Some Inference Problems for Inverse Gaussian Data

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Abstract

Some Inference Problems for Inverse Gaussian Data

Debaraj Sen

This thesis deals with some inference problems related with inverse Gaussian models. In Chapter 2, we investigate the properties of an estimator of mean of an inverse Gaussian population that is motivated from finite population sampling [see Chaubey and Dwivedi (1982)]. We demonstrate that when the coefficient of variation is large, the new estimator performs much better than the usual estimator of the mean, namely the sample average. In Chapter 3, we provide simple approximating formulae for the first four moments of the new estimator which may be used to approximate its finite sample distribution. Chapter 4 investigates some properties of the preliminary test estimator for mean of an IG population. Such an estimator was proposed and studied in detail in the statistical literature for Gaussian and other distributions [see Bancroft (1944), Ahmed (1992)]. Our conclusions for the inverse Gaussian model are similar to the case for Gaussian model. Next, in Chapter 5, overlap measures for two inverse Gaussian densities are studied on the lines of Mulekar and Mishra (1994, 2000).

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List of Symbols and Notations

Chapter 1

$IG(\mu,\lambda)$	:	Inverse Gaussian distribution with mean μ and dispersion λ.
$\bar{X}$	:	Sample mean.
$f(x;\mu,\lambda)$	:	Probability density function of $IG(\mu,\lambda)$.
$F(x;\mu,\lambda)$	:	Cumulative distribution function of $IG(\mu,\lambda)$.
$\varphi(t)$	:	Characteristic function of $IG(\mu,\lambda)$.
$K(t)$	:	Cumulant generating function $IG(\mu,\lambda)$.
β_1,β_2	:	Coefficients of Skewness and Kurtosis.

Chapter 2

C	:	Coefficient of Variation.
η	:	Square of the coefficient of variation.
$\tilde{\mu}=\frac{1}{1+\frac{\eta}{n}}\bar{X}$	:	Minimum mean square estimator for μ.
$\hat{\mu}$	:	Srivastava and Thompson estimator for μ.
$\tilde{\mu}_k$	:	Chaubey and Dwivedi estimator for μ.
$\hat{\mu}_k$	:	Proposed estimator for μ.
$L_j^{(r)}(x)$	:	Generalized Laguerre polynomial of order j with index r.

Chapter 3

ψ : $\frac{1}{\lambda}$.

η : $\frac{\mu}{\lambda}$ = Square of the coefficient of variation for $IG(\mu, \lambda)$.

Chapter 4

$\hat{\mu}_{PTE}$: Preliminary test estimator of mean.

CR : Critical region.

$\pi(\mu)$: Power function.

Chapter 5

ρ : Matusita's Measure.

δ : Morisita's Measure.

α^* : Pianka's Measure.

Δ : Weitzman's Measure.

$K_j(u)$: Modified Bessel function of order j.

$\Phi(.)$: Cumulative density function of standard normal distribution.

$\phi(.)$: Probability density function of standard normal distribution.

Chapter 1

Inverse Gaussian Model, Preliminaries and an Overview of the Thesis

1.1 Introduction

The normal distribution, because of its analytical elegance and the simplicity of the associated inference methods is ubiquitous in statistical modelling and data analysis. However, as reflected in Geary's (1947) provocative declaration, "Normality is a myth; there never was, and never will be a normal distribution", there have always been doubts and reservations about the common and uncritical use of the normality assumption. This is especially so when the data indicate pronounced skewness. For modelling such data, lognormal, gamma, Weibull and inverse Gaussian distributions are often recommended.

Chhikara and Folks (1989), in the introduction of their monograph on the inverse Gaussian family of distributions, argue that "Although the lognormal, gamma, and Weibull distributions enjoy extensive use in certain special areas, none of them allow for a wide range of statistical methods comparable to those based on the normal distribution" and illustrate the argument using "one and two-sample t-tests, analysis of variance, confidence intervals, regression analysis, and so on". They also note that "Although, some exact results exist for all of these distributions, they have not lent themselves to the development of a comprehensive statistical methodology for skewed data analysis." Their review paper read before the Royal Statistical Society in 1978, which highlighted the remarkable similarities between the properties and the inference procedures for the normal and the inverse Gaussian families, was received with considerable interest and enthusiasm. It lead to a widespread use of inverse Gaussian models for skewed data, a large follow-up research on the family and publication of the monographs by Chhikara and Folks (1989), Seshadri (1993), and an updated chapter in the recent edition of Johnson, Kotz and Balakrishnan (1994). Jørgensen's (1982) monograph presents a generalization of the inverse Gaussian family.

Natarajan and Mudholkar (2002) have catalogued some more analogues in addition to those brought out by Chhikara and Folks (1989). This distribution is now widely used in modelling positive and positively skewed data in such diverse areas of applied research as cardiology, hydrology, demography, linguistics, employment service, labor disputes and finance. The following references may be cited with respect to applications in different areas: Bhattacharyya and Fries (1982a,b) and Chhikara and Folks (1977) in reliability, Whitmore (1979) and Padgett and Tsai (1986), Eaton and Whitmore (1977) in social sciences, Banerjee and Bhattacharyya (1976) in marketing, Chaubey *et al.* (1998) in actuarial science, Marcus (1975) in traffic engineering, Lancaster (1972) in industrial management, Wise (1966) in cardiology, and Hasofer (1964) in civil engineering. In a recent paper, Huberman *et al.* (1998) use the data from America On Line, to provide an interesting application in the area of internet to the distribution of the number of links an internet user follows before the page value reaches a threshold. Chaubey (1991) and Chaubey, Nebebe and Chen (1996) have demonstrated the use of inverse Gaussian distribution as a super population model in finite population sampling. Further, Babu and Chaubey (1996) and Chaubey (2001) investigated Bootstrap method in the context of an inverse Gaussian regression model introduced by Bhattacharyya and Fries (1982a). Chaubey and Nebebe (1999) demonstrate the difficulties of a Bayesian analysis for one-way ANOVA using inverse Gaussian distribution in contrast to Gaussian distribution, where as Mudholkar, Natarajan and Chaubey (2001) proposed a goodness-of-fit test based on the following characterization due to Khatri (1962): A random sample $(X_1, X_2, ..., X_n)$ is from an inverse Gaussian population if and only if, $\bar{X} = \sum X_i/n$ and $V = \{\sum_{i=1}^{n}(1/X_i - 1/\bar{X})\}/n$ are independently distributed, assuming that the expected values of X, X^2, $1/X$ and $1/\sum X_i$ exist and are different from zero., [see also Seshadri (1983)], as in Lin and Mudholkar (1980) for the Gaussian distribution.

1.2 Preliminaries of the Inverse Gaussian Distribution

The inverse Gaussian distribution was obtained almost simultaneously and independently by Schrödinger (1915) and Smoluchowski (1915) as that of the first passage time in Brownian motion with positive drift. It appeared later in Hadwiger (1940), Halphen's work published as Dugué (1941) (see Seshadri and Law (1997)), Wald (1947) and Tweedie (1945). It was Tweedie who observed that the cumulant generating function of the inverse Gaussian distribution is the inverse of that of the Gaussian distribution and aptly gave its name in current usage. In what follows, we catalogue some basic properties relating to inverse Gaussian distribution, which can be found in Chhikara and Folks (1989).

The probability density function *(p.d.f.)* of the inverse Gaussian random variable X, with mean μ and dispersion parameter λ, denoted by $X \sim IG(\mu, \lambda)$, is given by

$$.f(x|\mu,\lambda) = \left\{\frac{\lambda}{2\pi x^3}\right\}^{1/2} \exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\}, \ x > 0, \ \mu > 0, \ \lambda > 0. \ (1.2.1)$$

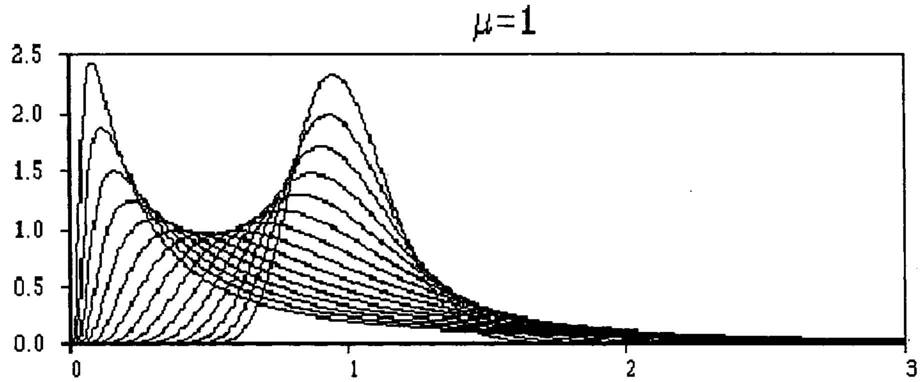

Figure 1.1: Graphs of Probability Densities of IG Distribution with Varying λ.

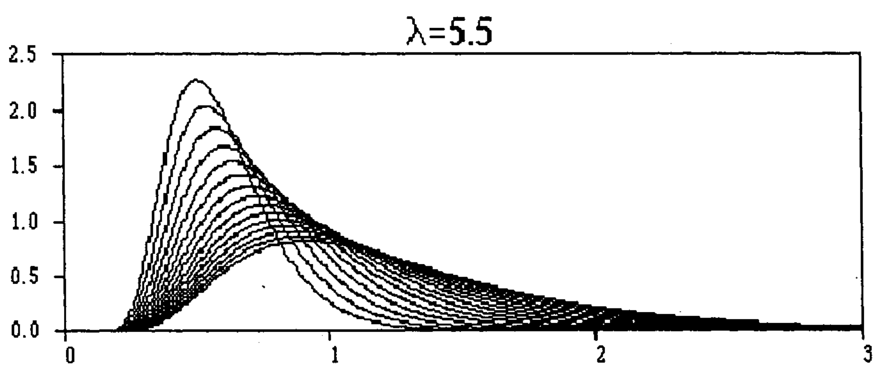

Figure 1.2: Graphs of Probability Densities of IG Distribution with Varying μ.

Figure 1.1 shows the graphs of probability density functions for $IG(\mu, \lambda)$ for $\mu = 1$ and varying λ, whereas Figure 1.2 displays those for $\lambda = 5.5$ with varying values of μ. These graphs clearly show that the inverse Gaussian distribution can be used to accommodate models covering a large range of shapes.

The cumulative distribution function *(c.d.f.)* of $IG(\mu, \lambda)$ can be written as

$$.F(x \,|\mu, \lambda) = \Phi \left\{ \sqrt{\frac{\lambda}{x}} \left(\frac{x}{\mu} - 1 \right) \right\} + \exp\{\frac{2\lambda}{\mu}\} \, \Phi \left\{ -\sqrt{\frac{\lambda}{x}} \left(\frac{x}{\mu} + 1 \right) \right\}, \quad (1.2.2)$$

where $\Phi(.)$ denotes the *c.d.f.* of a standard normal variable. The characteristic function of $IG(\mu, \lambda)$ is

$$\varphi(t) = \exp \left[\frac{\lambda}{\mu} \left\{ 1 - \left(1 - \frac{2\mu^2 it}{\lambda} \right)^{1/2} \right\} \right], \quad (1.2.3)$$

and the corresponding cumulant generating function is given by

$$K(t) = \frac{\lambda}{\mu} \left\{ 1 - \left(1 - \frac{2\mu^2 t}{\lambda} \right)^{1/2} \right\}. \quad (1.2.4)$$

Thus, for any integer $r \geq 1$, the r^{th} cumulant of $IG(\mu, \lambda)$ is given by

$$\kappa_r = 1 \times 3 \times 5 \cdots \times (2r - 3) \frac{\mu^{2r-1}}{\lambda^{r-1}}. \quad (1.2.5)$$

The first four cumulants of the $IG(\mu, \lambda)$ distribution are, therefore, given by

$$\kappa_1 = \mu, \ \kappa_2 = \frac{\mu^3}{\lambda}, \ \kappa_3 = \frac{3\mu^5}{\lambda^2}, \ \kappa_4 = \frac{15\mu^7}{\lambda^3}. \quad (1.2.6)$$

The coefficients of skewness and kurtosis of the family are

$$\sqrt{\beta_1} = 3\sqrt{\frac{\mu}{\lambda}}, \ \beta_2 = 3 + \frac{15\mu}{\lambda}, \quad (1.2.7)$$

and, hence, in the Pearson (β_1, β_2) plane the inverse Gaussian points fall on the straight line $\beta_2 = 3 + 5\beta_1/3$ which lies between the gamma (Type III) and reciprocal gamma (Type V) lines. In terms of (β_1, β_2), $IG(\mu, \lambda)$ family is very close to the lognormal family.

This thesis deals with some inference problems similar to the case of Gaussian data, when the population concerned follows an inverse Gaussian distribution. The following section presents an overview.

1.3 Overview of the Thesis

Chapter 2 is motivated by the problem of estimation of the mean of a finite population following an inverse Gaussian distribution. In many such applications, the coefficient of variation is assumed known, in which case the usual sample mean can be improved (Searles (1964)). This motivates our estimator, where we consider a plug-in estimator by replacing the coefficient of variation by its estimate. Such an estimator has been studied for the Gaussian case by Srivastava (1974) and Chaubey and Dwivedi (1982). We investigate how far these results hold for the inverse Gaussian case with respect to bias and mean square error relative to the sample mean. Chapter 3 investigates the finite sample distribution of the estimator introduced in Chapter 2. Here, we develop an approximation to moments intended to be used in an Edgeworth expansion as studied in Chaubey and Srivastava (1996) in the context of lognormal distribution.

Chapter 4 introduces a Preliminary Test Estimator for the mean of an inverse Gaussian population similar to that studied for the case of a Gaussian population by Bancroft (1944, 1963), Paul (1950), Huntsberger (1954a) and others. In practice, the preliminary test of significance uses the data in hand as an aid in determining an appropriate model for some subsequent inferences. Such tests are used in establishing a prior guess from an expert as a credible value which is then built-in the estimator.

Chapter 5 is devoted to the study of overlap measures of densities of two inverse Gaussian populations. Overlap measures have often been used to study similarity between two populations. This chapter is motivated from the study by Mulekar and Mishra (1994, 2000) concerning two Gaussian populations. Here we obtain parallel results to those obtained by Mulekar and Mishra (1994).

The final chapter outlines directions of further research with respect to IG populations regarding hypothesis tests for coefficients of variation, preliminary test estimation for $k(\geq 2)$ samples and generalization of overlap measures.

Chapter 2

Properties of An Estimator of Mean of An Inverse Gaussian Population

2.1 Introduction

Consider a population with mean μ and variance σ^2. It is well known that when the coefficient of variation (CV) $C = \frac{\sigma}{\mu}$ is known, the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, based on a random sample $X_1, X_2, ..., X_n$ can be improved by considering the class of estimators $\{k\bar{X}, k > 0\}$. The minimum mean square error estimator for μ in this situation is given by (see Searles (1964))

$$\tilde{\mu} = \frac{1}{1 + \frac{\eta}{n}} \bar{X}, \tag{2.1.1}$$

where η denotes the square of the coefficient of variation, namely,

$$\eta = C^2 = \frac{\sigma^2}{\mu^2}. \tag{2.1.2}$$

For a variety of populations, the sample mean is the best estimator in the sense of being unbiased and having smallest variance. On the other hand, the estimator given in Eq. (2.1.1) introduces a little bias but leads to significant reduction in the mean square error (MSE). This improved estimator, however, requires the exact value of the coefficient of variation. When this exact value of the coefficient of variation is not available, Srivastava (1974) and Thompson (1968) proposed estimators similar to that given in Eq. (2.1.1), but with η replaced by $\hat{\eta} = \frac{S^2}{\bar{X}^2}$, $\bar{X} > 0$, where S^2 is the sample variance, leading to the estimator

$$\hat{\mu} = \frac{\bar{X}}{1 + \frac{S^2}{n\bar{X}^2}}. \tag{2.1.3}$$

For $\bar{X} = 0$, the estimator is defined to be zero. This estimator loses the optimal property of the estimator in Eq. (2.1.1) Srivastava (1980) studied the finite sample bias and MSE properties of $\hat{\mu}$, for the case of a Gaussian population, where as Chaubey and Dwivedi (1982) studied similar properties of a modified estimator

$$\tilde{\mu}_k = \frac{\bar{X}}{1 + k\frac{\hat{\eta}}{n}}, \tag{2.1.4}$$

where k is a non-negative constant. Our aim is to investigate such properties for the case of an inverse Gaussian population. For a random sample $(X_1, X_2, ..., X_n)$, from an $IG(\mu, \lambda)$ population, a minimal sufficient statistic for

(μ, λ) is given by $(\bar{X}, n/\sum_{i=1}^{n}\frac{1}{X_i})$ (see Chhikara and Folks (1989), pp. 56). It is also interesting to note (Chhikara and Folks (1989), pp. 61) that

$$\bar{X} \sim IG(\mu, n\lambda) \quad \text{and} \quad \lambda \sum_{i=1}^{n}\left(\frac{1}{X_i} - \frac{1}{\bar{X}}\right) \sim \chi^2_{n-1}. \tag{2.1.5}$$

As such, $\bar{X}$ provides the best unbiased estimator of μ and

$$U = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{1}{X_i} - \frac{1}{\bar{X}}\right) \tag{2.1.6}$$

provides that for $\frac{1}{\lambda}$, and moreover they are independent. The square of the coefficient of variation η, in this case, is therefore given by $\eta = \frac{\mu}{\lambda}$ and its natural estimator is given by

$$\hat{\eta} = \bar{X}U. \tag{2.1.7}$$

Using the above estimator in Eq. (2.1.4), we aim to investigate the bias and MSE properties of the estimator

$$\hat{\mu}_k = \frac{\bar{X}}{1 + k\frac{\bar{X}U}{n}}. \tag{2.1.8}$$

When the estimation of mean concerns a normal distribution then unbiasedness may be an important criterion, but for skewed distributions robust estimation of the tail may be more important. However, this thesis does not address the robustness issue.

In Section 2, we develop some series expansions for the moments of $\hat{\mu}_k$, similar to those derived in Chaubey and Dwivedi (1982). Section 3 presents these moments in an univariate integral form that is found to be suitable for computation. The final section presents a comparison of the new estimator with the sample mean.

2.2 A Series Representation for the Moments of $\hat{\mu}_k$

Theorem 2.1 *For $0 < k < n-1$, the r^{th} moment of $\hat{\mu}_k = \frac{\bar{X}}{1+k\frac{\bar{X}U}{n}}$ may be given as the following convergent infinite series*

$$E(\hat{\mu}_k^r) = (n\lambda)^r \sum_{i=0}^{\infty} \frac{(r+i-1)!}{i!(r-1)!} \left(\frac{n-k-1}{n-1}\right)^i E\left(\frac{V^i}{(W+V)^{r+i}}\right),$$

where $V \sim \chi^2(n-1)$, $\frac{1}{W} \sim IG(n^{-1}\eta, 1)$ and $V \overset{ind.}{\sim} W$.

PROOF: We can write

$$\hat{\mu}_k = \frac{n\lambda\, Q}{1+k\frac{QV}{n-1}}, \tag{2.2.1}$$

where

$$Q = \frac{\bar{X}}{n\lambda} \text{ and } V = (n-1)\lambda U.$$

Further, letting $W = \frac{1}{Q}$, we find that

$$\hat{\mu}_k = \frac{n\lambda}{W + (\frac{k}{n-1})V} \tag{2.2.2}$$

$$= \frac{n\lambda}{W+V}\left[1 - \frac{n-k-1}{n-1}\frac{V}{W+V}\right]^{-1} \tag{2.2.3}$$

Since, $\left(\frac{n-k-1}{n-1}\right)\frac{V}{(W+V)} < 1$ for $0 < k < n-1$, using a negative binomial expansion gives the representation in the theorem.

□

The formula developed in Theorem 2.1 forms a basis for developing other computational formulae for exact moments of $\hat{\mu}_k$. First, we present a double series representation for the exact moments similar to the one developed in Srivastava (1980).

