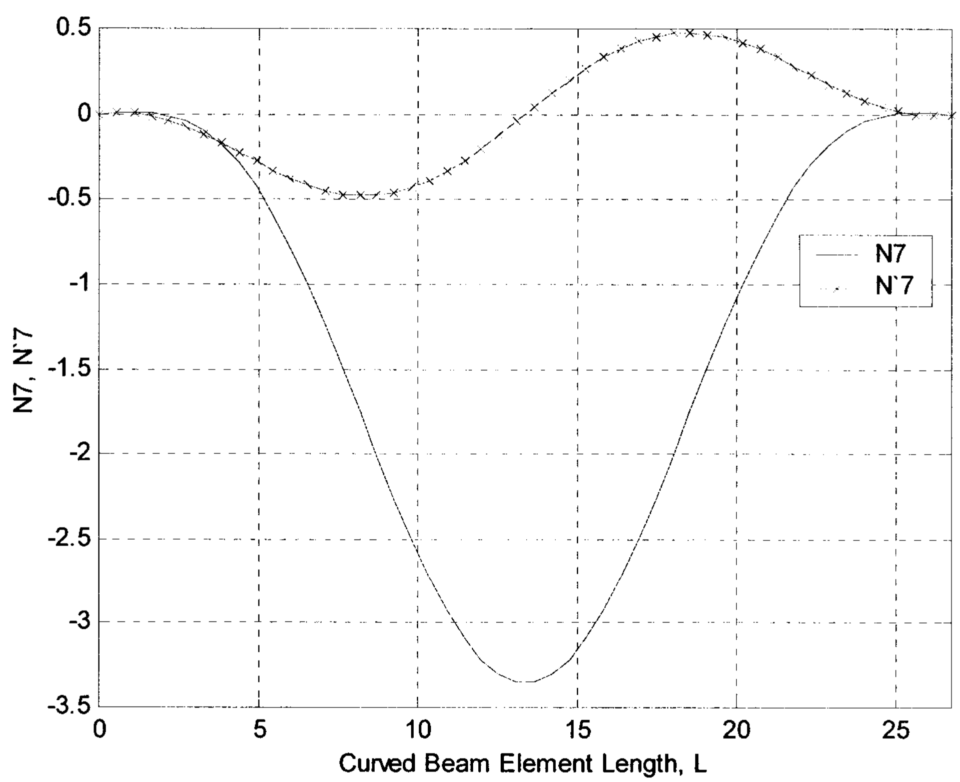


Figure 2.10 The Third Trigonometric Hierarchical Shape Function (N_7) and its Derivative (N_7')

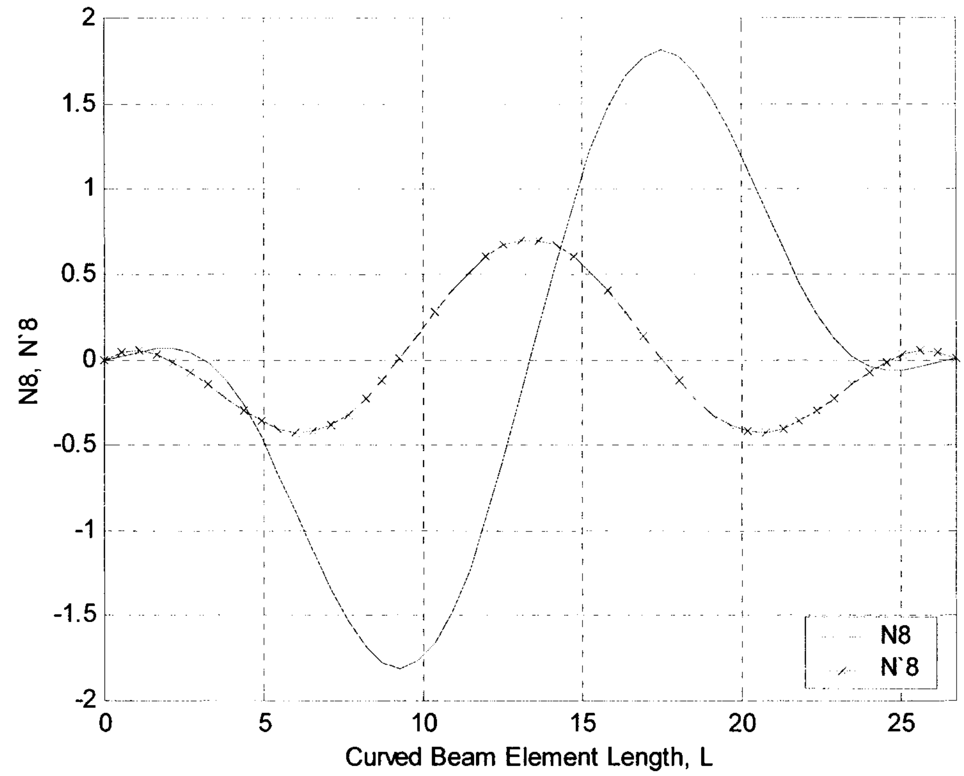


Figure 2.11 The Fourth Trigonometric Hierarchical Shape Function (N_8) and its Derivative (N_8')

The stiffness matrix by using two symmetric hierarchical terms is given below.

$$\begin{Bmatrix} X_1 \\ X_1' \\ Y_1 \\ M_1 \\ F_{1v} \\ F_{2v} \\ X_2 \\ X_2' \\ Y_2 \\ M_2 \\ F_{1w} \\ F_{2w} \end{Bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ & & [k_{12\times12}] & & \\ \\ \\ \\ \\ \\ \end{bmatrix} \begin{Bmatrix} v_1 \\ v_{s1} \\ w_1 \\ \theta_1 \\ v_{v5} \\ v_{v6} \\ v_2 \\ v_{s2} \\ w_2 \\ \theta_2 \\ w_{w5} \\ w_{w6} \end{Bmatrix} \tag{2.53}$$

As it is shown in Equation (2.53), hierarchical terms $v_{v5}, v_{v6}, w_{w5}, w_{w6}$ were added to the curved beam element for each of tangential (v) and radial (w) displacements. A total of 2, 4, 6, and 8 hierarchical terms were added for one symmetric trigonometric, two symmetric trigonometric, three symmetric trigonometric and four symmetric trigonometric hierarchical terms respectively. A considerable improvement was noted in the results after applying trigonometric hierarchical finite element model.

The results obtained using different symmetric hierarchical terms are given in Table 2.2. These results are then plotted against the number of elements for the comparison of different symmetric combinations in Figure 2.12.

Table 2.2 Central Deflection Calculated by using Symmetric Trigonometric Hierarchical Terms

8 D.O.F. Curved Beam Element										
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Conventional FEM		*Symmetric Trigonometric 1 HFEM term*		*Symmetric Trigonometric 2 HFEM terms*		*Symmetric Trigonometric 3 HFEM terms*		*Symmetric Trigonometric 4 HFEM terms*	
1	3	0.00429	5	0.00128	7	0.14079	9	0.14085	11	0.14148
2	7	0.11463	9	0.13330	11	0.13341	13	0.13360	15	0.13363
3	11	0.13715	13	0.13960	15	0.13961	17	0.13967	19	0.13969
4	15	0.14038	17	0.14090	19	0.14091	21	0.14093	23	0.14094
5	19	0.14112	21	0.14128	23	0.14128	25	0.14129	27	0.14129
6	23	0.14135	25	0.14141	27	0.14114	29	0.14141	31	0.14141
7	27	0.14143	29	0.14146	31	0.14146	33	0.14146	35	0.14146
8	31	0.14147	33	0.14149	35	0.14149	37	0.14149	39	0.14149
9	35	0.14149	37	0.14150	39	0.14150	41	0.14150	43	0.14150
10	39	0.14150	40	0.14151	43	0.14151	45	0.14151	47	0.14151
11	43	0.14151	45	0.14151	47	0.14151	49	0.14151	51	0.14151
12	47	0.14151	49	0.14151	51	0.14151	53	0.14151	55	0.14151

Analytical Solution: Central Deflection = 0.14152 in

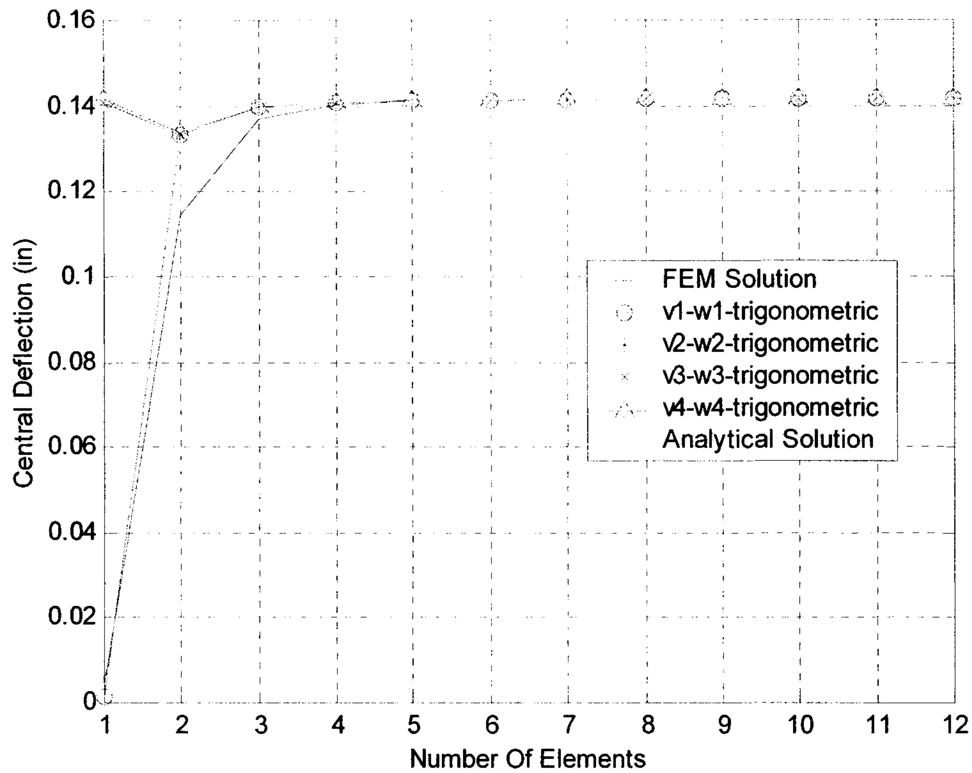


Figure 2.12 Comparison between the Solutions Obtained using Symmetric Trigonometric Hierarchical Terms

2.4.4 Discussion

Figure 2.12 shows that when 1 symmetric trigonometric hierarchical term is used the results show better convergence compared to conventional FEM except the 1-element mesh. When 2, 3, and 4 trigonometric hierarchical terms were used there was a considerable improvement in the results right from the very first element. The results obtained by using 2, 3, and 4 hierarchical terms are almost matching. The results of all HFEM and FEM models seem to converge at the 5-elements mesh.

The non-symmetric trigonometric terms were applied in the following way.

(i) $v_0 - w_n$ where n = 1, 2, 3, 4

which means that tangential displacement (v) function is provided with no trigonometric hierarchical function term and radial displacement (w) function is provided with 1, 2, 3, and 4 trigonometric hierarchical terms successively.

Similarly, the following cases were considered.

(ii) $v_1 - w_n$ where n = 0, 2, 3, 4

(iii) $v_2 - w_n$ where n = 0, 1, 3, 4

(iv) $v_3 - w_n$ where n = 0, 1, 2, 4

(v) $v_4 - w_n$ where n = 0, 1, 2, 3

The results obtained using these non-symmetric trigonometric hierarchical terms combinations are given in Tables 2.2, 2.3, 2.4, 2.5, 2.6, and 2.7. The values of central deflections are also plotted against the number of elements, which are shown in Figures (2.13 – 2.18).

Figure 2.13 shows that for only the 1st element the combinations ($v_0 - w_n$) display better result than conventional FEM results. All other elements show a little improvement and all the curves are almost matching. So it is clear from this figure that radial displacement (w) interpolation functions show little effect on the results because with the increase of hierarchical terms with radial displacement interpolation functions the improvement in the results was not that much significant.

Table 2.3 Central Deflection Calculated by using Non- Symmetric Trigonometric Hierarchical Terms ($v_0 - w_n$)

8 D.O.F. Curved Beam Element										
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Conventional FEM		*Non-Symmetric Trigonometric 0v-1w HFEM terms*		*Non-Symmetric Trigonometric 0v-2w HFEM terms*		*Non-Symmetric Trigonometric 0v-3w HFEM terms*		*Non-Symmetric Trigonometric 0v-4w HFEM terms*	
1	3	0.00429	4	0.07480	5	0.08009	6	0.10028	7	0.10033
2	7	0.11463	9	0.12185	11	0.12411	13	0.12418	15	0.12420
3	11	0.13715	14	0.13860	17	0.13969	20	0.13870	23	0.13870
4	15	0.14038	19	0.14084	23	0.14085	27	0.14085	39	0.14085
5	19	0.14112	24	0.14131	29	0.14131	34	0.14131	46	0.14131
6	23	0.14135	29	0.14144	35	0.14144	41	0.14144	54	0.14144
7	27	0.14143	34	0.14148	41	0.14148	48	0.14148	62	0.14148
8	31	0.14147	39	0.14150	46	0.14150	55	0.14150	70	0.14150
9	35	0.14149	44	0.14151	52	0.14151	62	0.14151	78	0.14151
10	39	0.14150	49	0.14151	58	0.14151	69	0.14151	86	0.14151
11	43	0.14151	54	0.14152	64	0.14152	76	0.14152	94	0.14152
12	47	0.14151	59	0.14152	70	0.14152	83	0.14152	102	0.14152

Analytical Solution: Central Deflection = 0.14152 in

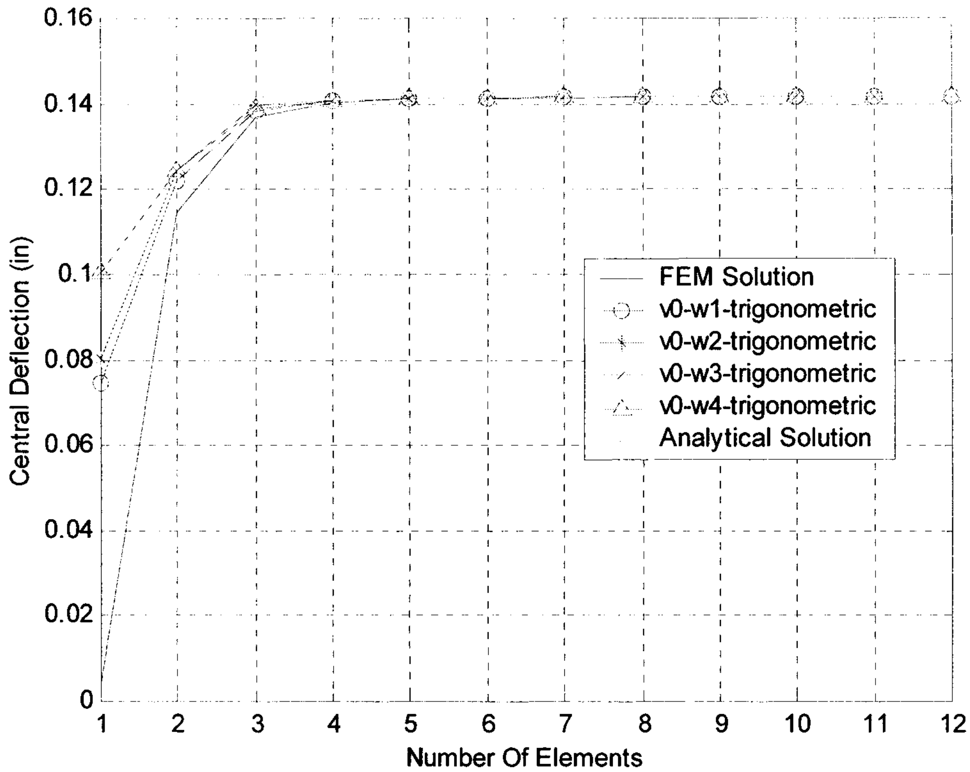


Figure 2.13 Comparison between the Results Corresponding to Non-Symmetric ($v_0 - w_n$) Trigonometric Hierarchical Terms

The results for the group of combinations ($v_1 - w_n$) are given in Table 2.4 and are plotted in Figure 2.14. The results show a little bit improvement for these combinations when one hierarchical term is added to the tangential (v) displacement function. For the first 4 or 5 elements the results are better compared to the results given by the combinations ($v_0 - w_n$) but after that results given by the combinations ($v_0 - w_n$) converge more rapidly.

Table 2.4 Central Deflection Calculated by using Non- Symmetric Trigonometric Hierarchical Terms ($v_1 - w_n$)

8 D.O.F. Curved Beam Element										
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Conventional FEM		*Non-Symmetric Trigonometric 1v-0w HFEM terms*		*Non-Symmetric Trigonometric 1v-2w HFEM terms*		*Non-Symmetric Trigonometric 1v-3w HFEM terms*		*Non-Symmetric Trigonometric 1v-4w HFEM terms*	
1	3	0.00429	4	0.00432	6	0.00128	7	0.02089	8	0.02089
2	7	0.11463	9	0.13394	12	0.13333	14	0.13610	16	0.13610
3	11	0.13715	14	0.14002	18	0.13961	21	0.14027	24	0.14027
4	15	0.14038	19	0.14104	24	0.14090	28	0.14112	32	0.14112
5	19	0.14112	24	0.14133	30	0.14128	35	0.14136	40	0.14136
6	23	0.14135	29	0.14143	36	0.14141	42	0.14145	48	0.14145
7	27	0.14143	34	0.14147	42	0.14146	49	0.14148	56	0.14148
8	31	0.14147	39	0.14149	48	0.14149	56	0.14150	64	0.14150
9	35	0.14149	44	0.14150	54	0.14150	63	0.14150	72	0.14150
10	39	0.14150	49	0.14151	60	0.14151	70	0.14151	80	0.14151
11	43	0.14151	54	0.14151	66	0.14157	77	0.14151	88	0.14151
12	47	0.14151	59	0.14151	72	0.14151	84	0.14151	96	0.14151

Analytical Solution: Central Deflection = 0.14152 in

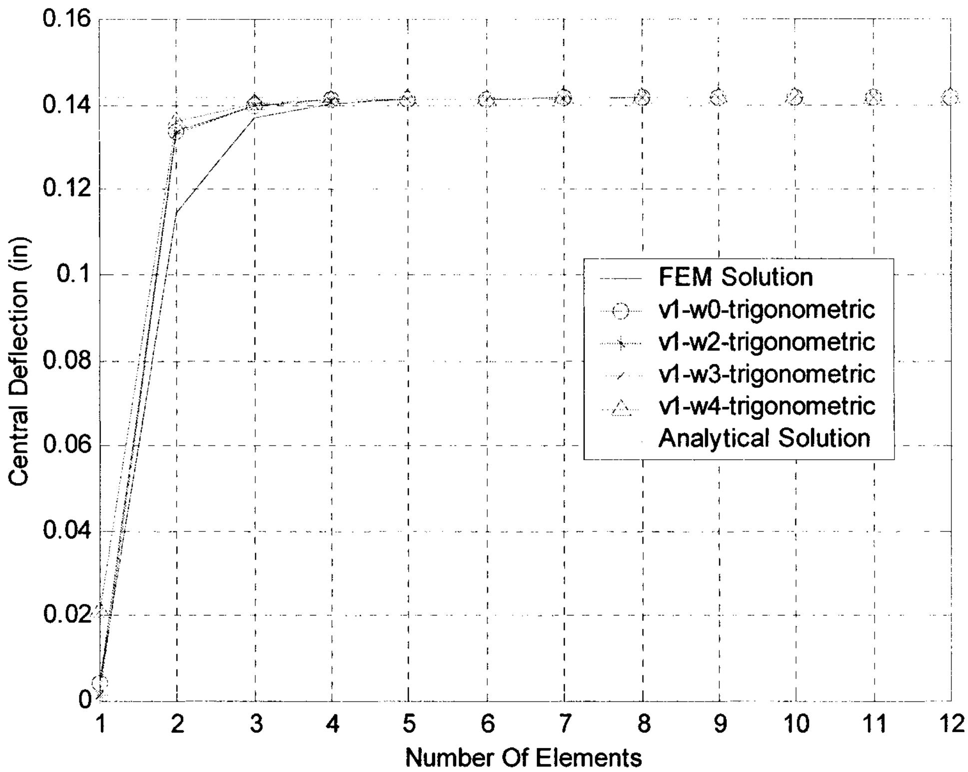


Figure 2.14 Comparison between the Results Corresponding to Non-Symmetric ($v_1 - w_n$) Trigonometric Hierarchical Terms

The results for the group of combinations ($v_2 - w_n$) are given in Table 2.5 and they are plotted in Figure 2.15. These combinations show some improvement from the previous group of combinations in terms of convergence of the results to the analytical solution. In particular, the combination ($v_2 - w_4$) provides convergence of the central deflection values right from the 1st element. The results for the rest of the combinations are also matching with each other very closely and their trend is the same as shown in Figure 2.14.

Table 2.5 Central Deflection Calculated by using Non- Symmetric Trigonometric Hierarchical Terms ($v_2 - w_n$)

8 D.O.F. Curved Beam Element										
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Conventional FEM		*Non-Symmetric Trigonometric* 2v-0w HFEM terms		*Non-Symmetric Trigonometric 2v-1w HFEM terms*		*Non-Symmetric Trigonometric 2v-3w HFEM terms*		*Non-Symmetric Trigonometric 2v-4w HFEM terms*	
1	3	0.00429	5	0.00432	6	0.00133	8	0.14084	9	0.14083
2	7	0.11463	11	0.13394	12	0.13341	15	0.13635	17	0.13636
3	11	0.13715	17	0.14002	18	0.13961	22	0.14029	25	0.14029
4	15	0.14038	23	0.14104	24	0.14091	29	0.14112	33	0.14112
5	19	0.14112	29	0.14133	30	0.14128	36	0.14136	41	0.14136
6	23	0.14135	35	0.14143	36	0.14141	43	0.14145	49	0.14150
7	27	0.14143	41	0.14147	42	0.14146	50	0.14148	57	0.14148
8	31	0.14147	47	0.14149	48	0.14149	57	0.14150	65	0.14150
9	35	0.14149	53	0.14150	54	0.14150	64	0.14150	73	0.14150
10	39	0.14150	59	0.14151	60	0.14151	71	0.14151	81	0.14151
11	43	0.14151	65	0.14151	66	0.14151	78	0.14151	89	0.14151
12	47	0.14151	71	0.14151	72	0.14151	85	0.14151	97	0.14151

Analytical Solution: Central Deflection = 0.14152 in

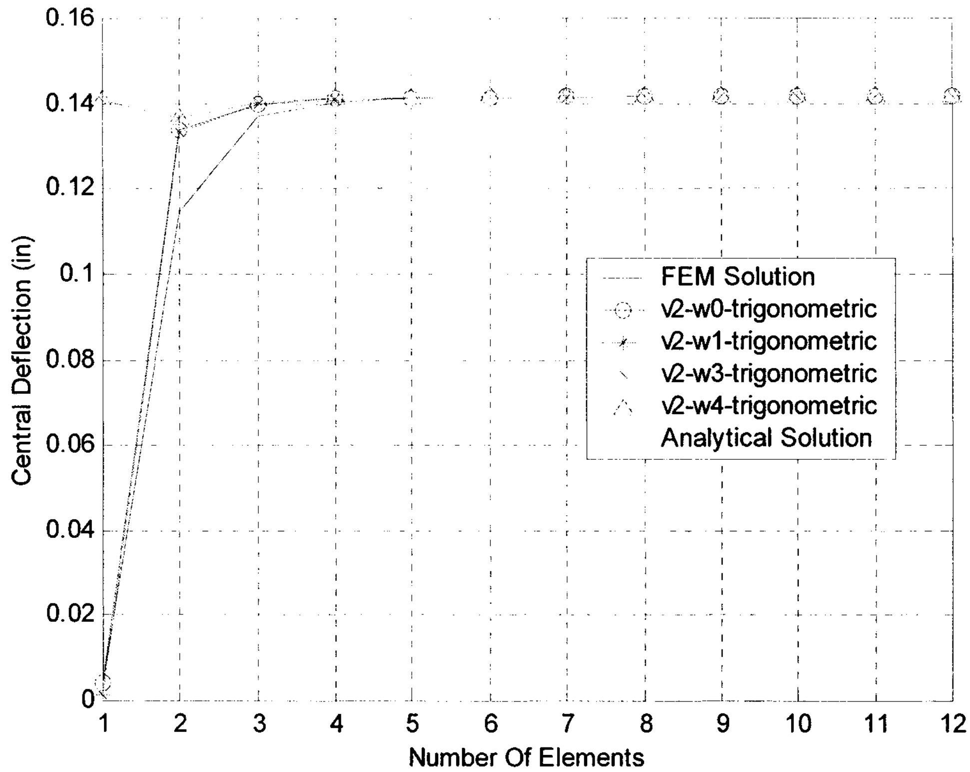


Figure 2.15 Comparison between the Results Corresponding to Non-Symmetric ($v_2 - w_n$) Trigonometric Hierarchical Terms

For the group of combinations ($v_3 - w_n$), the combinations ($v_3 - w_2$) and ($v_3 - w_4$) give results very similar to that of the results given by the combination ($v_2 - w_4$). The rest of the combinations give results closer to each other. It is also evident that with the addition of the hierarchical terms to the tangential displacement (v) function the results get better addition of each element.

The group of combinations ($v_4 - w_n$) provides better convergence of the results than all the previous combinations because of the addition of four hierarchical terms to the tangential displacement (v) function. The combination ($v_4 - w_2$) provides the best convergence among all the non-symmetric trigonometric combinations, which were used in the previously. The results given by this combination were very close to the analytical solution right from the 1-element mesh and they converge more rapidly than all the other combinations.

All the combinations seem to converge around the 4-elements mesh which is better than the case of the symmetric trigonometric hierarchical formulation.

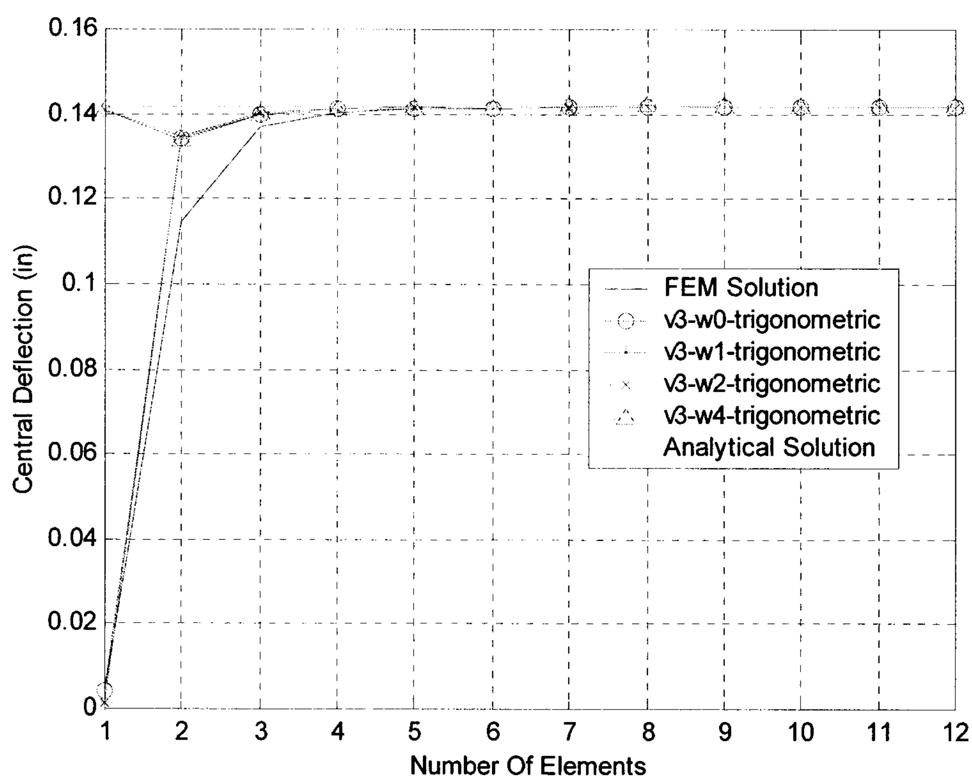


Figure 2.16 Comparison between the Results Corresponding to Non-Symmetric ($v_3 - w_n$) Trigonometric Hierarchical Terms

Table 2.6 Central Deflection Calculated by using Non- Symmetric Trigonometric Hierarchical Terms ($v_3 - w_n$)

8 D.O.F. Curved Beam Element										
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Conventional FEM		*Non-Symmetric Trigonometric 3v-0w HFEM terms*		*Non-Symmetric Trigonometric 3v-1w HFEM terms*		*Non-Symmetric Trigonometric 3v-2w HFEM terms*		*Non-Symmetric Trigonometric 3v-4w HFEM terms*	
1	3	0.00429	6	0.00432	7	0.00133	8	0.14082	9	0.14085
2	7	0.11463	13	0.13396	14	0.13464	15	0.13463	17	0.13360
3	11	0.13715	20	0.14003	21	0.14006	22	0.14005	25	0.13967
4	15	0.14038	27	0.14104	28	0.14109	29	0.14109	33	0.14093
5	19	0.14112	34	0.14133	35	0.14173	36	0.14137	41	0.14129
6	23	0.14135	41	0.14143	42	0.14146	43	0.14146	49	0.14141
7	27	0.14143	48	0.14147	49	0.14149	50	0.14144	57	0.14146
8	31	0.14147	55	0.14149	56	0.14150	57	0.14150	65	0.14194
9	35	0.14149	62	0.14150	63	0.14151	64	0.14151	73	0.14150
10	39	0.14150	69	0.14151	70	0.14151	71	0.14151	81	0.14151
11	43	0.14151	76	0.14151	77	0.14152	78	0.14152	89	0.14151
12	47	0.14151	83	0.14151	84	0.14151	85	0.14151	97	0.14151

Analytical Solution: Central Deflection = 0.14152 in

Table 2.7 Central Deflection Calculated by using Non- Symmetric Trigonometric Hierarchical Terms ($v_4 - w_n$)

8 D.O.F. Curved Beam Element										
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Conventional FEM		*Non-Symmetric Trigonometric 4v-0w HFEM terms*		*Non-Symmetric Trigonometric 4v-1w HFEM terms*		*Non-Symmetric Trigonometric 4v-2w HFEM terms*		*Non-Symmetric Trigonometric 4v-3w HFEM terms*	
1	3	0.00429	7	0.00432	8	0.00133	9	0.14133	10	0.14136
2	7	0.11463	15	0.13396	16	0.13470	17	0.14134	18	0.13363
3	11	0.13715	23	0.14003	24	0.14010	25	0.14007	26	0.13968
4	15	0.14038	31	0.14104	32	0.14111	33	0.14109	34	0.14093
5	19	0.14112	39	0.14133	40	0.14138	41	0.14137	42	0.14129
6	23	0.14135	47	0.14143	48	0.14146	49	0.14146	50	0.14141
7	27	0.14143	55	0.14147	56	0.14149	57	0.14149	58	0.14146
8	31	0.14147	63	0.14149	64	0.14151	65	0.14150	66	0.14149
9	35	0.14149	71	0.14150	72	0.14151	73	0.14151	74	0.14150
10	39	0.14150	79	0.14151	80	0.14151	81	0.14151	82	0.14151
11	43	0.14151	87	0.14151	88	0.14152	89	0.14152	90	0.14151
12	47	0.14151	95	0.14151	96	0.14152	97	0.14151	98	0.14151

Analytical Solution: Central Deflection = 0.14152 in

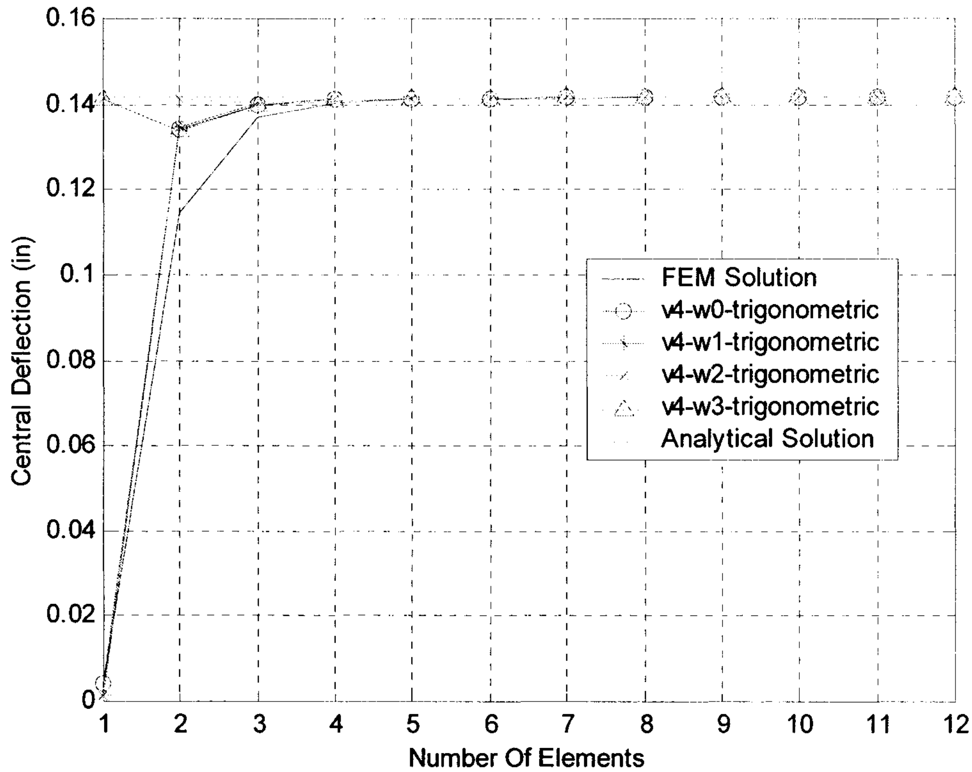


Figure 2.17 Comparison between the Results Corresponding to Non-Symmetric ($v_4 - w_n$) Trigonometric Hierarchical Terms

2.4.5 Illustrative Calculations for ($v_4 - w_2$) Combination

As it has been stated above that the combination ($v_4 - w_2$) provides the best combination for the given arch problem. If one element is used with such a combination of four hierarchical terms added to the tangential displacement (v) function and two terms added to the radial displacement (w) function, the stiffness matrix will be a 14×14 matrix, which is shown in Equation (2.54). For two elements the resultant matrix will be a 22×22 matrix. The order of the matrix will keep on increasing as we increase the number of elements.

The stiffness matrix and the equilibrium equation for one element are given below

$$\begin{Bmatrix} X_1 \\ X_1' \\ Y_1 \\ M_1 \\ F_{v5} \\ F_{v6} \\ F_{v7} \\ F_{v8} \\ X_2 \\ X_2' \\ Y_2 \\ M_2 \\ F_{w5} \\ F_{w6} \end{Bmatrix} = \left[k_{14\times 14} \right] \begin{Bmatrix} v_1 \\ v_{s1} \\ w_1 \\ \theta_1 \\ v_{v5} \\ v_{v6} \\ v_{v7} \\ v_{v8} \\ v_2 \\ v_{s2} \\ w_2 \\ \theta_2 \\ w_{w5} \\ w_{w6} \end{Bmatrix} \tag{2.54}$$

If one element is used to model half of the arch, the boundary conditions are

$$v_1 = w_1 = \left(\frac{\partial w}{\partial s} \right)_1 = v_2 = \left(\frac{\partial w}{\partial s} \right)_2 = 0 \tag{2.55}$$

After applying the boundary conditions the element matrix equation will be

$$\begin{Bmatrix} X_1' \\ F_{v5} \\ F_{v6} \\ F_{v7} \\ F_{v8} \\ X_2' \\ Y_2 \\ F_{w5} \\ F_{w6} \end{Bmatrix} = 10^7 \begin{bmatrix} 3.5615 & 0.01131 & 0.3267 & -1.3590 & -0.7794 & -0.8904 & -0.1573 & -0.0258 & 0.0855 \\ 0.1131 & 0.0084 & 0.0000 & -0.1297 & -0.0000 & -0.0031 & -0.0086 & 0.0000 & 0.0076 \\ 0.3267 & 0.0000 & 0.1362 & -0.0000 & -0.5322 & 0.3267 & 0.0000 & -0.0076 & 0.0000 \\ -1.3590 & -0.1297 & -0.0000 & 2.4731 & 0.0000 & 1.3590 & 0.1093 & -0.0000 & -0.1345 \\ -0.7794 & -0.0000 & -0.5322 & 0.0000 & 3.2417 & -0.7794 & 0.0000 & 0.0223 & 0.0000 \\ -0.8904 & -0.1131 & 0.3267 & 1.3590 & -0.7794 & 3.5615 & 0.1573 & -0.0258 & -0.0855 \\ -0.1573 & -0.0086 & 0.0000 & 0.1093 & 0.0000 & 0.1573 & 0.0344 & 0.0052 & -0.0067 \\ -0.0258 & 0.0000 & -0.0076 & -0.0000 & 0.0223 & -0.0258 & 0.0052 & 0.0017 & 0.0000 \\ 0.0855 & 0.0076 & 0.0000 & -0.1345 & 0.0000 & -0.0855 & -0.0067 & 0 & 0.0000 \end{bmatrix} \begin{Bmatrix} v_{s1} \\ v_{v5} \\ v_{v6} \\ v_{v7} \\ v_{v8} \\ v_{s2} \\ w_2 \\ w_{w5} \\ w_{w6} \end{Bmatrix} \tag{2.56}$$

Inverting the matrix gives

$$\begin{Bmatrix} v_{s1} \\ v_{v5} \\ v_{v6} \\ v_{v7} \\ v_{v8} \\ v_{s2} \\ w_2 \\ w_{w5} \\ w_{w6} \end{Bmatrix} = 10^{-3} \begin{bmatrix} 0.0001 & -0.0033 & -0.0006 & -0.0001 & -0.0000 & 0.0000 & 0.0001 & 0.0033 & -0.0000 \\ -0.0033 & 0.2383 & -0.0036 & 0.0088 & -0.0001 & 0.0013 & 0.0336 & -0.1443 & 0.0054 \\ -0.0006 & -0.0036 & 0.0086 & -0.0000 & 0.0009 & 0.0003 & -0.015 & 0.0666 & -0.0000 \\ -0.0001 & 0.0088 & -0.0000 & 0.0004 & -0.0000 & 0.0001 & 0.0001 & -0.0005 & -0.0005 \\ --0.0001 & -0.0001 & 0.0009 & -0.0000 & 0.0001 & -0.0000 & -0.0003 & 0.0012 & -0.0000 \\ 0.0000 & 0.0013 & 0.0003 & 0.0001 & -0.0000 & 0.0006 & -0.0083 & 0.0361 & 0.0000 \\ 0.0001 & 0.0336 & -0.0150 & 0.0001 & -0.0003 & -0.0083 & 0.1413 & -0.6069 & 0.0002 \\ 0.0003 & -0.1443 & 0.0666 & -0.0005 & 0.0012 & 0.0361 & -0.6069 & 2.6885 & -0.0010 \\ -0.0000 & 0.0054 & 0.0000 & -0.0005 & -0.0000 & 0.0000 & 0.0002 & -0.0010 & -0.0134 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1000 \\ 0 \\ 0 \end{Bmatrix}$$

(2.57)

which gives

$$w_2 = 0.1413 \text{ in}, \quad \text{with -0.13 \% error.}$$

Hence the hierarchical finite element formulation shows a great improvement in the convergence of the centre deflection values right from the 1-element mesh compared to the Conventional FEM which gives -97 % difference for the 1-element mesh.

2.4.6 Polynomial Hierarchical Formulation

In place of trigonometric functions that were used in the previous section we use polynomials that increase the degree of approximation of the displacement and rotation fields. The choice of the polynomials is governed by certain aspects. The chosen set should be complete. Polynomials that have the property that the set of the functions corresponding to an approximation of lower order constitutes a subset of the set of functions corresponding to a higher order approximation are particularly desirable. Also, the chosen function should not contribute to the displacement values at the element nodes. There is a wide array of polynomials that can be chosen from. In this work we have chosen the following set:

$$f_r(s) = s^{r+1}(s-L)^{r+1} \qquad\qquad r = 1, 2 \ldots M \qquad\qquad (2.58)$$

where L is the element length.

This function is chosen on the above mentioned basis and it fulfills the criteria when applied to the displacement as we will see in the following formulation.

2.4.7 Formulation based on Euler – Bernoulli Theory

Now, upon evaluating v, v_s, w, *and* θ at node 1 (i.e. at $s = 0$) and at node 2 (at $s = \mathrm{L}$) and evaluating the hierarchical term when r = 1, we get the following matrices:

$$\begin{Bmatrix} v_1 \\ v_{s1} \\ v_2 \\ v_{s2} \\ v_{v5} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} \qquad (2.59)$$

Similarly

$$\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_{w5} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{Bmatrix} \qquad (2.60)$$

The displacement field for the curved beam element is written as follows for this formulation:

For tangential displacement (v)

$$v(s) = N_1 v_1 + N_2 v_{s1} + N_3 v_2 + N_4 v_{s2} + \sum_{r=1}^{M} N_{r+4} A_r \tag{2.61}$$

Similarly for radial displacement (w)

$$w(s) = N_1 w_1 + N_2 \theta_1 + N_3 w_2 + N_4 \theta_2 + \sum_{r=1}^{M} N_{r+4} B_r \tag{2.62}$$

where

$$N_{r+1} = s^{r+1} (s - L)^{r+1} \qquad r = 1, 2, \ldots\ldots M \tag{2.63}$$

and A_r and B_r are the coefficients of the polynomial hierarchical terms.

The polynomial hierarchical shape functions are chosen such that,

$N_{r+4} = 0$ at $s = 0$ and $s = L$

$N'_{r+4} = 0$ at $s = 0$ and $s = L$

The above equations illustrate that the hierarchical shape functions provide zero displacement and zero slope at each end of the element. Again, it is important to mention that this property is highly significant, since these modes contribute only to the internal displacement field of the element, and do not therefore affect (i.e. over restrain) the displacements at the nodes.

The values of N_1, N_2, N_3, and N_4 are the same as calculated before and are given in Equation (2.8) while value of N_{r+5} is given by

$$N_{r+5} = (s)^{r+1}\left[(s-1)L\right]^{r+1} \tag{2.64}$$

These polynomial hierarchical functions are used in the same way as the trigonometric hierarchical functions were used for symmetric and non-symmetric combinations to calculate the central deflection.

2.4.8 Discussion and Conclusion

In the previous section the conventional and the hierarchical finite element methodologies have been described and the arch problem example has been solved to illustrate their applications. The HFEM displays superior results as compared to the conventional FEM. We did see how the results of the trigonometric hierarchical formulations compared with each other and within themselves in the previous section. Now the same comparison will be done for the polynomial hierarchical finite element formulation.

The results have been obtained using the conventional FEM and the trigonometric and polynomial formulations of the HFEM. These results are then compared with the analytical solution.

For symmetric polynomial hierarchical terms the results are given in Table 2.8 and plotted in Figure 2.18 as well. The results show improvement each time we add polynomial hierarchical terms to both tangential (v) and radial (w) displacement functions. After the 4-elements mesh all the curves converge to a single curve. The results obtained using symmetric trigonometric and polynomial hierarchical terms show that symmetric trigonometric hierarchical terms give better results than that of the results given by symmetric polynomial hierarchical terms.

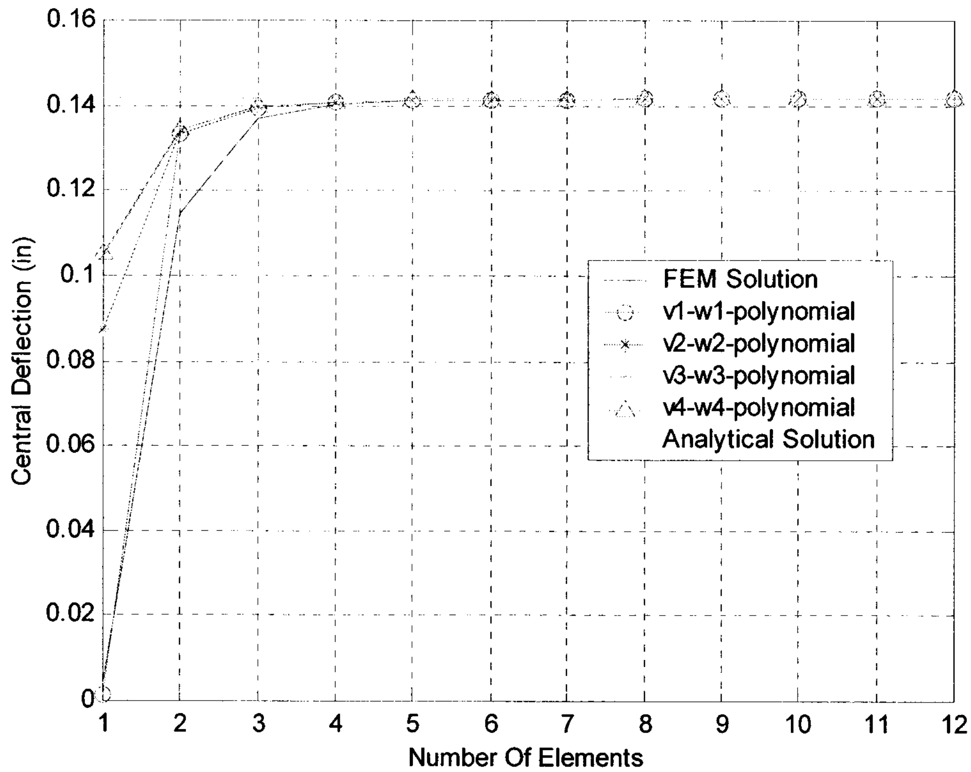


Figure 2.18 Comparison between the Results Corresponding to Symmetric Polynomial Hierarchical Terms

Table 2.8 Central Deflection Calculated by using Symmetric Polynomial Hierarchical Terms

8 D.O.F. Curved Beam Element										
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Conventional FEM		*Symmetric Polynomial 1 HFEM term*		*Symmetric Polynomial 2 HFEM terms*		*Symmetric Polynomial 3 HFEM terms*		*Symmetric Polynomial 4 HFEM terms*	
1	3	0.00429	5	0.00126	7	0.08719	9	0.10461	11	0.10507
2	7	0.11463	9	0.13318	11	0.13366	13	0.13395	15	0.13409
3	11	0.13715	13	0.13958	15	0.13967	17	0.13969	19	0.13970
4	15	0.14038	17	0.14090	19	0.14092	21	0.14093	23	0.14093
5	19	0.14112	21	0.14128	23	0.14128	25	0.14129	27	0.14129
6	23	0.14135	25	0.14141	27	0.14141	29	0.14141	31	0.14141
7	27	0.14143	29	0.14146	31	0.14146	33	0.14146	35	0.14146
8	31	0.14147	33	0.14149	35	0.14149	37	0.14149	39	0.14149
9	35	0.14149	37	0.14150	39	0.14150	41	0.14150	43	0.14150
10	39	0.14150	40	0.14151	43	0.14151	45	0.14151	47	0.14151
11	43	0.14151	45	0.14151	47	0.14151	49	0.14151	51	0.1415
12	47	0.14151	49	0.14151	51	0.14151	53	0.14151	55	0.14151

Analytical Solution: Central Deflection = 0.14152 in

Figure 2.19 shows that for only the 1st element the combinations ($v_0 - w_n$) give better results than that of the conventional FEM. All other elements show a little improvement and all the curves are almost matching. So it is clear from the figure that radial displacement (w) function show its little effect on the results because when hierarchical terms are added to the radial displacement (w) function, the improvement in the results was not that much significant.

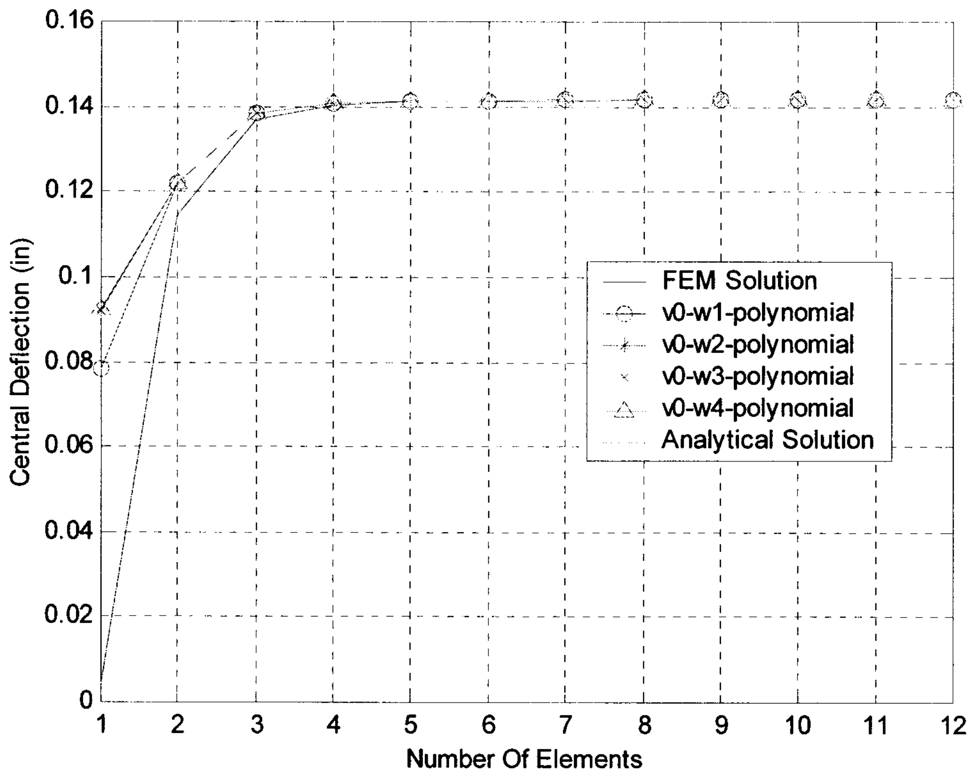


Figure 2.19 Comparison between the Results Corresponding to Non-Symmetric ($v_0 - w_n$) Polynomial Hierarchical Terms

Table 2.9 Central Deflection Calculated by using Non- Symmetric Polynomial Hierarchical Terms ($v_0 \; - w_n$)

8 D.O.F. Curved Beam Element										
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Conventional FEM		*Non-Symmetric Polynomial 0v-1w HFEM terms*		*Non-Symmetric Polynomial 0v-2w HFEM terms*		*Non-Symmetric Polynomial 0v-3w HFEM terms*		*Non-Symmetric Polynomial 0v-4w HFEM terms*	
1	3	0.00429	4	0.07845	5	0.09192	6	0.09227	7	0.09231
2	7	0.11463	9	0.12191	11	0.12193	13	0.12193	15	0.12193
3	11	0.13715	14	0.13861	17	0.13861	20	0.13861	23	0.13861
4	15	0.14038	19	0.14084	23	0.14084	27	0.14084	39	0.14084
5	19	0.14112	24	0.14131	29	0.14131	34	0.14131	46	0.14131
6	23	0.14135	29	0.14144	35	0.14144	41	0.14144	54	0.14144
7	27	0.14143	34	0.14148	41	0.14148	48	0.14148	62	0.14148
8	31	0.14147	39	0.14150	46	0.14150	55	0.14150	70	0.14150
9	35	0.14149	44	0.14151	52	0.14151	62	0.14151	78	0.14151
10	39	0.14150	49	0.14151	58	0.14151	69	0.14151	86	0.14151
11	43	0.14151	54	0.14152	64	0.14152	76	0.14152	94	0.14152
12	47	0.14151	59	0.14152	70	0.14152	83	0.14152	102	0.14152

Analytical Solution: Central Deflection = 0.14152 in

The results for the group of combinations ($v_1 - w_n$) are given in Table 2.4 and are plotted in Figure 2.20. The results show a little bit improvement for these combinations when hierarchical terms are added to the radial displacement (w) function. For these combinations the results given by polynomial hierarchical formulation are better than that of the trigonometric hierarchical formulation. This is contrary to the trends of the previous two types of combinations. The combination ($v_1 - w_4$) seems to have the results closest to the analytical solution except for the 1st element for this particular combination of hierarchical terms.

