Coherent States Based on the Euclidean Group

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Abstract

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This work is a contribution to the general theory of frames, using group representations. Specifically, we use the theory of generalized coherent states for semidirect product groups to construct coherent states for the three dimensional Euclidean group, starting from a a representation of this group, induced from a representation of the subgroup $H = \mathbb{R}^3 \times SO(2)$. Families of continuous coherent states are explicitly constructed for two specific choices of sections from the coset space $\Gamma = E(3)/(T_3 \times SO_3(2))$ to the group. We also discuss admissibility conditions for the existence of continuous frames for the general class of affine sections. Next we propose a discretization procedure for these continuous frames. Once again we obtain general admissibility conditions for the existence of frames and in particular, tight frames. Explicit examples are worked out.

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Contents

Introduction

In recent years wavelets have played an important role in research and technology. They where introduced by Morlet [23] in 1982 as a convenient tool to analyze seismic data. After a successful application in seismography, Morlet and Grossmann studied wavelets extensively [24] and developed the mathematical foundation for a wavelet theory. Later, Daubechies, Grossmann and Meyer [11] using wavelets, generalized the concept of a basis in a Hilbert space to frames. Subsequently, the theory of multiresolution analysis was developed by Mallat [21] and Meyer [22] for constructing frames, which essentially are suitably chosen families of vectors in a Hilbert space in terms of which arbitrary vectors can be decomposed, respecting some numerical stability and reconstructability conditions.

The reason of the popularity of wavelets lies, on the one hand, in the great variety of possible applications (signal analysis, numerical analysis, fractal imaging, telecommunications) and on the other, their mathematical simplicity and computational economy.

The motivation for this work arises from the possible use of the generalized wavelet transforms constructed here to signal analysis. The standard wavelet transform yields a time-scale representation of a signal, which is a more practical tool than the representation by the Fourier transform. This latter is a transform either only in time, or only in frequency. Generalized wavelet transforms are group theoretical extensions of the standard wavelet transform. In this thesis we approach this generalization via coherent states, noting that wavelets *are* coherent states associated with the affine group, that is the group of translations and dilations. The appearance of the translation and dilation parameters, jointly in the wavelet transform, enables one to analyze signals even at points of severe discontinuities and in fact, the wavelet transform is ideally suited to a study of discontinuities. Now, the affine group acts as a group of transformations on the real line, $\mathbb{R}$. Specifically, the two parameters of the group a, b, of dilation and translation, respectively, act in the manner $x \longrightarrow ax+b$. In other words, the signal itself is a function on $\mathbb{R}$, and the wavelet transform maps this function to a function of the group parameters (a, b). It then becomes of interest to see how one could analyze signals defined on other geometries by using other groups. This leads one to a generalization of wavelets/coherent states to coherent states associated to groups other than the affine group. The Poincaré group have already been considered [3], [5], [7], [17], and coherent states associated to this group have been constructed. A coherent states of the affine Galilei group also have been constructed [8]. In this thesis we chose the three dimensional Euclidean group. This choice will allow us to analyze signals on the sphere, S^2, by mapping them into functions on $S^2 \times \mathbb{R}^2$. In this manner, the signals are analyzed in terms of modulation and rotation parameters.

Before going into the computational details of this thesis, let us quickly retrace the historical development of those aspects of the theory of coherent states which are relevant to this thesis. Coherent states where "born" in the very early days of quantum mechanics. Schrödinger [28] studied a special kind of quantum states, which restored the classical behavior of the position operator of a quantum system, allowing the smooth transition from classical to quantum mechanics. He did not call them "coherent states" though. The term has its origin in quantum optics. R. Glauber used this special kind of states, which he termed "coherent states" in the description of coherent light beams emitted by lasers [14] [15]. But our approach to coherent states is based on group theory and the very useful identity, enjoyed by coherent states. The foundation to this approach was laid by J. Klauder [18]. He noticed a connection between the canonical coherent states and the unitary irreducible representation of the Weyl-Heisenberg group G_{WH}. The same connection was noticed by Gilmore [13] and Perelomov [26]. While Klauder in his approach used the whole group to index coherent states, Gilmore and Perelomov used as indices elements of the coset space $X = G_{WH}/Z$, where Z is the center of G_{WH} (i.e. the set of all elements commuting with every element of G_{WH}). A representation U of the Weyl-Heisenberg group is said to be "square integrable" if for some element η in the Hilbert space of the representation, the integral

$$\int_X | < U(g)\eta|\phi > |^2 d\nu(x), \; g \in G_{WH}, x = gZ,$$

is finite for every $\phi \in \mathfrak{H} = L^2(\mathbb{R}, dx)$. This is the starting point for the mathematical generalization of the theory of coherent states to other groups (G). In the Gilmore-Perelomov scheme, coherent states are define to be vectors

$$\eta_x = \{U(\sigma(x))\eta | x = gH_\eta \in X, g \in G\},$$

where U is a unitary irreducible representation of G on some appropriate Hilbert space, H_η is the subgroup of G leaving η fixed and $\sigma : X \to G$ is a suitably chosen section on the coset space G/H_η. But this method was not capable of handling some important groups in physics (like, for example, the Galilei, Poincaré and Euclidean groups). Another generalization was suggested by Ali, *et al.*[1] and eventually a much more general theory of coherent states was developed [4] which solves the problem for a large class of groups, in particular semidirect product groups (such as the Galilei, Poincaré and Euclidean groups). In this approach coherent states are constructed using a homogeneous space $X = G/H$, where H is suitable subgroup of G, not necessarily coinciding with H_η. Then one needs to find a section $\sigma : X \to G$ such that

$$\int_X | < U(\sigma(x))\eta | \phi > |^2 d\nu(x) < \infty \ ,$$

for all vectors ϕ in the representation space and for some vector η (called an *admissible vector*). It is assumed that the coset space G/H admits the *invariant* measure $d\nu$ under the natural action of the group. Coherent states are then defined to be the family of vectors η_x as x runs through the coset space G/H. The coherent state transform of a vector ϕ (the signal) is then the function $f(x) =< U(\sigma(x))\eta | \phi >$, defined now as function of the coset parameters x. In this thesis we shall use the term wavelet transform and coherent state transform interchangeably. The rest of this thesis is organized as follows. In Chapter 1 we describe in more details the construction of coherent states using the Gilmore-Perelomov method as well as more general method of Ali, *et al.* In that chapter we also set up the notation and terminology used in the thesis, and describe groups which are semidirect products of a real vector space V with S, a subgroup of the group $GL(V)$ of all nonsingular linear transformations of V. We construct induced representations for groups of this type, and describe the method of constructing coherent states using these representations. Chapter 2 is devoted to the three dimensional Euclidean group ($E(3)$), for which we give the explicit formulae for the representation induced from the trivial representation of $SO_3(2)$. The main results of this thesis are presented in chapters 3 and 4. In Chapter 3 we construct continuous coherent states associated with the Euclidean group for two various choices of sections, while in Chapter 4 we obtain discretized versions of these families.

Chapter 1

Preliminaries

In this chapter we present an overview of "coherent states", "induced representations" and "frames" - notions we will use in this thesis. We also describe the semidirect product groups, the type of group considered in this work.

In the first section we introduce the canonical coherent states - the first example of such states was used to study the transition from quantum to classical mechanics by Schrödinger [28].

In the next section we will concentrate on the group theoretical approach to coherent states, which is essential to this work. In section 3 we describe the method of finding the irreducible representations of the groups using the inducing technique of Mackey [20].

In section 4 we describe a semidirect-product groups, and using method from the previous section we find irreducible representations for this type of groups. We also discuss the construction of coherent states for semidirect product groups. In the last section of this chapter we discuss the concept of a "frame" - a very useful "substitute" for a basis in the Hilbert space.

1.1 Canonical Coherent States

The first example of coherent states was given by Schrödinger in 1926 [28]. He was interested in the transition from micro - to macro mechanics, or as we would say today, from quantum to classical mechanics. For the harmonic oscillator he proposed a class of normalized quantum states, which recover the classical behavior of the oscillator.

Let $\mathfrak{H}$ denote the Hilbert space of the states of the harmonic oscillator. We define the *creation* and *annihilation operators* as

$$a^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}), \qquad a = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}), \tag{1.1}$$

where $\hat{x}$, $\hat{p}$ are the position and momentum operators respectively, with a commutator $[\hat{x}, \hat{p}] = iI$. Such defined, the annihilation and creation operators satisfy the commutation relation

$$[a, a^\dagger] = \mathbf{1}.$$

In this setting the canonical coherent states $|z>$ are introduced as the "eigenstates" of the annihilation operator:

$$a|z> = z|z>,$$

where z is a complex eigenvalue. Following [19] we can express the states $|z>$ in terms of the eigenstates ($|n>$) of the number operator $N \equiv a^\dagger a$ as

$$|z>= e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} z^n |n> . \tag{1.2}$$

The factor $e^{-\frac{1}{2}|z|^2}$ is chosen such that the states $|z>$ are normalized, that is

$$< z|z >= 1.$$

But the states $|z>$ are not orthogonal, since

$$< z'|z >= \exp[-\frac{1}{2}|z|^2 - \frac{1}{2}|z'|^2 + z'^* z]$$

is a nowhere vanishing, continuous function of z and z', they form an overcomplete (i.e., the set remains complete upon removal of at least one vector) linearly dependent set. The important property of this states is that they give the resolution of the identity:

$$\frac{1}{\pi}\int d^2z|z>< z| = \sum_{n=0}^{\infty} |n>< n|; \qquad d^2z = d(Rez)d(Imz) = I.$$

and an arbitrary vector $|f>$ in the Hilbert space $\mathfrak{H}$ can be expanded as

$$|f>= \frac{1}{\pi}\int d^2z|z>< z|f> .$$

The coefficients of this expansion satisfy the equation

$$< z|f >= \frac{1}{\pi}\int d^2z' < z|z' >< z'|f >$$

so that $< z|z' >$ acts as a *reproducing kernel* .

The function $K : V \times V \to \mathbb{C}$ is a *reproducing kernel* if it satisfies the following properties:

1. $K(x,x) > 0,\ \forall x$ (positivity),

2. $K(x,y) = \overline{K(y,x)}$ (hermiticity),

3. $K(x,z) = \int_V K(x,y)K(y,z)d\nu(y)$ (idempotence),

and for any vector $f \in \mathfrak{H}_K \subset L^2(V, d\nu), \mathfrak{H}_K$ being the range of the projection operator defined by the kernel K,

$$f(x) = \int_V K(x,y)f(y)d\nu(y).$$

The quantum states $|z>$ (1.2) are parameterized by points of the classical phase space, because the expectation values of the position and momentum operators $\hat{x}, \hat{p}$ agree with the classical values. They also saturate the Heisenberg inequality:

$$\langle \triangle \hat{x} \rangle_z \langle \triangle \hat{p} \rangle_z = \frac{1}{2}\hbar,$$

where $\langle \triangle \hat{x} \rangle_z = [\langle z|\hat{x}^2|z\rangle - \langle z|\hat{x}|z\rangle^2]^{1/2}$.

1.2 The Standard Construction of Coherent States

In our work we are interested in coherent states associated with group representations. In this approach we construct coherent states in the following way.

Let G be a locally compact group, dg the left invariant Haar measure on G, and U a continuous, unitary, irreducible representation of G in some Hilbert space $\mathfrak{H}$. The representation U is called *square integrable* if there exists a vector $\eta \in \mathfrak{H}$ for which:

$$I(\eta, \phi) = \int_G |< U(g)\eta|\phi >|^2\, dg < \infty,\ \ \forall \phi \in \mathfrak{H} \qquad (1.3)$$

The vector η, normalized by $I(\eta, \eta) = 1$ is called *admissible.* Its orbit, $\mathcal{O} = \{\eta_g = U(g)\eta,\ g \in G\}$ under U, is an overcomplete family of vectors called *coherent states* associated with the representation U. For groups without square integrable representations, this condition is not satisfied, but a generalization was given by Perelomow [27] in which to weaken the admissibility condition, he considered a coset space $X = G/H_\eta$ with an invariant measure ν, H_η being the subgroup of G which leaves η invariant up to a phase:

$$U(h)\eta = e^{i\alpha(h)}\eta,\ h \in H_\eta.$$

Then the admissibility condition (1.3) changes to

$$I_X(\eta, \phi) = \int_X | < U(g)\eta|\phi > |^2\, d\nu(x) < \infty,\ \ \forall \phi \in \mathfrak{H}\ \text{ with } x \equiv gH_\eta. \tag{1.4}$$

In this case the representation U is called *square integrable mod* H_η .

1.2.1 The section σ

Since the integration in (1.4) is over a coset space X, it will be useful to represent g using elements of the coset space. To do this we take arbitrary Borel section $\sigma : X \to G$ in the principal fiber bundle $\pi : G \to X$, and write $g = \sigma(x)$. If condition (1.4) is satisfied for a given $\eta \in \mathfrak{H}$ we may define a set of coherent states

$$\mathcal{O}_\sigma = \{\eta_x = U(\sigma(x))\eta,\ x \in X\},$$

indexed by points $x \in X$. These coherent states constitute an overcomplete set of states, with all the nice properties of the canonical coherent states.

1. The CS system $\mathcal{O}_\sigma$ defines a resolution of identity

$$\int_X |\eta_x >< \eta_x| \, d\nu(x) = I$$

or equivalently, the linear map $W_\eta : \mathfrak{H} \rightarrow L^2(X, d\nu)$ defined by $(W_\eta \phi)(x) = < \eta_x | \phi >$, is an isometry onto a closed subspace $\mathfrak{H}_\eta$ of $L^2(X, d\nu)$.

2. The projection operator $P_\eta = W_\eta W_\eta^*$ on $\mathfrak{H}_\eta$ is an integral operator with the kernel $K(x', x) =< \eta_{x'} | \eta_x >$, which means $\mathfrak{H}_\eta = P_\eta \mathfrak{H}$ is a *reproducing kernel Hilbert space.*

3. The map W_η can be inverted on its range $\mathfrak{H}_\eta$ (because it is an isometry), and its inverse is the adjoint operator: $W_\eta^{-1} = W_\eta^*$ on $\mathfrak{H}_\eta$. This provides us with an *inversion formula:*

$$W_\eta^{-1} \phi = \int_X \phi(x) \eta_x d\nu(x), \quad \phi \in \mathfrak{H}_\eta.$$

But there are groups which do not possess square integrable representations in the above sense. A theory of generalized coherent states was developed [1]- [6], which solves the problem for the semidirect product groups. We describe this generalization in Section 1.4.

1.3 Induced Representations

In the construction of coherent states we use unitary irreducible representations of a group. Very effective way of obtaining such representations is inducing technique, where from the representation of a subgroup a representation of a whole group is induced. In the case of semidirect product groups of the type we are interested in every irreducible representation is obtained this way, thus it is worthwhile to describe this procedure, in some detail. In our description we will follow Mackay's inducing technique, as stated in [20].