Theorem 2.2 *For $0 < k < n-1$, the r^{th} moment of $\hat{\mu}_k$ may be represented as*

$$\begin{aligned} E(\hat{\mu}_k^r) &= \frac{(n\lambda)^r}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}\sum_{i=0}^{\infty}\sum_{j=o}^{\infty}\frac{(r+i-1)!}{i!(r-1)!}\left(\frac{n-k-1}{n-1}\right)^i\frac{1}{(\frac{\nu}{2}+i+j)} \\ & E\left(W^{\frac{\nu}{2}-r}L_j^{(-1/2)}\left(\frac{1}{2m^2W}\right)\right), \end{aligned} \tag{2.2.4}$$

where $1/W \sim IG(m,1), m = \frac{\eta}{n}, \nu = n-1$, and $L_j^{(r)}(x)$ denotes the generalized Laguerre polynomial of order j with index r given by

$$L_j^{(r)}(x) = \sum_{i=0}^{j}(-1)^j\frac{(j+r)!}{(j-i)!(j+r)!}\frac{x^i}{i!} \tag{2.2.5}$$

The expression for the first and second moments may now be obtained by putting $r=1$ and $r=2$ respectively, as given by

$$E(\hat{\mu}_k) = \frac{n\lambda}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{n-k-1}{n-1}\right)^i\frac{1}{(\frac{\nu}{2}+i+j)}E\left(W^{\frac{\nu}{2}-1}L_j^{(-1/2)}\left(\frac{1}{2m^2W}\right)\right), \tag{2.2.6}$$

$$E(\hat{\mu}_k^2) = \frac{(n\lambda)^2}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{n-k-1}{n-1}\right)^i\frac{(i+1)}{(\frac{\nu}{2}+i+j)}E\left(W^{\frac{\nu}{2}-2}L_j^{(-1/2)}\left(\frac{1}{2m^2W}\right)\right), \tag{2.2.7}$$

These expressions provide a convenient way to find the moments as the expectations in Eqs. (2.2.6) and (2.2.7) can be evaluated as follows,

$$E\left(W^{\frac{\nu}{2}-r}L_j^{(-\frac{1}{2})}\left(\frac{1}{2m^2W}\right)\right)=\sum_{s=0}^{j}\frac{(j-\frac{1}{2})!}{(j-s)!(s-\frac{1}{2})!}\left(\frac{1}{2m^2}\right)^s\frac{1}{s!}E\left[W^{\frac{\nu}{2}-s-r}\right], \tag{2.2.8}$$

The formulae for the r^{th} moment of W for any integral value of r can be obtained as (see Chhikara and Folks (1989)),

$$E(W^r)=\frac{1}{m^r}\sum_{s=0}^{r}\frac{(r+s)!}{s!(r-s)!}\left(\frac{m}{2}\right)^s, r=1,2,.. \tag{2.2.9}$$

PROOF: To derive the formula for the rth moment given in Eq. (2.2.4), we need to evaluate $E[V^i/(W+V)^{i+r}]$. Note that the joint distribution of (W,V) is given by

$$f(w,v)=\frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})\sqrt{2\pi w}}v^{\frac{\nu}{2}-1}e^{-\frac{v}{2}}e^{-\frac{(1-mw)^2}{2m^2w}}, w,v>0$$

hence

$$E\left[\frac{V^i}{(W+V)^{(i+r)}}\right]=\int_0^\infty\int_0^\infty\frac{v^i}{(w+v)^{i+r}}f(w,v)dwdv$$

$$=c\int_0^\infty\int_0^\infty\frac{v^{i+\frac{\nu}{2}-1}w^{-\frac{1}{2}}}{(w+v)^{i+r}}e^{-\frac{1}{2}\ (w+v)+\frac{1}{m^2(w+v)}+\frac{v}{m^2w(w+v)}}\ dwdv \tag{2.2.10}$$

where $c=\frac{e^{\frac{1}{m}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})\sqrt{2\pi}}$. Using the transformation $w=w_1(1-w_2), v=w_1w_2,\ 0<w_2<1,\ 0<w_1<\infty$, the above equation can be written as

$$E\left[\frac{V^i}{(W+V)^{(i+r)}}\right] = c\int_0^\infty\int_0^1 w_1^{\frac{\nu-1}{2}-r}w_2^{i+\frac{\nu}{2}-1}(1-w_2)^{-\frac{1}{2}}e^{-\frac{1}{2}\ w_1+\frac{1}{m^2w_1}} e^{-\frac{1}{2m^2w_1}\frac{w_2}{1-w_2}}dw_2dw_1 \tag{2.2.11}$$

Now using the generating function of the generalized Laguerre polynomials [see Szegö, 1967, (Eq. 5.1.9)] given by,

$$\sum_{j=0}^{\infty} L_j^{(r)}(x)t^j = (1-t)^{(-r-1)} \mathrm{e}^{-\frac{xt}{1-t}}, \qquad x > 0, \tag{2.2.12}$$

we can write, using $x = \frac{1}{(2m^2 w_1)}$,

$$(1-w_2)^{-\frac{1}{2}} \, \mathrm{e}^{-\frac{1}{2m^2 w_1}\frac{w_2}{1-w_2}} = \sum_{j=0}^{\infty} w_2^j L_j^{(-\frac{1}{2})}\left(\frac{1}{2m^2 w_1}\right).$$

Using the fact that

$$g(w_1) = \frac{1}{\sqrt{2\pi w_1}} \, \mathrm{e}^{\frac{1}{m}} \, \mathrm{e}^{-\frac{1}{2}(\frac{1}{m^2 w_1}+w_1)}$$

represents the probability density function of reciprocal of $IG(m, 1)$ random variable and

$$\int_0^{\infty} w_2^{\frac{\nu}{2}+i+j-1} dw_2 = \frac{1}{(\frac{\nu}{2}+i+j)},$$

in Eq. (2.2.11) together with Eq. (2.2.12) the result in Eq. (2.2.4) follows.

□

The representation given in the above theorem requires a large number of terms for computation and the truncation error is not stable. Such problems were also reported in Chaubey and Dwivedi (1982) for computation of the moments in the Gaussian case using a similar series representation. In addition, the computational complexity increases here, because, we also have to evaluate the value of Laguerre series coefficients and a large number of moments involving an inverse Gaussian random variable. The exact formula for moments given in Eq.

(2.2.9) is suitable only for sample sizes which are even. Hence, we follow the approach of Chaubey and Dwivedi (1982) to develop an alternative expression in the form of a univariate integral, which is found to be more appropriate for computational purposes.

2.3 An Univariate Integral Representation for $E(\hat{\mu}_k^r)$

The following lemma is useful in developing a univariate integral representation of $E(\hat{\mu}_k^r)$.

Lemma 2.1 *(Chaubey and Dwivedi, 1982) Let the random variable X_2 be positive almost everywhere and assume that the joint moment generating function of X_1, X_2*

$$M(t_1, t_2) = E[\exp(t_1 X_1 + t_2 X_2)],$$

exist in some neighbourhood of $(0,0)$ then for $r \geq 1$,.

$$E\left[\left(\frac{X_1}{X_2}\right)^r\right] = \frac{1}{\Gamma(r)} \int_{-\infty}^{0} (-t_2)^{r-1} \frac{\partial^r M(t_1, t_2)}{\partial t_1^r} \big|_{t_1=0} \, dt_2, \qquad r = 1, 2, ...$$

Proof: Since,

$$\frac{\partial^r M(t_1, t_2)}{\partial t_1^r} \big|_{t_1=0} = E[X_1^r e^{t_2 X_2}],$$

we have

$$\frac{1}{\Gamma(r)}\int_{-\infty}^{0}(-t_2)^{r-1}\frac{\partial^r M(t_1,t_2)}{\partial t_1^r}\Big|_{t_1=0}\,dt_2$$

$$\begin{aligned}
&= E\left[X_1^r\left(\frac{1}{\Gamma(r)}\int_{-\infty}^{0}(-t_2)^{r-1}e^{t_2X_2}dt_2\right)\right]\\
&= E\left[X_1^r\left(\frac{1}{\Gamma(r)}\int_{0}^{\infty}z^{r-1}e^{-X_2z}dz\right)\right]\\
&= E[X_1^rX_2^{-r}], r=1,2...,
\end{aligned}$$

because

$$\int_0^\infty z^{p-1}e^{-az}dz = a^{-p}\Gamma(p).$$

□

The above lemma helps us in establishing the following theorem.

Theorem 2.3 *The r^{th} raw moment of $\hat{\mu}_k$ is given by*

$$E(\hat{\mu}_k^r) = \frac{\mu^r}{\Gamma(r)(2c)^r}\int_0^1 g_r(w)dw \tag{2.3.1}$$

where, $g_r(w) = w^{r-1}(1-w)^{\frac{\nu}{2}-r-1}e^{\frac{n}{\eta}[1-(1+\frac{w\nu}{c(1-w)})^{\frac{1}{2}}]}\left(1+\frac{w\nu}{c(1-w)}\right)^{-\frac{1}{2}}$ *and* $c=\frac{k\eta}{n(n-1)}$.

PROOF: We can write $\hat{\mu}_k$ as

$$\hat{\mu}_k = \frac{\mu Z}{1+\frac{k\eta ZV}{n(n-1)}}, \tag{2.3.2}$$

where $Z=\frac{\bar{X}}{\mu}\sim IG(1,\frac{n}{\eta})$ and $V=(n-1)\lambda U\sim\chi^2_{n-1}, V \overset{ind}{\sim} Z$ *i.e.*

$$\frac{\hat{\mu}_k}{\mu} = \frac{Z}{1+cZV}, \tag{2.3.3}$$

where $c = \frac{k\eta}{n(n-1)}$. Now letting $Z_1 = Z$ and $Z_2 = 1 + cZV$, we find, the moment generating function of (Z_1, Z_2) as

$$
\begin{aligned}
M(t_1, t_2) &= E\left[e^{t_1 Z_1 + t_2 Z_2}\right] \\
&= E\left[e^{t_1 Z + t_2 + ct_2 ZV}\right] \\
&= e^{t_2} E\left[e^{t_1 Z + ct_2 ZV}\right] \\
&= e^{t_2} E_Z[e^{t_1 Z} E_{V|Z}(e^{ct_2 ZV})] \\
&= e^{t_2} E_Z\left[e^{t_1 Z}(1 - 2ct_2 Z)^{-\frac{\nu}{2}}\right]; \qquad -\infty < t_2 < \frac{1}{2cZ},
\end{aligned}
$$

because $M_{\chi^2_\nu}(t) = (1-2t)^{-\frac{\nu}{2}}$ for $t < \frac{1}{2}$. Therefore, using Lemma (2.1), we have

$$
\begin{aligned}
E\left(\frac{\hat{\mu}_k}{\mu}\right)^r &= E\left(\frac{Z_1}{Z_2}\right)^r \\
&= \frac{1}{\Gamma(r)} \int_{-\infty}^{0} (-t_2)^{r-1} \frac{\partial^r M(t_1, t_2)}{\partial t_1^r} |_{t_1=0}\, dt_2. \qquad (2.3.4)
\end{aligned}
$$

But,

$$
\frac{\partial^r M(t_1, t_2)}{\partial t_1^r} |_{t_1=0} = e^{t_2} E\left[Z^r (1 - 2ct_2 Z)^{-\frac{\nu}{2}})\right],
$$

hence, Eq. (2.3.4) becomes

$$
\begin{aligned}
E\left(\frac{\hat{\mu}_k}{\mu}\right)^r &= \frac{1}{\Gamma(r)} \int_{-\infty}^{0} (-t_2)^{r-1} e^{t_2} E_Z\left[Z^r(1 - 2ct_2 Z)^{-\frac{\nu}{2}}\right] dt_2 \\
&= \frac{1}{\Gamma(r)} \int_0^{\infty} t_2^{r-1} e^{-t_2} E_Z[Z^r (1 + 2ct_2 Z)^{-\frac{\nu}{2}}] dt_2 \\
&= \frac{1}{\Gamma(r)} \int_0^{\infty} t_2^{r-1} e^{-t_2} \int_0^{\infty} z^r (1 + 2ct_2 z)^{-\frac{\nu}{2}} f_z(z) dz dt_2
\end{aligned}
$$

Now, using the transformation $t_2 \rightarrow y = 2ct_2 z, dt_2 = \frac{dy}{2cz}$ gives

$$\begin{aligned} E\left(\frac{\hat{\mu}_k}{\mu}\right)^r &= \frac{1}{\Gamma(r)} \int_0^\infty \int_0^\infty e^{-\frac{y}{2cz}} \left(\frac{y}{2cz}\right)^{r-1} z^r (1+y)^{-\frac{\nu}{2}} f_Z(z) \frac{1}{2cz} dy dz \\ &= \frac{1}{\Gamma(r)(2c)^r} \int_0^\infty \int_0^\infty e^{-\frac{y}{2cz}} y^{r-1} (1+y)^{-\frac{\nu}{2}} f_Z(z) dy dz \\ &= \frac{1}{\Gamma(r)(2c)^r} \int_0^\infty y^{r-1} (1+y)^{-\frac{\nu}{2}} \left[\int_0^\infty e^{-\frac{y}{2cz}} f_Z(z) dz\right] dy \\ &= \frac{1}{\Gamma(r)(2c)^r} \int_0^\infty y^{r-1} (1+y)^{-\frac{\nu}{2}} e^{\frac{n}{\eta}[1-(1+\frac{y\eta}{cn})^{\frac{1}{2}}]} (1+\frac{y\eta}{cn})^{-\frac{1}{2}} dy \end{aligned}$$

Substituting, $y = \frac{w}{1-w}, dy = \frac{dw}{(1-w)^2}$, into the above integral we get

$$E\left(\frac{\hat{\mu}_k}{\mu}\right)^r = \frac{1}{\Gamma(r)(2c)^r} \int_0^1 w^{r-1} (1-w)^{\frac{\nu}{2}-r-1} e^{\frac{n}{\eta}[1-\left(1+\frac{w\nu}{k(1-w)}\right)^{\frac{1}{2}}]} \left(1 + \frac{w\nu}{k(1-w)}\right)^{-\frac{1}{2}} dw \tag{2.3.5}$$

$$= \frac{1}{\Gamma(r)(2c)^r} \int_0^1 g_r(w) dw \tag{2.3.6}$$

where, $g_r(w) = w^{r-1}(1-w)^{\frac{\nu}{2}-r-1} e^{\frac{n}{\eta}[1-\left(1+\frac{w\nu}{k(1-w)}\right)^{\frac{1}{2}}]} \left(1+\frac{w\nu}{k(1-w)}\right)^{-\frac{1}{2}}$. This completes the proof of Theorem 2.3.

□

2.4 Computations and Comparisons

We wish to evaluate the estimator proposed here in terms of its bias and mean square error. The criteria for comparison used are relative bias (RB) and relative mean square error (RMSE) as given below:

$$RB(\hat{\mu}_k) = \frac{E(\hat{\mu}_k) - \mu}{\mu} = \frac{E(\hat{\mu}_k)}{\mu} - 1, \tag{2.4.1}$$

$$RMSE(\hat{\mu}_k) = \frac{MSE(\hat{\mu}_k}{\mu^2} = \frac{E(\hat{\mu}_k^2)}{\mu^2} - 2\frac{E(\hat{\mu}_k)}{\mu} + 1. \qquad (2.4.2)$$

The values of k considered here are motivated by different estimators of λ. For the unbiased estimator of $\frac{1}{\lambda}$, $k = 1$; for the MLE of λ, $k = \frac{(n-1)}{n}$ and for the mode estimator $k = \frac{(n-1)}{(n-3)}$ (see Höglund (1974)). For various values of n and η, absolute relative biases are tabulated in Table 2.1, the relative MSE's are given in Table 2.2 and the relative efficiencies are given in Table 2.3.

For computational purpose, we need to evaluate the integral in Eq. (2.3.6). In practice, the computation of such type of functions arising in presents problems. In most of the situations, we may alleviate this problem by using the following technique. Let us denote w_0, the approximate value of w where $g(w)$ takes its maximum values. The value of w_0 can be found by plotting $g_r(w)$ against w, which may be computable. For the computation of the integral we use the area function in `R-code` given in Appendix A1.

This function works quite well, except for the cases where the function f may take extremely small values for a wide range of arguments. For example, if $f(x) < eps$ for x in an interval $(c, b), c < (a+b)/2$, and $f(a) = f(b) = 0$, the algorithm will produce a small but wrong value of the integral. To avoid such situations, we evaluate the integral over two intervals, (a, w_0) and (w_0, b), where w_0 is the approximate value of the argument where the function peaks. The approximate peak is obtained by taking the maximum of $g(w)$ evaluated over a grid of $w-$values.

Based on the graphs and tables referred to earlier, we draw the following conclusions, similar to those given in Chaubey and Dwivedi (1982):

(1) Relative mean square error decreases as n increases.

(2) There is a positive gain in efficiency of $\hat{\mu}_k$ over $\bar{X}$. Substantial gains in efficiency are achieved for small samples with large coefficient of variation.

(3) Efficiency and bias go in the same direction i.e. larger efficiency is accompanied by a larger bias, hence attention must be paid on the amount of bias in specific situations.

(4) Where we obtain positive gains in efficiency, estimator $\hat{\mu}_k$ seems preferable due to its higher efficiency.

(5) We see that, for $\eta \geq 0.1$, the gain in relative efficiency is always positive. Hence, we may wish to use $\hat{\mu}_k$ if a statistical test accepts the hypothesis $H_0 : \eta \geq 0.1$. This type of problem comes under the area of a preliminary test estimator. This is discussed in Chapter 4.

The integrand occurring in computations for the moments is very steep for small values of η. This necessitates special considerations in computations as mentioned in the previous section. For large sample sizes, adequate approximations may be developed. This is addressed in the next chapter.

Table 2.1: Absolute Relative Bias of $\hat{\mu}_k$

n	20	40	60	80	100
η		$k = \frac{n-1}{n}$			
00.01	0.000475	0.000243	0.000160	0.000123	0.000098
00.05	0.002374	0.001218	0.000819	0.000617	0.000495
00.10	0.004748	0.002437	0.001639	0.001234	0.000990
01.00	0.047167	0.024330	0.016373	0.012337	0.009897
05.00	0.216125	0.118380	0.080823	0.061228	0.049241
10.00	0.370854	0.222431	0.156437	0.120038	0.097181
		$k = 1.0$			
00.01	0.000499	0.000249	0.000162	0.000124	0.000098
00.05	0.002498	0.001249	0.000833	0.000624	0.000499
00.10	0.004996	0.002499	0.001666	0.001249	0.000999
01.00	0.049509	0.024938	0.016646	0.012491	0.009996
05.00	0.224471	0.120999	0.082069	0.061951	0.049713
10.00	0.381689	0.226641	0.158613	0.121353	0.098057
		$k = \frac{n-1}{n-3}$			
00.01	0.000559	0.000263	0.000168	0.000127	0.000101
00.05	0.002791	0.001317	0.000862	0.000641	0.000510
00.10	0.005579	0.002634	0.001725	0.001282	0.001020
01.00	0.054968	0.026247	0.017221	0.012812	0.010199
05.00	0.243308	0.126605	0.084682	0.063450	0.050682
10.00	0.405543	0.235567	0.163152	0.124071	0.099858

Table 2.2: Relative MSE of $\hat{\mu}_k$

n	20	40	60	80	100
η		$k = \frac{n-1}{n}$			
00.01	0.000499	0.000249	0.000163	0.000126	0.000098
00.05	0.002482	0.001246	0.000831	0.000623	0.000499
00.10	0.004930	0.002483	0.001659	0.001246	0.000997
01.00	0.043543	0.023254	0.015867	0.012045	0.009709
05.00	0.142442	0.088981	0.065685	0.052102	0.043163
10.00	0.221891	0.139673	0.107487	0.088371	0.075204
		$k = 1.0$			
00.01	0.000499	0.000249	0.000159	0.000126	0.000098
00.05	0.002478	0.001175	0.000830	0.000611	0.000499
00.10	0.004927	0.002481	0.001658	0.001246	0.000998
01.00	0.043385	0.023228	0.015858	0.012042	0.009706
05.00	0.142524	0.088742	0.065569	0.052042	0.043128
10.00	0.225273	0.139727	0.107333	0.088248	0.075119
		$k = \frac{n-1}{n-3}$			
00.01	0.000499	0.000249	0.000161	0.000124	0.000098
00.05	0.002477	0.001245	0.000830	0.000623	0.000499
00.10	0.004919	0.002479	0.001658	0.001246	0.000998
01.00	0.043077	0.023177	0.015845	0.012036	0.009703
05.00	0.143665	0.088299	0.065342	0.051922	0.043058
10.00	0.234333	0.140057	0.107064	0.088010	0.074951

Table 2.3: Relative Efficiency of $\hat{\mu}_k$

n	20	40	60	80	100
η		$k=\frac{n-1}{n}$			
00.01	1.001686	1.002981	1.025570	0.995324	1.018444
00.05	1.007201	1.003227	1.003286	1.002630	1.001806
00.10	1.014204	1.006774	1.004911	1.003611	1.002997
01.00	1.148303	1.075098	1.050378	1.037741	1.030005
05.00	1.755096	1.404798	1.268683	1.199572	1.158413
10.00	2.253357	1.789899	1.550580	1.414485	1.329710
		$k=1.0$			
00.01	1.001525	1.003235	1.051090	0.994706	1.020217
00.05	1.008752	1.063543	1.003412	1.022828	1.001809
00.10	1.014803	1.007596	1.004934	1.003481	1.002288
01.00	1.152468	1.076272	1.050962	1.038062	1.030256
05.00	1.754091	1.408572	1.270921	1.200955	1.159344
10.00	2.219526	1.789199	1.552805	1.416462	1.331222
		$k=\frac{n-1}{n-3}$			
00.01	1.001361	1.003458	1.032752	1.008956	1.024315
00.05	1.009150	1.004130	1.003673	1.003409	1.001416
00.10	1.016444	1.008288	1.005000	1.003523	1.002093
01.00	1.160706	1.078668	1.051841	1.038527	1.030645
05.00	1.740157	1.415652	1.275346	1.203725	1.161218
10.00	2.133719	1.784985	1.556695	1.420288	1.334206