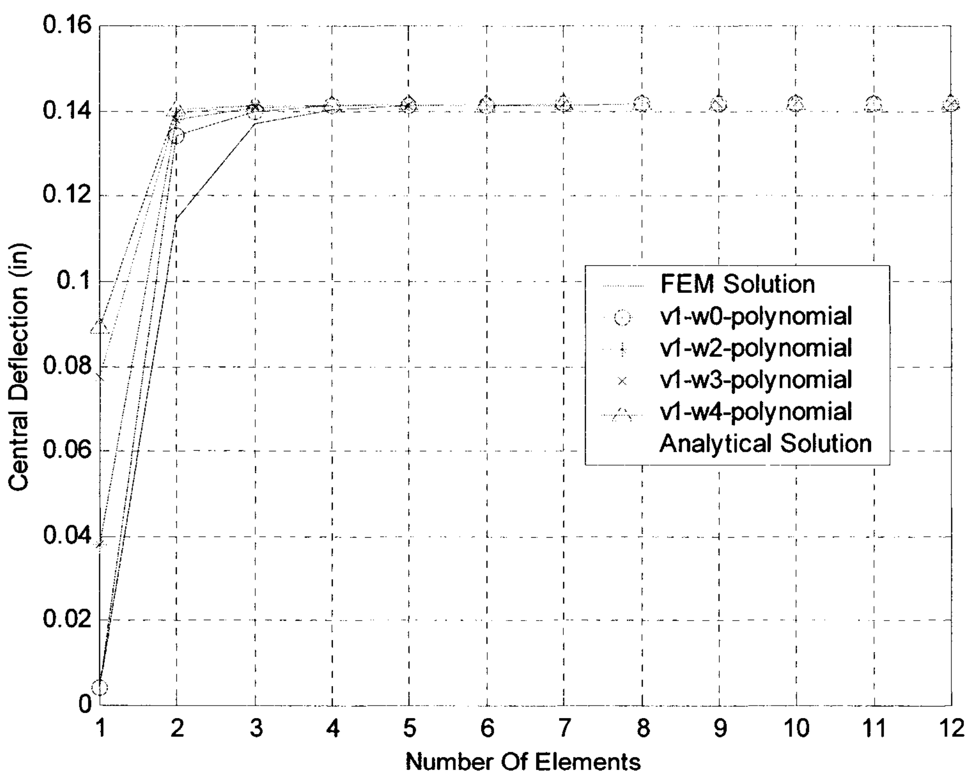


Figure 2.20 Comparison between the Results Corresponding to Non-Symmetric ($v_1 - w_n$) Polynomial Hierarchical Terms

Table 2.10 Central Deflection Calculated by using Non- Symmetric Polynomial Hierarchical Terms ($v_1 - w_n$)

8 D.O.F. Curved Beam Element										
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Conventional FEM		*Non-Symmetric Polynomial 1v-0w HFEM terms*		*Non-Symmetric Polynomial 1v-2w HFEM terms*		*Non-Symmetric Polynomial 1v-3w HFEM terms*		*Non-Symmetric Polynomial 1v-4w HFEM terms*	
1	3	0.00429	4	0.00432	6	0.03801	7	0.07721	8	0.08933
2	7	0.11463	9	0.13396	12	0.13800	14	0.13950	16	0.14022
3	11	0.13715	14	0.14003	18	0.14076	21	0.14111	24	0.14126
4	15	0.14038	19	0.14104	24	0.14128	28	0.14139	32	0.14144
5	19	0.14112	24	0.14133	30	0.14142	35	0.14147	40	0.14149
6	23	0.14135	29	0.14143	36	0.14147	42	0.14150	48	0.14150
7	27	0.14143	34	0.14147	42	0.14149	49	0.14150	56	0.14151
8	31	0.14147	39	0.14149	48	0.14151	56	0.14151	64	0.14151
9	35	0.14149	44	0.14150	54	0.14151	63	0.14151	72	0.14151
10	39	0.14150	49	0.14151	60	0.14151	70	0.14152	80	0.14152
11	43	0.14151	54	0.14151	66	0.14152	77	0.14152	88	0.14152
12	47	0.14151	59	0.14151	72	0.14151	84	0.14152	96	0.14152

Analytical Solution: Central Deflection = 0.14152 in

Figure 2.21 shows the results for the group of combinations ($v_2 - w_n$) and they are also given in Table 2.11. These results are even better than that of the results given by the previous combinations and start converging to the analytical solution much earlier than before. Trigonometric hierarchical formulation for these combinations give better results than polynomial hierarchical formulation for the initial few elements but after that results of polynomial hierarchical terms are only marginally better. The difference between the results corresponding to these two types of formulations is negligible.

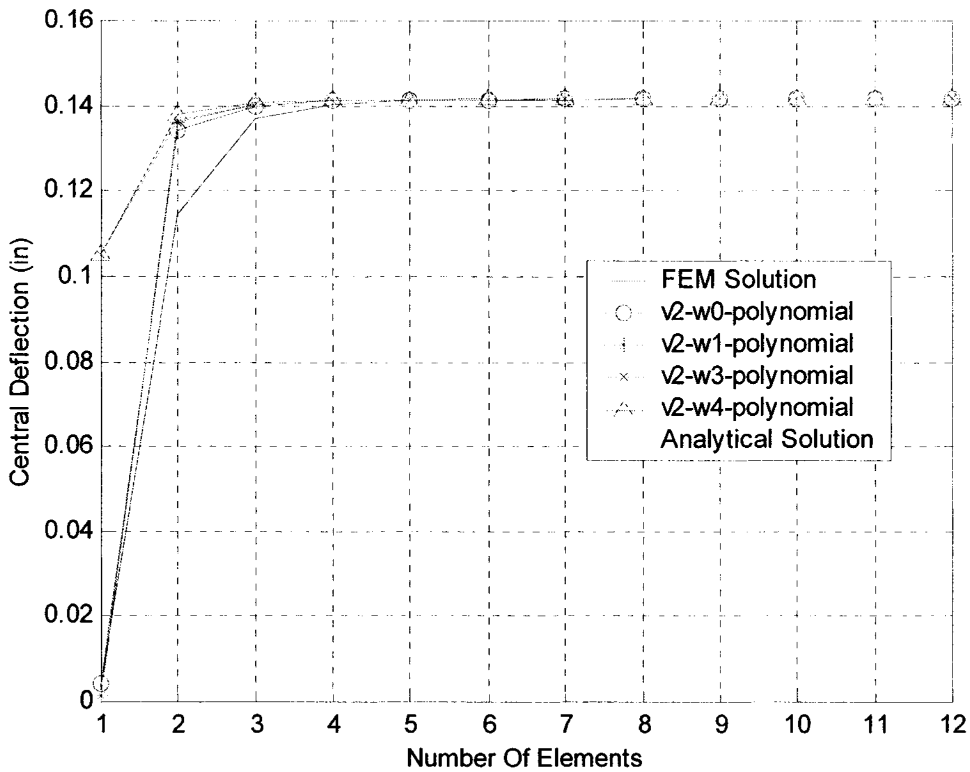


Figure 2.21 Comparison between the Results Corresponding to Non-Symmetric ($v_2 - w_n$) Polynomial Hierarchical Terms

Table 2.11 Central Deflection calculated by using Non- Symmetric Polynomial Hierarchical Terms ($v_2 - w_n$)

8 D.O.F. Curved Beam Element										
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Conventional FEM		*Non-Symmetric Polynomial 2v-0w HFEM terms*		*Non-Symmetric Polynomial 2v-1w HFEM terms*		*Non-Symmetric Polynomial 2v-3w HFEM terms*		*Non-Symmetric Polynomial 2v-4w HFEM terms*	
1	3	0.00429	5	0.00432	6	0.00126	8	0.10491	9	0.10511
2	7	0.11463	11	0.13396	12	0.13616	15	0.13671	17	0.13816
3	11	0.13715	17	0.14003	18	0.14040	22	0.14046	25	0.14080
4	15	0.14038	23	0.14104	24	0.14121	29	0.14120	33	0.14130
5	19	0.14112	29	0.14133	30	0.14142	36	0.14140	41	0.14144
6	23	0.14135	35	0.14143	36	0.14148	43	0.14147	49	0.14149
7	27	0.14143	41	0.14147	42	0.14150	50	0.14149	57	0.14150
8	31	0.14147	47	0.14149	48	0.14151	57	0.14151	65	0.14151
9	35	0.14149	53	0.14150	54	0.14151	64	0.14151	73	0.14151
10	39	0.14150	59	0.14151	60	0.14152	71	0.14151	81	0.14152
11	43	0.14151	65	0.14151	66	0.14152	78	0.14152	89	0.14152
12	47	0.14151	71	0.14151	72	0.14152	85	0.14151	97	0.14152

Analytical Solution: Central Deflection = 0.14152 in

The results for the group of combinations ($v_3 - w_n$) are shown in Figure 2.22 and these results are also given in Table 2.12. These combinations again give good results with the addition of the three hierarchical terms to the tangential displacement (v) function. The combination ($v_3 - w_1$) shows very good convergence at the 10-elements mesh. Only for the 1st element the trigonometric hierarchical formulation gives better results than the polynomial hierarchical formulation.

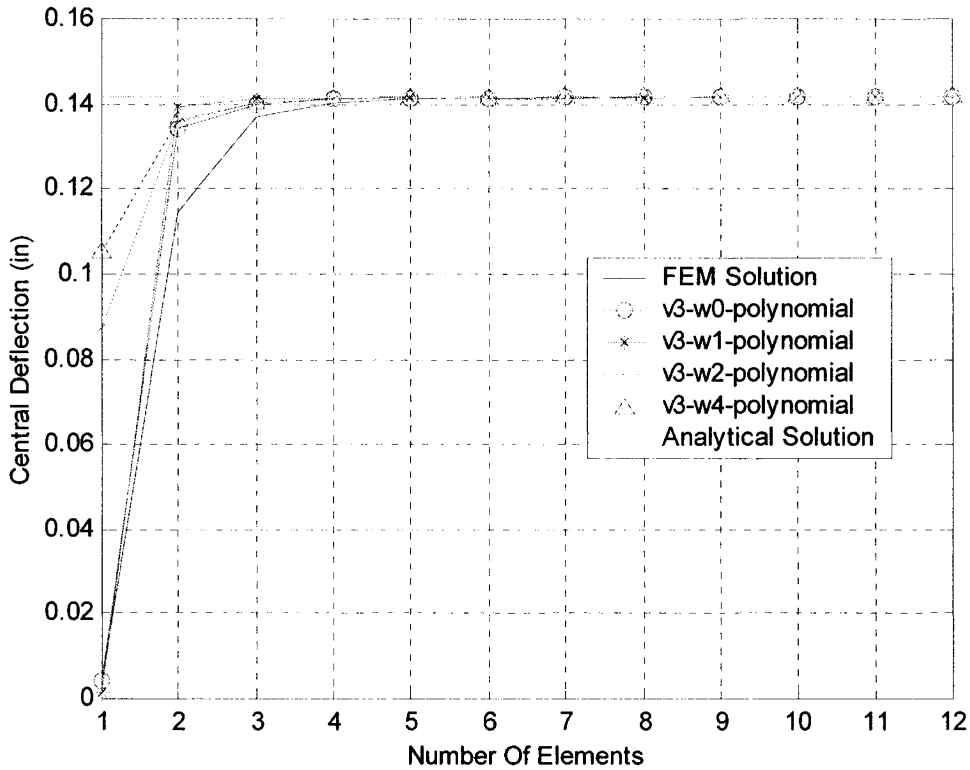


Figure 2.22 Comparison between the Results Corresponding to Non-Symmetric ($v_3 - w_n$) Polynomial Hierarchical Terms

Table 2.12 Central Deflection Calculated by using Non- Symmetric Polynomial Hierarchical Terms ($v_3 - w_n$)

8 D.O.F. Curved Beam Element										
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Conventional FEM		*Non-Symmetric Polynomial 3v-0w HFEM terms*		*Non-Symmetric Polynomial 3v-1w HFEM terms*		*Non-Symmetric Polynomial 3v-2w HFEM terms*		*Non-Symmetric Polynomial 3v-4w HFEM terms*	
1	3	0.00429	6	0.00432	7	0.00126	8	0.08719	9	0.10500
2	7	0.11463	13	0.13396	14	0.13932	15	0.13546	17	0.13591
3	11	0.13715	20	0.14003	21	0.14110	22	0.14026	25	0.14020
4	15	0.14038	27	0.14104	28	0.14142	29	0.14115	33	0.14111
5	19	0.14112	34	0.14133	35	0.14149	36	0.14139	41	0.14136
6	23	0.14135	41	0.14143	42	0.14151	43	0.14146	49	0.14145
7	27	0.14143	48	0.14147	49	0.14151	50	0.14149	57	0.14148
8	31	0.14147	55	0.14149	56	0.14142	57	0.14151	65	0.14150
9	35	0.14149	62	0.14150	63	0.14152	64	0.14151	73	0.14151
10	39	0.14150	69	0.14151	70	0.14152	71	0.14151	81	0.14151
11	43	0.14151	76	0.14151	77	0.14152	78	0.14152	89	0.14152
12	47	0.14151	83	0.14151	84	0.14152	85	0.14152	97	0.14151

Analytical Solution: Central Deflection = 0.1415 in

Table 2.13 Central Deflection Calculated by using Non- Symmetric Polynomial Hierarchical Terms ($v_4 - w_n$)

8 D.O.F. Curved Beam Element										
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Conventional FEM		*Non-Symmetric Polynomial 4v-0w HFEM terms*		*Non-Symmetric Polynomial 4v-1w HFEM terms*		*Non-Symmetric Polynomial 4v-2w HFEM terms*		*Non-Symmetric Polynomial 4v-3w HFEM terms*	
1	3	0.00429	7	0.00432	8	0.00126	9	0.08719	10	0.10461
2	7	0.11463	15	0.13396	16	0.14059	17	0.13796	18	0.13606
3	11	0.13715	23	0.14003	24	0.14136	25	0.14080	26	0.14021
4	15	0.14038	31	0.14104	32	0.14149	33	0.14132	34	0.14111
5	19	0.14112	39	0.14133	40	0.14151	41	0.14145	42	0.14136
6	23	0.14135	47	0.14143	48	0.14152	49	0.14149	50	0.14145
7	27	0.14143	55	0.14147	56	0.14151	57	0.14150	58	0.14148
8	31	0.14147	63	0.14149	64	0.14152	65	0.14151	66	0.14150
9	35	0.14149	71	0.14150	72	0.14152	73	0.14151	74	0.14151
10	39	0.14150	79	0.14151	80	0.14152	81	0.14152	82	0.14151
11	43	0.14151	87	0.14151	88	0.14152	89	0.14152	90	0.14152
12	47	0.14151	95	0.14151	96	0.14152	97	0.14152	98	0.14151

Analytical Solution: Central Deflection = 0.14152 in

Table 2.13 gives results for the combinations ($v_4 - w_n$) and the results are plotted in Figure 2.23. These combinations give the most accurate results. The combination ($v_4 - w_1$) provides the best convergence among all the polynomial hierarchical formulations. This combination of terms converges most rapidly at the 8-elements mesh. Again only for the 1st element the trigonometric hierarchical formulation is better than polynomial hierarchical formulation but for the rest of the elements the polynomial hierarchical terms give better results.

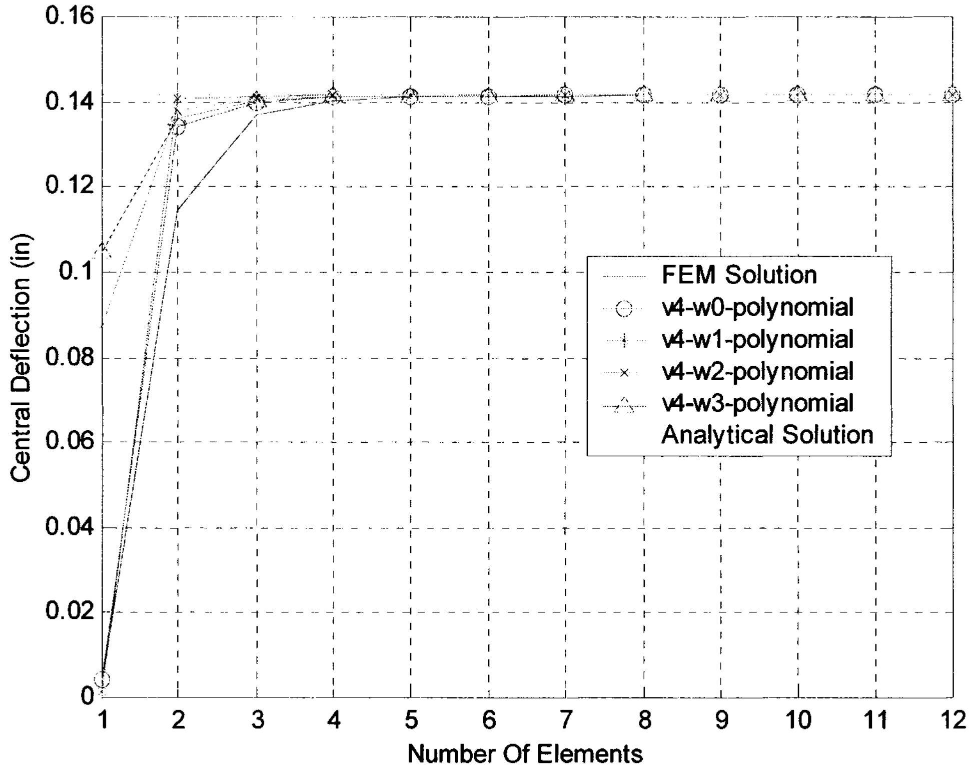


Figure 2.23 Comparison between the Results Corresponding to Non-Symmetric ($v_4 - w_n$) Polynomial Hierarchical Terms

To sum up, in this chapter the Hierarchical Finite Element Method has been presented and its formulation has been applied to Euler- Bernoulli curved beams made of isotropic materials. Two variations of the HFEM have been studied viz. Trigonometric and Polynomial HFEM. To start with, the conventional finite element formulation is presented and its derivation is detailed to stress the conceptual changes that are made in it for the HFEM. A semicircular arch problem has been solved using the conventional formulation so that a comparison can be made with regard to the HFEM results. The detailed formulation of the HFEM for both the trigonometric and the polynomial cases is also given to stress the major aspects of the method. Programs are developed in MATLAB® (for symbolic computation) software environment. The results obtained using the HFEM method are then compared with the results obtained using the conventional formulation and the analytical solutions.

Both the forms of HFEM are found to give highly accurate results. Results can be achieved to any desired degree of accuracy by simply increasing the number of hierarchical terms in each element. Trigonometric formulation gives better results than the conventional formulation and gives even better results than polynomial formulation for some of the initial elements, although polynomial formulation yields the best results among the three.

Now we have laid the foundation for the application of HFEM in the analysis of 1-D curved composite structures. The inspiring results for isotropic materials should lead us to similar computational efficiency for structures made of composite materials. In the next chapter, we shall explore the applications to composite curved structures using the HFEM methodology.

Chapter 3

Hierarchical Finite Element Formulation for Curved Composite Beams

3.1 Introduction

In a broad sense the word "composite" means "made of two or more different parts". A composite material consists of an assemblage of two materials of different nature completing and allowing us to obtain a material of which the set of performance characteristics is better than that of the components taken separately.

In most general case a composite material consists of one or more discontinuous phases distributed in one continuous phase. In the case of several discontinuous phases of different nature the composite is said to be a hybrid. The discontinuous phase is usually harder and with mechanical properties superior to those of the continuous phase. The continuous phase is called the *matrix*. The discontinuous phase is called the *reinforcement*, or *reinforcing material.* Composite materials, especially laminated composites are being increasingly used in the aerospace and automobile industries. This is mainly because these materials exhibit high strength-to-weight and stiffness-to-weight ratios.

The application to isotropic curved beams in the preceding chapter showed us that the hierarchical finite element formulation performs much better than the conventional finite element method in terms of faster convergence and use of less number of elements. In composite structures, the in-plane strains and stresses in different plies of the laminate are functions of the curvature of the laminate, in accordance with the classical laminated plate theory [48]. As a result, the continuity of the in–plane stresses and strains in each ply of the laminate depends upon the continuity of curvature across adjacent elements. This continuity is not enforced and guaranteed in the conventional finite element formulation, which requires the use of many elements to obtain reasonable accuracy. The use of many elements results in the presence of corresponding discontinuities. In the case of variable-thickness composite laminates, additional complexities arise due to the presence of drop-off plies. Hierarchical finite element method (HFEM) makes it possible to model a structure using very few elements. In some cases the use of two or three elements provides accurate solutions. These features of the HFEM make it an attractive choice to overcome the limitations associated with the conventional finite element formulations in the analysis of the composite curved beams.

3.2 Constitutive Equations for Laminated Cylindrical Plates

Consider a cylindrical plate of constant radius R as illustrated in Figure 3.1. As in the case of flat plates, the thickness and in-plane dimensions are denoted by h, a, and b respectively. The displacements in the *x, s,* and *z* directions are denoted by *u, v,* and *w,* respectively [48]. Assumptions are as follows:

1. The plate is constructed of an arbitrary number of orthotropic layers bonded together. The orthotropic axes of material symmetry, however, of an individual layer need not to coincide with the x-s axes of the cylindrical plate.
2. The plate is thin, i.e., the thickness *h* is smaller than the other physical dimensions.
3. The dimensions *u, v,* and *w* are small compared to the plate thickness.
4. In-plane strains $\varepsilon_x, \varepsilon_s$, and ε_{xs} are small compared to unity.
5. The radius of the plate *R* is much larger than the thickness *h*.
6. In order to include in-plane force effects, nonlinear terms in the equations of motion involving products of stresses and plate slopes are retained. All other nonlinear terms are neglected.
7. Transverse shear strains ε_{xs} and ε_{sz} are negligible.
8. Tangential displacements *u* and *v* are linear function of the *z* coordinate.
9. The transverse normal strain ε_z is negligible.
10. Each ply obeys Hooke's law.
11. The plate has constant thickness.
12. Rotatory inertia terms are negligible.
13. There are no body forces.
14. Transverse shear stresses τ_{xz} and τ_{sz} vanish on the surfaces $z = \pm h/2$.

The strain-displacement relations for general thin shells are well known (see for example, Refs. 46 and 49) from classical theory of elasticity which are applicable to the coordinate system shown in Figure 3.1 are as follows.

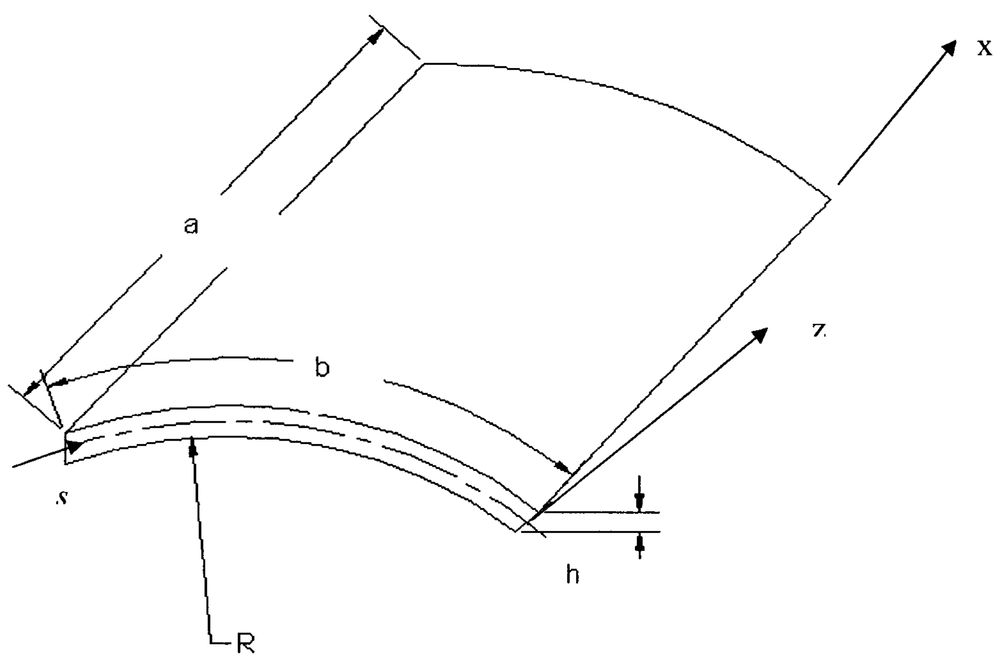


Figure 3.1 Nomenclature of Curved Laminated Plate

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u^0}{\partial x} - z\,\frac{\partial^2 w}{\partial x^2} \\
\varepsilon_s &= \frac{\partial v^0}{\partial s} + \frac{w}{R} + \frac{z}{R}\frac{\partial v}{\partial s} - \frac{z}{(1+z/R)}\,\frac{\partial^2 w}{\partial s^2} \\
\varepsilon_{sx} &= (1+\frac{z}{R})\,\frac{\partial v^0}{\partial s} + \frac{\partial u^0}{\partial s} - z\,(2+\frac{z}{R})\,\frac{\partial^2 w}{\partial x \partial s}
\end{aligned} \tag{3.1}$$

where u^0 and v^0 are the axial and tangential displacements of the mid plane, respectively. Since the plane is shallow $(R >> h)$, z/R is small compared to unity. Thus,

$$(1+z/R) \approx 1 \tag{3.2}$$

and Equation (3.1) can be written in the form

$$\begin{aligned}
\varepsilon_x &= \varepsilon^0{}_x + z\kappa_x \\
\varepsilon_s &= \varepsilon^0{}_s + z\kappa_s \\
\varepsilon_{sx} &= \varepsilon^0{}_{sx} + z\kappa_{sx}
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
&\varepsilon^0{}_x = \frac{\partial u^0}{\partial x} \quad , \quad \varepsilon^0{}_s = \frac{\partial v^0}{\partial s} + \frac{w}{R} \quad , \quad \varepsilon^0{}_{sx} = \frac{\partial u^0}{\partial s} + \frac{\partial v^0}{\partial x} \\
&\kappa_x = -\frac{\partial^2 w}{\partial x^2} \quad , \quad \kappa_s = \frac{1}{R}\frac{\partial v}{\partial s} - \frac{\partial^2 w}{\partial x^2} \quad , \quad \kappa_{sx} = -2\frac{\partial^2 w}{\partial x \partial s}
\end{aligned} \tag{3.4}$$

The ply constitutive relations are as follows

$$\begin{bmatrix} \sigma_s^{(k)} \\ \sigma_x^{(k)} \\ \sigma_{sx}^{(k)} \end{bmatrix} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & Q_{16}^{(k)} \\ Q_{21}^{(k)} & Q_{22}^{(k)} & Q_{26}^{(k)} \\ Q_{16}^{(k)} & Q_{26}^{(k)} & Q_{66}^{(k)} \end{bmatrix} \begin{bmatrix} \varepsilon_s \\ \varepsilon_x \\ \varepsilon_{sx} \end{bmatrix} \tag{3.5}$$

where Q_{ij} are the reduced stiffnesses for the plane stress.

We define force and moment resultants as follows [50].

$$(N_x, N_s, N_{sx}) = \int_{-h/2}^{h/2} (\sigma_x^{(k)}, \sigma_s^{(k)}, \sigma_{sx}^{(k)})\, dz \tag{3.6}$$

$$(M_x, M_s, M_{sx}) = \int_{-h/2}^{h/2} (\sigma_x^{(k)}, \sigma_s^{(k)}, \sigma_{sx}^{(k)})\, z\, dz \tag{3.7}$$

Combining Equations (3.3) and (3.5), substituting the results into Equation (3.6) and performing the integrations, we arrive at the laminate constitutive relations which are

$$
\begin{bmatrix} N_s \\ N_x \\ N_{sx} \\ \hdashline M_s \\ M_x \\ M_{sx} \end{bmatrix} = \left[\begin{array}{ccc:ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hdashline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array}\right] \begin{bmatrix} \varepsilon_s^0 \\ \varepsilon_x^0 \\ \varepsilon_{sx}^0 \\ \hdashline \kappa_s \\ \kappa_x \\ \kappa_{sx} \end{bmatrix} \tag{3.8}
$$

or in practical form

$$
\begin{bmatrix} N \\ \hdashline M \end{bmatrix} = \left[\begin{array}{c:c} A & B \\ \hdashline B & D \end{array}\right] \begin{bmatrix} \varepsilon^0 \\ \hdashline \kappa \end{bmatrix} \tag{3.9}
$$

where A_{ij}, B_{ij} and D_{ij} are the stiffnesses defined as

$$
(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}^{(k)} (1, z, z^2)\, dz \tag{3.10}
$$

3.3 Energy Formulation of Laminate Theory

The energy theorems can be used to obtain a variational formulation of the governing equations of laminates. This formulation, associated with the boundary conditions, provides the bases for the development of approximate solutions of the mechanical behavior of laminates. The energy theorems are also the bases for the analysis of laminates by finite element method.

3.3.1 Strain Energy for the Analysis of 1-D Curved Laminate

The strain energy, U, for curved plate in terms of an x ,s ,z coordinate system is given by the relationship [48].

$$U=\frac{1}{2}\iiint(\sigma_s\varepsilon_s+\sigma_x\varepsilon_x+\sigma_z\varepsilon_z+\sigma_{sz}\varepsilon_{sz}+\sigma_{xz}\varepsilon_{xz}+\sigma_{sx}\varepsilon_{sx})dx\,ds\,dz \tag{3.11}$$

where the triple integration is performed over the volume of the body. Taking into account the basic assumption of laminated plate theory i.e., $\varepsilon_z=\varepsilon_{xz}=\varepsilon_{sz}=0$ and the we find that the Equation (3.10) becomes

$$U=\frac{1}{2}\iiint(Q_{11}^{(k)}\varepsilon_s^2+2Q_{12}^{(k)}\varepsilon_x\varepsilon_s+2Q_{16}^{(k)}\varepsilon_x\varepsilon_{xs}+2Q_{26}^{(k)}\varepsilon_s\varepsilon_{xs}+Q_{22}^{(k)}\varepsilon_x^2+Q_{66}^{(k)}\varepsilon_{xs}^2)dx\,ds\,dz \tag{3.12}$$

Substituting the kinematic relations, Equations (3.3) and (3.4), into Equation (3.12) and integrating with respect to z, we obtain the following strain energy relationship:

$$U=\frac{1}{2}\iint\left\{A_{22}\left(\frac{\partial u^0}{\partial x}\right)^2+2A_{12}\frac{\partial u^0}{\partial x}\left(\frac{\partial v^0}{\partial s}+\frac{w}{R}\right)+A_{11}\left[\frac{\partial v^0}{\partial s}\left(\frac{\partial v^0}{\partial s}+\frac{w}{R}\right)+\left(\frac{w}{R}\right)^2\right]\right.$$

$$\left[A_{16}\frac{\partial u^0}{\partial s}+A_{26}\left(\frac{\partial v^0}{\partial s}+\frac{w}{R}\right)\right]\left(\frac{\partial u^0}{\partial s}+\frac{\partial v^0}{\partial x}\right)+A_{66}\left(\frac{\partial u^0}{\partial s}+\frac{\partial v^0}{\partial x}\right)^2-B_{22}\frac{\partial u^0}{\partial x}\frac{\partial^2 w}{\partial x^2}$$

$$\left[\left(\frac{\partial v^0}{\partial s}+\frac{w}{R}\right)\frac{\partial^2 w}{\partial x^2}+\frac{\partial u^0}{\partial x}\frac{\partial^2 w}{\partial s^2}\right]-2B_{11}\left(\frac{\partial v^0}{\partial s}+\frac{w}{R}\right)\frac{\partial^2 w}{\partial s^2}-2B_{16}\left[\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial u^0}{\partial s}+\frac{\partial v^0}{\partial x}\right)\right]$$

$$+\frac{2B_{11}}{R}\left(\frac{\partial v}{\partial s}\right)^2+\frac{2B_{11}}{R^2}\left[\left(\frac{\partial v}{\partial s}\right)\left(\frac{\partial^2 w}{\partial s^2}\right)\right]-4B_{16}\left[\frac{\partial u^0}{\partial x}\frac{\partial^2 w}{\partial x\partial s}\right]-2B_{16}\left[\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial u^0}{\partial s}+\frac{\partial v^0}{\partial x}\right)\right]$$

$$+4B_{16}\frac{\partial u^0}{\partial x}\frac{\partial^2 w}{\partial x\partial s}-2B_{26}\left[\frac{\partial^2 w}{\partial s^2}\left(\frac{\partial u^0}{\partial s}+\frac{\partial v^0}{\partial x}\right)+2\left(\frac{\partial v^0}{\partial s}+\frac{w}{R}\right)\frac{\partial^2 w}{\partial x\partial s}\right]$$

$$-4D_{66}\frac{\partial^2 w}{\partial x\partial s}\left(\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial s}\right)+D_{22}\left(\frac{\partial^2 w}{\partial x^2}\right)+2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial s^2}-\frac{2D_{11}}{R}\left(\frac{\partial v}{\partial s}\right)\left(\frac{\partial^2 w}{\partial s^2}\right)$$

$$+\frac{D_{11}}{R^2}\left(\frac{\partial v}{\partial s}\right)^2+4\left(D_{16}\frac{\partial^2 w}{\partial x^2}+D_{26}\frac{\partial^2 w}{\partial s^2}\right)\frac{\partial^2 w}{\partial x\partial s}+D_{11}\left(\frac{\partial^2 w}{\partial s^2}\right)^2+4D_{66}\left(\frac{\partial^2 w}{\partial x\partial s}\right)^2\ \Big\}\,dx\,ds$$

$$(3.13)$$

For a 1-D problem all the terms with x will be neglected. We are assuming a symmetric laminate so all the terms with the coefficients of B matrix are also neglected. Consequently, we will be left with the following equation.

$$U=\frac{1}{2}\int_{s=0}^{s=L}\left\{A_{11}\left[\left(\frac{\partial v}{\partial s}\right)^2+\left(\frac{w}{R}\right)^2\right]+\frac{2A_{11}}{R}\left(w\frac{\partial v}{\partial s}\right)+D_{11}\left(\frac{\partial^2 w}{\partial s^2}\right)^2+\frac{D_{11}}{R^2}\left(\frac{\partial v}{\partial s}\right)^2\right.$$

$$\left.-\frac{2D_{11}}{R}\left[\left(\frac{\partial v}{\partial s}\right)\left(\frac{\partial^2 w}{\partial s^2}\right)\right]\right\}ds \qquad (3.14)$$

3.4 Cubic-Cubic Circularly Curved Composite Beam Finite Element

We have considered and analyzed the cubic-cubic curved beam finite element (Figure 2.2) by using the conventional and hierarchical finite element formulations in the previous chapter. Now we will use the same curved beam element for the semi-circular arch problem (section 2.2.5, Figures 2.2(a) and 2.2(b)) by first considering the laminates with isotopic layers and then by using different configurations of the laminate. Both conventional and hierarchical finite element formulations will be used.

Total strain energy for the curved beam as discussed before is

$$U = U_{vv} + U_{vw} + U_{ww} \tag{3.15}$$

where

$$\begin{aligned} U_{vv} &= \frac{A_{11}}{2}\int_0^L (v')^2\, ds + \frac{D_{11}}{2R^2}\int_0^L (v')^2\, ds \\ U_{vw} &= \frac{A_{11}}{R}\int_0^L (v')(w) ds - \frac{D_{11}}{R}\int_0^L (v')(w'') ds \\ U_{ww} &= \frac{A_{11}}{2R^2}\int_0^L (w)^2\, ds + \frac{D_{11}}{2}\int_0^L (w'')^2\, ds \end{aligned} \tag{3.16}$$

The energy expressions U_{vv}, U_{vw}, and U_{ww} are associated with axial, axial-flexural coupling, and flexural behaviors, respectively.

3.4.1 Stiffness Equations

Substituting the displacement functions for v and w, Equation (2.1), into the energy expressions, Equation (3.16) and then performing partial differentiations of the strain energy with respect to each of the eight degrees of freedom, the 8×8 stiffness matrix equations for the element are obtained.

$$
\begin{Bmatrix} X_1 \\ X_1' \\ X_2 \\ X_2' \\ \cdots \\ Y_1 \\ M_1 \\ Y_2 \\ M_2 \end{Bmatrix} = \left[\begin{array}{c|c} [k_{vv\,4\times4}] & [k_{vw\,4\times4}] \\ \hline [k_{wv\,4\times4}] & [k_{ww\,4\times4}] \end{array} \right] \begin{Bmatrix} v_1 \\ v_{s1} \\ v_2 \\ v_{s2} \\ \cdots \\ w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \tag{3.17}
$$

where X_1' and X_2' are the counterpart generalized forces in inch-pounds associated with the degrees of freedom v_{s1} and v_{s2}, respectively. The coefficients in the 4×4 sub-matrices are obtained as

$$
\begin{aligned}
k_{vv_{ij}} &= \int_0^L \left(\frac{A_{11}}{2} + \frac{D_{11}}{2R^2} \right) N_i' N_j' \, ds \\
k_{vw_{ij}} = k_{wv_{ji}} &= \int_0^L \left\{ \left[\left(\frac{A_{11}}{R} \right) N_i' \, N_j \right] - \left[\left(\frac{D_{11}}{R} \right) N_i' N_j'' \right] \right\} ds \\
k_{ww_{ij}} &= \int_0^L \left\{ \left[\left(\frac{A_{11}}{R^2} \right) N_i \, N_j \right] - \left[D_{11} \, N_i'' N_j'' \right] \right\} ds
\end{aligned} \tag{3.18}
$$

3.4.2 Laminates with Isotropic Layers

The reduced stiffness matrix of an isotropic layer is given by the relation [50].

$$
Q = \begin{bmatrix} \frac{E}{1-\upsilon^2} & \frac{\upsilon E}{1-\upsilon^2} & 0 \\ \frac{\upsilon E}{1-\upsilon^2} & \frac{E}{1-\upsilon^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\upsilon)} \end{bmatrix} \tag{3.19}
$$

The stiffness constants of a laminate made of n isotropic layers with different properties are then given by the following relations:

$$A_{11}=\sum_{k=1}^{n}\frac{E_k e_k}{1-\upsilon_k^2},\qquad A_{12}=\sum_{k=1}^{n}\frac{\upsilon_k E_k e_k}{1-\upsilon_k^2},\qquad A_{16}=0,$$
$$A_{22}=A_{11},\quad A_{26}=0,\quad A_{66}=\sum_{k=1}^{n}\frac{E_k e_k}{2(1+\upsilon_k)},$$

$$B_{11}=\sum_{k=1}^{n}\frac{E_k e_k z_k}{1-\upsilon_k^2},\qquad B_{12}=\sum_{k=1}^{n}\frac{\upsilon_k E_k e_k z_k}{1-\upsilon_k^2},\qquad B_{16}=0,$$
$$B_{22}=B_{11},\quad B_{26}=0,\quad B_{66}=\sum_{k=1}^{n}\frac{E_k e_k z_k}{2(1+\upsilon_k)},$$

$$D_{11}=\sum_{k=1}^{n}\frac{E_k}{1-\upsilon_k^2}\left(e_k z_k^2+\frac{e_k^3}{12}\right)$$
$$D_{12}=\sum_{k=1}^{n}\frac{\upsilon_k E_k}{1-\upsilon_k^2}\left(e_k z_k^2+\frac{e_k^3}{12}\right)$$

$$D_{22}=D_{11},\qquad D_{16}=0,\qquad D_{26}=0,$$
$$D_{66}=\sum_{k=1}^{n}\frac{E_k}{2(1+\upsilon_k)}\left(e_k z_k^2+\frac{e_k^3}{12}\right)\tag{3.20}$$

where

e_k = thickness of the ply k

z_k = distance of the ply k from the middle plane

3.4.2.1 *Curved Composite Beam Example*

In order to evaluate the validity of the strain energy Equation (3.14) the curved beam example as shown in Figures 2.2(a) and 2.2(b) was solved again by using the finite element method and for a laminate of isotropic layers by using the A and D matrix coefficients A_{11} and D_{11} in Equation (3.14), and by using the relations for laminates for isotropic layers given in Equation (3.20). The boundary conditions and stiffness matrix equations for this arch problem are given by Equations (2.19) and (3.18). The parameters are defined as

$$E = 10^7 \; psi., \quad I = 1/12 \; in^4., \quad A = 1 \times 1 \; in^2., \quad P = 1000 \; lb., \quad R = 17 \; in.,$$

The interpolation functions are also the same as were given in the previous chapter (Equation 2.6). The results are given and compared in Table 3.1. These results are plotted in Figure 3.2 as well, which show that the difference between the conventional FEM solution using the curved beam element and the element made of isotropic laminate is 10.88 %. These results validate the strain energy equation for 1-D composite curved beam element. The difference of 10.88 % is due to fact that for unit width bending and stretching coefficients D_{11} and A_{11} have $1-\upsilon^2$ terms in their denominators (3.16) which the terms EA and EI do not possess (2.14b), which is the reason for this error as explained below:

$$\begin{aligned} \frac{EA}{b} &= \frac{E_k\left(\sum_{k=1}^{n} e_k\right)}{1-\upsilon^2} \quad \rightarrow \quad Eh = \frac{E_k h}{1-\upsilon^2} \\ \frac{EI}{b} &= \frac{E_k\left(\sum_{k=1}^{n} e_k^3\right)}{12(1-\upsilon^2)} \quad \rightarrow \quad \frac{Eh^3}{12} = \frac{E_k h^3}{12(1-\upsilon^2)} \end{aligned} \tag{3.21}$$

Table 3.1 Central Deflections Calculated using Curved Beam Element and Element Made of Isotropic laminate

Comparison Study of Arch Problem				
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Curved Beam Element		*Isotropic Laminate's Case*	
1	3	0.00429	3	0.00383
2	7	0.11463	7	0.10215
3	11	0.13715	11	0.12222
4	15	0.14038	15	0.12509
5	19	0.14112	19	0.12575
6	23	0.14135	23	0.12595
7	27	0.14143	27	0.12603
8	31	0.14147	31	0.12607
9	35	0.14149	35	0.12608
10	39	0.14150	39	0.12609
11	43	0.14151	43	0.12610
12	47	0.14151	47	0.12610

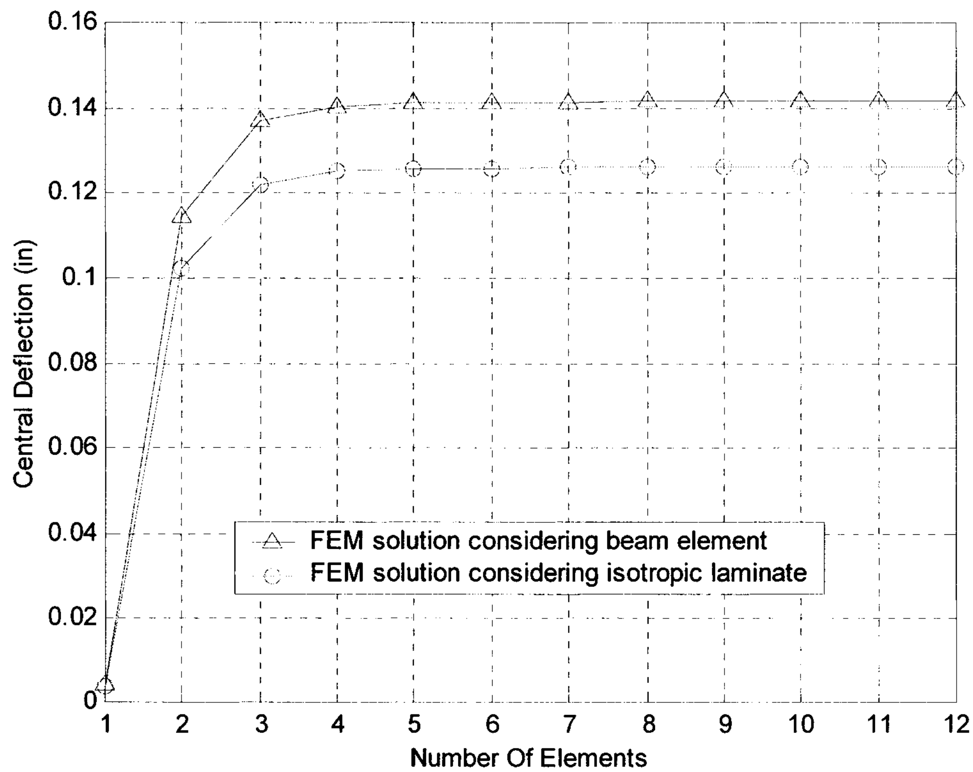


Figure 3.2 FEM Solutions Considering Beam and Isotropic Laminate

3.5 The Hierarchical Finite Element Formulation for Composite Curved Beam

The hierarchical finite element formulation for the composite beam proceeds in the same way as the procedure described in the previous chapter for the isotropic curved beam. The difference being that now it is applied to the strain energy equation for a composite curved beam given by Equation (3.16) instead of Equation (2.14b) and that the *EA* and *EI* terms for unit width are replaced by the terms A_{11} and D_{11} respectively. The salient steps in the HFEM formulation are mentioned below:

The tangential (v) and transverse (w) displacements are approximated as

$$
\begin{aligned}
v(s) &= a_1 + a_2 s + a_3 s^2 + a_4 s^3 + \sum_{r=1}^{N} a_{r+4} \sin[\delta_r s] \\
w(s) &= c_1 + c_2 s + c_3 s^2 + c_4 s^3 + \sum_{r=1}^{N} c_{r+4} \sin[\delta_r s]
\end{aligned}
\tag{3.22}
$$

where

$$\delta_r = \frac{r\pi}{L}, \qquad r = 1, 2, 3 \ldots N$$

The derivation is similar to the one described in section 2.3.1.2. It gives us the following expressions for the displacement fields v and w

For tangential displacement (v)

$$v = N_1 v_1 + N_2 v_{s1} + N_3 v_2 + N_4 v_{s2} + \sum_{r=1}^{N} N_{r+4} v_{vr+4} \tag{3.23}$$

Similarly for radial displacement (w)

$$w = N_1 w_1 + N_2 \theta_1 + N_3 w_2 + N_4 \theta_2 + \sum_{r=1}^{N} N_{r+4} w_{wr+4} \tag{3.24}$$

where the shape functions are as follows

$$\begin{aligned} N_1 &= 1 - 3\xi^2 + 2\xi^3 \\ N_2 &= L\left(\xi - 2\xi^2 + \xi^3\right) \\ N_3 &= 3\xi^2 - 2\xi^3 \\ N_4 &= L\left(-\xi^2 + \xi^3\right) \end{aligned} \tag{3.25}$$

and

$$N_{r+4} = -\delta_r s + \left(2\delta_r + (-1)^r \delta_r\right)s^2 + \left(-\delta_r - (-1)^r \delta_r\right)s^3 + \sin[\delta_r s] \tag{3.26}$$

where

$$\delta_r = \frac{r\pi}{L}, \qquad r = 1, 2, 3 \ldots N$$

The expressions for the shape functions are the same as detailed in the previous chapter. The finite element model for the composite curved beam is obtained by making use of the Equation (3.14) and the shape functions given by the Equations (3.25) and (3.26). The element stiffness matrixes are then assembled by combining all the 4 submatrices given in Equations (3.17) and (3.18) through the usual overlay procedure.

The polynomial hierarchical finite element formulation for the composite curved beam would differ from the above trigonometric formulation in the nature of the hierarchical shape functions chosen for the formulation. The shape functions for the polynomial formulations were described in the previous chapter and will be applied to the strain energy Equation (3.14) for the composite curved beam.

Hence, the displacement fields for the curved beam element would be, for tangential displacement (v)

$$v\left(s\right)=N_1v_1+N_2v_{s1}+N_3v_2+N_4v_{s2}+\sum_{r=1}^{M}N_{r+4}A_r \tag{3.27}$$

and for radial displacement (w)

$$w\left(s\right)=N_1w_1+N_2\theta_1+N_3w_2+N_4\theta_2+\sum_{r=1}^{M}N_{r+4}B_r \tag{3.28}$$

where

$$N_{r+1}=s^{r+1}\left(s-L\right)^{r+1}\ , \qquad r=1,2,3\ldots M \tag{3.29}$$

A_r and B_r are the coefficients of the polynomial hierarchical terms.

The polynomial hierarchical shape functions are chosen such that,

$$N_{r+4}=0 \qquad at\quad s=0 \quad \text{and} \quad s=L$$

$$N'_{r+4}=0 \qquad at\quad s=0 \quad \text{and} \quad s=L$$

The above equations illustrate that the functions provide zero displacement and zero slope at each end of the element. Again, it is important to mention that this property is highly significant, since these modes contribute only to the internal displacement field of the element, and do not therefore affect (i.e. over restrain) the displacements at the nodes.