Let G be a locally compact group, H a closed subgroup. Let X denote the coset space G/H. Let us assume that X carries an invariant measure ν. Now consider all measurable functions $f : G \to \mathbb{C}$, such that

$$f(gh) = f(g) \qquad \forall h \in H \tag{1.5}$$

$$\int_X |f(g)|^2 \, d\nu(\pi(g)) \; < \infty \tag{1.6}$$

where $\pi : G \to G/H$, $\pi(g) = gH$ is a projection on the coset space X. Because of the property (1.5), f is actually a function on X and (1.6) is well defined. With the scalar product defined as

$$< f_1|f_2 >= \int_X \overline{f_1(g)} f_2(g) \, d\nu(\pi(g)), \tag{1.7}$$

the set of all such functions constitutes a closed Hilbert space, denoted by $\mathfrak{H}$. On this Hilbert space we define the left representation of G as

$$\big(U(g)f\big)(g') = f(g^{-1}g').$$

1.3.1 Inducing from representation of a subgroup

Now we would like to generalize this construction to include cases were the representation of the subgroup is given. Suppose H has a unitary representation $h \to L(h)$ on some Hilbert space $\mathfrak{K}$. This time we consider the set of all functions $f : G \to \mathfrak{K}$, satisfying

$$f(gh) = L(h)^{-1} f(g) \qquad\qquad \forall h \in H \tag{1.8}$$

$$\int_X < f(g)|f(g) >_{\mathfrak{K}} \, d\nu(\pi(g)) \, < \infty \tag{1.9}$$

where $< | >_{\mathfrak{K}}$ denotes the scalar product in $\mathfrak{K}$. Again, (1.9) is well defined, since

$$< f_1(gh)|f_2(gh) >_{\mathfrak{K}} = < L(h)^{-1} f_1(g)|L(h)^{-1} f_2(g) >_{\mathfrak{K}} = < f_1(g)|f_2(g) >_{\mathfrak{K}}$$

the second equality being the result of the unitarity of L. With the scalar product defined as

$$< f_1|f_2 > = \int_X < f_1(g)|f_2(g) >_{\mathfrak{K}} \, d\nu(\pi(g)) \tag{1.10}$$

the set of all functions satisfying (1.8) and (1.9) constitutes a Hilbert space, which we call $\mathfrak{H}^L$. We can define a representation U^L of G on $\mathfrak{H}^L$as

$$\big(U^L(g)f\big)(g') = f(g^{-1}g'). \tag{1.11}$$

We call this representation, the representation induced from the representation L of the subgroup H.

1.3.2 Inducing with quasi-invariant measure

We can reformulate this construction for the case in which the coset space X does not carry an invariant measure, but only a quasi-invariant measure ν.

The measure ν is said to be *quasi-invariant* if ν is equivalent to ν_g for all $g \in G$, where the measure ν_g is obtained by the natural action of a group element g on ν:

$$\nu_g(\triangle) = \nu(g\triangle),$$

for $\triangle \in \mathcal{B}(X)$ being a Borel set on the coset space X. The Radon-Nikodym derivative of ν_g with respect to ν :

$$\lambda(g, x) = \frac{d\nu_g(x)}{d\nu(x)}, \tag{1.12}$$

is a *cocycle* for the quasi-invariant measures and satisfies the following properties:

$$\begin{aligned} \lambda(g_1 g_2, x) &= \lambda(g_1, x)\lambda(g_2, g_1^{-1}x), \\ \lambda(e, x) &= 1. \end{aligned} \tag{1.13}$$

A *section* on X is a map $\sigma : X \to G$, satisfying $\pi(\sigma(x)) = x$, for all $x \in X$. Where π denotes the projection onto the coset space X;

$$\pi : G \to X, \qquad \pi(g) = gH.$$

For a fixed Borel section $\sigma : X \to G$, for $g \in G$ and $x \in X$ we may write the action of the group section as

$$g\sigma(x) = \sigma(gx)h(g, x), \text{ where } h(g, x) = \sigma(gx)^{-1}g\sigma(x) \in H. \tag{1.14}$$

Here $h : G\times X \to H$ is again a cocycle, and $h'(g,x) = [h(g^{-1},x)]^{-1}$ satisfies conditions similar to (1.13):

$$\begin{aligned} h'(g_1g_2,x) &= h'(g_1,x)h'(g_2,g_1^{-1}x), \\ h'(e,x) &= e. \end{aligned} \tag{1.15}$$

for all $g_1, g_2 \in G$ and all $x \in X$.

Now suppose that H has a unitary representation $h \to L(h)$, for $h \in H$, on some Hilbert space $\mathfrak{K}$. We can define the map $B : G \times X \to \mathcal{U}(\mathfrak{K})$ (where $\mathcal{U}(\mathfrak{K})$ denotes the group of all unitary operators on $\mathfrak{K}$) as

$$B(g,x) = [\lambda(g,x)]^{1/2}L(h(g^{-1},x))^{-1} \tag{1.16}$$

which satisfies the cocycle conditions (1.13) for all $g_1, g_2 \in G$ and $x \in X$, with 1 being replaced by the identity operator $\mathbf{1}_{\mathfrak{K}}$ on $\mathcal{U}(\mathfrak{K})$ in this case, λ being the Radon-Nikodym derivative given by(1.12), and h a cocycle satisfying conditions (1.15). Now let us consider the Hilbert space $\tilde{\mathfrak{H}}^L = \mathfrak{K} \otimes L^2(X, d\nu)$ of the functions $\varphi : X \to \mathfrak{K}$, which are square integrable in the norm

$$\|\varphi\|^2_{\tilde{\mathfrak{H}}^L} = \int_X \|\varphi(x)\|^2_{\mathfrak{K}} d\nu(x). \tag{1.17}$$

In this case, as have been shown in [4], the representation $g \to \tilde{U}^L(g)$ induced from the representation L is given by the following formula:

$$\left(\tilde{U}^L(g)\varphi\right)(x) = B(g,x)\varphi(g^{-1}x) = \left[[\lambda(g,x)]^{1/2}L(h(g^{-1},x))^{-1})\right]\varphi(g^{-1}x), \varphi \in \tilde{\mathfrak{H}}^L \tag{1.18}$$

This representation is unitarily equivalent to the representation U^L given by (1.11). To see that let us denote by x the element in the coset space $X = G/H$, $x = gH$. Next we define an unitary map $V : \tilde{\mathfrak{H}}^L \to \mathfrak{H}^L$ by

$$(V\varphi)(g) = f(g) = L(h)^{-1}\varphi(x), \;\; x \in X \tag{1.19}$$

where the element $h \in H$ is determined from the decomposition $g = \sigma(x)h$, which, for given section, is unique. Then writing $U^L(g) = V\tilde{U}^L(g)V^{-1}$, and computing $(V\varphi')(g)$, for $\varphi'(g') = (\tilde{U}^L(g)\varphi)(x)$, where $\tilde{U}^L$ is the representation given by (1.18), we get

$$(V\tilde{U}^L(g)\varphi)(x) = \lambda(g,x)^{1/2} f(g^{-1}g'), \tag{1.20}$$

with $g' = xh$, and λ being Radon-Nicodym derivative of ν_g with respect to ν, as given in (1.12). For invariant measure $\lambda(g,x) = 1$ and we have induced representation U^L given by (1.11).

1.4 Semidirect Product Groups

In our work we are interested in the semidirect product groups of the type $G = V \rtimes S$, where V is an n-dimensional real vector space, and V a subgroup of $GL(V)$ - the group of all nonsingular linear transformations of V. The multiplication law in a group of this type is defined as

$$(v_1, s_1)(v_2, s_2) = (v_1 + s_1 v_2, s_1 s_2),$$

where $v_1, v_2 \in V$, $s_1, s_2 \in S$, and sv indicates the natural action of S on the vector space V, that is $(v,s) \to sv$, where $v \in V$ and $s \in S$. We denote by V^* the dual of

V, where duality is expressed by the usual pairing:

$$< k, v >, k \in V^*, v \in V.$$

The dual action of S on V^*: $(k, s) \to sk, k \in V^*$, defined as

$$< sk, v >=< k, s^{-1}v >$$

is the coadjoint action.

Let $k_0 \in V^*$ be a fixed element. Then by $\mathcal{O}^*$ we denote the orbit of k_0 under the action of S, and

$$\mathcal{O}^* = \{k \in V^* | k = sk_0, s \in S\}. \tag{1.21}$$

At any point k of that orbit we may consider the tangent space $T_k\mathcal{O}^*$, which can be identified with a subspace of V^*, and the cotangent space $T_k^*\mathcal{O}^*$, the dual space of $T_k\mathcal{O}^*$, which can be identified with a subspace of V. If by S_0 we denote the stability subgroup of k_0 ($s \in S_0$ if $sk_0 = k_0$), then $S/S_0 \simeq \mathcal{O}^*$. Let us also denote by N_0 the *annihilator* of the tangent space $T_{k_0}\mathcal{O}^*$ in V, that is :

$$N_0 = \{v \in V | < p, v >= 0, \forall p \in T_{k_0}\mathcal{O}^*\}$$

and by N_0^* its dual,

$$N_0^* = \{p \in V^* | < p, v >= 0, \forall v \in T_{k_0}^*\mathcal{O}^*\}$$

Then we can decompose our vector space V as $N_0 \oplus T_{k_0}^*\mathcal{O}^*$, and any element v of V can be uniquely written as $v = n + t$, with $n \in N_0$ and $t \in T_{k_0}^*\mathcal{O}^*$. S_0 leaves N_0

invariant, and $N_0 \rtimes S_0$ is a subgroup of G. Let Γ denote the left coset space

$$\Gamma = G/H_0, \;\; H_0 = N_0 \rtimes S_0. \tag{1.22}$$

1.4.1 Induced representations of semidirect product groups

To obtain the irreducible representations of $G = V \rtimes S$ we will use the inducing construction introduced in Section 1.3. To start, let us consider the vector in the dual space of V, $k_0 \in V^*$, and given by (1.21) its orbit $\mathcal{O}^*$ under the action of S. Then an one-dimensional representation of V is defined by the associated unitary character χ of the abelian subgroup V:

$$\chi(v) = \exp(-i < k_0, v >), \qquad v \in V. \tag{1.23}$$

Let $s \longrightarrow L(s)$ be a unitary irreducible representation of S_0, the stability subgroup of k_0, carried on some Hilbert space $\mathfrak{K}$. By χL we denote the unitary irreducible representation of $V \rtimes S_0$ carried by $\mathfrak{K}$, and given by:

$$\chi L(v, s) = \exp(-i < k_0, v >) L(s). \tag{1.24}$$

A representation of G will be induced from the representation χL.

The coset space $\Gamma = G/(V \rtimes S_0)$ is isomorphic to the orbit $\mathcal{O}^*$ given by (1.21), and as in Section 1.3, we need a section from the orbit to the group. We denote that section by σ and define it as

$$\sigma : \mathcal{O}^* \rightarrow G, \;\; \sigma(k) = (0, \Lambda(k)), \tag{1.25}$$

where $\Lambda : \mathcal{O}^* \to S$ is a global Borel section, such that

$$\Lambda(k_0) = e \text{ - identity element of } S$$

$$\Lambda(k)k_0 = k,\ k \in \mathcal{O}^*.$$

Because $s \in S$ can be uniquely written as $s = \Lambda(k)s_0$, where $k \in \mathcal{O}^*$ and $s_0 \in S_0$, the coset decomposition of the element of the group $(v, s) \in G$ is given by

$$(v, s) = (0, \Lambda(k))(\Lambda(k)^{-1}v, s_0). \tag{1.26}$$

The action of G on $\sigma(p) \in G$, $p \in \mathcal{O}^*$, given by (1.14) will be given by

$$(v, s)(0, \Lambda(p)) = (0, \Lambda(sp))(\Lambda(sp)^{-1}v, \Lambda(sp)^{-1}s\Lambda(p))$$

where,as shown in [4]

$$(\Lambda(sp)^{-1}v, \Lambda(sp)^{-1}s\Lambda(p)) = h((v, s), p),\ \ h : G \times \mathcal{O}^* \to V \rtimes S_0,$$

and

$$\Lambda(sp)^{-1}s\Lambda(p) = h_0(s, p), \qquad h_0 : S \times \mathcal{O}^* \to S_0$$

are cocycles.

We can show that the representation of the element $(h(v, s)^{-1}, p)$ is given by

$$\chi L(h(v, s)^{-1}, p)) = \exp(-i < k, v >)L(h_0(s^{-1}, p)), \tag{1.27}$$

and following Chapter 1, Section 1.3 the representation of G induced from χL , on the Hilbert space $\widetilde{\mathfrak{H}}^L = \mathfrak{K} \otimes L^2(\mathcal{O}^*, d\nu)$ is given by

$$(U^{\chi L}(v, s)f)(k) = \exp\left(-i < k, v >\right)L(h_0(s^{-1}, k))^{-1}f(s^{-1}k). \tag{1.28}$$

1.4.2 Coherent states of the semidirect product

The representation $U^{\chi L}$ given by (1.28) is usually not square integrable over the whole group. By taking the group element as $\sigma_{\mathcal{P}}(q,p) = (\Lambda(p)q, \Lambda(p))$, where $\sigma_{\mathcal{P}} : T_{k_0}\mathcal{O}^* \times \mathcal{O}^* \to G$ is called the *principal section*, we restrict the representation to the coset space Γ given by (1.22). This will allow us to find the conditions under which $U^{\chi L}$ will be square integrable over the coset space. Using the irreducible representation $U^{\chi L}$, let us define the set of vectors

$$\eta^i_{\sigma_{\mathcal{P}}(q,p)} = U^{\chi L}(\Lambda(p)q, \Lambda(p))\eta^i, \tag{1.29}$$

where $(q,p) \in T^*_{k_0}\mathcal{O}^* \times \mathcal{O}^*$ and $\eta^i, i = 1, 2, \dots N < \infty$, are linearly independent vectors in $\widetilde{\mathfrak{H}}^L = \mathfrak{K} \otimes L^2(\mathcal{O}^*, d\nu)$ supported on the open set $O(k_0)$ containing k_0.

We also define the positive, bounded operator on $\widetilde{\mathfrak{H}}^L$ as

$$F = \sum_{i=1}^{N} |\eta^i >< \eta^i| \tag{1.30}$$

We assume that F is invariant under the action of the stability subgroup S_0, i.e. it satisfies the condition

$$U^{\chi L}(0, s_0) F U^{\chi L}(0, s_0)^* = F, \ \forall s_0 \in S_0. \tag{1.31}$$

Using the explicit form of the representation $U^{\chi L}$ given by (1.28) we find

$$(U^{\chi L}(0, s_0)\eta^i)(k) = \exp(i < k, 0 >) L(h_0(s_0^{-1}, k))^{-1} \eta^i(s_0^{-1}k) = (L(s_0)\eta^i)(k). \tag{1.32}$$

Thus assumption (1.31) implies

$$L(s_0) F L(s_0)^* = F. \tag{1.33}$$

For arbitrary elements $\phi, \psi \in \widetilde{\mathfrak{H}}^L$, consider the formal integral,

$$I_{\phi,\psi} = \sum_{i=1}^{N} \int_{T^*_{k_0}\mathcal{O}^* \times \mathcal{O}^*} \langle \phi | \eta^i_{\sigma_{\mathcal{P}}(q,p)} \rangle \langle \eta^i_{\sigma_{\mathcal{P}}(q,p)} | \psi \rangle d\mu(q,p) \tag{1.34}$$

$$= \int_{T^*_{k_0}\mathcal{O}^* \times \mathcal{O}^*} \langle \phi | U^{\chi L}(\sigma_{\mathcal{P}}(q,p)) F U^{\chi L}(\sigma_{\mathcal{P}}(q,p))^* \psi \rangle d\mu(q,p)$$

where $d\mu(p,q)$ is an invariant measure on $T^*_{k_0}\mathcal{O}^* \times \mathcal{O}^*$. The detailed calculation of $d\mu$ may be found in Chapter 10, Section 10.2 of [4] and here we just present the highlights of that construction.