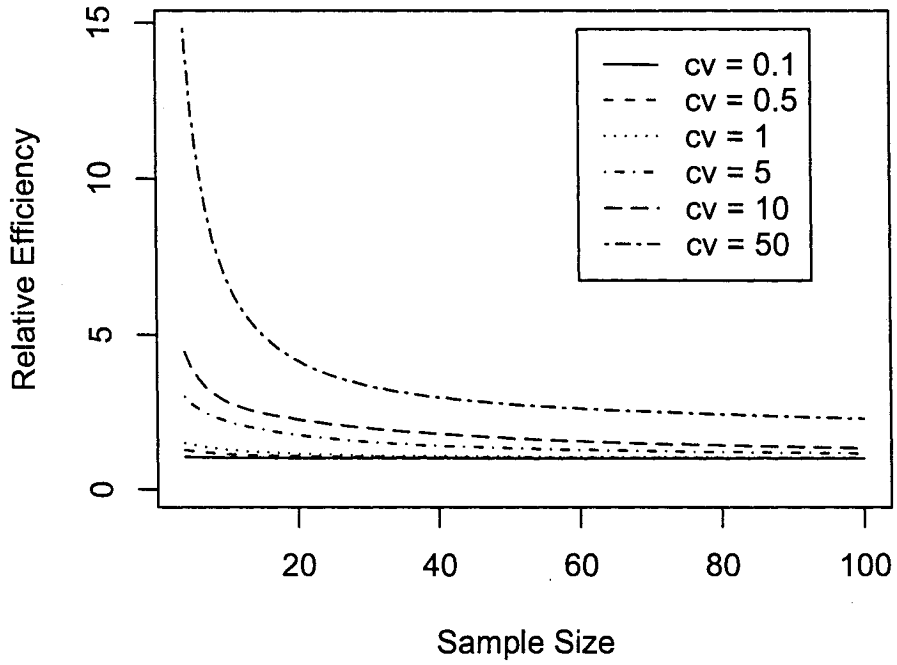

Figure 2.1: Relative Efficiency of $\hat{\mu}_k, k = \frac{n-1}{n}$

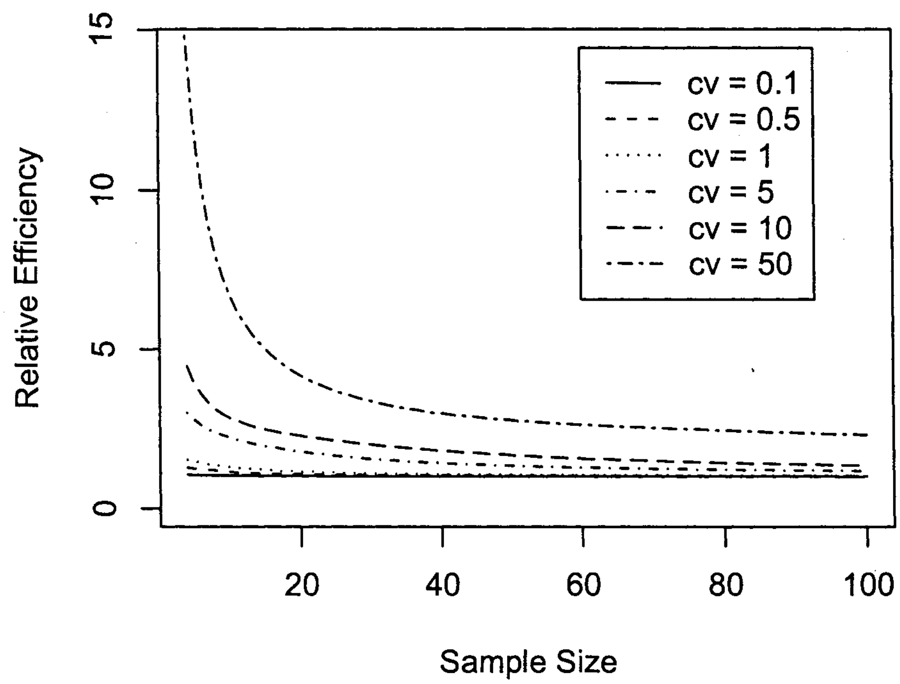

Figure 2.2: Relative Efficiency of $\hat{\mu}_k, k = 1$

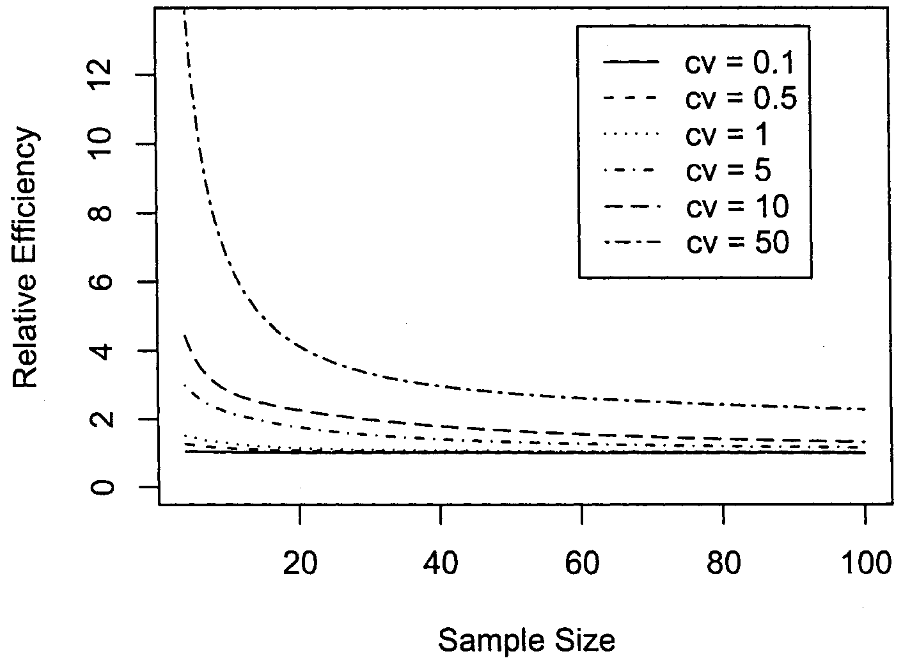

Figure 2.3: Relative Efficiency of $\hat{\mu}_k, k = \frac{n-1}{n-3}$

Chapter 3

Finite Sampling Distribution of $\hat{\mu}_k$

3.1 Introduction

For an *i.i.d.* sample $X_1, ..., X_n$ from an $IG(\mu, \mu/\eta)$ population, it follows from large sample theory that

$$\sqrt{n/\eta}\frac{(\bar{X}-\mu)}{\mu} \overset{L}{\sim} N(0,1) \quad \text{as} \quad n \to \infty.$$

In this case, however, the large sample distribution is not important because, we have the exact distribution, namely

$$\bar{X} \sim IG(\mu, n/\eta). \tag{3.1.1}$$

Since, $\hat{\eta} = \bar{X}U$ in (2.1.7) is a strongly consistent estimator of η, it follows that as $n \longrightarrow \infty$,

$$\left|\frac{1}{1+k\frac{\bar{X}U}{n}} - \frac{1}{1+k\frac{\mu\eta}{n}}\right| \stackrel{a.s.}{\longrightarrow} 0, \tag{3.1.2}$$

hence,

$$\left|\hat{\mu}_k - \frac{1}{1+k\frac{\mu\eta}{n}}\bar{X}\right| \stackrel{a.s.}{\longrightarrow} 0. \tag{3.1.3}$$

Furthermore, it is known from Chhikara and Folks (1989, pp. 13) that if $X \sim IG(\mu, \lambda)$, then for any constant $a > 0,\ aX \sim IG(a\mu, a\lambda)$. This together with Eq. (3.1.1) provides the following large sample approximation to the distribution of $\hat{\mu}_k$ as

$$\hat{\mu}_k \stackrel{L}{\sim} IG\left(\frac{n}{(n+k\mu\eta)}\mu, \frac{n}{(n+k\mu\eta)}(n/\eta)\right).$$

In finite samples, we may still use this approximation, especially for moderate sample sizes, however the moments may be quite different to the large sample moments, unless the coefficient of variation is small. We will actually see later that closeness of the approximate moments to large sample moments depends on the value of the ratio η/n. Furthermore, the moments are useful in developing Edgeworth series expansion for the probability distribution of the statistic of interest [see Chaubey and Srivastava (1996)].

3.2 Approximate Moments of $\hat{\mu}_k$

3.2.1 First Method

Here, we follow the method developed in Chaubey and Srivastava (1996), to develop approximations to moments of $\hat{\mu}_k$ in terms of powers of $1/n$. This requires expressing the estimator $\hat{\mu}_k$ in (2.1.8)in terms of the standardized random variables $\varepsilon_1 = \frac{\sqrt{n}(\bar{X}-\mu)}{\sqrt{\frac{\mu^3}{\lambda}}}$ and $\varepsilon_2 = \frac{\sqrt{(n-1)}(U-\psi)}{\sqrt{2}\psi}$, where $\psi = \frac{1}{\lambda}$. We can then express, $\bar{X}$ in terms of ε_1 and ε_2 as, $\bar{X} = \frac{1}{\psi}(\eta + \sqrt{\eta^3}\frac{\varepsilon_1}{\sqrt{n}})$ and $U = \psi + \sqrt{2}\psi\varepsilon_2(n-1)^{-\frac{1}{2}}$ where again $\psi = \frac{1}{\lambda}$ and $\eta = \frac{\mu}{\lambda}$.

Expanding $(n-1)^{-\frac{1}{2}}$ in a power series in n we may further write

$$U = \psi[1 + \sqrt{2}\varepsilon_2(\frac{1}{\sqrt{n}} + \frac{1}{2n\sqrt{n}} + \frac{3}{8n^2\sqrt{n}} + ...)]$$

So

$$\bar{X}U = \frac{1}{\psi}(\eta + \sqrt{\eta^3}\frac{\varepsilon_1}{\sqrt{n}})\psi[1 + \sqrt{2}\varepsilon_2(\frac{1}{\sqrt{n}} + \frac{1}{2n\sqrt{n}} + \frac{3}{8n^2\sqrt{n}} + ...)]$$

$$\approx \eta\left[1 + \frac{\sqrt{2}\varepsilon_2}{\sqrt{n}} + \frac{\varepsilon_2}{n\sqrt{2n}} + \frac{3\varepsilon_2}{4n^2\sqrt{2n}} + \frac{\sqrt{\eta}\varepsilon_1}{\sqrt{n}} + \sqrt{\eta}\left(\frac{\sqrt{2}\varepsilon_1\varepsilon_2}{n} + \frac{\varepsilon_1\varepsilon_2}{\sqrt{2}n^2} + \frac{3\varepsilon_1\varepsilon_2}{4\sqrt{2}n^3}\right)\right]$$

Now, $\hat{\mu}_k$ can be expressed in terms of ε_1 and ε_2 as

$$\hat{\mu}_k = \mu(1 + \sqrt{\eta}\frac{\varepsilon_1}{\sqrt{n}})(1 + \frac{k\eta}{n}(1+X))^{-1} \qquad (3.2.1)$$

where

$$X = e_{-\frac{1}{2}} + e_{-1} + e_{-\frac{3}{2}} + e_{-2} + e_{-\frac{5}{2}} + e_{-3} + O_p(n^{-3}) \qquad (3.2.2)$$

and

$$e_{-\frac{1}{2}} = \sqrt{\frac{2}{n}}\varepsilon_2 + \sqrt{\frac{\eta}{n}}\varepsilon_1, e_{-1} = \frac{\sqrt{2\eta}}{n}\varepsilon_1\varepsilon_2, e_{-\frac{3}{2}} = \frac{\varepsilon_2}{\sqrt{2}n\sqrt{n}}$$
$$e_{-2} = \frac{\sqrt{\eta}\varepsilon_1\varepsilon_2}{\sqrt{2}n^2}, e_{-\frac{5}{2}} = \frac{3\varepsilon_2}{4\sqrt{2}n^2\sqrt{n}}, e_{-3} = \frac{3\varepsilon_1\varepsilon_2}{4\sqrt{2}n^3}$$

Expanding further the expression in the exponent we get

$$\begin{aligned}\hat{\mu}_k &= \mu[1 + \frac{\sqrt{\eta}}{\sqrt{n}}\varepsilon_1 - \frac{k\eta}{n}(1+X) - \frac{k\eta\sqrt{\eta}}{n\sqrt{n}}\varepsilon_1(1+X) + \frac{k^2\eta^2}{n^2}(1+X)^2 \\ &\quad + \frac{k^2\eta^2\sqrt{\eta}}{n^2\sqrt{n}}\varepsilon_1(1+X)^2 - \frac{k^3\eta^3}{n^3}(1+X)^3 + ...] \end{aligned} \tag{3.2.3}$$

Thus up to order $O_p(n^{-3})$ we have

$$\frac{(\hat{\mu}_k - \mu)}{\mu} = \xi_{-\frac{1}{2}} + \xi_{-1} + \xi_{-\frac{3}{2}} + \xi_{-2} + \xi_{-\frac{5}{2}} + \xi_{-3} \tag{3.2.4}$$

where

$$\xi_{-\frac{1}{2}} = \sqrt{\frac{\eta}{n}}\varepsilon_1, \xi_{-1} = -\frac{k\eta}{n}, \xi_{-\frac{3}{2}} = \frac{\sqrt{2}k\eta\varepsilon_2}{n\sqrt{n}}\frac{2k\eta\sqrt{\eta}\varepsilon_1}{n\sqrt{n}}$$
$$\xi_{-2} = -\frac{2\sqrt{2}k\eta\sqrt{\eta}\varepsilon_1\varepsilon_2}{n^2} - \frac{k\eta^2\varepsilon_1^2}{n^2} + \frac{k^2\eta^2}{n^2},$$
$$\xi_{-\frac{5}{2}} = -\frac{k\eta\varepsilon_2}{\sqrt{2}n^2\sqrt{n}} - \frac{\sqrt{2}k\eta^2\varepsilon_1^2\varepsilon_2}{n^2\sqrt{n}} + \frac{2\sqrt{2}k^2\eta^2\varepsilon_2}{n^2\sqrt{n}} + \frac{3k^2\eta^2\sqrt{\eta}\varepsilon_1}{n^2\sqrt{n}}$$
$$\xi_{-3} = -\frac{\sqrt{2}k\eta\sqrt{\eta}\varepsilon_1\varepsilon_2}{n^3} + \frac{6\sqrt{2}k^2\eta^2\sqrt{\eta}\varepsilon_1\varepsilon_2}{n^3} + \frac{2k^2\eta^2\varepsilon_2^2}{n^3} + \frac{3k^2\eta^3\varepsilon_1^2}{n^3} - \frac{k^3\eta^3}{n^3}$$

Approximate expression for $m_r(\hat{\mu}_k) = E(\hat{\mu}_k - \mu)^r; r = 1,2,3,4$ up to order $O(n^{-3})$ are obtained from the above expansion,

$$m_1(\hat{\mu}_k) = \mu E[\xi_{-\frac{1}{2}} + \xi_{-1} + \xi_{-\frac{3}{2}} + \xi_{-2} + \xi_{-\frac{5}{2}} + \xi_{-3}]$$

$$
\begin{aligned}
m_2(\hat{\mu}_k) &= \mu^2 E[\xi^2_{-\frac{1}{2}} + \xi^2_{-1} + \xi^2_{-\frac{3}{2}} + 2\xi_{-\frac{1}{2}}\xi_{-1} + 2\xi_{-\frac{1}{2}}\xi_{-\frac{3}{2}} + 2\xi_{-\frac{1}{2}}\xi_{-2} + 2\xi_{-\frac{1}{2}}\xi_{-\frac{5}{2}} \\
&\quad + 2\xi_{-1}\xi_{-\frac{3}{2}} + 2\xi_{-1}\xi_{-2}]
\end{aligned}
$$

$$
m_3(\hat{\mu}_k) = \mu^3 E[\xi^3_{-\frac{1}{2}} + \xi^3_{-1} + 3\xi^2_{-\frac{1}{2}}\xi_{-1} + 3\xi^2_{-\frac{1}{2}}\xi_{-\frac{3}{2}} + 3\xi^2_{-\frac{1}{2}}\xi_{-2} + 3\xi_{-\frac{1}{2}}\xi^2_{-1} + 6\xi_{-\frac{1}{2}}\xi_{-1}\xi_{-\frac{3}{2}}]
$$

$$
m_4(\hat{\mu}_k) = \mu^4 E[\xi^4_{-\frac{1}{2}} + \xi^2_{-\frac{1}{2}}\xi_{-\frac{3}{2}} + 6\xi^2_{-\frac{1}{2}}\xi^2_{-1}]
$$

Now we evaluate the following expectations to be able to obtain approximate expressions as

$E(\xi_{-\frac{1}{2}}) = 0, E(\xi_{-1}) = -\frac{k\eta}{n}, E(\xi_{-\frac{3}{2}}) = 0, E(\xi_{-2}) = \frac{k^2\eta^2}{n^2}, E(\xi_{-\frac{5}{2}}) = 0,$

$E(\xi_{-1}\xi_{-\frac{3}{2}}) = 0, E(\xi_{-3}) = \frac{2k^2\eta^2}{n^3} + \frac{3k^2\eta^3}{n^3} - \frac{k^3\eta^3}{n^3}, E(\xi^2_{-\frac{1}{2}}) = \frac{\eta}{n}, E(\xi^2_{-1}) = \frac{k^2\eta^2}{n^2},$

$E(\xi^2_{-\frac{3}{2}}) = \frac{2k^2\eta^2}{n^3} + \frac{4k^2\eta^3}{n^3}, E(\xi_{-\frac{1}{2}}\xi^2_{-1}) = 0, E(\xi^3_{-\frac{1}{2}}) = \frac{3\eta^2}{n^2}, E(\xi^3_{-1}) = -\frac{k^3\eta^3}{n^3},$

$E(\xi^2_{-\frac{1}{2}}\xi_{-1}) = -\frac{k\eta^2}{n^2}, E(\xi_{-\frac{1}{2}}\xi_{-1}) = 0, E(\xi_{-\frac{1}{2}}\xi_{-\frac{3}{2}}) = -\frac{2k\eta^2}{n^2}, E(\xi_{-\frac{1}{2}}\xi_{-2}) = -\frac{3k\eta^3}{n^3},$

$E(\xi^2_{-\frac{1}{2}}\xi_{-\frac{3}{2}}) = -\frac{6k\eta^3}{n^3}, E(\xi_{-\frac{1}{2}}\xi_{-\frac{5}{2}}) = \frac{3k^2\eta^3}{n^3}, E(\xi_{-1}\xi_{-2}) = \frac{k^2\eta^3}{n^3} - \frac{k^3\eta^3}{n^3},$

$E(\xi^4_{-\frac{1}{2}}) = \frac{3\eta^2}{n^2} + \frac{15\eta^3}{n^3}, E(\xi^2_{-\frac{1}{2}}\xi_{-2}) = -3\frac{k\eta^3}{n^3} + \frac{k^2\eta^3}{n^3}, E(\xi^3_{-\frac{1}{2}}\xi_{-1}) = -\frac{3k\eta^3}{n^3},$

$E(\xi^2_{-\frac{1}{2}}\xi_{-\frac{3}{2}}) = -6\frac{k\eta^3}{n^3}, E(\xi^2_{-\frac{1}{2}}\xi^2_{-1}) = \frac{k^2\eta^3}{n^3}, E(\xi_{-\frac{1}{2}}\xi_{-1}\xi_{-\frac{3}{2}}) = 2\frac{k^2\eta^3}{n^3}.$

Substituting appropriate expectations gives

$$
\begin{aligned}
m_1(\hat{\mu}_k) &= -\frac{k\eta}{n} + \frac{k^2\eta^2}{n^2} - \frac{k\eta^2}{n^2} + \frac{2k^2\eta^2}{n^3} + \frac{3k^2\eta^3}{n^3} - \frac{k^3\eta^3}{n^3} \\
m_2(\hat{\mu}_k) &= \frac{\eta}{n} + \frac{k^2\eta^2}{n^2} - \frac{4k\eta^2}{n^2} + \frac{2k^2\eta^2}{n^3} + \frac{12k^2\eta^3}{n^3} - \frac{2k^3\eta^3}{n^3} - \frac{6k\eta^3}{n^3}. \\
m_3(\hat{\mu}_k) &= \frac{3\eta^2}{n^2} - \frac{3k\eta^2}{n^2} - \frac{k^3\eta^3}{n^3} - \frac{27k\eta^3}{n^3} + \frac{15k^2\eta^3}{n^3} \\
m_4(\hat{\mu}_k) &= \frac{3\eta^2}{n^2} + \frac{15\eta^3}{n^3} - \frac{12k\eta^3}{n^3} + \frac{6k^2\eta^3}{n^3}
\end{aligned}
$$

3.2.2 Second Method

Here, we want to express $\hat{\mu}_k$ in terms of its asymptotic limit $\tilde{\mu}_k$. This requires, therefore, the variations in $\hat{\eta}$ with respect to η. Hence, we develop first a representation of $\hat{\mu}_k$ in terms of $\bar{X}$. Since $\bar{X} \sim IG(\mu, n\lambda)$ then $(n-1)\lambda U \sim \chi^2_{(n-1)}$. Also

$E(\bar{X}) = \mu, E(\bar{X}^2) = \frac{\mu^3}{n\lambda} + \mu^2, E(\bar{X}^3) = \mu^3 + \frac{3\mu^4}{n\lambda} + \frac{3\mu^5}{n^2\lambda^2}$

$E(\bar{X}^4) = \mu^4 + \frac{6\mu^5}{n\lambda} + \frac{15\mu^6}{n^2\lambda^2} + \frac{15\mu^7}{n^3\lambda^3}, E(U) = \frac{1}{\lambda}, E(U^2) = \frac{1}{\lambda^2}(1 + \frac{2}{n} + \frac{2}{n^2} + \frac{2}{n^3} + ...)$

$E(U^3) = \frac{1}{\lambda^3}(1 + \frac{6}{n} + \frac{11}{n^2} + \frac{9}{n^3} + ...), E(U^4) = \frac{1}{\lambda^4}(1 + \frac{3}{n} + \frac{25}{n^2} + \frac{38}{n^3} + ...)$

Let $\hat{\eta} = \bar{X}U$ then

$$E(\hat{\eta}) = \eta, V(\hat{\eta}) = (\frac{\mu^3}{n\lambda} + \mu^2)\frac{1}{\lambda^2}(1 + \frac{2}{n-1}) - \eta^2, \text{ where } \eta = \frac{\mu}{\lambda}$$

Which can be written as

$$\begin{aligned} V(\hat{\eta}) &= (\frac{\eta^3}{n} + \eta^2)(1 + \frac{2}{n-1}) - \eta^2 \\ &= \frac{\eta^3}{n}(1 + \frac{2}{n-1} + \frac{2n}{(n-1)\eta}) \\ &= \frac{\eta^3}{n}[\frac{n+1}{n-1} + \frac{2}{\eta}(1 + \frac{1}{n} + \frac{1}{n^2} + ...)] \\ &\approx \frac{\eta^3}{n} + \frac{2\eta^2}{n} \\ &= \frac{\eta^2(2+\eta)}{n} \end{aligned}$$

So, $V(\sqrt{n}\hat{\eta}) = \eta^2(2+\eta)$. Consider, $c_1 = \sqrt{\eta^2(2+\eta)}$ and $\varepsilon = \frac{\sqrt{n}(\hat{\eta}-\eta)}{\sqrt{\eta^2(2+\eta)}}$ then,

$$\hat{\eta} = \eta + \frac{\varepsilon c_1}{\sqrt{n}}.$$

Now $\hat{\mu}_k$ in terms of $\bar{X}$ can be written as

$$\begin{aligned}
\hat{\mu}_k &= \frac{\bar{X}}{(1+\frac{k\bar{X}U}{n})} \\
&= \frac{\bar{X}}{(1+\frac{k\hat{\eta}}{n})} \\
&= \frac{\bar{X}}{(1+\frac{k\eta}{n})}\left[1+\frac{\frac{k(\hat{\eta}-\eta)}{n}}{(1+\frac{k\eta}{n})}\right]^{-1} \\
&= \frac{\bar{X}}{(1+\frac{k\eta}{n})}\left[1+\frac{kc_1\varepsilon}{\sqrt{n}(n+k\eta)}\right]^{-1} \\
&= \frac{\bar{X}}{(1+\frac{k\eta}{n})}\left[1-\frac{kc_1\varepsilon}{\sqrt{n}(n+k\eta)}+\frac{k^2c_1^2\varepsilon^2}{n(n+k\eta)^2}-\frac{k^3c_1^3\varepsilon^3}{n\sqrt{n}(n+k\eta)^3}+...\right]
\end{aligned}$$

Therefore

$$\hat{\mu}_k = \frac{n\bar{X}}{(n+k\eta)}-\frac{k\sqrt{n}c_1\varepsilon\bar{X}}{(n+k\eta)^2}+\frac{k^2c_1^2\varepsilon^2\bar{X}}{(n+k\eta)^3}-\frac{k^3c_1^3\varepsilon^3\bar{X}}{\sqrt{n}(n+k\eta)^4}+... \quad (3.2.5)$$