Composite curved beam example shown in Figure 3.3 will be solved and modeled by using four hierarchical trigonometric and polynomial elements. The numbers of trigonometric and polynomial hierarchical terms per element were used in each and every possible combination for both tangential (v) and transverse (w) displacements. A kind of parametric study was conducted in terms of the use of trigonometric and polynomial hierarchical terms per element. First, symmetric hierarchical terms were used for both tangential (v) and transverse (w) displacements i.e. $v_1 - w_1, v_2 - w_2, v_3 - w_3, v_4 - w_4$. Second, each of these hierarchical terms were used for every possible combination like $v_0 - w_n, v_1 - w_n, v_2 - w_n, v_3 - w_n$ and $v_4 - w_n$, where n = 1, 2, 3, 4.

3.6 Approximate Solution for Composite Curved Beam by Ritz Method

In this section approximate solution in conjunction with Ritz method [50] is discussed. In the case of composite curved beam the strain energy equation for 1-D problem is given by the Equation (3.14):

$$U_d = \frac{1}{2}\int_{s=0}^{s=L}\left\{A_{11}\left[\left(\frac{\partial v}{\partial s}\right)^2 + \left(\frac{w}{R}\right)^2\right] + \frac{2A_{11}}{R}\left(w\frac{\partial v}{\partial s}\right) + D_{11}\left(\frac{\partial^2 w}{\partial s^2}\right)^2 + \frac{D_{11}}{R^2}\left(\frac{\partial v}{\partial s}\right)^2\right.$$

$$\left.-\frac{2D_{11}}{R}\left[\left(\frac{\partial v}{\partial s}\right)\left(\frac{\partial^2 w}{\partial s^2}\right)\right]\right\}ds \tag{3.30}$$

The potential energy owed to the transverse point load P at the free end is

$$W_f = P(w)_L \tag{3.31}$$

The approximate solution is given by single summation series:

$$w_0(s) = \sum_{m=1}^{M} A_m S_m(s) \tag{3.32}$$

$$v_0(s) = \sum_{n=1}^{N} B_n S_n(s)$$

The functions $S_m(s)$ and $S_n(s)$ have to form function bases for polynomials, trigonometric functions and hyperbolic functions and are chosen to satisfy the boundary conditions. The coefficients A_m and B_n are next determined by the stationary conditions, which are written as:

$$\frac{\partial \widetilde{U}}{\partial A_m} = 0, \qquad \text{or} \qquad \frac{\partial \widetilde{U}_d}{\partial A_m} = \frac{\partial \widetilde{W}_f}{\partial A_m} \tag{3.33}$$

$$\frac{\partial \widetilde{U}}{\partial B_n} = 0, \qquad \text{or} \qquad \frac{\partial \widetilde{U}_d}{\partial B_n} = \frac{\partial \widetilde{W}_f}{\partial B_n} \tag{3.34}$$

The $\widetilde{U}_d$ and $\widetilde{W}_f$ are the strain energy and the potential energy owed to the transverse point load, obtained by substituting the approximate expressions for the deflections into Equations (3.30) and (3.31) respectively. The calculation of the approximate strain energy requires us to express the terms

$\left(\frac{w}{R}\right)^2$, $\left(w\frac{\partial v}{\partial s}\right)$, $\left(\frac{\partial^2 w}{\partial s^2}\right)^2$, $\left(\frac{\partial v}{\partial s}\right)^2$, $\left(\frac{\partial v}{\partial s}\right)$ $\left(\frac{\partial^2 w}{\partial s^2}\right)$ in terms of A_m and B_n.

For Example:

$$\frac{\partial^2 w}{\partial s^2} = \sum_{m=1}^{M} A_m \frac{d^2 S_m(s)}{ds^2} \tag{3.35}$$

$$\frac{\partial v}{\partial s} = \sum_{n=1}^{N} B_n \frac{dS_n}{ds} \tag{3.36}$$

Whence

$$\left(\frac{\partial^2 w}{\partial s^2}\right)^2 = \sum_{m=1}^{M} \sum_{i=1}^{M} A_m A_i \frac{d^2 S_m(s)}{ds^2} \frac{d^2 S_i(s)}{ds^2} \tag{3.37}$$

$$\left(\frac{\partial v}{\partial s}\right)^2 = \sum_{n=1}^{N} \sum_{j=1}^{N} B_n B_j \frac{dS_n(s)}{ds} \frac{dS_j(s)}{ds} \tag{3.38}$$

and

$$\frac{1}{2} \frac{\partial}{\partial A_m} \left(\frac{\partial^2 w}{\partial s^2}\right)^2 = \sum_{m=1}^{M} \sum_{i=1}^{M} A_i \frac{d^2 S_m(s)}{ds^2} \frac{d^2 S_i(s)}{ds^2} \tag{3.39}$$

$$\frac{1}{2} \frac{\partial}{\partial B_n} \left(\frac{\partial v}{\partial s}\right)^2 = \sum_{j=1}^{N} \sum_{n=1}^{N} B_j \frac{dS_n(s)}{ds} \frac{dS_j(s)}{ds} \tag{3.40}$$

Integration of these terms $\frac{\partial^2 w}{\partial s^2}$ and $\frac{\partial v}{\partial s}$ yields

$$\frac{1}{2} \int_{s=0}^{s=L} \left(\frac{\partial^2 w}{\partial s^2}\right)^2 ds = \sum_{m=1}^{M} \sum_{i=1}^{M} A_i \int_{s=0}^{s=L} \frac{d^2 S_m(s)}{ds^2} \frac{d^2 S_i(s)}{ds^2} ds \tag{3.41}$$

$$\frac{1}{2} \int_{s=0}^{s=L} \left(\frac{\partial v}{\partial s}\right)^2 ds = \sum_{j=1}^{N} \sum_{n=1}^{N} B_j \int_{s=0}^{s=L} \frac{dS_n(s)}{ds} \frac{dS_j(s)}{ds} ds \tag{3.42}$$

The left hand sides of the Equations (3.33) and (3.34) can be put in the practical form as follows:

$$\frac{\partial \widetilde{U}_d}{\partial A_m} = \int_{s=0}^{s=L} \left\{ \frac{A_{11}}{R^2} \left[\sum_{i=1}^{M} A_i S_m S_i \right] + \frac{A_{11}}{R} \left[\sum_{n=1}^{N} B_n \frac{dS_n}{ds} S_m \right] + D_{11} \left[\sum_{i=1}^{M} A_i \frac{d^2 S_m}{ds^2} \frac{d^2 S_i}{ds^2} \right] \right.$$

$$\left. - \frac{D_{11}}{R} \left[\sum_{n=1}^{N} B_n \frac{d^2 S_m}{ds^2} \frac{dS_n}{ds} \right] \right\} ds \tag{3.43}$$

$$\frac{\partial \widetilde{U}_d}{\partial B_n} = \int_{s=0}^{s=L} \left\{ A_{11} \left[\sum_{j=1}^{N} B_j \frac{dS_n}{ds} \frac{dS_j}{ds} \right] + \frac{A_{11}}{R} \left[\sum_{M=1}^{M} A_m \frac{dS_n}{ds} S_m \right] + \frac{D_{11}}{R^2} \left[\sum_{j=1}^{N} B_j \frac{dS_n}{ds} \frac{dS_j}{ds} \right] \right.$$

$$\left. - \frac{D_{11}}{R} \left[\sum_{M=1}^{M} A_m \frac{d^2 S_m}{ds^2} \frac{dS_n}{ds} \right] \right\} ds \tag{3.44}$$

Equations (3.44) and (3.45) finally become

$$\frac{\partial \widetilde{U}_d}{\partial A_m} = \int_{s=0}^{s=L} \Bigg\{ \left[\frac{A_{11}}{R^2} \sum_{i=1}^{M} (S_m\ S_i) + D_{11} \sum_{i=1}^{M} \left(\frac{d^2 S_m}{ds^2} \frac{d^2 S_i}{ds^2} \right) \right] A_i$$

$$- \left[\frac{D_{11}}{R} \sum_{n=1}^{N} \left(\frac{d^2 S_m}{ds^2} \frac{dS_n}{ds} \right) + \frac{A_{11}}{R} \sum_{n=1}^{N} \left(\frac{dS_n}{ds} S_m \right) \right] B_n \Bigg\} ds \qquad (3.45)$$

$$\frac{\partial \widetilde{U}_d}{\partial B_n} = \int_{s=0}^{s=L} \Bigg\{ \left[\frac{A_{11}}{R} \sum_{M=1}^{M} \left(\frac{dS_n}{ds} S_m \right) - \frac{D_{11}}{R} \sum_{M=1}^{M} \left(\frac{d^2 S_m}{ds^2} \frac{dS_n}{ds} \right) \right] A_m$$

$$+ A_{11} \left[\sum_{j=1}^{N} \left(\frac{dS_n}{ds} \frac{dS_j}{ds} \right) + \frac{D_{11}}{R^2} \sum_{\mathrm{j}=1}^{\mathrm{N}} \left(\frac{dS_n}{ds} \frac{dS_j}{ds} \right) \right] \mathrm{B_j} \Bigg\} \mathrm{ds} \qquad (3.46)$$

where

$m,\ i = 1, 2, 3 \ldots\ldots\ldots\ldots M,$

$n,\ j = 1, 2, 3 \ldots\ldots\ldots\ldots N,$

3.6.1 Curved Beam Example Based on Euler – Bernoulli Theory

Uniform composite curved beam with fixed-free boundary condition is shown in Figure 3.2. It is made of NCT-301 Graphite Epoxy material. The deterministic material properties of the NCT-301 material are given as:

$E_1 = 129.43\ GPa,\ \ E_2 = 7.99\ GPa,\ \ \upsilon_{21} = 0.021,\ \ G_{12} = 4.28\ GPa$

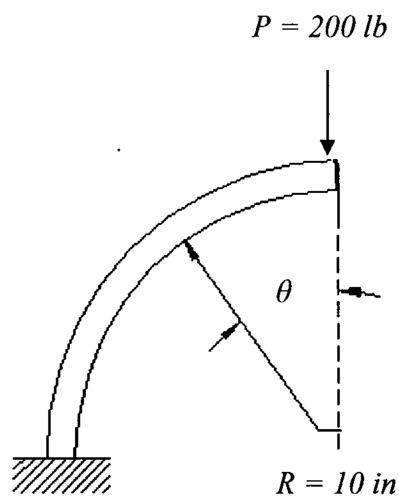


Figure 3.3 Fixed-Free Composite Curved Beam

The geometric properties of the beam are: length $L = 10\times\pi/2$; individual ply thickness (e_k) = 0.125 mm. There are 32 plies in the laminate and the configuration of the laminate is $[0/90]_{8s}$. The laminate thickness of 4 mm is obtained by multiplying the total number of plies with the individual ply thickness.

3.6.1.1 Fixed-Free Composite Curved Beam

We will discuss the case of composite curved beam fixed at one end and free at the other end as shown in Figure 3.3, subjected to a point load at the free end. As the curved beam is fixed at one end and free at the other end the boundary conditions are:

For tangential displacement (v)

for edge $s = 0$:

$$v_1 = 0, \qquad \left.\frac{\partial v}{\partial s}\right|_1 = v_{s1} \neq 0 \tag{3.47}$$

for edge $s = L$

$$v_2 \neq 0, \qquad \left.\frac{\partial v}{\partial s}\right|_2 = v_{s2} = 0 \tag{3.48}$$

For radial displacement (w)

for edge $s = 0$:

$$w_1 = 0, \qquad \left.\frac{\partial w}{\partial s}\right|_1 = \theta_1 = 0 \tag{3.49}$$

for edge $s = L$

$$w_2 \neq 0, \qquad \left.\frac{\partial w}{\partial s}\right|_2 = \theta_2 \neq 0 \tag{3.50}$$

3.6.1.2 *Solution Approximated by Displacement Functions*

For the transverse displacement (w) we choose the trigonometric functions in the form:

$$S_m(s) = s^m \left(\cos \frac{m\pi s}{L} \right) \tag{3.51}$$

For the tangential displacement (v) we choose the polynomial displacement functions in the form:

$$S_n(s) = \left(\frac{s}{L} \right)^{2n} - \left(\frac{2ns}{L} \right) \tag{3.52}$$

These functions satisfy the boundary conditions in Equations (3.47), (3.48), (3.49), and (3.50).

In the case where $m = n = 2$ the system of Equations (3.45) and (3.46) to calculate A_m and B_n becomes

$$\begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ a_{31} & a_{32} & b_{31} & b_{32} \\ a_{41} & a_{42} & b_{41} & b_{42} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ 0 \\ 0 \end{Bmatrix} \tag{3.53}$$

$$\begin{gathered} [ab]\{AB\} = [F] \\ \{AB\} = [ab]^{-1}[F] \end{gathered} \tag{3.54}$$

As there is no force being applied in the tangential direction and the only force available is in the transverse direction the matrix due to work done has zeros in the last two rows. The values of A_1 and A_2 will be substituted in Equation (3.32) to calculate the values of transverse deflection at $s = L$.

The results are given in Table 3.2 and plotted as well in Figure 3.4. The results show an improvement in the transverse deflection as we increase the value of m in the displacement function. For the first two values of m the improvement is almost linear and then there is a jump in the value for the third value of m. For the next four values the improvement between them is almost same. For the last two values of m the transverse deflection value is same. Normally for a curved structure, we stop adding more terms in the Ritz solution if the % age difference between the two consecutive values is less than 5 %. The approximate value for the transverse deflection is obtained for $m = 7$ which is $0.0568 in$.

Table 3.2 Ritz Method Solution for $[0/90]_{8s}$ Laminate for Fixed-Free Boundary Condition

Value of m	*Centre Deflection (in)*
1	0.0041
2	0.0072
3	0.0329
4	0.0421
5	0.0583
6	0.0568
7	0.0568

3.7 Solution to the Composite Curved Beam by HFEM

The above curved beam example will now be solved by using both conventional and hierarchical finite element methods and the results will be compared with the solution obtained by the Ritz method.

For the HFEM, both the sub-formulations, viz. trigonometric and polynomial formulations are used to obtain the results. While solving the problem with either formulation, the beams are discretized such that the numbers of degrees of freedom used in the analysis by HFEM and by conventional FEM are comparable. This is done to make a comparison between the two formulations vis-à-vis the number of elements required, the number of nodal degrees of freedom to obtain the desired accuracy.

The analysis is based on the strain Energy Equation (3.14) developed for 1-D curved beams. Tables (3.4 - 3.9) show the results for fixed-free composite curved beam based on the above strain energy equation for mid-plane symmetric composite laminate having the configuration $[0/90]_{8s}$. Comparison is made between the polynomial and trigonometric HFEM and the conventional FEM formulations and the results are then compared with the approximate solutions which were obtained using Ritz Method.

The results for conventional FEM are given in Table 3.3. The results show that the conventional FEM results show a smooth improvement in results as we increase the number of elements, resulting in almost linear curve as shown in Figures (3.5 - 3.11). The results show a considerable improvement in the results when we move from 1-element mesh to 2-elements mesh which is the same trend as we noted for the isotropic curved beam element in the previous chapter. After the 2-elements mesh there was a constant improvement in the results until we reached the approximate solution at the 8-elements mesh.

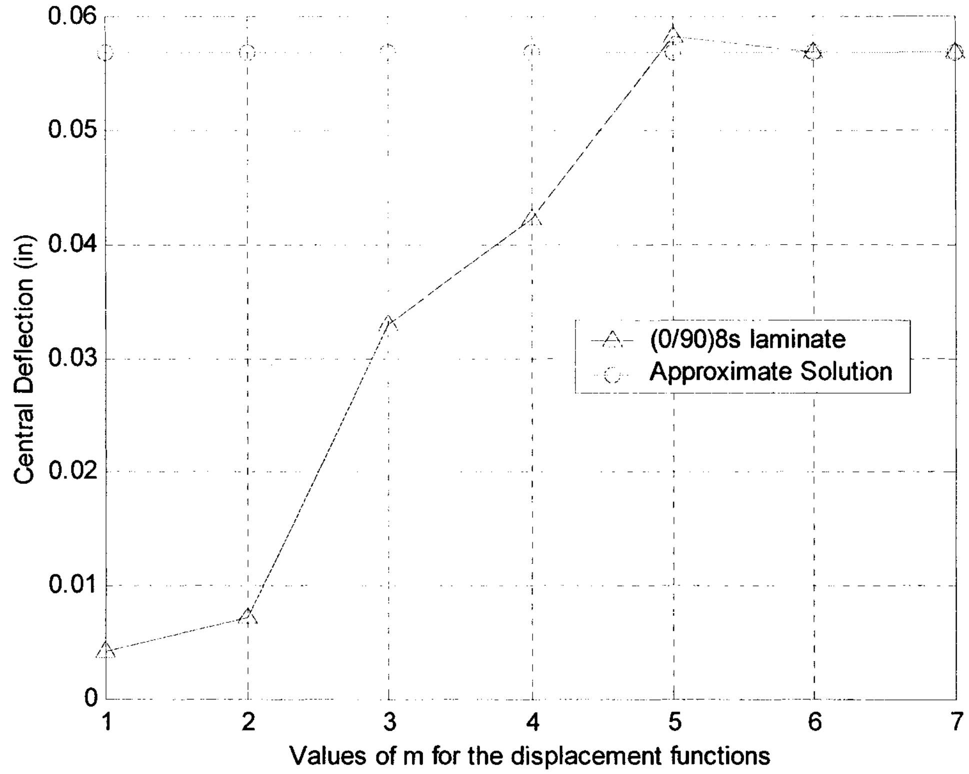


Figure 3.4 Improvement in the Ritz Method Solution with Values of m

For Hierarchical Finite Element method, first symmetric hierarchical trigonometric and polynomial terms $v_1 - w_1, v_2 - w_2, v_3 - w_3, v_4 - w_4$, will be used and then unsymmetrical hierarchical terms will be used for each and every possible combination like $v_0 - w_n, v_1 - w_n, v_2 - w_n, v_3 - w_n$ and $v_4 - w_n$ where n = 1, 2, 3, 4.

The results of the hierarchical finite element formulation are given in Tables (3.4 - 3.9) and are also plotted in Figures (3.5 - 3.11).

Table 3.3 Conventional FEM Solution for Fixed-Free $[0/90]_{8S}$ laminate

Composite Curved Beam with Fixed-Free Boundary Condition			
Number of Elements	*Number of D.O.F.*	*Centre Deflection (in)*	*Error (%)*
1	4	0.0057	89.96
2	8	0.0218	61.62
3	12	0.0264	53.52
4	16	0.0307	45.95
5	20	0.0360	36.62
6	24	0.0421	25.88
7	28	0.0485	14.61
8	32	0.0551	2.96

3.8 Discussion and Conclusion

The Hierarchical Finite Element Method developed and applied to isotropic curved beams in the previous chapter has been applied in this chapter to uniform thickness composite curved beams. The uniform-thickness composite curved beams have been modeled using the 1-D cylindrical laminated plate theory. Both the forms of HFEM are applied, and contrary to the case of isotropic curved beams the polynomial HFEM gives better results than the trigonometric form. Results for the Euler-Bernoulli beams have been presented.

Application of the hierarchical finite element method to composite beams, as in the case of isotropic curved beams, yields the same advantage of numerical efficiency and faster convergence. Less number of elements is required to model and obtain precise answers for analysis of composite curved beams. The graphs in Figures (3.7-3.16) give us a comparison of the convergence of the number of elements required by trigonometric HFEM, polynomial HFEM and the conventional formulation to get the approximate solution. There is a substantial reduction in the number of elements required to obtain the results that are almost the same as the approximate solution by Ritz method.

For symmetric Polynomial and Trigonometric Hierarchical terms the results are plotted in Figures 3.5 and 3.6. These results are given in Table 3.4. The results show that polynomial hierarchical formulation gives better results than the trigonometric hierarchical formulation for the symmetric hierarchical terms. The results are converged more rapidly by the polynomial hierarchical terms than by the trigonometric hierarchical terms when a hierarchical term is added to both tangential (v) and radial (w) displacement functions. When 3 and 4 symmetric polynomial terms were used the resulting stiffness matrix becomes ill-conditioned and hence, their solutions become inaccurate for the 1-element mesh. When 4 symmetric polynomial hierarchical terms were used, we reached the result by using just one element mesh. Figures 3.5 and 3.6 show a considerable improvement in the results when we increase the number of symmetric hierarchical terms from one term to four terms for both polynomial and trigonometric hierarchical formulations.

Table 3.4 Central Deflection Calculated by using Symmetric Polynomial and Trigonometric Hierarchical Terms for Fixed- Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0/90]_{8s}$ Laminate

Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Symmetric Polynomial 1 HFEM term		*Symmetric Polynomial 2 HFEM terms*		*Symmetric Polynomial 3 HFEM terms*		*Symmetric Polynomial 4 HFEM terms*	
1	4	0.0127	6	0.0281	8	0.0422	10	0.0578
2	8	0.0304	11	0.0443	14	0.0583		
3	12	0.0395	16	0.0572				
4	16	0.0504						
5	20	0.0620						

Number of elements	*Symmetric Trigonometric 1 HFEM term*		*Symmetric Trigonometric 2 HFEM terms*		*Symmetric Trigonometric 3 HFEM terms*		*Symmetric Trigonometric 4 HFEM terms*	
1	4	0.0125	6	0.0184	8	0.0250	10	0.0309
2	8	0.0298	11	0.0387	14	0.0490	18	0.0603
3	12	0.0386	16	0.0543	20	0.0709		
4	16	0.0491	21	0.0710				
5	20	0.0604						

Approximate Solution by Ritz method: Central Deflection = 0.0568 in

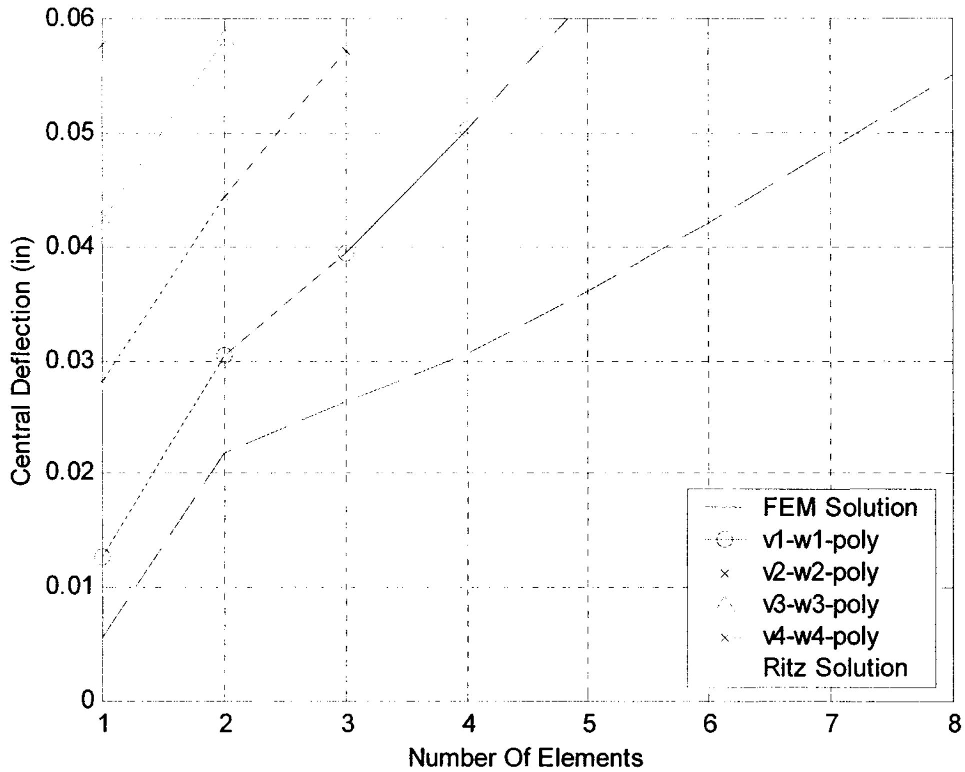


Figure 3.5 Comparison between the Solutions Obtained using Symmetric Polynomial Hierarchical Terms for $[0/90]_{8s}$ Laminate

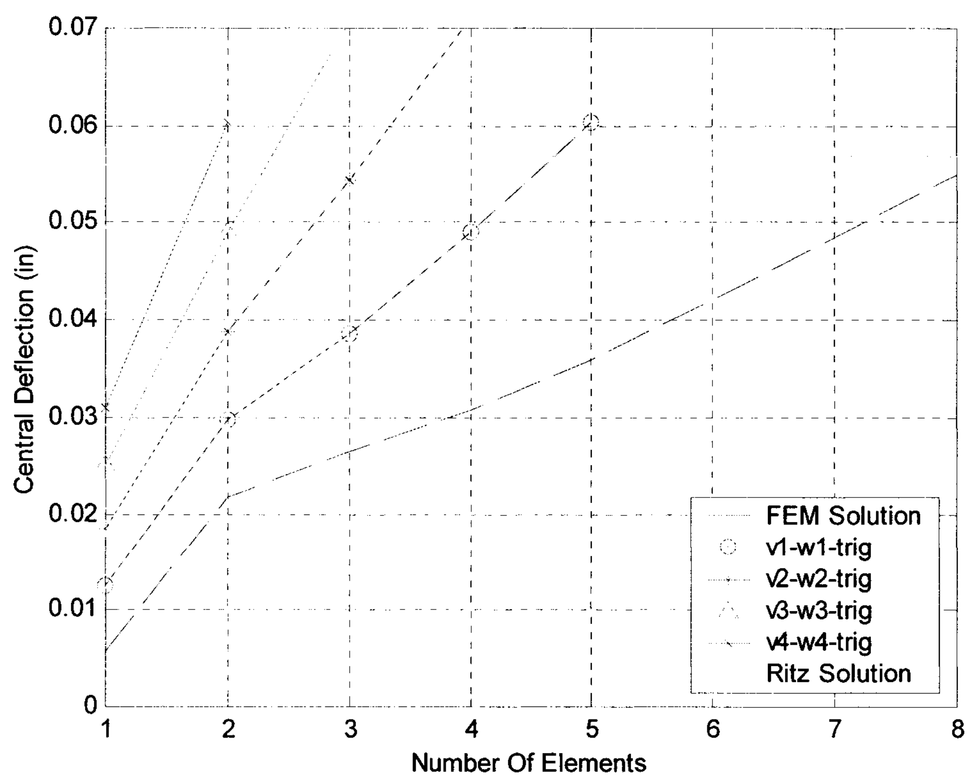


Figure 3.6 Comparison between the Solutions Obtained using Symmetric Trigonometric Hierarchical Terms for $[0/90]_{8s}$ Laminate

The results for the group of combinations ($v_0 - w_n$) of both polynomial and trigonometric hierarchical terms are almost identical to that of the conventional FEM reaching the approximate solution at the 8-elements mesh as shown in Table 3.5 and Figures 3.7 and 3.8. So it is evident that despite increasing the number of hierarchical terms associated with transverse displacement (w) function the results did not get better showing its little effect on the results. The results also show that for the combination ($v_0 - w_n$) the results given by the trigonometric hierarchical terms are almost identical to that of the results given by the polynomial hierarchical formulation. For both polynomial and trigonometric hierarchical formulations all the combinations of this group ($v_0 - w_n$) give almost same results. But as we increase the number of tangential displacement (v) hierarchical terms, a great improvement in the results is observed showing its greater effect on the results and it will be shown in the coming figures.

The Figures (3.9-3.16) show greater effect of hierarchical terms added to the tangential displacement (v) function. As shown in Figures 3.9 and 3.10, the results get better for the combinations ($v_1 - w_n$) than that of the previous group of combinations ($v_0 - w_n$). In present case the results were almost reached by using just 5-elements mesh instead of 8-elements mesh used in the previous case showing considering the fact that only one hierarchical term was added to the tangential displacement (v) function for both polynomial and trigonometric hierarchical formulations. The results given by all the combinations ($v_1 - w_n$) of polynomial hierarchical formulation are similar to each other. Generally, polynomial hierarchical formulation gives better results that that of trigonometric hierarchical formulation.

Table 3.5 Central Deflection Calculated by using Non- Symmetric Polynomial and Trigonometric Hierarchical Terms ($v_0 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0/90]_{8s}$ Laminate

Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Conventional (FEM)		*Non-Symmetric Polynomial v0-w1 HFEM terms*		*Non-Symmetric Polynomial v0-w2 HFEM terms*		*Non-Symmetric Polynomial v0-w3 HFEM terms*		*Non-Symmetric Polynomial v0-w4 HFEM terms*	
1	4	0.0057	5	0.0108	6	0.0124	7	0.0127	8	0.0131
2	8	0.0218	10	0.0266	12	0.0271	14	0.0273	16	0.0273
3	12	0.0264	15	0.0281	18	0.0288	21	0.0288	24	0.0288
4	16	0.0307	20	0.0314	24	0.0317	28	0.0317	32	0.0317
5	20	0.0360	25	0.0364	30	0.0365	35	0.0365	40	0.0365
6	24	0.0421	30	0.0423	36	0.0424	42	0.0424	48	0.0424
7	28	0.0485	35	0.0486	42	0.0487	49	0.0487	56	0.0487
8	32	0.0551	40	0.0552	48	0.0552	56	0.0552	64	0.0552

Number of Elements	*Conventional (FEM)*		*Non-Symmetric Trig v0-w1 HFEM terms*		*Non-Symmetric Trig v0-w2 HFEM terms*		*Non-Symmetric Trig v0-w3 HFEM terms*		*Non-Symmetric Trig v0-w4 HFEM terms*	
1	4	0.0057	5	0.0106	6	0.0139	7	0.0162	8	0.0182
2	8	0.0218	10	0.0268	12	0.0268	14	0.0270	16	0.0275
3	12	0.0264	15	0.0283	18	0.0289	21	0.0294	24	0.0296
4	16	0.0307	20	0.0314	24	0.0323	28	0.0326	32	0.0326
5	20	0.0360	25	0.0364	30	0.0371	35	0.0372	40	0.0372
6	24	0.0421	30	0.0423	36	0.0427	42	0.0428	48	0.0428
7	28	0.0485	35	0.0486	42	0.0489	49	0.0490	56	0.0490
8	32	0.0551	40	0.0552	48	0.0554	56	0.0554	64	0.0554

Approximate Solution by Ritz method: Central Deflection = 0.0568 in

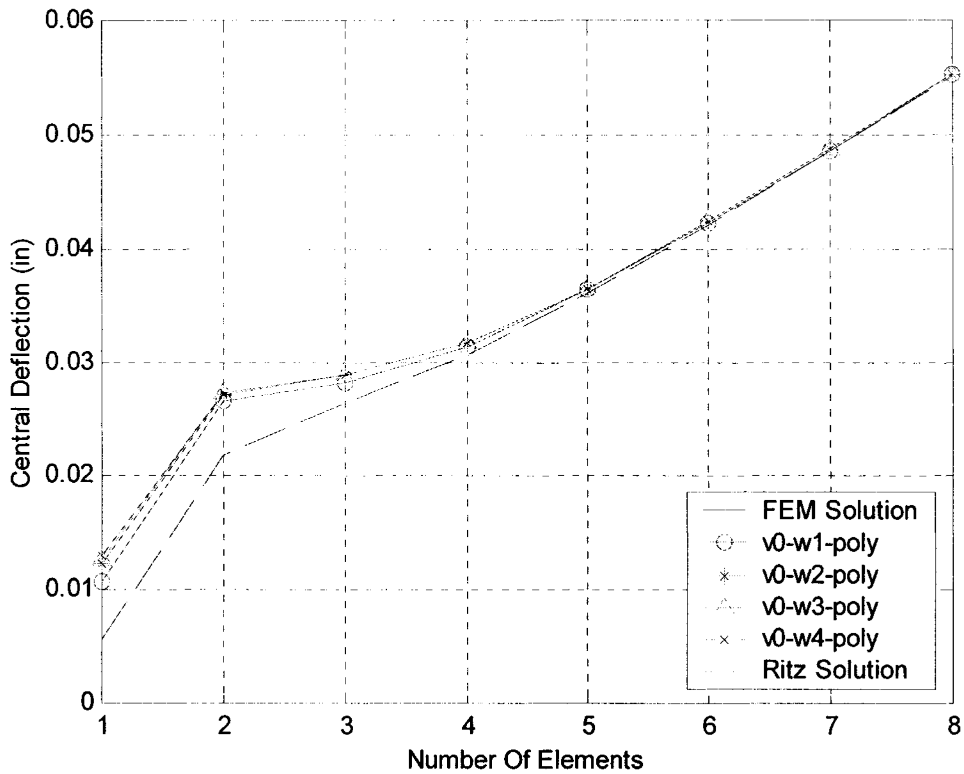


Figure 3.7 Comparison between the Solutions Obtained using Non-Symmetric ($v_0 - w_n$) Polynomial Hierarchical Terms for $[0/90]_{8s}$ Laminate

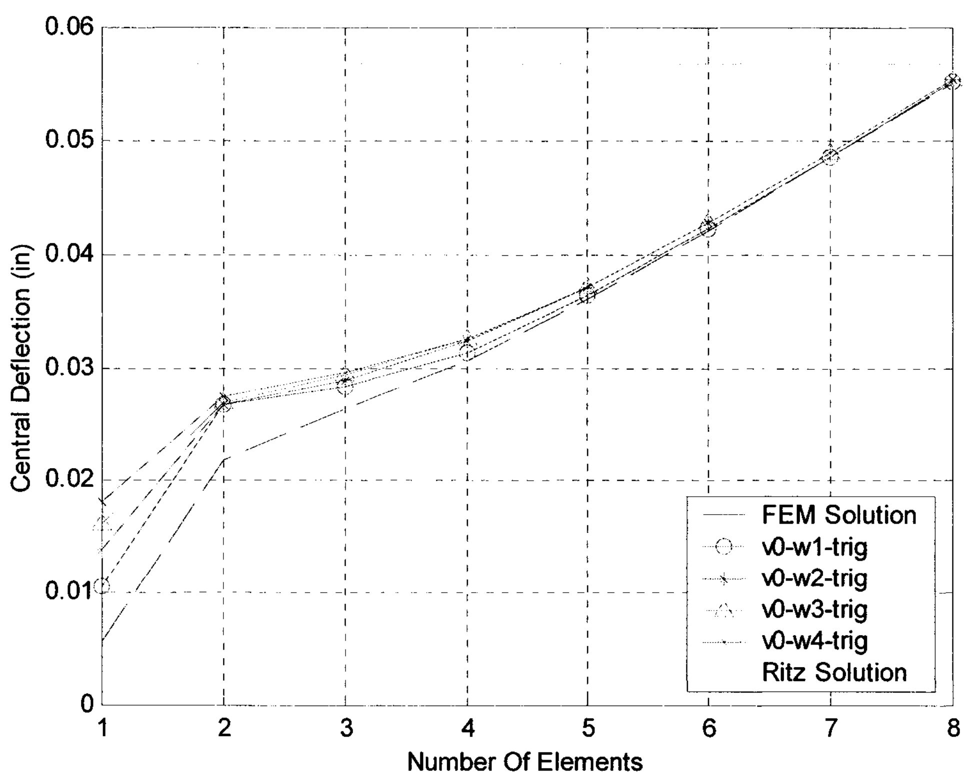


Figure 3.8 Comparison between the Results Obtained using Non-Symmetric ($v_0 - w_n$) Trigonometric Hierarchical Terms for $[0/90]_{8s}$ Laminate

Table 3.6 Central Deflection Calculated by using Non- Symmetric polynomial and Trigonometric Hierarchical Terms ($v_1 - w_n$) for Fixed-Free Boundary Condition

8. D.O.F Composite Curved Beam Element for $[0/90]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Non-Symmetric Polynomial v1-w0 HFEM terms		*Non-Symmetric Polynomial v1-w2 HFEM terms*		*Non-Symmetric Polynomial v1-w3 HFEM terms*		*Non-Symmetric Polynomial v1-w4HFEM terms*	
1	5	0.0128	7	0.0173	8	0.0211	9	0.0222
2	10	0.0249	12	0.0310	13	0.0313	14	0.0313
3	15	0.0374	17	0.0406	18	0.0407	19	0.0407
4	20	0.0498	22	0.0513	23	0.0514	24	0.0514
5	25	0.0623	27	0.0629	28	0.0630	29	0.0631

Number of Elements	*Non-Symmetric Trigonometric v1-w0 HFEM terms*		*Non-Symmetric Trigonometric v1-w2 HFEM terms*		*Non-Symmetric Trigonometric v1-w3 HFEM terms*		*Non-Symmetric Trigonometric v1-w4HFEM terms*	
1	5	0.0123	7	0.0184	8	0.0219	9	0.0228
2	10	0.0242	12	0.0321	13	0.0325	14	0.0328
3	15	0.0363	17	0.0393	18	0.0401	19	0.0403
4	20	0.0484	22	0.0494	23	0.0501	24	0.0502
5	25	0.0605	27	0.0605	28	0.0611	29	0.0612

Approximate Solution by Ritz method: Central Deflection = 0.0568 in

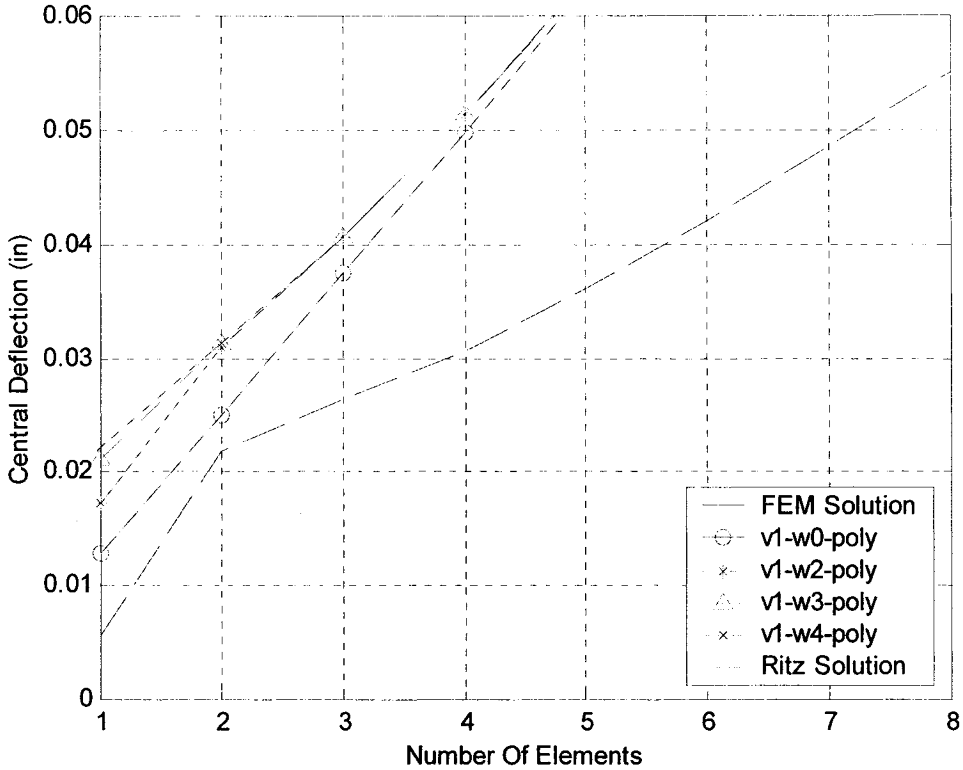


Figure 3.9 Comparison between the Results Obtained using Non-Symmetric ($v_1 - w_n$) Polynomial Hierarchical Terms for $[0/90]_{8s}$ Laminate

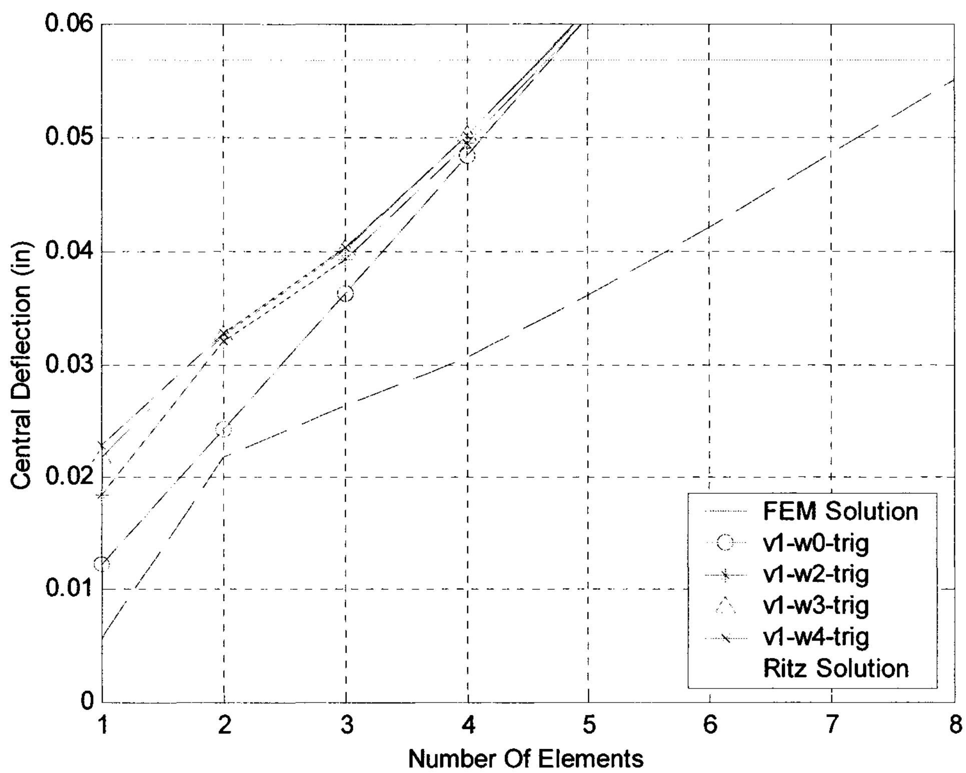


Figure 3.10 Comparison between the Results Obtained using Non-Symmetric ($v_1 - w_n$) Trigonometric Hierarchical Terms for $[0/90]_{8s}$ Laminate

The results for the group of combinations ($v_2 - w_n$) are given in Table 3.7 and are plotted in Figures 3.11 and 3.12. The results given by the polynomial hierarchical terms converge better than that of the trigonometric hierarchical terms. For both polynomial and trigonometric formulations the results converge to the approximate solution around 3-elements mesh that shows an improvement in the results from the previous group of combinations ($v_1 - w_n$). The results given by the trigonometric hierarchical formulation show little improvement each time we add hierarchical terms to the transverse displacement (w) function. The curves for the trigonometric hierarchical formulation almost converge to a single curve where as the curves for polynomial hierarchical formulation show more variations among them.

The results for the group of combinations ($v_3 - w_n$) get even better with the addition of three hierarchical terms to the transverse displacement (v) function. Polynomial hierarchical formulation also shows a bit more variation in the results when different hierarchical terms are added to the radial displacement (w) function reaching the approximate solution at the 2-elements mesh for the combination ($v_3 - w_4$). The results for the polynomial hierarchical formulation also show an improvement when hierarchical terms are added to the radial displacement (w) function. Trigonometric hierarchical terms once again show very little variation and look like converging to one curve and reaching the approximate solution at the 3-elements mesh. So it is evident that polynomial hierarchical terms give better results than the trigonometric terms.

Table 3.7 Central Deflection Calculated by Non- Symmetric Polynomial and Trigonometric Hierarchical Terms ($v_2 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0/90]_{8s}$ Laminate

Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Non-Symmetric Polynomial v2-w0 HFEM terms		*Non-Symmetric Polynomial v2-w1 HFEM terms*		*Non-Symmetric Polynomial v2-w3 HFEM terms*		*Non-Symmetric Polynomial v2-w4HFEM terms*	
1	6	0.0217	7	0.0216	9	0.0304	10	0.0304
2	12	0.0367	13	0.0418	15	0.0448	16	0.0450
3	18	0.0542	19	0.0530	21	0.0574	22	0.0579
4	24	0.0722	25	0.0662	27	0.0716	28	0.0723

Number of Elements	*Non-Symmetric Trigonometric v2-w0 HFEM terms*		*Non-Symmetric Trigonometric v2-w1 HFEM terms*		*Non-Symmetric Trigonometric v2-w3 HFEM terms*		*Non-Symmetric Trigonometric v2-w4HFEM terms*	
1	6	0.0179	7	0.0183	9	0.0232	10	0.0261
2	12	0.0358	13	0.0358	15	0.0400	16	0.0404
3	18	0.0536	19	0.0535	21	0.0548	22	0.0552
4	24	0.0715	25	0.0711	27	0.0714	28	0.0717

Approximate Solution by Ritz method: Central Deflection = 0.0568 in

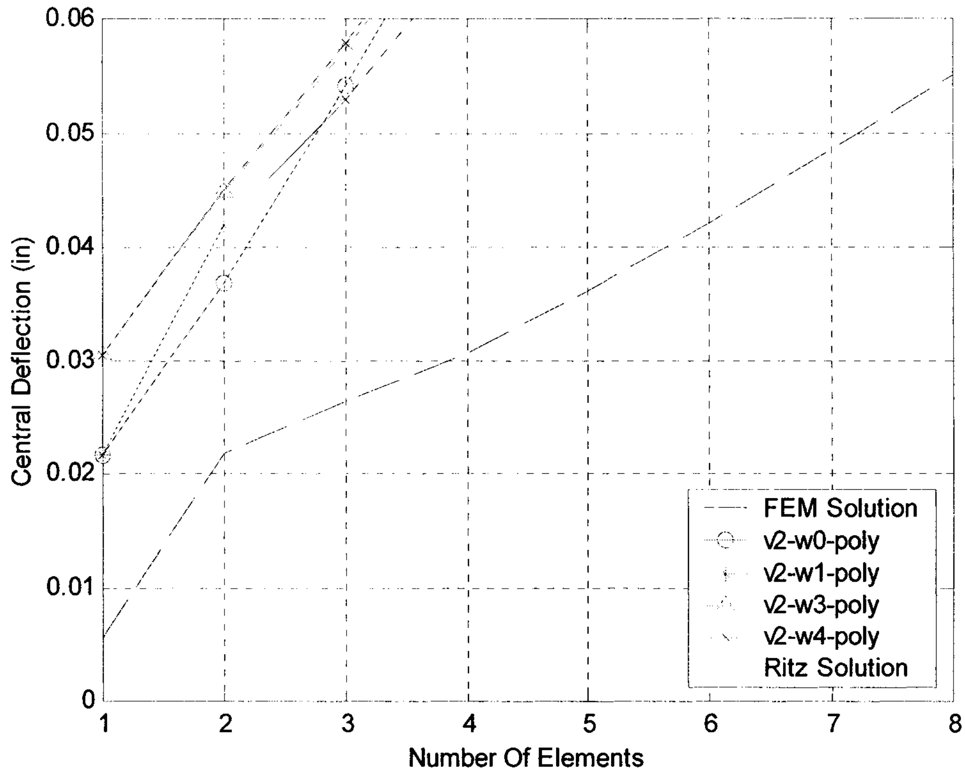


Figure 3.11 Comparison between the Results Obtained using Non-Symmetric ($v_2 - w_n$) Polynomial Hierarchical Terms for $[0/90]_{8s}$ Laminate

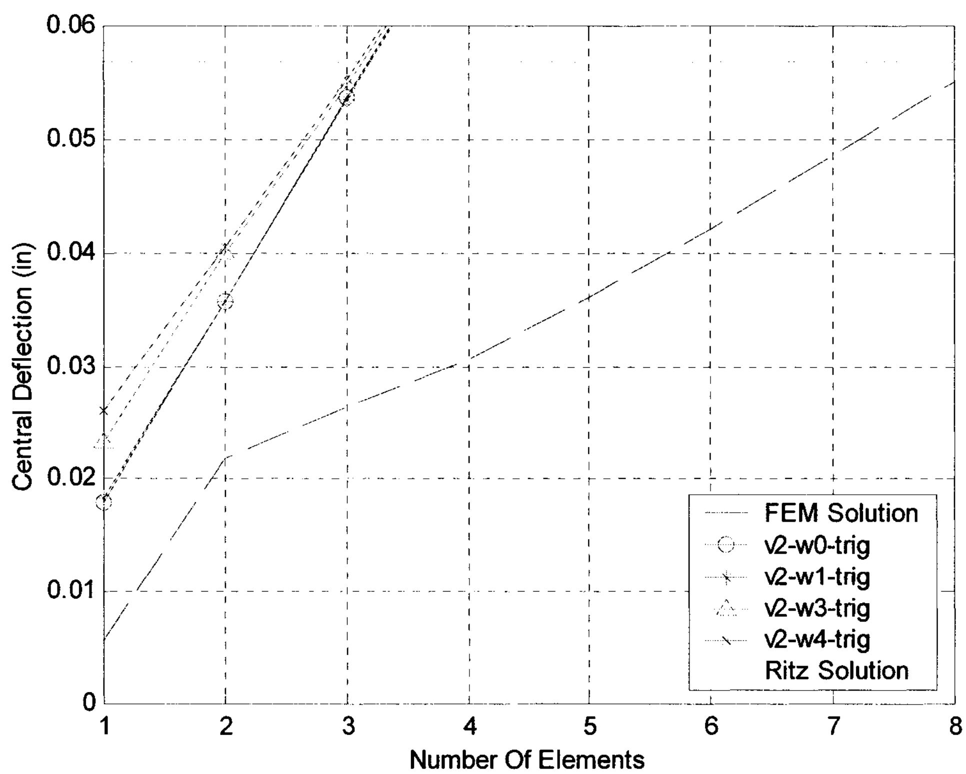


Figure 3.12 Comparison between the Results Obtained using Non-Symmetric ($v_2 - w_{(n)}$) Trigonometric Hierarchical Terms for $[0/90]_{8s}$ Laminate

Table 3.8 Central Deflection Calculated by Non- Symmetric Polynomial and Trigonometric Hierarchical Terms ($v_3 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0/90]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Non-Symmetric Polynomial v3-w0 HFEM terms		*Non-Symmetric Polynomial v3-w1 HFEM terms*		*Non-Symmetric Polynomial v3-w2 HFEM terms*		*Non-Symmetric Polynomial v3-w4HFEM terms*	
1	7	0.0337	8	0.0336	9	0.0402	11	0.0424
2	14	0.0503	15	0.0483	16	0.0536	18	0.0586
3	21	0.0716	22	0.0574	23	0.0669		

Number of Elements	*Non-Symmetric Trigonometric v3-w0 HFEM terms*		*Non-Symmetric Trigonometric v3-w1 HFEM terms*		*Non-Symmetric Trigonometric v3-w2 HFEM terms*		*Non-Symmetric Trigonometric v3-w4HFEM terms*	
1	7	0.0240	8	0.0246	9	0.0246	11	0.0287
2	14	0.0478	15	0.0513	16	0.0473	18	0.0497
3	21	0.0717	22	0.0729	23	0.0698	25	0.0712

Approximate Solution by Ritz method: Central Deflection = 0.0568 in

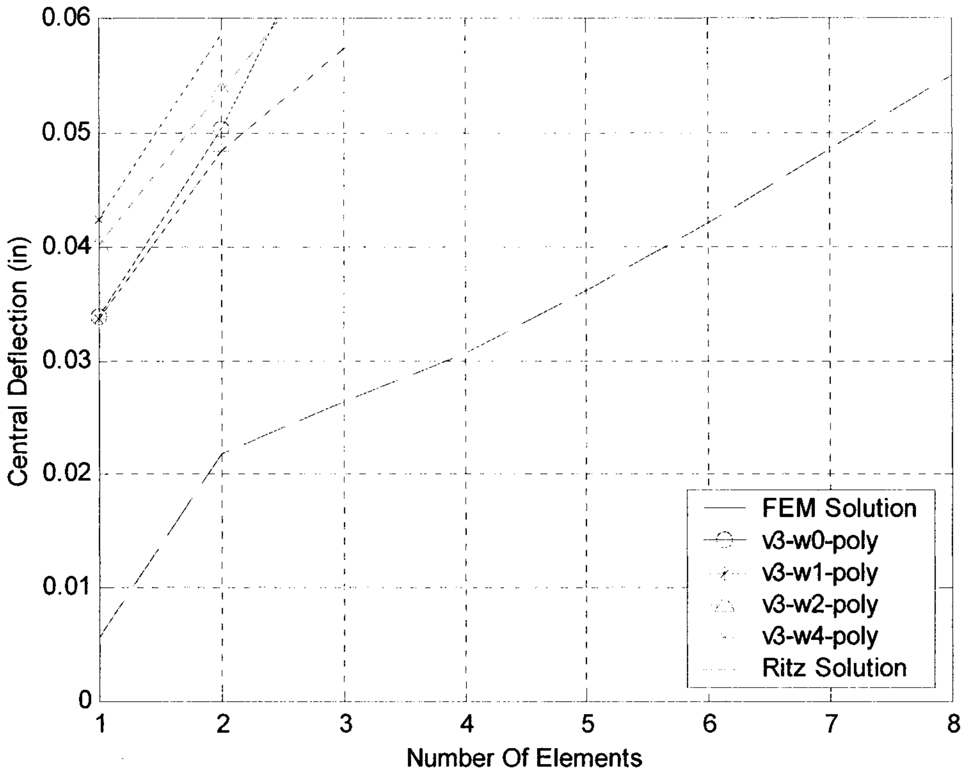


Figure 3.13 Comparison between the Results Obtained using Non-Symmetric ($v_3 - w_n$) Polynomial Hierarchical Terms for $[0/90]_{8s}$ Laminate

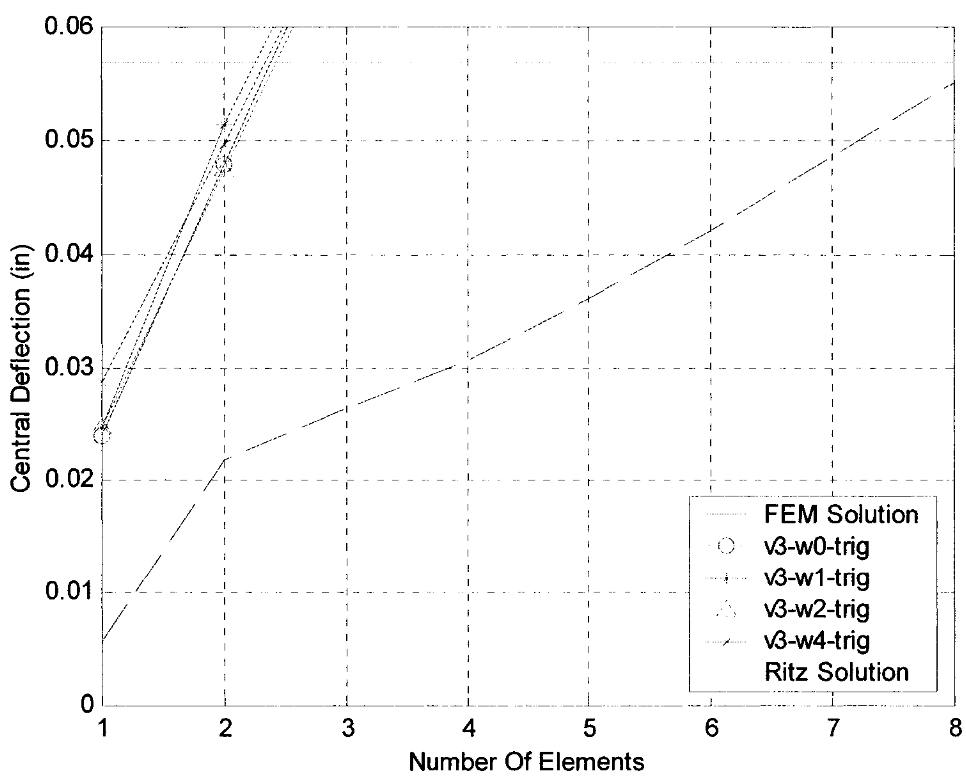


Figure 3.14 Comparison between the Results Obtained using Non-Symmetric ($v_3 - w_n$) Trigonometric Hierarchical Terms for $[0/90]_{8s}$ Laminate

The combinations $(v_4 - w_n)$ with maximum number of hierarchical terms associated with the tangential displacement (v) function and no hierarchical term with the transverse displacement (w) function display the most rapid convergence of the results. Only one element mesh is required for the combination $(v_4 - w_3)$ to reach the approximate Ritz solution. Trigonometric hierarchical terms once again show very little variation but this time convergence is faster than the previous combinations and the approximate solution was reached at the 2-elements mesh.