Since the coset space Γ is isomorphic to the cotangent bundle $T^*\mathcal{O}^*$, it comes equipped with a nondegenerate two-form invariant under the action of G. This two-form may be expressed locally on an open subset of the orbit:$U \subset \mathcal{O}^*$ as

$$\Omega = \sum_{i=1}^{m} dv^i_p \wedge dp_i, \tag{1.35}$$

where $\{dp_i\}_{i=1}^m$ is the dual basis in $T^*_p\mathcal{O}^*$ and $T^*_p\mathcal{O}^* \ni v_p = \sum_{i=1}^m v^i_p dp_i$, $v^i_p \in \mathbb{R}$. The associated left-invariant measure on $T^*\mathcal{O}^*$ is then, locally

$$d\omega = dv^1_p \wedge dv^2_p \wedge \ldots \wedge dv^m_p \wedge dp_1 \wedge dp_2 \wedge \ldots \wedge dp_m, \tag{1.36}$$

For our purposes, we find a Borel isomorphism Λ such that $v_p = \Lambda(p)q \in T^*_p\mathcal{O}^*$, which allows us to work with $T^*_{k_0}\mathcal{O}^* \times \mathcal{O}^*$ instead of $T^*\mathcal{O}^*$. Denoting by dq the Lebesgue measure on $T^*_{k_0} \simeq \mathbb{R}^m$, we may write,

$$dv_p = f(p)dq,$$

with $f(p)$ a positive and nonzero function on U. Our additional assumption is that

$\mathcal{O}^*$ carries an invariant measure $d\nu$, which on U may be written as

$$d\nu(p) = m(p) dp_1 \wedge dp_2 \wedge \ldots \wedge dp_m,$$

where $m(p)$ is a measurable function on $\mathcal{O}^*$, which is positive and nonzero on U. With all that the invariant measure $d\omega$ transforms locally on U to

$$d\mu(q,p) = \frac{f(p)}{m(p)} dq d\nu(p).$$

Hence globally on $T^*_{k_0}\mathcal{O}^*$, we may write

$$d\mu(q,p) = \rho(p) dq d\nu(p),$$

where ρ is a measurable function that is positive and nonzero almost everywhere on $\mathcal{O}^*$.

Now using the explicit form of the representation $U^{\chi L}$, given by (1.28), and measure $d\mu(p,q)$ obtained as described, our formal integral (1.34) is as follows:

$$\begin{aligned} I_{\phi,\psi} &= \sum_{i=1}^{N} \int_{\mathcal{O}^* \times \mathcal{O}^* \times T^*_{k_0}\mathcal{O}^* \times \mathcal{O}^*} \left\langle \phi(\Lambda(p)k) | L(h_0(\Lambda(p)^{-1}, \Lambda(p)k))^{-1} \eta^i(k) \right\rangle_{\mathfrak{K}} \\ &\qquad \times \left\langle \eta^i(k') | L(h_0(\Lambda(p)^{-1}, \Lambda(p)k)) \psi(\Lambda(p)k' \right\rangle_{\mathfrak{K}} \\ &\qquad \times e^{i<k-k',q>} d\nu(k) d\nu(k') d\mu(q,p). \end{aligned} \tag{1.37}$$

The factor $e^{i<k-k',q>}$ produces a δ -measure type of integral with respect to q. Then the integration over k' results in

$$
\begin{aligned}
I_{\phi,\psi} &= (2\pi)^m \sum_{i=1}^{N} \int_{\mathcal{O}^*} d\nu(k) \int_{\mathcal{O}^*} d\nu(p) m(\Lambda(p)^{-1}k)\rho(p) \\
&\quad \times \left\langle \phi(k) | L(h_0(\Lambda(p)^{-1}, k))^{-1} \eta^i(\Lambda(p)^{-1}k) \right\rangle_{\mathfrak{K}} \\
&\quad \times \left\langle \eta^i(\Lambda(p)^{-1}k) | L(h_0(\Lambda(p)^{-1}, k)) \psi(k) \right\rangle_{\mathfrak{K}}. \\
&= (2\pi)^m \sum_{i=1}^{N} \int_{\mathcal{O}^*} d\nu(k) \int_{\mathcal{O}^*} d\nu(p) m(\Lambda(p)^{-1}k)\rho(p) \\
&\quad \times \left\langle \phi(k) | (U^{\chi L}(0, \Lambda(p))\eta^i)(k) \right\rangle_{\mathfrak{K}} \left\langle (U^{\chi L}(0, \Lambda(p))\eta^i)(k) | \psi(k) \right\rangle_{\mathfrak{K}}. \quad (1.38)
\end{aligned}
$$

Now we can define a formal operator $A_{\sigma_{\mathcal{P}}}$ on the Hilbert space $\mathfrak{K} \otimes L^2(\mathcal{O}^*, d\nu)$ by

$$
(A_{\sigma_{\mathcal{P}}}\phi)(k) = \big(A_{\sigma_{\mathcal{P}}}(k)\big)\phi(k), \tag{1.39}
$$

where the integral

$$
\begin{aligned}
A_{\sigma_{\mathcal{P}}}(k) &= (2\pi)^m \sum_{i=1}^{N} \int_{\mathcal{O}^*} d\nu(p) m(\Lambda(p)^{-1}k)\rho(p) \\
&\quad \times |(U^{\chi L}(0, \Lambda(p))\eta^i)(k)\rangle_{\mathfrak{K}\,\mathfrak{K}}\langle (U^{\chi L}(0, \Lambda(p))\eta^i)(k)| \quad (1.40)
\end{aligned}
$$

is a measurable function on $\mathcal{O}^*$. Then the square integrability condition for $U^{\chi L}$ is, that (1.40) is bounded and bounded away from zero. Then $A_{\sigma_{\mathcal{P}}}$ becomes a bounded operator with bounded inverse and we may write

$$
I_{\phi,\psi} = < \phi | A_{\sigma_{\mathcal{P}}} \psi >, \tag{1.41}
$$

and

$$
A_{\sigma_{\mathcal{P}}} = \sum_{i=1}^{N} \int_{T^*_{k_0}\mathcal{O}^* \times \mathcal{O}^*} |\eta^i_{\sigma_{\mathcal{P}}(q,p)}\rangle \langle \eta^i_{\sigma_{\mathcal{P}}(q,p)}| d\mu(q,p). \tag{1.42}
$$

where the set of vectors

$$\eta^i_{\sigma_{\mathcal{P}}(q,p)} = U^{\chi L}(\Lambda(p)q, \Lambda(p))\eta^i, i = 1, 2, 3, \ldots.., N, \ \ (q,p) \in T^*_{k_0}\mathcal{O}^* \times \mathcal{O}^*$$

constitute a set of coherent states.

1.4.3 A general section

The principal section is not the only one for which coherent states can be constructed. We would like to derive the conditions for square integrability of representation $U^{\chi L}$ (1.28) for more general choice of section.

Any section can be expressed in terms of the principal section $\sigma_{\mathcal{P}}$ in the following manner:

$$\sigma(q,p) = \sigma_{\mathcal{P}}(q,p)(n(q,p), s_0(q,p)), \tag{1.43}$$

where $n : T_{k_0}\mathcal{O}^* \times \mathcal{O}^* \to N_0$, $s_0 : T_{k_0}\mathcal{O}^* \times \mathcal{O}^* \to S_0$ are Borel functions. We will be interested in a class of sections for which n and s_0 are

$$\begin{aligned} n(q,p) &= \Theta(p)q + \Xi(p) \\ s_0(q,p) &= s_0(p), \end{aligned} \tag{1.44}$$

where Θ, Ξ, s_0 depend on p only, and for fixed $p, \Theta(p) : V \to V$ is a linear map with kernel being N_0. This function is also assumed to be differentiable on the open dense set on which the global Borel section $\Lambda(p)$ is differentiable. Ξ introduces only a phase factor and we may set it equal to zero, without loss of generality. Taking such n and

s_0 into (1.43) we get

$$\sigma(q,p) = \big(\Lambda(p)q + \Lambda(p)\Theta(p)q + \Lambda(p)\Xi(p), \Lambda(p)s_0(p)\big), \tag{1.45}$$

which we can also write as

$$\sigma(q,p) = \big(\Lambda(p)(I_V + \Theta(p))q + \Lambda(p)\Xi(p), \Lambda(p)s_0(p)\big), \tag{1.46}$$

where I_V denotes the identity operator on V. This type of sections described in [4] are called *admissible affine sections* if $I_{V^*} + \Theta(p)^*$ (I_{V^*} is the identity element on V^* and $\Theta(p)^*$ adjoint map to $\Theta(p)$) maps $O(k_0)$, open set containing k_0, into itself for each $p \in \mathcal{O}^*$, and the Jacobian $J(p,k)$ of the map $I_{V^*} + \Theta(p)^*$ restricted to $O(k_0)$ does not equal zero anywhere:

$$det[J(p,k)] \neq 0, \;\; p \in \mathcal{O}^*, \;\; k \in O(k_0).$$

For such sections we can again build the family of coherent states:

$$\eta^i_{\sigma_{\mathcal{P}}(q,p)} = U^{\chi L}(\Lambda(p)q, \Lambda(p))\eta^i, \tag{1.47}$$

where again $\eta^i \in \widetilde{\mathfrak{H}} = \mathfrak{K} \otimes L^2(\mathcal{O}^*, d\nu)$, $i = 1, 2, ..., N$ are smooth functions on $\mathcal{O}^*$ with supports contained in $O(k_0)$ and satisfying the invariance condition (1.31) under the action of the subgroup S_0. The representation $U^{\chi L}$ (1.28) is then square integrable mod(H_0, σ) and the operator

$$A_\sigma = \sum_{i=1}^{N} \int_{T^*_{k_0}\mathcal{O}^* \times \mathcal{O}^*} |\eta^i_{\sigma(q,p)}\rangle\langle\eta^i_{\sigma(q,p)}| d\mu(q,p). \tag{1.48}$$

is a multiplication operator

$$(A_\sigma \phi)(k) = A_\sigma(k)\phi(k) \tag{1.49}$$

with the multiplying function

$$\begin{aligned} A_\sigma(k) &= (2\pi)^m \sum_{i=1}^{N} \int_{\mathcal{O}^*} |(U^{\chi L}(0, \Lambda(p))\eta^i)(k)\rangle_{\mathfrak{K}} \\ &\times \ {}_{\mathfrak{K}}\langle (U^{\chi L}(0, \Lambda(p))\eta^i)(k)| \frac{m(\Lambda(p)^{-1}k}{|det[J(p,k)]|} \rho(p) d\nu(p). \end{aligned} \tag{1.50}$$

If there exist two nonzero, positive numbers a and b, $a < b$ for which:

$$a < (2\pi)^m \sum_{i=1}^{N} \int_{\mathcal{O}^*} ||\eta^i(\Lambda(p)^{-1}k)||^2_{\mathfrak{K}} \frac{m(\Lambda(p)^{-1}k}{|det[J(p,k)]|} \rho(p) d\nu(p) < b, \tag{1.51}$$

the operator A_σ (1.48) defines a frame. For some admissible affine sections the operator A_σ might never be a multiple of the identity.

1.5 Frames

The concept of a *frame* was introduced in the context of non-harmonic Fourier series by Duffin and Schaeffer in 1952 [12], and later reviewed in Young [32], Daubechies [10] and Heil and Walnut [16].

The frames are an alternative to the orthonormal bases. They are sets of non independent vectors, which may be used as the components in the in expansion of any vector ϕ in given space. The difference with the base is that expansion of ϕ is not unique (which is the case, when we expand using orthonormal basis), but requirements of the frame are much less restrictive than the requirements of the basis. Here we use a general definition of the frame which will be most suitable for our calculations. Let $\mathfrak{H}$ be a separable Hilbert space, and X a locally compact space with a regular Borel measure ν.

Definition 1.5.1 *A set of vectors* $\{\eta_x^i\}_{i=1}^N$ *in* $\mathfrak{H}$ *is a* frame *if for all* $x \in X$ *the vectors* $\{\eta_x^i\}, i = 1, 2, \ldots, N$, *are linearly independent, and there exist two numbers* $A, B > 0$ *such that for all* $\phi \in \mathfrak{H}$ *one has*

$$A\|\phi\|^2 \leq \sum_{i=1}^{N} \int_X |<\eta_x^i|\phi>|^2 d\nu(x) \leq B\|\phi\|^2 \tag{1.52}$$

The numbers A, B are called *frame bounds.* The frame is called *tight* if A = B, and the frame is *exact* if it fails to be a frame after removal of any single element from the set. In general, the set of vectors $\{\eta_x^i\}_{i=1}^N$ is not an orthogonal basis. If X is a discrete space with ν being a counting measure, our frame condition becomes

$$A\|\phi\|^2 \leq \sum_{i=1}^{N} \sum_{x \in X} |<\eta_x^i|\phi>|^2 \leq B\|\phi\|^2 \tag{1.53}$$

and the set $\{\eta_x^i\}_{i=1}^N$ is called *discrete frame*, more commonly used in the literature. Otherwise, if X is not discrete, we call our frame *continuous.* Now, if the set $\{\eta_x^i\}_{i=1}^N$ is a frame, we can define the *resolution operator* T by

$$T = \sum_{i=1}^{N} \int_X |\eta_x^i><\eta_x^i| d\nu(x) \tag{1.54}$$

or in the discrete case

$$T = \sum_{i=1}^{N} \sum_{x \in X} |\eta_x^i><\eta_x^i| \tag{1.55}$$

The frame condition (1.52) (or (1.53) in the discrete case) is equivalent to stating that T and T^{-1} are bounded positive operators on $\mathfrak{H}$, and the frame is tight, if T is a multiple of the identity, T $= \lambda I$, for some positive λ.

1.5.1 The resolution operator

Now let us concentrate on discrete frames with $N = 1$, that is the set $\{\eta_x\}$, where $x \in X$, X being a discrete space. Our frame condition (1.53) then reduces to

$$A\|\phi\|^2 \leq \sum_{x\in X} | < \eta_x|\psi > |^2 \leq B\|\phi\|^2, \forall\phi \in \mathfrak{H}. \tag{1.56}$$

and the definition of the resolution operator given by (1.55) reduces to

$$T = \sum_{x\in X} |\eta_x >< \eta_x|.$$

Let us define the *frame operator* F as

$$F : \phi \longrightarrow \{< \eta_x|\phi >\}, \qquad \phi \in \mathfrak{H}.$$

Then the resolution operator is $T = F^*F$ and the condition (1.56) may be written as

$$AI \leq F^*F \leq BI, \tag{1.57}$$

where I is the identity operator. This implies, that T is invertible, and T^{-1} is obtained from the condition

$$B^{-1}I \leq (F^*F)^{-1} \leq A^{-1}I$$

We can define $\{T^{-1}\eta_x\} = \{\tilde{\eta}_x\}$, which is also a frame, with the frame bounds B^{-1} and A^{-1} respectively, and the frame operator $\tilde{F} = F(F^*F)^{-1}$. The set $\{\tilde{\eta}_x\}$ is called the *dual* or *reciprocal frame* of the frame $\{\eta_x\}$.