Now for computing the moments $m_r(\hat{\mu}_k)$ of order $O(n^{-3})$ we list below some expectations;

$$\begin{aligned}
E(\varepsilon\bar{X}) &= E\left[\frac{\sqrt{n}(\hat{\eta}-\eta)\bar{X}}{c_1}\right] \\
&= \frac{\mu\eta^2}{\sqrt{n}c_1} \\
E(\varepsilon^2\bar{X}) &= E\left[\frac{n(\hat{\eta}-\eta)^2\bar{X}}{c_1^2}\right] \\
&= \frac{\eta^2\mu}{c_1^2}\left[2+\eta+\frac{2}{n}+\frac{6\eta}{n}+\frac{3\eta^2}{n}\right] \\
E(\varepsilon^3\bar{X}) &= E\left[\frac{\sqrt{n}^3(\hat{\eta}-\eta)^3\bar{X}}{c_1^3}\right] \\
&= \frac{3\sqrt{n}\mu\eta^4}{c_1^3}
\end{aligned}$$

$$
\begin{aligned}
E(\varepsilon\bar{X}^2) &= E\left[\frac{\sqrt{n}(\hat{\eta}-\eta)\bar{X}^2}{c_1}\right] \\
&= \frac{\sqrt{n}}{c_1}\left[\frac{2\mu^2\eta^3}{n}+\frac{3\mu^2\eta^3}{n^2}\right] \\
E(\varepsilon\bar{X}^3) &= \frac{\sqrt{n}}{c_1}\left[\frac{3\mu^3\eta^2}{n}+\frac{12\mu^3\eta^3}{n^2}+\frac{15\mu^3\eta^4}{n^3}\right] \\
E(\varepsilon^2\bar{X}^2) &= \frac{n}{c_1^2}\left[2\mu^2\eta^2+\frac{\mu^2\eta^3}{n}+\frac{9\mu^2\eta^4+12\mu^2\eta^3+2\mu^2\eta^2}{n^2}\right. \\
&\quad \left.+\frac{30\mu^2\eta^4+15\mu^2\eta^5+12\mu^2\eta^3+2\mu^2\eta^2}{n^3}\right]
\end{aligned}
$$

Now for order $O(n^{-3})$

$$
\begin{aligned}
m_1\left(\frac{\hat{\mu}_k}{\mu}\right) &= \frac{n}{(n+k\eta)}-\frac{k\eta^2}{(n+k\eta)^2}+\frac{k^2\eta^2}{(n+k\eta)^3}(2+\eta) \\
m_2\left(\frac{\hat{\mu}_k}{\mu}\right) &= \frac{1+\frac{\eta}{n}}{(1+\frac{k\eta}{n})^2}+\frac{3k^2}{(1+\frac{k\eta}{n})^4}\left(\frac{2\eta^2}{n^2}+\frac{\eta^3}{n^3}\right)-\frac{2k}{(1+\frac{k\eta}{n})^3}\left(\frac{2\eta^2}{n^2}+\frac{3\eta^3}{n^3}\right) \\
m_3\left(\frac{\hat{\mu}_k}{\mu}\right) &= \frac{n^3}{(n+k\eta)^3}\left(1+\frac{3\eta}{n}+\frac{3\eta^2}{n^2}\right)-\frac{3kn^4}{(n+k\eta)^4}\left(\frac{3\eta^2}{n^2}+\frac{12\eta^3}{n^3}\right) \\
m_4\left(\frac{\hat{\mu}_k}{\mu}\right) &= \frac{n^4}{(n+k\eta)^4}\left(1+\frac{6\eta}{n}+\frac{15\eta^2}{n^2}+\frac{15\eta^3}{n^3}\right)
\end{aligned}
$$

3.2.3 Third Method

Here, we treat $\hat{\mu}_k$ as a function of two independent statistics $\bar{X}$ and U and use Taylor's series expansion around (μ, ψ). Writing

$$
\hat{\mu}_k = \frac{\bar{X}}{1+\frac{k\bar{X}U}{n}} = g(\bar{X}, U)
$$

and using Taylor's expansion of $g(\bar{X},U)$ around (μ,ψ), we have

$$
\begin{aligned}
g(\bar{X},U) =& g(\mu,\psi) + (\bar{X}-\mu)g_{\mu}^{(1)}(\mu,\psi) + (U-\psi)g_{\psi}^{(1)}(\mu,\psi) + \frac{1}{2}(\bar{X}-\mu)^2 g_{\mu}^{(2)}(\mu,\psi) + \\
& \frac{1}{2}(U-\psi)^2 g_{\psi}^{(2)}(\mu,\psi) + (\bar{X}-\mu)(U-\psi)g_{\mu\psi}^{(2)}(\mu,\psi) + \frac{1}{3!}(\bar{X}-\mu)^3 g_{\mu}^{(3)}(\mu,\psi) + \\
& \frac{1}{3!}(U-\psi)^3 g_{\psi}^{(3)}(\mu,\psi) + \ldots\ldots..
\end{aligned}
\tag{3.2.6}
$$

where $g_u^{(i)} = \frac{\partial^i g(.,.)}{\partial u^i}$ and $g_{uv}^{(i+j)} = \frac{\partial^{(i+j)} g(.,.)}{\partial u^i v^j}$. We have

$$
g_{\mu}^{(1)}(\mu,\psi) = \frac{1}{(1+\frac{k\eta}{n})^2},\ g_{\mu}^{(2)}(\mu,\psi) = \frac{-\frac{2k\psi}{n}}{(1+\frac{k\eta}{n})^3},\ g_{\mu}^{(3)}(\mu,\psi) = \frac{\frac{6k^2\psi^2}{n^2}}{(1+\frac{k\eta}{n})^4}
$$

$$
g_{\psi}^{(1)}(\mu,\psi) = \frac{-\frac{k\mu^2}{n}}{(1+\frac{k\eta}{n})^2},\ g_{\psi}^{(2)}(\mu,\psi) = \frac{\frac{2k^2\mu^3}{n^2}}{(1+\frac{k\eta}{n})^3},\ g_{\psi}^{(3)}(\mu,\psi) = \frac{-\frac{6k^3\mu^4}{n^3}}{(1+\frac{k\eta}{n})^4}
$$

Also

$E(U) = \frac{1}{\lambda}, E(\bar{X}-\mu)^2 = \frac{\mu^3\psi}{n}, E(\bar{X}-\mu)^3 = \frac{3\mu^5\psi}{n^2\lambda^2}, E(\bar{X}^3) = \mu^3 + \frac{3\mu^4}{n\lambda} + \frac{3\mu^5}{n^2\lambda^2},$

$E(U^2) = \frac{n+1}{(n-1)\lambda^2}, E(U-\psi)^2 = \frac{2\psi^2}{n-1}, E(U-\psi)^3 = \frac{8\psi^3}{(n-1)^2},$

$E(\bar{X}^4) = \mu^4 + \frac{6\mu^5}{n\lambda} + \frac{15\mu^6}{n^2\lambda^2} + \frac{15\mu^7}{n^3\lambda^3}.$

Using the above expressions in Eq. (3.2.6), we can obtain,

$$
\begin{aligned}
E(\frac{\hat{\mu}_k}{\mu}) =& \frac{1}{1+\frac{k\eta}{n}} - \frac{\frac{k\eta^2}{n^2}}{(1+\frac{k\eta}{n})^3} + \frac{\frac{2k^2\eta^2}{n^2(n-1)}}{(1+\frac{k\eta}{n})^3}\left(1+\frac{3\eta}{n}+\frac{3\eta^2}{n^2}\right) + \frac{\frac{3k^2\eta^4(n+1)}{n^4(n-1)}}{(1+\frac{k\eta}{n})^4} - \\
& \frac{\frac{8k^3\eta^3}{n^3(n-1)^2}}{(1+\frac{k\eta}{n})^4}\left(1+\frac{6\eta}{n}+\frac{15\eta^2}{n^2}+\frac{15\eta^3}{n^3}\right),
\end{aligned}
\tag{3.2.7}
$$

$$
\begin{aligned}
E\left(\frac{\hat{\mu}_k}{\mu}\right)^2 =& \frac{1}{(1+\frac{k\eta}{n})^2} + \frac{\frac{k^2(n+1)}{n^2(n-1)}}{(1+\frac{k\eta}{n})^6}\left(\frac{15\eta^5}{n^3}+\frac{3\eta^4}{n^2}\right) - \frac{\frac{2k\eta^2}{n^2}}{(1+\frac{k\eta}{n})^4} + \frac{\frac{4k^2}{n^2(n-1)}}{(1+\frac{k\eta}{n})^4} \\
& \left(\eta^2+\frac{3\eta^3}{n}+\frac{3\eta^4}{n^2}\right) + \frac{\frac{6k^2\eta^4(n+1)}{n^2(n-1)}}{(1+\frac{k\eta}{n})^5} - \frac{\frac{16k^3}{n^3(n-1)^2}}{(1+\frac{k\eta}{n})^5}\left(\eta^3+\frac{6\eta^4}{n}+\frac{15\eta^5}{n^2}+\frac{15\eta^6}{n^3}\right).
\end{aligned}
\tag{3.2.8}
$$

3.3 Comparison of Approximations

The three approximations developed above are compared in Tables 3.1, 3.2 and 3.3 along with the corresponding errors plotted in Figures 3.1, 3.2 and 3.3. It is seen that all the three approximations are qualitatively the same. The approximations are good for values of $\eta \leq 1$, and improve further as the sample size increases. Conversely, the value of η increases, the approximations lose their accuracy.

Table 3.1: First Approximation for the Mean of $\hat{\mu}_k$

n	20	40	60	80	100
η		$k=\frac{n-1}{n}$			
00.01	0.999525	0.999756	0.999836	0.999877	0.999901
00.05	0.997625	0.998781	0.999181	0.999383	0.999505
00.10	0.995251	0.997563	0.998361	0.998766	0.999010
01.00	0.952838	0.975669	0.983625	0.987662	0.990102
05.00	0.794080	0.882247	0.919294	0.938807	0.950776
10.00	0.766953	0.787776	0.845579	0.880584	0.903067
		$k=1.0$			
00.01	0.999500	0.999750	0.999833	0.999875	0.999900
00.05	0.997501	0.998750	0.999167	0.999375	0.999500
00.10	0.995003	0.997500	0.998333	0.998750	0.999000
01.00	0.950500	0.975063	0.983352	0.987508	0.990004
05.00	0.787500	0.879688	0.918056	0.938086	0.950300
10.00	0.775000	0.784375	0.843519	0.879297	0.902200
		$k=\frac{n-1}{n-3}$			
00.01	0.999441	0.999737	0.999828	0.999872	0.999898
00.05	0.997208	0.998683	0.999138	0.999359	0.999490
00.10	0.994419	0.997366	0.998275	0.998718	0.998979
01.00	0.945053	0.973753	0.982778	0.987188	0.989800
05.00	0.773353	0.874224	0.915461	0.936591	0.949332
10.00	0.799191	0.777301	0.839233	0.876639	0.900419

Table 3.2: Second Approximation for the Mean of $\hat{\mu}_k$

n	20	40	60	80	100
η		$k = \frac{n-1}{n}$			
00.01	0.997626	0.998782	0.999181	0.999383	0.999505
00.05	0.997625	0.998781	0.999181	0.999383	0.999505
00.10	0.995251	0.997563	0.998361	0.998766	0.999010
01.00	0.952784	0.975666	0.983624	0.987662	0.990103
05.00	0.779727	0.881102	0.919047	0.938726	0.950736
10.00	0.610986	0.773891	0.842432	0.879512	0.902607
		$k = 1.0$			
00.01	0.999500	0.999750	0.999833	0.999875	0.999900
00.05	0.997501	0.998750	0.999167	0.999375	0.999500
00.10	0.995003	0.997500	0.998333	0.998750	0.999000
01.00	0.950437	0.975058	0.983351	0.987508	0.990004
05.00	0.771200	0.878464	0.917797	0.938001	0.950265
10.00	0.600000	0.769600	0.840233	0.878189	0.901728
		$k = \frac{n-1}{n-3}$			
00.01	0.999441	0.999737	0.999828	0.999872	0.999898
00.05	0.997206	0.998683	0.999138	0.999359	0.999490
00.10	0.994418	0.997366	0.998275	0.998718	0.998979
01.00	0.944967	0.973748	0.982777	0.987187	0.989800
05.00	0.751983	0.872820	0.915178	0.936500	0.949294
10.00	0.575988	0.760506	0.835646	0.875455	0.899921

Table 3.3: Third Approximation for the Mean of $\hat{\mu}_k$

n	20	40	60	80	100
η		$k = \frac{n-1}{n}$			
00.01	0.999525	0.999756	0.999836	0.999876	0.999901
00.05	0.997625	0.998781	0.999181	0.999383	0.999505
00.10	0.995251	0.997562	0.998361	0.998765	0.999010
01.00	0.952794	0.975666	0.983623	0.987661	0.990102
05.00	0.779883	0.881115	0.919049	0.938726	0.950736
10.00	0.611357	0.773931	0.842441	0.879515	0.902608
		$k = 1.0$			
00.01	0.999500	0.999750	0.999833	0.999875	0.999900
00.05	0.997625	0.998781	0.999181	0.999383	0.999505
00.10	0.995002	0.997500	0.998333	0.998750	0.999000
01.00	0.950448	0.975059	0.983351	0.987507	0.990003
05.00	0.771368	0.878477	0.917800	0.938002	0.950265
10.00	0.600389	0.769641	0.840243	0.878192	0.901729
		$k = \frac{n-1}{n-3}$			
00.01	0.999441	0.999736	0.999827	0.999871	0.999897
00.05	0.997625	0.998781	0.999181	0.999383	0.999505
00.10	0.994418	0.997365	0.998275	0.998717	0.998979
01.00	0.944980	0.973748	0.982777	0.987187	0.989800
05.00	0.752178	0.872835	0.915180	0.936501	0.949294
10.00	0.576422	0.760550	0.835656	0.875458	0.899922

Table 3.4: Exact Mean of $\hat{\mu}_k$

n	20	40	60	80	100
η		$k = \frac{n-1}{n}$			
00.01	0.999881	0.999939	0.999959	0.999969	0.999975
00.05	0.999405	0.999695	0.999795	0.999846	0.999876
00.10	0.998808	0.999389	0.999589	0.999691	0.999752
01.00	0.987708	0.993796	0.995853	0.996886	0.997507
05.00	0.932453	0.967048	0.978343	0.983893	0.987184
10.00	0.858495	0.930558	0.954772	0.966611	0.973579
		$k = 1.0$			
00.01	0.999875	0.999937	0.999958	0.999969	0.999975
00.05	0.999374	0.999687	0.999792	0.999844	0.999875
00.10	0.998746	0.999374	0.999583	0.999687	0.999749
01.00	0.987072	0.993638	0.995783	0.996846	0.997482
05.00	0.929224	0.966236	0.977985	0.983693	0.987057
10.00	0.852534	0.928935	0.954047	0.966205	0.973319
		$k = \frac{n-1}{n-3}$			
00.01	0.999860	0.999934	0.999957	0.999968	0.999975
00.05	0.999300	0.999670	0.999784	0.999839	0.999872
00.10	0.998598	0.999340	0.999568	0.999679	0.999745
01.00	0.985576	0.993296	0.995635	0.996765	0.997429
05.00	0.921738	0.964487	0.977232	0.983277	0.986794
10.00	0.838935	0.925449	0.952525	0.965362	0.972786

Table 3.5: First Approximation for the Variance of $\hat{\mu}_k$

n	20	40	60	80	100
η		$k = \frac{n-1}{n}$			
00.01	0.000499	0.000250	0.000167	0.000125	0.000099
00.05	0.002483	0.001245	0.000832	0.000624	0.000499
00.10	0.004930	0.002482	0.001659	0.001245	0.000997
01.00	0.043409	0.023244	0.015870	0.012047	0.009708
05.00	0.127910	0.086893	0.065157	0.051915	0.043084
10.00	0.225094	0.126506	0.102763	0.086420	0.074278
		$k = 1.0$			
00.01	0.000499	0.000249	0.000167	0.000125	0.000099
00.05	0.002482	0.001245	0.000831	0.000624	0.000499
00.10	0.004928	0.002481	0.001658	0.001245	0.000997
01.00	0.043250	0.023219	0.015861	0.012043	0.009706
05.00	0.131250	0.086719	0.065046	0.051855	0.043050
10.00	0.275000	0.128126	0.102778	0.086328	0.074200
		$k = \frac{n-1}{n-3}$			
00.01	0.000499	0.000249	0.000167	0.000125	0.000099
00.05	0.002481	0.001245	0.000831	0.000624	0.000499
00.10	0.004923	0.002481	0.001658	0.001245	0.000997
01.00	0.042945	0.023167	0.015849	0.012036	0.009702
05.00	0.142272	0.086462	0.064832	0.051737	0.042981
10.00	0.412309	0.132302	0.102912	0.086164	0.074050

Table 3.6: Second Approximation for the Variance of $\hat{\mu}_k$

n	20	40	60	80	100
η		$k = \frac{n-1}{n}$			
00.01	0.000500	0.000250	0.000167	0.000125	0.000100
00.05	0.002509	0.001253	0.000835	0.000626	0.000500
00.10	0.005034	0.002510	0.001672	0.001253	0.001002
01.00	0.051775	0.025802	0.017085	0.012753	0.010169
05.00	0.198301	0.126035	0.087165	0.065804	0.052604
10.00	0.155588	0.202547	0.160065	0.127111	0.104065
		$k = 1.0$			
00.01	0.000500	0.000250	0.000167	0.000125	0.000100
00.05	0.002511	0.001253	0.000835	0.000626	0.000500
00.10	0.005043	0.002512	0.001672	0.001253	0.001002
01.00	0.052413	0.025902	0.017117	0.012767	0.010177
05.00	0.202051	0.127249	0.087658	0.066048	0.052741
10.00	0.158519	0.204516	0.161147	0.127732	0.104447
		$k = \frac{n-1}{n-3}$			
00.01	0.000501	0.000250	0.000166	0.000125	0.000100
00.05	0.002517	0.001254	0.000835	0.000626	0.000501
00.10	0.005066	0.002514	0.001673	0.001253	0.001002
01.00	0.054081	0.026130	0.017187	0.012797	0.010192
05.00	0.211390	0.129982	0.088727	0.066565	0.053029
10.00	0.165844	0.208889	0.163476	0.129047	0.105248

Table 3.7: Third Approximation for the Variance of $\hat{\mu}_k$

n	20	40	60	80	100
η		$k = \frac{n-1}{n}$			
00.01	0.001498	0.000749	0.000499	0.000375	0.000299
00.05	0.007465	0.003741	0.002496	0.001872	0.001498
00.10	0.014860	0.007464	0.004984	0.003741	0.002994
01.00	0.136593	0.071458	0.048397	0.036589	0.029414
05.00	0.451594	0.292866	0.212042	0.165691	0.135855
10.00	0.361796	0.441026	0.355788	0.291466	0.245488
		$k = 1.0$			
00.01	0.001498	0.000749	0.000499	0.000374	0.000299
00.05	0.007463	0.003740	0.002496	0.001872	0.001498
00.10	0.014853	0.007463	0.004983	0.003740	0.002994
01.00	0.135939	0.071371	0.048369	0.036578	0.029408
05.00	0.438968	0.291012	0.211455	0.165433	0.135719
10.00	0.312338	0.434505	0.353680	0.290527	0.244989
		$k = \frac{n-1}{n-3}$			
00.01	0.001498	0.000749	0.000499	0.000374	0.000299
00.05	0.007459	0.003740	0.002496	0.001873	0.001498
00.10	0.014836	0.007461	0.004983	0.003740	0.002994
01.00	0.134419	0.071182	0.048314	0.036554	0.029396
05.00	0.410273	0.287041	0.210222	0.164899	0.135442
10.00	0.196578	0.420603	0.349274	0.288585	0.243964

Table 3.8: Exact Variance of $\hat{\mu}_k$

n	20	40	60	80	100
η		$k = \frac{n-1}{n}$			
00.01	0.000499	0.000249	0.000167	0.000125	0.000099
00.05	0.002495	0.001249	0.000833	0.000624	0.000499
00.10	0.004976	0.002494	0.001664	0.001249	0.000999
01.00	0.047553	0.024380	0.016391	0.012344	0.009900
05.00	0.185896	0.108681	0.076127	0.058471	0.047434
10.00	0.261041	0.183794	0.137054	0.108429	0.089461
		$k = 1.0$			
00.01	0.000500	0.000249	0.000166	0.000126	0.000099
00.05	0.002493	0.001249	0.000832	0.000624	0.000500
00.10	0.004975	0.002494	0.001664	0.001248	0.000999
01.00	0.047430	0.024364	0.016386	0.012342	0.009899
05.00	0.183280	0.108305	0.076013	0.058423	0.047409
10.00	0.253937	0.182465	0.136621	0.108241	0.089363
		$k = \frac{n-1}{n-3}$			
00.01	0.000499	0.000250	0.000166	0.000125	0.000099
00.05	0.002494	0.001249	0.000833	0.000626	0.000501
00.10	0.004971	0.002493	0.001663	0.001249	0.000999
01.00	0.047142	0.024331	0.016376	0.012338	0.009897
05.00	0.177351	0.107499	0.075773	0.058322	0.047357
10.00	0.238394	0.179637	0.135714	0.074410	0.062688

3.4 Empirical Approximation to Moments

Method 1 provides a polynomial approximation in η/n to the moments of $\hat{\mu}_k$. This may be used to develop a working approximation for these moments using the least square polynomial approximation. We have investigated different polynomial approximations using ordinary least squares and weighted least squares (weight $\propto$ sample size) using the correct moments for sample sizes $n = 10(10)100$. The following approximations are obtained which are correct up to 4 decimals.

$$
\begin{aligned}
E[(\hat{\mu}_{(1)})/\mu] &= 0.996 - 0.935(\frac{\eta}{n}) + 1.13(\frac{\eta}{n})^2 - 0.274(\frac{\eta}{n})^3 + 0.0171(\frac{\eta}{n})^4 \\
E[(\hat{\mu}_{(1)}/\mu)^2] &= 1.01 - 1.14(\frac{\eta}{n}) + 1.36(\frac{\eta}{n})^2 - 0.330(\frac{\eta}{n})^3 + 0.0205(\frac{\eta}{n})^4 \\
E[(\hat{\mu}_{(1)}/\mu)^3] &= 0.942 - 1.47(\frac{\eta}{n}) + 1.92(\frac{\eta}{n})^2 - 0.474(\frac{\eta}{n})^3 + 0.0296(\frac{\eta}{n})^4 \\
E[(\hat{\mu}_{(1)}/\mu)^4] &= 1.03 - 1.74(\frac{\eta}{n}) + 2.28(\frac{\eta}{n})^2 - 0.563(\frac{\eta}{n})^3 + 0.0352(\frac{\eta}{n})^4
\end{aligned}
$$

These are obtained by polynomial least square fits. The residuals from the regression are plotted in Figures 3.1-3.4. They show decreasing dispersion with sample size. This feature was used in weighted least squares polynomial fits with weights inversely proportional to the sample size. However, there were negligible differences in the resulting estimates. Hence, we have reported the results of the simple least square fit only.