So the combinations $(v_4 - w_3)$ and $(v_4 - w_4)$ seem to be the fast converging combinations for the polynomial hierarchical and trigonometric hierarchical formulations respectively.

Table 3.9 Central Deflection Calculated by Non- Symmetric Polynomial and Trigonometric Hierarchical Herms ($v_4 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0/90]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Non-Symmetric Polynomial v4-w0 HFEM terms		*Non-Symmetric Polynomial v4-w1 HFEM terms*		*Non-Symmetric Polynomial v4-w2 HFEM terms*		*Non-Symmetric Polynomial v4-w3HFEM terms*	
1	8	0.0490	9	0.0489	10	-	11	0.0574
2	16	0.0664	17	0.0503	18	0.0554		
3			25	0.0578	26	0.0679		

Number of Elements	*Non-Symmetric Trigonometric v4-w0 HFEM terms*		*Non-Symmetric Trigonometric v4-w1 HFEM terms*		*Non-Symmetric Trigonometric v4-w2 HFEM terms*		*Non-Symmetric Trigonometric v4-w3HFEM terms*	
1	8	0.0302	9	0.0302	10	0.0302	11	0.0307
2	16	0.0602	17	0.0595	18	0.0591	19	0.0594

Approximate Solution by Ritz method: Central Deflection = 0.0568 in

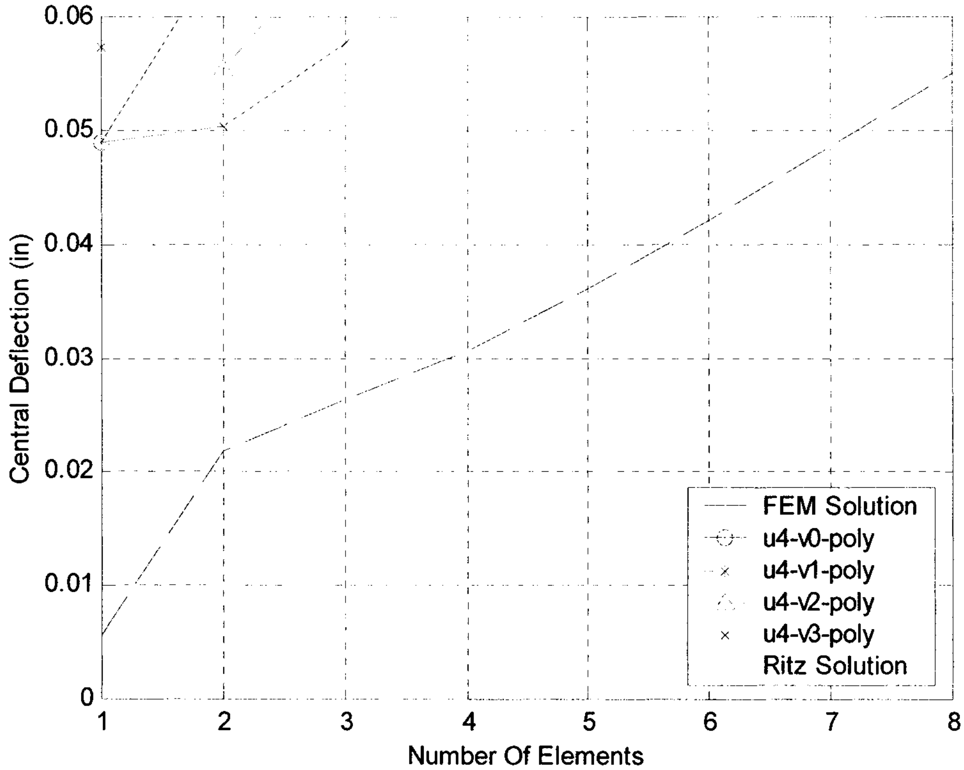


Figure 3.15 Comparison between the Results Obtained using Non-Symmetric ($v_4 - w_n$) Polynomial Hierarchical Terms for $[0/90]_{8s}$ Laminate

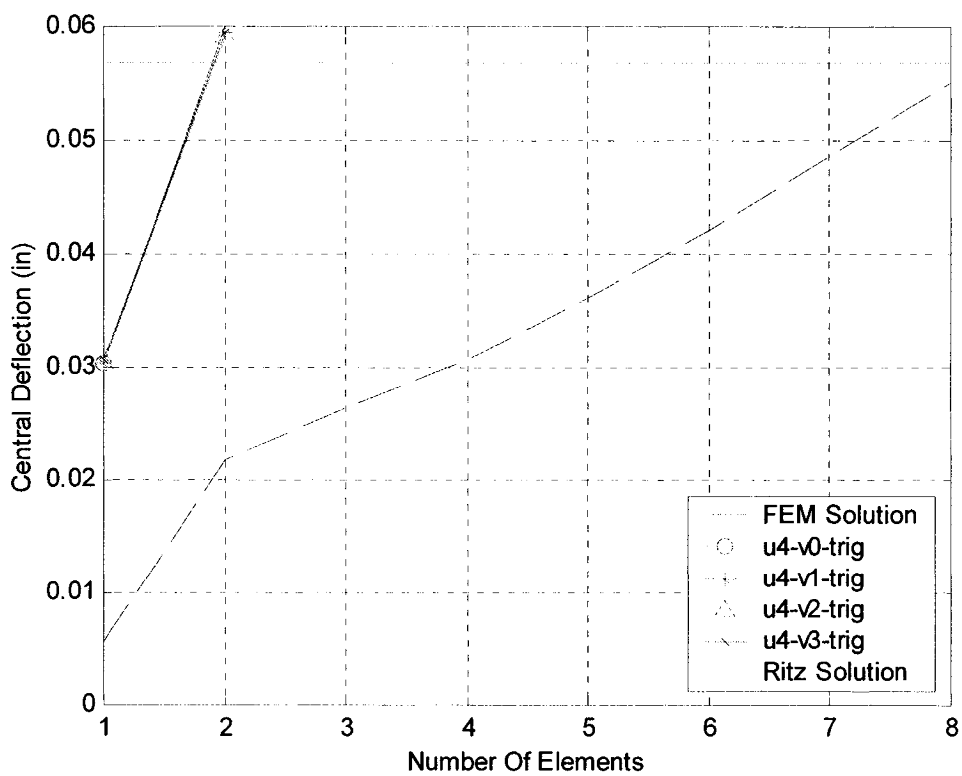


Figure 3.16 Comparison between the Results Obtained using Non-Symmetric ($v_4 - w_n$) Trigonometric Hierarchical Terms for $[0/90]_{8s}$ Laminate

Chapter 4

Buckling Analysis of Curved Beams Made of Isotropic and Composite Materials using HFEM

4.1 Introduction

Buckling generally occurs when the component is loaded in compression. A simple way to describe the buckling phenomenon is to use an example of an ideally straight bar with uniform and axisymmetrical cross section subjected to a compressive force along the centre axis of the bar. Under such a force, the bar will be slightly shortened but remain straight with no bending. If a small lateral force such as a breeze is applied, the beam will be bent infinitesimally but will return to its original straight form when the breeze disappears. If the axial force is gradually increased, a condition will be reached in which a small lateral force will cause a deflection which remains when the lateral force disappears. Such an instable phenomenon is called buckling and the critical force is called buckling load or Euler load. Buckling usually occurs when the compressive stress is well below the material's stress limit [46].

In linear mechanics of deformable bodies, displacements are proportional to loads. The essence of buckling, however, is a disproportionate increase in displacement resulting from a small increase in load. Consequently, buckling analysis is a subtopic of nonlinear rather than linear mechanics. Nonlinearity in mechanics of deformable bodies is either physical or geometrical; i.e., it enters the theory either in the stress-strain relations or in expressions representing the influence of rotations [51].

Only with extensive construction of truss railway bridges did buckling problems become of practical importance. Due to advances in high-strength-material technology, the structural members used have become increasingly thinner and lighter and thus buckling problems have become increasing concern. Buckling can happen to structures in many forms, such as columns, truss members, components of thin-walled beams and plate girders, walls, arches, and shell roofs. Buckling can also happen to torispherical shells under internal pressure. In aerospace structures, minimum-weight design is an important criterion so that the structures are made of skins and thin members. The buckling problem is a predominant one [46]. In this chapter the primary concern is focused on curved beams made of isotropic and composite materials. The same arch problem will be used for the purpose as in the 2^{nd} chapter.

4.2 Formulation of a Curved Beam Finite Element With Constant Axial Force

4.2.1 Energy Expressions

The strain energy for a curved beam with uniform cross section has been given in Chapter 2 as

$$U_b = \frac{EA}{2}\int \varepsilon^2\, ds + \frac{EI}{2}\int \kappa^2\, ds \tag{4.1}$$

where ε and κ are the axial strain and curvature of the middle surface, respectively, with

$$\varepsilon = \frac{\partial v}{\partial s} + \frac{w}{R} = v' + \frac{w}{R} \tag{4.2}$$

$$\kappa = \frac{1}{R}\frac{\partial v}{\partial s} - \frac{\partial^2 w}{\partial s^2} = \frac{1}{R}v' - w'' \tag{4.3}$$

Substituting Equations (4.2) and (4.3) into Equation (4.1) gives

$$U_b = U_{vv} + U_{vw} + U_{ww} \tag{4.4a}$$

where

$$U_{vv} = \frac{EA}{2}\int_0^L (v')^2\, ds + \frac{EI}{2R^2}\int_0^L (v')^2\, ds$$

$$U_{vw} = \frac{EA}{R}\int_0^L v'\, w\, ds - \frac{EI}{R}\int_0^L v'\, w''\, ds \tag{4.4b}$$

$$U_{ww} = \frac{EA}{2R^2}\int_0^L (w)^2\, ds + \frac{EI}{2}\int_0^L (w'')^2\, ds$$

The energy expressions U_{vv}, U_{vw}, and U_{ww} are associated with axial, axial-flexural coupling, and flexural behaviors, respectively.

4.2.2 Thin Ring Deformation Theory

The kinematic relations for a thin ring are shown in Figure 4.1. For simplicity, the ring cross section is assumed to be axisymmetric and only in-plane bending is considered.

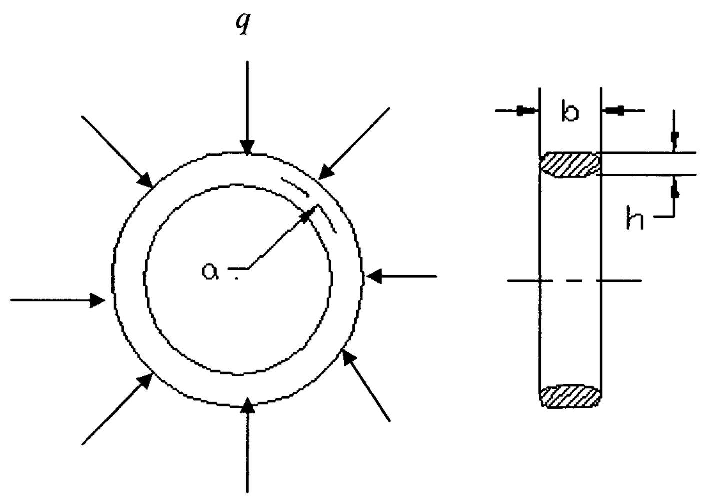


Figure 4.1 The Circular Ring Subjected to Uniform External Pressure

The constant a represents the radius of the undeformed centroidal surface and the maximum thickness h is taken to be much smaller than a. Points in the ring are referred to polar coordinates R and θ, as shown in the sketch of the undeformed ring in Figure 4.2. For convenience, an additional coordinate variable is defined by the relation $z \equiv R - a$. Thus z is measured positive outward from the centroidal surface.

Consider a circumferential line element of length ds referred to rectangular Cartesian coordinates x and y, as shown in Figure 4.3. After deformation the length of the line element is ds^*, and the element is referred to new coordinates x^* and y^*. Let v and w, denote components of the displacements of the displacement vector in θ and z directions, respectively [51].

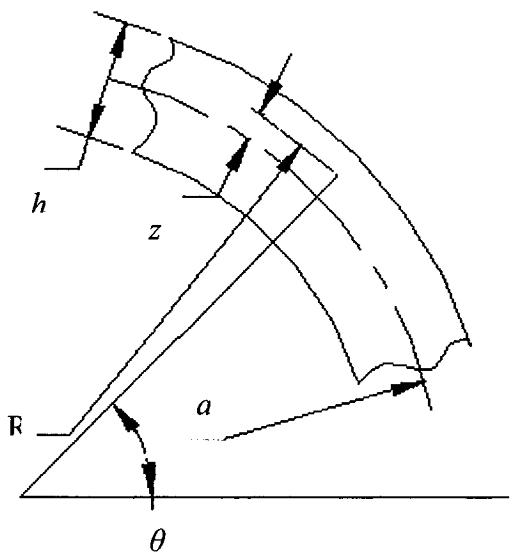


Figure 4.2 Coordinate System

Then from Figure 4.3,

$$\begin{aligned} x &= R\cos\theta \\ y &= R\sin\theta \end{aligned} \tag{4.5}$$

$$\begin{aligned} x^* &= R\cos\theta - v\sin\theta + w\cos\theta \\ y^* &= R\sin\theta + v\cos\theta + w\sin\theta \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \frac{dx^*}{d\theta} &= -R\sin\theta - v'\sin\theta - v\cos\theta + w'\cos\theta - w\sin\theta \\ \frac{dy^*}{d\theta} &= R\cos\theta + v'\cos\theta - v\sin\theta + w'\sin\theta + w\cos\theta \end{aligned} \tag{4.7}$$

where $v' \equiv dv/d\theta$

and $w' \equiv dw/d\theta$

In terms of polar coordinates,

$$(ds)^2 = (R\,d\theta)^2 \tag{4.8}$$

and

$$(ds^*)^2 = (dx^*)^2 + (dy^*)^2 \tag{4.9}$$

$$(ds^*)^2 = \left[R^2 + 2R(v'+w) + (v'+w)^2 + (v-w')^2\right](d\theta)^2 \tag{4.10}$$

$$(ds^*)^2 = \left[1 + \frac{2(v'+w)}{R} + \left(\frac{v'+w}{R}\right)^2 + \left(\frac{v-w'}{R}\right)^2\right](R\,d\theta)^2 \tag{4.11}$$

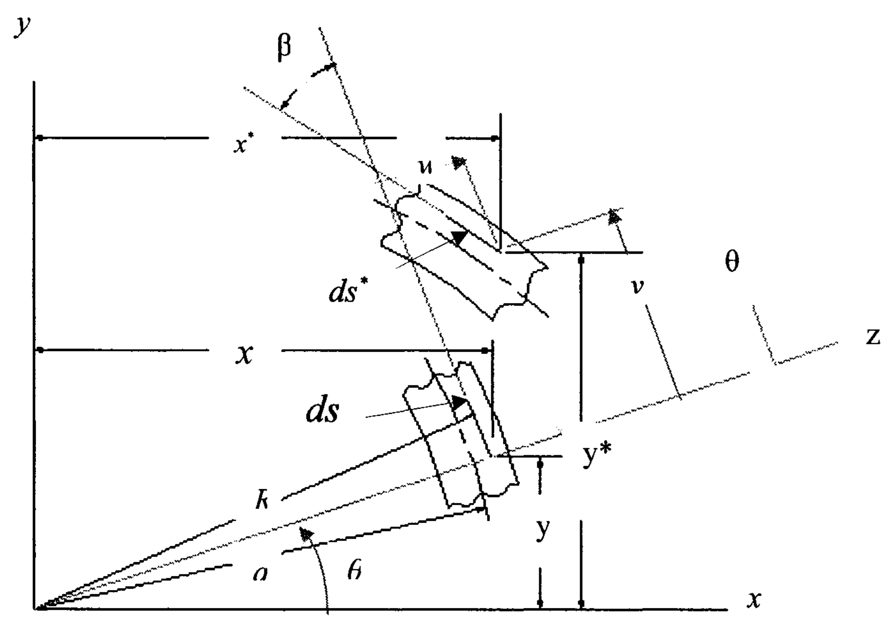


Figure 4.3 Circumferential Line Elements Before and After Deformation

The work done by an axial force P due to bending of the curved beam can be derived by considering the beam shown in Figure 4.4. Due to lateral deflection of the curved beam from its initial position, free end B is displaced by a small amount and length of the circumferential line element changes from ds to ds^*. This displacement is equal to the difference between the lengths of the circumferential line element before and after displacement.

We first consider the difference between the length of the circumferential element ds^* and the corresponding circumferential line element ds of the curved beam as shown in Figure 4.3.

$$ds^* - ds = \left(1 + \frac{2(v'+w)}{R} + \left(\frac{v'+w}{R}\right)^2 + \left(\frac{v-w'}{R}\right)^2\right)^{1/2} R\,d\theta - R\,d\theta \tag{4.12}$$

Suppose $$x = \frac{2(v'+w)}{R} + \left(\frac{v'+w}{R}\right)^2 + \left(\frac{v-w'}{R}\right)^2$$

$$\Delta = [1+x]^{1/2} R\,d\theta - R\,d\theta \tag{4.13}$$

$$\Delta = \left[1 + \frac{1}{2}x - \frac{1}{4}x^2 + \ldots\ldots\ldots\right] R\,d\theta - R\,d\theta \tag{4.14}$$

$$\Delta \approx \frac{1}{2}x \tag{4.15}$$

Its fourth order is certainly too small to be included. The displacement of the free end B is the integration of $(ds^* - ds)$ through the curved beam length L.

$$\Delta = \frac{1}{2}\int_0^L \left[\frac{2(v'+w)}{R} + \left(\frac{v'+w}{R}\right)^2 + \left(\frac{v-w'}{R}\right)^2\right] ds \tag{4.16}$$

Thus the work done by the axial force P due to the free end displacement of B is

$$W_n = \frac{P}{2}\int_0^L \left[\frac{2(v'+w)}{R} + \left(\frac{v'+w}{R}\right)^2 + \left(\frac{v-w'}{R}\right)^2\right] ds \tag{4.17}$$

where $v' = \partial v / \partial s$, $w' = \partial w / \partial s$ and the axial force P is positive when in compression.

$$W_n = \frac{P}{2}\int_0^L \left[\frac{2v'}{R} + \frac{2w}{R} + \frac{v'^2}{R^2} + \frac{w^2}{R^2} + \frac{2v'w}{R^2} + \frac{v^2}{R^2} + \frac{w'^2}{R^2} - \frac{2vw'}{R^2}\right] ds \tag{4.18}$$

$$W_n = W_{vv} + W_{vw} + W_{ww} \tag{4.19a}$$

where

$$\begin{aligned} W_{vv} &= \frac{P}{2}\int_0^L \left[\frac{2v'}{R} + \frac{v'^2}{R^2} + \frac{v^2}{R^2}\right] ds \\ W_{vw} &= \frac{P}{2}\int_0^L \left[\frac{2v'w}{R^2} - \frac{2vw'}{R^2}\right] ds \\ W_{ww} &= \frac{P}{2}\int_0^L \left[\frac{2w}{R} + \frac{w^2}{R^2} + \frac{w'^2}{R^2}\right] ds \end{aligned} \tag{4.19b}$$

4.2.3 Interpolation Functions

The same interpolation functions for tangential (v) and radial (w) displacements for the curved beam finite element used in chapter 2 will be used for the buckling analysis as well.

$$v(s) = N_1 v_1 + N_2 v_{s1} + N_3 v_2 + N_4 v_{s2} \tag{4.20}$$

$$w(s) = N_1 w_1 + N_2 \theta_1 + N_3 w_2 + N_4 \theta_2 \tag{4.21}$$

where the interpolation functions are

$$
\begin{aligned}
N_1 &= 1-3\left(\frac{s}{L}\right)^2+2\left(\frac{s}{L}\right)^3 \\
N_2 &= x-2\left(\frac{s^2}{L}\right)+\left(\frac{s^3}{L^2}\right) \\
N_3 &= 3\left(\frac{s}{L}\right)^2-2\left(\frac{s}{L}\right)^3 \\
N_4 &= -\left(\frac{s^2}{L}\right)+\left(\frac{s^3}{L^2}\right)
\end{aligned}
\tag{4.22}
$$

4.2.4 Basic Stiffness and Incremental Stiffness Matrices

Substituting the deflection functions (4.18) and (4.19) into Equations (4.4b) and (4.19b) and then performing partial differentiations of the energy expressions U and W with respect to each of the eight degrees of freedom (2.15), the equations for the 8×8 stiffness and incremental stiffness matrices of the element are obtained [46].

$$
[[k]-P[n]]\{d\} = 0 \tag{4.23}
$$

where $[k]$ is the basic stiffness matrix associated with the bending deflection; $[n]$ is called the incremental stiffness matrix associated with the effect of the axial force P on bending deflection.

The coefficients in 4×4 sub-matrices in [k] are obtained as given in Equation (2.16).

$$
\begin{aligned}
k_{vv_{ij}} &= \int_0^L EA\left(1+\frac{\alpha}{R^2}\right)N_i\, N_j'\, ds \\
k_{vw_{ij}} &= k_{wv_{ji}} = \int_0^L \frac{EA}{R}\left(N_i\, N_j' - \alpha N_i'\, N_j''\right) ds \\
k_{ww_{ij}} &= \int_0^L EA\left(\frac{N_i\, N_j}{R^2}+\alpha N_i''\, N_j''\right) ds
\end{aligned}
\tag{4.24}
$$

where the primes indicate differentiation with respect to s and $\alpha = EI/EA$.

By substituting the interpolation functions (4.22) into Equation (4.19b), the coefficients in [n] can be derived. Energy expressions W_{vv} and W_{ww} each have one linear term $(\frac{2v'}{R})$ and $(\frac{2w}{R})$ respectively. After substituting the interpolation functions in these linear terms and performing the integration through the curved beam length L, we will get a constant value. As a result of this integration Equation (4.23) will become as:

$$Constant + \;\; \left[[k] - P[n]\right]\{d\} \;=\; 0 \tag{4.25}$$

as

$$Constant \;\neq 0 \tag{4.26}$$

Equation (4.19b) will be modified after discarding those linear terms in energy expressions W_{vv} and W_{ww}.

$$\begin{aligned} W_{vv} &= \frac{P}{2}\int_0^L\left[\frac{v'^2}{R^2}+\frac{v^2}{R^2}\right]ds \\ W_{vw} &= \frac{P}{2}\int_0^L\left[\frac{2v'w}{R^2}-\frac{2vw'}{R^2}\right]ds \\ W_{ww} &= \frac{P}{2}\int_0^L\left[\frac{w^2}{R^2}+\frac{w'^2}{R^2}\right]ds \end{aligned} \tag{4.27}$$

The coefficients in 4×4 sub-matrices in $[n]$ can be derived as

$$\begin{aligned} n_{ij_{vv}} &= \frac{P}{R^2}\int_0^L\left[N_i' N_j' + N_i N_j\right]ds \\ n_{ij_{vw}} &= n_{ji_{wv}} = \frac{P}{R^2}\int_0^L\left[N_i' N_j + N_i N_j'\right]ds \\ n_{ij_{ww}} &= \frac{P}{R^2}\int_0^L\left[N_i N_j + N_i' N_j'\right]ds \end{aligned} \tag{4.28}$$

where the primes indicate differentiation with respect to s.

Because the incremental stiffness matrix [n] contains P, it is often referred to as the *initial stress matrix*. Because this matrix contains L but no EI, it sometimes referred to as the *geometric stiffness matrix*.

4.3 Formulation for a Curved Beam using Ritz Method

In this section approximate solution in conjunction with Ritz method [50] is discussed. The strain energy expression for a curved beam which is in a special reduced form of that for a thin shell [68] is given by Equation (2.14b).

$$U_b = \frac{1}{2}\int_{s=0}^{s=L}\left\{ EA\left[\left(\frac{\partial v}{\partial s}\right)^2 + \left(\frac{w}{R}\right)^2\right] + \frac{2EA}{R}\left(w\frac{\partial v}{\partial s}\right) + EI\left(\frac{\partial^2 w}{\partial s^2}\right)^2 + \frac{EI}{R^2}\left(\frac{\partial v}{\partial s}\right)^2\right.$$

$$\left. -\frac{2EI}{R}\left[\left(\frac{\partial v}{\partial s}\right)\left(\frac{\partial^2 w}{\partial s^2}\right)\right]\right\}ds \tag{4.29}$$

The potential energy owed to the axial force P at the free end is

$$W_n = \frac{P}{2}\int_0^L\left[\frac{v'^2}{R^2} + \frac{w^2}{R^2} + \frac{2v'w}{R^2} + \frac{v^2}{R^2} + \frac{w'^2}{R^2} - \frac{2vw'}{R^2}\right]ds \tag{4.30}$$

The approximate solution is expanded in a single summation series

$$w(s) = \sum_{m=1}^{M} A_m S_m(s) \tag{4.31}$$

$$v(s) = \sum_{n=1}^{N} B_n S_n(s) \tag{4.32}$$

The functions $S_m(s)$ and $S_n(s)$ have to form functional bases for polynomials, trigonometric functions and hyperbolic functions and are chosen to satisfy the boundary conditions. The coefficients A_m are next determined by the stationary conditions, which are written as:

$$\frac{\partial U}{\partial A_m} = 0, \qquad \text{or} \qquad \frac{\partial U_b}{\partial A_m} = \frac{\partial W_n}{\partial A_m} \tag{4.33}$$

$$\frac{\partial U}{\partial B_n} = 0, \qquad \text{or} \qquad \frac{\partial U_b}{\partial B_n} = \frac{\partial W_n}{\partial B_n} \tag{4.34}$$

U_b and W_n are the strain energy and the potential energy owed to the axial force, obtained by substituting the approximate expressions for the deflections into Equations (4.29) and (4.30) respectively.

The left hand sides of the Equations (4.32) and (4.33) can be put in the practical form as follows:

$$\frac{\partial U_b}{\partial A_m} = \int_{s=0}^{s=L} \left\{ \frac{EA}{R^2}\left[\sum_{i=1}^{M} A_i\, S_m\, S_i\right] + \frac{EA}{R}\left[\sum_{n=1}^{N} B_n \frac{dS_n}{ds} S_m\right] + EI\left[\sum_{i=1}^{M} A_i \frac{d^2 S_m}{ds^2}\frac{d^2 S_i}{ds^2}\right] \right.$$
$$\left. - \frac{EI}{R}\left[\sum_{n=1}^{N} B_n \frac{d^2 S_m}{ds^2}\frac{dS_n}{ds}\right] \right\} ds \tag{4.35}$$

$$\frac{\partial U_b}{\partial B_n} = \int_{s=0}^{s=L} \left\{ EA\left[\sum_{j=1}^{N} B_j \frac{dS_n}{ds}\frac{dS_j}{ds}\right] + \frac{EA}{R}\left[\sum_{M=1}^{M} A_m \frac{dS_n}{ds} S_m\right] + \frac{EI}{R^2}\left[\sum_{j=1}^{N} B_j \frac{dS_n}{ds}\frac{dS_j}{ds}\right] \right.$$
$$\left. - \frac{EI}{R}\left[\sum_{M=1}^{M} A_m \frac{d^2 S_m}{ds^2}\frac{dS_n}{ds}\right] \right\} ds \tag{4.36}$$

$$\frac{\partial W_n}{\partial A_m} = \frac{P}{2}\int_0^L\{\frac{2}{R^2}\left[\sum_{i=1}^{M} A_i\, S_m\, S_i\right] + \frac{2}{R^2}\left[\sum_{n=1}^{N} B_n\, S_m \frac{dS_n}{ds}\right] + \frac{2}{R^2}\left[\sum_{i=1}^{M} A_i \frac{dS_m}{ds}\frac{dS_i}{ds}\right]$$

$$-\frac{2}{R^2}\left[\sum_{n=1}^{N} B_n \frac{dS_m}{ds} S_n\right]\}\; ds \tag{4.37}$$

$$\frac{\partial W_n}{\partial B_n} = \frac{P}{2}\int_0^L\{\frac{2}{R^2}\left[\sum_{j=1}^{N} B_j \frac{dS_n}{ds}\frac{dS_j}{ds}\right] + \frac{2}{R^2}\left[\sum_{m=1}^{M} A_m\, S_m \frac{dS_n}{ds}\right] + \frac{2}{R^2}\left[\sum_{j=1}^{N} B_j\, S_n\, S_j\right]$$

$$-\frac{2}{R^2}\left[\sum_{m=1}^{M} A_m \frac{dS_m}{ds} S_n\right]\}\; ds \tag{4.38}$$

Equations (4.34), (4.35), (4.36) and (4.37) finally become

$$\frac{\partial U_b}{\partial A_m} = \int_{s=0}^{s=L}\{\;\left[\;\frac{EA}{R^2}\sum_{i=1}^{M}\left(S_m\; S_i\right) + \;EI\sum_{i=1}^{M}\left(\frac{d^2 S_m}{ds^2}\;\frac{d^2 S_i}{ds^2}\right)\;\right]\; A_i$$

$$-\left[\;\frac{EI}{R}\;\sum_{n=1}^{N}\;\left(\frac{d^2 S_m}{ds^2}\;\frac{dS_n}{ds}\right) + \;\frac{EA}{R}\;\sum_{n=1}^{N}\left(\;\frac{dS_n}{ds}\; S_m\;\right)\right]\; B_n\}ds \tag{4.39}$$

$$\frac{\partial U_b}{\partial B_n} = \int_{s=0}^{s=L}\{\;\left[\frac{EA}{R}\sum_{M=1}^{M}\left(\;\frac{dS_n}{ds}\;\; S_m\;\right) - \;\frac{EI}{R}\sum_{M=1}^{M}\left(\frac{d^2 S_m}{ds^2}\;\frac{dS_n}{ds}\right)\right]\; A_m$$

$$+\;EA\left[\sum_{j=1}^{N}\;\left(\;\frac{dS_n}{ds}\;\frac{dS_j}{ds}\;\right) + \;\frac{EI}{R^2}\;\sum_{\mathrm{j=1}}^{\mathrm{N}}\;\left(\;\frac{dS_n}{ds}\;\frac{dS_j}{ds}\;\right)\right]\;\mathrm{B_j}\;\;\}\mathrm{ds} \tag{4.40}$$

$$\frac{\partial W_n}{\partial A_m} = P\int_0^L\{\left[\;\frac{1}{R^2}\left(\sum_{i=1}^{M} S_m\, S_i\right) + \frac{1}{R^2}\left(\sum_{i=1}^{M}\frac{dS_m}{ds}\frac{dS_i}{ds}\right)\right]\; A_i$$

$$+\left[\;\frac{1}{R^2}\left(\sum_{n=1}^{N} S_m \frac{dS_n}{ds}\right) - \frac{1}{R^2}\left(\sum_{n=1}^{N}\frac{dS_m}{ds}\; S_n\right)\right]\;\; B_n\}\;\; ds \tag{4.41}$$

$$\frac{\partial W_n}{\partial B_n} = P \int_0^L \left\{ \left[\frac{1}{R^2} \left(\sum_{j=1}^{N} S_m \frac{dS_n}{ds} \right) - \frac{1}{R^2} \left(\sum_{m=1}^{M} \frac{dS_m}{ds} S_n \right) \right] A_m \right.$$

$$\left. + \left[\frac{1}{R^2} \left(\sum_{j=1}^{N} \frac{dS_n}{ds} \frac{dS_j}{ds} \right) + \frac{1}{R^2} \left(\sum_{j=1}^{N} S_n \, S_j \right) \right] B_i \right\} ds \qquad (4.42)$$

where

$m, i = 1, 2, 3 \ldots\ldots\ldots M,$

$n, j = 1, 2, 3 \ldots\ldots\ldots N,$

4.4 Fixed-Free Curved Beam Example: Analytical Solution

Uniform curved beam with fixed-free boundary condition is shown in Figure 4.4. The parameters are defined as $A = 1 \times 1\ in^2$, $R = 20\ in$, $I = 1/12\ in^4$, and $E = 10^7\ psi$.

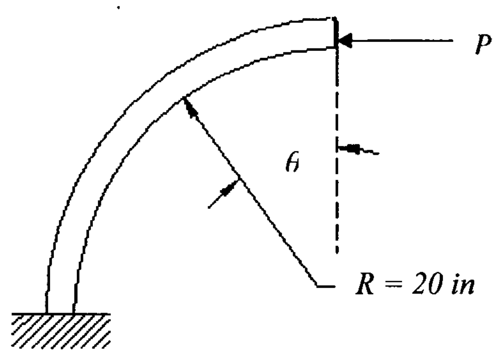


Figure 4.4 Fixed-Free Curved Beam

4.4.1 Solution by Ritz Method

We will discuss the case of a curved beam fixed at one end and free at the other end as shown in Figure 4.4 subjected to an axial force at the free end. As the curved beam is fixed at one end and free at the other end the boundary conditions are:

For tangential displacement (v)

for edge $s = 0$:

$$v_1 = 0, \qquad \left.\frac{\partial v}{\partial s}\right|_1 = v_{s1} \neq 0 \tag{4.43}$$

for edge $s = L$

$$v_2 \neq 0, \qquad \left.\frac{\partial v}{\partial s}\right|_2 = v_{s2} = 0 \tag{4.44}$$

For radial displacement (w)

for edge $s = 0$:

$$w_1 = 0, \qquad \left.\frac{\partial w}{\partial s}\right|_1 = \theta_1 = 0 \tag{4.45}$$

for edge $s = L$

$$w_2 \neq 0, \qquad \left.\frac{\partial w}{\partial s}\right|_2 = \theta_2 \neq 0 \tag{4.46}$$

4.4.1.1 Solution Approximated by Displacement Functions

For the transverse displacement (w) we choose the polynomials in the form:

$$S_m(s) = \left(\frac{s}{L}\right)^2 \left(\frac{s}{L}\right)^{m-1} \tag{4.47}$$

For the tangential displacement (v) we choose the polynomial displacement functions in the form:

$$S_n(s) = \left(\frac{s}{L}\right)^{2n} - \left(\frac{2ns}{L}\right) \tag{4.48}$$

These functions satisfy the boundary conditions in Equations (4.43), (4.44), (4.45), and (4.46). In the case where $m = n = 2$ the system of equations to calculate the critical buckling load becomes

$$\left[\begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ a_{31} & a_{32} & b_{31} & b_{32} \\ a_{41} & a_{42} & b_{41} & b_{42} \end{bmatrix} - P\begin{bmatrix} c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{22} & d_{22} \\ c_{31} & c_{32} & d_{31} & d_{32} \\ c_{41} & c_{42} & d_{41} & d_{42} \end{bmatrix}\right] = 0 \tag{4.49}$$

where a_{ij} and b_{ij} represent the coefficients of the $[k]$ matrix, whereas c_{ij} and d_{ij} represent the coefficients of the $[n]$ matrix.

Table 4.1 Solutions for Fixed-Free Curved Beam using Ritz Method

Value of m & n	*Critical Buckling Load (lb)*
1	0.638×10^7
2	0.246×10^6
3	64842
4	54920
5	44330
6	44046
7	40910
8	40539
9	38495
10	37684
11	36094
13	34788
14	34084
15	34691

The results for the Ritz solution for the given curved beam example shown in Figure 4.4 are given in Table 4.1. The results show improvement in the critical buckling load value as we increase the values of m and n in their displacement functions. The value shows a greater difference initially but as we increase the values of m and n, the difference gets smaller and smaller. The approximate critical buckling load value given by Ritz method is 34691 lb.

4.4.2 Solution using Eight Degrees of Freedom (D.O.F.) Curved Beam Element

If one element is used to model the curved beam shown in Figure 4.4, the boundary conditions are

$$v_1 = w_1 = \left(\frac{\partial w}{\partial s}\right)_1 = \left(\frac{\partial v}{\partial s}\right)_2 = 0 \tag{4.50}$$

After applying the boundary conditions and from Equations (4.23), (4.24) and (4.28), the basic stiffness and incremental stiffness equations can be obtained as

$$\left[\begin{bmatrix} k_{22} & k_{25} & k_{27} & k_{28} \\ k_{52} & k_{55} & k_{57} & k_{58} \\ k_{72} & k_{75} & k_{77} & k_{78} \\ k_{82} & k_{85} & k_{87} & k_{88} \end{bmatrix} - P \begin{bmatrix} n_{22} & n_{25} & n_{27} & n_{28} \\ n_{52} & n_{55} & n_{57} & n_{58} \\ n_{72} & n_{75} & n_{77} & n_{78} \\ n_{82} & n_{85} & n_{87} & n_{88} \end{bmatrix}\right] \begin{Bmatrix} v_{s1} \\ v_2 \\ w_2 \\ \theta_2 \end{Bmatrix} = 0 \tag{4.51}$$

where k_{ij} and n_{ij} represent the coefficients of 8×8 stiffness and incremental stiffness matrices respectively.

The results for calculating the critical buckling load for 8 D.O.F. curved beam element have been given in Table 4.2 which show that at the 35^{th} element, the critical buckling load value becomes almost equal to the approximate value given by the Ritz method solution with a 0.32 % error. For the first value there is an error of 98.94 % which suddenly reduces to 31.73 % when two elements were used. As we increase the number of elements the difference becomes smaller and smaller and finally ends up with a difference of 0.32 %.

Table 4.2 8 D.O.F. Curved Beam Finite Element Solution for Fixed-Free Boundary Condition

Number of Elements	*Number of D.O.F.*	*Critical Buckling Load (lb)*	*Error (%)*
1	4	0.32643×10^7	98.94
2	8	50818	31.73
3	12	44551	22.13
4	16	43722	20.66
5	20	43355	19.98
6	24	43044	19.41
7	28	42729	18.81
8	32	42420	18.22
9	36	42109	17.62
10	40	41780	16.97
12	48	41144	15.68
14	56	40524	14.39
16	64	40007	13.29
18	72	39127	11.34
20	80	38752	10.48
22	88	38191	9.16
24	96	37849	8.34
25	100	37358	7.14
30	120	36043	3.75
31	124	35934	3.46
33	132	35294	1.71
34	136	35057	1.04
35	140	34801	0.32

4.5 The Hierarchical Finite Element Formulation for Isotropic Curved Beam

The hierarchical finite element formulation for the isotropic curved beam for buckling analysis will be used in the same way as described in chapter 2. In the hierarchical formulation, we modify the approximating functions (i) by adding trigonometric functions (2.33) and (ii) by adding polynomial functions (2.58).

Hence, the displacement field for the element, in terms of the nodal degrees of freedom and the hierarchical degrees of freedom, can now be written as,

for tangential displacement (v)

$$v = N_1 v_1 + N_2 v_{s1} + N_3 v_2 + N_4 v_{s2} + \sum_{r=1}^{N} N_{r+4} v_{vr+4} \tag{4.52}$$

and similarly for radial displacement (w)

$$w = N_1 w_1 + N_2 \theta_1 + N_3 w_2 + N_4 \theta_2 + \sum_{r=1}^{N} N_{r+4} w_{wr+4} \tag{4.53}$$

We will proceed by using the same combinations of the hierarchical terms. Firstly, symmetric polynomial and trigonometric hierarchical terms will be used with both tangential (v) and radial (w) displacement functions. Secondly, non-symmetric hierarchical terms will be used for both displacement functions by trying each and every possible combination of these hierarchical terms.

4.6 Discussion and Conclusion

The results for symmetric trigonometric and polynomial hierarchical formulations for calculating critical buckling load for curved beams with fixed-free boundary conditions are given in Table 4.3. When we increase the symmetric hierarchical terms from one hierarchical term to four hierarchical terms for trigonometric hierarchical terms, the results show better convergence as we increase the number of elements. The convergence for each of these terms is very small and similar to the results given by the conventional FEM. For symmetric polynomial hierarchical terms when we increase the number of elements, at one point the results given by two, three and four hierarchical terms converge to one value and they all give the same results. The results given by polynomial hierarchical terms are not much different from that of the results given by trigonometric terms except that for some of the initial elements the convergence is better with polynomial terms but when we increase the number of elements, the results with the trigonometric terms get better. Four symmetric trigonometric hierarchical terms give good results among all the terms i.e. 33949 *lb*. For symmetric polynomial hierarchical terms, results given by all the terms are same i.e. 34182 *lb*.

The results for the (v_0 $- w_n$) combinations with trigonometric hierarchical terms are given in Table 4.4. When we increase the number of trigonometric hierarchical terms with the radial displacement (w), the results get better for the first few elements with each addition of the hierarchical terms from 1 term to 4 terms. At one point, for one specific element, all the hierarchical terms for the combinations (v_0 $- w_n$) give the same result.

Table 4.3 Critical Buckling Load Calculated by using Symmetric Trigonometric and Polynomial Hierarchical Terms for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Symmetric Trigonometric 1 HFEM term		*Symmetric Trigonometric 2 HFEM terms*		*Symmetric Trigonometric 3 HFEM terms*		*Symmetric Trigonometric 4 HFEM terms*	
1	6	92272	8	44510	10	43639	12	43603
2	11	43913	14	43606	17	42947	20	42436
3	16	43333	20	42589	24	41871	28	41344
4	21	42789	26	41881	31	41231	36	40761
5	26	42316	32	41340	38	40775	44	40390
15	76	38991	92	38354	108	38279	124	38235
20	101	38724	122	37227	143	37195	164	37173
25	126	36477	152	36116	78	36100	104	36087
30	151	35307	182	35023	113	35013	144	35004
35	176	34188	112	33958	148	33952	184	33949
	Symmetric Polynomial 1 HFEM term		*Symmetric Polynomial 2 HFEM terms*		*Symmetric Polynomial 3 HFEM terms*		*Symmetric Polynomial 4 HFEM terms*	
1	6	90160	8	81992	10		12	
2	11	43927	14	43159	17	42316	20	42325
3	16	43299	20	42596	24	42016	28	42247
4	21	42739	26	42160	31	41693	36	41997
5	26	42260	32	41774	38	41383	44	41687
15	76	38963	92	38915	108	38660	124	38660
20	101	37681	122	37665	143	37665	164	37665
25	126	36466	152	36460	78	36460	104	36460
30	151	35300	182	35297	113	35297	144	35297
35	176	34183	112	34182	148	34182	184	34182

Approximate Solution by Ritz Method: Critical Buckling Load = 34691 *lb*

Table 4.4 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric Hierarchical Terms ($v_0 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 0v-1w HFEM terms		*Non-Symmetric Trigonometric 0v-2w HFEM terms*		*Non-Symmetric Trigonometric 0v-3w HFEM terms*		*Non-Symmetric Trigonometric 0v-4w HFEM terms*	
1	5	0.15567×10^6	6	0.15131×10^6	7	0.12137×10^6	8	0.12089×10^6
2	10	49315	12	48410	14	48409	16	48391
3	15	44291	18	44250	21	44246	24	44245
4	20	43636	24	43627	28	43625	32	43625
5	25	4320	30	4316	35	43315	40	43315
6	30	43026	36	43023	42	43023	48	43023
7	35	42722	42	42720	49	42720	56	42720
8	40	42410	48	42409	56	42409	64	42409
9	45	42095	54	42094	63	42094	72	42094
10	50	41778	60	41778	70	41778	80	41778
11	55	41463	66	41462	77	41462	88	41462
12	60	41148	72	41148	84	41148	96	41148
13	65	40837	78	40836	91	40836	104	40836
14	70	40527	84	40527	98	40527	112	40527
15	75	40221	90	40221	105	40221	120	40221
20	100	38739	120	38739	140	38739	160	38739
25	125	37341	150	37341	175	37341	200	37341
30	150	36025	180	36025	220	36025	240	36025
35	175	34789	210	34789	255	34789	280	34789

Approximate Solution by Ritz Method: Critical Buckling Load = 34691 *lb*

The results for the group of combinations ($v_1 - w_n$) are given in Table 4.5. With the increase of number of hierarchical terms associated with the radial displacement (w) function we get better results. There is a large difference in the critical buckling load values among the combinations ($v_1 - w_0$), ($v_1 - w_2$), ($v_1 - w_3$) and ($v_1 - w_4$) of hierarchical terms when one element mesh was used. The results get better with the increase of the number of elements. After the 5- elements mesh, the convergence for the combination ($v_1 - w_0$) is better than all the other combinations. The critical buckling load value is 34677 *lb* for this combination and this value is obtained at the 20-elements mesh compared to 35-elements mesh for the conventional FEM.

Table 4.6 gives results of the trigonometric hierarchical terms for the combinations ($v_2 - w_n$). When we increase the number of elements, results for the 1st element get better for the group of combinations ($v_2 - w_n$). The results given by the combinations ($v_2 - w_3$) and ($v_2 - w_4$) are closer to each other. The combinations ($v_2 - w_0$) and ($v_2 - w_1$) give better convergence compared to all the other combinations of this group. The critical buckling load values for these combinations are obtained at the 15-elements mesh and 25-elements mesh respectively compared to 35-elemnts mesh for the conventional FEM.