Now any vector in the Hilbert space $\mathfrak{H}$, $\phi \in \mathfrak{H}$ can be written as

$$\phi(k) = \sum_{x \in X} < \eta_x | \phi > \tilde{\eta}_x(k) \tag{1.58}$$

where the above expansion converges strongly in $\mathfrak{H}$, which means $\tilde{F}^* F = I$. Thus duality of the two frames may be expressed by the equation

$$\sum_x |\eta_x >< \tilde{\eta}_x| = \sum_x |\tilde{\eta}_x >< \eta_x| = I. \tag{1.59}$$

As was mentioned at the beginning of this section a good frame can be used in place of an orthonormal basis to be used in reconstruction of ϕ from $< \eta_x | \phi >$. To proceed with the reconstruction we need to know $\tilde{\eta}_x = (F^* F)^{-1} \eta_x$. If A and B (frame bounds) are close to each other, then by (1.57) $F^* F$ is close to $\frac{A+B}{2} I$, thus the inverse $(F^* F)^{-1}$ is close to $\frac{2}{A+B} I$ and $\tilde{\eta}_x$ is close to $\frac{2}{A+B} \eta_x$. Then ϕ can be written as

$$\phi = \sum_{x \in X} < \eta_x | \phi > \eta_x + R\phi \tag{1.60}$$

where $R = I - \frac{2}{A+B} F^* F$ and $-\frac{B-A}{B+A} I \leq R \leq \frac{B-A}{B+A} I$ which implies $\|R\| \leq \frac{B-A}{B+A} = \frac{B/A-1}{B/A+1}$. If $B/A - 1 \ll 1$ we can drop the rest term $R\phi$ from the expansion (1.60), and we have a reconstruction formula for ϕ accurate up to an error of $\frac{B/A-1}{B/A+1} \|\phi\|$.

Chapter 2

The Euclidean Group

The goal of our study was to construct coherent states for the Euclidean group. We present the results in the next two chapters. In Chapter 3 we construct continuous coherent states, and in Chapter 4 the discrete ones. But first we set up the "stage". In this chapter we describe the three-dimensional Euclidean group $E(3)$ as the semidirect product of a real, three-dimensional vector space with the group of rotations in this vector space. We will also construct the irreducible representations of $E(3)$ using the Mackey's induction method [20] which we presented in Section 1.4 of Chapter 1.

2.1 Notation and Set Up

The Euclidean group the semidirect product group described in Section 1.4: $E(n) = \mathbb{R}^n \rtimes SO(n)$, where $\mathbb{R}^n$ is the n-dimensional, real vector space, and $SO(n)$ is the group of all rigid rotations in $\mathbb{R}^n$ about the origin. Elements of $SO(n)$ are $n \times n$

real orthogonal matrices. In our work we concentrate on $n = 3$. Every element of $E(3)$ may be written uniquely in the form $g = (\underline{b}, R)$, where $\underline{b} \in \mathbb{R}^3$ and $R \in SO(3)$, $RR^T = R^T R = I_3$. The group multiplication is given by

$$g_1 g_2 = (\underline{b}_1, R_1)(\underline{b}_2, R_2) = (\underline{b}_1 + R_1 \underline{b}_2, R_1 R_2) \tag{2.1}$$

where $R_1 \underline{b}_2 = \underline{b}' \in \mathbb{R}^3$ is the natural action of the matrix R on the vector $\underline{b} \in \mathbb{R}^3$. The identity element of the group is $e = (\underline{0}, I)$, and the inverse of the element g is given by $g^{-1} = (-R^{-1}\underline{b}, R^{-1})$.

Let $H = T_3 \otimes SO_3(2)$ be the subgroup of $E(3)$, where $SO_3(2)$ is the group of rotations about the z-axis and T_3 the group of translations along z-axis (the index 3 indicates the axis z). The homogenous space $\Gamma = E(3)/H$ is isomorphic to the coadjoint orbit generated by the vector $(\underline{0}, \hat{k}) = (0,0,0;0,0,1) \in T^*\mathcal{O}^*$, where the orbit $\mathcal{O}^*$ is the two dimensional sphere S^2. Let us write $\underline{b} \in \mathbb{R}^3$ as $(\underline{b}^{\perp}, b_3)$, and the rotation $R \in SO(3)$ in 3-space as $R_{\phi_1} R_\theta R_{\phi_2}$, where

$$R_{\phi_i} = \begin{pmatrix} \cos\phi_i & -\sin\phi_i & 0 \\ \sin\phi_i & \cos\phi_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ - rotation about z-axis}, i = 1, 2 \tag{2.2}$$

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \text{ - rotation about x-axis} \tag{2.3}$$

as given in [25]. We can thus express the rotation matrix as $R = R(\varphi, \theta)R_{\phi_2}$, where $\varphi = \phi_1 - \frac{\pi}{2}$ and

$$R(\varphi,\theta) = \begin{pmatrix} -\sin\varphi & -\cos\theta\cos\varphi & \sin\theta\cos\varphi \\ \cos\varphi & -\cos\theta\sin\varphi & \sin\theta\sin\varphi \\ 0 & \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\phi_1 & -\cos\theta\sin\phi_1 & \sin\theta\sin\phi_1 \\ \sin\phi_1 & \cos\theta\cos\phi_1 & -\sin\theta\cos\phi_1 \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Note also, that we can make the association $R(\varphi, \theta) = R(\hat{n})$, where $\hat{n}$ is the unit vector

$$\hat{n} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \sin\theta\sin\phi_1 \\ -\sin\theta\cos\phi_1 \\ \cos\theta \end{pmatrix}.$$

Thus we can write $\hat{n} = R(\hat{n})\,\hat{k}$ with $\hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Now, let us take an element of the Euclidean group and decompose it as follows:

$$\mathbb{E}(3) \ni (\underline{b}, R) = \Big((\underline{b}'^{\perp}, 0), R(\hat{n})\Big)\Big((\underline{0}, \alpha), R_{\phi_2}\Big) \Rightarrow \underline{b} = (\underline{b}'^{\perp}, 0) + R(\hat{n})\alpha, \tag{2.4}$$

where $\Big((\underline{b}'^{\perp}, 0), R(\hat{n})\Big)$ is en element from the coset space Γ and $\Big((\underline{0}, \alpha), R_{\phi_2}\Big)$ an element of the subgroup $H = T_3 \otimes SO_3(2)$ of $E(3)$. Thus the orthogonal part of $\underline{b}$ (first two coordinates) is given by

$$\underline{b}^{\perp} = \underline{b}'^{\perp} + \alpha\hat{n}^{\perp} \Rightarrow \underline{b}'^{\perp} = \underline{b}^{\perp} - \alpha\hat{n}^{\perp}$$

and the third coordinate is

$$b_3 = \hat{n}_3\alpha \Rightarrow \alpha = \frac{b_3}{\hat{n}_3}.$$

Because every element of $E(3)$ decomposes as stated in (2.4), we can parameterize $\Gamma = E(3)/T_3 \otimes SO(2)$ by $(\underline{q}, \hat{p})$, where $\underline{q} = (q_1, q_2, 0)$ has vanishing third coordinate, as in $\left((\underline{b}'^{\perp}, 0), R(\hat{n})\right)$ and $\hat{p} = (p_1, p_2, p_3)$, such that $\hat{p}^2 = 1$ ($\hat{p}$ is a point on the sphere of radius 1). Such a $(\underline{q}, \hat{p})$ is an element of the cotangent boundle T^*S^2, to which Γ is isomorphic.

The action of $E(3)$ on Γ is computed by first writing

$$(\underline{b}_0, R_0)(\underline{q}, \hat{p}) = (\underline{q}'', \hat{p}'') \qquad \underline{q}'' = \underline{b}_0 + R_0\underline{q} \qquad \hat{p}'' = R_0\hat{p} = R_0R(\hat{p}). \tag{2.5}$$

Note, that $\hat{p} = R(\hat{p})\underline{k}$, that is, with each $\hat{p}$ we can associate a rotation $R(\theta, \phi)$ acting on $\underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Following this action, we decompose the new group element as before. Thus according to the coset decomposition

$$(\underline{q}'', \hat{p}'') = \left(\underline{q}', R(\hat{p}')\right)\left((0, \alpha'), R_{\phi_2'}\right) = \sigma_0(\underline{q}'\hat{p}')\left((0, \alpha'), R_{\phi_2'}\right),$$

where σ_0 denotes the basic section, mapping from the coset space to the group $\sigma_0 : \Gamma \longrightarrow \mathbb{E}(3)$ and is defined as

$$\sigma_0(\underline{q}, \hat{p}) = \left((q_1, q_2, 0), R(\hat{p})\right) \in \mathbb{E}(3), \tag{2.6}$$

and $\left((0, \alpha'), R_{\phi_2'}\right)$ is an element of the subgroup $H = T_3 \otimes SO_3(2)$. We can relate any other section $\sigma : \Gamma \longrightarrow \mathbb{E}(3)$ to the section σ_0 by

$$\sigma(\underline{q}, \hat{p}) = \sigma_0(\underline{q}, \hat{p})\left((f(\underline{q}, \hat{p}), 0), \mathcal{R}(\underline{q}, \hat{p})\right),$$

where f and $\mathcal{R}$ are smooth functions of $\underline{q}$ and $\hat{p}$.

Going back to our decomposition (2.4)and comparing with (2.5) we write

$$(\underline{b}_0 + R_0\underline{q}, R_0\hat{p}) = \big((q'_1, q'_2, 0), R(\hat{p}')\big)\big((0, \alpha'), R_{\phi'_2}\big). \tag{2.7}$$

According to the group action the right-hand side of the above equation becomes

$$\big((q'_1, q'_2, 0) + R(\hat{p}')(0, \alpha'), R(\hat{p}')R_{\phi'_2}\big),$$

and we see, that $\hat{p}' = R_0\hat{p} = R_0 R(\hat{p})$, and the third component of the vector $\underline{b}$ becomes $b_{03} + [R_0\underline{q}]_3 = 0 + [R_0\hat{p}]_3\alpha'$, which implies

$$\alpha = \frac{b_{03} + [R_0\underline{q}]_3}{[R_0\hat{p}]_3}.$$

Also $R_0 R(\hat{p}) = R(\hat{p}')R_{\phi'_2}$ implies

$$R_{\phi'_2} = R(R_0\hat{p})^{-1} R_0 R(\hat{p}).$$

This gives the following transformation rule for the element of the coset space Γ under the action of $E(3)$:

$$\underline{q} \longrightarrow \underline{q}' = \underline{b}_0^{\perp} + (R_0\underline{q})^{\perp} - \frac{b_{03} + (R_0\underline{q})_3}{R_0(\hat{p})_3}(R_0\hat{p})^{\perp}. \tag{2.8}$$

and

$$\hat{p} \longrightarrow \hat{p}' = R_0\hat{p} \tag{2.9}$$

2.2 Induced Representations of the Euclidean Group

To obtain irreducible representations of the Euclidean group we will use the method described in Chapter 1, Section 1.4. Let us fix $\underline{k}_0 \in (\mathbb{R}^3)^*$ in the dual space, to be the vector $(0,0,1)$. Then the orbit generated under the action of $SO(3)$ on $\underline{k}_0$ is the two dimensional unit sphere, S^2, and the stability subgroup $S_0 = \{R|R\underline{k}_0 = \underline{k}_0\}$ will consist of all rotations about the z-axis, that is $SO_3(2)$ (the index 3 indicates the z-axis). The character $\chi(\underline{b}) = \exp(-i < \hat{n}, \underline{b} >)$, where $\hat{n} \in (\mathbb{R}^3)^*$ defines the one-dimensional representation of $\mathbb{R}^3$. Rotations about the $z-$axis may be represented as $L(R_\theta) = L_n(\theta) = e^{in\theta}$, where $R_\theta \in SO_3(2)$ is a rotation about the $z-$axis by the angle θ and n is an arbitrary, fixed integer. Thus the representation from which we will be inducing representations of the Euclidean group has the form

$$\chi L_n(\underline{b}, \theta) = e^{(-i<\hat{n},\underline{b}>)}e^{in\theta} \tag{2.10}$$

Following the construction in Chapter 1, Section 1.4 we now need a section σ, from the orbit to the group, as given by (1.25). Let k be an element of the orbit

$$\mathcal{O}^* = \{k|k = Rk_0, R \in SO(3)\} = S^2,$$

then $\sigma(k) = (0, R(k))$, $R : S^2 \to SO(3)$ being a rotation such that $R(k)k_0 = k$. We can decompose the element of the Euclidean group as follows:

$$(\underline{b}, R) = (0, R(k))(R(k)^{-1}\underline{b}, R_3) \tag{2.11}$$

where $R_3 \in SO_3(2)$ is an element of the stability subgroup of k_0. The action of $E(3)$ on $\sigma(p)$, $p \in \mathcal{O}^*$ is then

$$(\underline{b}_0, R_0)(0, R(p)) = (0, R(R_0p)(R(R_0p)^{-1}\underline{b}, R(R_0p)^{-1}RR(p)), \tag{2.12}$$

as given in Chapter 1, Section 1.4, where

$$(R(R_0p)^{-1}\underline{b}, R(R_0p)^{-1}RR(p)) = h((\underline{b}_0, R_0), p)$$

is a cocycle ($h : E(3) \times \mathcal{O}^* \longrightarrow \mathbb{R}^3 \rtimes SO_3(2)$), and

$$R(R_0p)^{-1}RR(p) = h_0(R, p)$$

is another cocycle ($h_0 : SO(3) \times \mathcal{O}^* \longrightarrow SO_3(2)$), both satisfying conditions (1.15). Using the above notation, we calculate

$$\chi L_n(h(\underline{b}_0, R_0)^{-1}, p)) = \exp\left(-i < k, \underline{b}_0 >\right) L_n(h_0(R_0^{-1}, p)), \tag{2.13}$$

where, as before $L_n(R(\theta)) = e^{in\theta}$, and consequently

$$L_n(h_0(R^{-1}, p)) = L_n(R(R_0p)) = e^{in\theta'},$$

where $\theta' = \theta + \theta_0$, θ_0 being the angle corresponding to the rotation R_0. Thus following (1.28) the representation of E(3) induced from the representation χL of the subgroup $H = \mathbb{R}^3 \times SO_3(2)$ is given by

$$(U^{\chi L}(\underline{b}, R)f)(k) = \exp\left(-i < k, \underline{b} >\right) L(h_0(R^{-1}, k))^{-1} f(R^{-1}k), \tag{2.14}$$

and is carried by the Hilbert space $\tilde{\mathfrak{H}}^L = \mathfrak{K} \otimes L^2(S^2, d\nu)$, of functions f, which are square integrable on S^2 with the rotationally invariant measure $d\nu = \frac{dk_1 dk_2}{k_3}$.

In our calculation we choose the trivial representation of $SO_3(2)$, that is $n = 0$, so that $e^{in\theta} = 1$. The other choices of n will only introduce a phase factor. Also for $\hat{n}$ we choose the vector $(0,0,1)$, thus $\chi(\underline{b}) = \exp(-ib_3)$. This implies that the representation we are inducing from is $(\chi L)(\underline{b},\theta) = \exp(-ib_3)$, and the induced representation assumes the form:

$$\left(U^{\chi L}(\underline{b},R)f\right)(k) = e^{-ik\cdot\underline{b}} f(R^{-1}k). \tag{2.15}$$

Chapter 3

Coherent States of E(3)

This chapter is devoted to the construction of coherent states for the three-dimensional Euclidean group. We start with a detailed calculation of the invariant measure on the coset space $\Gamma = E(3)/H$, where $H = T_3 \otimes SO_3(2)$, T_3 and $SO_3(2)$ being the groups of translations along the z-axis and rotations about the z-axis respectively.