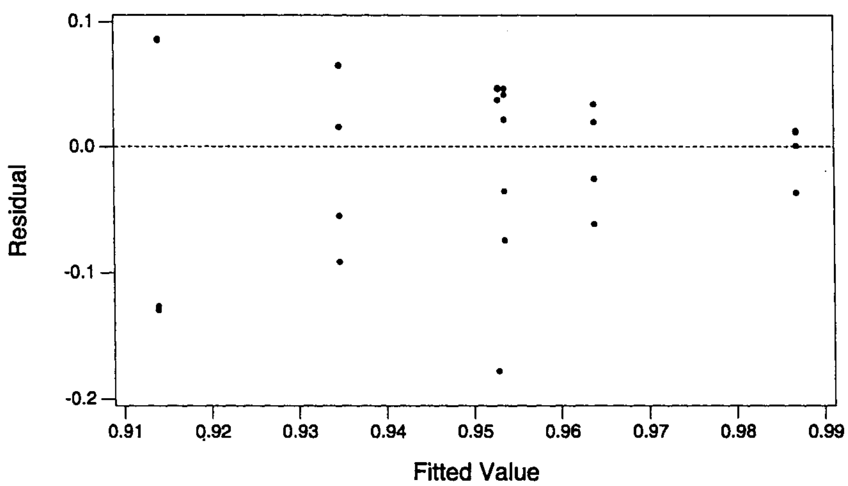

Figure 3.1: Residuals for the Approximation of the First Moment of $\frac{\hat{\mu}_1}{\mu}$

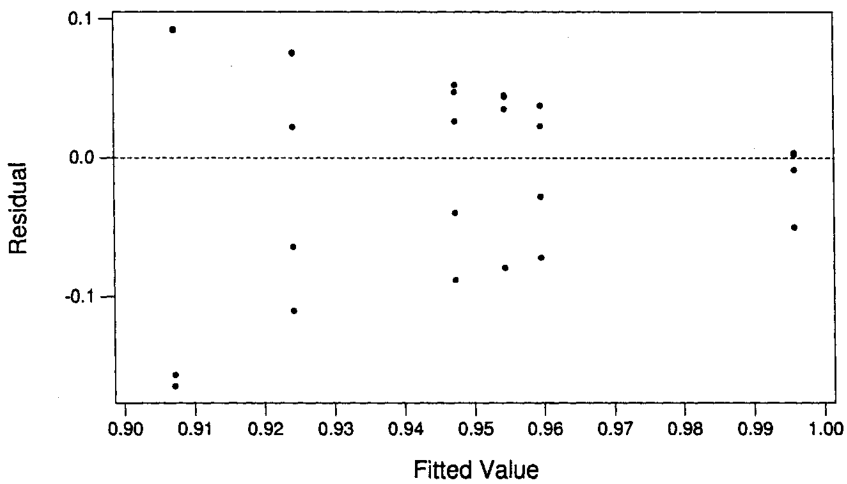

Figure 3.2: Residuals for the Approximation of the Second Moment of $\frac{\hat{\mu}_1}{\mu}$

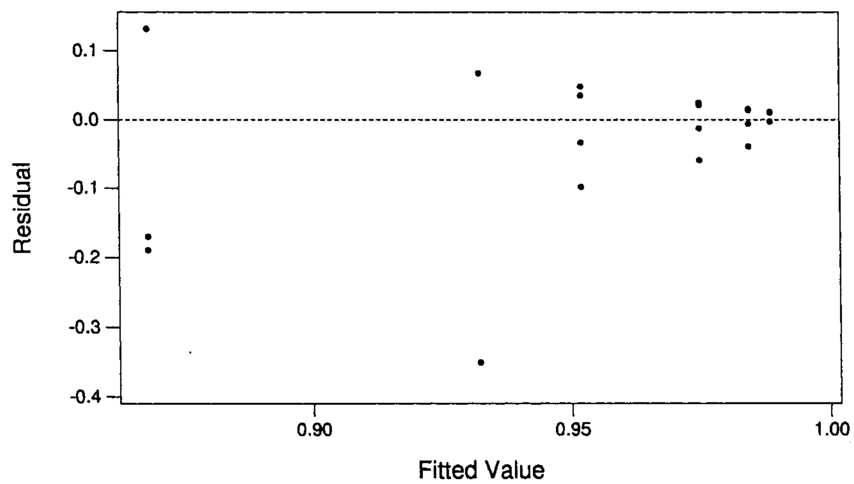

Figure 3.3: Residuals for the Approximation of the Third Moment of $\frac{\hat{\mu}_1}{\mu}$

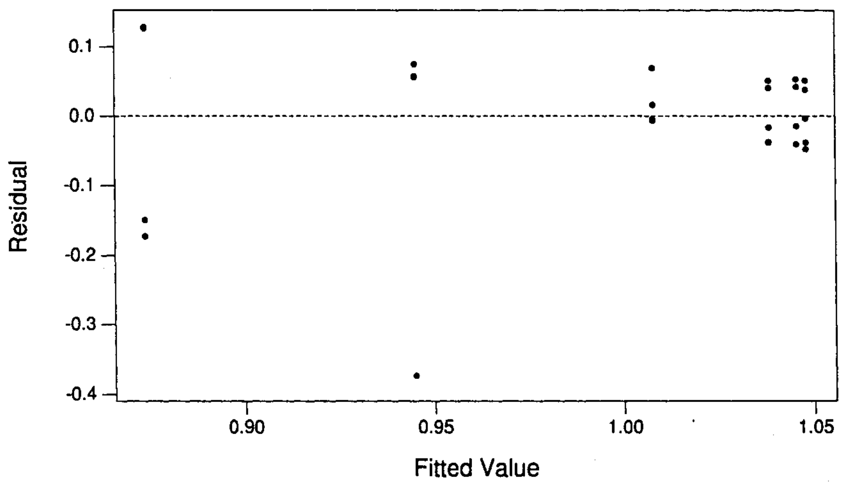

Figure 3.4: Residuals for the Approximation of the Fourth Moment of $\frac{\hat{\mu}_1}{\mu}$

Chapter 4

Preliminary Test Estimator of the Mean of an Inverse Gaussian Population

4.1 Introduction

Tests of hypothesis are often used to validate a given model. Such tests are referred to as preliminary tests of significance. In practice, the accepted value of a parameter under a null hypothesis may be considered the true value. Bancroft (1944) proposed to use such prior guesses to be used in place of the usual estimator if the prior guess is ascertained using a test of hypothesis, otherwise the traditional estimator is to be used. The resulting estimator is termed as the Preliminary Test Estimator (PTE) or simply a *testimator*. To fix the basic idea behind this procedure, let us consider estimating the mean μ of some infinite population. Suppose, μ_0 is the prior guess of the parameter μ. For a given sample, let $\bar{X}$ be the sample mean, then under a variety of situations/models, it is a "good" estimator of μ. However, there may be strong evidence in favor of μ_0, in that case, the statistician should choose μ_0 as the proper estimator. If the evidence is taken from the sample based on a test statistic T, such an estimator may be represented as

$$\hat{\mu}_{PTE} = \bar{X} I_{T \in CR} + \mu_0 I_{T \notin CR}$$

where CR denotes the critical region for testing $H_0 : \mu = \mu_0$ vs. $H_0 : \mu \neq \mu_0$, based on a test statistic T.

Bancroft (1944) considered the case of a Gaussian population and showed that such estimators may provide large gains in efficiency especially if the true value of the parameter is near the hypothesized value. They further provided guidelines for choosing the level of significance. This method has been adapted in various other situations by Bancroft (1963), Paul A. E. (1950), Huntsburger (1954a), Arnold and Katti (1972), Bock *et al.* (1973), Ghosh and Sinha (1988), Upadhyay *et al.*, Yancey *et al.* (1989), Han (1978) and many others.

In this chapter, we investigate the performance of the PTE of the mean in a single population inverse Gaussian set-up. Section 2 considers the case of a known dispersion parameter λ and Section 3 considers the unknown case. Section 4 presents the relative bias and relative MSE properties of the resulting estimator.

4.2 Preliminary Test Estimation of the Mean with a Prior Guess

4.2.1 Known λ Case

As explained above, the PTE requires testing about the prior guess. In this case, we first test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. In this case, the uniformly most powerful unbiased test is given in the form of the critical region:

$$CR = \{\bar{x} : \bar{x} < k_1 \text{ } or \text{ } \bar{x} > k_2\},$$

where k_1, k_2 are determined from the conditions

$$\int_{k_1}^{k_2} g(t)dt = 1 - \alpha \quad \text{and} \quad \int_{k_1}^{k_2} tg(t)dt = \mu_0(1 - \alpha)$$

and g is the pdf of $\bar{x}$. Chhikara and Folks (1989) show that this is equivalent to considering the test statistic

$$Z = \frac{\sqrt{n\lambda}(\bar{x} - \mu_0)}{\mu_0\sqrt{\bar{x}}}$$

and corresponding critical region,

$$|Z| > z_{1-\alpha/2},$$

where $z_{1-\alpha/2}$, is the $100(1 - \alpha/2)\%$ percentiles of the standard normal distribution. Using the above critical region, the constants k_1 and k_2 can be found as,

$$k_1 = \left[\frac{\mu_0 c_1 + \sqrt{\mu_0^2 c_1^2 + 4\mu_0 n\lambda}}{2\sqrt{n\lambda}}\right]^2$$

and

$$k_2 = \left[\frac{\mu_0 c_2 + \sqrt{\mu_0^2 c_2^2 + 4\mu_0 n\lambda}}{2\sqrt{n\lambda}} \right]^2,$$

where $c_1 = -z_{1-\alpha/2}$ and $c_2 = z_{1-\alpha/2}$. Now the computation of a preliminary test estimator of the mean μ is given by

$$\begin{aligned} \hat{\mu}_z &= \bar{X} I_{\bar{X} \in CR} + \mu_0 I_{\bar{X} \notin CR} \\ &= \bar{X} - (\bar{X} - \mu_0) I_{[k_1 < \bar{X} < k_2]}. \end{aligned} \quad (4.2.1)$$

4.2.2 Unknown λ Case

For unknown λ, the UMP-unbiased test for $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ is given in the form of the critical region:

$$CR = \{\bar{X} < k_3 \; or \; \bar{X} > k_4\},$$

where k_3 and k_4 are determined by

$$\int_{k_3}^{k_4} h(u|v)du = 1 - \alpha \quad \text{and} \quad \int_{k_3}^{k_4} uh(u|v)du = (1-\alpha) \int_{-\infty}^{\infty} uh(u|t)du$$

and $h(u|v)$ denotes the conditional density function of $\bar{X}$ given V. Chhikara and Folks (1989) show that it is equivalent to consider the statistic,

$$T = \frac{\sqrt{n-1}(\bar{X} - \mu_0)}{\mu_0 \sqrt{V\bar{X}}},$$

where

$$V = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{X_i} - \frac{1}{\bar{X}}\right)$$

and corresponding critical region,

$$\left|\frac{\sqrt{(n-1)}(\bar{X}-\mu_0)}{\mu_0\sqrt{(\bar{X}V)}}\right| > t_{1-\frac{\alpha}{2}},$$

were $t_{1-\frac{\alpha}{2}}$, is the $100(1-\frac{\alpha}{2})\%$ percentiles of the student's t distribution with $(n-1)$ degrees of freedom. This gives k_3 and k_4 in terms $t_{1-\frac{\alpha}{2}}$, as

$$k_3 = \left[\frac{\mu_0 c_1\sqrt{V}+\sqrt{\mu_0^2 c_1^2 V + 4\mu_0(n-1)}}{2\sqrt{n-1}}\right]^2$$

and

$$k_4 = \left[\frac{\mu_0 c_2\sqrt{V}+\sqrt{\mu_0^2 c_2^2 V + 4\mu_0(n-1)}}{2\sqrt{n-1}}\right]^2.$$

And hence, the PTE of μ in this case is given by

$$\begin{aligned}\hat{\mu_t} &= \bar{X} I_{\bar{X}\in CR} + \mu_0 I_{\bar{X}\notin CR} \\ &= \bar{X} - (\bar{X}-\mu_0) I_{[k_3<\bar{X}<k_4]}. \end{aligned} \tag{4.2.2}$$

In order to judge the performance of PTE, we need compute its bias and MSE. This is explained in the following section.

4.3 Bias and the MSE of PTE's

4.3.1 Known λ Case

The moments of the PTE with known λ depend only on the distribution of $\bar{X}$. The following propositions will be used in computing the bias and MSE for the known λ case.

Proposition 4.1 *The power function for the test* $H_0 : \mu = \mu_0$ *against* $H_1 : \mu \neq \mu_0$ *is*

$$\pi(\mu) = 1 - \Pr[k_1 < \bar{x} < k_2|\mu],$$

which may be written as

$$\pi(\mu) = 1 - F(k_2; \mu, n\lambda) + F(k_1; \mu, n\lambda), \tag{4.3.1}$$

where $F(.; \mu, \lambda)$ *denotes the cumulative distribution function of* $IG(\mu, \lambda)$.

Now we provide a figure for the power function for $n = 16$, $\mu_0 = 1$, $\lambda = 1$.

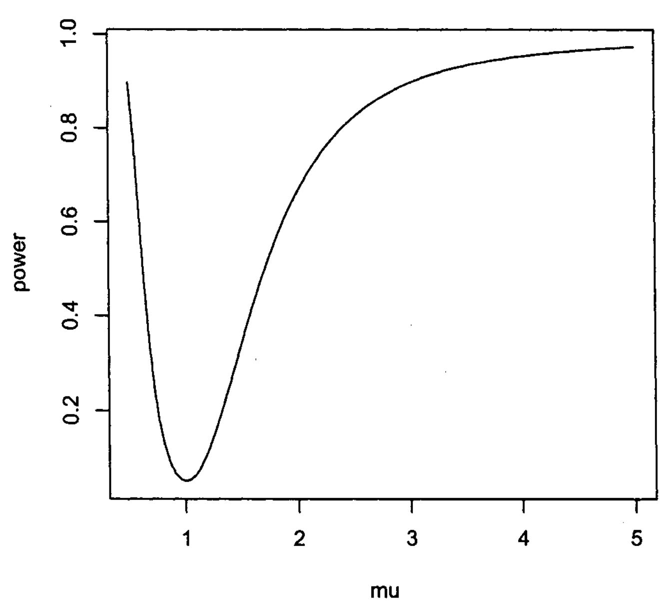

Figure 4.1: Power function for the UMP unbiased test in IG case, $H_0 : \mu = 1$

Proposition 4.2 *The expressions for the bias and the MSE of $\hat{\mu}_z$, are respectively given by*

$$Bias(\hat{\mu}_z) = \mu_0[1 - \pi(\mu)] - \int_{k_1}^{k_2} w f_{\bar{X}}(w)dw, \tag{4.3.2}$$

and

$$MSE(\hat{\mu}_z) = \frac{\mu^3}{n\lambda} + 2\mu Bias(\hat{\mu}_z) + \mu_0^2(1 - \pi(\mu)) - \int_{k_1}^{k_2} w^2 f_{\bar{X}}(w)dw \tag{4.3.3}$$

where $\pi(\mu)$ is the power function of the UMP-unbiased test for testing $H_0 : \mu = \mu_0$ vs. $H_0 : \mu \neq \mu_0$, and $B(\hat{\mu}_z)$ denotes the bias of $\hat{\mu}_z$ given in Eq. (4.3.4).

PROOF: Using the expression in Eq. (4.2.1), a straight forward calculation provides,

$$E(\hat{\mu}_z) = \mu - \int_{k_1}^{k_2} (w - \mu_0) f_{\bar{X}}(w)dw,$$

and the expression for the bias follows, noting that

$$\pi(\mu) = 1 - \Pr[k_1 < \bar{X} < k_2|\mu] = \int_{k_1}^{k_2} f_{\bar{X}}(w)dw.$$

To simplify the MSE expression, using Eq. (4.2.1), we note that

$$(\hat{\mu}_z - \mu)^2 = (\bar{X} - \mu)^2 + [\mu_0^2 - \bar{X}^2 + 2\mu(\bar{X} - \mu_0)]I_{k_1 \leq \bar{X} \leq k_2},$$

and hence, the result follows. □

4.3.2 Unknown λ Case

We note in this case that the critical region depends on the values of V. Hence, for computing the moments of $\hat{\mu}_t$, first we compute the conditional moments $b(v) = E[(\hat{\mu}_t - \mu)|V = v]$, and $m(v) = E[(\hat{\mu}_t - \mu)^2|V = v]$ using the Proposition

(4.2) replacing k_1, k_2 by $k_3(v) \simeq k_3, k_4(v) \simeq k_4$. Hence, we obtain the following expressions for bias and the MSE in the unknown case.

Proposition 4.3 *The expressions for the bias and the MSE of $\hat{\mu}_t$, are respectively given by*

$$Bias(\hat{\mu}_t) = \int_0^{\infty} b(v) f_V(v) dv$$

and

$$MSE(\hat{\mu}_t) = \int_0^{\infty} m(v) f_V(v) dv,$$

where $f_V(v)$ is the probability density function of the Chi-square distribution with $(n-1)$ degrees of freedom.

4.4 A Numerical Comparison of the Estimators

The above formulae are used to compute the bias and the MSE for various sample sizes and different values of μ. The integrals involved were computed using the Splus `integrate` function. The value of λ was fixed at 1. Figures 1.2, 1.3, 1.4, 1.5, 1.6, 1.7 summarize these computations, however, some representative values are tabulated.

Table 4.1 gives the bias of $\hat{\mu}_z$ for $n = 16$ and Table 4.2 presents that for $n = 20$ for two different values of α, namely, 1% and 5%. Similarly Tables 4.3 and 4.4 present those for unknown λ case. Table 4.5 presents the relative MSE's for $n = 16$ where as Table 4.6 presents the relative efficiencies of the PTE's $\hat{\mu}_z$ and $\hat{\mu}_t$ with respect to the sample mean for $\alpha = 5\%$. To visualize the effect of the preliminary test on the bias and to assess the gain in efficiency, we plot the relative bias and relative efficiency (with respect to $\bar{X}$) for different sets of parameters, sample sizes and significance levels.

Based on these graphs and tables, we draw the following conclusions:

(1) Bias decreases as n increases.

(2) When $\mu \leq 1$ then bias increases, for $1 < \mu \leq 1.5$ then bias decreases but when $1.5 < \mu$ then again bias increases. As α increases the bias also increases.

(3) For fixed μ, the bias increases as μ_0 increases, but for fixed μ_0 the bias decreases as μ increases.

(4) The maximum possible loss of efficiency increases for $\mu = 1$ and $\mu \geq 1.5$, but when $\mu = 1.5$ the efficiency decreases.

(4) The effective difference of efficiency is greater when α increases.

(5) The results indicate that the IG estimator is effective in reducing the maximum loss of efficiency and increasing the effective difference.

(6) By examination of the values in graphs and tables, it will be seen that when λ is known, the preliminary test of significance controls the bias well for larger values of μ, resulting in substantial gains in relative efficiency.