Table 4.5 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric Hierarchical Terms ($v_1 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 1v-0w HFEM terms		*Non-Symmetric Trigonometric 1v-2w HFEM terms*		*Non-Symmetric Trigonometric 1v-3w HFEM terms*		*Non-Symmetric Trigonometric 1v-4w HFEM terms*	
1	5	0.16141×10^7	7	89121	8	75856	9	75836
2	10	44922	12	43891	14	43803	16	43796
3	15	43552	17	43329	20	43313	23	43311
4	20	42822	22	42788	26	42765	30	42764
5	25	42199	27	42315	32	42256	37	42255
6	30	41610	32	41884	38	41778	44	41778
7	35	41039	37	41485	44	41329	51	41329
8	40	40482	42	41114	50	40909	58	40909
9	45	39938	47	40767	56	40514	65	40514
10	50	39406	52	40439	62	40143	72	40143
11	55	38885	57	40127	68	39794	79	39794
12	60	38375	62	39829	74	39463	96	39463
13	65	37876	67	39542	80	39148	103	39148
14	70	3788	72	39263	86	38847	110	38847
15	75	36911	77	38991	92	38557	117	38557
20	100	34677	102	37698	122	37232	152	37232
25			127	36477	152	36033	187	36033
30			152	35307	182	34905	222	34905

Approximate Solution by Ritz Method: Critical Buckling Load = 34691 *lb*

Table 4.6 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric Hierarchical Terms (v_2 $- w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 2v-0w HFEM terms		*Non-Symmetric Trigonometric 2v-1w HFEM terms*		*Non-Symmetric Trigonometric 2v-3w HFEM terms*		*Non-Symmetric Trigonometric 2v-4w HFEM terms*	
1	6	0.1056×10^7	7	90316	9	44503	10	44138
2	12	44330	13	43635	16	43496	18	43488
3	18	42686	19	42510	23	42579	26	42555
4	24	41701	25	41688	30	41861	34	41814
5	30	40836	31	41008	37	41270	42	41195
6	36	40016	37	40399	44	40760	50	40657
7	42	39228	43	39841	51	40311	58	40184
8	48	38466	49	39327	58	39914	66	39767
9	54	37729	55	38850	65	39556	74	39395
10	60	37016	61	38403	72	39231	82	39059
15	90	33783	91	36461	107	37883	120	37708
20			121	34780	142	36738	158	36589
25			151	32236	177	35658	196	35536
30					212	34612	134	34512

Approximate Solution by Ritz Method: Critical Buckling Load = 34691 *lb*

The results of the trigonometric hierarchical terms for the group of combinations ($v_3 - w_n$) are given in Table 4.7. These results follow the same pattern of the combinations ($v_2 - w_n$). The combination ($v_3 - w_0$) provides better convergence for the values of critical buckling load compared to all other combinations of this group. The number of elements mesh used to get the critical buckling load value has been reduced to 10 compared to 15 for the combination ($v_2 - w_0$).

The results of the trigonometric hierarchical terms for the combinations ($v_4 - w_n$) are given in Table 4.8. Once again the combination ($v_4 - w_0$) provides the best convergence among all the combinations of this group. The convergence for this combination significantly improves and 9-elements mesh is used to reach the approximate Ritz solution. As the number of trigonometric hierarchical terms is increased associated with the radial displacement (w) function the convergence becomes less fast.

Critical buckling load calculated by using polynomial hierarchical formulation for all the combinations as used for the trigonometric hierarchical functions are given in Tables 4.9, 4.10, 4.11, 4.12 and 4.13. The results for the combinations ($v_0 - w_n$) are given in Table 4.9. These results show a little difference between the critical buckling load values calculated by all the combinations of this group. All of these combinations give a single value of the critical buckling load which obtained by using 35-elements mesh.

Table 4.7 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric Hierarchical Terms ($v_3 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 3v-0w HFEM terms		*Non-Symmetric Trigonometric 3v-1w HFEM terms*		*Non-Symmetric Trigonometric 3v-2w HFEM terms*		*Non-Symmetric Trigonometric 3v-4w HFEM terms*	
1	7	0.77371×10^6	8	87203	9	43746	11	43954
2	14	4375	15	42816	16	42804	19	42942
3	21	41827	22	41747	23	41856	27	41842
4	28	40588	29	41168	30	41395	35	41174
5	35	39489	36	40684	37	41045	43	40690
6	42	38456	43	40220	44	40730	51	40302
7	49	37473	50	39762	57	40431	59	39976
8	56	36534	57	39305	58	40140	67	39691
9	63	35636	64	38849	65	39852	75	39434
10	70	34778	71	38393	72	39564	83	39196
15			106	36166	107	38109	123	38112
20			141	34080	142	36649	163	37052
25					177	35227	203	35982
30					212	33867	243	34916

Approximate Solution by Ritz Method: Critical Buckling Load = 34691 *lb*

Table 4.8 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric Hierarchical Terms ($v_4 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 4v-0w HFEM terms		Non-Symmetric Trigonometric 4v-1w HFEM terms		Non-Symmetric Trigonometric 4v-2w HFEM terms		Non-Symmetric Trigonometric 4v-3w HFEM terms	
1	8	0.60985×10^6	9	85663	10	43701	11	43593
2	16	43110	17	42219	18	42304	19	42457
3	24	40957	25	40947	26	41448	27	41453
4	32	39488	33	40220	34	41102	35	40924
5	40	38179	41	39623	42	40862	43	40582
6	48	6960	49	39060	50	40630	51	40307
7	56	35811	57	38509	58	40386	59	40061
8	64	34726	65	37964	66	40125	67	39825
9	72	33702	73	37423	74	39850	75	39594
10			81	36887	82	39562	83	39363
15			121	34304	122	38019	123	38190
20					162	36437	163	36992
25					202	34907	203	35800
30							243	34633

Approximate Solution by Ritz Method: Critical Buckling Load = 34691 *lb*

The results for the group of combinations ($v_1 - w_n$) of polynomial hierarchical formulation are given in Table 4.10. These results show that when we increase the number of hierarchical terms, while keeping one polynomial hierarchical term with the tangential displacement (v), critical buckling load convergence becomes better with each addition of the element. The combination ($v_1 - w_0$) provides the best convergence in this group of combinations. The critical buckling load value is obtained at the 20-elements mesh.

The results given by the combinations ($v_2 - w_n$) of polynomial hierarchical formulation are given in Table 4.11. The convergence of calculating critical buckling load for the combinations ($v_2 - w_1$), ($v_2 - w_3$) and ($v_2 - w_4$) is almost similar for all of these mentioned combinations. The combination ($v_2 - w_0$) gives the best results among all these combinations and the value of critical buckling load given is reached at the 14-elements mesh compared to 20-elements mesh in the previous case.

The combinations ($v_3 - w_n$) give results which are given in Table 4.12. When we increase the number of polynomial hierarchical terms with the radial displacement (w) function for the first few elements, we see an improvement in the results. The combinations ($v_3 - w_2$) and ($v_3 - w_4$) give critical buckling load values that are very close to each other. The combination ($v_3 - w_0$) provides better convergence than the previous combinations. The approximate Ritz solution is obtained at just 10-elements mesh.

Table 4.9 Critical Buckling Load Calculated by using Non- Symmetric Polynomial Hierarchical Terms ($v_0 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Polynomial 0v-1w HFEM terms		*Non-Symmetric Polynomial 0v-2w HFEM terms*		*Non-Symmetric Polynomial 0v-3w HFEM terms*		*Non-Symmetric Polynomial 0v-4w HFEM terms*	
1	5	0.14807×10^6	6	0.12145×10^6	7		8	
2	10	49316	12	49313	14	49312	16	49312
3	15	44292	18	44287	21	44287	24	44287
4	20	43637	24	43634	28	43634	32	43634
5	25	43321	30	43320	35	43320	40	43320
6	30	43026	36	43025	42	43025	48	43025
7	35	42722	42	43721	49	43721	56	43721
8	40	42410	48	42410	56	42410	64	42410
9	45	42095	54	42095	63	42095	72	42095
10	50	41779	60	41779	70	41779	80	41779
11	55	41463	66	41463	77	41463	88	41463
12	60	41148	72	41148	84	41148	96	41148
13	65	40837	78	40837	91	40837	104	40837
14	70	40527	84	40527	98	40527	112	40527
15	75	40221	90	40221	105	40221	120	40221
20	100	38739	120	38739	140	38739	160	38739
25	125	37341	150	37341	175	37341	200	37341
30	150	36025	180	36025	220	36025	240	36025
35	175	34789	210	34789	255	34789	280	34789

Approximate Solution by Ritz Method: Critical Buckling Load = 34691 *lb*

Table 4.10 Critical Buckling Load Calculated by using Non- Symmetric Polynomial Hierarchical Terms ($v_1 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Polynomial 1v-0w HFEM terms		*Non-Symmetric Polynomial 1v-2w HFEM terms*		*Non-Symmetric Polynomial 1v-3w HFEM terms*		*Non-Symmetric Polynomial 1v-4w HFEM terms*	
1	5	0.1542×10^7	7	56730	8		9	
2	10	44945	12	43759	14	43695	16	43671
3	15	43521	17	43269	20	43260	23	43256
4	20	42762	22	42699	26	42688	30	42683
5	25	42119	27	42154	32	42121	37	42107
6	30	41513	32	41636	38	41572	44	41543
7	35	40927	37	41146	44	41044	51	40997
8	40	40356	42	40683	50	40539	58	40470
9	45	39799	47	40248	56	40056	65	39963
10	50	39254	52	39838	62	39596	72	397476
11	55	38721	57	39452	68	39159	79	39008
12	60	38200	62	39087	74	38742	96	38559
13	65	37691	67	38742	80	38346	103	38130
14	70	37193	72	38414	86	37969	110	37718
15	75	36706	77	38102	92	37609	117	37324
20	100	34432	102	36714	122	36035	152	35588
25			127	35512	152	34735	187	34165
30			152	34414				

Approximate Solution by Ritz Method: Critical Buckling Load = 34691 *lb*

Table 4.11 Critical Buckling Load Calculated by using Non- Symmetric Polynomial Hierarchical Terms ($v_2 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Polynomial 2v-0w HFEM terms		*Non-Symmetric Polynomial 2v-1w HFEM terms*		*Non-Symmetric Polynomial 2v-3w HFEM terms*		*Non-Symmetric Polynomial 2v-4w HFEM terms*	
1	6	0.12595×10^{7}	7		9		10	
2	12	44324	13	43022	16	43025	18	43009
3	18	42680	19	42707	23	42543	26	42533
4	24	41681	25	42460	30	42143	34	42082
5	30	40804	31	42145	37	41763	42	41656
6	36	39976	37	41795	44	41409	50	41264
7	42	39181	43	41425	51	41070	58	40893
8	48	38413	49	41044	58	40739	66	40537
9	54	37671	55	40657	65	40414	74	40189
10	60	36953	61	40267	72	40092	82	39846
11	66	36259	67	39877	79	39771	90	39507
12	72	35587	73	39489	86	39453	98	39170
13	78	34938	79	39103	93	39137	106	38835
14	84	34310	85	38721	100	38824	114	38562
15			91	38342	107	38513	120	38172
20			121	36519	142	37009	158	36561
25			151	34826	177	35610	196	35038
30					212	34321	134	33618

Approximate Solution by Ritz Method: Critical Buckling Load = 34691 *lb*

Table 4.12 Critical Buckling Load Calculated by using Non- Symmetric Polynomial Hierarchical Terms ($v_3 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Polynomial 3v-0w HFEM terms		*Non-Symmetric Polynomial 3v-1w HFEM terms*		*Non-Symmetric Polynomial 3v-2w HFEM terms*		*Non-Symmetric Polynomial 3v-4w HFEM terms*	
1	7	0.66639×10^7	8	0.32535×10^7	9		11	
2	14	43622	15	42726	16	43105	19	42550
3	21	41834	22	42691	23	42585	27	42285
4	28	40634	29	42386	30	42138	35	42004
5	35	39551	36	41926	37	41745	43	41887
6	42	38529	43	41418	44	41388	51	41375
7	49	37554	50	40894	57	41049	59	41066
8	56	36623	57	40367	58	40719	67	40761
9	63	35732	64	39844	65	40395	75	40458
10	70	34879	71	39327	72	40074	83	40156
15			106	36875	107	38479	123	38663
20			141	34660	142	36920	163	37203
25					177	35428	203	35800
30					212	34017	243	34471

Approximate Solution by Ritz Method: Critical Buckling Load = 34691 *lb*

The results for the combinations ($v_4 - w_n$) of polynomial hierarchical formulation are given in Table 4.13. The results given by the combination ($v_4 - w_0$) provides the best convergence among all the combinations tried before and the critical buckling load value for this combination is obtained by using just 9-elements mesh compared to 35-elements mesh for the conventional FEM. This shows a significant improvement in the convergence of the critical buckling load values. Generally, for the initial first few elements, when we increase the number of polynomial hierarchical terms with the radial displacement (w) function, convergence for the critical buckling load value gets better. As we proceed the combination ($v_4 - w_0$) gives the best convergence among all the other combinations.

The critical buckling load was first calculated by the conventional finite element method and the results were compared with the Ritz method. We got the critical buckling load value at the 35-elements mesh with a 0.32 % error. Then hierarchical finite element analysis was used to improve the analysis.

For symmetric polynomial and trigonometric hierarchical functions i.e. when same numbers of terms were used with both tangential (v) and radial (w) displacement functions the results were better than the conventional FEM for the initial few elements but when the number of elements is increased there was a little difference between the results of both the methods. The non-symmetric polynomial and trigonometric hierarchical terms show a different trend than the symmetric terms. When no hierarchical

Table 4.13 Critical Buckling Load Calculated by using Non- Symmetric Polynomial Hierarchical Terms ($v_4 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Polynomial 4v-0w HFEM terms		*Non-Symmetric Polynomial 4v-1w HFEM terms*		*Non-Symmetric Polynomial 4v-2w HFEM terms*		*Non-Symmetric Polynomial 4v-3w HFEM terms*	
1	8	0.66639×10^{7}	9		10	51070	11	
2	16	42808	17	42669	18	42636	19	42408
3	24	40947	25	42560	26	42410	27	42266
4	32	39600	33	42057	34	42102	35	42002
5	40	38350	41	41425	42	41740	43	41687
6	48	37164	49	40770	50	41361	51	41376
7	56	36039	57	40117	58	40970	59	41067
8	64	34974	65	39473	66	40572	67	40762
9	72	33965	73	38842	74	40169	75	40458
10			81	38226	82	39765	83	40157
15			121	35368	122	37766	123	38661
20			161	32867	162	35873	163	37194
25					202	34118	203	35780
30								34434

Approximate Solution by Ritz Method: Critical Buckling Load = 34691 *lb*

term was used with the tangential displacement (v) function, there was no change in the convergence of the results compared to the results given by the conventional FEM. Moreover, the values of critical buckling load for both polynomial and trigonometric hierarchical formulations at one particular element converged to a single value. So the results given by all the four different combinations ($v_0 - w_1$), ($v_0 - w_2$), ($v_0 - w_3$) and ($v_0 - w_4$) for polynomial and trigonometric hierarchical formulations were same.

The results get better each time we increase the number of hierarchical terms with the tangential displacement (v) function. Normally for one type of group of combinations e.g. ($v_0 - w_n$), ($v_1 - w_n$), ($v_2 - w_n$), ($v_3 - w_n$) and ($v_4 - w_n$), the combinations ($v_n - w_0$) with no hierarchical term with radial displacement (w) function show better convergence compared to all the other combinations. The combination ($v_4 - w_0$) with most number of hierarchical terms with tangential displacement (v) function and no hierarchical term with radial displacement (w) function gives the best convergence and the approximate Ritz solution is reached by using just 9-elements mesh compared to 35-elements mesh for the conventional FEM.

4.7 Cubic-Cubic Circularly Curved Composite Beam Finite Element

The same curved beam element as used previously in Section 4.4.2 will be used for composite curved beam as well. It will first be applied to the laminates with isotopic layers and then for different configurations of the laminate, both conventional and hierarchical finite element methods will be used.

4.7.1 Energy Expressions

The strain energy expression for a composite curved beam for a 1-D problem was derived (3.14) in Chapter 3.

$$U_b = \frac{1}{2}\int_{s=0}^{s=L}\left\{A_{11}\left[\left(\frac{\partial v}{\partial s}\right)^2 + \left(\frac{w}{R}\right)^2\right] + \frac{2A_{11}}{R}\left(w\frac{\partial v}{\partial s}\right) + D_{11}\left(\frac{\partial^2 w}{\partial s^2}\right)^2 + \frac{D_{11}}{R^2}\left(\frac{\partial v}{\partial s}\right)^2\right.$$

$$\left. - \frac{2D_{11}}{R}\left[\left(\frac{\partial v}{\partial s}\right)\left(\frac{\partial^2 w}{\partial s^2}\right)\right]\right\}ds \tag{4.54}$$

Total strain energy for the curved beam as discussed before is

$$U_b = U_{vv} + U_{vw} + U_{ww} \tag{4.55}$$

where

$$\begin{aligned} U_{vv} &= \frac{A_{11}}{2}\int_0^L (v')^2\,ds + \frac{D_{11}}{2R^2}\int_0^L (v')^2\,ds \\ U_{vw} &= \frac{A_{11}}{R}\int_0^L (v')(w)ds - \frac{D_{11}}{R}\int_0^L (v')(w'')ds \\ U_{ww} &= \frac{A_{11}}{2R^2}\int_0^L (w)^2\,ds + \frac{D_{11}}{2}\int_0^L (w'')^2\,ds \end{aligned} \tag{4.56}$$

The stiffness equation as derived in Chapter 3 would be

$$\begin{aligned} k_{vv_{ij}} &= \int_0^L \left(\frac{A_{11}}{2} + \frac{D_{11}}{2R^2}\right) N_i' N_j'\,ds \\ k_{vw_{ij}} = k_{wv_{ji}} &= \int_0^L \left\{\left[\left(\frac{A_{11}}{R}\right) N_i' N_j\right] - \left[\left(\frac{D_{11}}{R}\right) N_i' N_j''\right]\right\}ds \\ k_{ww_{ij}} &= \int_0^L \left\{\left[\left(\frac{A_{11}}{R^2}\right) N_i N_j\right] - \left[D_{11} N_i'' N_j''\right]\right\}ds \end{aligned} \tag{4.57}$$

The work done by the axial force P at the free end of the curved beam shown in Figure 4.4 and the equations for the incremental stiffness matrix will be same as given by the Equations (4.19a), (4.19b) and (4.28).

4.7.2 Laminate with Isotropic Layers

The stiffness constants of a laminate made of n isotropic layers with different properties are then given by the relations described in Chapter 3 by Equation (3.20).

4.7.2.1 Curved Beam Example: Analytical Solution

In order to evaluate the validity of the strain energy Equation (4.54) the curved beam example as shown in Figure 4.4 was solved again by using the finite element method and for a laminate of isotropic layers by using the coefficients A_{11} and D_{11} of the A and D matrices in Equation (4.54), and by using the relations for laminates with isotropic layers given in Equation (3.20). The boundary conditions and stiffness matrix equations for this arch problem are given by Equations (4.43), (4.44), (4.45), (4.46) and (4.57). The parameters are defined as:

$E = 10^7\ psi.,\qquad I = 1/12\ in^4.,\qquad A = 1\times 1\ in^2.,\qquad R = 20\ in.,$

The results for the critical buckling load of the laminate with isotropic layers calculated by the conventional FEM are given in Table 4.14. These results show an error of 10.88 % which validates the strain energy (4.54) and the work done expressions (4.18).

The difference of 10.88 % is due to fact that for unit width bending and stretching coefficients D_{11} and A_{11} have $1-\upsilon^2$ terms in their denominators (3.16) which terms EA and EI do not possess (2.12b).

Table 4.14 Critical Buckling Load Calculated by FEM Solution for the Laminate with Isotropic Layers

Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Conventional FEM Solution using Beam Element		*Conventional FEM Solution for Laminate with Isotropic layers*	
1	4	0.32643×10^7	4	0.36632×10^7
2	8	50818	8	57031
3	12	44551	12	49993
4	16	43722	16	49061
5	20	43355	20	48657
6	24	43044	24	48300
7	28	42729	28	47592
8	32	42420	32	47598
9	36	42109	36	47240
10	40	41780	40	46889
12	48	41144	48	46178
14	56	40524	56	45574
16	64	40007	64	44817
18	72	9127	72	43929
20	80	38752	80	43348
22	88	38191	88	42752
24	96	37849	96	42231
25	100	37358	100	42140
30	120	36043	104	40123
35	140	34801	124	44749
36	144	35672	140	37910

4.8 Approximate Solution for Composite Curved Beam by Ritz Method

In this section approximate solution in conjunction with Ritz method [50] is discussed. In the case of composite curved beam the strain energy equation (4.54) will be used.

4.8.1 Composite Curved Beam Example

Uniform composite curved beam with fixed-free boundary condition is shown in Figure 4.4. It is made of NCT-301 Graphite Epoxy material. The deterministic material properties of the NCT-301 material are given as:

$$E_1 = 129.43\ GPa,\ \ E_2 = 7.99\ GPa,\ \ \upsilon_{21} = 0.021,\ \ G_{12} = 4.28\ GPa,\ R = 15\ in$$

The geometric properties of the beam are: length (L) $= 15 \times \pi / 2$; individual ply thickness (e_k) = 0.125 mm. There are 32 plies in the laminate and the configuration of the laminate is $[0/90]_{8s}$. The laminate thickness of 4 mm is obtained by multiplying the total number of plies with the individual ply thickness.

4.8.1.1 Fixed-Free Composite Curved Beam

We will discuss the case of composite curved beam fixed at one end and free at the other end as shown in Figure 4.4 subjected to an axial force P at the free end. As the curved beam is fixed at one end and free at the other end the boundary conditions will be the same as used in Equations (4.43), (4.44), (4.45) and (4.46). Displacement functions for tangential (v) and radial (w) displacements will also be the same as given in Equations (4.47) and (4.48).

The comparison of the results for the critical buckling load for $[0/90]_{8s}$ composite laminate by using conventional FEM and Ritz method are given in Table 4.15. Both forms of the solution show good convergence of the results. The final value of the critical buckling load, calculated by both of these methods does not differ by more than 2% of error which is quite a good accuracy for the curved beam problem.

Table 4.15 Ritz Method and Conventional Solutions for $[0/90]_{8s}$ Laminate for Fixed-Free Boundary Condition

$[0/90]_{8s}$ Laminate					
Ritz Method		**Conventional FEM Solution using 8 D.O.F. Composite Curved Beam Element**			
Value of m	Critical Buckling Load (lb)	Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Error (%)
1	0.984×10^6	1	4	0.509×10^6	99.93
2	23524	2	8	711.68	49.82
3	3000.7	3	12	427.34	15.73
4	1366.2	4	16	375.76	4.16
5	609.59	5	20	361.86	0.48
6	449.96	6	24	357.22	-0.81
7	395.77	7	28	355.79	-1.22
8	380.59	8	32	355.22	-1.38
9	367.69	9	36	354.60	-1.56
10	365.63	10	40	354.63	-1.55
12	360.13	11	44	354.31	-1.64
14	360.65	12	48	354.14	-1.69

4.9 The Hierarchical Finite Element Formulation for Composite Curved Beam

The hierarchical finite element formulation for calculating critical buckling load for the curved composite beam proceeds in the same way as described in the previous sections for the isotropic curved beam. The difference is, that the strain energy equation is now applied for a composite curved beam (4.54) and that the *EA* and *EI* terms for unit width are replaced by the A_{11} and D_{11} respectively. Composite curved beam example of Figure 4.4 will be solved and modeled by using four hierarchical trigonometric and polynomial elements.

4.10 Discussion and Conclusion

The results for the symmetric polynomial and trigonometric hierarchical formulations for a curved composite beam for fixed-free boundary condition are given in Tables 4.16 and 4.17. Values of critical buckling load, given by symmetric polynomial hierarchical formulation in the start are better as compared to the conventional FEM. The convergence of the critical buckling load values gets better when we increase the number of polynomial hierarchical terms with both tangential (v) and radial (w) displacement functions. The results for the four symmetric polynomial hierarchical terms converged best among all the combinations of this group. The critical buckling load value is reached by using just 4-elements mesh.

The results for the symmetric trigonometric hierarchical formulation follow the same trends as observed in the case of symmetric polynomial hierarchical terms. As we increase the number of symmetric trigonometric hierarchical terms, the convergence of the critical buckling load values becomes better and faster. The best value of critical buckling load for symmetric trigonometric hierarchical terms is given by four symmetric trigonometric hierarchical terms. This value is obtained by using 4-elements mesh compared to 12-elements mesh for conventional FEM.

For non-symmetric trigonometric hierarchical formulation, we will start with the (v_0 $- w_n$) group of combinations. The results given by this group are given in Table 4.18. This group gives larger values of critical buckling load when one element was used. This value suddenly drops to a far lower value with the introduction of the 2-elements mesh. After the 6-elements mesh, all the combinations except the combination (v_0 $- w_1$) converge to a single value and all the combinations give the same results. The critical buckling load value given by this group is 354.28 *lb* and this value is obtained by using 11-elements mesh.

The results given by the group of combinations (v_1 $- w_n$) of trigonometric hierarchical terms are given in Table 4.19. In this group one trigonometric hierarchical term is used with the tangential displacement (v) function while we increase the number of hierarchical terms with the radial displacement (w) function. The results get better for the first few elements with the increase of trigonometric hierarchical terms added to the radial displacement (w) function. When two, three and four trigonometric hierarchical terms are added to the radial displacement (w) function almost same results are obtained. Three and four hierarchical terms converge to a single value at the 4-elements mesh. Generally this group of non-symmetric hierarchical terms gives almost same values of critical buckling load and this value is reached at the 7-elements mesh showing a little improvement from the previous group of combinations (v_0 $- w_n$).

Table 4.16 Critical Buckling Load Calculated by using Symmetric Polynomial Hierarchical Terms for Fixed-Free Boundary Condition

8 D.O.F. Curved Composite Beam Element for $[0/90]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Symmetric Polynomial 1 HFEM term		*Symmetric Polynomial 2 HFEM terms*		*Symmetric Polynomial 3 HFEM terms*		*Symmetric Polynomial 4 HFEM terms*	
1	6	721.28	8	-	10	-	12	-
2	11	362.49	14	360.40	17	359.55	20	358.86
3	16	356.73	20	356.10	24	355.53	28	355.05
4	21	355.39	26	354.82	31	354.35	36	353.97
5	26	354.82	32	354.21				
6	31	354.46						
7	36	354.16						

Approximate Solution by Ritz Method: Critical Buckling Load = 360.13 *lb*

Table 4.17 Critical Buckling Load Calculated by using Symmetric Trigonometric Hierarchical Terms for Fixed-Free Boundary Condition

8 D.O.F. Curved Composite Beam Element for $[0/90]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Symmetric Trigonometric 1 HFEM term		*Symmetric Trigonometric 2 HFEM terms*		*Symmetric Trigonometric 3 HFEM terms*		*Symmetric Trigonometric 4 HFEM terms*	
1	6	785.05	8	391.53	10	372.24	12	-
2	11	364.13	14	360.84	17	359.25	20	359.15
3	16	356.63	20	350.53	24	356.12	28	355.76
4	21	355.34	26	354.95	31	354.48	36	353.98
5	26	354.82	32	354.25	38	353.67		
6	31	354.48						
7	36	354.20						

Approximate Solution by Ritz Method: Critical Buckling Load = 360.13 *lb*

The results for the group of combinations ($v_2 - w_n$) are given in Table 4.20. This group starts with a very good accuracy of the results. The results given by the all the combinations of this group show improvement in the convergence of the critical buckling load values when number of hierarchical terms are increased with the radial displacement (w) function. Generally, all of these combinations give similar results and the critical buckling load values are converged better than that of the group of combinations ($v_1 - w_n$). The approximate Ritz solution is reached by using just 5-elements mesh.

The group of combinations ($v_3 - w_n$) gives results that are given in Table 4.21. Again, these combinations start with a good accuracy and the results show better convergence of the critical buckling load values than that of the previous group of combinations. All the combinations of this group give results closer to each other. The critical buckling load value given by this group is reached by using only 5-elements mesh.

The results given by the combinations ($v_4 - w_n$) are given in Table 4.22. Values of the critical buckling load get better with each addition of the element mesh. All the combinations of this group give almost similar results. This group of combinations gives the best convergence of the critical buckling load values than all the combinations used before for the trigonometric hierarchical formulation. The approximate Ritz solution is reached by using just 4-elements mesh. The critical buckling load value given by 1-element mesh is not given due to ill-conditioned matrix.

Table 4.18 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric Hierarchical Terms ($v_0 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Curved Composite Beam Element for $[0/90]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 0v-1w HFEM terms		*Non-Symmetric Trigonometric 0v-2w HFEM terms*		*Non-Symmetric Trigonometric 0v-3w HFEM terms*		*Non-Symmetric Trigonometric 0v-4w HFEM terms*	
1	5	13887	6	13421	7	3252.5	8	3232.6
2	10	700.60	12	539.81	14	535.62	16	529.11
3	15	424.28	18	401.72	21	401.28	24	401.20
4	20	375.40	24	371.19	28	371.06	32	371.05
5	25	361.76	30	360.88	35	360.85	40	360.84
6	30	357.32	36	357.11	42	357.10	48	357.10
7	35	355.69	42	355.63	49	355.63	56	355.63
8	40	355.01	48	354.99	56	354.98	64	354.98
9	45	354.66	54	354.65	63	354.65	72	354.65
10	50	354.45	60	354.44	70	354.44	80	354.44
11	55	354.28	66	354.27	77	354.28	88	354.28

Approximate Solution by Ritz Method: Critical Buckling Load = 360.13 *lb*

Table 4.19 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric Hierarchical Terms ($v_1 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Curved Composite Beam Element for $[0/90]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 1v-0w HFEM terms		*Non-Symmetric Trigonometric 1v-2w HFEM terms*		*Non-Symmetric Trigonometric 1v-3w HFEM terms*		*Non-Symmetric Trigonometric 1v-4w HFEM terms*	
1	5	-	7	716.61	8	708.38	9	707.81
2	10	3364.43	12	362.67	14	359.82	16	359.71
3	15	356.87	17	356.56	20	356.10	23	356.08
4	20	355.42	22	355.33	26	355.13	30	355.13
5	25	354.86	27	354.81	32	354.72	37	354.72
6	30	354.50	32	354.47	38	354.43	44	354.43
7	35	354.21	37	354.19	44	354.17	51	354.17

Approximate Solution by Ritz Method: Critical Buckling Load = 360.13 *lb*

Table 4.20 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric Hierarchical Terms ($v_2 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Curved Composite Beam Element for $[0/90]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 2v-0w HFEM terms		*Non-Symmetric Trigonometric 2v-1w HFEM terms*		*Non-Symmetric Trigonometric 2v-3w HFEM terms*		*Non-Symmetric Trigonometric 2v-4w HFEM terms*	
1	6	-	7	784.19	9	377.29	10	362.82
2	12	364.28	13	360.55	16	359.50	18	358.47
3	18	356.47	19	356.57	23	355.84	26	355.80
4	24	354.92	25	354.97	30	354.72	34	354.71
5	30	354.25	31	354.25	37	354.16	42	354.15

Approximate Solution by Ritz Method: Critical Buckling Load = 360.13 *lb*

Table 4.21 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric Hierarchical Terms ($v_3 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Curved Composite Beam Element for $[0/90]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 3v-0w HFEM terms		*Non-Symmetric Trigonometric 3v-1w HFEM terms*		*Non-Symmetric Trigonometric 3v-2w HFEM terms*		*Non-Symmetric Trigonometric 3v-4w HFEM terms*	
1	7	-	8	723.86	9	373.24	11	371.80
2	14	363.07	15	359.17	16	359.24	19	359.19
3	21	356.14	22	356.03	23	356.03	27	356.11
4	28	354.49	29	354.47	30	354.41	35	354.48
5	35	253.67	36	253.62	37	253.63	43	253.66

Approximate Solution by Ritz Method: Critical Buckling Load = 360.13 *lb*

Table 4.22 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric Hierarchical Terms ($v_4 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Curved Composite Beam Element for $[0/90]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 4v-0w HFEM terms		*Non-Symmetric Trigonometric 4v-1w HFEM terms*		*Non-Symmetric Trigonometric 4v-2w HFEM terms*		*Non-Symmetric Trigonometric 4v-3w HFEM terms*	
1	8	-	9	723.21	10	356.65	11	355.74
2	16	362.83	17	359.06	18	359.11	19	359.18
3	24	355.75	25	355.63	26	355.64	27	355.77
4	32	353.97	33	353.86	34	353.90	35	353.99

Approximate Solution by Ritz Method: Critical Buckling Load = 360.13 *lb*

The best result after considering all the different combinations of symmetric and non-symmetric trigonometric formulation is given by the combination ($v_4 \; - w_1$) and the given critical buckling load value is 353.86 *lb*. This value of critical buckling load is a little bit better than the value given by the Ritz method, which is 360.13 *lb* and this value is obtained by using 4-elements mesh comparing to 12-elements mesh for the conventional FEM.

The results for the non-symmetric polynomial hierarchical formulation for $[0/90]_{8s}$ laminate are given in Tables 4.23, 4.24, 4.25, 4.26 and 4.27. The group of combinations ($v_0 \; - w_n$) starts with lesser accuracy as compared to the same group with trigonometric hierarchical formulation. At the 3-elements mesh, all the combinations of this group except the combination ($v_0 \; - w_1$) converge to a single value and finally the results given by all these combinations are same. The critical buckling load value given by this group is reached by using 9-elements mesh. This value is the same as given by the trigonometric hierarchical formulation.

The results for the group of combinations ($v_1 \; - w_n$) are given in Table 4.24. These values show that when we increase the number of hierarchical terms associated with radial displacement (w) function, we get better convergence. Generally, all the combinations of this group give almost similar results. The final value of critical buckling load 354.10 *lb* is obtained by using only 6-elements mesh compared to 12-elements mesh for the conventional FEM.

Table 4.23 Critical Buckling Load Calculated by using Non- Symmetric Polynomial Hierarchical Terms ($v_0 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Curved Composite Beam Element for $[0/90]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Polynomial 0v-1w HFEM terms		*Non-Symmetric Polynomial 0v-2w HFEM terms*		*Non-Symmetric Polynomial 0v-3w HFEM terms*		*Non-Symmetric Polynomial 0v-4w HFEM terms*	
1	5	121450	6	-	7	-	8	-
2	10	699.52	12	693.51	14	693.32	16	693.28
3	15	424.31	18	423.67	21	423.64	24	423.64
4	20	375.42	24	375.23	28	375.23	32	375.23
5	25	361.76	30	361.71	35	361.71	40	361.71
6	30	357.32	36	357.30	42	357.30	48	357.30
7	35	355.69	42	355.68	49	355.68	56	355.68
8	40	355.01	48	355.00	56	355.00	64	355.00
9	45	354.66	54	354.66	63	354.66	72	354.66
10	50	354.45	60	354.44	70	354.44	80	354.44
11	55	354.28	66	354.28	77	354.28	88	354.28

Approximate Solution by Ritz Method: Critical Buckling Load = 360.13 *lb*

Table 4.24 Critical Buckling Load Calculated by using Non- Symmetric Polynomial Hierarchical Terms ($v_1 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Curved Composite Beam Element for $[0/90]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Polynomial 1v-0w HFEM terms		*Non-Symmetric Polynomial 1v-2w HFEM terms*		*Non-Symmetric Polynomial 1v-3w HFEM terms*		*Non-Symmetric Polynomial 1v-4w HFEM terms*	
1	5	-	7	633.81	8	-	9	-
2	10	363.64	12	359.13	14	357.79	16	357.33
3	15	350.89	17	355.83	20	355.39	23	355.24
4	20	355.45	22	355.03	26	354.90	30	354.85
5	25	354.86	27	354.67	32	354.62	37	354.60
6	30	354.49	32	354.38	38	354.36	44	354.35
7	35	354.18	37	354.12	44	354.11	51	354.10

Approximate Solution by Ritz Method: Critical Buckling Load = 360.13 *lb*

The results for the group of combinations ($v_2 - w_n$) are given in Table 4.25. These values show better convergence each time we increase the number of hierarchical terms associated with the radial displacement (w) function. The results given are almost same for all the combinations. There is a very little difference between the critical buckling load values given by the trigonometric and polynomial hierarchical formulations. The critical buckling load value using polynomial hierarchical formulation is obtained by using 5-elements mesh.

The results for the group of combinations ($v_3 - w_n$) are given in Table 4.26. Critical buckling load value calculated by using one element is not mentioned in the table because of the ill conditioned matrix. The combination ($v_3 - w_4$) provides the best convergence of the critical buckling load values among all the other combinations of this group. The critical buckling load value given by this group is obtained by using just 4-elements mesh compared to 12-elements mesh for the conventional FEM.

The results for the group of combinations ($v_4 - w_n$) using polynomial hierarchical formulations are given in Table 4.27. The combination with one hierarchical term added to the radial displacement (w) function provides the best convergence of the results. The value of critical buckling load given by this combination is 353.13 *lb*. This value is reached by using 4-elemnts mesh. All the rest of combinations give results that are almost similar to each other.

Table 4.25 Critical Buckling Load Calculated by using Non- Symmetric Polynomial Hierarchical Terms ($v_2 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Curved Composite Beam Element for $[0/90]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Polynomial 2v-0w HFEM terms		*Non-Symmetric Polynomial 2v-1w HFEM terms*		*Non-Symmetric Polynomial 2v-3w HFEM terms*		*Non-Symmetric Polynomial 2v-4w HFEM terms*	
1	6	0.11413×10^6	7	-	9	-	10	-
2	12	363.99	13	360.80	16	358.99	18	358.15
3	18	356.54	19	355.82	23	355.62	26	355.31
4	24	354.99	25	354.75	30	354.62	34	354.51
5	30	354.29	31	354.22	37	354.12	42	354.08

Approximate Solution by Ritz Method: Critical Buckling Load = 360.13 *lb*

Table 4.26 Critical Buckling Load Calculated by using Non- Symmetric Polynomial Hierarchical Terms ($v_3 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Curved Composite Beam Element for $[0/90]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Polynomial 3v-0w HFEM terms		*Non-Symmetric Polynomial 3v-1w HFEM terms*		*Non-Symmetric Polynomial 3v-2w HFEM terms*		*Non-Symmetric Polynomial 3v-4w HFEM terms*	
1	7	-	8	-	9	-	11	-
2	14	363.09	15	359.01	16	360.38	19	358.65
3	21	356.18	22	354.88	23	356.07	27	355.28
4	28	354.52	29	354.29	30	354.64	35	354.23

Approximate Solution by Ritz Method: Critical Buckling Load = 360.13 *lb*

Table 4.27 Critical Buckling Load Calculated by using Non- Symmetric polynomial Hierarchical Terms ($v_4 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0/90]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Polynomial 4v-0w HFEM terms		*Non-Symmetric Polynomial 4v-1w HFEM terms*		*Non-Symmetric Polynomial 4v-2w HFEM terms*		*Non-Symmetric Polynomial 4v-3w HFEM terms*	
1	8	0.118×10^7	9	-	10	-	11	-
2	16	362.75	17	357.54	18	359.05	19	358.25
3	24	355.79	25	354.28	26	355.49	27	355.07
4	32	354.05	33	354.10	34	354.37	35	354.17

Approximate Solution by Ritz Method: Critical Buckling Load = 360.13 *lb*

It is evident from the above shown results that with the increase of the polynomial hierarchical terms associated with the tangential displacement (v) function the convergence of the critical buckling load value improves and we get better results when we use more hierarchical terms with the tangential displacement (v) function and less hierarchical terms with the radial displacement (w) function. Critical buckling load values keep on converging when we increase the number of elements. The approximate solution given by the Ritz method is reached by using just 4-elements mesh compared to 12-elements mesh for the conventional FEM.

Chapter 5

Parametric Study on Curved Composite Beams

5.1 Introduction

In the previous chapters, the hierarchical finite element analysis methodology for both the isotropic and composite curved beams has been developed. Both the forms of HFEM viz. polynomial and trigonometric forms were developed. The HFEM was applied first to the curved composite beam to find out the central deflection. The HFEM was then applied for the buckling analysis of the composite curved beam.

Major considerations in the analysis of curved composite beams are laminate configuration, ply orientation and boundary conditions. We now intend to conduct a comprehensive parametric study for the curved composite beams. The material chosen is NCT-301 Graphite – Epoxy. The material properties are given as:

$$E_1 = 129.43\ GPa,\ \ E_2 = 7.99\ GPa,\ \ \upsilon_{21} = 0.021,\ \ G_{12} = 4.28\ GPa$$

The specifications of the composite laminate and their geometric properties are detailed in each case. All the cases are solved using both conventional and hierarchical finite element formulation.

The mid-plane curved laminate beam is analyzed considering all types of variations: variations in the boundary conditions, variations in the stacking sequences and variations in the hierarchical terms combinations. Where applicable, results are plotted for both the conventional and the hierarchical formulations and suitable comparisons are made. After each figure, appropriate interpretations are provided to explain how these variations affect the central deflection and critical buckling load for the curved beams. For example, how the variations in the boundary conditions are related to the global degrees of freedom and how this will affect the final results one investigated. Also, a comparison between the results obtained using both the formulations versus the number of elements is done with the help of figures.

Finally, overall conclusions that relate to the two kinds of formulations and different combinations of the hierarchical terms are provided that serves as factors to be considered in calculating the optimal results. These conclusions can guide the designer on the choice of different parameters involved in the analysis such as boundary conditions and stacking sequences.

5.2 Parametric Study on Composite Curved Beams

A kind of parametric study was conducted in chapter 3 for $[0/90]_{8s}$ laminate when different symmetric and non-symmetric combinations of four trigonometric and polynomial hierarchical terms were used with tangential (v) and radial (w) displacement interpolation functions. In this way the most optimum combination of the hierarchical terms was figured out for both trigonometric and polynomial formulations.

5.2.1 The Effect of Laminate Configuration

In this section, laminate configurations are chosen differently to see the effect of different fiber orientations on the values of central deflection for a fixed-free beam. Central deflections are calculated for the following types of laminates that have the configurations $[0]_{16s}$, $[90]_{16s}$, angle ply $[\pm 45]_{8s}$ and quasi-isotropic $[0/90/+45/-45]_{4s}$. The values are determined by using trigonometric and polynomial hierarchical finite elements in different combinations to model the beam. The same input data is used as in the previous chapter except that the kind of laminate configuration is chosen differently each time as described above to see the effect of different fiber orientations on the composite curved beams.

Ritz method will be used to get the approximate solution for each of these laminate configurations. Approximate solutions were obtained using the same procedure as given in section 3.5.1. Then both conventional and hierarchical finite element formulations will be used to get the results and finally these values will be compared with the approximate solution given by the Ritz method.

Both symmetric and non-symmetric trigonometric and polynomial hierarchical formulations will be used to analyze the same curved beam example given in chapter 3. We will see the effect of internal degrees of freedom in terms of improvement in the results. We will try to figure out such symmetric and non-symmetric combinations of trigonometric and polynomial hierarchical terms that will give the best convergence to reach the results.

5.2.1.1 $[0]_{16s}$ *Laminate*

The approximate solution for the central deflection for the $[0]_{16s}$ laminate is 0.0338 *in* and is reached by using the 10-elements mesh of the conventional finite element method. Hierarchical finite element analysis for the $[0]_{16s}$ laminate is done by following the same procedure as described previously.

There is a significant improvement in the results when we use symmetric hierarchical terms with tangential (v) and radial (w) displacement functions. When one symmetric hierarchical term was added to both tangential (v) and radial (w) displacement functions, the number of elements mesh required to give the most accurate results, reduced to 6 which was 10 for the conventional FEM. The results keep on getting better when the number of symmetric hierarchical terms were increased from one to four.

The results for the group of combinations ($v_0 - w_n$) follow the same trends as observed in the case of $[0/90]_{8s}$ laminate. These combinations do not show much improvement in terms of the convergence of the results in comparison to the conventional FEM solution. These combinations also require the 10-elements mesh to get the results.

The central deflection values of both polynomial and trigonometric hierarchical terms for the group of combinations ($v_1 - w_n$) do not show much difference of the results in between them. In this case, only the 6-elements mesh was required to get the desired results. Polynomial hierarchical terms give better results that that of trigonometric hierarchical terms.

A significant improvement in the convergence of the results for the group of combinations ($v_2 - w_n$) is observed when we use two hierarchical terms with the tangential displacement (v) function. The 4-elements mesh proved to be enough to get the desired results for both polynomial and trigonometric hierarchical formulations where final value of central deflection given by both these formulations is almost same.

The results for the group of combinations ($v_3 - w_n$) show an improvement each time we increase the number of hierarchical terms with the radial displacement (w) function. For these combinations only the 3-elements mesh was enough to produce the desired results. Initially, polynomial hierarchical terms produced better results than trigonometric hierarchical terms but at the end both of them gave almost same results with a minor difference in between them.

The results for the last group of combinations ($v_4 - w_n$) for $[0]_{16s}$ laminate are given in Table 5.1. The combination ($v_4 - w_0$) of both polynomial and trigonometric hierarchical terms gives the best results by using minimum number of elements. In this case only the 2-elements mesh was enough to produce the desired results for polynomial terms and the 3-elements mesh was used for trigonometric hierarchical terms. These results are also plotted in Figures 5.1 and 5.2. By comparison, trigonometric hierarchical terms produce results better than polynomial hierarchical terms. The results for trigonometric hierarchical terms given by all the combinations of this group show very little difference among themselves. Polynomial hierarchical terms show a lot of variations among the results given by different combinations of this group for $[0]_{16s}$ laminate.

Table 5.1 Central Deflection Calculated by using Non- Symmetric Polynomial and Trigonometric Hierarchical Herms ($v_4 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0]_{16s}$ Laminate								
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection(in) (in)	Number of D.O.F.	Centre Deflection(in) (in)	Number of D.O.F.	Centre Deflection(in) (in)
	Non-Symmetric Polynomial v4-w0 HFEM terms		*Non-Symmetric Polynomial v4-w1 HFEM terms*		*Non-Symmetric Polynomial v4-w2 HFEM terms*		*Non-Symmetric Polynomial v4-w3 HFEM terms*	
1	8	0.0260	9	0.0259	10	0.0295	11	0.0306
2	16	0.0352	17	0.0271	18	0.0296	19	0.0325
3			25	0.0311	26	0.0364	27	0.0389
4			33	0.0374				

Number of Elements	*Non-Symmetric Trigonometric v4-w0 HFEM terms*		*Non-Symmetric Trigonometric v4-w1 HFEM terms*		*Non-Symmetric Trigonometric v4-w2 HFEM terms*		*Non-Symmetric Trigonometric v4-w3 HFEM terms*	
1	8	0.0160	9	0.0160	10	0.0160	11	0.0165
2	16	0.0319	17	0.0316	18	0.0314	19	0.0315
3	24	0.0479	25	0.0469	26	0.0460	27	0.0464

Approximate solution by Ritz method: Centre Deflection = 0.0338 *in*

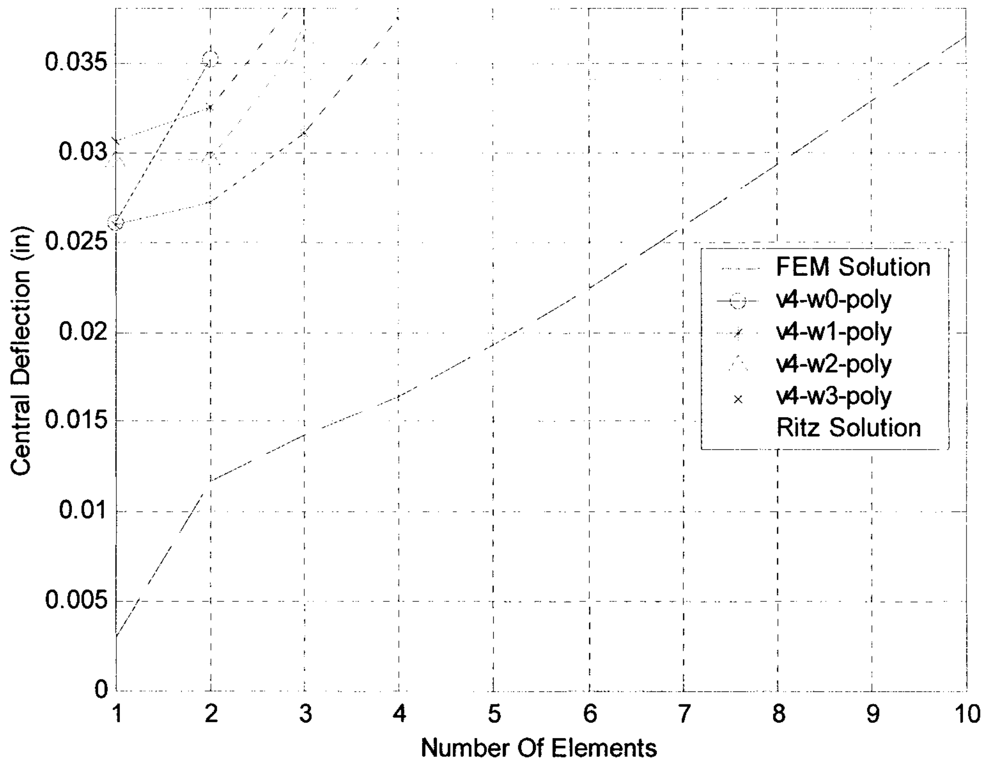


Figure 5.1 Comparison between the Results Obtained using Non-Symmetric ($v_4 - w_n$) Polynomial Hierarchical Terms for $[0]_{16s}$ Laminate

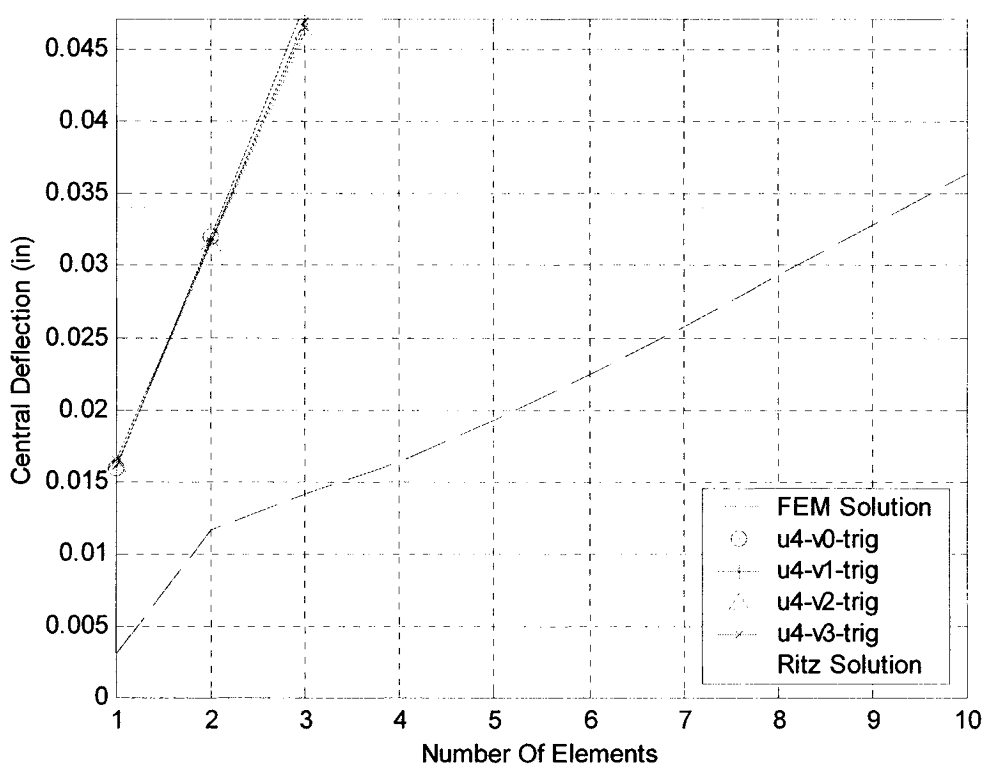


Figure 5.2 Comparison between the Results Obtained using Non-Symmetric ($v_4 - w_n$) Trigonometric Hierarchical Terms for $[0]_{16s}$ Laminate

5.2.1.2 $[90]_{16s}$ *Laminate*

The approximate solution for the central deflection given by the Ritz method is 0.5741 *in*. This value was obtained by using the 10- elements mesh of the conventional finite element method.