In section 3.2 we give an example of the construction of two families of coherent states, for two different choices of sections. We will also discuss the conditions under which coherent states exist for a general choice of an admissible affine section $\sigma(q, p)$.

3.1 Invariant Measure on the Coset Space

The coset space Γ is equipped with a measure invariant under the group action. In this section we find that invariant measure $d\mu(\underline{q}, \hat{p})$ following the general method given in [4], which we described in Section 1.4 of Chapter 1. In our calculation of the measure we use the short notation for wedge products $da \wedge db \wedge ... \wedge dz$, writing them simply as $dadb...dz$.

Let $R_0 = R_{\alpha_1} R_\beta R_{\alpha_2}$ be an element of the rotation group $SO(3)$, where R_{α_i} is a rotation about the z-axis by the angle α_i, and R_β a rotation about x-axis by the angle β. The matrix representations of these rotations are given by (2.2) and (2.3).

Let us take an element from the coset space $(\underline{q}, \hat{p}) \in \Gamma$, where $\underline{q} = (q_1, q_2, 0)$, and $\hat{p}$ is such that $|\hat{p}|^2 = 1$, as we introduced in Section 2.1 of Chapter 2. Then the action of the rotation matrix R_0 on $\underline{q}$ gives us:

$$R_0 \underline{q} = R_{\alpha_1} R_\beta R_{\alpha_2} \underline{q} = R_{\alpha_1} R_\beta \underline{a} = R(\hat{p}_0) \underline{a} = \begin{pmatrix} \cos\alpha_1 a_1 - \cos\beta \sin\alpha_1 a_2 \\ \sin\alpha_1 a_1 + \cos\beta\cos\alpha_1 a_2 \\ \sin\beta a_2 \end{pmatrix}, \tag{3.1}$$

where the vector $\underline{a} \in \mathbb{R}^3$ is the result of the rotation of $\underline{q}$ about the z-axis, through the angle α_2, and

$$\underline{a} = R_{\alpha_2} \underline{q} = (a_1, a_2, 0),$$

($R_{\alpha_2} \in SO_3(2)$ is an element of the stability subgroup S_0). In (3.1) we also made an association of the rotation $R_{\alpha_1} R_\beta$ with $R(\hat{p}_0)$, where

$$\hat{p}_0 = R_{\alpha_1} R_\beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\beta \sin\alpha_1 \\ -\sin\beta\cos\alpha_1 \\ \cos\beta \end{pmatrix}$$

which was discussed in Section 2.1 of Chapter 2.

Going back to the transformation rule for $\underline{q}$ under the group action given in (2.8)

$$\underline{q}' = \underline{b}_0^{\perp} + (R_0\underline{q})^{\perp} - \frac{b_{03} + (R_0\underline{q})_3}{R_0(\hat{p})_3}(R_0\hat{p})^{\perp}.$$

The components of $\underline{q}' = R_0\underline{q}$ are

$$q_1' = b_{01} + a_1\cos\alpha_1 - a_2(\cos\beta\sin\alpha_1 + \frac{\sin\beta\hat{p}_1'}{\hat{p}_3'}) - \frac{b_{03}\hat{p}_1'}{\hat{p}_3'}$$

$$q_2' = b_{02} + a_1\sin\alpha_1 + a_2(\cos\beta\cos\alpha_1 - \frac{\sin\beta\hat{p}_2'}{\hat{p}_3'}) - \frac{b_{03}\hat{p}_2'}{\hat{p}_3'}$$

Taking differentials gives us

$$dq_1' = da_1\cos\alpha_1 - da_2(\cos\beta\sin\alpha_1 + \frac{\sin\beta\hat{p}_1'}{\hat{p}_3'}) + \ \{\text{terms with } dp_i\ \}$$

$$dq_2' = da_1\sin\alpha_1 + da_2(\cos\beta\cos\alpha_1 - \frac{\sin\beta\hat{p}_2'}{\hat{p}_3'}) + \ \{\text{terms with } dp_i\ \}$$

$$\Rightarrow dq_1'dq_2' = \big(\cos\beta\cos^2\alpha_1 - \frac{\sin\beta\cos\alpha_1\hat{p}_2'}{\hat{p}_3'} + \cos\beta\sin^2\alpha_1 + \frac{\sin\beta\sin\alpha_1\hat{p}_1'}{\hat{p}_3'}\big)da_1da_2$$

$$+ \ \{\text{ terms with at least one of } dp_i\text{'s }\}\ .$$

(We are omitting explicit expressions for terms involving dp_i's, because they drop out when wedge products are taken. Eventually we will be interested in the measure $dq_1'dq_2'd\hat{p}_1'd\hat{p}_2'$, the wedge product of $dq_1'dq_2'$ with $d\hat{p}_1'd\hat{p}_2'$, and because of the antisymmetry property of the wedge product the terms in $dq_1'dq_2'$ involving dp_i's will vanish after taking the wedge product with $d\hat{p}_1'd\hat{p}_2'$.)

That is the $d\underline{q}$ part of the measure. We still have the differentials related to $\hat{p}$. Since $\hat{p}' = R_0\hat{p}$ is the transformation on the sphere, the invariant measure for that transformation is

$$\sin\beta\, d\beta d\alpha_1 = \frac{d\hat{p}_1 d\hat{p}_2}{\hat{p}_3}.$$

In view of the invariance

$$\frac{d\hat{p}_1' d\hat{p}_2'}{\hat{p}_3'} = \frac{d\hat{p}_1 d\hat{p}_2}{\hat{p}_3} \qquad \Rightarrow \qquad d\hat{p}_1' d\hat{p}_2' = \frac{\hat{p}_3'}{\hat{p}_3} d\hat{p}_1 d\hat{p}_2. \tag{3.2}$$

Now we are ready to calculate $dq_1' dq_2' d\hat{p}_1' d\hat{p}_2'$. We will use the fact that $da_1 da_2 = dq_1 dq_2$, which is the result of the simple calculation: $\underline{a} = R_{\alpha_2}\underline{q}$, where R_{α_2} is given by (2.2) and $\underline{q} = (q_1, q_2, 0)$. Applying that, we get

$$da_1 da_2 = (\cos^2 \alpha_2 + \sin^2 \alpha_2) dq_1 dq_2 = dq_1 dq_2.$$

Using this fact and relation (3.2) for $d\hat{p}_1' d\hat{p}_2'$ we calculate the invariant measure to be

$$dq_1' dq_2' d\hat{p}_1' d\hat{p}_2' = \left(\cos\beta + \frac{\sin\beta}{\hat{p}_3'}(\sin\alpha_1 \hat{p}_1' - \cos\alpha_1 \hat{p}_2')\right) \frac{\hat{p}_3'}{\hat{p}_3} dq_1 dq_2 dp_1 dp_2$$

$$= \frac{1}{\hat{p}_3}(\hat{p}_3' \cos\beta + \hat{p}_1' \sin\beta \sin\alpha_1 - \hat{p}_2' \sin\beta \cos\alpha_1) dq_1 dq_2 dp_1 dp_2 = \frac{\hat{p}_0 \cdot \hat{p}'}{\hat{p}_3} dq_1 dq_2 dp_1 dp_2.$$

But $\hat{p}_0 = R_0 \hat{k}$ and $\hat{p}' = R_0 \hat{p}$, so that

$$\hat{p}_0 \cdot \hat{p}' = (R_0 \hat{k}) \cdot (R_0 \hat{p}) = \hat{k} \cdot \hat{p} = \hat{p}_3.$$

This shows, that

$$d\mu(\underline{q}, \hat{p}) = dq_1 dq_2 dp_1 dp_2 \tag{3.3}$$

is an invariant measure on Γ.

3.2 Construction of Coherent States

In the construction of the coherent states for the three dimensional Euclidean group we will follow the general method presented in [4] which we described in Section 1.4 of Chapter 1.

Let us construct the family of vectors in Hilbert space:

$$\eta_{\underline{q},\hat{p}}(\underline{k}) = U^{\chi L}\big(\sigma(\underline{q},\hat{p})\big)\eta(\underline{k}), \tag{3.4}$$

where $U^{\chi L}(\underline{b}, R)$ is the unitary irreducible representation of $E(3)$ given by (2.15), carried by the Hilbert space $\tilde{\mathfrak{H}} = L^2(\mathcal{O}^*, d\nu)$. By taking $U^{\chi L}(\sigma(\underline{q},\hat{p}))$ we restrict the representation to the coset space Γ, from which an element $(\underline{q},\hat{p})$ is taken.

In Chapter 2 we showed, that for $E(3)$ the orbits $\mathcal{O}^*$ are isomorphic to the two dimensional unit sphere: $\mathcal{O}^* \simeq S^2$. We need to choose $\eta \in \widetilde{\mathfrak{H}} = L^2(S^2, d\nu)$, satisfying the invariance condition (1.31). For the Euclidean group which we are considering

$$\big(U^{\chi L}(0, R_3)\eta\big)(\underline{k}) = \eta(R_3^{-1}\underline{k}), \tag{3.5}$$

where $R_3 \in SO_3(2)$, is an element of the stability subgroup i.e the rotation about the z-axis. Defining the operator F as $F = |\eta >< \eta|$, condition (1.31) implies the following condition on η:

$$|\eta(R_3^{-1}\underline{k})|^2 = |\eta(\underline{k})|^2. \tag{3.6}$$

By choosing η as a function of the third variable only, $\eta(\underline{k}) = \eta(k_3)$, we have a vector satisfying (3.6). We also need an open set containing k_0, on which η is supported. For such set $O(k_0)$ we need to choose a subset of S^2 on which the third coordinate does not vanish. Let us choose the upper hemisphere:

$$O(k_0) = \{k \in S^2 | k_3 > 0\}. \tag{3.7}$$

Our family of vectors, given by (3.4) also depends on the choice of the section σ. We give examples of the construction of coherent states for two choices of section: $\sigma_0(\underline{q},\hat{p}) = \big((\underline{q}^{\perp},0), R(\hat{p})\big)$, and $\sigma_{\mathcal{P}}(\underline{q},\hat{p}) = \big(R(\hat{p})(\underline{q},0), R(\hat{p})\big)$ called the principal section. We also find admissibility condition for general choice of affine section.

3.2.1 The section σ_0

As a first example let us take for a section $\sigma(\underline{q},\hat{p}) = \sigma_0(\underline{q},\hat{p}) = \big((\underline{q}^{\perp},0), R(\hat{p})\big)$. Then our family of vectors is

$$\eta_{\underline{q},\hat{p}}(\underline{k}) = e^{i\underline{q}\cdot\underline{k}}\eta\big(R(\hat{p})^{-1}\,\underline{k}\big) \tag{3.8}$$

Theorem 3.2.1 *Let E(3) be the Euclidean group, $U^{\chi L}$ its unitary irreducible representation induced from the trivial representation $\chi L = e^{ib_3}$ of the subgroup $H = \mathbb{R}^3 \rtimes SO_3(2)$. Let $\Gamma = E(3)/H$ denote the quotient space with the invariant measure $d\mu$ given by (3.3). For the choice of the vector $\eta \in \widetilde{\mathfrak{H}} = L^2(S^2, d\nu)$, which is a smooth functions on S^2 with support contained in the open set $\mathcal{O}(k_0)$, as given above, the formal operator*

$$A = \int_{\Gamma} |\eta_{\underline{q},\hat{p}}\rangle\langle\eta_{\underline{q},\hat{p}}|\, d\mu$$

is multiple of the identity $A = \lambda I$, where $\lambda = (2\pi)^2 \int |\eta(p)|^2 dp$

Proof. Let Φ, Ψ be arbitrary functions on $\widetilde{\mathfrak{H}} = L^2(S^2, d\nu)$. We consider the formal integral:

$$I_{\Phi,\Psi} = \int_{\Gamma} \langle\Phi|\eta_{\underline{q},\hat{p}}\rangle\langle\eta_{\underline{q},\hat{p}}|\Psi\rangle\, d\mu \tag{3.9}$$

$$= \int_\Gamma \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \overline{\Phi(\underline{k})}\, e^{i\underline{q}(\underline{k}-\underline{k}')}\, \eta\big(R(\hat{p})^{-1}\,\underline{k}\big)\, \overline{\eta\big(R(\hat{p})^{-1}\,\underline{k}'\big)}\, \Psi(\underline{k}')\, d\mu(\underline{q},\hat{p})\, d\nu(\underline{k})\, d\nu(\underline{k}')$$

where $d\nu(\underline{k}) = \frac{dk_1 dk_2}{k_3}$, and $d\mu$ is given by (3.3).

To perform the integration we use the Dirac distribution

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iy(x-x_0)} dy \tag{3.10}$$

with the property:

$$\int \delta(x - x_0) f(x) dx = f(x_0) \tag{3.11}$$

To perform the integration with respect dq_1, dq_2 in (3.9), we assume that $\Phi, \Psi \in \tilde{\mathcal{D}}$, where $\tilde{\mathcal{D}}$ is the space of infinitely differentiable complex functions on S^2. Then by (3.10) we get two integrations of delta type: $2\pi\delta(k_1 - k_1')$ as the result of the integration with respect to dq_1 and $2\pi\delta(k_2 - k_2')$, from the integration with respect to dq_2. Thus the property (3.11) will make the integration with respect the measure $d\nu(\underline{k}') = \frac{dk_1' dk_2'}{k_3'}$ result in

$$(2\pi)^2 \int \overline{\Phi(\underline{k})}\, \Psi(\underline{k})\, |\eta\big(R(\hat{p})^{-1}\underline{k}\big)|^2\, \frac{1}{k_3}\, d\nu(\underline{k})\, d\hat{p}_1\, d\hat{p}_2. \tag{3.12}$$

Since $\eta(\underline{k}) = \eta(k_3)$ is a function of the third variable only, and because

$$\big(R(\hat{p})^{-1}\underline{k}\big)_3 = k_1 \sin\phi \sin\theta - k_2 \cos\phi \sin\theta + k_3 \cos\theta = \hat{p} \cdot \underline{k} = \underline{k} \cdot \hat{p}. \tag{3.13}$$

then

$$\eta\Big(\big(R(\hat{p})^{-1}\,\underline{k}\big)_3\Big) = \eta\Big(\big(R(\underline{k})^{-1}\,\hat{p}\big)_3\Big).$$

where

$$\big(R(\underline{k})^{-1}\,\hat{p}\big)_3 = \hat{p}_1 \sin\phi \sin\theta - \hat{p}_2 \cos\phi \sin\theta + \hat{p}_3 \cos\theta \tag{3.14}$$