Table 4.1: Bias of $\hat{\mu}_z$ for $n = 16$

μ	$\mu_0 = 1$	$\mu_0 = 2$	$\mu_0 = 3$	$\mu_0 = 1$	$\mu_0 = 2$	$\mu_0 = 3$
		$\alpha = 1\%$			$\alpha = 5\%$	
0.5	0.135169	0.001663	0.000017	0.033591	0.000014	0.000000
1.0	0.000000	0.666944	0.701347	0.000000	0.33155	0.165811
1.5	-0.274586	0.459349	1.209603	-0.157975	0.361778	0.755871
2.0	-0.231339	0.000000	0.918142	-0.106140	0.000000	0.720122
2.5	-0.151435	-0.404113	0.469159	-0.061421	-0.284178	0.382874
3.0	-0.099347	-0.651850	0.000000	-0.037523	-0.409260	0.000000
3.5	-0.068564	-0.756061	-0.429436	-0.024774	-0.434245	-0.316430
4.0	-0.049948	-0.530559	-0.552975	-0.017527	-0.415125	-0.535160
4.5	-0.038148	-0.745209	-1.032303	-0.013118	-0.381317	-0.668068
5.0	-0.030303	-0.700645	-1.202236	-0.010269	-0.345466	-0.738977

Table 4.2: Bias of $\hat{\mu}_z$ for $n = 20$

μ	$\mu_0 = 1$	$\mu_0 = 2$	$\mu_0 = 3$	$\mu_0 = 1$	$\mu_0 = 2$	$\mu_0 = 3$
		$\alpha = 1\%$			$\alpha = 5\%$	
0.5	.073691	.000545	.000004	0.013155	0.000001	0.000000
1.0	.000000	.556286	.394361	0.000000	0.232133	0.060848
1.5	-.025092	.453104	1.010851	-0.138439	0.349072	0.618549
2.0	-.017663	.000000	.901902	-0.075309	0.000000	0.685998
2.5	-.097573	-.399575	.467579	-0.036206	-0.278504	0.379441
3.0	-.055609	-.625242	-.00000	-0.019052	-0.383887	0.000000
3.5	-.034233	-.696693	-.428009	-0.011162	-0.387064	-0.314403
4.0	-.022715	-.682034	.767887	-0.007169	-0.351793	-0.523547
4.5	-.060613	-.631524	-1.00486	-0.004956	-0.308290	-0.640477
5.0	-.011960	-.571377	-1.15-792	-0.003631	-0.267625	-0.692917

Table 4.3: Bias of $\hat{\mu}_t$ for $n = 16$

μ	$\mu_0 = 1$	$\mu_0 = 2$	$\mu_0 = 3$	$\mu_0 = 1$	$\mu_0 = 2$	$\mu_0 = 3$
		$\alpha = 1\%$			$\alpha = 5\%$	
0.5	0.220419	0.056914	0.0194914	0.063415	0.001149	0.000075
1.0	0.000000	0.754592	1.051109	0.000000	0.412151	0.321653
1.5	-0.320748	0.465899	1.277217	-0.185189	0.375385	0.848951
2.0	-0.322579	0.000000	0.931265	-0.139115	0.000000	0.747992
2.5	-0.238804	-0.425646	0.473027	-0.086274	-0.306782	0.393048
3.0	-0.169686	-0.726898	0.000000	-0.055011	-0.462131	0.000000
3.5	-0.123476	-0.893645	-0.444393	-0.037340	-0.509779	-0.335851
4.0	-0.093282	-0.960589	-0.827869	-0.026917	-0.502651	-0.582566
4.5	-0.073113	-0.966792	-1.134207	-0.020412	-0.472982	-0.745113
5.0	-0.059190	-0.940561	-1.363285	-0.016134	-0.436640	-0.842185

Table 4.4: Bias of $\hat{\mu}_t$ for $n = 20$

μ	$\mu_0 = 1$	$\mu_0 = 2$	$\mu_0 = 3$	$\mu_0 = 1$	$\mu_0 = 2$	$\mu_0 = 3$
		$\alpha = 1\%$			$\alpha = 5\%$	
0.5	0.154391	0.009589	0.001245	0.027164	0.000031	0.000002
1.0	0.000000	0.686583	0.776284	0.000000	0.298308	0.138373
1.5	-0.303374	0.465515	1.225797	-0.159564	0.361638	0.706735
2.0	-0.261062	0.000000	0.928041	-0.096314	0.000000	0.712842
2.5	-0.16503	-0.426797	0.475417	-0.049367	-0.296960	0.388071
3.0	-0.102495	-0.712976	0.000001	-0.027020	-0.426232	0.000000
3.5	-0.066754	-0.848150	-0.447743	-0.016238	-0.445135	-0.330139
4.0	-0.046032	-0.878382	-0.829270	-0.010609	-0.416043	-0.562023
4.5	-0.033447	-0.851539	-1.124489	-0.007423	-0.372609	-0.702414
5.0	-0.025408	-0.799343	-1.333576	-0.005486	-0.328976	-0.774574

Table 4.5: Relative MSE of $\hat{\mu}_z$ and $\hat{\mu}_t$ for $n = 16$, $\alpha = 5\%$

μ	$\mu_0 = 1$	$\mu_0 = 2$	$\mu_0 = 3$	$\mu_0 = 1$	$\mu_0 = 2$	$\mu_0 = 3$
		Known λ			Unknown λ	
0.5	0.120303	0.031362	0.031250	0.186201	0.039450	0.032097
1.0	0.017443	0.454858	0.471961	0.015725	0.523701	0.800750
1.5	0.131528	0.119756	0.647558	0.133363	0.118039	0.689564
2.0	0.165866	0.034885	0.242931	.175505	0.031451	0.241816
2.5	0.181158	0.116823	0.065896	0.189792	0.109330	0.063769
3.0	0.202221	0.209199	0.052328	0.208383	0.204601	0.047176
3.5	0.227908	0.273657	0.114466	0.232186	0.275439	0.104433
4.0	0.256054	0.316686	0.197398	0.259088	0.323892	0.185390
4.5	0.285477	0.348612	0.275665	0.287699	0.359163	0.265356
5.0	0.315591	0.375782	0.341917	0.317272	0.387956	0.335461

Table 4.6: Relative Efficiency of $\hat{\mu}_z$ and $\hat{\mu}_t$ for $n = 16$, $\alpha = 5\%$

μ	$\mu_0 = 1$	$\mu_0 = 2$	$\mu_0 = 3$	$\mu_0 = 1$	$\mu_0 = 2$	$\mu_0 = 3$
		Known λ			Unknown λ	
0.5	0.259759	0.996401	0.999997	0.167829	0.792141	0.973609
1.0	3.583147	0.137405	0.132426	3.974425	0.119343	0.078052
1.5	0.712775	0.782836	0.144774	0.702970	0.794223	0.135955
2.0	0.753619	3.583147	0.514549	0.712229	3.974425	0.516921
2.5	0.862507	1.337489	2.371157	0.823269	1.429157	2.450233
3.0	0.927205	0.896272	3.583147	0.899786	0.916418	3.974425
3.5	0.959816	0.799356	1.911042	0.942130	0.794186	2.094635
4.0	0.976356	0.789425	1.266476	0.964922	0.771862	1.348508
4.5	0.985193	0.806770	1.020260	0.977581	0.783069	1.059895
5.0	0.990206	0.831599	0.913964	0.984956	0.805503	0.931552

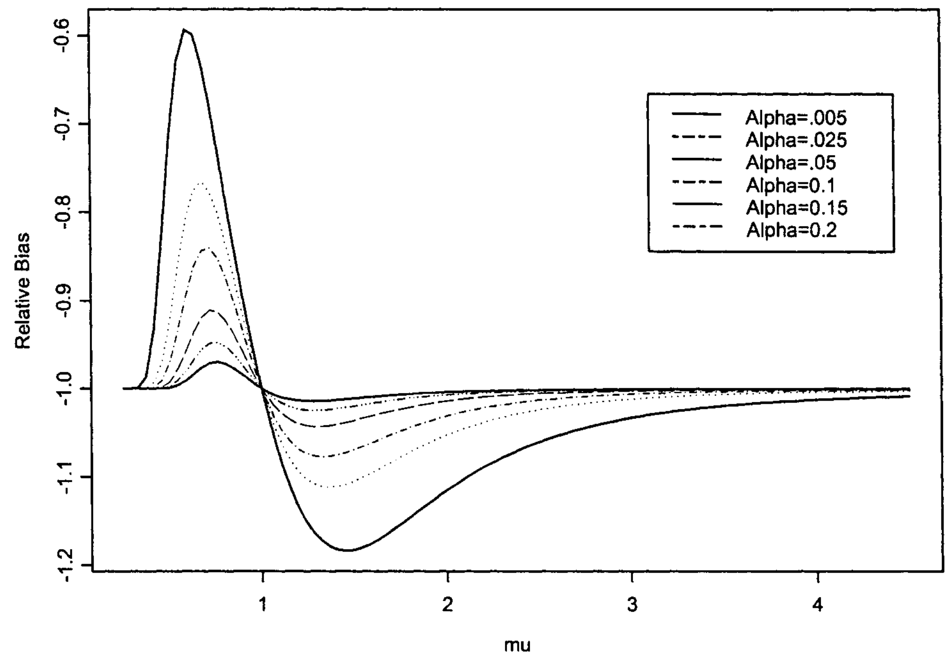

Figure 4.2: Relative Bias of $\hat{\mu}_z$

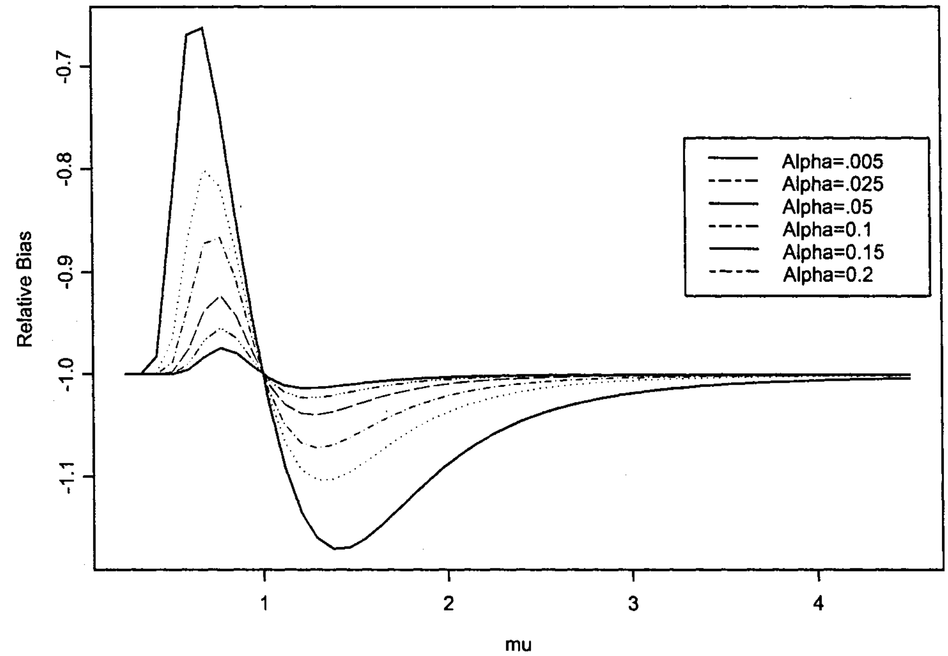

Figure 4.3: Relative Bias of $\hat{\mu}_t$

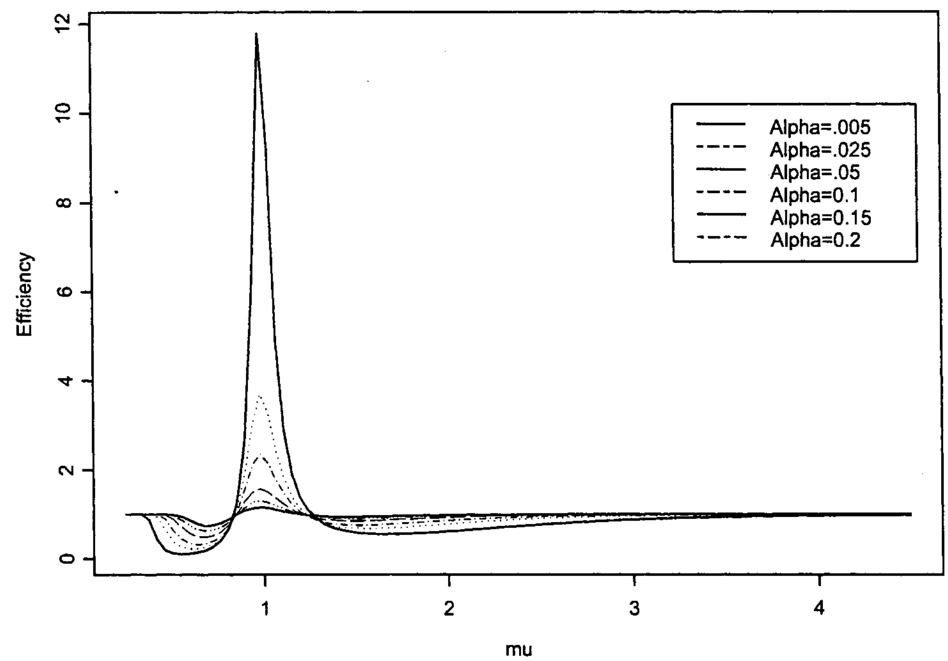

Figure 4.4: Relative Efficiency of $\hat{\mu}_z$

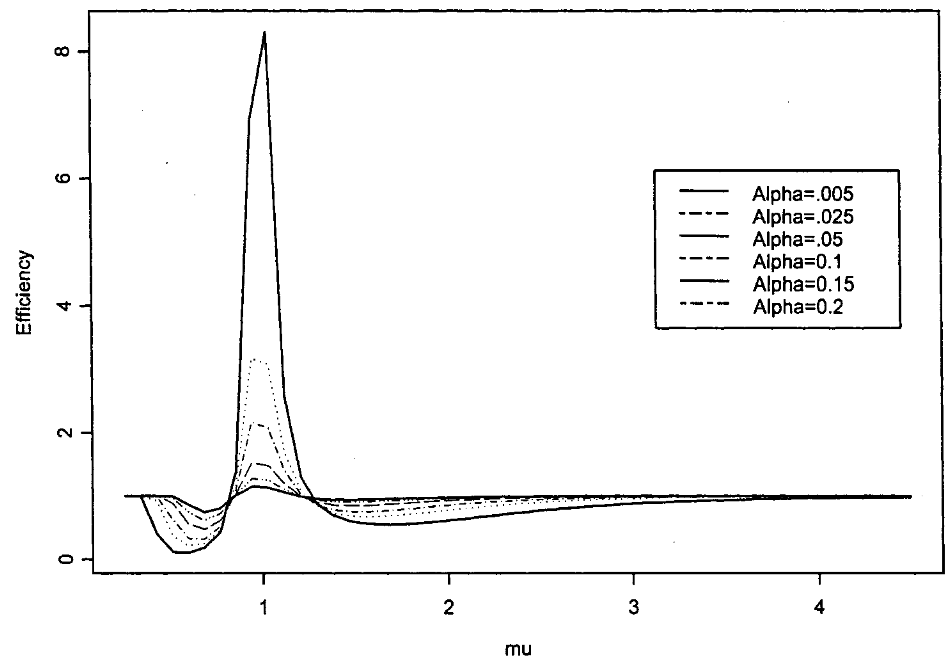

Figure 4.5: Relative Efficiency of $\hat{\mu}_t$

Chapter 5

Inference on overlap in two inverse Gaussian populations: equal means case

5.1 Introduction

Let F_1 and F_2 be two distribution functions with the corresponding density functions with respect to the Lebesgue measure. Four commonly used measures that describe the closeness between F_1 and F_2 are described below;

Matusita's Measure:

$$.\rho = \int \sqrt{f_1(x)f_2(x)}dx. \tag{5.1.1}$$

Morisita's Measure:

$$.\delta = \frac{2\int f_1(x)f_2(x)dx}{\int[f_1(x)]^2dx + \int[f_2(x)]^2dx}. \tag{5.1.2}$$

Pianka's Measure:

$$.\alpha^* = \frac{\int f_1(x)f_2(x)dx}{\sqrt{\int[f_1(x)]^2dx\int[f_2(x)]^2dx}}. \tag{5.1.3}$$

Weitzman's Measure:

$$.\Delta = \int min(f_1(x), f_2(x))dx. \qquad (5.1.4)$$

These measure are widely studied in the literature [see Mulekar and Mishra (1994, 2000)] when the two densities correspond to the normal case. In this note we consider inference for these measures for the case of inverse Gaussian densities. Due to the wide use in applications, as mentioned in Chapter 1, it is of interest to study the properties of the overlap coefficients for two inverse Gaussian populations in contrast to two Gaussian populations. In Section 2 the expressions for the measures described above are derived under various conditions together with their properties along the lines in Mulekar and Mishra (1994). In Section 3 we provide their maximum likelihood estimates along with approximate variances and covariances. Some striking similarities to the Gaussian case are noted. In Section 4, we present a confidence interval estimation for these coefficients and finally in Section 5 a simulation study is carried out to compare different methods for finding the confidence intervals.

5.2 Properties of Different Overlap Measures

Let $f_i(x)$ denote the inverse Gaussian density with mean μ and dispersion parameter λ_i. Then using the form of the $IG(\mu, \lambda)$ density from Eq. (1.2.1.), we get

$$f(x_i) = \left\{\frac{\lambda_i}{2\pi x^3}\right\}^{\frac{1}{2}} exp\left\{-\frac{\lambda_i}{2\mu^2 x}(x-\mu)^2\right\}, \qquad i = 1, 2, x_i \geq 0.$$

Now

$$\{f_1(x)f_2(x)\}^{\frac{1}{2}} = \sqrt{\frac{\sqrt{\lambda_1\lambda_2}}{\lambda}} f(x), \tag{5.2.1}$$

where $f(x)$ denotes an inverse Gaussian density with mean μ and dispersion parameter λ and

$$\lambda = \frac{\lambda_1 + \lambda_2}{2}.$$

Hence, we have from Eq. (5.1.1).

$$\begin{aligned}\rho &= \sqrt{2}\sqrt{\frac{\sqrt{\lambda_1\lambda_2}}{\lambda_1 + \lambda_2}} \\ &= \sqrt{\frac{2C}{1+C^2}},\end{aligned} \tag{5.2.2}$$

where

$$C^2 = \frac{\lambda_2}{\lambda_1}.$$

For deriving the form of the Morisita's measure, we recall the definition of the modified Bessel function of the third kind of order j (see Abramowitz and Stegun (1972) for its properties), given by

$$K_j(u) = \frac{1}{2}\int_0^\infty t^{j-1} exp\{-\frac{u}{2}(t + t^{-1})\}dt. \tag{5.2.3}$$

Then, for $\chi,\psi \geq 0$,

$$\int_0^\infty x^{\lambda-1} exp[-\frac{1}{2}(\psi x + \chi x^{-1})]dx = \frac{2K_\lambda(\sqrt{\chi\psi})}{(\frac{\psi}{\chi})^{\lambda/2}}. \tag{5.2.4}$$

Now

$$f_1(x)f_2(x) = \frac{\sqrt{\lambda_1\lambda_2}}{2\pi} x^{-3} exp\frac{-\lambda(x-\mu)^2}{2\mu^2 x}. \tag{5.2.5}$$

Using the above definition, with $\lambda = \frac{\lambda_1+\lambda_2}{2}$, we have

$$\begin{aligned} \int_0^\infty f_1(x)f_2(x)dx &= \frac{(\lambda_1\lambda_2)^{\frac{1}{2}}}{2\pi} exp\left(\frac{2\lambda}{\mu}\right) \int_0^\infty x^{-3} exp\left[-\frac{1}{2}\left(\frac{\lambda x}{\mu^2} + \lambda x^{-1}\right)\right] dx \\ &= \frac{(\lambda_1\lambda_2)^{\frac{1}{2}}}{\pi\mu^2} exp\left(\frac{2\lambda}{\mu}\right) K_2\left(\frac{2\lambda}{\mu}\right). \qquad (5.2.6) \end{aligned}$$

Hence, the Morisita's measure of overlap as given in Eq. (5.1.2). becomes,

$$\begin{aligned} \delta &= \frac{\frac{2(\lambda_1\lambda_2)^{\frac{1}{2}}}{\mu^2} exp\left(\frac{2\lambda}{\mu}\right) K_2\left(\frac{2\lambda}{\mu}\right)}{\frac{\lambda_1}{\mu^2} exp\left(\frac{2\lambda_1}{\mu}\right) K_2(\frac{2\lambda_1}{\mu}) + \frac{\lambda_2}{\mu^2} exp\left(\frac{2\lambda_2}{\mu}\right) K_2(\frac{2\lambda_2}{\mu})} \\ &= \frac{2(\lambda_1\lambda_2)^{1/2}\exp(2\lambda/\mu)K_2(2\lambda/\mu)}{\lambda_1\exp(2\lambda_1/\mu)K_2(2\lambda_1/\mu) + \lambda_2\exp(2\lambda_2/\mu)K_2(2\lambda_2/\mu)}. \qquad (5.2.7) \end{aligned}$$

We also get from Eq. (5.1.3)., Pianka's overlap measure as

$$\begin{aligned} \alpha^* &= \frac{(\lambda_1\lambda_2)^{\frac{1}{2}}\exp(\frac{\lambda_1+\lambda_2}{\mu})K_2(\lambda/\mu)}{(\lambda_1\lambda_2)^{\frac{1}{2}}\exp(\frac{\lambda_1+\lambda_2}{\mu})\sqrt{K_2(\frac{2\lambda_1}{\mu})K_2(\frac{2\lambda_2}{\mu})}} \\ &= \frac{K_2(2\lambda/\mu)}{\sqrt{K_2(2\lambda_1/\mu)K_2(2\lambda_2/\mu)}}. \qquad (5.2.8) \end{aligned}$$

The Weitzman's measure is obtained by evaluating the common area under the two intersecting curves $y = f_1(x)$ and $y = f_2(x)$. Let the intersecting points be given by $x_1 < x_2$, then for $\lambda_1 < \lambda_2$, (f_{λ_1} has shorter tails than f_{λ_1})

$$\begin{aligned} \Delta &= F_2(x_1) + F_1(x_2) - F_1(x_1) + 1 - F_2(x_2) \\ &= 1 - \{F_2(x_2) - F_2(x_1)\} + \{F_1(x_2) - F_1(x_1)\}, \qquad (5.2.9) \end{aligned}$$

where F_i denotes the cumulative distribution function corresponding to f_i. We note that the distribution function corresponding to a $IG(\mu, \lambda)$, distribution as given by Eq. (1.2.2). can be written as

$$F(x) = \Phi(y) + e^{2\lambda/\mu}\Phi\left(-\sqrt{y^2 + \frac{4\lambda}{\mu}}\right), \qquad x > 0, \qquad (5.2.10)$$

where $y = \sqrt{\frac{\lambda}{x}}\frac{x-\mu}{\mu}$ and $\Phi(.)$ denotes the *cdf* of the standard normal distribution (see Chhikara and Folks (1989), Eq. 2.16).

To compute the intersection's point x_1 and x_2 we have to solve $f_1(x) = f_2(x)$ as

$$\lambda_1^{\frac{1}{2}} exp\left(-\frac{\lambda_1(x-\mu)^2}{2\mu^2 x}\right) = \lambda_2^{\frac{1}{2}} exp\left(-\frac{\lambda_2(x-\mu)^2}{2\mu^2 x}\right),$$

$$\frac{1}{2}\ln\lambda_1 - \frac{\lambda_1(x-\mu)^2}{2\mu^2} = \frac{1}{2}\ln\lambda_2 - \frac{\lambda_2(x-\mu)^2}{2\mu^2},$$

$$\frac{(x-\mu)^2}{\mu^2 x} = \frac{\ln\frac{\lambda_1}{\lambda_2}}{\lambda_1 - \lambda_2}.$$

Since $\frac{\lambda_1}{\lambda_2} = c^2$ then

$$\frac{\lambda_2(x-\mu)^2}{\mu^2 x} = \frac{2\ln c}{c^2 - 1}.$$

It can be easily seen that the points of intersection satisfy

$$\frac{\sqrt{\lambda_2}}{\mu}\frac{x_1-\mu}{\sqrt{x_1}} = -b,$$

$$\frac{\sqrt{\lambda_2}}{\mu}\frac{x_2-\mu}{\sqrt{x_2}} = b,$$

where

$$b = \sqrt{\frac{-2\ln C}{1-C^2}}. \tag{5.2.11}$$

Now

$$F_2(x_1) = \phi(y_1) + exp(\frac{2\lambda_2}{\mu})\phi(-\sqrt{(}y_1^2 + \frac{4\lambda_2}{\mu})),$$

where

$$y_1 = \frac{\sqrt{(}\lambda_2)(x_1-\mu)}{\mu\sqrt{(}x_1)} = -b.$$

Further

$$F_2(x_2) = \phi(y_2) + exp(\frac{2\lambda_2}{\mu})\phi(-\sqrt{(}y_2^2 + \frac{4\lambda_2}{\mu})), \qquad x_2 > 0,$$

where

$$y_2 = \frac{\sqrt(\lambda_2)(x_2 - \mu)}{\mu\sqrt(x_2)} = b.$$

Therefore

$$F_2(x_1) = 1 - \phi(b) + exp(\frac{2\lambda_2}{\mu})\phi(-\sqrt(b^2 + \frac{4\lambda_2}{\mu})), \qquad x_1 > 0,$$

and

$$F_2(x_2) = \phi(b) + exp(\frac{2\lambda_2}{\mu})\phi(-\sqrt(b^2 + \frac{4\lambda_2}{\mu})), \qquad x_2 > 0.$$

Similarly, we can write

$$F_1(x_2) = \phi(cb) + exp(\frac{2\lambda_1}{\mu})\phi(-\sqrt(c^2b^2 + \frac{4\lambda_1}{\mu})), \qquad x_2 > 0.$$

and

$$F_1(x_1) = 1 - \phi(cb) + exp(\frac{2\lambda_1}{\mu})\phi(-\sqrt(c^2b^2 + \frac{4\lambda_1}{\mu})), \qquad x_1 > 0,$$

Thus, we have

$$\begin{aligned} F_1(x_2) - F_1(x_1) &= 2\Phi(Cb) - 1, \\ F_2(x_2) - F_2(x_1) &= 2\Phi(b) - 1, \end{aligned}$$

and hence assuming $C < 1$, we get

$$\Delta = 1 - 2\Phi(b) + 2\Phi(Cb). \tag{5.2.12}$$

For $C > 1$, we can similarly show that

$$\Delta = 1 + 2\Phi(b) - 2\Phi(Cb). \tag{5.2.13}$$

Note that the above expression also tallies with the expression in the Gaussian case. We can now establish the following properties.