The results for the symmetric polynomial and trigonometric hierarchical terms for $[90]_{16s}$ laminate show that we reach the Ritz solution at the 6-elements mesh, when we use one symmetric polynomial and trigonometric hierarchical term, showing a lot of improvement compared with the conventional FEM solution. The results become better with polynomial hierarchical formulation than with trigonometric hierarchical formulation when we increase the number of symmetric hierarchical terms.

The results for non-symmetric polynomial and trigonometric hierarchical formulations for the group of combinations ($v_0 - w_n$) for $[90]_{16s}$ laminate show that this group of combinations also requires the 10-elements mesh of HFEM, as in the case of conventional FEM to reach the approximate solution.

With the addition of just one hierarchical term to the tangential displacement (v) function, the results for the group of combinations ($v_1 - w_n$) show a considerable improvement from the previous group of combinations ($v_0 - w_1$). The approximate solution is reached by using 6-elements mesh of HFEM.

The results for the group of combinations ($v_2 - w_n$) show further improvement in the results, when two hierarchical terms are added to the tangential displacement (v) function. With this type of combinations, the approximate solution was reached at the 4-elements mesh. The results given by polynomial hierarchical formulation are better than that of the results given by trigonometric hierarchical formulation for all the number of elements used for this group of combinations.

The results for the group of combinations ($v_3 - w_n$) show even more improvement in the results and we reach the approximate solution using just 3-elements mesh. It was observed that trigonometric hierarchical terms show very little difference in the results for all the combinations of this group.

The results for the polynomial and trigonometric hierarchical formulations for the group of combinations ($v_4 - w_n$) for $[90]_{16s}$ laminate are given in Table 5.2. These results show that with the addition of one more term to the tangential displacement (v) function, the convergence becomes even faster. Polynomial and trigonometric hierarchical terms ($v_4 - w_0$) with maximum number of hierarchical terms with the tangential displacement (v) function and with the minimum number of hierarchical terms with the radial displacement (w) function are the best combinations among all the combinations tried for the $[90]_{16s}$ laminate. Figures 5.2 and 5.3 show that curves for polynomial hierarchical terms give more variation among themselves. Curves for trigonometric hierarchical terms are converging to a single curve showing fewer variations among themselves.

Table 5.2 Central Deflection Calculated by using Non- Symmetric Polynomial and Trigonometric Hierarchical Herms ($v_4 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[90]_{16s}$ Laminate								
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection(in) (in)	Number of D.O.F.	Centre Deflection(in) (in)	Number of D.O.F.	Centre Deflection(in) (in)
	Non-Symmetric Polynomial v4-w0 HFEM terms		*Non-Symmetric Polynomial v4-w1 HFEM terms*		*Non-Symmetric Polynomial v4-w2 HFEM terms*		*Non-Symmetric Polynomial v4-w3 HFEM terms*	
1	8	0.4212	9	0.4206	10	0.4773	11	0.4957
2	16	0.5724	17	0.4398	18	0.4811	19	0.5272
3	24	0.7780	25	0.5039	26	0.5907	27	0.6316
4			33	0.6072				

Number of Elements	*Non-Symmetric Trigonometric v4-w0 HFEM terms*		*Non-Symmetric Trigonometric v4-w1 HFEM terms*		*Non-Symmetric Trigonometric v4-w2 HFEM terms*		*Non-Symmetric Trigonometric v4-w3 HFEM terms*	
1	8	0.2598	9	0.2603	10	0.2605	11	0.2649
2	16	0.5177	17	0.5129	18	0.5092	19	0.5116
3	24	0.7762	25	0.7600	26	0.7459	27	0.7525

Approximate solution by Ritz method: Centre Deflection = 0.5741 *in*

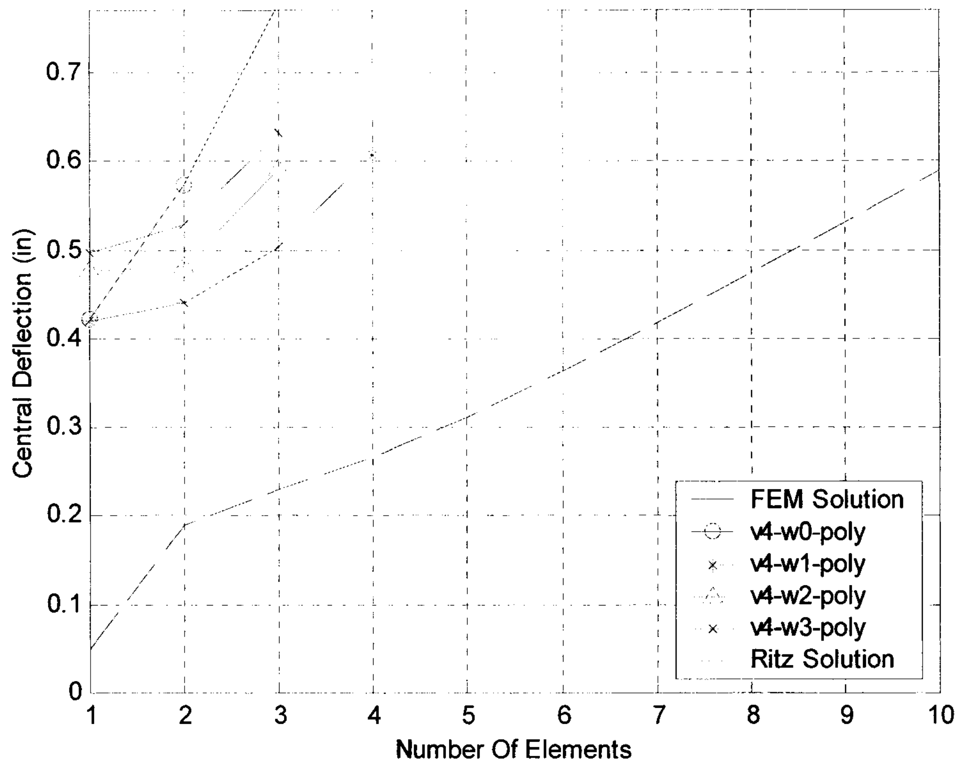


Figure 5.3 Comparison between the Results Obtained using Non-Symmetric ($v_4 - w_n$) Polynomial Hierarchical Terms for $[90]_{16s}$ Laminate

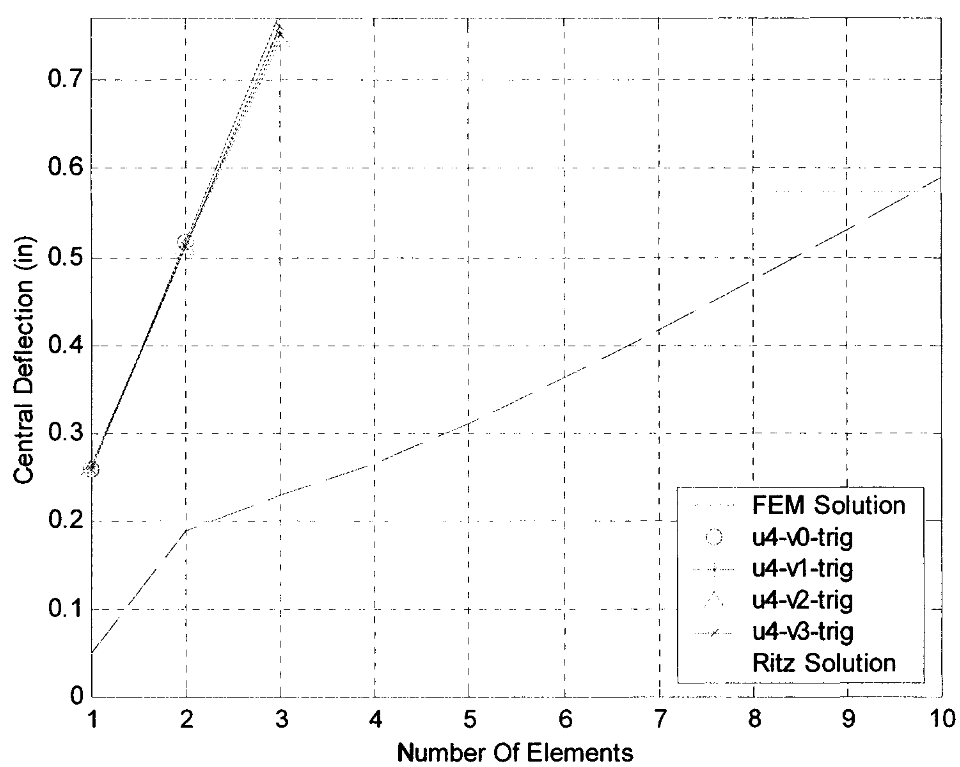


Figure 5.4 Comparison between the Results Obtained using Non-Symmetric ($v_4 - w_n$) Trigonometric Hierarchical Terms for $[90]_{16s}$ Laminate

5.2.1.3 $[\pm 45]_{8s}$ *Laminate*

The approximate solution given by the Ritz method is 0.0955 *in*. Conventional FEM took 8-elements mesh to reach the approximate Ritz solution for $[\pm 45]_{8s}$ laminate. The % age error between the two solutions is just 0.73 %.

The results for the symmetric polynomial and trigonometric hierarchical terms for $[\pm 45]_{8s}$ laminate show an improvement when symmetric hierarchical terms are used. The results show greater convergence when 3 and 4 symmetric hierarchical terms are added to both tangential (v) and radial (w) displacement functions. For 4 symmetric polynomial hierarchical terms only one element mesh was required to reach the approximate Ritz solution. Four symmetric trigonometric hierarchical terms took 2-elements mesh to reach the approximate solution.

The results for the group of combinations ($v_0 - w_n$) of both polynomial and trigonometric hierarchical terms show an improvement as compared to the conventional FEM. After the 5th element all the values of central deflection converge to a single curve giving almost similar results at the 8-elements mesh.

The results show further improvement with the addition of one hierarchical term to the tangential displacement (v) function for the group of combinations ($v_1 - w_n$). Only the 5-elements mesh is used to get the desired results instead of the 8-elements mesh as in the previous case.

The final solution for the group of combinations ($v_2 - w_n$) is reached when 3-elements mesh is used with the addition of two hierarchical terms to the tangential displacement (v) function. The results get better when we increase the number of hierarchical terms with the radial displacement (w) function.

The results for the group of combinations ($v_3 - w_n$) show an improvement in the central deflection values in comparison to the results given by the previous combinations. The combination ($v_3 - w_4$) gives the best results among all the combinations in this group for polynomial hierarchical functions by using just 2-elements mesh.

The results for the group of combinations ($v_4 - w_n$) are given in Table 5.3. The values for this group of combinations are plotted in Figures 5.5 and 5.6. The central deflection values show that we get the best results when four hierarchical terms are used with the tangential displacement (v) function among all the other combinations for the $[\pm 45]_{8s}$ laminate. The combination ($v_4 - w_2$) of the polynomial hierarchical terms starts right from the approximate solution given by the Ritz method. This is a significant improvement in the result by considering the fact that conventional FEM solution gave the same result at the 8-elements mesh. The combination ($v_4 - w_3$) even started with the value more than the approximate solution given by the Ritz method. This combination is the best for both polynomial and trigonometric hierarchical formulations. All of the combinations of this group converge to a single value as shown in Figure 5.6.

Table 5.3 Central Deflection Calculated by using Non- Symmetric Polynomial and Trigonometric Hierarchical Herms ($v_4 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for [+45/-45]$_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Non-Symmetric Polynomial v4-w0 HFEM terms		*Non-Symmetric Polynomial v4-w1 HFEM terms*		*Non-Symmetric Polynomial v4-w2 HFEM terms*		*Non-Symmetric Polynomial v4-w3 HFEM terms*	
1	8	0.0843	9	0.0842	10	0.0955	11	0.0992
2	16	0.1144	17	0.0879	18	0.0962	19	0.1054
3			25	0.1007				

Number of Elements	*Non-Symmetric Trigonometric v4-w0 HFEM terms*		*Non-Symmetric Trigonometric u4-v1 HFEM terms*		*Non-Symmetric Trigonometric v4-w2 HFEM terms*		*Non-Symmetric Trigonometric v4-w3 HFEM terms*	
1	8	0.0519	9	0.0520	10	0.0520	11	0.0529
2	16	0.1035	17	0.1025	18	0.1018	19	0.1022

Approximate solution by Ritz method: Centre Deflection = 0.0955 in

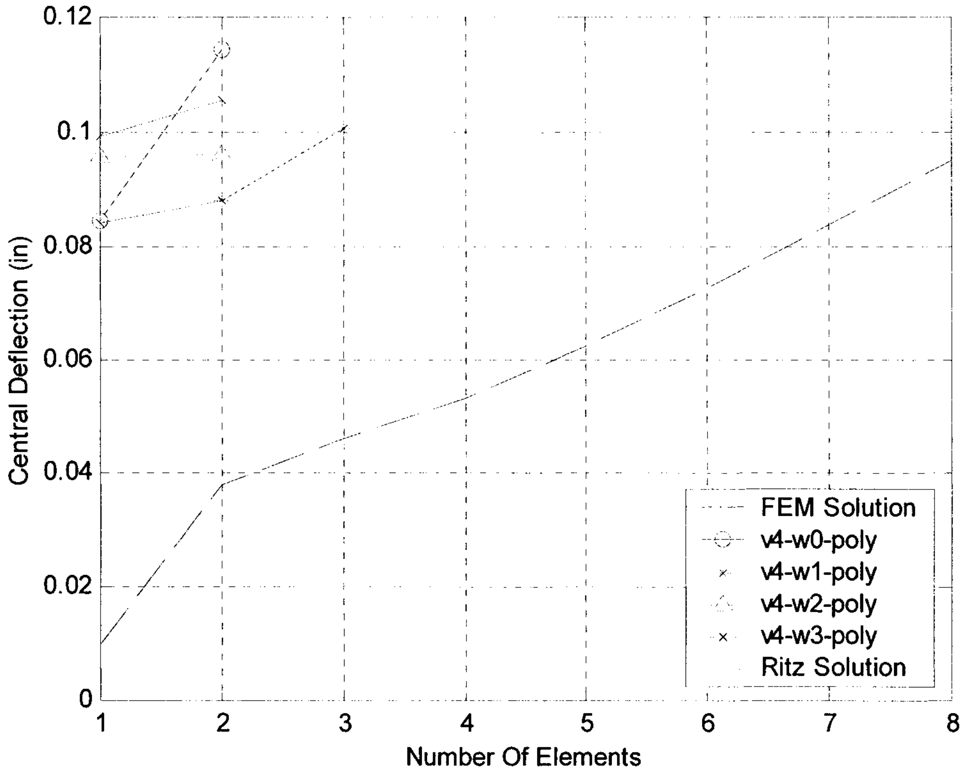


Figure 5.5 Comparison between the Results Obtained using Non-Symmetric ($v_4 - w_n$) Polynomial Hierarchical Terms for $[\pm 45]_{8s}$ Laminate

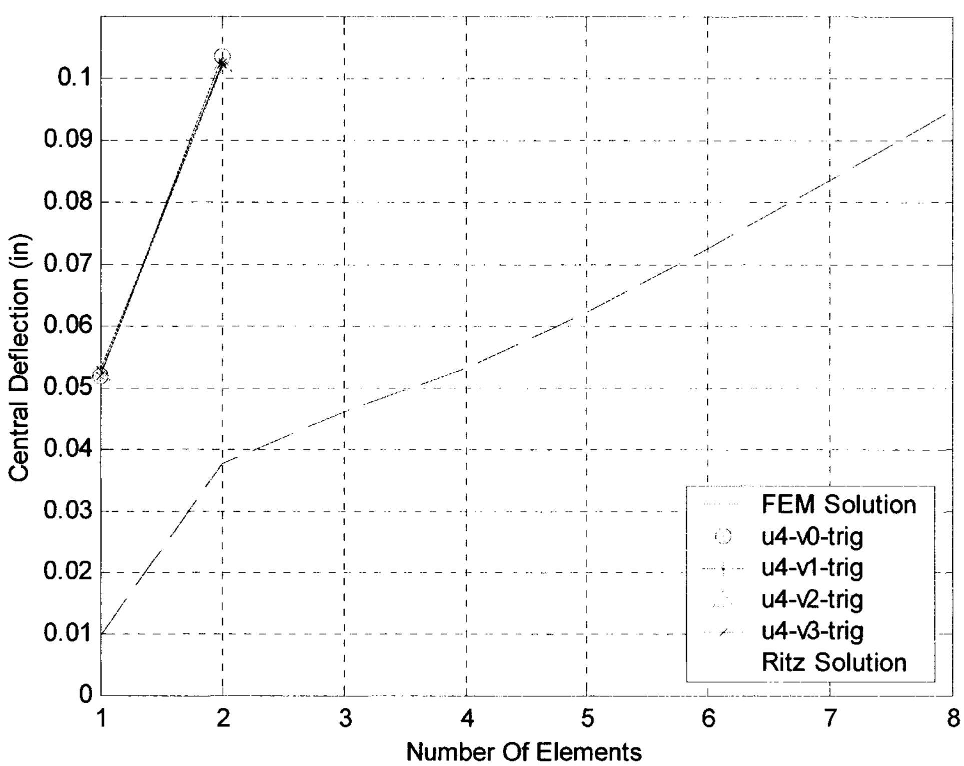


Figure 5.6 Comparison between the Results Obtained using Non-Symmetric ($v_4 - w_n$) Trigonometric Hierarchical Terms for $[\pm 45]_{8s}$ Laminate

5.2.1.4 $[0/90/+45/-45]_{4s}$ Symmetric Quasi-Isotropic Laminate

Symmetric Quasi-isotropic laminates have ten stiffnesses. All the B_{ij} terms for a symmetric quasi-isotropic laminate are zero. There will be four in the extensional stiffnesses: A_{11}, A_{22}, A_{33}, A_{12}; $A_{13} = A_{23} = 0$ and there will be six in the bending stiffnesses: D_{11}, D_{22}, D_{33}, D_{12}, D_{13}, D_{23} .

So the laminate stiffnesses for the symmetric quasi-isotropic laminates are:

$$\left[\begin{array}{ccc|ccc} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & D_{11} & D_{12} & D_{13} \\ 0 & 0 & 0 & D_{12} & D_{22} & D_{23} \\ 0 & 0 & 0 & D_{13} & D_{23} & D_{33} \end{array}\right] \tag{5.1}$$

The curved composite laminate consisting of a total of 32 plies, each having a thickness of 0.125 mm will be used for the analysis. The resultant A and D matrices are as follows:

$$\begin{bmatrix} 1.2500\times10^6 & 0.3932\times10^6 & 0 & 0 & 0 & 0 \\ 0.3932\times10^6 & 1.2500\times10^6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4284\times10^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0029\times10^6 & 0.0007\times10^6 & 0.0001\times10^6 \\ 0 & 0 & 0 & 0.0007\times10^6 & 0.0026\times10^6 & 0.0001\times10^6 \\ 0 & 0 & 0 & 0.0001\times10^6 & 0.0001\times10^6 & 0.0008\times10^6 \end{bmatrix} \tag{5.2}$$

Equation (5.2) confirms all the properties of the symmetric quasi-isotropic laminate.

For the conventional FEM, it took 9-elements mesh to reach the approximate solution given by the Ritz method for the quasi-isotropic laminate. The %age error reduces when number of elements is increased. At the 8-elements mesh % age error reduces to 1.69 %.

The central deflection values show a great improvement for quasi-isotropic laminate in terms of the convergence of the results when symmetric hierarchical terms were used. The number of elements mesh used reduced to 5 for one symmetric hierarchical term comparing to 9 for the conventional FEM solution. The results get even better when the numbers of hierarchical terms are increased from 1 to 4. Four symmetric polynomial and trigonometric hierarchical terms use only one and two elements mesh to reach the approximate solution.

For the group of combinations ($v_0 - w_n$) no hierarchical term was used with the tangential displacement (v) function. Number of hierarchical terms associated with the radial displacement (w) function is increased from one term to four terms. All these combinations used the 9-elements mesh to reach the approximate solution as in the case of conventional FEM. Both polynomial and trigonometric hierarchical terms show a slower convergence.

The results for the group of combinations ($v_1 - w_n$) show a considerable improvement in the convergence of the elements from the 9-elements mesh to the 5-elements mesh when only one hierarchical term was added to the tangential displacement

(v) function. The results get better each time we increase the number of hierarchical terms with the radial displacement (w) function. The convergence for this combination is almost linear. All the other combinations behave similar to each other, giving similar results.

The results for the group of combinations ($v_2 - w_n$) show that the results given by polynomial hierarchical terms are better than that of the results given by trigonometric hierarchical terms for all the combinations of this group.

The results for the group of non-symmetric combinations ($v_3 - w_n$) show a further improvement in the convergence of the results. The combination ($v_3 - w_4$) gives the desired results just after the 2-elements mesh for polynomial hierarchical formulation and after the 3-elements mesh for trigonometric hierarchical formulation.

The results for the group of combinations ($v_4 - w_n$) are given in Table 5.4. These values are plotted in Figures 5.7 and 5.8. These combinations use the maximum number of hierarchical terms with the tangential displacement (v) function. The combination ($v_4 - w_0$) for polynomial hierarchical terms is the best combination among all other combinations of this group for the quasi-isotropic laminate. The results for trigonometric hierarchical terms for all the combinations are similar to each other as shown in Figure 5.7. Once again, the combination ($v_4 - w_0$) is the best combination for trigonometric hierarchical terms and uses 2-elements mesh to reach the approximate Ritz solution.

Table 5.4 Central Deflection Calculated by using Non- Symmetric Polynomial and Trigonometric Hierarchical Herms ($v_4 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0/90/+45/-45]_{4s}$ Laminate								
Number of Elements	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)	Number of D.O.F.	Centre Deflection (in)
	Non-Symmetric Polynomial v4-w0 HFEM terms		*Non-Symmetric Polynomial v4-w1 HFEM terms*		*Non-Symmetric Polynomial v4-w2 HFEM terms*		*Non-Symmetric Polynomial v4-w3 HFEM terms*	
1	8	0.0621	9	0.0620	10	0.0704	11	0.0731
2	16	0.0840	17	0.0647	18	0.0708	19	0.07759
3	24	0.1143	25	0.0742	26	0.0869	27	0.0929
4			33	0.0894	34	0.1045	35	0.1121
Number of Elements	*Non-Symmetric Trigonometric v4-w0 HFEM terms*		*Non-Symmetric Trigonometric v4-w1 HFEM terms*		*Non-Symmetric Trigonometric v4-w2 HFEM terms*		*Non-Symmetric Trigonometric v4-w3 HFEM terms*	
1	8	0.0382	9	0.0382	10	0.0383	11	0.0389
2	16	0.0761	17	0.0754	18	0.0748	19	0.0752
3	24	0.1141	25	0.1117	26	0.1097	27	0.1106

Approximate solution by Ritz method: Centre Deflection = 0.0899 in

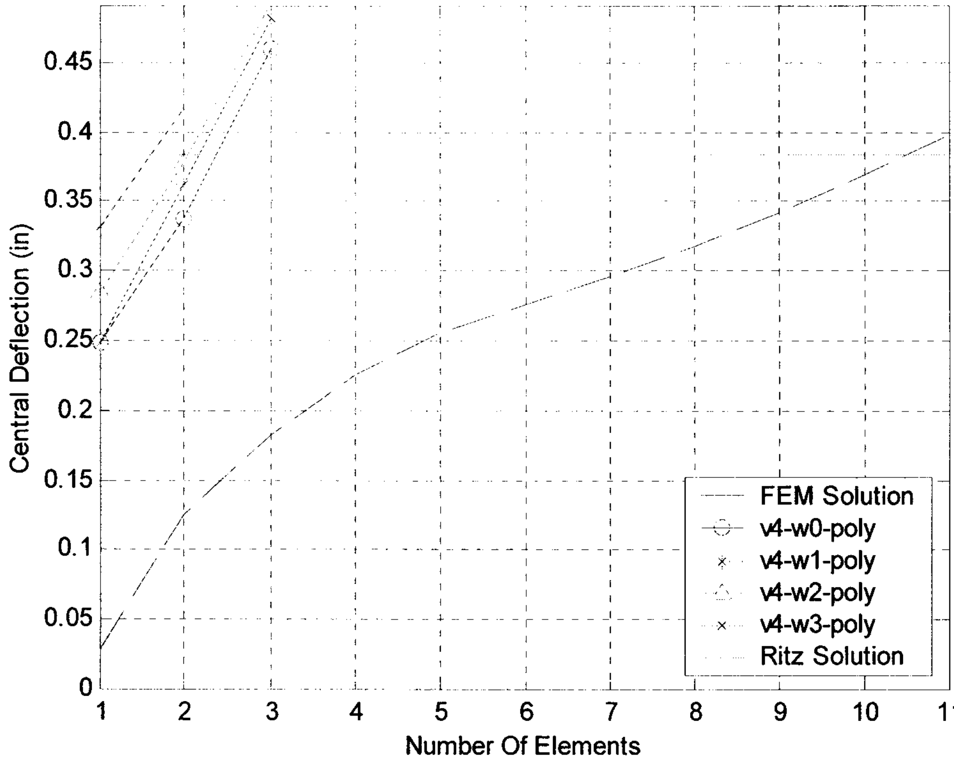


Figure 5.7 Comparison between the Results Obtained using Non-Symmetric ($v_4 - w_n$) Trigonometric Hierarchical Terms for [0 / 90 / +45 / -45]$_{4s}$ Laminate

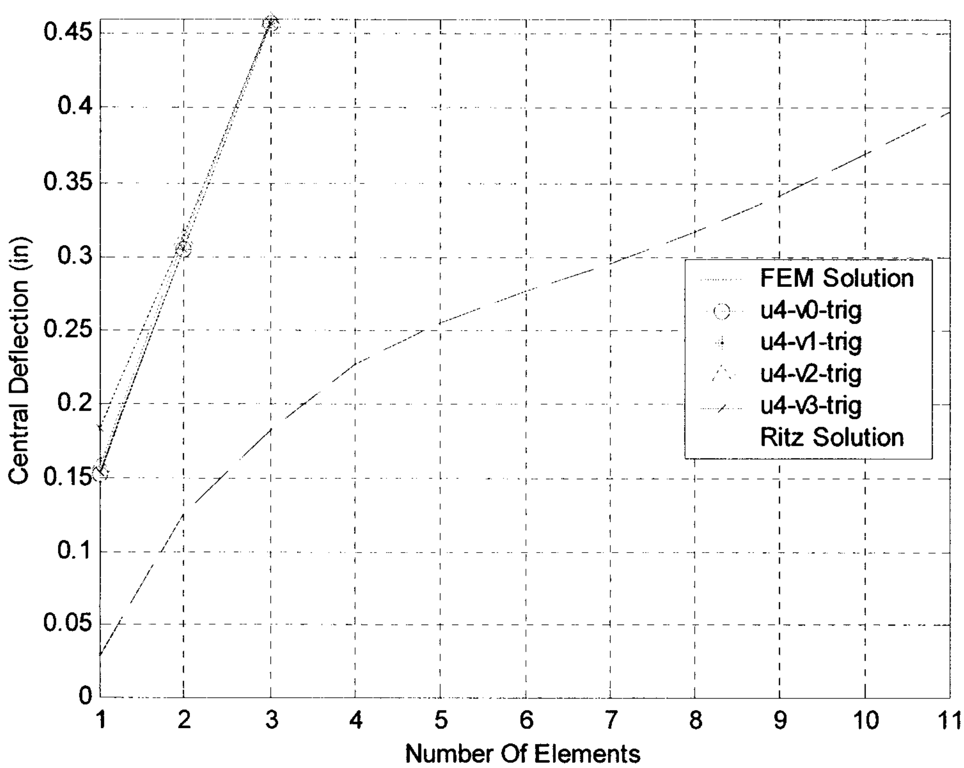


Figure 5.8 Comparison between the Results Obtained using Non-Symmetric ($v_4 - w_n$) Trigonometric Hierarchical Terms for $[0/90/+45/-45]_{4s}$ Laminate

5.2.2 Conclusion

5.2.2.1 $[0]_{16s}$ Laminate and $[90]_{16s}$ Laminate

For these two laminate configurations the hierarchical terms behave almost similarly. The combinations (v_0 $-$ w_n) for both of these laminates give results similar to that of conventional FEM with the same number of elements mesh used. The results do not show much improvement even when we increase the number of elements with the radial displacement (w) function. The results get better when we increase the number of hierarchical terms with the tangential displacement (v) function. This fact shows the greater effect of hierarchical terms with the tangential displacement (v) function. The best combination for both polynomial and trigonometric hierarchical terms for both of these two laminates is ($v_4 - w_0$). Trigonometric hierarchical terms give better results when we use less number of hierarchical terms with the tangential displacement (v) function. As we increase the number of hierarchical terms, results with polynomial hierarchical terms get better.

5.2.2.2 $[\pm 45]_{8s}$ Angle Ply Laminate

The conventional finite element method gives the most accurate result at the 8-elements mesh. As in the previous case, the results given by trigonometric hierarchical formulation are better than that given by polynomial hierarchical formulation when we use no hierarchical terms with the tangential displacement (v) function. In general, it is observed that with the increase of the number of trigonometric hierarchical terms with the tangential displacement (v) function, the values of central deflection for all the combinations of one type of group try to converge to a single curve regardless of how many hierarchical terms are being used with the radial displacement (w) function. The results given by one type of group e.g. ($v_2 - w_n$), ($v_3 - w_n$) or ($v_4 - w_n$) for trigonometric hierarchical functions are almost similar to each other.

The best results are given by the ($v_4 - w_0$) combination for both polynomial and trigonometric hierarchical formulations. The results obtained by this combination reached the approximate solution by using just the 2-elements mesh in comparison to the conventional FEM when the 8-elements mesh was used to get the same result.

5.2.2.3 $[0/90/+45/-45]_{4s}$ *Symmetric Quasi-Isotropic Laminate*

NCT-301 graphite epoxy was used in the symmetric quasi-isotropic laminate. The properties of the laminate are verified in Equation (5.1).

The results given by the group of combinations ($v_0 - w_n$) and ($v_1 - w_n$), for both polynomial and trigonometric hierarchical formulations behave similarly. They differ little in the results and trends. The best results for both polynomial and trigonometric hierarchical functions are obtained when four hierarchical terms were used with the tangential displacement (v) and no hierarchical term was used with the radial displacement (w) interpolation functions i.e.($v_4 - w_0$).

5.3 Parametric Study on Buckling of Composite Curved Beam

The same curved beam element as used previously in chapter 4 for the buckling analysis in Section 4.4.2 will be used in this section for the parametric study of the composite curved beams. This study will be conducted in terms of different configurations of the laminate, different boundary conditions and using both conventional and hierarchical finite element methods. Ritz method will be used to get the approximate solution for each of these laminate configurations.

5.3.1 The Effect of Laminate Configuration

In this section, different laminate configurations are chosen to see the effect of different fiber orientations on the values of critical buckling loads for a fixed-free composite curved beam. Critical buckling loads are calculated for the following types of laminates that have the configurations $[0]_{16s}$, $[90]_{16s}$, angle-ply $[\pm 45]_{8s}$ and quasi-isotropic $[0/90/+45/-45]_{4s}$. The results obtained by using different combinations of the hierarchical terms will be compared with the approximate solution obtained by the Ritz method.

5.3.1.1 $[0]_{16s}$ *Laminate*

The comparison of the results by Ritz method and the conventional FEM solution is given in Table 5.5. Ritz solution converged to a single value at m = 18. When 9-elements mesh was used, conventional FEM solution also reached this approximate solution. The % age difference between these two solutions is -0.6 %.

Table 5.5 Ritz Method and Conventional FEM Solutions for $[0]_{16s}$ Laminate with Fixed-Free Boundary Condition

$[0]_{16s}$ Laminate					
Ritz Method		**Conventional FEM Solution using 8 D.O.F. Composite Curved Beam Element**			
Value of m	Critical Buckling Load (lb)	Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Error (%)
1	0.18537×10^{7}	1	4	96045	99.35
2	44184	2	8	1263.3	50.83
3	5450	3	12	748.71	17.04
4	2467.9	4	16	657.15	5.48
5	1078.5	5	20	630.67	1.51
6	788.61	6	24	622.52	0.22
7	691.82	7	28	619.05	-0.34
8	663.99	8	32	617.50	-0.59
9	641.48	9	36	617.43	-0.6
10	637.85				
11	611.60				
12	625.99				
13	625.71				
14	621.66				
18	621.15				

The results for $[0]_{16s}$ laminate for symmetric combinations of both trigonometric and polynomial terms are given in Table 5.6. The results show that for trigonometric hierarchical terms, critical buckling load values get better when we increase the number

Table 5.6 Critical Buckling Load Calculated by using Symmetric Trigonometric and Polynomial Hierarchical Terms for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0]_{16s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Symmetric Trigonometric 1 HFEM term		*Symmetric Trigonometric 2 HFEM terms*		*Symmetric Trigonometric 3 HFEM terms*		*Symmetric Trigonometric 4 HFEM terms*	
1	6	1373	8	686.39	10	650.45	12	619.02
2	11	634.15	14	628.08	17	625.14	20	624.99
3	16	620.55	20	620.42	24	619.74	28	619.16
4	21	618.31	26	617.70	31	616.94	36	616.13
	Symmetric Polynomial 1 HFEM term		*Symmetric Polynomial 2 HFEM terms*		*Symmetric Polynomial 3 HFEM terms*		*Symmetric Polynomial 4 HFEM terms*	
1	6	1254.8	8	1135.5	10	1146.8	12	
2	11	631.07	14	627.36	17	733.23	20	624.78
3	16	620.73	20	619.69	24	660.01	28	617.95
4	21	618.39	26	617.46	31	640.39	36	616.07

Approximate Solution by Ritz Method: Critical Buckling Load = 621.15 *lb*

of hierarchical terms with both tangential (v) and radial (w) displacement interpolation functions. Four symmetric trigonometric hierarchical terms associated with both tangential (v) and radial (w) displacement interpolation functions produce best convergence among this group of symmetric combinations. Generally, as we increase the number of hierarchical terms with tangential (v) and radial (w) displacement functions, the convergence for the critical buckling load values gets better. But three symmetric polynomial hierarchical terms with both the displacement functions produce least good results among all the combinations of this group. The best result is produced by the combination ($v_4 - w_4$) by using 4-elements mesh.

The results for the group of non-symmetric combinations ($v_0 - w_n$) of both trigonometric and polynomial hierarchical terms are given in Table 5.7. The results show that for the first few elements the convergence for the critical buckling load values gets better each time when the number of trigonometric hierarchical terms associated with the radial displacement (w) interpolation function is increased. Polynomial hierarchical terms give similar results for all the combinations of this group. All the combinations of this group give the same result after the 5-elements mesh. Critical buckling load value given by polynomial hierarchical formulation is also the same as given by trigonometric hierarchical formulation.

Table 5.7 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_0 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0]_{16s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 0v-1w HFEM terms		*Non-Symmetric Trigonometric 0v-2w HFEM terms*		*Non-Symmetric Trigonometric 0v-3w HFEM terms*		*Non-Symmetric Trigonometric 0v-4w HFEM terms*	
1	5	26075	6	25198	7	5976.7	8	5940
2	10	1234.1	12	948.39	14	940.62	16	927.40
3	15	743.35	18	701.61	21	700.82	24	700.64
4	20	655.37	24	647.28	28	647.04	32	647.01
5	25	630.38	30	628.65	35	628.59	40	628.58
25	125	613.49	150	613.48	175	613.49	200	613.49
	Non-Symmetric Polynomial 0v-1w HFEM terms		*Non-Symmetric Polynomial 0v-2w HFEM terms*		*Non-Symmetric Polynomial 0v-3w HFEM terms*		*Non-Symmetric Polynomial 0v-4w HFEM terms*	
1	5	23366	6	4697.9	7		8	
2	10	1232.2	12	1220.8	14	1220.5	16	1220.4
3	15	743.41	18	742.27	21	742.21	24	742.21
4	20	655.39	24	655.04	28	655.09	32	655.09
5	25	630.38	30	630.29	35	630.29	40	630.29
25	125	613.49	150	613.49	175	613.49	200	613.49

Approximate Solution by Ritz Method: Critical Buckling Load = 621.15 *lb*

The results produced by the group of combinations ($v_1 - w_n$) are given in Table 5.8. For the first few elements, the results show an improvement each time we increase the number of trigonometric hierarchical terms associated with the radial displacement (w) interpolations function. As the number of elements is increased the best convergence for the critical buckling load values is obtained by the combination ($v_1 - w_0$). Once again, similar trend is followed by the polynomial hierarchical terms. For polynomial hierarchical terms, the best convergence for the critical buckling load value is obtained by the combination ($v_1 - w_0$). This group of combination uses 4-elements mesh to reach the approximate Ritz solution.

The results obtained by using the group of non-symmetric combinations ($v_2 - w_n$) of both trigonometric and polynomial hierarchical terms are given in Table 5.9. The results get better with the increase of number of hierarchical terms associated with the radial displacement (w) interpolation function. As the number of elements is increased the results for the combination ($v_2 - w_4$) of both trigonometric and polynomial hierarchical terms are best converged and 3-elements mesh is used to reach the approximate central deflection value given by the Ritz method. The critical buckling load values given by this group of combinations for trigonometric and polynomial hierarchical terms are 619.12 *lb* and 618.32 *lb* respectively.

Table 5.8 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_1 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0]_{16s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 1v-0w HFEM terms		*Non-Symmetric Trigonometric 1v-2w HFEM terms*		*Non-Symmetric Trigonometric 1v-3w HFEM terms*		*Non-Symmetric Trigonometric 1v-4w HFEM terms*	
1	5	0.48×10^6	7	1246.8	8	1233.5	9	1232.5
2	10	634.23	12	631.44	14	626.31	16	626.11
3	15	620.95	17	620.44	20	619.63	23	619.61
4	20	618.43	22	618.28	26	617.95	30	617.94
	Non-Symmetric Polynomial 1v-0w HFEM terms		*Non-Symmetric Polynomial 1v-2w HFEM terms*		*Non-Symmetric Polynomial 1v-3w HFEM terms*		*Non-Symmetric Polynomial 1v-4w HFEM terms*	
1	5	0.45×10^6	7	738.54	8	0.50×10^6	9	
2	10	632.66	12	625.06	14	622.71	16	621.89
3	15	620.96	17	619.16	20	618.38	23	618.13
4	20	618.48	22	617.76	26	617.53	30	617.45

Approximate Solution by Ritz Method: Critical Buckling Load = 621.15 *lb*

The results for the group of combinations ($v_3 - w_n$) of both trigonometric and polynomial hierarchical terms are given in Table 5.10. The results show almost similar results for all the combinations of this group for trigonometric hierarchical terms. As the number of elements is increased the result for the combination ($v_3 - w_0$) gets better. Same is the case with polynomial hierarchical terms as well. The best results among all the combinations of this group are produced by the combination ($v_3 - w_{10}$). The critical buckling load value is 617.64 *lb* for polynomial hierarchical terms which is better than that of the value given by trigonometric hierarchical terms, which is 619.60 *lb* after using 3-elements mesh.

The results for the group of combinations ($v_4 - w_n$) of both trigonometric and polynomial hierarchical terms are given in Table 5.11. These results show that the combination ($v_4 - w_1$) produces better results than the other two combinations of this group for both trigonometric and polynomial hierarchical formulations. The critical buckling load value given by this combination of both trigonometric and polynomial hierarchical formulations is 618.97 *lb* and 616.59 *lb* respectively. All the combinations show little difference in the critical buckling load values for polynomial hierarchical terms. The critical buckling load value is obtained by using just 3-elements mesh.

Table 5.9 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_2 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0]_{16s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 2v-0w HFEM terms		*Non-Symmetric Trigonometric 2v-1w HFEM terms*		*Non-Symmetric Trigonometric 2v-3w HFEM terms*		*Non-Symmetric Trigonometric 2v-4w HFEM terms*	
1	6		7	1371.6	9	658.70	10	631.76
2	12	634.00	13	627.53	16	625.79	18	623.82
3	18	620.30	19	620.48	23	619.21	26	619.12
	Non-Symmetric Polynomial 2v-0w HFEM terms		*Non-Symmetric Polynomial 2v-1w HFEM terms*		*Non-Symmetric Polynomial 2v-3w HFEM terms*		*Non-Symmetric Polynomial 2v-4w HFEM terms*	
1	7		8		9		11	
2	14	631.78	15	625.18	16	627.30	19	624.31
3	21	619.81	22	617.64	23	619.64	27	618.32

Approximate Solution by Ritz Method: Critical Buckling Load = 621.15 *lb*

Table 5.10 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_3 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0]_{16s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 3v-0w HFEM terms		*Non-Symmetric Trigonometric 3v-1w HFEM terms*		*Non-Symmetric Trigonometric 3v-2w HFEM terms*		*Non-Symmetric Trigonometric 3v-4w HFEM terms*	
1	7		8	1259.7	9	652.24	11	649.65
2	14	631.75	15	625.00	16	625.13	19	625.04
3	21	619.74	22	619.60	23	619.60	27	619.72
	Non-Symmetric Polynomial 3v-0w HFEM terms		*Non-Symmetric Polynomial 3v-1w HFEM terms*		*Non-Symmetric Polynomial 3v-2w HFEM terms*		*Non-Symmetric Polynomial 3v-4w HFEM terms*	
1	7		8		9		11	
2	14	631.78	15	625.18	16	627.30	19	624.31
3	21	619.81	22	617.64	23	619.64	27	618.32

Approximate Solution by Ritz Method: Critical Buckling Load = 621.15 *lb*

Table 5.11 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_4 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0]_{16s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 4v-0w HFEM terms		*Non-Symmetric Trigonometric 4v-1w HFEM terms*		*Non-Symmetric Trigonometric 4v-2w HFEM terms*		*Non-Symmetric Trigonometric 4v-3w HFEM terms*	
1	8		9	1258.6	10	620.72	11	
2	16	631.35	17	624.86	18	624.95	19	625.05
3	24	619.12	25	618.97	26	618.97	27	619.19
	Non-Symmetric Polynomial 4v-0w HFEM terms		*Non-Symmetric Polynomial 4v-1w HFEM terms*		*Non-Symmetric Polynomial 4v-2w HFEM terms*		*Non-Symmetric Polynomial 4v-3w HFEM terms*	
1	8	0.22×10^7	9		10		11	
2	16	631.23	17	622.64	18	624.99	19	623.64
3	24	619.18	25	616.59	26	618.67	27	617.95

Approximate Solution by Ritz Method: Critical Buckling Load = 621.15 *lb*

5.3.1.2 $[90]_{16s}$ *Laminate*

The critical buckling load values given by Ritz method and the conventional FEM solution are given in Table 5.12. The results for Ritz method give the same values of critical buckling load at the 11^{th} and 13^{th} value of m which is an excellent convergence. The conventional FEM solution also shows a very good convergence of the results after the 6-elements mesh. The difference in the values of critical buckling load given by these two methods is very small i.e. -0.34 %.

Table 5.12 Ritz Method and Conventional FEM Solutions for $[90]_{16s}$ Laminate with Fixed-Free Boundary Condition

$[90]_{16s}$ Laminate					
Ritz Method		Conventional FEM Solution using 8 D.O.F. Composite Curved Beam Element			
Value of m	Critical Buckling Load (lb)	Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Error (%)
1	11444	1	4	59291	99.93
2	2727.6	2	8	77.98	50.55
3	336.44	3	12	46.20	16.53
4	152.35	4	16	40.53	4.85
5	66.57	5	20	38.97	1.04
6	48.68	6	24	38.43	-0.34
7	42.70	7	28	38.20	-0.93
8	40.99	8	32	38.16	-1.04
9	39.60	9	36	38.14	-1.1
10	39.37	10	40	38.06	-1.33
11	38.56	11	44	38.01	-1.45
12	38.67				
13	38.56				

The results for the critical buckling loads calculated by using symmetric trigonometric and polynomial hierarchical terms are given in Table 5.13. The results for

Table 5.13 Critical Buckling Load Calculated by using Symmetric Trigonometric and Polynomial Hierarchical Terms for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[90]_{16s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Symmetric Trigonometric 1 HFEM term		*Symmetric Trigonometric 2 HFEM terms*		*Symmetric Trigonometric 3 HFEM terms*		*Symmetric Trigonometric 4 HFEM terms*	
1	6	84.75	8	42.37	10	40.15	12	38.21
2	11	39.14	14	38.77	17	38.59	20	38.58
3	16	38.30	20	38.30				
4	21	38.17						
	Symmetric Polynomial 1 HFEM term		*Symmetric Polynomial 2 HFEM terms*		*Symmetric Polynomial 3 HFEM terms*		*Symmetric Polynomial 4 HFEM terms*	
1	6	77.45	8	70.09	10	71.40	12	
2	11	38.95	14	38.72	17	45.26	20	38.56
3	16	38.31	20	38.25	24	40.74		
4	21	38.17			31	39.53		
5					38	39.06		
6					45	38.82		
7					52	38.69		
8					59	38.54		

Approximate Solution by Ritz Method: Critical Buckling Load = 38.56 *lb*

the symmetric trigonometric hierarchical terms show that the convergence for the critical buckling load values gets better as we increase the number of hierarchical terms with the tangential (v) and radial (w) displacement functions. The results for three and four symmetric hierarchical terms are almost similar. The critical buckling load value calculated for this group of symmetric combinations reaches Ritz solution using 2-elements mesh. The results for the symmetric polynomial hierarchical terms show that four symmetric polynomial hierarchical terms give best results in this group among all the other symmetric combinations. One and two symmetric hierarchical terms give better results than that of three symmetric hierarchical terms. Critical bulking load value given by this group of combinations is same as given by symmetric trigonometric hierarchical terms and uses the same number of elements.

The results for the group of combinations ($v_0 - w_n$) of both trigonometric and polynomial hierarchical terms are given in Table 5.14. Both trigonometric and polynomial hierarchical terms show great similarity of the results given by all the combinations of this group. For the first few elements the results get better when we increase the number of hierarchical terms associated with the radial displacement (w) interpolation function of both trigonometric and polynomial hierarchical terms. As the number of elements is increased with both trigonometric and polynomial hierarchical terms, the results converge to a single value and the final result given by both hierarchical formulations is the same. Both polynomial and trigonometric hierarchical formulations use 6-elements mesh to reach the approximate Ritz Solution.

Table 5.14 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_0 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[90]_{16s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 0v-1w HFEM terms		*Non-Symmetric Trigonometric 0v-2w HFEM terms*		*Non-Symmetric Trigonometric 0v-3w HFEM terms*		*Non-Symmetric Trigonometric 0v-4w HFEM terms*	
1	5	1609.7	6	1555.5	7	368.96	8	366.7
2	10	76.18	12	58.54	14	58.06	16	57.25
3	15	45.88	18	43.31	21	43.26	24	43.25
4	20	40.45	24	39.95	28	39.94	32	39.94
5	25	38.91	30	38.80	35	38.80	40	38.80
6	30	38.40	36	38.38	42	38.37	48	38.37
	Non-Symmetric Polynomial 0v-1w HFEM terms		*Non-Symmetric Polynomial 0v-2w HFEM terms*		*Non-Symmetric Polynomial 0v-3w HFEM terms*		*Non-Symmetric Polynomial 0v-4w HFEM terms*	
1	5	1442.4	6	290.01	7		8	
2	10	76.06	12	75.36	14	75.34	16	75.34
3	15	45.89	18	45.82	21	45.81	24	45.81
4	20	40.45	24	40.43	28	40.43	32	40.43
5	25	38.91	30	38.91	35	38.91	40	38.91
6	30	38.40	36	38.40	42	38.40	48	38.40

Approximate Solution by Ritz Method: Critical Buckling Load = 38.56 *lb*

The results for the group of combinations ($v_1 - w_n$) are given in Table 5.15. These results show that when the number of hierarchical terms is increased with the radial displacement (w) function, the value of critical buckling load gets better. Convergence for the polynomial hierarchical terms is better than that of the trigonometric hierarchical terms. The results keep on getting better with the increase of number of elements. All the combinations of the polynomial hierarchical terms give results that show little difference in critical buckling load values. However, the combination ($v_1 - w_3$) gives better results than all the other combinations of this group. This combination uses 3-elements mesh to get reach the approximate central deflection given by the Ritz method.

The results for the group of combinations ($v_2 - w_n$) are given in Table 5.16. These results are shown for both trigonometric and polynomial hierarchical interpolation functions. The values of critical buckling load associated with the polynomial hierarchical terms start with better convergence than the values given by the trigonometric hierarchical terms. The best critical buckling load value is given by the combination ($v_2 - w_4$). This combination uses almost 2-elements mesh to reach the approximate Ritz solution which is a great improvement in the results compared to the conventional FEM where 11-elements mesh were used to reach the approximate Ritz solution.