We want to perform the integration over $d\hat{p}_1\, d\hat{p}_2$. Since

$$d\nu(\hat{p}) = \frac{d\hat{p}_1 d\hat{p}_2}{\hat{p}_3}$$

is a rotationally invariant measure, we write $d\hat{p}_1\, d\hat{p}_2 = \hat{p}_3\, d\nu(\hat{p})$. Change of variables $\hat{p}' \to R(\underline{k})^{-1}\, \hat{p}$ gives us

$$I_{\Phi,\Psi} = (2\pi)^2 \int \Phi(\underline{k})\, \Psi(\underline{k}) \mid \eta(\hat{p}'_3) \mid^2 \frac{1}{k_3} d\nu(\underline{k}) \hat{p}_3\, d\nu(\hat{p}'). \tag{3.15}$$

where

$$\hat{p}_3 = (R(\underline{k})\hat{p}')_3 = \hat{p}'_1 \sin\phi_2 \sin\theta + \hat{p}'_2 \cos\phi_2 \sin\theta + \hat{p}'_3 \cos\theta$$

and $\hat{p}'_1 \in [-1, 1]$, $\hat{p}'_2 \in [-1, 1]$, $\hat{p}_3 = \sqrt{1 - \hat{p}_1^2 - \hat{p}_2^2}$. Also notice that $\underline{k}$ is a vector on the unit sphere, $\underline{k} = \begin{pmatrix} \sin\theta \sin\phi \\ -\sin\theta\cos\phi \\ \cos\theta \end{pmatrix}$, thus $k_3 = \cos\theta$. Now the integral (3.15) becomes

$$I_{\Phi,\Psi} = (2\pi)^2 \int \Phi(\underline{k})\, \Psi(\underline{k}) \frac{\mid \eta(\hat{p}'_3) \mid^2}{\cos\theta} d\nu(\underline{k}) (\hat{p}'_1 \sin\phi \sin\theta - \hat{p}'_2 \cos\phi \sin\theta + \hat{p}'_3 \cos\theta)\, d\hat{p}'_1\, d\hat{p}'_2. \tag{3.16}$$

The integration with respect $d\hat{p}'_1\, d\hat{p}'_2$ will reduce the first two terms of $\hat{p}_3$ to zeros and we will be left with

$$\begin{aligned} I_{\Phi,\Psi} &= (2\pi)^2 \int \Phi(\underline{k})\, \Psi(\underline{k}) \mid \eta(\hat{p}') \mid^2 \frac{1}{\cos\theta} d\nu(\underline{k}) (\hat{p}'_3 \cos\theta)\, d\nu(\hat{p}') \\ &= 2\pi \int \Phi(\underline{k})\, \Psi(\underline{k}) \mid \eta(\hat{p}'_3) \mid^2\, d\nu(\underline{k})\, d\hat{p}'_1\, d\hat{p}'_2 \end{aligned}$$

where $\int \mid \eta(\hat{p}') \mid^2\, d\hat{p}'_1\, d\hat{p}'_2 = P_3$ is a constant, which depends on the choice of η. For example, if we make the simplest possible choice, $\eta = 1$, we obtain $P_3 = 4$. For

$\eta(\hat{p}_3) = \sqrt{\hat{p}_3}, P_3 = \frac{4}{3}$. Thus,

$$I_{\Phi,\Psi} =< \Phi|A\Psi >= (2\pi)^2 P_3 < \Phi|\Psi >$$

for a smooth functions $\Phi, \Psi \in S^2$. We can extend this result by the continuity of the scalar product to any $\tilde{\Phi}, \tilde{\Psi} \in L^2(S^2)$, because $\tilde{\mathcal{D}}$ is a dense subspace of $L^2(S^2)$. This implies that our operator $A = \int_\Gamma |\eta_{\underline{q},\hat{p}}\rangle\langle\eta_{\underline{q},\hat{p}}|\, d\mu$ is the stated multiple of the identity.

■

3.2.2 The principal section $\sigma_{\mathcal{P}}$

Now let us try a different choice of the section. Our next section can be expressed as a product of basic section with an element of the subgroup (in our case $T_3 \otimes SO_3(2)$). Let

$$\sigma(\underline{q}, \hat{p}) = \sigma_{\mathcal{P}}(\underline{q}, \hat{p}) = \sigma_0(\underline{q}, \hat{p})\big((\underline{0}, \alpha), \mathbb{I}_3\big) = \big(R(\hat{p})(\underline{q}, 0), R(\hat{p})\big),$$

where σ_0 is defined by (2.6). We calculate that for this choice of the section,

$$\alpha = -[R(\hat{p})^{-1}(\underline{q}, 0)]_3.$$

Now the set of vectors $\tilde{\eta}_{\underline{q},\hat{p}}$ is given as

$$\tilde{\eta}_{\underline{q},\hat{p}}(\underline{k}) = e^{iR(\hat{p})\underline{q}\cdot\underline{k}}\eta\big(R(\hat{p})^{-1}\,\underline{k}\big) = e^{i\underline{q}\cdot R(\hat{p})^{-1}\underline{k}}\eta\big(R(\hat{p})^{-1}\,\underline{k}\big). \tag{3.17}$$

For this choice of section we may prove a theorem similar to (3.2.1).

Theorem 3.2.2 *Let $E(3), U^{\chi L}, \Gamma$ be as in the Theorem 3.2.1. Let $\eta \in \mathfrak{H}$ be a smooth function supported on the upper hemisphere (3.7), satisfying the invariance condition*

(3.6) be given by

$$\eta(\underline{k}) = k_3^{1/2} B^{1/2}(k_3), \tag{3.18}$$

where $B(k_3)$ is a bounded multiplication operator: $(B\Psi)(\underline{k}) = B^{1/2}(k_3)\Psi(\underline{k})$, and $\|B\| = sup|B(\underline{k})|, 0 < \|B\| < \infty$. Then for the $\tilde{\eta}_{\underline{q},\hat{p}}$ given by (3.17), the formal operator

$$A = \int_\Gamma |\tilde{\eta}_{\underline{q},\hat{p}}\rangle\langle\tilde{\eta}_{\underline{q},\hat{p}}| d\mu(\underline{q},\hat{p}),$$

is bounded with bounded inverse. If $\eta(k_3) = \sqrt{k_3}$, the operator A is a multiple of the identity.

Proof. Consider the formal integral $\tilde{I}_{\Phi\Psi} = \langle\Phi|A\Psi\rangle$. Then

$$\tilde{I}_{\Phi\Psi} = \int_\Gamma \langle\Phi|\tilde{\eta}_{\underline{q},\hat{p}}\rangle\langle\tilde{\eta}_{\underline{q},\hat{p}}|\Psi\rangle \, d\mu \tag{3.19}$$

$$= \int_\Gamma \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \overline{\Phi(\underline{k})}\, e^{iR(\hat{p})\underline{q}(\underline{k}-\underline{k}')}\, \eta\big(R(\hat{p})^{-1}\,\underline{k}\big)\, \overline{\eta\big(R(\hat{p})^{-1}\,\underline{k}'\big)}\, \Psi(\underline{k}')\, d\mu(\underline{q},\hat{p})\, d\nu(\underline{k})\, d\nu(\underline{k}')$$

$$= \int_\Gamma \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \overline{\Phi(\underline{k})}\, e^{i\underline{q}R(\hat{p})^{-1}(\underline{k}-\underline{k}')}\, \eta\big(R(\hat{p})^{-1}\,\underline{k}\big)\, \overline{\eta\big(R(\hat{p})^{-1}\,\underline{k}'\big)}\, \Psi(\underline{k}')\, d\mu(\underline{q},\hat{p})\, d\nu(\underline{k})\, d\nu(\underline{k}').$$

By setting $\hat{k} = R(\hat{p})^{-1}\underline{k}$ and performing the integration with respect $dq_1\, dq_2$ as in the proof of the Theorem 3.2.1, we again have two delta functions, $2\pi\,\delta(\hat{k}_1 - \hat{k}_1')$ and $2\pi\,\delta(\hat{k}_2 - \hat{k}_2')$, which make the integration with respect to the measure $d\nu(\underline{k}') = d\nu(\hat{k}') = \frac{d\hat{k}_1' d\hat{k}_2'}{\hat{k}_3'}$ result in

$$\tilde{I}_{\Phi\Psi} = 2\pi \int_{\mathbb{S}^2_+ \times \mathbb{S}^2} \overline{\Phi(R(\hat{p})\hat{k})}\, \Psi(R(\hat{p})\hat{k})\, |\eta(\hat{k})|^2\, \frac{1}{\hat{k}_3}\, d\nu(\hat{k})\, d\hat{p}_1\, d\hat{p}_2. \tag{3.20}$$

Now let us take an explicit form of η, given by (3.18), as a function of only the third component of the vector

$$\eta(\hat{k}) = \begin{cases} \sqrt{\hat{k}_3} A^{1/2}(\hat{k}_3) & \text{for} \quad \hat{k}_3 > 0, \\ 0 & \qquad\qquad \text{otherwise.} \end{cases} \tag{3.21}$$

Then the integral (3.20) becomes

$$\begin{aligned} \tilde{I}_{\Phi\Psi} &= 2\pi \int_{\mathbb{S}^2_+ \times \mathbb{S}^2} \overline{\Phi(R(\hat{p})\hat{k})}\, \Psi(R(\hat{p})\hat{k})\, \hat{k}_3 \,\frac{A(\hat{k}_3)}{\hat{k}_3}\, d\nu(\hat{k})\, d\hat{p}_1\, d\hat{p}_2 \\ &= 2\pi \int_{\mathbb{S}^2_+ \times \mathbb{S}^2} \overline{\Phi(\underline{k})}\, \Psi(\underline{k}) A([R(\hat{p})^{-1}\underline{k}]_3) d\nu(\underline{k})\, d\hat{p}_1\, d\hat{p}_2. \end{aligned}$$

As we already calculated in (3.13) $[R(\hat{p})^{-1}\underline{k}]_3 = \hat{p} \cdot \underline{k}$, thus we can write

$$\tilde{I}_{\Phi\Psi} = 2\pi \int_{\mathbb{S}^2_+ \times \mathbb{S}^2} \Phi(\underline{k})\Psi(\underline{k}) d\nu(\underline{k}) A(\hat{p} \cdot \underline{k}) d\hat{p}_1 d\hat{p}_2 = A(\underline{k}) < \Phi|\Psi >, \tag{3.22}$$

where $A(\underline{k}) = 2\pi \int A(\hat{p} \cdot \underline{k}) d\hat{p}_1 d\hat{p}_2$ is a bounded operator, with lower bound $a > 0$, and upper bound $b < \infty$.

To show the case of the tight frame we just substitute $\eta(\hat{k}) = \sqrt{\hat{k}_3}$, for $\hat{k}_3 > 0$, into (3.20) to obtain

$$\begin{aligned} \tilde{I}_{\Phi\Psi} &= 2\pi \int_{\mathbb{S}^2_+ \times \mathbb{S}^2} \overline{\Phi(R(\hat{p})\hat{k})}\, \Psi(R(\hat{p})\hat{k})\, \hat{k}_3 \,\frac{1}{\hat{k}_3}\, d\nu(\hat{k})\, d\hat{p}_1\, d\hat{p}_2 \\ &= 2\pi \int_{\mathbb{S}^2_+ \times \mathbb{S}^2} \overline{\Phi(\underline{k})}\, \Psi(\underline{k}) d\nu(\underline{k})\, d\hat{p}_1\, d\hat{p}_2 = 2\pi \int < \Phi|\Psi > d\hat{p}_1\, d\hat{p}_2 \\ &= 4\pi^2 < \Phi|\Psi >, \end{aligned} \tag{3.23}$$

which means $A = \int_\Gamma |\eta(\underline{q}, \hat{p})\rangle\langle\eta(\underline{q}, \hat{p})|\, d\mu$ is a multiple of the identity, and set of vectors $\{\tilde{\eta}_{\underline{q},\hat{p}}\}$ constitutes a tight frame. ∎

3.2.3 A general section

We presented two examples of sections for which coherent states can be constructed. Now let us generalize: let $\sigma(q,p) = \sigma_{\mathcal{P}}(q,p)(n(q,p), s_0(q,p))$ be an admissible affine section we described in Section 1.4 of Chapter 1, where $n(q,p)$ and $s_0(q,p)$ are the functions given by (1.44), that is

$$\begin{aligned} n(q,p) &= \Theta(p)q + \Xi(p) \\ s_0(q,p) &= s_0(p). \end{aligned} \tag{3.24}$$

Because of the results in [4] we know that the form of Θ is of biggest interest for us. Let us start with a most general Θ :

$$\Theta(p) = \begin{pmatrix} a(p) & b(p) & c(p) \\ d(p) & e(p) & f(p) \\ g(p) & h(p) & i(p) \end{pmatrix}, \tag{3.25}$$

where $a, b, ..., i$ are real functions of p. From now we will write $a, b, ...i$ instead of $a(p), b(p), ..., i(p)$. Our first constraint on Θ is that for the fixed p, Ker $(\Theta(p)) = N_0$. In our case this means $\mathrm{Ker}(\Theta(p)) = \mathbb{R}$. Since our map acts on the vectors from the upper hemisphere, we have:

$$\Theta(p) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow c = f = i = 0. \tag{3.26}$$

Hence

$$\Theta(p) = \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 0 \end{pmatrix}. \tag{3.27}$$

Thus the map adjoint to $\Theta(p)$, which satisfies the condition $\mathrm{Ker}\Theta(p) = \mathbb{R}$ is

$$\Theta(p)^* = \Theta(p)^T = \begin{pmatrix} a & d & g \\ b & e & h \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.28}$$

From Section 10.3, Chapter 10 of [4] we know that the map $T^*(p) = I_{(\mathbb{R}^3)^*} + \Theta(p)^*$ has to map the set $O(k_0)$ to itself, which gives us the next constraint on Θ:

$$T^*(p)k = k', \qquad k, k' \in O(k_0), \tag{3.29}$$

which for $k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in O(k_0)$ gives $k' = \begin{pmatrix} g \\ h \\ 1 \end{pmatrix}$. Because $k' \in O(k_0)$ implies $k'_3 = \sqrt{1 - (k'_1)^2 - (k'_2)^2} = \sqrt{1 - g^2 - h^2} = 1$ thus $g = h = 0$, and

$$T(p)^* = I_{(\mathbb{R}^3)^*} + \Theta(p)^* = \begin{pmatrix} 1+a & d & 0 \\ b & 1+e & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.30}$$

The Jacobian $J(p, k)$ of this map restricted to $O(k_0)$ should not be zero anywhere. Because of (3.29) and $T(p)^*$ given by (3.30) we have:

$$\begin{aligned} k_1' &= (1+a)k_1 + dk_2, \\ k_2' &= bk_1 + (1+e)k_2, \\ k_3' &= k_3. \end{aligned}$$

Restriction of $T(p)^*$ to the upper hemisphere $O(k_0)$ means we have only 2 independent coordinates, and the third coordinate satisfies the condition

$$k_3' = \sqrt{1 - (k_1')^2 - (k_2')^2}.$$

Thus the transformation $k \longrightarrow k'$ will produce

$$dk_1' dk_2' = \det|J(p, k)| dk_1 dk_2, \tag{3.31}$$

where

$$J(p, k) = \begin{pmatrix} 1+a & d \\ b & 1+e \end{pmatrix} \tag{3.32}$$

and

$$\det|J(p, k)| = (1+a)(1+e) - bd.$$

We also have $k_3' = k_3$ which gives us condition

$$\sqrt{1 - (k_1')^2 - (k_2')^2} = \sqrt{1 - k_1^2 - k_2^2}$$

or

$$(k_1')^2 + (k_2')^2 = k_1^2 + k_2^2.$$

This is satisfied if

$$k_1' = \cos\phi_0 k_1 - \sin\phi_0 k_2,$$

$$k_2' = \sin\phi_0 k_1 + \cos\phi_0 k_2,$$

where ϕ_0 represents the angle of rotation about $z-$ axis. Then

$$\det|J(p,k)| = (1+a)(1+e) - bd = \cos^2\phi_0 + \sin^2\phi_0 = 1.$$

After applying all restrictions on the map $\Theta(p)$ we see that the section

$$\sigma(q,p) = \sigma_{\mathcal{P}}(q,p)(n(q,p), s_0(q,p))$$

given by (1.43) will be affine admissible section for the Euclidean group if the function $\Theta(p)$ from (3.24) has the form

$$\Theta(p) = \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.33}$$

where $a = e = \cos\phi_0 - 1, b = -d = \sin\phi_0$ and $(1+a)(1+e) - bd = (1+a)^2 + b^2 = \cos^2\phi_0 + \sin^2\phi_0 = 1$. Thus the map

$$T(p) = I_{\mathbb{R}^3} + \Theta(p) = \begin{pmatrix} \cos\phi_0 & -\sin\phi_0 & 0 \\ \sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.34}$$

is the rotation about the z-axis. Thus our section assumes the following form:

$$\sigma(q,p) = \big(R(p)(T(p))q + R(p)\Xi(p), R(p)s_0(p)\big), \tag{3.35}$$

with $T(p)$ as in (3.34).