Lemma 5.1 *(i) For ρ and Δ, OVL measures, $OVL(C) = OVL(1/C)$.*

(ii) ρ and Δ do not depend on μ, however α^* and λ depend on μ though the parameters λ_1/μ and λ_2/μ.

(iii) $0 \leq OVL \leq 1$.

Proof: The form of ρ and Δ are similar to those obtained in the case of the Gaussian populations with equal means. Thus, we need only prove these properties for the measures α^* and λ only. For this purpose, we shall need the following lemma on log-convexity of the Bessel function $K_a(x)$.

Lemma 5.2 *Let $g(x) = \log K_a(x)$, then we have $g''(x) > 0$, for $x > 0$, i.e. the function $K_a(x)$ is log-convex.*

Proof: Clearly,

$$g''(x) = \frac{K_a''(x)}{K_a(x)} - \left[\frac{K_a'(x)}{K_a(x)}\right]^2. \tag{5.2.14}$$

Since, the function $K_\lambda(x)$ is a solution to the modified Bessel equation

$$y'' + \frac{1}{x}y' - (1 + \frac{a^2}{x^2})y = 0,. \tag{5.2.15}$$

the first term in equation Eq. (5.2.14) becomes

$$\frac{K_a''(x)}{K_a(x)} = -\frac{K_a'(x)}{xK_a(x)} + (1 + \frac{a^2}{x^2}). \tag{5.2.16}$$

We get for a solution y to Eq. (5.2.15)

$$\frac{1}{2}\frac{\partial}{\partial x}[(xy')^2] = x^2y'y'' + x(y')^2 = (x^2 + a^2)yy',$$

i.e.,

$$\frac{\partial}{\partial x}[(xy')^2] = (x^2 + a^2)\frac{\partial}{\partial x}y^2. \tag{5.2.17}$$

Since, K_a and K_a' vanish exponentially as $x \to \infty$, we can write Eq. (5.2.17) by using integration by parts as,

$$[t^2(K_a'(t))^2]_x^\infty = [(t^2 + a^2)K_a^2(t)]_x^\infty - 2\int_x^\infty tK_a^2(t)dt. \tag{5.2.18}$$

Using the above equation we can rewrite the second term of Eq. (5.2.14) as

$$\left[\frac{K_a'(x)}{K_a(x)}\right]^2 = (1 + \frac{a^2}{x^2}) - \frac{2}{x^2}\int_x^\infty t\left(\frac{K_a(t)}{K_a(x)}\right)^2 dt. \tag{5.2.19}$$

Therefore, Eq. (5.2.14) becomes

$$g''(x) = -\frac{K_a'(x)}{xK_a(x)} + \frac{2}{x^2}\int_x^\infty t\left(\frac{K_a(t)}{K_a(x)}\right)^2 dt. \tag{5.2.20}$$

Since,

$$K_a(x) = -\frac{1}{2}[K_{a-1}(x) + K_{a+1}(x)],$$

we note that $K_a'(x) \geq 0$ and we conclude that $g''(x) \geq 0$, and the result in the lemma follows.

Using the above lemma, therefore, it follows that,

$$\log K_2(\lambda_1 + \lambda_2) \leq \frac{1}{2}\log K_2(2\lambda_1) + \frac{1}{2}\log K_2(2\lambda_2).$$

This shows that

$$K_2(2\lambda) \leq K_2^{1/2}(2\lambda_1)K_2^{1/2}(2\lambda_2).$$

Table 5.1: Dependence of λ and α^* on μ

	Values of λ		
μ	0.1	2	100
$\lambda_1 = 1, C^2 = 1.5$	0.9825295	0.9598499	0.9412238
$\lambda_1 = 2, C^2 = 1.5$	0.9836466	0.9679075	0.9418291
	Values of α^*		
μ	0.1	2	100
$\lambda_1 = 1, C^2 = 1.5$	0.9856234	0.9637654	0.9600027
$\lambda_1 = 2, C^2 = 1.5$	0.9876336	0.9685528	0.9600106

It shows that $0 \leq \alpha^* \leq 1$.

Again using the log-convexity of K_2, we have for λ,

$$\begin{aligned}\sqrt{\lambda_1\lambda_2}\exp(\frac{\lambda_1+\lambda_2}{\mu})K_2(\frac{2\lambda}{\mu}) &\leq \left[\lambda_1\exp(2\frac{\lambda_1}{\mu})K_2(2\frac{\lambda_1}{\mu})\lambda_2\exp(2\frac{\lambda_2}{\mu})K_2(2\frac{\lambda_2}{\mu})\right]^{1/2} \\ &\leq \frac{\lambda_1\exp(2\frac{\lambda_1}{\mu})K_2(2\frac{\lambda_1}{\mu} + \lambda_2\exp(2\frac{\lambda_2}{\mu})K_2(2\frac{\lambda_2}{\mu})}{2},\end{aligned}$$

the latter inequality following from AM-GM inequality, which in turn proves that $0 \leq \delta \leq 1$.

The dependence of the measures α^* and λ on μ is demonstrated by some calculations given below.

Due to the dependence of δ and α^*, on μ, in contrast to ρ and Δ, the latter measures, namely ρ and Δ, may be preferred. As mentioned in Lemma 5.1 (ii), α^* and δ depend on λ_1/μ and λ_2/μ, we may reparametrize the family in terms of the parameters, $\eta_1 = \mu/\lambda_1$ and $\eta_2 = \mu/\lambda_2$, which are the coefficients of variation (CV) for the two populations. Now we have $C^2 = \eta_2/\eta_1$. Mulekar and Mishra (1994) also noted that $\Delta \leq \rho$ and $\lambda \leq \rho$, for the Gaussian case. We may also establish the following parallel result for the inverse Gaussian case.

Lemma 5.3 *For ρ, δ, α^* and Δ defined in equations Eq. (5.1.1)-Eq. (5.1.4), we have (i) $\Delta \leq \rho$ and (ii) $\delta \leq \alpha^*$, equality holding in case $\eta_1 = \eta_2$.*

Proof: The proof of the first part follows from Lemma 2(ii) of Mulekar and Mishra (1994). The proof of the second part is as follows.

It is easy to see that

$$\delta = \alpha^* \left[\frac{2r}{1+r^2}\right], \tag{5.2.21}$$

where

$$r \equiv r(\eta_1, \eta_2) = \left[\frac{\eta_1}{\eta_2}\frac{K_2(2/\eta_2)}{K_2(2/\eta_1)}\right]^{1/2} \exp\left\{-\left(\frac{1}{\eta_1} - \frac{1}{\eta_2}\right)\right\}. \tag{5.2.22}$$

Since, $\max_{r \geq 0}\left[\frac{2r}{1+r^2}\right] = 1$, part (ii) of the above lemma follows. In the next section, we address the estimation of these measures based on two independent samples from inverse Gaussian populations with the same mean.

5.3 Estimation of OVL Measures

Parallel results to those for the two normal populations can be established as in Mulekar and Mishra (1994) for the inverse Gaussian populations with common mean. This is based on the following results (see Chhikara and Folks (1989, Chap. 5)). Suppose that $(X_{ij}; j = 1, ..., n_i; i = 1, 2)$ denote independent observations from two independent IG populations. An unbiased estimator of λ_i^{-1} is given by

$$U_i = \frac{1}{n_i - 1} \sum_{i=1}^{n} \left(\frac{1}{X_i} - \frac{1}{\bar{X}} \right). \tag{5.3.1}$$

Then we may define an estimate of C^2 as

$$\hat{C}^2 = \frac{U_1}{U_2}.$$

Therefore, since,

$$(n_i - 1)\lambda_i U_i \sim \chi^2_{n_i - 1}. \tag{5.3.2}$$

replacing S_i^2 by U_i, Lemma 3 of Mulekar and Mishra (1994) follows, which is given below.

Lemma 5.4

$$E(\hat{C}^2) = \gamma_1 C^2, \quad \mathrm{Var}(\hat{C}^2) = \gamma_2 C^4,$$

where

$$\gamma_1 = \frac{n_2 - 1}{n_2 - 3}, \quad \gamma_2 = \frac{(n_2 - 1)^2 (n_1 + 1)}{(n_1 - 1)(n_2 - 3)(n_2 - 5)} - \gamma_1^2.$$

The results in the above lemma become apparent, by noting that

$$\hat{C}^2 = \frac{C^2 \nu_2}{\nu_1} \frac{\chi^2_{\nu_1}}{\chi^2_{\nu_2}},$$

where $\nu_i = n_i - 1$, and the chi-squares appearing in the above equations are independent.

Consider a function $g(\theta)$ of some parameter θ, and let $\hat{\theta}$ be an almost sure consistent estimate of θ, then the mean and variance of $g(\hat{\theta})$ may be obtained using the linear Taylor approximation

$$g(\hat{\theta}) \approx g(\theta) + (\hat{\theta} - \theta) g'(\theta). \tag{5.3.3}$$

Then

$$E[g(\hat{\theta})] = g(\theta) + E(\hat{\theta} - \theta)g'(\theta).$$

For the estimator of $\hat{\rho}$, we let $\theta = C^2$, we have

$$\hat{\rho} = g(\hat{\theta}), \quad g(\theta) = \sqrt{2}\theta^{1/4}(1+\theta)^{-1/2}.$$

Since, in this case,

$$\frac{g'(\theta)}{g(\theta)} = \frac{1}{4\theta} - \frac{1}{2}\frac{1}{1+\theta} = \frac{1-\theta}{4\theta(1+\theta)},$$

from Eq. (5.3.3) we have

$$E(\hat{\rho}) - \rho = E(\hat{\theta} - \theta)g'(\theta) = \frac{(\gamma_1 - 1)\rho}{4}\frac{1-C^2}{1+C^2} \tag{5.3.4}$$

and

$$\begin{aligned}\mathrm{Var}(\hat{\rho}) = \mathrm{Var}(\hat{\theta})[g'(\theta)^2] &= \frac{\gamma_2}{16}\frac{(1-C^2)^2}{(1+C^2)^2}. \\ &= \frac{\gamma_2}{16}\frac{(1-\theta)^2}{(1+\theta)^2}.\end{aligned} \tag{5.3.5}$$

Also

$$E(\hat{\rho}) - \rho = (\gamma_1 - 1)C^2 g'(\theta),$$

and

$$C^2 g'(\theta) = \frac{Bias}{\gamma_1 - 1}.$$

The above can also be expressed as

$$\mathrm{Var}(\hat{\rho}) = \frac{\gamma_2}{(\gamma_1 - 1)^2} Bias^2(\hat{\rho}). \tag{5.3.6}$$

We have from Eq. (5.2.12), for $0 < C < 1$,

$$g(\theta) = \Delta = 1 - 2\Phi(b) + 2\Phi(Cb),$$

where

$$b = \sqrt{\frac{-\ln\theta}{1-\theta}}$$
$$= (1-\theta)^{-\frac{1}{2}}(\ln(\frac{1}{\theta})^{\frac{1}{2}}.$$

Now

$$g'(\theta) = -2\phi(b)\frac{\partial b}{\partial\theta} + 2\phi(Cb)\frac{\partial(Cb)}{\partial\theta},$$

but

$$\frac{\partial b}{\partial\theta} = \frac{1}{2}\left(-\frac{\ln\theta}{1-\theta}\right)^{\frac{1}{2}}\left[\frac{\theta\ln\theta+1-\theta}{\theta(1-\theta)\ln\theta}\right]$$
$$\frac{\partial(Cb)}{\partial\theta} = \frac{1}{2}Cb\left[\frac{1}{\theta(1-\theta)} + \frac{1}{\theta\ln\theta}\right]$$

Therefore

$$\begin{aligned} E\hat{\Delta} &= \Delta + (\gamma_1 - 1)\theta\left[\phi(Cb)Cb\left(\frac{1}{C^2(1-C^2)} + \frac{1}{C^2\ln C^2}\right)\right. \\ &\quad \left. -\phi(b)b\left(\frac{1}{1-C^2} + \frac{1}{C^2\ln C^2}\right)\right] \\ &= \Delta + (\gamma_1 - 1)\left\{(C\phi(Cb) - \phi(b))\left(\frac{C^2b^2-1}{b(1-C^2)}\right) + Cb\phi(Cb)\right\} \end{aligned}$$

$$Bias(\hat{\Delta}) = (\gamma_1 - 1)\left\{(C\phi(Cb) - \phi(b))\left(\frac{C^2b^2-1}{b(1-C^2)}\right) + Cb\phi(Cb)\right\}$$

and

$$\text{Var}(\hat{\Delta}) = \frac{\gamma_2}{(\gamma_1-1)^2}Bias^2(\hat{\Delta}).$$

This is the same result obtained in Theorem of Section 3 of Mulekar and Mishra (1994) about the bias and variance of $\hat{\rho}$. Proceeding in a similar fashion, since $\hat{\Delta}$ also involves only $\hat{C}^2$ as the random variable, we get the same result as given in Theorem of Mulekar and Mishra (1994) in Section 3. This is reproduced below.

Theorem 5.1 *Suppose that $\hat{\rho}$ and $\hat{\Delta}$ are the estimates of ρ and Δ, respectively by substituting $\hat{C}^2$ for C^2, then for $n_1 > 1, n_2 > 5$, we have approximate expressions for the bias and variance of $\hat{\rho}$ and $\hat{\Delta}$ given by*

$$\begin{aligned}
Bias(\hat{\rho}) &= \frac{(\gamma_1 - 1)\rho}{4}\frac{1-C^2}{1+C^2}, \\
Var(\hat{\rho}) &= \frac{\gamma_2}{(\gamma_1-1)^2}Bias^2(\hat{\rho}), \\
Bias(\hat{\Delta}) &= (\gamma_1 - 1)\left\{(C\phi(cb) - \phi(b))\left(\frac{C^2b^2-1}{b(1-C^2)}\right) + Cb\phi(Cb)\right\} I_C, \\
Var(\hat{\Delta}) &= \frac{\gamma_2}{(\gamma_1-1)^2}Bias^2(\hat{\Delta}),
\end{aligned}$$

where

$$I_C = \begin{cases} 1 & \text{for } 0 < C < 1 \\ -1 & \text{for } C > 1. \end{cases}$$

Corollaries to the theorem for $n_1 = n_2$ also remain valid. Next we compute the approximate mean and variance for the other two measures.

The estimates of α^* and δ require estimation of η_1 and η_2. These are given by

$$\hat{\eta}_i = \bar{X}U_i, \qquad i = 1, 2, \tag{5.3.7}$$

where $\bar{X} = \frac{n_1\bar{X}_1 + n_2\bar{X}_2}{n_1+n_2}$. Consider, $\hat{\eta}_1 = \bar{X}U_1$. Since $\nu_1 U_1 \lambda_1 \sim \chi^2_{\nu_1}$ then

$$E(U_1^2) = \frac{1}{(\nu_1\lambda_1)^2}E(\chi^2_{\nu_1}) = \frac{\nu_1+2}{\nu_1\mu^2}\eta_1^2,$$

and

$$E(\bar{X}^2) = Var(\bar{X}) + \mu^2 = \mu^2\left(w_1^2\frac{\eta_1}{n_1} + w_2^2\frac{\eta_2}{n_2}\right) + \mu^2.$$

Then

$$E(\hat{\eta}_1^2) = E(\bar{X}^2)E(U_1^2) = \frac{\nu_1+2}{\nu_1}\eta_1^2\left(w_1^2\frac{\eta_1}{n_1} + w_2^2\frac{\eta_2}{n_2} + 1\right).$$

Therefore

$$Var(\hat{\eta}_1) = \frac{\nu_1+2}{\nu_1}\eta_1^2\left(w_1^2\frac{\eta_1}{n_1} + w_2^2\frac{\eta_2}{n_2}\right) + \frac{2}{\nu_1}\eta_1^2 = V_1^2.$$

Similarly

$$Var(\hat{\eta}_2) = \frac{\nu_2+2}{\nu_2}\eta_2^2\left(w_1^2\frac{\eta_1}{n_1} + w_2^2\frac{\eta_2}{n_2}\right) + \frac{2}{\nu_2}\eta_2^2 = V_2^2.$$

Also

$$V_{12} = E(\hat{\eta}_1 - \eta_1)(\hat{\eta}_2 - \eta_2) = \eta_1\eta_2\left(w_1^2\frac{\eta_1}{n_1} + w_2^2\frac{\eta_2}{n_2}\right).$$

Lemma 5.5 *$\hat{\eta}_i$ is unbiased for η_i and*

$$\begin{aligned} Var(\hat{\eta}_i) &= \frac{n_i+1}{n_i-1}\eta_i^2\left(\omega^2\frac{\eta_1}{n_1} + (1-\omega)^2\frac{\eta_2}{n_2}\right) + \frac{2}{n_1-1}\eta_i^2 \equiv V_i^2, \\ Cov(\hat{\eta}_1, \hat{\eta}_2) &= \eta_1\eta_2\left(\omega^2\frac{\eta_1}{n_1} + (1-\omega)^2\frac{\eta_2}{n_2}\right) \equiv V_{12}, \end{aligned}$$

where $w = \frac{n_1}{n_1+n_2}$.

The above lemma follows from the independence of $\bar{X}_i, U_i, for i = 1, 2$, and their distributional properties. Using the first order Taylor expansion in two variables, we obtain the following theorem.

Theorem 5.2 *Suppose that $\hat{\alpha}^*$ and $\hat{\delta}$ are the estimates of α^* and δ, respectively, by substituting $\hat{\eta}_1$ and $\hat{\eta}_2$ for η_1 and η_2 respectively, then for $n_1 > 2, n_2 > 2$, we have that $\hat{\alpha}^*$ and $\hat{\delta}$ are approximately unbiased with approximate variances given by*

$$\begin{aligned} Var(\hat{\alpha}^*) &= V_1^2 g_1^2 + V_2^2 g_2^2 + 2V_{12}g_1g_2, \\ Var(\hat{\delta}) &= V_1^2 q_1^2 + V_2^2 q_2^2 + 2V_{12}q_1q_2, \end{aligned}$$

where

$$g_i = \frac{\alpha^*}{\eta_i^2}\left[\frac{K_2'(2/\eta_i)}{K_2(2/\eta_i)} - \frac{K_2'(\eta_1^{-1}+\eta_2^{-1})}{K_2(\eta_1^{-1}+\eta_2^{-1})}\right],$$

$$q_i = \frac{2r}{1+r^2}\left[g_i - \delta_i\alpha^*\frac{r^2-1}{\eta_i^2(r^2+1)}\left(\frac{\eta_i}{2} + \frac{K_2'(2/\eta_i)}{K_2(2/\eta_i)} + 1\right)\right],$$

where $\delta_i = 1$, *for* $i = 1$, *and* -1, *for* $i = 2$, *and* V_1^2, V_2^2 *and* V_{12} *are as given in Lemma 3.2. The derivative* $K_2'(x)$ *may be computed from the formula*

$$K_a'(x) = -\frac{1}{2}\left(K_{a-1}(x) + K_{a+1}(x)\right), \qquad x > 0,$$

and r *is as given in Eq. (5.2.22)*

Proof:

Consider $\hat{\eta}_i = \hat{\mu}U_i$, for $i = 1, 2$, where $U_i = \frac{1}{n_i-1}\sum_{j=1}^{n}(\frac{1}{X_{ij}} - \frac{1}{\bar{X}_i})$ and $(n_i - 1)U_i\lambda_i \sim \chi^2_{n_i-1}$ for $i = 1, 2$. Now Eq.(5.2.8) can be written as

$$\hat{\alpha}^* = \frac{K_2(2\hat{\lambda}/\hat{\mu})}{\sqrt{K_2(2\hat{\lambda}_1/\hat{\mu})K_2(2\hat{\lambda}_2/\hat{\mu})}} = g(\hat{\eta}_1, \hat{\eta}_2).$$

By using Taylor's theorem,

$$\hat{\alpha}^* = g(\eta_1, \eta_2) + (\hat{\eta}_1 - \eta_1)\frac{\partial g}{\partial \eta_1} + (\hat{\eta}_2 - \eta_2)\frac{\partial g}{\partial \eta_2} + \ldots$$

Then it can be easily shown that

$$E(\hat{\alpha}^*) \approx g(\eta_1, \eta_2)$$

and

$$Var(\hat{\alpha}^*) \approx E(\hat{\eta}_1 - \eta_1)^2\left(\frac{\partial g}{\partial \eta_1}\right)^2 + E(\hat{\eta}_2 - \eta_2)^2\left(\frac{\partial g}{\partial \eta_2}\right)^2 + 2\left(\frac{\partial g}{\partial \eta_1}\right)\left(\frac{\partial g}{\partial \eta_2}\right)E(\hat{\eta}_1 - \eta_1)(\hat{\eta}_2 - \eta_2).$$