Table 5.15 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_1 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[90]_{16s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 1v-0w HFEM terms		*Non-Symmetric Trigonometric 1v-2w HFEM terms*		*Non-Symmetric Trigonometric 1v-3w HFEM terms*		*Non-Symmetric Trigonometric 1v-4w HFEM terms*	
1	5	29665	7	76.96	8	76.15	9	76.08
2	10	39.15	12	38.98	14	38.66	16	38.65
3	15	38.33	17	38.30	20	38.25	23	38.25
	Non-Symmetric Polynomial 1v-0w HFEM terms		*Non-Symmetric Polynomial 1v-2w HFEM terms*		*Non-Symmetric Polynomial 1v-3w HFEM terms*		*Non-Symmetric Polynomial 1v-4w HFEM terms*	
1	5	28352	7	45.59	8	31164	9	
2	10	39.05	12	38.58	14	38.44	16	38.39
3	15	38.33	17	38.22	20	38.17	23	38.10

Approximate Solution by Ritz Method: Critical Buckling Load = 38.56 *lb*

Table 5.16 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_2 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[90]_{16s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 2v-0w HFEM terms		*Non-Symmetric Trigonometric 2v-1w HFEM terms*		*Non-Symmetric Trigonometric 2v-3w HFEM terms*		*Non-Symmetric Trigonometric 2v-4w HFEM terms*	
1	6		7	84.67	9	40.66	10	39.00
2	12	39.13	13	38.73	16	38.63	18	38.51
3	18	38.29	19	38.30	23	38.22	26	38.22
	Non-Symmetric Polynomial 2v-0w HFEM terms		*Non-Symmetric Polynomial 2v-1w HFEM terms*		*Non-Symmetric Polynomial 2v-3w HFEM terms*		*Non-Symmetric Polynomial 2v-4w HFEM terms*	
1	6		7	49.00	9		10	
2	12	39.03	13	38.78	16	38.57	18	38.48
3	18	38.29	19	38.22	23	38.20	26	38.16

Approximate Solution by Ritz Method: Critical Buckling Load = 38.56 *lb*

The results for the group of combinations ($v_3 - w_n$) are given in Table 5.17. Once again, the results converge to a single value for the trigonometric hierarchical interpolation functions right after the 1st element for all the combinations of this group. The convergence for the trigonometric hierarchical terms is better than that of the polynomial hierarchical terms. Critical buckling load values given by the polynomial hierarchical terms differ slightly with each other. The best value of critical buckling load given by this group for both trigonometric and polynomial hierarchical terms is by the combination ($v_3 - w_0$) and is reached by using 2-elements mesh.

The results for the group of combinations ($v_4 - w_n$) of both trigonometric and polynomial hierarchical terms are given in Table 5.18. The convergence for polynomial hierarchical terms is better than that of trigonometric hierarchical terms. Once again the best value of critical buckling load is obtained when we use no trigonometric and polynomial hierarchical term with the radial displacement (w) interpolation function. So the combination ($v_4 - w_0$) gives the best result in this group for both trigonometric and polynomial hierarchical terms. The critical buckling load value given by this combination is exactly 38.56 *lb* and is obtained by using 2-elements mesh.

Table 5.17 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_3 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[90]_{16s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 3v-0w HFEM terms		*Non-Symmetric Trigonometric 3v-1w HFEM terms*		*Non-Symmetric Trigonometric 3v-2w HFEM terms*		*Non-Symmetric Trigonometric 3v-4w HFEM terms*	
1	7		8	77.76	9	40.26	11	40.10
2	14	39.00	15	38.58	16	38.59	19	38.58
3	21	38.25	22	38.25	23	38.25	27	38.25
	Non-Symmetric Polynomial 3v-0w HFEM terms		*Non-Symmetric Polynomial 3v-1w HFEM terms*		*Non-Symmetric Polynomial 3v-2w HFEM terms*		*Non-Symmetric Polynomial 3v-4w HFEM terms*	
1	7		8		9		11	
2	14	39.00	15	38.59	16	38.72	19	38.54
3	21	38.26	22	38.12	23	38.25	27	38.17

Approximate Solution by Ritz Method: Critical Buckling Load = 38.56 *lb*

Table 5.18 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_4 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[90]_{16s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 4v-0w HFEM terms		*Non-Symmetric Trigonometric 4v-1w HFEM terms*		*Non-Symmetric Trigonometric 4v-2w HFEM terms*		*Non-Symmetric Trigonometric 4v-3w HFEM terms*	
1	8		9	77.69	10	38.31	11	38.21
2	16	38.97	17	38.57	18	38.58	19	38.58
3	24	38.22						
	Non-Symmetric Polynomial 4v-0w HFEM terms		*Non-Symmetric Polynomial 4v-1w HFEM terms*		*Non-Symmetric Polynomial 4v-2w HFEM terms*		*Non-Symmetric Polynomial 4v-3w HFEM terms*	
1	8	0.13×10^6	9		10		11	
2	16	38.96	17	38.43	18	38.58	19	38.49
3	24	38.22	25	38.06	26	38.19	27	38.14

Approximate Solution by Ritz Method: Critical Buckling Load = 38.56 *lb*

5.3.1.3 $[+45/-45]_{8s}$ *Laminate*

The results for the comparison between Ritz method and the conventional FEM solutions for the $[+45/-45]_{8s}$ laminate are given in Table 5.19. At the 13th value of m Ritz method gives very good convergence for the critical buckling load value. The conventional FEM solution converged almost to a single value at the 18-elements mesh. The % age difference between the two solutions is -1.5 % which is good for a curved beam problem. Approximate solution for the critical buckling load value given by the Ritz method is 194.23 *lb*.

Table 5.19 Ritz Method and Conventional FEM Solutions for $[+45/-45]_{8s}$ Laminate with Fixed-Free Boundary Condition

$[+45/-45]_{8s}$ Laminate					
Ritz Method		**Conventional FEM Solution using 8 D.O.F. Composite Curved Beam Element**			
Value of m	Critical Buckling Load (lb)	Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Error (%)
1	0.57×10^6	1	4	29655	99.35
2	13642	2	8	390.02	50.52
3	1682.8	3	12	231.22	16.53
4	762.00	4	16	202.93	4.89
5	333.00	5	20	194.70	0.87
6	243.50	6	24	192.22	-0.41
7	213.61	7	28	191.15	-0.97
8	205.02	8	32	190.71	-1.2
9	198.07	9	36	190.45	-1.34
10	196.83	10	40	190.54	-1.29
11	193.52	15	60	190.47	-1.33
12	194.10	16	64	190.29	-1.42
13	194.23	17	68	190.15	-1.5
		18	72	190.16	-1.49

The results for the group of symmetric combinations of both trigonometric and polynomial hierarchical terms for $[+45/-45]_{8s}$ laminate are given in Table 5.20. The results show a continuous improvement in the results when more symmetric trigonometric hierarchical terms are added to both tangential (v) and radial (w) displacement function. Four symmetric trigonometric hierarchical terms give the best convergence among all the other symmetric combinations of this group. For polynomial hierarchical terms, once again four symmetric polynomial hierarchical terms provide the best combination for the convergence of the critical buckling load values. Three symmetric polynomial hierarchical terms provide the least good combination in comparison to the trigonometric hierarchical terms where one symmetric trigonometric hierarchical term was the least good combination. Both trigonometric and polynomial hierarchical formulations use 4-elements mesh to reach the approximate Ritz solution.

The results for the group of non-symmetric combinations ($v_0 - w_n$) of both trigonometric and polynomial hierarchical terms for $[+45/-45]_{8s}$ laminate are given in Table 5.21. For trigonometric hierarchical terms, the results get better when we increase the number of hierarchical terms associated with the radial displacement (w) function from zero to four. All the combinations for the trigonometric hierarchical formulation give almost similar results. Polynomial hierarchical terms follow the same trend and all the combinations again converge to a single value. Both formulation use 5-elements mesh to reach the approximate critical buckling load value given by the Ritz solution.

Table 5.20 Critical Buckling Load Calculated by using Symmetric Trigonometric and Polynomial Hierarchical Terms for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[+45/-45]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Symmetric Trigonometric 1 HFEM term		*Symmetric Trigonometric 2 HFEM terms*		*Symmetric Trigonometric 3 HFEM terms*		*Symmetric Trigonometric 4 HFEM terms*	
1	6	423.93	8	211.94	10	200.84	12	191.13
2	11	195.80	14	193.93	17	193.02	20	192.98
3	16	191.61	20	191.56	24	191.36	28	191.18
4	21	190.91	26	190.72	31	190.49	36	190.24
	Symmetric Polynomial 1 HFEM term		*Symmetric Polynomial 2 HFEM terms*		*Symmetric Polynomial 3 HFEM terms*		*Symmetric Polynomial 4 HFEM terms*	
1	6	387.43	8	350.61	10	356.70	12	
2	11	194.85	14	193.71	17	226.40	20	192.91
3	16	191.66	20	191.34	24	203.79	28	190.81
4	21	190.94	26	190.65	31	197.73	36	190.22

Approximate Solution by Ritz Method: Critical Buckling Load = 194.23 *lb*

Table 5.21 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_0 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[+45/-45]_{8s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 0v-1w HFEM terms		*Non-Symmetric Trigonometric 0v-2w HFEM terms*		*Non-Symmetric Trigonometric 0v-3w HFEM terms*		*Non-Symmetric Trigonometric 0v-4w HFEM terms*	
1	5	8051.0	6	7780.2	7	1845.4	8	1834.1
2	10	381.05	12	292.83	14	290.43	16	286.35
3	15	229.52	18	216.63	21	216.39	24	216.34
4	20	194.64	24	199.86	28	199.78	32	199.78
5	25	192.09	30	194.11	35	194.09	40	194.61
	Non-Symmetric Polynomial 0v-1w HFEM terms		*Non-Symmetric Polynomial 0v-2w HFEM terms*		*Non-Symmetric Polynomial 0v-3w HFEM terms*		*Non-Symmetric Polynomial 0v-4w HFEM terms*	
1	5	7214.5	6	1450.5	7		8	
2	10	380.46	12	376.95	14	376.85	16	376.83
3	15	229.54	18	229.19	21	229.17	24	229.17
4	20	202.36	24	202.25	28	202.25	32	202.25
5	25	194.64	30	194.61	35	194.61	40	194.61

Approximate Solution by Ritz Method: Critical Buckling Load = 194.23 *lb*

The results for the group of non-symmetric combinations ($v_1 - w_n$) are given in Table 5.22. These results show that the convergence gets better when we add trigonometric hierarchical terms associated with the radial displacement (w) function. But when the number of elements is increased the convergence for the combination ($v_1 - w_4$) becomes the best among all the other combinations of this group. Polynomial hierarchical terms once again tend to converge to a single value with a small difference in between them. The same combination with four polynomial hierarchical terms added to the radial displacement (w) function gives the best convergence and uses 3-elements mesh to give the best convergence.

The results for the group of combinations ($v_2 - w_n$) are given in Table 5.23. The results tend to converge to a single value for trigonometric hierarchical terms. When we increase the number of elements, the combination with most number of trigonometric hierarchical terms associated with the radial displacement (w) interpolation function i.e. ($v_2 - w_4$) gives the best convergence for the critical buckling load values. Polynomial hierarchical terms also follow the same trend and the combination ($v_2 - w_4$) gives the best value of the critical buckling load. The 3-elements mesh is used for this group of combinations to get the desired results.

Table 5.22 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_1 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[+45/-45]_{8s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 1v-0w HFEM terms		*Non-Symmetric Trigonometric 1v-2w HFEM terms*		*Non-Symmetric Trigonometric 1v-3w HFEM terms*		*Non-Symmetric Trigonometric 1v-4w HFEM terms*	
1	5	0.14×10^6	7	384.97	8	380.88	9	380.56
2	10	195.83	12	194.97	14	193.38	16	193.32
3	15	191.73	17	191.57	20	191.32	23	191.31
	Non-Symmetric Polynomial 1v-0w HFEM terms		*Non-Symmetric Polynomial 1v-2w HFEM terms*		*Non-Symmetric Polynomial 1v-3w HFEM terms*		*Non-Symmetric Polynomial 1v-4w HFEM terms*	
1	5	0.14×10^6	7	228.04	8	0.15×10^6	9	
2	10	195.35	12	193.00	14	192.27	16	192.02
3	15	191.73	17	191.18	20	190.94	23	190.86

Approximate Solution by Ritz Method: Critical Buckling Load = 194.23 *lb*

Table 5.23 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_2 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[+45/-45]_{8s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 2v-0w HFEM terms		*Non-Symmetric Trigonometric 2v-1w HFEM terms*		*Non-Symmetric Trigonometric 2v-3w HFEM terms*		*Non-Symmetric Trigonometric 2v-4w HFEM terms*	
1	6		7	423.50	9	203.38	10	195.07
2	12	195.76	13	193.76	16	193.22	18	192.62
3	18	191.53	19	191.58	23	191.19	26	191.17
	Non-Symmetric Polynomial 2v-0w HFEM terms		*Non-Symmetric Polynomial 2v-1w HFEM terms*		*Non-Symmetric Polynomial 2v-3w HFEM terms*		*Non-Symmetric Polynomial 2v-4w HFEM terms*	
1	6		7	245.00	9		10	
2	12	195.22	13	193.97	16	192.94	18	192.48
3	18	191.56	19	191.19	23	191.08	26	190.91

Approximate Solution by Ritz Method: Critical Buckling Load = 194.23 *lb*

The results for the group of non-symmetric combinations ($v_3 - w_n$) for the $[+45/-45]_{8s}$ laminate are given in Table 5.24. For trigonometric hierarchical terms, as all the combinations of this group give similar results and use 3-elemnts mesh convergence to give the critical buckling load value. For polynomial hierarchical terms, when we increase the number of polynomial hierarchical terms the results get better. But afterwards, the combination ($v_3 - w_1$) gives the best value of critical buckling load which is slightly less than the value given by the trigonometric hierarchical terms.

The results for the group of combinations ($v_4 - w_n$) of both trigonometric and polynomial hierarchical terms are given in Table 5.25. The trend followed by this group of combinations is the same as in the previous case for both trigonometric and polynomial hierarchical terms. The best critical buckling load value is given by the combination ($v_4 - w_1$) for polynomial hierarchical formulation while all the combinations of this group for trigonometric hierarchical formulation give similar results. Generally, the convergence gets better when we use more hierarchical terms with the tangential displacement (v) function.

Table 5.24 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_3 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[+45/-45]_{8s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 3v-0w HFEM terms		*Non-Symmetric Trigonometric 3v-1w HFEM terms*		*Non-Symmetric Trigonometric 3v-2w HFEM terms*		*Non-Symmetric Trigonometric 3v-4w HFEM terms*	
1	7		8	388.95	9	201.39	11	200.59
2	14	195.06	15	192.98	16	193.02	19	192.99
3	21	191.36	22	191.31	23	191.31	27	191.35
	Non-Symmetric Polynomial 3v-0w HFEM terms		*Non-Symmetric Polynomial 3v-1w HFEM terms*		*Non-Symmetric Polynomial 3v-2w HFEM terms*		*Non-Symmetric Polynomial 3v-4w HFEM terms*	
1	7		8		9		11	
2	14	195.07	15	193.04	16	193.69	19	192.77
3	21	191.38	22	190.71	23	191.32	27	190.92

Approximate Solution by Ritz Method: Critical Buckling Load = 194.23 *lb*

Table 5.25 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_4 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[+45/-45]_{8s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 4v-0w HFEM terms		*Non-Symmetric Trigonometric 4v-1w HFEM terms*		*Non-Symmetric Trigonometric 4v-2w HFEM terms*		*Non-Symmetric Trigonometric 4v-3w HFEM terms*	
1	8		9		10	191.66	11	191.16
2	16	194.94	17	192.94	18	192.96	19	193
3	24	191.16	25	191.12	26	191.12	27	191.18
	Non-Symmetric Polynomial 4v-0w HFEM terms		*Non-Symmetric Polynomial 4v-1w HFEM terms*		*Non-Symmetric Polynomial 4v-2w HFEM terms*		*Non-Symmetric Polynomial 4v-3w HFEM terms*	
1	8		9		10		11	
2	16	194.9	17	192.25	18	192.98	19	192.56
3	24	191.18	25	190.38	26	191.03	27	190.8

Approximate Solution by Ritz Method: Critical Buckling Load = 194.23 *lb*

5.3.1.4 $[0/90/+45/-45]_{4s}$ *Quasi-Isotropic Laminate*

The results for the comparison between Ritz method and the conventional FEM solutions are given in Table 5.26. Conventional FEM results show good convergence at the 9-elements mesh. Ritz solution seems to be converging at m = 14.

Table 5.26 Ritz Method and Conventional FEM Solutions for Quasi-Isotropic Laminate with Fixed-Free Boundary Condition

Quasi-Isotropic Laminate					
Ritz Method		Conventional FEM Solution using 8 D.O.F. Composite Curved Beam Element			
Value of m	Critical Buckling Load (lb)	Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Error (%)
1	0.778×10^6	1	4	0.403×10^6	99.99
2	18619	2	8	579.07	56.36
3	2398.1	3	12	345.35	24.8
4	1093.7	4	16	304.14	10.45
5	491.02	5	20	292.59	-0.41
6	363.47	6	24	289.30	-1.58
7	319.95	7	28	288.18	-3.29
8	307.85	8	32	287.32	-4.19
9	297.44	9	36	287.24	-3.04
10	295.83				
11	295.73				
12	287.73				
14	287.90				

The results for the symmetric trigonometric and polynomial hierarchical formulations for the quasi-isotropic laminate are given in Table 5.27. These results show an improvement in the results when the number of trigonometric hierarchical terms is added to the radial displacement (w) interpolation function. All the four symmetric combinations show very good convergence. This convergence gets better with the addition of each symmetric trigonometric hierarchical term. The combination with maximum number of symmetric hierarchical terms associated with the tangential (v) and radial (w) displacement interpolation functions i.e. (v_4 $- w_4$) gives the best value of critical buckling load which is 286.52 *lb*. This critical buckling load value is obtained by using just 4-elements mesh compared to 9-elements mesh for the conventional FEM. Symmetric polynomial hierarchical terms also follow the same trend. The critical buckling load value given by the symmetric polynomial hierarchical terms is 286.53 *lb* and is given by the same combination (v_4 $- w_4$) using 4-elements mesh. Generally, results given by both symmetric trigonometric and polynomial hierarchical terms show little difference in the critical buckling load values.

The results given by the non-symmetric combinations (v_0 $- w_n$) of both trigonometric and polynomial hierarchical terms are given in Table 5.28. These results show that when we increase the number of trigonometric hierarchical terms associated with the radial displacement (w) interpolation function, we see an improvement in the results. As the number of elements is increased, the results given by all the combinations of this group tend to converge to a single value, so all the combinations of this group give the same result. The critical buckling load given by this group of combinations is 287.08 *lb* and is given by all the combinations of the group (v_0 $- w_n$). Polynomial hierarchical terms also follow the same trend and all the combinations give the same result while converging to a single value of critical buckling load which is 287.09 *lb*.

Table 5.27 Critical Buckling Load Calculated by using Symmetric Trigonometric and Polynomial Hierarchical Terms for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for Quasi Isotropic Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Symmetric Polynomial 1 HFEM term		*Symmetric Polynomial 2 HFEM terms*		*Symmetric Polynomial 3 HFEM terms*		*Symmetric Polynomial 4 HFEM terms*	
1	6	583.91	8	-	10	-	12	-
2	11	293.40	14	291.72	17	291.02	20	290.45
3	16	288.78	20	288.26	24	287.79	28	287.39
4	21	287.69	26	287.22	31	286.83		
5	26	287.23						
	Symmetric Trigonometric 1 HFEM term		*Symmetric Trigonometric 2 HFEM terms*		*Symmetric Trigonometric 3 HFEM terms*		*Symmetric Trigonometric 4 HFEM terms*	
1	6	634.60	8	316.28	10	300.98	12	287.94
2	11	294.69	14	292.07	17	290.81	20	290.72
3	16	288.69	20	288.61	24	288.27	28	287.97
4	21	287.65	26	287.33	31	286.94	36	286.52
5	26	287.22						

Approximate Solution by Ritz Method: Critical Buckling Load = 287.73 *lb*

Table 5.28 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms (v_0 $- w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for Quasi Isotropic Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 0v-1w HFEM terms		*Non-Symmetric Trigonometric 0v-2w HFEM terms*		*Non-Symmetric Trigonometric 0v-3w HFEM terms*		*Non-Symmetric Trigonometric 0v-4w HFEM terms*	
1	5	10993	6	10624	7	2591.5	8	2575.7
2	10	565.04	12	435.75	14	432.42	16	427.40
3	15	342.76	18	324.84	21	324.49	24	324.42
4	20	303.61	24	300.29	28	300.19	32	300.18
5	25	292.73	30	292.04	35	292.02	40	292.01
6	30	289.20	84	289.04	98	289.03	112	289.03
7	35	287.91	90	287.87	105	287.86	120	287.86
8	40	287.37	96	287.35	112	287.35	128	287.35
	Non-Symmetric Polynomial 0v-1w HFEM terms		*Non-Symmetric Polynomial 0v-2w HFEM terms*		*Non-Symmetric Polynomial 0v-3w HFEM terms*		*Non-Symmetric Polynomial 0v-4w HFEM terms*	
1	5	9856.9	6	2065.7	7	-	8	-
2	10	565.17	12	559.43	14	559.27	16	559.25
3	15	342.79	18	342.27	21	342.25	24	342.25
4	20	303.62	24	303.47	28	303.47	32	303.47
5	25	292.73	30	292.69	35	292.69	40	292.69
6	30	289.20	84	289.19	98	289.19	112	289.19
7	35	287.91	90	287.91	105	287.91	120	287.91
8	40	287.37	96	287.37	112	287.37	128	287.37

Approximate Solution by Ritz Method: Critical Buckling Load = 287.73 *lb*

The results given by the non-symmetric combinations ($v_1 - w_n$) for the quasi-isotropic laminate are given in Table 5.29. The results show exactly the same trend as observed in the last group of combinations ($v_0 - w_n$). These results also show an improvement each time, we add a trigonometric or polynomial hierarchical term associated with the radial displacement (w) interpolation function. The combination ($v_1 - w_0$) for the polynomial hierarchical interpolation function used 6-elements mesh while all the other combinations of this group used 5-elements mesh to reach the approximate solution of critical buckling load. Results given by both polynomial and trigonometric hierarchical formulations do not differ much in the critical buckling load values.

The results for the group of combinations ($v_2 - w_n$) are given in Table 5.30. Once again for both trigonometric and polynomial hierarchical terms, trends are same for all the combinations of this group. The convergence for the critical buckling load values shows improvement when we increase the number of hierarchical terms with the radial displacement (w) function. With maximum number of trigonometric or polynomial hierarchical terms added to the radial displacement (w) interpolation function, we get the best convergence for the critical buckling load values. The critical buckling load value given by the combination ($v_2 - w_4$) for polynomial hierarchical terms is reached by using just 4-elements mesh. Results given by trigonometric hierarchical terms for all the combinations of this group show little difference among them.

Table 5.29 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_1 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for Quasi Isotropic Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 1v-0w HFEM terms		*Non-Symmetric Trigonometric 1v-2w HFEM terms*		*Non-Symmetric Trigonometric 1v-3w HFEM terms*		*Non-Symmetric Trigonometric 1v-4w HFEM terms*	
1	5	-	7	580.13	8	573.31	9	572.85
2	10	294.99	12	293.54	14	291.25	16	291.16
3	15	288.90	17	288.64	20	288.26	23	288.25
4	20	287.72	22	287.64	26	287.49	30	287.48
5	25	287.26	27	287.22	32	287.15	37	287.14
	Non-Symmetric Polynomial 1v-0w HFEM terms		*Non-Symmetric Polynomial 1v-2w HFEM terms*		*Non-Symmetric Polynomial 1v-3w HFEM terms*		*Non-Symmetric Polynomial 1v-4w HFEM terms*	
1	5	-	7	515.39	8	-	9	-
2	10	294.38	12	290.70	14	289.61	16	289.25
3	15	288.91	17	288.04	20	287.69	23	287.58
4	20	287.74	22	287.40	26	287.30	30	287.26
5	25	287.26	27	287.10	32	287.06	37	287.05

Approximate Solution by Ritz Method: Critical Buckling Load = 287.73 *lb*

Table 5.30 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_2 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for Quasi Isotropic Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 2v-0w HFEM terms		*Non-Symmetric Trigonometric 2v-1w HFEM terms*		*Non-Symmetric Trigonometric 2v-3w HFEM terms*		*Non-Symmetric Trigonometric 2v-4w HFEM terms*	
1	6	-	7	633.89	9	305.13	10	293.64
2	12	294.87	13	291.84	16	290.98	18	290.17
3	18	288.57	19	288.64	23	288.05	26	288.02
4	24	287.30	25	287.34	30	287.14	34	287.13
	Non-Symmetric Polynomial 2v-0w HFEM terms		*Non-Symmetric Polynomial 2v-1w HFEM terms*		*Non-Symmetric Polynomial 2v-3w HFEM terms*		*Non-Symmetric Polynomial 2v-4w HFEM terms*	
1	6	-	7	-	9	-	10	-
2	12	294.17	13	292.01	16	290.58	18	289.90
3	18	288.62	19	288.03	23	287.87	26	287.62
4	24	287.36	25	287.16	30	287.06	34	286.97

Approximate Solution by Ritz Method: Critical Buckling Load = 287.73 *lb*

The results given by the group of non-symmetric combinations ($v_3 - w_n$) of both trigonometric and polynomial hierarchical terms are given in Table 5.31. These results show that once again all the combinations of this group show an improvement in the results when we increase the number of hierarchical terms with the radial displacement (w) function. The final values of critical buckling load given by both trigonometric and polynomial hierarchical terms are obtained by the combinations ($v_3 - w_1$) and ($v_3 - w_4$) respectively by using 4-elements mesh.

The results for the group of non-symmetric combinations ($v_4 - w_n$) of both trigonometric and polynomial hierarchical formulations are given in Table 5.32. The trend of these results once again is similar to the one observed in the previous group of combinations. All the combinations of this group for both trigonometric and polynomial hierarchical terms show an improvement in the convergence of the critical buckling load values when more hierarchical terms are used with the radial displacement (w) interpolation function. The final value of the critical buckling load is given by the combination ($v_4 - w_1$) for both polynomial and trigonometric hierarchical formulations.

Table 5.31 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms $(v_3 - w_n)$ for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for Quasi Isotropic Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 3v-0w HFEM terms		*Non-Symmetric Trigonometric 3v-1w HFEM terms*		*Non-Symmetric Trigonometric 3v-2w HFEM terms*		*Non-Symmetric Trigonometric 3v-4w HFEM terms*	
1	7	-	8	385.96	9	301.78	11	300.62
2	14	293.91	15	290.74	16	290.80	19	290.76
3	21	288.29	22	288.20	23	288.20	27	288.26
4	28	286.94	29	286.88	30	286.88	35	286.94
	Non-Symmetric Polynomial 3v-0w HFEM terms		*Non-Symmetric Polynomial 3v-1w HFEM terms*		*Non-Symmetric Polynomial 3v-2w HFEM terms*		*Non-Symmetric Polynomial 3v-4w HFEM terms*	
1	7	-	8	-	9	-	11	-
2	14	293.93	15	290.56	16	291.70	19	290.30
3	21	288.32	22	287.26	23	288.23	27	287.59
4	28	286.97			30	287.07	35	286.74

Approximate Solution by Ritz Method: Critical Buckling Load = 287.73 *lb*

Table 5.32 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_4 - w_n$) for Fixed-Free Boundary Condition

8 D.O.F. Composite Curved Beam Element for Quasi Isotropic Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 4v-0w HFEM terms		*Non-Symmetric Trigonometric 4v-1w HFEM terms*		*Non-Symmetric Trigonometric 4v-2w HFEM terms*		*Non-Symmetric Trigonometric 4v-3w HFEM terms*	
1	8	-	9	585.42	10	288.69	11	287.96
2	16	293.71	17	290.66	18	290.69	19	290.75
3	24	287.97	25	287.87	26	287.87	27	287.98
4	32	286.52	33	286.42	34	286.46	35	286.53
	Non-Symmetric Polynomial 4v-0w HFEM terms		*Non-Symmetric Polynomial 4v-1w HFEM terms*		*Non-Symmetric Polynomial 4v-2w HFEM terms*		*Non-Symmetric Polynomial 4v-3w HFEM terms*	
1	8	-	9	-	10	-	11	-
2	16	293.64	17	289.36	18	290.62	19	289.97
3	24	288.00	25	286.77	26	287.76	27	287.41
4	32	286.58			34	286.85	35	286.69

Approximate Solution by Ritz Method: Critical Buckling Load = 287.73 *lb*

5.3.2 The Effect of Boundary Conditions

To consider the effect of different boundary conditions on the critical buckling load values given by HFEM, the same composite curved beam as used in section 4.4 will now be analyzed in this section with fixed-fixed boundary condition.

All the parameters which were used for the previous case of Fixed-Free boundary condition will be used in this case except the radius i.e. $R = 15$ *in*. Approximate solution will be obtained by the Ritz Method and then we will generalize our results by using the hierarchical finite element formulation.

5.3.2.1 $[0]_{16s}$ *Laminate*

The comparison of the results obtained by using the conventional FEM and Ritz method solutions for fixed-fixed boundary condition is given in Table 5.33. The results show very good convergence for both the conventional FEM and Ritz method solutions. At m = 10, the critical buckling load value converges exactly to a single value. The difference in the results between the two solutions is just 0.006 %.

The results for the group of symmetric combinations of both trigonometric and polynomial hierarchical terms for $[0]_{16s}$ laminate are given in Table 5.34. The results show an improvement in the convergence of the critical buckling load values when we add symmetric hierarchical terms to both tangential (v) and radial (w) displacement interpolation functions. This improvement in the convergence of the results continues with the increase of the number of elements. Results given by trigonometric hierarchical terms show an improvement for the convergence of the critical buckling load values for the initial few elements. After the 5-elements mesh all the combinations converge to a single value and the final critical buckling load value given by all the combinations is same. Similar trends are observed for polynomial hierarchical terms. For the first few elements, an improvement in the results with the increase of the symmetric polynomial hierarchical terms is observed. When the number of elements is increased all the results given by this group of combinations converge to a single value. The critical buckling load value given by this group of symmetric combinations is 16589 *lb*.

Table 5.33 Ritz Method and Conventional FEM Solutions for $[0]_{16s}$ Laminate with Fixed-Fixed Boundary Condition

$[0]_{16s}$ Laminate					
Ritz Method		**Conventional FEM Solution using 8 D.O.F. Composite Curved Beam Element**			
Value of m	Critical Buckling Load (lb)	Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Error (%)
1	0.291×10^{7}	1	2	0.118×10^{7}	99.86
2	0.147×10^{7}	2	6	0.231×10^{6}	92.82
3	0.137×10^{6}	3	10	34921	52.50
4	45870	4	14	19843	16.40
5	17010	5	18	17710	6.34
6	16855	6	22	17059	2.76
7	16591	7	26	16810	1.32
8	16590	8	30	16702	0.68
9	16588	9	34	16652	0.38
10	16588	10	38	16624	0.22
		11	42	16612	0.14
		12	46	16602	0.08
		13	50	16597	0.05
		14	54	16595	0.04
		15	58	16593	0.03
		16	62	16589	0.006

Table 5.34 Critical Buckling Load Calculated by using Symmetric Trigonometric and Polynomial Hierarchical Terms for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0]_{16s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Symmetric Trigonometric 1 HFEM term		*Symmetric Trigonometric 2 HFEM terms*		*Symmetric Trigonometric 3 HFEM terms*		*Symmetric Trigonometric 4 HFEM terms*	
1	4	0.117×10^{8}	6	0.214×10^{7}	8	18217	10	18253
2	9	24844	12	19763	15	19735	18	19692
3	14	17292	18	17157	22	17147	26	17146
4	19	16992	24	16987	29	16985	34	16985
5	24	16776	30	16775	36	16774	42	16673
10	49	16600	60	16600	71	16600	90	16600
15	74	16590	90	16590	106	16590	130	16590
16	79	16589	96	16589	113	16589	138	16589
17	83	16589	102	16589	120	16589	146	16589
	Symmetric Polynomial 1 HFEM term		*Symmetric Polynomial 2 HFEM terms*		*Symmetric Polynomial 3 HFEM terms*		*Symmetric Polynomial 4 HFEM terms*	
1	4	0.117×10^{8}	6	0.221×10^{7}	8	0.159×10^{7}	10	-
2	9	24813	12	23012	15	22888	18	22804
3	14	17257	18	17246	22	17237	26	17230
4	19	16994	24	16989	29	16985	34	16982
5	24	16777	30	16675	36	16774	42	16673
10	49	16600	60	16600	71	16600	90	16600
15	74	16590	90	16590	106	16590	130	16590
16	79	16589	96	16589	113	16589	138	16589
17	83	16589	102	16589	120	16589	146	16589

Approximate Solution by Ritz Method: Critical Buckling Load = 16589 *lb*

The results for the non-symmetric combinations ($v_0 - w_n$) of both trigonometric and polynomial hierarchical terms are given in Table 5.35. The results show an improvement in the convergence of the critical buckling load values for the first few elements when we increase the number of hierarchical terms associated with the radial displacement (w) function. The results given by the combination ($v_0 - w_1$) of both trigonometric and polynomial hierarchical terms are same. After the 7-elements mesh the results given by both trigonometric and polynomial hierarchical formulation for all the combinations of this group converge to a single value and the critical buckling load value given by all the combinations is same. The critical buckling load value given by both polynomial and trigonometric hierarchical terms is 16589 *lb*.

The results for the group of non-symmetric combinations ($v_1 - w_n$) of both trigonometric and polynomial hierarchical terms are given in Table 5.36. The convergence for this group is better than that of the previous group of combinations ($v_0 - w_n$) for both polynomial and trigonometric hierarchical terms. Four trigonometric hierarchical terms added to the radial displacement (w) interpolation function provide the best convergence for the critical buckling load values among all the combinations of this group for both polynomial and trigonometric hierarchical formulations. The given value of critical buckling load is 16589 *lb*. Polynomial and trigonometric hierarchical formulations use 10-elements mesh and 15-elements mesh respectively to reach the approximate Ritz solution compared to 18-elements mesh for the conventional FEM.

Table 5.35 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms $(v_0 - w_n)$ for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0]_{16s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 0v-1w HFEM terms		*Non-Symmetric Trigonometric 0v-2w HFEM terms*		*Non-Symmetric Trigonometric 0v-3w HFEM terms*		*Non-Symmetric Trigonometric 0v-4w HFEM terms*	
1	3	0.21×10^7	4	0.15×10^7	5	0.15×10^7	6	0.10×10^7
2	8	55550	10	35806	12	29523	14	29132
3	13	32661	16	26096	19	25989	22	25894
4	18	19556	22	19021	26	12995	30	18993
5	23	17602	28	17507	34	17501	38	17500
10	48	16616	58	16616	69	16616	72	16616
15	73	16591	88	16591	104	16591	112	16591
16	78	16590	94	16590	111	16590	120	16590
17	83	16589	100	16589	118	16589	128	16589
	Non-Symmetric Polynomial 0v-1w HFEM terms		*Non-Symmetric Polynomial 0v-2w HFEM terms*		*Non-Symmetric Polynomial 0v-3w HFEM terms*		*Non-Symmetric Polynomial 0v-4w HFEM terms*	
1	3	0.22×10^7	4	0.15×10^7	5	0.12×10^7	6	0.11×10^7
2	8	53213	10	41985	12	41910	14	41857
3	13	32601	16	32509	19	32502	22	32502
4	18	19550	22	19524	26	19524	30	19524
5	23	17600	28	17595	34	17594	38	17594
10	48	16616	58	16616	69	16616	72	16616
15	73	16591	88	16591	104	16591	112	16591
16	78	16590	94	16590	111	16590	120	16590
17	83	16589	100	16589	118	16589	128	16589

Approximate Solution by Ritz Method: Critical Buckling Load = 16589 *lb*

Table 5.36 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_1 - w_n$) for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0]_{16s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 1v-0w HFEM terms		*Non-Symmetric Trigonometric 1v-2w HFEM terms*		*Non-Symmetric Trigonometric 1v-3w HFEM terms*		*Non-Symmetric Trigonometric 1v-4w HFEM terms*	
1	3	0.11×10^8	5	0.15×10^7	6	0.15×10^7	7	51221
2	8	30170	10	24737	12	22981	14	22971
3	13	17322	15	17233	18	17031	21	17030
4	18	16912	20	16986	24	16858	28	16858
5	23	16728	25	16774	30	16709	35	16709
10	48	16597	50	16600	60	16595	70	16595
15	73	16590	75	16590	90	15689	77	15689
16	78	16589	80	16589				
17	83	16589	90	16589				
	Non-Symmetric Polynomial 1v-0w HFEM terms		*Non-Symmetric Polynomial 1v-2w HFEM terms*		*Non-Symmetric Polynomial 1v-3w HFEM terms*		*Non-Symmetric Polynomial 1v-4w HFEM terms*	
1	3	0.11×10^8	5	0.20×10^7	6	0.17×10^7	7	0.21×10^7
2	8	29954	10	23035	12	22003	14	21549
3	13	17303	15	16989	18	16869	21	16829
4	18	16908	20	16774	24	16697	28	16668
5	23	16726	25	16624	30	16630	35	16616
10	48	16597	50	16591	60	16590	70	16589
15	73	16590	75	16589	90			
16	78	16589						
17	83	16589						

Approximate Solution by Ritz Method: Critical Buckling Load = 16589 *lb*

The results for the groups of combinations ($v_2 - w_n$) and ($v_3 - w_n$) of both trigonometric and polynomial hierarchical formulations are given in Tables 5.37 and 5.38. These two groups of combinations show great similarity in the results. These results show a continuous improvement for the initial few elements with the addition of each trigonometric hierarchical term to the radial displacement (w) interpolation function. These two groups of combinations show very little difference in the results. The results given by both trigonometric and polynomial hierarchical formulations for both of these two groups of combinations are similar to each other. Both formulations use 15-elements mesh to reach the Ritz solution. The results given by trigonometric hierarchical terms for the first few elements are better than that of the results given by the polynomial hierarchical terms.

The results for the non-symmetric combinations ($v_4 - w_n$) of both trigonometric and polynomial hierarchical terms are given in Table 5.39. With the addition of each trigonometric hierarchical term to the radial displacement (w) interpolation function, we get an improvement in the results. Polynomial hierarchical terms give better results than that of trigonometric hierarchical terms. The combination ($v_4 - w_1$) provides the best combination for the polynomial hierarchical formulation reaching the approximate solution by using 10-elements mesh compared to 18-elements mesh for the conventional FEM. The results given by the trigonometric hierarchical terms by all the combinations of this group converge to a single value when the number of elements is increased reaching the Ritz solution by using 15-elements mesh.

Table 5.37 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms $(v_2 - w_n)$ for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0]_{16s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 2v-0w HFEM terms		*Non-Symmetric Trigonometric 2v-1w HFEM terms*		*Non-Symmetric Trigonometric 2v-3w HFEM terms*		*Non-Symmetric Trigonometric 2v-4w HFEM terms*	
1	6	0.11×10^8	7	0.11×10^8	9	0.24×10^6	10	0.24×10^6
2	12	30141	13	19944	16	18552	18	18547
3	18	17322	19	17160	23	16977	26	16961
4	24	16911	25	16988	30	16838	34	16836
5	30	16782	31	16776	37	16705	42	16705
10	60	16597	61	16600	72	16595	82	16595
15	90	16589	91	16590	107	16589	122	16589
	Non-Symmetric Polynomial 2v-0w HFEM terms		*Non-Symmetric Polynomial 2v-1w HFEM terms*		*Non-Symmetric Polynomial 2v-3w HFEM terms*		*Non-Symmetric Polynomial 2v-4w HFEM terms*	
1	6	0.11×10^8	7	0.11×10^8	9	0.16×10^7	10	0.24×10^7
2	12	29954	13	24621	16	22939	18	22369
3	18	17303	19	17243	23	17084	26	16976
4	24	16908	25	16965	30	16861	34	16781
5	30	16726	31	16762	37	16709	42	16670
10	60	16597	61	16598	72	16595	82	16593
15	90	16589	91	16589	107	16589	122	16589
16	96	16589						

Approximate Solution by Ritz Method: Critical Buckling Load = 16588 *lb*

Table 5.38 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_3 - w_n$) for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0]_{16s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 3v-0w HFEM terms		*Non-Symmetric Trigonometric 3v-1w HFEM terms*		*Non-Symmetric Trigonometric 3v-2w HFEM terms*		*Non-Symmetric Trigonometric 3v-4w HFEM terms*	
1	5	0.11×10^8	6	0.11×10^8	7	0.21×10^7	9	16911
2	12	29960	13	19840	14	19750	17	19731
3	19	17304	20	17152	21	17152	25	17138
4	26	16908	27	16982	28	16981	33	16984
5	33	16727	34	16772	35	16772	41	16775
10	68	16597	69	16699	70	16599	81	16600
14	96	16590	97	16590	98	16590	113	16591
15	103	16590	104	16590	105	16590	121	16590
16	110	16589						
	Non-Symmetric Polynomial 3v-0w HFEM terms		*Non-Symmetric Polynomial 3v-1w HFEM terms*		*Non-Symmetric Polynomial 3v-2w HFEM terms*		*Non-Symmetric Polynomial 3v-4w HFEM terms*	
1	5	0.11×10^8	6	0.11×10^8	7	0.22×10^7	9	0.18×10^7
2	12	29954	13	24426	14	22821	17	22840
3	19	17303	20	17166	21	17195	25	17129
4	26	16908	27	16889	28	16950	33	16904
5	33	16726	34	16723	35	16750	41	16732
10	68	16597	69	16694	70	16597	81	16597
14	96	16590	97	16589	98	16590	113	16590
15	103	16590	104		105	16590	121	16590
16	110	16589	111		112	16589	129	16589

Approximate Solution by Ritz Method: Critical Buckling Load = 16588 *lb*

Table 5.39 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_4 - w_n$) for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[0]_{16s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 4v-0w HFEM terms		*Non-Symmetric Trigonometric 4v-1w HFEM terms*		*Non-Symmetric Trigonometric 4v-2w HFEM terms*		*Non-Symmetric Trigonometric 4v-3w HFEM terms*	
1	6	0.11×10^8	7	0.11×10^8	8	0.21×10^7	9	18217
2	14	29959	15	19772	16	19675	17	19691
3	22	17304	23	17151	24	17151	25	17147
4	30	16908	31	16982	32	16981	33	16985
5	38	16727	39	16771	40	16771	41	16775
10	68	16597	79	16599	80	16599	81	16600
15	98	16590	119	16590	120	16590	121	16589
16	104	16589						
	Non-Symmetric Polynomial 4v-0w HFEM terms		*Non-Symmetric Polynomial 4v-1w HFEM terms*		*Non-Symmetric Polynomial 4v-2w HFEM terms*		*Non-Symmetric Polynomial 4v-3w HFEM terms*	
1	6	0.11×10^8	7	-	8	-	9	-
2	14	29954	15	24018	16	22643	17	22786
3	22	17303	23	17047	24	17064	25	17134
4	30	16908	31	16796	32	16840	33	16906
5	38	16726	39	16674	40	16696	41	16734
10	68	16597	79	16590	80	16593	81	16597
15	98	16590			120	16589	121	16590
16	104	16589					129	16589

Approximate Solution by Ritz Method: Critical Buckling Load = 16588 *lb*

5.3.2.2 $[90]_{16s}$ Laminate

The results for the comparison between the conventional FEM and Ritz method solutions are given in Table 5.40. Ritz method shows very good convergence for critical buckling load values for the $[90]_{16s}$ laminate at m = 10. Conventional FEM also shows an excellent convergence of the results reaching the same value of critical buckling load given by the Ritz solution at the 17-elements mesh.

Table 5.40 Ritz Method and Conventional FEM Solutions for $[90]_{16s}$ Laminate with Fixed-Fixed Boundary Condition

$[90]_{16s}$ Laminate					
Ritz Method		**Conventional FEM Solution using 8 D.O.F. Composite Curved Beam Element**			
Value of m	Critical Buckling Load (lb)	Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Error (%)
1	0.18×10^6	1	2	0.73×10^6	99.86
2	91343	2	6	14276	92.83
3	8472.8	3	10	2155.8	52.50
4	2831.7	4	14	1224.9	16.40
5	1050.0	5	18	1093.3	6.34
6	1040.5	6	22	1053.1	2.76
7	1024.2	7	26	1037.7	1.32
8	1024.1	8	30	1031.1	0.69
9	1024.0	9	34	1028.0	0.39
10	1024.0	10	38	1026.3	0.22
		11	58	1025.4	0.14
		12	62	1024.8	0.07
		13	66	1024.5	0.04
		14	70	1024.3	0.02
		15	74	1024.1	0.009
		17	78	1024.0	0.00

The results for the symmetric combinations of both trigonometric and polynomial hierarchical terms for the $[90]_{16s}$ laminate are given in Table 5.41. The results given by both trigonometric and polynomial hierarchical terms are same. These results show that when we increase the number of trigonometric hierarchical terms associated with the tangential (v) and radial (w) displacement interpolation functions, we get an improvement in the convergence of the critical buckling load values. For the first few elements, the results show an improvement with the increase of both symmetric trigonometric polynomial hierarchical terms associated with the tangential (v) and radial (w) displacement functions. When the number of elements is increased, all the values of critical buckling load converge to a single value for all the combinations of this group. The given critical buckling load value is 1024.1 *lb*.

The results for the non-symmetric combinations ($v_0 \; - w_n$) of both trigonometric and polynomial terms are given in Table 5.42. These results follow the same trends as observed in the case of symmetric hierarchical terms. We observe a continuous improvement in the results each time, we add a trigonometric and polynomial hierarchical term to the radial displacement (w) interpolation function. The results given by both trigonometric and polynomial hierarchical terms once again converge to a single value when the number of hierarchical terms is increased. Generally, the results given by trigonometric hierarchical formulation start with a better convergence of the critical buckling load values that that of the polynomial hierarchical formulation. The critical buckling load value given by both trigonometric and polynomial hierarchical terms is 1024.1 *lb*.

Table 5.41 Critical Buckling Load Calculated by using Symmetric Trigonometric and Polynomial Hierarchical Terms for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[90]_{16s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Symmetric Trigonometric 1 HFEM term		*Symmetric Trigonometric 2 HFEM terms*		*Symmetric Trigonometric 3 HFEM terms*		*Symmetric Trigonometric 4 HFEM terms*	
1	4	0.72×10^6	6	0.13×10^6	8	1124.6	10	1126.8
2	9	1533.7	12	1220.1	15	1218.3	18	1215.6
3	14	1067.5	18	1059.2	22	1058.6	26	1058.5
4	19	1049.0	24	1048.6	29	1048.5	34	1048.5
5	24	1035.6	30	1035.6	37	1035.6	42	1035.6
13	64	1024.3	78	1024.3	92	1024.3	114	1024.3
14	69	1024.2	84	1024.2	99	1024.2	122	1024.2
15	74	1024.1	90	1024.1	108	1024.1	130	1024.1
	Symmetric Polynomial 1 HFEM term		*Symmetric Polynomial 2 HFEM terms*		*Symmetric Polynomial 3 HFEM terms*		*Symmetric Polynomial 4 HFEM terms*	
1	4	0.72×10^6	6	0.13×10^6	8	98417	10	-
2	9	1531.8	12	1420.6	15	1413.0	18	1407.7
3	14	1066.4	18	1064.6	22	1064.1	26	1063.7
4	19	1049.1	24	1048.8	29	1048.5	34	1048.4
5	24	1035.7	30	1035.6	37	1035.5	42	1035.5
13	64	1024.3	78	1024.3	92	1024.3	114	1024.3
14	69	1024.2	84	1024.2	99	1024.2	122	1024.2
15	74	1024.1	90	1024.1	108	1024.1	130	1024.1

Approximate Solution by Ritz Method: Critical Buckling Load = 1024.0 *lb*

Table 5.42 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_0 - w_n$) for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[90]_{16s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 0v-1w HFEM terms		*Non-Symmetric Trigonometric 0v-2w HFEM terms*		*Non-Symmetric Trigonometric 0v-3w HFEM terms*		*Non-Symmetric Trigonometric 0v-4w HFEM terms*	
1	3	0.13×10^6	4	95713	5	95713	6	65418
2	8	3429.3	10	2210.4	12	1822.5	14	1798.4
3	13	2016.3	16	1611.0	19	1604.4	22	1604.1
4	18	1207.3	22	1174.2	26	1172.6	30	1172.5
5	23	1086.6	28	1080.8	34	1080.4	38	1080.3
10	48	1025.8	58	1025.8	69	1025.8	72	1025.8
15	73	1024.2	88	1024.2	104	1024.2	112	1024.2
16	78	1024.1	94	1024.1	111	1024.1	120	1024.1
17	83	1024.1	100	1024.1	118	1024.1	128	1024.1
	Non-Symmetric Polynomial 0v-1w HFEM terms		*Non-Symmetric Polynomial 0v-2w HFEM terms*		*Non-Symmetric Polynomial 0v-3w HFEM terms*		*Non-Symmetric Polynomial 0v-4w HFEM terms*	
1	3	0.13×10^6	4	98406	5	75475	6	72093
2	8	3285.0	10	25918	12	2587.2	14	2583.9
3	13	2012.5	16	2006.9	19	2006.4	22	2006.4
4	18	1206.8	22	1205.3	26	1205.3	30	1205.3
5	23	1086.5	28	1086.2	34	1086.2	38	1086.2
10	48	1025.8	58	1025.8	69	1025.8	72	1025.8
15	73	1024.2	88	1024.2	104	1024.2	112	1024.2
16	78	1024.1	94	1024.1	111	1024.1	120	1024.1
17	83	1024.1	100	1024.1	118	1024.1	128	1024.1

Approximate Solution by Ritz Method: Critical Buckling Load = 1024.0 *lb*

The results for the non-symmetric combinations ($v_1 - w_n$) are given in Table 5.43. We get better convergence for the critical buckling load values when we use more polynomial hierarchical terms associated with the radial displacement (w) interpolation function. The combination ($v_1 - w_4$) provides the best convergence of critical buckling load values reaching the approximate solution by using just 10-elements mesh compared to 17-elements mesh for the conventional FEM. The results given by trigonometric hierarchical terms show some improvement in the convergence of the critical buckling load values for the first few elements. As we increase the number of elements, the results converge to a single value and all the combinations of this group for the trigonometric hierarchical terms give the same value of the critical buckling load.

The results for the group of combinations ($v_2 - w_n$) are given in Table 5.44. The results given by the combinations ($v_2 - w_0$) and ($v_1 - w_0$) give exactly the same. The results get better for this group of combinations when we increase the number of polynomial hierarchical terms with the radial displacement (w) function. The combination ($v_2 - w_4$) provides the best convergence of the results in this group reaching the approximate solution by using 12-elements mesh. Trigonometric hierarchical terms again show the same trend and all the combinations of this group converge to a single value of critical buckling load which is the same as obtained previously. The results show that the convergence is better for trigonometric hierarchical terms for the first few elements compared to the previous group of combinations ($v_1 - w_n$).