As in the other choices of sections we define set of vectors $\hat{\eta}_{\sigma q,p}(k) = U^{\chi L}(\sigma(q,p))\eta(k)$, where $\eta(k)$ satisfies condition (3.6). Thus for the section given by (3.35) the set $\{\hat{\eta}_{\sigma(q,p)}\}$ is

$$\hat{\eta}_{\sigma(q,p)}(k) = e^{i(R(p)T(p)q+R(p)\Xi(p))k}\eta\big(R(\hat{p})^{-1}\,k\big) = e^{i(T(p)q+\Xi(p))R(p)^{-1}k}\eta\big(s_0^{-1}R(p)^{-1}\,k\big) \tag{3.36}$$

Again we can consider the formal integral

$$\hat{I}_{\Phi\Psi} = \int_\Gamma \langle\Phi|\hat{\eta}_{q,p}\rangle\langle\hat{\eta}_{q,p}|\Psi\rangle\, d\mu \tag{3.37}$$

$$= \int_\Gamma\int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \overline{\Phi(k)}\, e^{i(T(p)q+\Xi(p))R(p)^{-1}(k-k')}\,\eta\big(s_0^{-1}R(p)^{-1}\,k\big)$$
$$\times\overline{\eta\big(s_0^{-1}R(p)^{-1}\,k'\big)}\,\Psi(k')\,d\mu(q,p)\,d\nu(k)\,d\nu(k')$$
$$= \int_\Gamma\int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \overline{\Phi(\underline{k})}\, e^{iT(p)qR(p)^{-1}(k-k')}\, e^{i\Xi(p)R(p)^{-1}(k-k')}\eta\big(s_0^{-1}R(p)^{-1}\,k\big)$$
$$\times\overline{\eta\big(s_0^{-1}R(p)^{-1}\,k'\big)}\,\Psi(k')\,d\mu(q,p)\,d\nu(k)\,d\nu(k').$$

We can change variable $q \to q'$ by setting $q' = T(p)q$. Because $T(p)$ is a rotation about the z-axis, measure $d\mu$ is invariant under such change and we have

$$\hat{I}_{\Phi\Psi} = \int_\Gamma\int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \overline{\Phi(\underline{k})}\, e^{iq'R(p)^{-1}(k-k')}\, e^{i\Xi(p)R(p)^{-1}(k-k')}\eta\big(s_0^{-1}R(p)^{-1}\,k\big)$$
$$\times\overline{\eta\big(s_0^{-1}R(p)^{-1}\,k'\big)}\,\Psi(k')\,d\mu(q',p)\,d\nu(k)\,d\nu(k').$$

By setting $\hat{k} = R(p)^{-1}k$ and performing the integration with respect $dq_1'\,dq_2'$ as in the proof of the Theorem 3.2.1, we again have two delta functions, $2\pi\delta(\hat{k}_1 - \hat{k}_1')$

and $2\pi\delta(\hat{k}_2 - \hat{k}'_2)$, which make the integration with respect to the measure $d\nu(k') = d\nu(\hat{k}') = \frac{d\hat{k}'_1 d\hat{k}'_2}{\hat{k}'_3}$ result in

$$\tilde{I}_{\Phi\Psi} = (2\pi)^2 \int_{\mathbb{S}^2_+ \times \mathfrak{D}} \overline{\Phi(R(\hat{p})\hat{k})}\, \Psi(R(\hat{p})\hat{k})\, |\eta(s_0^{-1}\hat{k})|^2 \frac{1}{\hat{k}_3}\, d\nu(\hat{k})\, d\hat{p}_1\, d\hat{p}_2. \tag{3.38}$$

where $\mathfrak{D} = \{p_1^2 + p_2^2 \leq 1\}$.

(Factor $e^{i\Xi(p)R(p)^{-1}(k-k')} = 1$ for $k = k'$.) Because our choice of η was such that it is invariant under the rotation about z-axis (condition (3.6)) and s_0 is such rotation, we have

$$\begin{aligned}
\tilde{I}_{\Phi\Psi} &= (2\pi)^2 \int_{\mathbb{S}^2_+ \times \mathfrak{D}} \overline{\Phi(R(\hat{p})\hat{k})}\, \Psi(R(\hat{p})\hat{k})\, |\eta(\hat{k})|^2 \frac{1}{\hat{k}_3}\, d\nu(\hat{k})\, d\hat{p}_1\, d\hat{p}_2. \\
&= (2\pi)^2 \int_{\mathbb{S}^2_+ \times \mathfrak{D}} \overline{\Phi(k)}\, \Psi(k)\, |\eta(R(\hat{p})^{-1}k)|^2 \frac{1}{(R(\hat{p})^{-1}k)_3}\, d\nu(k)\, d\hat{p}_1\, d\hat{p}_2. \quad (3.39)
\end{aligned}$$

Thus we have a multiplication operator $\hat{A}$:

$$\hat{A}\Phi(k) = A(k)\Phi(k),$$

and admissibility condition is that

$$A(k) = (2\pi)^2 \int_{\mathfrak{D}} \frac{|\eta(R(\hat{p})^{-1}k)|^2}{(R(\hat{p}^{-1}k)_3}\, d\hat{p}_1\, d\hat{p}_2. \tag{3.40}$$

is a bounded function with a bounded inverse, that is, there should exist a, b such that

$$0 < a \leq A(k) \leq b < \infty.$$

Then $\hat{A} = \int_\Gamma |\hat{\eta}_{\sigma(q,p)} >< \eta(\hat{q,} p)| d\mu$ is a bounded operator with bounded inverse. Hence the vectors $\hat{\eta}_{\sigma(q,p)}$ form a frame.

Our general section reduces to $\sigma_0(q,p)$ if we choose $T(p) = R(p)^{-1}$, $\Xi(p) = 0$ and $s_0 = I$ in (3.35). To obtain principal section $\sigma_{\mathcal{P}}(q,p)$, we chose $T(p) = I_{\mathbb{R}^3}$, and again $\Xi(p) = 0$ and $s_0 = I$.

Chapter 4

Discretization of the Coherent States

The calculations in the previous chapter were for the parameterization of a coset space Γ with the continuous parameters $(\underline{q}, \hat{p})$. But in practice, numerical calculations of the integral require a restriction of the continuous set to a discrete subset. Because the continuous set of vectors was overcomplete, we are able to choose a countable set of points, such that the resolution of the identity still be satisfied. We want the discrete set of points $(\underline{q}_{ij}\hat{p}_{ln})$ in Γ, for which the operator

$$T = \sum_{\underline{q}_{ij}, \hat{p}_{ln}}^{\infty} |\eta_{\underline{q}_{ij}, \hat{p}_{ln}}\rangle\langle\eta_{\underline{q}_{ij}, \hat{p}_{ln}}|$$

will be bounded with bounded inverse. By analyzing this operator we will explore conditions under which the set of vectors $\{\eta_{\underline{q}_{ij}, \hat{p}_{ln}}\}$ forms a discrete frame for the same choices of the sections as in Chapter 3.

4.1 Discrete Coherent States for E(3)

Let us chose from the elements $(\underline{q}, \hat{p})$ of the coset space $\Gamma = E(3)/(T_3 \otimes SO_3(2))$ a discrete set of points $(\underline{q}_{ij}, \hat{p}_{ln})$ in the following manner:

$$\underline{q}_{ij} = (q_i, q_j, 0) = 2\pi(\frac{i}{L_1}, \frac{j}{L_2}, 0), \tag{4.1}$$

where L_1, L_2 are constant distances between any two consecutive q_i's and q_j's respectively, and

$$\hat{p}_{ln} = (p_l, p_n, \sqrt{1 - p_l^2 - p_n^2}) \tag{4.2}$$

thus the $\hat{p}_{ln}$'s are restricted to elements with positive third coordinate.

Let η, like in the continuous case in Chapter 3, be a function supported on the upper hemisphere (the set $O(k_0)$ given by (3.7)) and satisfy the condition (3.6).

Now we can define a discrete family of vectors as

$$\eta_{\underline{q}_{ij}, \hat{p}_{ln}}(k) = \widetilde{U}^{\chi L}(\sigma(\underline{q}_{ij}, \hat{p}_{ln}))\eta(k), \tag{4.3}$$

where $\widetilde{U}^{\chi L}$ is the unitary representation of the Euclidean group given by (1.28), and for the choices of section explored in Chapter 3 we will study the convergence of the operator

$$T = \sum_{\underline{q}_{ij}, \hat{p}_{ln}}^{\infty} |\eta_{\underline{q}_{ij}, \hat{p}_{ln}}\rangle\langle\eta_{\underline{q}_{ij}, \hat{p}_{ln}}|$$

4.1.1 The section σ_0

Let us start with the section $\sigma_0(\underline{q}_{ij}, \hat{p}_{ln}) = (\underline{q}_{ij}, R(\hat{p}_{ln}))$. For such a section our discrete family (4.3) assumes the form

$$\hat{\eta}_{\underline{q}_{ij},\hat{p}_{ln}}(k) = e^{i\underline{q}_{ij}\cdot\underline{k}}\eta\big(R(\hat{p}_{ln})^{-1}\,\underline{k}\big) \tag{4.4}$$

To study the convergence properties of the operator T, we consider the formal sum $S_{\Phi\Psi} =< \Phi|T\Psi >$, where Φ, Ψ are arbitrary vectors in $\widetilde{\mathfrak{H}}^L$. Thus, writing $\hat{\eta}_{\underline{q}_{ij},\hat{p}_{ln}}$ explicitly we have

$$S_{\Phi\Psi} = \sum_{\underline{q}_{ij},\hat{p}_{ln}} \int_{\mathbb{S}^2\times\mathbb{S}^2} \overline{\Phi(\underline{k})}\, e^{i\underline{q}_{ij}\cdot(\underline{k}-\underline{k}')}\,\eta\big(R(\hat{p}_{ln})^{-1}\,\underline{k}\big)\overline{\eta\big(R(\hat{p}_{ln})^{-1}\,\underline{k}'\big)}\,\Psi(\underline{k}')\,d\nu(\underline{k})\,d\nu(\underline{k}'). \tag{4.5}$$

Using the distributional identity

$$\frac{1}{L}\sum_{n=-\infty}^{\infty} e^{i\frac{2\pi}{L}n(k-k')} = \delta(k-k')$$

we can perform the summation over i and j which results in two delta functions: $\delta(k_1 - k_1')$ and $\delta(k_2 - k_2')$. Using this delta function we are able to perform the integration over k ($d\nu(k) = \frac{dk_1 dk_2}{k_3}$), which gives us

$$\begin{aligned} S_{\Phi\Psi} &= L_1L_2\sum_{\hat{p}_{ln}}\int_{\mathbb{S}^2}\overline{\Phi(\underline{k})}\,|\eta\big(R(\hat{p}_{ln})^{-1}\,\underline{k}\big)|^2\,\Psi(\underline{k})\,\frac{1}{k_3}d\nu(\underline{k}) \qquad (4.6)\\ &= \int_{\mathbb{S}^2}\overline{\Phi(\underline{k})}\,\Psi(\underline{k})\,d\nu(\underline{k})\,\sum_{\hat{p}_{ln}}\frac{L_1L_2}{k_3}|\eta\big(R(\hat{p}_{ln})^{-1}\,\underline{k}\big)|^2 \end{aligned}$$

This implies that the operator T is a multiplication operator, the function by which we multiply being of the form

$$T(k) = \sum_{\hat{p}_{ln}}\frac{L_1L_2}{k_3}|\eta\big(R(\hat{p}_{ln})^{-1}\,\underline{k}\big)|^2. \tag{4.7}$$

For choice of η satisfying the invariance condition (3.6), for example $\eta(k) = \sqrt{k_3}$ we get

$$T(k) = \sum_{\hat{p}_{ln}} \frac{L_1 L_2}{k_3} |\sqrt{p_l k_1 + p_n k_2 + p_3 k_3}|^2, \text{ where } p_3 = \sqrt{1 - p_l^2 - p_n^2} \tag{4.8}$$

due to relation $\left(R(p_{ln})^{-1}k\right)_3 = p_{ln} \cdot k$. An vector $k \in O(k_0)$ is an element in the upper hemisphere and $k_3 \in (0,1]$, thus the function $T(k)$ is not bounded (when $k_3 \to 0$ the numerator of $T(k)$ does not approach zero, and $T(k)$ becomes infinite), and the operator T is not bounded. Hence the family $\{\hat{\eta}_{\underline{q}_{ij}, \hat{p}_{ln}}\}$ given by (4.4) does not constitute a frame.

4.1.2 The principal section $\sigma_{\mathcal{P}}$

The second choice of the section is the principal section $\sigma_{\mathcal{P}}(\underline{q}_{ij}, \hat{p}_{ln}) = \left(R(\hat{p}_{ln})\underline{q}_{ij}, R(\hat{p}_{ln})\right)$. Then the set of vectors (4.3) is

$$\eta_{\underline{q}_{ij}, \hat{p}_{ln}} = U\left(\sigma_{\mathcal{P}}(\underline{q}_{ij}, \hat{p}_{ln})\right)\eta(\underline{k}) = e^{iR(\hat{p}_{ln})\underline{q}_{ij} \cdot \underline{k}} \eta\left(R(\hat{p}_{ln})^{-1}\, \underline{k}\right) = e^{i\underline{q}_{ij} \cdot R(\hat{p}_{ln})^{-1}\underline{k}} \eta\left(R(\hat{p}_{ln})^{-1}\, \underline{k}\right) \tag{4.9}$$

and for this choice we have a frame, which is the result of the next theorem.