But

$$\frac{\partial g(\eta_1,\eta_2)}{\partial \eta_1} = \frac{\alpha^*}{\eta_1^2}\left[\frac{K_2'\left(\frac{2}{\eta_1}\right)}{K_2\left(\frac{2}{\eta_1}\right)} - \frac{K_2'\left(\frac{1}{\eta_1}+\frac{1}{\eta_2}\right)}{K_2\left(\frac{1}{\eta_1}+\frac{1}{\eta_2}\right)}\right] = g_1,$$

$$\frac{\partial g(\eta_1,\eta_2)}{\partial \eta_2} = \frac{\alpha^*}{\eta_2^2}\left[\frac{K_2'\left(\frac{2}{\eta_2}\right)}{K_2\left(\frac{2}{\eta_2}\right)} - \frac{K_2'\left(\frac{1}{\eta_1}+\frac{1}{\eta_2}\right)}{K_2\left(\frac{1}{\eta_1}+\frac{1}{\eta_2}\right)}\right] = g_2,$$

and the result for $Var(\hat{\alpha}^*)$ follows. To find the variance of $\hat{\delta}$, we can write the Eq.(5.2.7) as

$$\delta = \frac{2K_2(\frac{1}{\lambda_1}+\frac{1}{\lambda_2})}{(\frac{\lambda_1}{\lambda_2})^{\frac{1}{2}}\exp(\frac{1}{\lambda_2}-\frac{1}{\lambda_1})K_2(\frac{2}{\lambda_2}) + (\frac{\lambda_2}{\lambda_1})^{\frac{1}{2}}\exp(\frac{1}{\lambda_1}-\frac{1}{\lambda_2})K_2(\frac{2}{\lambda_1})}$$

$$= \frac{2\alpha^*(K_2(\frac{2}{\lambda_1})K_2(\frac{2}{\lambda_2}))^2}{(\frac{\lambda_1}{\lambda_2})^{\frac{1}{2}}\exp(\frac{1}{\lambda_2}-\frac{1}{\lambda_1})K_2(\frac{2}{\lambda_2}) + (\frac{\lambda_2}{\lambda_1})^{\frac{1}{2}}\exp(\frac{1}{\lambda_1}-\frac{1}{\lambda_2})K_2(\frac{2}{\lambda_1})}$$

$$= \frac{2\alpha^*}{(\frac{\lambda_1}{\lambda_2})^{\frac{1}{2}}\exp(\frac{1}{\lambda_2}-\frac{1}{\lambda_1})\left(\frac{K_2(\frac{2}{\lambda_2})}{K_2(\frac{2}{\lambda_1})}\right)^{\frac{1}{2}} + (\frac{\lambda_2}{\lambda_1})^{\frac{1}{2}}\exp(\frac{1}{\lambda_1}-\frac{1}{\lambda_2})\left(\frac{K_2(\frac{2}{\lambda_1})}{K_2(\frac{2}{\lambda_2})}\right)^{\frac{1}{2}}}.$$

Consider

$$r = \left(\frac{\lambda_1}{\lambda_2}.\frac{K_2(\frac{2}{\lambda_2})}{K_2(\frac{2}{\lambda_1})}\right)^{\frac{1}{2}}\exp(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}).$$

Then

$$\delta = \alpha^*\left[\frac{2r}{1+r^2}\right]. \tag{5.3.8}$$

Suppose

$$q(\lambda_1,\lambda_2) = \left[\frac{2r}{1+r^2}\right],$$

then

$$\frac{\partial q(\lambda_1,\lambda_2)}{\partial \lambda_1} = -\frac{2(r^2-1)}{(r^2+1)^2}.\frac{\partial r(\lambda_1,\lambda_2)}{\partial \lambda_1}.$$

But

$$\frac{\partial r(\lambda_1,\lambda_2)}{\partial \lambda_1} = \frac{r}{\lambda_1^2}\left[\frac{1}{2}\lambda_1 + \frac{K_2'(\frac{2}{\lambda_1})}{K_2(\frac{2}{\lambda_1})} + 1\right].$$

Similarly

$$\frac{\partial r(\lambda_1,\lambda_2)}{\partial \lambda_2} = -\frac{r}{\lambda_2^2}\left[\frac{1}{2}\lambda_2 + \frac{K_2'(\frac{2}{\lambda_2})}{K_2(\frac{2}{\lambda_2})} + 1\right].$$

Therefore

$$\frac{\partial \delta}{\partial \lambda_1} = \frac{2r}{1+r^2}\left[g_1 - \alpha^* \frac{r^2-1}{\lambda_1^2(r^2+1)}\left(\frac{\lambda_1}{2} + \frac{K_2'(2/\lambda_1)}{K_2(2/\lambda_1)} + 1\right)\right] = q_1$$

and

$$\frac{\partial \delta}{\partial \lambda_2} = \frac{2r}{1+r^2}\left[g_1 + \alpha^* \frac{r^2-1}{\lambda_2^2(r^2+1)}\left(\frac{\lambda_2}{2} + \frac{K_2'(2/\lambda_2)}{K_2(2/\lambda_2)} + 1\right)\right] = q_2,$$

and the result for $Var(\hat{\delta})$ follows.

5.4 A Numerical Study

In this section we perform a numerical study comparing the approximate formulae for the bias and the mse derived in the previous section for different OVL measures. Random samples of size n are generated from two inverse Gaussian distributions with $\mu_i = 1, i = 1,2, \lambda_1 = 1, \lambda_2 = C^2\lambda_1$. The exact mean and variance is approximated by using 1000 replications for n taking values 10, 25, 50, 100, 200 and 500 and for values of C taking values 0.2, 0.5 and 0.8. The choice of these values have been motivated by the paper of Mulekar and Mishra (1994). All the computations have been performed using the R Software where the random number generation for the inverse Gaussian distribution has been performed using the function `rinvgauss`. The random samples are generated using the algorithm due to Michael *et al.* (1976). The R-code for this random number generator appears in Appendix A2. The Bessel function required in the computation of the OVL measures is computed using the function `Bessel.f`, the `R-code` appears in Appendix A3. Tables 5.2 and 5.3 summarize the results of the bias and the mse respectively, for various combinations of n and C. The conclusions are similar to those mentioned in Mulekar and Mishra (1994) for Gaussian populations.

From the expressions for the bias of the estimators of ρ and Δ, it is clear that these are approximately zero when $C = 1$. In our numerical study, generally the actual OVL's are found to be underestimated; only for small values of C and small sample sizes, they show overestimation. For sample sizes larger than 50, the bias is fairly close to zero. Matusita's measure has less bias than others but Weitzman's measure has the largest bias. As C approaches 1, then the standard deviation of $\hat{\Delta}$ increases. The bias decreases as sample size increases, as expected and the MSE goes to zero. The approximations for the bias and the MSE seem to be appropriate for sample sizes larger than 50.

Table 5.2: Bias of Estimators of OVL Measures Based on 1000 Samples

Note: Approximate values are in brackets.

n	10	25	50	100	200	500
			Matusita's Measure ρ			
0.2	0.00335	0.00107	-0.00172	0.00232	0.00067	-0.00021
	(0.02586)	(0.00823)	(0.00385)	(0.00187)	(0.00092)	(0.00036)
0.5	-0.01067	-0.00807	-0.00427	-0.00142	-0.00055	0.00031
	(0.03833)	(0.01219)	(0.00571)	(0.00277)	(0.00136)	(0.00054)
0.8	-0.02574	-0.01081	-0.00471	-0.00180	-0.00113	-0.00062
	(0.01959)	(0.00623)	(0.00292)	(0.00141)	(0.00069)	(0.00028)
			Weitzman's Measure Δ			
0.2	0.01070	-0.00479	-0.00169	0.00019	-0.00003	-0.00033
	(0.03904)	(0.01242)	(0.00581)	(0.00282)	(0.00139)	(0.00055)
0.5	-0.00705	-0.00171	0.00134	-0.00086	0.00067	-0.00045
	(0.06149)	(0.01957)	(0.00915)	(0.00444)	(0.00218)	(0.00087)
0.8	-0.00181	-0.00416	-0.00139	-0.00372	0.00007	-0.00070
	(0.06828)	(0.02173)	(0.01017)	(0.00493)	(0.00243)	(0.00096)
			Morisita's Measure δ			
0.2	0.05030	0.02395	0.01040	0.00732	0.00276	0.00050
	—	—	—	—	—	—
0.5	0.00975	0.00520	0.01025	0.00095	0.00041	0.00268
	—	—	—	—	—	—
0.8	-0.06794	-0.02788	-0.01225	-0.00577	-0.00477	-0.00212
	—	—	—	—	—	—
			Pianka's Measure α^*			
0.2	0.03713	0.01987	0.00732	0.00408	0.00201	0.00049
	—	—	—	—	—	—
0.5	0.00960	0.00987	0.00432	-0.00161	-0.00082	-0.00024
	—	—	—	—	—	—
0.8	-0.06855	-0.02735	-0.01278	-0.00716	-0.00380	-0.00168
	—	—	—	—	—	—

Table 5.3: MSE of Estimators of OVL Measures Based on 1000 Samples

Note: Approximate values are in brackets.

n	10	25	50	100	200	500
			Matusita's Measure ρ			
0.2	0.01062	0.00347	0.00166	0.00079	0.00041	0.00016
	(0.01029)	(0.00193)	(0.00079)	(0.00035)	(0.00017)	(0.00006)
0.5	0.00759	0.00314	0.00143	0.00078	0.00034	0.00013
	(0.02262)	(0.00425)	(0.00174)	(0.00078)	(0.00037)	(0.00015)
0.8	0.00296	0.00075	0.00030	0.00014	0.00006	0.00002
	(0.00591)	(0.00111)	(0.00045)	(0.00021)	(0.00009)	(0.00004)
			Weitzman's's Measure Δ			
0.2	0.00647	0.00205	0.00098	0.00045	0.00020	0.00009
	(0.05967)	(0.00021)	(0.00004)	(0.00002)	(0.00001)	(0.00001)
0.5	0.02023	0.00620	0.00329	0.00173	0.00083	0.00032
	(0.14811)	(0.00053)	(0.00009)	(0.00002)	(0.00001)	(0.00001)
0.8	0.02508	0.00910	0.00455	0.00235	0.00123	0.00042
	(0.18258)	(0.00065)	(0.00012)	(0.00003)	(0.00001)	(0.00001)
			Morisita's Measure δ			
0.2	0.01329	0.00371	0.00148	0.00079	0.00034	0.00012
	(0.06679)	(0.01997)	(0.00895)	(0.00423)	(0.00205)	(0.00081)
0.5	0.04076	0.01896	0.01096	0.00554	0.00266	0.00099
	(0.27539)	(0.09654)	(0.04619)	(0.02259)	(0.01117)	(0.00444)
0.8	0.01492	0.00528	0.00240	0.00109	0.00063	0.00024
	(0.03605)	(0.01303)	(0.00630)	(0.00310)	(0.00154)	(0.00061)
			Pianka's Measure α^*			
0.2	0.01258	0.00431	0.00184	0.00088	0.00042	0.00017
	(0.00623)	(0.00181)	(0.00078)	(0.00037)	(0.00018)	(0.00007)
0.5	0.03721	0.01828	0.01002	0.00497	0.00254	0.00092
	(0.05743)	(0.02011)	(0.00962)	(0.00470)	(0.00233)	(0.00092)
0.8	0.01433	0.00470	0.00236	0.00113	0.00055	0.00022
	(0.01301)	(0.00470)	(0.00227)	(0.00112)	(0.00055)	(0.00022)

Chapter 6

Directions for Further Research

6.1 Tests for Coefficient of Variation

It is seen that the coefficient of variation plays an important role in the problems considered earlier. There have been numerous tests involving the coefficient of variation for the one sample and the many samples case. Gupta *et al.* (1986) considered an exact test for the normal mean for known coefficient of variation where as Gupta and Liang (1987) considered the problem of selecting the best unknown mean amongst the normal populations with constant coefficient of variation. Pagurova and Orlov (1984) and Miller (1991) presented and studied some tests for the coefficient of variation in normal populations. Recently, Rao and Vidya (1992) have evaluated the performance of a test for the coefficient of variation in the normal population. Testing equality of coefficient of variations for two independent normal samples and for a bivariate normal sample was considered in Bhoj and Ahsanullah (1993a, 1993b). Nomachi (1983) considered the estimation of the common mean utilizing their coefficients of variation for

$k \geq 2$ normal populations where as Samanta (1984) considered the problem of estimating k means when the coefficients of variation are considered equal but unknown.

On the other hand, in the case of IG population, estimation of mean with known coefficient of variation has been considered by Joshi and Shah (1991) and Farsipour (1997) where as Hsieh (1990) has studied the likelihood ratio test in the one sample case and Choi and Kim (2003) has considered testing equality of coefficients of variation of two inverse Gaussian populations. Here we show that the most powerful invariant test in a single sample case, for simple null hypothesis vs. a simple alternative coincides with the Likelihood Ratio test derived by Hsieh (1990).

We will find it convenient to parametrize the density of $IG(\mu, \lambda)$ in terms of (μ, ϑ), where ϑ denotes the inverse of square of coefficient of variation denoted by

$$\vartheta = \frac{\lambda}{\mu}.$$

The testing problem about the coefficient of variation C can be stated in terms of the parameter ϑ. Given a realization $(\mathbf{x})$ of the random sample $\mathbf{X} = (\mathbf{X_1}, \mathbf{X_2}, ..., \mathbf{X_n})$ from $IG(\mu, \mu\vartheta)$, we find that the likelihood function $\ell(\mu, \vartheta|\mathbf{x})$ factors as follows;

$$\begin{aligned} \ell(\mu, \vartheta|\mathbf{x}) &= \prod_{i=1}^{n} \left[\mathrm{e}^{\vartheta} \left(\frac{\mu\vartheta}{2\vartheta\pi \mathrm{x}_i^3} \right)^{1/2} \right] \exp\left\{ -\frac{\vartheta}{2\mu} \sum \mathrm{x}_i - \frac{\mu\vartheta}{2} \sum \frac{1}{\mathrm{x}_i} \right\} \\ &= g_\theta(T).h(\mathbf{x}) \end{aligned}$$

where

$$g_\theta(T) = \prod_{i=1}^{n} \left[e^{\vartheta} \left(\frac{\mu\vartheta}{2\vartheta\pi} \right)^{1/2}_{\cdot} \right] \exp\left\{ -\frac{\vartheta}{2\mu} \sum \mathrm{x_i} - \frac{\mu\vartheta}{2} \sum \frac{1}{\mathrm{x_i}} \right\}$$

is a function of parameters only through

$$T(x) = (\sum x_i, \sum \frac{1}{x_i}),$$

and

$$h(x) = \prod_{i=1}^{n} \frac{1}{x_i^{3/2}},$$

is a function of the data only. Thus, by the Neyman's factorization theorem, it follows that the bivariate statistic $(\sum X_i, \sum \frac{1}{X_i})$ is sufficient. Note also that $IG(\mu, \lambda)$ is a convex exponential family (see Seshadri (1993), Prop. 2.6), $T(X)$ is complete sufficient. Furthermore, this family is closed under the group $G_c = \{g_c\}$, where $g_c(y) = cy, c > 0$, we may consider the reduction of the data as a maximal invariant under scale changes given by

$$Y = (Y_1, Y_2, ..., Y_n),$$

where

$$\begin{aligned} Y_i &= X_i/X_{i+1},\ i = 1, ..., n-1, \\ Y_n &= X_n. \end{aligned}$$

The distribution for Y has been worked out in Jörgenson (1982) (see Eq. 3.21) for the generalized inverse Gaussian distribution, whence, the joint pdf of Y is given by

$$\frac{n^{-n/2}}{2^n K_{-1/2}(\vartheta)^n} \prod_{i=1}^{n} y_i^{-\frac{i}{2}-1} \exp\left[-\frac{1}{2}\left(\mu\vartheta \sum_{k=1}^{n}\prod_{i=k}^{n} y_i^{-1} + \frac{\vartheta}{\mu}\sum_{k=1}^{n}\prod_{i=k}^{n} y_i\right)\right]. \tag{6.1.1}$$

Integrating out y_n, the joint density of the maximal invariant $(Y_1, Y_2, ..., Y_{n-1})$ is given by (see Eq. 3.23 of Jörgenson (1982));

$$\frac{K_{n/2}\vartheta T_1(y)T_2(y))}{2^{n-1}K_{1/2}^{n}(\vartheta)} \prod_{i=1}^{n} y_i^{-\frac{i}{2}-1} \left(\frac{T_1(y)}{T_2(y)}\right)^{-\frac{n}{2}} \tag{6.1.2}$$

where

$$T_1(y) = \sum_{k=1}^{n}\prod_{i=k}^{n} y_i^{-1}, \; T_2(y) = \sum_{k=1}^{n}\prod_{i=k}^{n} y_i.$$

The above analysis shows that the distribution of the maximal invariant $(Y_1, Y_2, ..., Y_{n-1})$ depends on the maximal invariant in the parameter space, namely ϑ. Jörgenson (1982) shows that the statistic $T = X_. X_{-1}$, where $X_. = \sum X_i$ and $X_- = \sum \frac{1}{X_i}$, is maximal invariant in the space of sufficient statistics, and since $T_1(y)T_2(y) = \sqrt{T}$, the distribution of T depends only on ϑ and the most powerful test of $H_0 : \vartheta = \vartheta_0$ against a simple alternative may be based on the distribution of the test statistic T. The distribution of T is not simple, however, it is shown in Jörgenson (1982) (see Eq. 5.6), that the ratio of the non-null pdf to that of the null pdf of T is given by

$$U(t; \vartheta_0, \vartheta) = \frac{K_{n/2}(\vartheta t)}{K_{1/2}^{n}(\vartheta)} \frac{K_{1/2}^{n}(\vartheta_0)}{K_{n/2}(t\vartheta_0)}.$$

This shows that there is no uniformly most powerful test for a composite alternative. The most powerful test for simple null vs. simple alternative, $H_0 : \vartheta = \vartheta_0$

vs. $H_1 : \vartheta = \vartheta_1$ is based on the test statistic T, since $K_a(tu)$ is monotone function of t, the critical region is given by

$$CR : \{X : T(X) \geq t_\alpha\},$$

where t_α is obtained from

$$\Pr[T \geq t_\alpha | \vartheta = \vartheta_0] = \alpha.$$

The exact percentage points t_α can be obtained from those of the modified statistic

$$T^* = \frac{1}{(n-1)\vartheta_0}\bar{X}(\sum(\frac{1}{X_i} - \frac{1}{\bar{X}}) = ZV,$$

where $Z \sim IG(1, \vartheta_0)$, and $(n-1)V \sim \chi^2_{n-1}$. Hence the distribution function of T^* under H_0 is given by

$$F_{T^*}(x) = \int_0^\infty G(x/z)f(z)dz.$$

This equation can be used for evaluation of the exact percentage points of T^*, however, Hsieh (1990) provides some approximations for the distribution of $1/T^*$ as well as some selected exact percentage points for some odd sample sizes.

6.2 Preliminary Test Estimator Using Coefficient of Variation

Preliminary tests of significance on η may be used to propose new, hopefully, improved estimators of mean. For example, suppose we accept a null hypothesis,

$H_0 : \eta \geq \eta_0$, we use the estimator proposed in Chapter 2, otherwise, we may use the sample mean as an estimator of the mean of the IG population.

The other direction in the use of the preliminary test estimator is that of Ahmed (1995) which considered the problem of estimation of coefficient of variation when it is *a priori* suspected that the two coefficients of variations are the same.

6.3 Preliminary Test Estimation in k Samples

PTE's have been adapted and studied in depth to shrinkage estimators for multi-sample case in a series of papers by Ahmed (1992), Ahmed and Saleh (1988), Ahmed and Saleh (1999), Saleh and Sen (1985a, 1985b, 1986, 1987a, 1987b) and Saleh and Han (1990). The case of more than two IG populations for estimating means or coefficients of variation may also be of interest where procedures similar to those studied in the normal case, as mentioned above may be investigated.

6.4 Overlap Measures in Unequal Means Case

Mulekar and Mishra (1997) have considered the inference procedures for overlap measures in the Gaussian case when the corresponding means are not the same. It is of interest to study such procedures for inverse Gaussian populations.

Appendix

A1. R-code for Area function

```
area<-function(f,a,b,...,fa=f(a,...),fb=f(b,...),
        limit=50,eps=1.0e-06)
        {
##Program to integrate a function f using recursive Simpson's
##rule eps is the absolute target error limit is max. number of
##iterations.

    h<-b-a
    d<-(b+a)/2
    fd<-f(d,...)
    a1<-((fa+fb)*h)/2
    a2<-((fa+4*fd+fb)*h)/6
    if ( abs(a1-a2) < eps )
        return(a2)
    if (limit ==0){
    warning(paste("recursion limit reached near x= ",d))
    return(a2)
    }
    Recall(f,a,d,...,fa=fa,fb=fd,limit=limit-1,eps=eps)+
    Recall(f,d,b,...,fa=fd,fb=fb,limit=limit-1,eps=eps)
}
```

A2. R-code for the Generation of IG Random Numbers

```
rinvgauss <- function(n, mu = stop("no shape arg"), lambda = 1) {

##Reference: Michael, Schucany and Haas, R.W. (1976). Generating
##random variables using transformation with multiple roots,
##American Statistician, vol. 30, 88-90.

  if(any(mu<=0)) stop("mu must be positive")
  if(any(lambda<=0)) stop("lambda must be positive")
  if(length(n)>1) n <- length(n)
  if(length(mu)>1 && length(mu)!=n) mu <- rep(mu,length=n)
  if(length(lambda)>1 && length(lambda)!=n)
  lambda <- rep(lambda,length=n)
  y2 <- rchisq(n,1)
  u <- runif(n)
  r1 <- mu/(2*lambda) * (2*lambda + mu*y2 -
  sqrt(4*lambda*mu*y2 + mu^2*y2^2))
  r2 <- mu^2/r1
  ifelse(u < mu/(mu+r1), r1, r2)
}
```

A3. R-code for Computing Bessel Functions of the Third Kind

```
   Bessel.f<-function(u,a)
##Bessel function of order a: {
   Bessel.kernel<-function(x,uk,ak)0.5*x^(ak-1.0)*
   exp(-0.5*uk*(x+(1/x)))
   integrate(Bessel.kernel,lower=0,upper=Inf,ak=a,uk=u)$value
}
```

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