Table 5.43 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms $(v_1 - w_n)$ for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[90]_{16s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 1v-0w HFEM terms		*Non-Symmetric Trigonometric 1v-2w HFEM terms*		*Non-Symmetric Trigonometric 1v-3w HFEM terms*		*Non-Symmetric Trigonometric 1v-4w HFEM terms*	
1	3	0.72×10^6	5	9356.5	6	9356.5	7	3162.0
2	8	1862.5	10	1527.1	12	1418.7	14	1418.1
3	13	1069.3	15	1063.9	18	1051.3	21	1051.3
4	18	1044.0	20	1048.6	24	1040.7	28	1040.7
5	23	1032.6	25	1035.5	30	1031.5	35	1031.5
10	48	1024.6	50	1024.7	60	1024.5	70	1024.5
11	53	1024.4	55	1024.5	66	1024.3	77	1024.3
15	73	1024.1	75	1024.1	90	1024.1	105	1024.1
	Non-Symmetric Polynomial 1v-0w HFEM terms		*Non-Symmetric Polynomial 1v-2w HFEM terms*		*Non-Symmetric Polynomial 1v-3w HFEM terms*		*Non-Symmetric Polynomial 1v-4w HFEM terms*	
1	3	0.72×10^6	5	0.12×10^6	6	0.11×10^6	7	0.13×10^6
2	8	1849.2	10	1422.0	12	1358.3	14	1330.3
3	13	1068.2	15	1048.8	18	1041.4	21	1038.9
4	18	1043.8	20	1035.5	24	1030.7	28	1029.0
5	23	1032.6	25	1028.7	30	1026.6	35	1025.8
10	48	1024.6	50	1024.3	60	1024.1	70	1024.1
11	53	1024.4	55	1024.2	90	1024.1		
15	73	1024.1						

Approximate Solution by Ritz Method: Critical Buckling Load = 1024.0 *lb*

Table 5.44 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_2 - w_n$) for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[90]_{16s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 2v-0w HFEM terms		*Non-Symmetric Trigonometric 2v-1w HFEM terms*		*Non-Symmetric Trigonometric 2v-3w HFEM terms*		*Non-Symmetric Trigonometric 2v-4w HFEM terms*	
1	6	0.72×10^6	7	0.72×10^6	9	14851	10	14851
2	12	1860.7	13	1231.2	16	1145.3	18	1145.0
3	18	1069.3	19	1059.3	23	1048.0	26	1047.0
4	24	1044.0	25	1048.7	30	1039.5	34	1039.4
5	30	1032.6	31	1035.6	37	1031.2	42	1031.2
10	60	1024.6	61	1024.7	72	1024.5	82	1024.5
12	72	1024.3	73	1024.4	86	1024.2	98	1024.2
15	90	1024.0	91	1024.1				
	Non-Symmetric Polynomial 2v-0w HFEM terms		*Non-Symmetric Polynomial 2v-1w HFEM terms*		*Non-Symmetric Polynomial 2v-3w HFEM terms*		*Non-Symmetric Polynomial 2v-4w HFEM terms*	
1	6	0.72×10^6	7	0.72×10^6	9	0.10×10^6	10	0.14×10^6
2	12	1849.2	13	1519.9	16	1416.1	18	1380.9
3	18	1068.2	19	1064.5	23	1054.7	26	1048.0
4	24	1043.8	25	1047.3	30	1040.8	34	1035.9
5	30	1032.6	31	1034.7	37	1031.5	42	1029.1
10	60	1024.6	61	1024.6	72	1024.5	82	1024.3
12	72	1024.3	73	1024.3	86	1024.1	98	1024.1
15	90	1024.0	91	1024.0				

Approximate Solution by Ritz Method: Critical Buckling Load = 1024.0 *lb*

The results for the group of non-symmetric combinations ($v_3 - w_n$) of both trigonometric and polynomial hierarchical terms are given in Table 5.45. We observe a continuous improvement in the results with the addition of each polynomial hierarchical term to the radial displacement (w) interpolation function. The convergence for this group of combinations is worse compared to the previous group of combinations ($v_1 - w_n$). The best critical buckling load value is given by the combination ($v_3 - w_1$) by using 12-elements mesh. All the combinations of this group for trigonometric hierarchical formulation converge to a single value as we increase the number of elements. The critical buckling load value given by the trigonometric hierarchical formulation is reached by using 15-elements mesh.

The results for the non-symmetric combinations ($v_4 - w_n$) for both trigonometric and polynomial hierarchical terms are given in Table 5.46. The results for the polynomial hierarchical terms show very little difference in the critical buckling load values given by all the combinations of ($v_4 - w_n$) of this group. All these combinations give results that almost converge to a single value. The best critical buckling load value given by this group of combinations is obtained by the combination ($v_4 - w_1$) using 11-elements mesh. Critical buckling load values given by the trigonometric hierarchical terms once again do not show any change in the results and the final value given by this group is obtained by using the same number of elements as in all the previous cases. For the first few elements the convergence for the trigonometric hierarchical terms is better than that of the polynomial hierarchical terms.

Table 5.45 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_3 - w_n$) for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[90]_{16s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 3v-0w HFEM terms		*Non-Symmetric Trigonometric 3v-1w HFEM terms*		*Non-Symmetric Trigonometric 3v-2w HFEM terms*		*Non-Symmetric Trigonometric 3v-4w HFEM terms*	
1	5	0.72×10^{6}	6	0.72×10^{8}	7	0.13×10^{6}	9	1044.0
2	12	1849.5	13	1224.8	14	1219.2	17	1218.1
3	19	1068.2	20	1058.9	21	1058.8	25	1058.0
4	26	1043.8	27	1048.3	28	1048.3	33	1048.5
5	33	1032.6	34	1035.4	35	1035.3	41	1035.5
10	68	1024.6	69	1024.7	70	1024.7	81	1024.7
12	82	1024.3	83	1024.3	84	1024.3	97	1024.3
15	103	1024.1	104	1024.1	105	1024.1	121	1024.1
	Non-Symmetric Polynomial 3v-0w HFEM terms		*Non-Symmetric Polynomial 3v-1w HFEM terms*		*Non-Symmetric Polynomial 3v-2w HFEM terms*		*Non-Symmetric Polynomial 3v-4w HFEM terms*	
1	5	0.72×10^{6}	6	0.72×10^{8}	7	0.13×10^{6}	9	0.11×10^{6}
2	12	1849.2	13	1507.9	14	1408.8	17	1410.0
3	19	1068.2	20	1059.7	21	1061.5	25	1057.4
4	26	1043.8	27	1042.6	28	1046.4	33	1043.5
5	33	1032.6	34	1032.3	35	1034.0	41	1032.9
10	68	1024.6	69	1024.4	70	1024.6	81	1024.6
12	82	1024.3	83	1024.1	84	1024.3	97	1024.3
15	103	1024.1			105	1024.1	121	1024.1

Approximate Solution by Ritz Method: Critical Buckling Load = 1024.0 *lb*

Table 5.46 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_4 - w_n$) for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[90]_{16s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 4v-0w HFEM terms		*Non-Symmetric Trigonometric 4v-1w HFEM terms*		*Non-Symmetric Trigonometric 4v-2w HFEM terms*		*Non-Symmetric Trigonometric 4v-3w HFEM terms*	
1	6	0.72×10^6	7	0.72×10^6	8	0.13×10^6	9	1124.6
2	14	1849.5	15	1220.6	16	1214.6	17	1215.6
3	22	1068.2	23	1058.8	24	1058.8	25	1058.5
4	30	1043.8	31	1048.3	32	1048.3	33	1048.5
5	38	1032.6	39	1035.3	40	1035.3	41	1035.6
10	68	1024.6	79	1024.7	80	1024.7	81	1024.7
15	98	1024.1	119	1024.1	120	1024.1	121	1024.2
	Non-Symmetric Polynomial 4v-0w HFEM terms		*Non-Symmetric Polynomial 4v-1w HFEM terms*		*Non-Symmetric Polynomial 4v-2w HFEM terms*		*Non-Symmetric Polynomial 4v-3w HFEM terms*	
1	6	0.72×10^6	7	-	8	-	9	-
2	14	1849.5	15	1482.7	16	1397.8	17	1406.7
3	22	1068.2	23	1052.4	24	1053.4	25	1057.8
4	30	1043.8	31	1036.8	32	1039.6	33	1043.6
5	38	1032.6	39	1029.3	40	1030.7	41	1033.0
10	78	1024.6	79	1024.2	80	1024.4	81	1024.6
12	94	1024.3	95	1024.1	96	1024.2	97	1024.2
15	118	1024.1			120	1024.1	121	1024.1

Approximate Solution by Ritz Method: Critical Buckling Load = 1024.0 *lb*

5.3.2.3 $[+45/-45]_{8s}$ *Laminate*

The comparison of the results between the Ritz method and the conventional FEM solutions for the $[+45/-45]_{8s}$ laminate is given in Table 5.47. At m = 10 the critical buckling load value converges exactly to the same value as given by m = 9 which is the perfect convergence for the Ritz solution. Conventional FEM solution also gives an excellent convergence with a difference of just 0.01 % between the two solutions.

Table 5.47 Ritz Method and Conventional FEM Solutions for $[+45/-45]_{8s}$ Laminate with Fixed-Fixed Boundary Conditions

$[45/-45]_{8s}$ Laminate					
Ritz Method		**Conventional FEM Solution using 8 D.O.F. Composite Curved Beam Element**			
Value of m	Critical Buckling Load (lb)	Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Error (%)
1	0.90×10^6	1	2	0.36×10^7	99.86
2	0.45×10^6	2	6	71406	92.83
3	42378	3	10	10783	52.50
4	14163	4	14	6126.9	16.40
5	5252.0	5	18	5468.2	6.30
6	5204.3	6	22	5267.1	2.76
7	5122.8	7	26	5190.4	1.32
8	5122.4	8	30	5157.1	0.68
9	5121.8	9	34	5141.2	0.38
10	5121.8	10	38	5133.4	0.23
		15	58	5123.1	0.02
		17	62	5122.3	0.009
		18	66	5122.8	0.01
		19	70	5122.4	0.01

The results for the symmetric combinations of both trigonometric and polynomial hierarchical terms for the $[+45/-45]_{8s}$ laminate are given in Table 5.48. These results show a small improvement for both trigonometric and polynomial hierarchical terms in the results with the addition of symmetric trigonometric hierarchical terms to the tangential (v) and radial (w) displacement interpolation functions. The best critical buckling load value is obtained by using 19-elements mesh which is the same as observed in the case of conventional FEM. The results for both trigonometric and polynomial hierarchical formulations converge similarly for all the combinations of this group.

The results for the non-symmetric combinations ($v_0 \; - w_n$) of both trigonometric and polynomial hierarchical terms are given in Table 5.49. These results follow the same trends as observed in the previous case of symmetric hierarchical terms for both trigonometric and polynomial hierarchical formulations for all the combinations of this group. These results show that when we increase the number of hierarchical terms associated with the radial displacement (w) interpolation function, we get a little improvement in the convergence of the results. As we increase the number of elements, the results given by all the combinations converge to a single critical buckling load value. All the combinations of both trigonometric and polynomial hierarchical formulations use 18-elements mesh to reach the approximate Ritz solution.

Table 5.48 Critical Buckling Load Calculated by using Symmetric Trigonometric and Polynomial Hierarchical Terms for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[+45/-45]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Symmetric Trigonometric 1 HFEM term		*Symmetric Trigonometric 2 HFEM terms*		*Symmetric Trigonometric 3 HFEM terms*		*Symmetric Trigonometric 4 HFEM terms*	
1	4	0.36×10^{7}	6	0.66×10^{6}	8	5624.8	10	5635.9
2	9	7671.0	12	6102.3	15	6093.4	18	6080.2
3	14	5339.2	18	5297.6	22	5294.6	26	5294.3
4	19	5246.6	24	5245.0	29	5244.5	34	5244.5
5	24	5179.8	30	5179.7	36	5179.6	42	5179.1
10	49	5125.5	60	5125.4	71	5125.4	90	5125.4
15	74	5122.4	90	5122.4	106	5122.4	130	5122.5
18	88	5122.1	108	5122.1	127	5122.1	154	5122.1
19	93	5122.0	114	5122.0	134	5122.0	162	5122.0
	Symmetric Polynomial 1 HFEM term		*Symmetric Polynomial 2 HFEM terms*		*Symmetric Polynomial 3 HFEM terms*		*Symmetric Polynomial 4 HFEM terms*	
1	4	0.36×10^{7}	6	0.68×10^{6}	8	0.49×10^{6}	10	-
2	9	7661.5	12	7105.5	15	7067.2	18	7041.0
3	14	5334.0	18	5325.0	22	5322.1	26	5320.1
4	19	5247.3	24	5245.7	29	5244.5	34	5243.6
5	24	5180.1	30	5179.6	36	5179.3	42	5179.1
10	49	5125.5	60	5125.4	71	5125.5	90	5125.4
15	74	5122.5	90	5122.5	106	5122.5	130	5122.5
18	88	5122.1	108	5122.1	127	5122.1	154	5122.1
19	93	5122.0	114	5122.0	134	5122.0	162	5122.0

Approximate Solution by Ritz Method: Critical Buckling Load = 5121.8 *lb*

Table 5.49 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_0 - w_n$) for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[+45/-45]_{8s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 0v-1w HFEM terms		*Non-Symmetric Trigonometric 0v-2w HFEM terms*		*Non-Symmetric Trigonometric 0v-3w HFEM terms*		*Non-Symmetric Trigonometric 0v-4w HFEM terms*	
1	3	0.66×10^6	4	0.47×10^6	5	0.47×10^6	6	0.32×10^6
2	8	17152	10	11056	12	9115.7	14	8995.1
3	13	10085	16	8057.6	19	8024.6	22	8023.0
4	18	6038.3	22	5873.2	26	5865.1	30	5864.4
5	23	5435.0	28	5405.6	34	5403.6	38	5403.5
10	48	5130.6	58	5130.6	69	5130.6	72	5130.6
15	73	5122.7	88	5122.7	104	5122.7	112	5122.7
17	83	5122.2	100	5122.2	118	5122.2	128	5122.2
18	88	5122.1	106	5122.1	125	5122.1	136	5122.1
	Non-Symmetric Polynomial 0v-1w HFEM terms		*Non-Symmetric Polynomial 0v-2w HFEM terms*		*Non-Symmetric Polynomial 0v-3w HFEM terms*		*Non-Symmetric Polynomial 0v-4w HFEM terms*	
1	3	0.68×10^6	4	0.49×10^6	5	0.37×10^6	6	0.36×10^6
2	8	16430	10	12963	12	12941	14	12924
3	13	10066	16	10038	19	10036	22	10036
4	18	6036.3	22	6028.4	26	6028.3	30	6026.3
5	23	5434.4	28	5432.6	34	5432.6	38	5432.6
10	48	5130.6	58	5130.6	69	5130.6	72	5130.6
15	73	5122.7	88	5122.7	104	5122.7	112	5122.7
17	83	5122.2	100	5122.2	118	5122.2	128	5122.2
18	88	5122.1	106	5122.1	125	5122.1	136	5122.1

Approximate Solution by Ritz Method: Critical Buckling Load = 5121.8 *lb*

The results for the group of combinations of both trigonometric and polynomial hierarchical terms ($v_1 - w_n$) are given in Table 5.50. In this case, trigonometric and polynomial hierarchical terms have been increased from zero to one with the tangential displacement (v) interpolation function. This addition of one hierarchical term has a positive effect on the convergence of the critical buckling load values. This time all the combinations show good convergence of the results with a small difference in the critical buckling load values given by all the combinations of this group. The combination ($v_1 - w_4$) of polynomial hierarchical terms provides the best convergence of the results and the approximate Ritz solution is reached by using just 11-elements compared to 19-elements mesh for the conventional FEM. As the number of trigonometric hierarchical terms is increased associated with the radial displacement (w) interpolation function, all the values of the critical buckling load converge to a single value for all the combinations of this group. Trigonometric hierarchical terms use 18-elements mesh to reach the Ritz solution.

Table 5.51 gives the results for the group of combinations ($v_2 - w_n$) of both trigonometric and polynomial hierarchical terms. These results show similar trends as observed in the previous group of combinations ($v_0 - w_n$). Both polynomial and trigonometric hierarchical formulations show improvement in the results when we increase the number of hierarchical terms with the radial displacement (w) function. As the number of elements is increased the critical buckling load values converge to a single value. The combination ($v_2 - w_4$) provides the best converged value of critical buckling load for both formulations and is reached by using 14-elements mesh.

Table 5.50 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms $(v_1 - w_n)$ for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[+45/-45]_{8s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 1v-0w HFEM terms		*Non-Symmetric Trigonometric 1v-2w HFEM terms*		*Non-Symmetric Trigonometric 1v-3w HFEM terms*		*Non-Symmetric Trigonometric 1v-4w HFEM terms*	
1	3	0.36×10^7	5	46798	6	46798	7	15815
2	8	9315.6	10	7637.9	12	7095.7	14	7092.8
3	13	5348.5	15	5321.1	18	5258.5	21	5258.4
4	18	5221.8	20	5244.8	24	5205.1	28	5205.1
5	23	5165.0	25	5179.4	30	5159.3	35	5159.3
10	48	5124.6	50	5125.4	60	5124.1	70	5124.1
15	73	5122.3	75	5122.4	90	5122.2	105	5122.2
17	83	5122.1	85	5122.2	102	5122.0	119	5122.0
18	88	5122.0	90	5122.1				
	Non-Symmetric Polynomial 1v-0w HFEM terms		*Non-Symmetric Polynomial 1v-2w HFEM terms*		*Non-Symmetric Polynomial 1v-3w HFEM terms*		*Non-Symmetric Polynomial 1v-4w HFEM terms*	
1	3	0.36×10^7	5	0.63×10^6	6	0.55×10^6	7	0.66×10^6
2	8	9249.0	10	7112.3	12	6793.9	14	6653.6
3	13	5342.7	15	5245.7	18	5208.6	21	5196.2
4	18	5220.6	20	5179.2	24	5155.4	28	5146.7
5	23	5164.6	25	5145.3	30	5134.7	35	5130.6
10	48	5124.6	50	5123.1	60	5122.5	70	5122.2
11	53	5123.7	55	5123.7	66	5122.2	77	5122.1
12	58	5123.1	60	5122.4	72	5122.1	84	5122.0
15	73	5122.3	75	5122.0				

Approximate Solution by Ritz Method: Critical Buckling Load = 5121.8 *lb*

Table 5.51 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_2 - w_n$) for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[+45/-45]_{8s}$ Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 2v-0w HFEM terms		*Non-Symmetric Trigonometric 2v-1w HFEM terms*		*Non-Symmetric Trigonometric 2v-3w HFEM terms*		*Non-Symmetric Trigonometric 2v-4w HFEM terms*	
1	6	0.36×10^7	7	0.36×10^7	9	74282	10	74282
2	12	9306.5	13	6158.2	16	5728.4	18	5726.8
3	18	5348.5	19	5298.4	23	5241.9	26	5236.9
4	24	5221.7	25	5245.2	30	5199.1	34	5198.5
5	30	5164.9	31	5179.7	37	5158.0	42	5157.8
10	60	5124.6	61	5125.4	72	5124.1	82	5122.9
14	84	5122.5	85	5122.7	100	5122.3	114	5122.1
15	90	5122.3	91	5122.4	107	5122.2	122	5122.0
17	102	5122.1	103	5122.2	121	5122.0		
	Non-Symmetric Polynomial 2v-0w HFEM terms		*Non-Symmetric Polynomial 2v-1w HFEM terms*		*Non-Symmetric Polynomial 2v-3w HFEM terms*		*Non-Symmetric Polynomial 2v-4w HFEM terms*	
1	6	0.36×10^7	7	0.36×10^7	9	0.52×10^6	10	0.74×10^6
2	12	9249.0	13	7602.1	16	7082.7	18	6906.9
3	18	5342.7	19	5324.2	23	5275.1	26	5241.8
4	24	5220.6	25	5238.1	30	5206.0	34	5181.3
5	30	5164.6	31	5175.4	37	5159.2	42	5147.3
10	60	5124.6	61	5124.9	72	5124.0	82	5123.2
14	84	5122.5	85	5122.4	100	5122.3	114	5122.1
15	90	5122.3	91	5122.3	107	5122.2		
17	102	5122.1	103	5122.1	121	5122.1		

Approximate Solution by Ritz Method: Critical Buckling Load = 5121.8 *lb*

The results for the group of non-symmetric combinations ($v_3 - w_n$) of both trigonometric and polynomial hierarchical terms are given in Table 5.52. In this case, there is a little improvement in the convergence of the results when we increase the number of polynomial hierarchical terms with the tangential displacement (v) interpolation function. The combination ($v_3 - w_1$) provides the best converged value of critical buckling load and the approximate Ritz solution is reached by using 14-elements mesh. Trigonometric hierarchical formulation as usual give one value of critical buckling loads which is same for all the combinations. This value of critical buckling load is obtained by using 16-elements mesh. Generally, the convergence is improved for this group of combinations when three hierarchical terms are added with the tangential displacement (v) function.

The results for the group of combinations ($v_4 - w_n$) for the $[+45/-45]_{8s}$ laminate are given in Table 5.53. The addition of four trigonometric and polynomial hierarchical terms to the tangential displacement (v) interpolation function has a positive effect on the convergence of the results. Best convergence in this group is obtained by the combination ($v_4 - w_1$) for polynomial hierarchical formulation and the critical buckling load value given by this combination is obtained by using just 12-elements mesh compared to 19-elements mesh for the conventional FEM. As we increase the number of elements for trigonometric hierarchical terms the critical buckling load values converge to a single value. All the combinations of this group for trigonometric hierarchical formulation use 18-elements mesh to reach the approximate Ritz solution.

Table 5.52 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_3 - w_n$) with Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[+45/-45]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 3v-0w HFEM terms		*Non-Symmetric Trigonometric 3v-1w HFEM terms*		*Non-Symmetric Trigonometric 3v-2w HFEM terms*		*Non-Symmetric Trigonometric 3v-4w HFEM terms*	
1	5	0.36×10^7	6	0.36×10^7	7	0.66×10^6	9	5221.5
2	12	9250.6	13	6125.9	14	6098.1	17	6092.4
3	19	5342.8	20	5296.1	21	5295.8	25	5291.8
4	26	5220.6	27	5243.5	28	5243.1	33	5244.2
5	33	5164.6	34	5178.6	35	5178.5	41	5179.5
10	68	5124.6	69	5125.2	70	5125.2	81	5125.4
15	103	5122.3	104	5122.4	105	5122.4	121	5122.4
16	110	5122.2	111	5122.2	112	5122.2	129	5122.2
17	117	5122.1	118	5122.1	119	5122.1	137	5122.1
	Non-Symmetric Polynomial 3v-0w HFEM terms		*Non-Symmetric Polynomial 3v-1w HFEM terms*		*Non-Symmetric Polynomial 3v-2w HFEM terms*		*Non-Symmetric Polynomial 3v-4w HFEM terms*	
1	5	0.36×10^7	6	0.36×10^7	7	0.68×10^6	9	0.58×10^6
2	12	9249.0	13	7541.8	14	7046.3	17	7052.4
3	19	5342.7	20	5300.3	21	5309.4	25	5289.0
4	26	5220.6	27	5241.7	28	5233.6	33	5219.3
5	33	5164.6	34	5163.5	35	5172.0	41	5166.4
10	68	5124.6	69	5123.6	70	5124.5	81	5124.5
15	103	5122.3	104	5122.0	105	5122.2	121	5122.2
16	110	5122.2			112	5122.1	129	5122.1
17	117	5122.1			119	5122.0	137	5122.0

Approximate Solution by Ritz Method: Critical Buckling Load = 5121.8 *lb*

Table 5.53 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_4 - w_n$) with Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for $[+45/-45]_{8s}$ Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 4v-0w HFEM terms		*Non-Symmetric Trigonometric 4v-1w HFEM terms*		*Non-Symmetric Trigonometric 4v-2w HFEM terms*		*Non-Symmetric Trigonometric 4v-3w HFEM terms*	
1	6	0.36×10^7	7	0.36×10^7	8	0.66×10^6	9	5624.8
2	14	9250.5	15	6105.0	16	6075.1	17	6080.0
3	22	5342.8	23	5295.6	24	5295.7	25	5294.5
4	30	5220.6	31	5243.3	32	5243.0	33	5244.5
5	38	5164.6	39	5178.3	40	5178.3	41	5179.5
10	78	5124.6	79	5125.2	80	5125.2	81	5125.4
12	94	5123.1	95	5123.2	96	5123.2	97	5123.5
15	118	5123.3	119	5122.4	120	5122.4	121	5122.4
16	126	5123.2	127	5122.2	128	5122.2	129	5122.3
	Non-Symmetric Polynomial 4v-0w HFEM terms		*Non-Symmetric Polynomial 4v-1w HFEM terms*		*Non-Symmetric Polynomial 4v-2w HFEM terms*		*Non-Symmetric Polynomial 4v-3w HFEM terms*	
1	6	0.36×10^7	7	-	8	-	9	-
2	14	9249.0	15	7415.9	16	6991.6	17	7035.7
3	22	5342.7	23	5263.7	24	5268.9	25	5290.6
4	30	5220.6	31	5185.9	32	5199.7	33	5220.0
5	38	5164.6	39	5148.3	40	5155.2	41	5166.9
10	78	5124.6	79	5122.5	80	5123.5	81	5124.5
12	94	5123.1	95	5122.0	96	5122.6	97	5123.0
15	118	5123.3			120	5122.1	121	5122.3
16	126	5123.2					129	5122.1

Approximate Solution by Ritz Method: Critical Buckling Load = 5121.8 *lb*

5.3.2.4 $[0/90/+45/-45]_{4s}$ *Quasi-Isotropic Laminate*

The results for the comparison of critical buckling load values with fixed-fixed boundary condition for the quasi-isotropic laminate by using the conventional FEM and the Ritz method solutions are given in Table 5.54. Ritz solution gives very good convergence at m = 10. The results show that last two values of critical buckling load are exactly same. Conventional FEM solution also gives results with good convergence. The difference between the two solutions is just 0.005 %.

Table 5.54 Ritz Method and Conventional FEM Solutions for Quasi-Isotropic Laminate with Fixed-Fixed Boundary Condition

Quasi-Isotropic Laminate					
Ritz Method		**Conventional FEM Solution using 8 D.O.F. Composite Curved Beam Element**			
Value of m	Critical Buckling Load (lb)	Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Error (%)
1	0.12×10^7	1	2	0.49×10^7	99.84
2	0.62×10^6	2	6	98437	92.16
3	58399	3	10	15448	50.03
4	20562	4	14	9119.6	15.35
5	7908.1	5	18	8200.9	5.87
6	7840.2	6	22	7921.5	2.55
7	7720.9	7	26	7815.3	1.23
8	7720.5	8	30	7768.8	0.63
9	7719.5	9	34	7747.1	0.36
10	7719.5	10	38	7735.2	0.20
		15	58	7722.0	0.03
		16	62	7719.0	-0.006
		17	66	7720.1	0.005

The results for the symmetric combinations of both trigonometric and polynomial hierarchical formulations for the quasi-isotropic laminate are given in Table 5.55. These results show that when we increase the number of both trigonometric and polynomial hierarchical terms with the tangential (v) and radial (w) displacement interpolation functions, the convergence becomes a little better. Symmetric trigonometric hierarchical terms for the first few elements give better convergence than that of symmetric polynomial hierarchical terms. After the 8-elements mesh all the results for the critical buckling loads of both trigonometric and polynomial hierarchical terms converge to a single value. The critical buckling load values given by all the symmetric combinations are same. The given critical buckling load value reaches the approximate Ritz solution at the 18-elements mesh.

The results for the non-symmetric combinations ($v_0 - w_n$) of both trigonometric and polynomial hierarchical terms for the quasi-isotropic laminate are given in Table 5.56. In this case no hierarchical term was used with the tangential displacement (v) interpolation function and 1 to 4 hierarchical terms were used with the radial displacement (w) interpolation function. These results show similar trends as observed in the previous case of symmetric polynomial and trigonometric hierarchical formulations. All the combinations of the group ($v_0 - w_n$) show that when we increase the number of elements, the critical buckling load values for both trigonometric and polynomial hierarchical formulations converge to a single value. All the combinations of this group use 17-elements mesh to reach the approximate Ritz solution.

Table 5.55 Critical Buckling Load Calculated by using Symmetric Trigonometric and Polynomial Hierarchical Terms for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for Quasi Isotropic Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Symmetric Trigonometric 1 HFEM term		*Symmetric Trigonometric 2 HFEM terms*		*Symmetric Trigonometric 3 HFEM terms*		*Symmetric Trigonometric 4 HFEM terms*	
1	4	0.49×10^7	6	0.90×10^6	8	8405.7	10	8421.1
2	9	11459	12	9194.3	15	9182.3	18	9163.5
3	14	8041.6	18	7983.6	22	7979.8	26	7979.4
4	19	7907.2	24	7905.0	29	7904.3	34	7904.2
5	24	7806.8	30	7806.7	36	7806.5	42	7866.4
10	49	7725.1	60	7725.1	71	7725.0	90	7725.0
15	74	7720.6	90	7720.6	106	7720.6	130	7720.5
18	88	7720.0	108	7720.0	127	7720.0	154	7720.0
19	93	7719.9	114	7719.9	134	7719.9	162	7719.9
20	98	7719.8	120	7719.8	141	7719.8	170	7719.8
	Symmetric Polynomial 1 HFEM term		*Symmetric Polynomial 2 HFEM terms*		*Symmetric Polynomial 3 HFEM terms*		*Symmetric Polynomial 4 HFEM terms*	
1	4	0.49×10^7	6	0.93×10^6	8	0.67×10^6	10	-
2	9	11440	12	10657	15	10600	18	10562
3	14	8034.8	18	8022.5	22	8018.3	26	8015.4
4	19	7908.3	24	7905.9	29	7904.0	34	7902.6
5	24	7807.3	30	7806.5	36	7806.0	42	7805.7
10	49	7725.1	60	7725.1	71	7725.0	90	7725.0
15	74	7720.6	90	7720.6	106	7720.6	130	7720.5
18	88	7720.0	108	7720.0	127	7720.0	154	7720.0
19	93	7719.9	114	7719.9	134	7719.9	162	7719.9
20	98	7719.8	120	7719.8	141	7719.8	170	7719.8

Approximate Solution by Ritz Method: Critical Buckling Load = 7719.5 *lb*

Table 5.56 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_0 - w_n$) for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for Quasi Isotropic Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 0v-1w HFEM terms		*Non-Symmetric Trigonometric 0v-2w HFEM terms*		*Non-Symmetric Trigonometric 0v-3w HFEM terms*		*Non-Symmetric Trigonometric 0v-4w HFEM terms*	
1	3	0.92×10^6	4	0.65×10^6	5	0.65×10^6	6	0.44×10^6
2	8	24143	10	16020	12	13431	14	13290
3	13	14480	16	11885	19	11843	22	11839
4	18	8986.1	22	8776.0	26	8765.2	30	8764.2
5	23	8150.3	28	8147.3	34	8111.3	38	8111.1
10	48	7731.6	58	7731.5	69	7731.5	72	7731.5
15	73	7720.8	88	7720.8	104	7720.8	112	7720.8
18	88	7720.0	106	7720.0	125	7720.0	136	7720.0
19	93	7719.8	112	7719.8	132	7719.8	144	7719.8
20	98	7719.8	118	7719.8	139	7719.8	152	7719.8
	Non-Symmetric Polynomial 0v-1w HFEM terms		*Non-Symmetric Polynomial 0v-2w HFEM terms*		*Non-Symmetric Polynomial 0v-3w HFEM terms*		*Non-Symmetric Polynomial 0v-4w HFEM terms*	
1	3	0.92×10^6	4	0.67×10^6	5	0.52×10^6	6	0.50×10^6
2	8	23176	10	18812	12	18768	14	18749
3	13	14456	16	14423	19	14421	22	14421
4	18	8983.1	22	8973.1	26	8973.0	30	8973.0
5	23	8149.5	28	8147.3	34	8147.3	38	8147.3
10	48	7731.6	58	7731.6	69	7731.6	72	7731.6
15	73	7720.8	88	7720.8	104	7720.8	112	7720.8
18	88	7720.0	106	7720.0	125	7720.0	136	7720.0
19	93	7719.8	112	7719.8	132	7719.8	144	7719.8
20	98	7719.8	118	7719.8	139	7719.8	152	7719.8

Approximate Solution by Ritz Method: Critical Buckling Load = 7719.5 *lb*

The results for the non-symmetric combinations ($v_1 - w_n$) of both trigonometric and polynomial hierarchical terms are given in Table 5.57. In this case, we use one hierarchical term with the tangential displacement (w) interpolation function and this produced the results better than that of the previous case. The combination ($v_1 - w_4$) for the polynomial hierarchical terms once again gives the best results for all the combinations of this group. The critical buckling load value obtained by this combination is reached by using 11-elements mesh compared to 17-elements mesh for the conventional FEM. Trigonometric hierarchical formulation follows similar trends and when we increase the number of trigonometric hierarchical terms with the radial displacement (w) interpolation function we get a little better convergence. As we increase the number of elements the critical buckling load values given by all the combinations of this group converge to a single value.

The results for the non-symmetric combinations ($v_2 - w_n$) of both trigonometric and polynomial hierarchical terms for quasi-isotropic laminate are given in Table 5.58. These results tend to follow the same trends as observed in the case of combinations ($v_0 - w_n$). In this case, two hierarchical terms are used with the tangential displacement (v) interpolation function. This addition of two hierarchical terms gives us better convergence when we add hierarchical terms to the radial displacement (w) interpolation function. As we increase the number of elements, the critical buckling load values converge to a single value. All the combinations of this group of both trigonometric and polynomial hierarchical terms give the critical buckling load value by using 16-elements mesh.

Table 5.57 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_1 - w_n$) for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for Quasi Isotropic Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 1v-0w HFEM terms		*Non-Symmetric Trigonometric 1v-2w HFEM terms*		*Non-Symmetric Trigonometric 1v-3w HFEM terms*		*Non-Symmetric Trigonometric 1v-4w HFEM terms*	
1	3	0.49×10^7	5	64381	6	64381	7	22585
2	8	14031	10	11411	12	10628	14	10623
3	13	8060.4	15	8015.0	18	7922.2	21	7922.1
4	18	7870.1	20	7904.5	24	7844.0	28	7844.0
5	23	7784.6	25	7806.2	30	7775.9	35	7775.9
10	48	7723.8	50	7725.1	60	7723.0	70	7723.0
11	53	7722.4	55	7723.3	66	7721.9	77	7721.9
12	58	7721.6	60	7722.1	72	7721.2	84	7721.2
15	73	7720.4	75	7720.6	90	7720.2	105	7720.2
18	88	7719.9	90	7720.0	108	7719.9	126	7719.9
	Non-Symmetric Polynomial 1v-0w HFEM terms		*Non-Symmetric Polynomial 1v-2w HFEM terms*		*Non-Symmetric Polynomial 1v-3w HFEM terms*		*Non-Symmetric Polynomial 1v-4w HFEM terms*	
1	3	0.49×10^7	5	0.86×10^6	6	0.75×10^6	7	0.91×10^6
2	8	13940	10	10642	12	10155	14	9951.4
3	13	8052.5	15	7901.8	18	7846.7	21	7828.1
4	18	7868.5	20	7804.6	24	7769.1	28	7756.0
5	23	7784.1	25	7754.7	30	7738.8	35	7732.6
10	48	7723.8	50	7721.6	60	7720.6	70	7720.2
11	53	7722.4	55	7720.9	66	7720.2	77	7720.0
12	58	7721.6	60	7720.5	72	7720.0		
15	73	7720.4	75	7719.9				
18	88	7719.9						

Table 5.58 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_2 - w_n$) for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for Quasi Isotropic Laminate								
No of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 2v-0w HFEM terms		*Non-Symmetric Trigonometric 2v-1w HFEM terms*		*Non-Symmetric Trigonometric 2v-3w HFEM terms*		*Non-Symmetric Trigonometric 2v-4w HFEM terms*	
1	6	0.49×10^7	7	0.49×10^7	9	0.71×10^6	10	0.10×10^6
2	12	13940	13	11352	16	10610	18	10335
3	18	8052.5	19	8019.8	23	7947.2	26	7897.1
4	24	7868.5	25	7893.5	30	7845.3	34	7808.2
5	30	7784.1	31	7799.8	37	7775.6	42	7757.7
10	60	7723.8	61	7724.1	72	7722.9	82	7721.0
14	84	7720.6	85	7720.5	100	7720.3	114	7719.9
15	90	7720.4	91	7720.3	107	7720.1	122	
16	96	7720.2	97	7720.0	114	7720.0	130	
	Non-Symmetric Polynomial 2v-0w HFEM terms		*Non-Symmetric Polynomial 2v-1w HFEM terms*		*Non-Symmetric Polynomial 2v-3w HFEM terms*		*Non-Symmetric Polynomial 2v-4w HFEM terms*	
1	6	0.49×10^7	7	0.49×10^7	9	0.71×10^6	10	0.10×10^6
2	12	13940	13	11352	16	10610	18	10335
3	18	8052.5	19	8019.8	23	7947.2	26	7897.1
4	24	7868.5	25	7893.5	30	7845.3	34	7808.2
5	30	7784.1	31	7799.8	37	7775.6	42	7757.7
10	60	7723.8	61	7724.1	72	7722.9	82	7721.0
14	84	7720.6	85	7720.5	100	7720.3	114	7719.9
15	90	7720.4	91	7720.3	107	7720.1		
16	96	7720.2	97	7720.0	114	7720.0		

Approximate Solution by Ritz Method: Critical Buckling Load = 7719.5 *lb*

The results for the non-symmetric combinations ($v_3 - w_n$) of both trigonometric and polynomial hierarchical terms for the quasi-isotropic laminate are given in Table 5.59. These results show that this group of combinations behaves similarly to that of the previous group of combinations ($v_2 - w_n$). But within the group of these combinations, the convergence gets better with the increase of each trigonometric and polynomial hierarchical term associated with the radial displacement (w) function. The best critical buckling load value for polynomial hierarchical formulation is given by the combination ($v_3 - w_1$) by using 14-elements mesh. Trigonometric hierarchical formulation once again follows similar trends as before and all the combinations of this group give one value of critical buckling load by using 17-elements mesh.

The results for the group of combinations ($v_4 - w_n$) for both trigonometric and polynomial hierarchical interpolation terms are given in Table 5.60. There is a gradual improvement in the convergence of the critical buckling load values with polynomial hierarchical terms. The combination ($v_4 - w_1$) provides the best convergence of the critical buckling load value among all the other combinations of this group for polynomial hierarchical formulation. This value is obtained at the 11-elements mesh compared to 17-elements mesh for the conventional FEM. All the combinations of this group for trigonometric hierarchical formulation give the same value of critical buckling load after converging to a single value as we increase the number of elements.

Table 5.59 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_3 - w_n$) for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for Quasi Isotropic Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 3v-0w HFEM terms		*Non-Symmetric Trigonometric 3v-1w HFEM terms*		*Non-Symmetric Trigonometric 3v-2w HFEM terms*		*Non-Symmetric Trigonometric 3v-4w HFEM terms*	
1	5	0.49×10^7	6	0.49×10^7	7	0.90×10^6	9	7864.0
2	12	13942	13	9229.1	14	9187.1	17	9180.8
3	19	8052.7	20	7981.8	21	7981.4	25	7975.5
4	26	7868.5	27	7902.3	28	7901.8	33	7903.9
5	33	7784.1	34	7804.7	35	7804.6	41	7806.4
10	68	7723.8	69	7724.7	70	7723.0	81	7725.0
14	96	7720.6	97	7720.7	98	7720.4	113	7720.9
15	103	7720.4	104	7720.4	105	7720.2	121	7720.5
16	110	7720.2	111	7720.2	112	7720.2	129	7720.3
17	117	7720.0	118	7720.0	119	7720.0	137	7720.1
	Non-Symmetric Polynomial 3v-0w HFEM terms		*Non-Symmetric Polynomial 3v-1w HFEM terms*		*Non-Symmetric Polynomial 3v-2w HFEM terms*		*Non-Symmetric Polynomial 3v-4w HFEM terms*	
1	5	0.49×10^7	6	0.49×10^7	7	0.93×10^6	9	0.80×10^6
2	12	13940	13	11254	14	10562	17	10571
3	19	8052.5	20	7982.1	21	7998.3	25	7968.6
4	26	7868.5	27	7856.2	28	7886.6	33	7865.6
5	33	7784.1	34	7780.6	35	7794.4	41	7786.5
10	68	7723.8	69	7722.1	70	7723.6	81	7752.7
14	96	7720.6	97	7720.0	98	7720.5	113	7722.3
15	103	7720.4			105	7720.2	121	7720.1
16	110	7720.2			112	7720.0	129	7720.0
17	117	7720.0						

Table 5.60 Critical Buckling Load Calculated by using Non- Symmetric Trigonometric and Polynomial Hierarchical Terms ($v_4 - w_n$) for Fixed-Fixed Boundary Condition

8 D.O.F. Composite Curved Beam Element for Quasi Isotropic Laminate								
Number of Elements	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)	Number of D.O.F.	Critical Buckling Load (lb)
	Non-Symmetric Trigonometric 4v-0w HFEM terms		*Non-Symmetric Trigonometric 4v-1w HFEM terms*		*Non-Symmetric Trigonometric 4v-2w HFEM terms*		*Non-Symmetric Trigonometric 4v-3w HFEM terms*	
1	6	0.49×10^7	7	0.49×10^7	8	0.90×10^6	9	-
2	14	13942	15	9199.8	16	9154.9	17	9163.2
3	22	8052.7	23	7981.1	24	7981.3	25	7979.6
4	30	7868.5	31	7902.1	32	7901.6	33	7904.2
5	38	7784.1	39	7804.3	40	7804.4	41	7806.4
10	78	7723.8	79	7724.6	80	7724.7	81	7725.0
12	94	7721.6	95	7721.9	96	7721.9	97	7721.1
15	118	7720.6	119	7720.4	120	7720.4	121	7720.5
16	126	7720.2	127	7720.2	128	7720.2	129	7720.3
17	134	7720.0	135	7720.0	136	7720.0	137	7720.1
	Non-Symmetric Polynomial 4v-0w HFEM terms		*Non-Symmetric Polynomial 4v-1w HFEM terms*		*Non-Symmetric Polynomial 4v-2w HFEM terms*		*Non-Symmetric Polynomial 4v-3w HFEM terms*	
1	6	0.49×10^7	7	-	8	-	9	-
2	14	13940	15	11046	16	10470	17	10546
3	22	8052.5	23	7925.3	24	7937.1	25	7970.7
4	30	7868.5	31	7811.9	32	7835.3	33	7866.4
5	38	7784.1	39	7757.4	40	7769.3	41	7787.2
10	78	7723.8	79	7720.6	80	7722.1	81	7723.7
12	94	7721.6	95	7719.9	96	7720.7	97	7721.5
15	118	7720.6			120	7720.0	121	7720.3
16	126	7720.2					129	7720.1
17	134	7720.0					137	7720.0

5.4 Conclusion

Based on the results obtained in the preceding sections for different variations in terms of different laminate configurations and different boundary conditions in the buckling analysis of the composite curved beams, we can summarize some key conclusions in the following discussion.

First, critical buckling load values are calculated for fixed-free boundary condition for four different laminate configurations $[0]_{16s}$, $[90]_{16s}$, angle ply $[\pm 45]_{8s}$ and quasi-isotropic $[0/90/+45/-45]_{4s}$ laminate. The results for the fixed-free boundary condition show that, though trigonometric hierarchical formulation gives better convergence of the critical buckling load than that of polynomial hierarchical formulation, the difference in the results given by these two types of formulations is not much significant. Even in some cases, polynomial hierarchical terms give results that are slightly better than that of trigonometric hierarchical terms. Generally, the non-symmetric combination with maximum number of hierarchical terms associated with the tangential (v) displacement function and no hierarchical term used with the radial (w) displacement function i.e. ($v_4 - w_0$) provides the best convergence of the critical buckling load values for both trigonometric and polynomial hierarchical formulations. With regard to the fiber orientations, the $[0]_{16s}$ laminate has the maximum stiffness and gives maximum values of critical buckling load for both trigonometric and polynomial hierarchical formulations. $[90]_{16s}$ laminate gives the minimum value of the critical buckling load. The critical buckling load values for the $[+45/-45]_{8s}$ and quasi-isotropic laminate are closer to each other. The critical buckling load value given by quasi-isotropic laminate is more than the value given by the $[+45/-45]_{8s}$ laminate.

The results for all the above mentioned laminate configurations for fixed-fixed boundary condition give very interesting results which are different from the results given by fixed-free boundary condition. The first observation is that critical buckling load values for the fixed-fixed boundary condition are a lot higher than the values given by the same laminate with fixed-free boundary condition which could be an important factor in designing the structures for buckling loads. For fixed-fixed boundary condition, difference between the results given by the trigonometric and polynomial hierarchical terms is not much significant. Polynomial hierarchical formulation gives far better and faster convergence for the critical buckling load values compared to the trigonometric hierarchical formulation. Contrary to the fixed-free boundary condition, the combination with one trigonometric hierarchical term associated with the tangential (v) displacement function and maximum number of hierarchical terms with the radial (w) displacement function i.e. ($v_1 - w_4$) provides the best convergence of the critical buckling load values for all the laminates. Trigonometric hierarchical formulation also shows very different behavior; for all the above mentioned laminate configurations all the symmetric and non-symmetric combinations give results that converge to a single value when we increase the number of elements showing no change in the convergence of the critical buckling load values for all the different combinations used in the buckling analysis.

Chapter 6

Conclusions

In the present thesis the hierarchical finite element formulations for the analysis of composite curved beam have been developed. The developed formulations have been adapted so as to be applicable and appropriate to 1-D thin composite curved beams. Central deflection values of isotropic and composite curved beams are calculated using hierarchical finite element formulation. The buckling analysis of composite curved beam has been conducted using the developed formulations. Two sub-formulations of the Hierarchical Finite Element Method (HFEM), viz. trigonometric and polynomial formulations, have been considered and their applications to 1-D thin composite curved beams have been made.

Prior to the introduction of the HFEM formulation, the conventional finite element formulation has been derived in detail to systematically bring out the efficiency, accuracy and the advantage of the HFEM formulation. This has been done to make evident the basic aspects of the conventional method and the enhancement that are made in it through the HFEM.

The conventional finite element model for the curved beam structure possesses four degrees of freedom at each nodal point: a tangential displacement v, a derivative of the tangential displacement $\partial v / \partial s$ or $v_{,s}$, a radial displacement w and a derivative of the radial displacement $\partial w / \partial s$ or w_{s} or slope θ so as to satisfy the geometric boundary conditions. The hierarchical finite element formulation enhances the capability of the element by making the degree of the approximating function to tend to infinity. This is done by making use of trigonometric and polynomial functions. The four cubic displacement modes used for each of the tangential (v) and radial (w) displacements used in the conventional formulation are retained. The higher order modes are selected from a variety of polynomial and trigonometric functions. Accordingly, the stiffness and mass matrices are set up.

The programming involving symbolic computation is done using MATLAB® software. A comparison between the results obtained using the conventional and the hierarchical formulations are inherent in all the cases. Results obtained by the hierarchical finite element formulation are also validated by using the approximate solution given by the Ritz method. To elaborate on the analysis in the present thesis, a parametric study using both types of the formulations is provided.

The parametric study considers various changes in the composite laminate to demonstrate their influences on the central deflection and on critical buckling loads in the buckling analysis. These changes include the change in the boundary conditions, change in the laminate configuration and change in the internal degrees of freedom.

The study carried out in this thesis is considered to be of importance to the mechanical designer, who designs and develops composite curved structures to withstand buckling loads. The important and principal conclusions are:

1) The hierarchical formulation developed in this thesis is much more efficient than the conventional formulation. Throughout the thesis, we have seen that the HFEM uses far less number of elements than the conventional FEM and gives more accurate answers much faster. In terms of the system degrees of freedom (D.O.F.) of the curved beam structure, the HFEM gives more accurate answers than the conventional FEM having the same number of system D.O.F. The inclusions of the internal degrees of freedom in the HFEM, lends more freedom to the inside of the laminate thereby increasing its efficiency to model the curved beam. Hence, more accurate results are obtained much faster using less number of elements and degrees of freedom.
2) Both the hierarchical sub-formulations perform better than the conventional formulation, both in terms of accuracy and speed of convergence. Also significant is that both the sub-formulations use less number of elements and system degrees of freedom to reach a more accurate answer. The trigonometric sub-formulation is more accurate than the polynomial sub-formulation for calculating the central deflection as demonstrated in the thesis. For the fixed-free boundary condition in the buckling analysis, trigonometric sub-formulation performs marginally better than the polynomial sub-formulation. For fixed-fixed boundary condition, polynomial sub-formulation

performs poorly while trigonometric sub-formulation performs far better to calculate the critical buckling load.

3) The parametric study performed on the composite curved beam gives a comprehensive understanding of its behavior under different physical conditions. The values of central deflection and critical buckling load calculated for composite curved beam are different for different laminate configurations and internal degrees of freedom. Central deflection values of curved composite beams are calculated for fixed-free boundary condition while the values of critical buckling load are calculated for fixed-free and fixed-fixed boundary condition. For fixed-free boundary condition, the optimum critical buckling load value given by the combination that provides the best convergence for both trigonometric and polynomial hierarchical formulations for one type of laminate shows small difference in critical buckling load value in comparison to the Ritz solution. For fixed-fixed boundary condition we observe the same response for both trigonometric and polynomial hierarchical formulations by all the laminates where the converged critical buckling load value matches the approximate Ritz solution better than the fixed-free boundary condition.

The primary contributions have been mentioned in the respective chapters and are summarized as follows:

1) The trigonometric and polynomial hierarchical finite element formulations have been proposed and applied in the analysis of isotropic and composite curved beams for calculating the central deflection and critical buckling load.
2) A set of polynomials is proposed for the polynomial hierarchical formulation that fares better than the conventional FEM formulation.
3) Both forms of the hierarchical formulations, viz. trigonometric and polynomial sub-formulations have been applied to the analysis of the composite curved beam using all possible symmetric and non-symmetric combinations.
4) Buckling analysis of composite curved beam for calculating the critical buckling load is carried out using both the sub-formulations and two boundary conditions.

The following recommendations may be considered for future studies:

i) The HFEM can be applied to the stress analysis of the composite curved beams.
ii) The HFEM considered in this thesis has been used to analyze a 1-D composite curved beam. This study can be extended to a 3-D composite cylindrical shell.

iii) The HFEM considered in this thesis uses cubic-cubic circularly curved beam finite element. There are many other circularly curved beam finite elements available e.g. cubic v and quintic w, quintic v and cubic w, quintic v and quintic w, constant strain and linear curvature, etc. which can be used for the further study to obtain even better results.

iv) Curved beam finite element with non uniform cross sections or variable radii of the curvatures can also be included in the future work.

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