Theorem 4.1.1 *Let E(3) be the Euclidean group, $U^{\chi L}$ its unitary irreducible representation induced from the trivial representation $\chi L = e^{ib_3}$ of the subgroup $H = \mathbb{R}^3 \rtimes SO_3(2)$. Let $\Gamma = E(3)/H$ the quotient space with counting measure, be parameterized by the points $(\underline{q}_{ij}, \hat{p}_{ln})$ where $\underline{q}_{ij}$ and $\hat{p}_{ln}$ are given by (4.1) and (4.2)* *respectively. Let the vector*

$$\eta(k) = \begin{cases} \sqrt{k_3} \text{ if } & k_3 \in \mathcal{O}(k_0), \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

satisfy the invariance condition (3.6). Then the operator

$$T = \sum_{\underline{q}_{ij}, \hat{p}_{ln}}^{\infty} |\eta_{\underline{q}_{ij}, \hat{p}_{ln}}\rangle\langle\eta_{\underline{q}_{ij}, \hat{p}_{ln}}| \tag{4.11}$$

with $\eta_{\underline{q}_{ij}, \hat{p}_{ln}}$ given by (4.9) is a multiple of the identity.

Proof. Let us consider the formal sum

$$S_{\phi\Psi} = \langle\Phi|T\Psi\rangle = \sum_{\underline{q}_{ij}, \hat{p}_{ln}}^{\infty} \langle\Phi|\eta_{\underline{q}_{ij}, \hat{p}_{ln}}\rangle\langle\eta_{\underline{q}_{ij}, \hat{p}_{ln}}|\Psi\rangle$$

$$= \sum_{\underline{q}_{ij}, \hat{p}_{ln}} \int_{\mathbb{S}^2} \overline{\Phi(\underline{k})}\, e^{iR(\hat{p}_{ln})\underline{q}_{ij}\cdot(\underline{k}-\underline{k}')}\, \eta\big(R(\hat{p}_{ln})^{-1}\,\underline{k}\big)\overline{\eta\big(R(\hat{p}_{ln})^{-1}\,\underline{k}'\big)}\, \Psi(\underline{k}')\, d\nu(\underline{k})\, d\nu(\underline{k}').$$

Taking $\underline{q}_{ij} = (q_i, q_j, 0)$ as given by (4.1) and using

$$\frac{1}{L}\sum_{n=-\infty}^{\infty} e^{i\frac{2\pi}{L}n(x-x')} = \delta(x-x')$$

(where $x = R(\hat{p}_{ln})^{-1}\underline{k}$ and $d\nu(x) = d\nu(\underline{k})$) we can perform the summation over i and j and obtain

$$\sum_{\hat{p}_{ln}} \int_{\mathbb{S}^2} \overline{\Phi(R(\hat{p}_{ln})x)}\, L_1\, L_2\delta(x_1 - x_1')\delta(x_2 - x_2')\, \eta(x)\, \overline{\eta(x')}\, \Psi(R(\hat{p}_{ln})x')\, d\nu(x)\, d\nu(x')$$

The integration with respect to $d\nu(x') = \frac{dx_1' dx_2'}{x_3'}$, leaves us with

$$\sum_{\hat{p}_{ln}} \int_{\mathbb{S}^2} \overline{\Phi(R(\hat{p}_{ln})x)}\, L_1\, L_2\, |\eta(x)|^2\, \Psi(R(\hat{p}_{ln})x)\, \frac{1}{x_3}\, d\nu(x) \tag{4.12}$$

Changing back to the variable $\underline{k} = R(\hat{p}_{ln})x$, (4.12) becomes

$$\sum_{\hat{p}_{ln}} \int_{\mathbb{S}^2} \overline{\Phi(\underline{k})}\, L_1\, L_2\, |\eta(R(\hat{p}_{ln})^{-1}\underline{k})|^2\, \Psi(\underline{k})\, \frac{1}{(R(\hat{p}_{ln})^{-1}\underline{k})_3}\, d\nu(\underline{k})$$

By taking the explicit form of η, in (4.10), our formal sum becomes

$$S_{\Phi\Psi} = \sum_{\hat{p}_{ln}} \int_{\mathbb{S}^2} \overline{\Phi(\underline{k})}\, L_1\, L_2\, \Psi(\underline{k})\, d\nu(\underline{k}) = L_1 L_2 \sum_{\hat{p}_{ln}} < \Phi|\Psi >,$$

and because $\hat{p}_{ln} \in S^2$, S^2 being a compact set, the sum is over the finite number of elements, thus it gives finite result. Let us say $\sum_{\hat{p}_{ln}} 1 = N$, thus

$$< \Phi|T\Psi >= L_1 L_2 N < \Phi|\Psi >,$$

thus $T = L_1 L_2 N I$ is a multiple of the identity and elements $\tilde{\eta}_{\underline{q}_{ij}, \hat{p}_{ln}}$ constitute the discrete tight frame. ■

4.1.3 A general section

Now let us consider, like in the continuous case, an admissible affine section:

$$\sigma(q_{ij}, p_{ln}) = \big(R(p_{ln})(T(p_{ln}))q_{ij} + R(p_{ln})\Phi(p_{ln}), R(p_{ln})s_0(p_{ln})\big),$$

where q_{ij}, p_{ln} are the discrete parameters in the coset space $\Gamma = E(3)/(T_3 \otimes SO_3(2))$ as given by (4.1) and (4.2) respectively, and $T(p_{ln}) = I + \Theta(p_l n)$ is the discrete version of (3.34). We can choose a discrete family of vectors in the Hilbert space $\widetilde{\mathfrak{H}}^L = L^2(S^2, d\nu)$ as

$$\eta_{q_{ij}, p_{ln}}(k) = e^{i(T(p_{ln}))q_{ij} + \Phi(p_{ln}))\cdot R(p_{ln})^{-1}k}\, \eta\big(s_0(p_{ln})^{-1} R(p_{ln})^{-1}\, k\big) \qquad (4.13)$$

and by studying the formal operator

$$\mathfrak{T} = \sum_{q_{ij},p_{ln}} |\eta_{q_{ij},p_{ln}} >< \eta_{q_{ij},p_{ln}}| \tag{4.14}$$

we determine the conditions under which the set of vectors in (4.13) constitutes a frame.

Again we consider a sum

$$\begin{aligned}
S_{\Upsilon\Psi} &= <\Upsilon|\mathfrak{T}\Psi> \\
&= \sum_{q_{ij},p_{ln}} \int_{\mathbb{S}^2\times\mathbb{S}^2} \overline{\Upsilon(k)}\, e^{iq_{ij}\cdot T(p_{ln})^{-1}R(p_{ln})^{-1}(k-k')} e^{i\Phi(p_{ln})R(p_{ln})^{-1}(k-k')} \qquad (4.15)\\
&\times\ \eta\big(s_0(p_{ln})^{-1}R(p_{ln})^{-1}\,k\big)\overline{\eta\big(s_0(p_{ln})^{-1}R(p_{ln})^{-1}\,k'\big)}\,\Psi(k')\,d\nu(k)\,d\nu(k') \\
&= \sum_{q_{ij}p_{ln}} \int_{\mathbb{S}^2\times\mathbb{S}^2} \overline{\Upsilon(R(p_{ln})T(p_{ln})\hat{k})}\, e^{iq_{ij}\cdot(\hat{k}-\hat{k}')} e^{i\Phi(p_{ln})T(p_{ln})(\hat{k}-\hat{k}')} \qquad (4.16)\\
&\times\ \eta\big(s_0(p_{ln})^{-1}T(p_{ln})\,\hat{k}\big)\overline{\eta\big(s_0(p_{ln})^{-1}T(p_{ln})\,\hat{k}'\big)}\,\Psi(R(p_{ln})T(p_{ln})\hat{k}')\,d\nu(\hat{k})\,d\nu(\hat{k}'),
\end{aligned}$$

the last equation being the result of the change of variable $\hat{k} = T(p_{ln})^{-1}R(p_{ln})^{-1}k$ which leaves the measure $d\nu(k)$ invariant. Now we are able to perform the summation over q_{ij} as well as the integration with respect $\hat{k}'$ using the same identity as in the case of the principal section:

$$\frac{1}{L}\sum_{n=-\infty}^{\infty} e^{i\frac{2\pi}{L}n(x-x')} = \delta(x-x').$$

Thus

$$S_{\Upsilon\Psi} = \sum_{p_{ln}} L_1L_2 \int_{\mathbb{S}^2} \overline{\Upsilon(R(p_{ln})T(p_{ln})\hat{k})}\frac{|\eta\big(s_0(p_{ln})^{-1}T(p_{ln})\,\hat{k}\big)|^2}{\hat{k}_3}\Psi(R(p_{ln})T(p_{ln})\hat{k})\,d\nu(\hat{k}) \tag{4.17}$$

Because we chosen η to be invariant under the action of s_0 (that is, η satisfies the condition (3.6)), we get

$$S_{\Upsilon\Psi} = \sum_{p_{ln}} L_1 L_2 \int_{\mathbb{S}^2} \overline{\Upsilon(R(p_{ln})T(p_{ln})\hat{k})} \frac{|\eta(T(p_{ln})\hat{k}_3)|^2}{\hat{k}_3} \Psi(R(p_{ln})T(p_{ln})\hat{k})\, d\nu(\hat{k}) \tag{4.18}$$

$$= \sum_{p_{ln}} L_1 L_2 \int_{\mathbb{S}^2} \overline{\Upsilon(k)} \frac{|\eta\big(R(p_{ln})^{-1}k)_3\big)|^2}{\big(T(p_{ln})^{-1}R(p_{ln})^{-1}k\big)_3} \Psi(k)\, d\nu(k) \tag{4.19}$$

Thus again $\mathfrak{T}$ is a multiplication operator:

$$\mathfrak{T}\Psi(k) = \mathcal{T}(k)\Psi(k),$$

by the function

$$\mathcal{T}(k) = \sum_{p_{ln}} \frac{|\eta(R(p_{ln})^{-1}k)|^2}{\big(T(p_{ln})^{-1}R(p_{ln})^{-1}k\big)_3}. \tag{4.20}$$

In order to yield a frame, this function has to satisfy an admissibility condition $0 < a \leq \mathcal{T} \leq b < \infty$ for some positive numbers a, b in order for $\mathfrak{T}$ to constitute discrete frame.

To recover the previously given sections we chose $T(p_{ln}) = R(p_{ln})^{-1}$ for σ_0 and $T(p_{ln}) = I$ for $\sigma_{\mathcal{P}}$. $s_0(p_{ln}) = I, \Phi(p_{ln}) = 0$ in both cases.

Conclusion

Summarizing our results in this thesis, we have constructed continuous coherent states associated with the three dimensional Euclidean group for two choices of sections:

(1) for $\sigma_0(\underline{q},\hat{p}) = \big((\underline{q}^{\perp},0),R(\hat{p})\big)$ they are of the form $\eta_{\underline{q},\hat{p}}(\underline{k}) = e^{i\underline{q}\cdot\underline{k}}\eta\big(R(\hat{p})^{-1}\,\underline{k}\big)$ as given by (3.8)

(2) for the principal section, $\sigma_{\mathcal{P}}(\underline{q},\hat{p}) = \big(R(\hat{p})(\underline{q},0),R(\hat{p})\big)$, they are given by $\tilde{\eta}_{\underline{q},\hat{p}}(\underline{k}) = e^{iR(\hat{p})\underline{q}\cdot\underline{k}}\eta\big(R(\hat{p})^{-1}\,\underline{k}\big)$ (formula (3.17)).

We have also found that the general admissible affine section is of the form

$$\sigma(q,p) = \big(R(p)(T(p))q + R(p)\Xi(p), R(p)s_0(p)\big). \tag{4.21}$$

where the function $T(p)$ (3.34) is a rotation about the z-axis.

As for the discrete case, for the choice of $\underline{q}_{ij} = (q_i,q_j,0) = 2\pi(\frac{i}{L_1},\frac{j}{L_2},0)$ and $\hat{p}_{ln} = (p_l,p_n,\sqrt{1-p_l^2-p_n^2})$ as given by (4.1) and (4.2) respectively, the principal section $\sigma_{\mathcal{P}}$ leads to the discrete set of vectors constituting the family of coherent states. The admissibility condition for the general affine section is that the operator $\mathfrak{T}$ given by (4.14) be a multiplication operator $\mathfrak{T}\Psi(k) = \mathcal{T}(k)\Psi(k)$, where the multiplication function is given by

$$\mathcal{T}(k) = \sum_{p_{ln}} \frac{|\eta(R(p_{ln})^{-1}k)|^2}{\left(T(p_{ln})^{-1}R(p_{ln})^{-1}k\right)_3}. \tag{4.22}$$

This function has to be bounded above and below, i.e., there should exist a, b such that $0 < a \leq \mathcal{T}(k) \leq b < \infty$. As we have seen in Chapter 4 such $a, b > 0$ can be easily found for right choice of section σ and admissible vector η.

As mentioned in the Introduction of this thesis, the construction obtained here can be used to analyze signals on the sphere. This has potential applications to the analysis of geophysical data or medical brain imaging as well as other data which may be given on the sphere. The signals defined on the sphere have been considered before. B. Torresani [31] built continuous "position-frequency" representations of signals on spheres. His approach was based on rotations and modulations. In this thesis we obtained similar results using a group-theoretical approach. Our coherent states are associated with the unitary representations of the Euclidean group. Also a group-theoretical derivation of the continuous wavelet transform on S^2 was considered before [9]. In that work J.-P. Antoine and P. Vandergheynst used translations and dilations on the sphere. Their coherent states are associated to the representation of the Lorentz group $SO_o(3,1)$. The parameter space X in this case is the product of $SO(3)$ with $\mathbb{R}_*^+$.

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Appendix A

The Dirac Distribution

In many integrations in our work we have used the property

$$< \delta_{(x_0)}, f >= \int \delta(x - x_0) f(x) dx = \int \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iy(x-x_0)} dy f(x) dx = f(x_0), \quad \text{(A.1)}$$

so in this appendix we give short summary of the theory of distribution based on work of L. Schwartz [29], [30]. Let us define the vector space $\mathcal{D}$ as the space of complex functions on $\mathbb{R}^n$ which are infinitely differentiable and have bounded supports. Then a *distribution* T is a continuous linear functional on the vector space $\mathcal{D}$. This means that to each $\phi \in \mathcal{D}$, T assigns a complex number $T(\phi)$, also denoted by $< T, \phi >$, with the properties

$$T(\phi_1 + \phi_2) = T(\phi_1) + T(\phi_2),$$

$$T(\lambda\phi) = \lambda T(\phi), \text{ where } \lambda \text{ is any complex constant,}$$

If ϕ_j converges to ϕ as $j \to \infty$ in the sens of the topology of $\mathcal{D}$,

the complex numbers $T(\phi_j)$ converge to the complex number $T(\phi)$ as $j \to \infty$.

$$\tag{A.2}$$

One important example of a distribution we are interested in is the *Dirac distribution* δ. It is defined as

$$\delta : \mathcal{D} \to \mathbb{C}, < \delta, f >= f(0) \tag{A.3}$$

or $\delta_{(x_0)}$ as

$$< \delta_{(x_0)}, f >= f(x_0). \tag{A.4}$$

In one dimension we can write

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iy(x-x_0)} dy, \tag{A.5}$$

where $\delta(x - x_0)$ is the symbolic device called a "Dirac function" (not a "true" function), satisfying

$$\begin{aligned} &\delta(x - x_0) = 0, \text{ for } x \neq x_0, \\ &\delta(x - x_0) = \infty, \\ &\textstyle\int_{\mathbb{R}} \delta(x - x_0) dx = 1. \end{aligned} \tag{A.6}$$

Then for $f \in \mathcal{D}$:

$$< \delta_{(x_0)}, f >= \int \delta(x - x_0) f(x) dx = \int \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iy(x-x_0)} dy f(x) dx = f(x_0). \tag{A.7}$$