On Periodic and Markovian Non-homogeneous Poisson Processes and Their Application in Risk Theory

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Abstract

On Periodic and Markovian Non-homogeneous Poisson Processes
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Periodic non-homogeneous Poisson processes and Poisson models under Markovian environments are studied in this thesis. By accounting for periodic seasonal variations and random fluctuations in the underlying risk, these models generalize the classical homogeneous Poison risk model.

Non–homogenous Poisson processes with periodic claim intensity rates are proposed for the claim counting process of risk theory. We introduce a doubly periodic Poisson model with short and long–term trends. Beta–type intensity functions are presented as illustrations.

Doubly periodic Poisson models are appropriate when the seasonality does not repeat the exact same short–term pattern every year, but has a peak intensity that varies over a longer period. This reflects periodic environments like those forming hurricanes, in alternating El Niño/La Niña years. The properties of the model and the statistical inference of the model parameters are discussed. An application of the model to the dataset of Atlantic Hurricanes Affecting the United States (1899-2000) is discussed in detail.

Further we introduce a periodic regime–switching Cox risk model by considering both, seasonal variations and stochastic fluctuations in the claims intensity. The intensity process, governed by a periodic function with a random peak level, is proposed.

The periodic intensity function follows a deterministic pattern in each short–term period, and is illustrated by a beta–type function. A finite–state Markov chain defines the level process, explaining the random effect due to different underlying risk years.

The properties of this regime–switching claim counting process are discussed in detail. By properly defining the Lundberg coefficient, Lundberg–type bounds for finite time ruin probabilities in the two–state risk model case are derived. A detailed derivation of the likelihood function and the maximum likelihood estimates of the model parameters is also given. Statistical applications of the model to the Atlantic hurricanes affecting the United States dataset are discussed under two different level classifications schemes.

The Markov-modulated risk model is considered to reflect a risk process or insurance business alternating between a finite number of Poisson models. Here we assume that the claim inter–arrivals, claim severities and premiums of the model are influenced by an external Markovian environment. The effect of this external environment may be characterized, at any time, by a state variable, representing for example, certain types of epidemics, a variety of weather conditions or of different states of the economy.

The ruin problem in the case of a two–state Markov–modulated risk model is studied. Given the initial state of the environment, systems of Laplace transforms of the non–ruin probabilities and the distributions of the severity of ruin are established. Then explicit formulas for these probabilities are derived, when the initial reserve is zero, or when both claim severity distributions are from the rational family. Numerical illustrations are also given.

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Notation and List of Symbols

$\mathbb{R}$	$(-\infty, \infty)$
$\mathbb{R}^+$	$(0, \infty)$
$\mathbb{C}$	complex plane
$\mathbb{N}$	set of non-negative integers
$\mathbb{N}^+$	set of positive integers
Ω	the whole sample space
$\mathcal{A}$	sigma algebra on Ω
$\Re(s)$	real part of a complex number s
NHP	non–homogeneous Poisson process
C.NHP.	compound non–homogeneous Poisson process
HDF	hazard distribution function
PHDF	periodic hazard distribution function
G3B(p, q, ϵ)	generalized 3-parameter beta function
$\mathcal{R}_f^+$	family of the rational distributions on $\mathbb{R}^+$
K_n	family of distributions, a subclass of $\mathcal{R}_f^+$
r.v.	random variable
i.i.d.	independent and identically distributed
$\bar{F}(x)$	$1 - F(x)$, the tail of distribution function F

pdf	probability density function of a r.v.
cdf	cumulative distribution function of a r.v.
pgf	probability generating function
mgf	moment generating function
$\hat{f}(s)$	Laplace transform of f
$f * g(t)$	convolution of functions f and g at t
MLE	maximum likelihood estimation
$\square$	end of proof

Introduction

Consider the risk process

$$U(t) = u + c\,t - \sum_{j=1}^{N(t)} X_j\ , \qquad t \geq 0\ , \tag{i}$$

where u is the initial value, c is the (constant) premium rate, $\{N(t);\, t \geq 0\}$ is a point process which models the number of claims arriving within the time interval $[0, t)$ and X_j is the j-th claim size. When $\{N(t);\, t \geq 0\}$ is a Poisson process with (constant) intensity λ and the claim sizes X_j, $j \geq 1$, are i.i.d. and independent of N, then (i) is known as the classical (homogeneous Poisson) risk model, which has been investigated extensively in the actuarial literature.

The classical risk model is not realistic in some practical situations. Two main modifications are made here. First, a non–homogeneous Poisson (NHP) process is used to model "size fluctuations" in the claim intensity of a risk subject to seasonality. Then, a Cox process, also called doubly stochastic Poisson process and a natural extension of the NHP process, is used to characterize the underlying "risk fluctuations" in the claims intensity [see Grandell (1991)].

As a more general time–dependent model, NHP processes are introduced to model claim frequency. Since their intensity rate $\lambda(t)$ is a function of time, the corresponding claim counting process has independent but not necessarily stationary increments. Due to this additional mathematical difficulty, the actuarial literature on NHP processes is rather limited, compared to that of the classical homogeneous Poisson process.

NHP processes are commonly used in non–actuarial applications, especially in the reliability analysis of repairable systems. For instance, an age model is proposed in Guo and Love (1992) for modeling imperfect repairable systems operating under a NHP environment. Maximum likelihood estimates are also derived for the parameters of interest. Recently, Saldanha et al. (2001) presents an application of the NHP point process to the reliability analysis of service water pumps of a nuclear power plant. Pulcini (2001) also deals with the reliability modeling of the failure process of large and complex repairable equipment, whose failure intensity shows a bathtub type non–monotonic behavior, by using a NHP process approach.

For actuarial applications of NHP processes, Berg and Haberman (1994) use a non–homogeneous Markov birth process, of which the NHP is a special case, to predict trends such as the time to the next claim or the expected total number of claims in a year for a life insurance portfolio. The predictions are obtained by employing Bayesian revision procedures for inference models on the claim occurrence process.

Beard et al. (1984) and Daykin et al. (1994) claim that the risk process is often subject to continual changes in risk propensity. The long–term systematic slow–changing trends, more or less irregular cyclical up– and downturns and the short–term random variations affect the number of claims. The model to be employed then is to consider a suitably defined time–dependent function $\lambda(t)$ or a stochastic process $\{\lambda(t);\ t \geq 0\}$, instead of the Poisson parameter λ. These authors propose a formula of the type $\lambda(t) = \lambda_g(t)[1 + d(t)]\mathbf{q}(t)$, which may be appropriate to describe the effect on the Poisson parameter for year t, where $\lambda_g(t)$ is the trend adjusted expected number of claims in year t, $d(t)$ introduces deviations from the normal trend and $\mathbf{q}(t)$ indicates short–term variation of the risk propensity.

In practice, natural phenomena evolving in a periodic environment, or under seasonal conditions, affect insurance claims. For example, weather factors are known to affect automobile or fire insurance claims, while seasonal snow storms in the north and hurricanes or floods in the south affect property–casualty insurance. A periodic time–dependent intensity rate is a reasonable model for the claim frequency in such situations. With some types of intensity function, it can also be tractable, even for the corresponding aggregate claim process.

The similarities between intensity and failure rate functions, used in reliability maintenance models under a minimal repair policy, help exploring different applications of NHP processes. Chukova et al. (1993) shows that a random variable X with almost–lack–of–memory

$$P\{X > x + c \mid X > c\} = P\{X > x\}, \qquad \text{for some } c, \text{ and all } x,$$

has a periodic hazard rate (intensity) function of period c, $h_X(t) = \frac{f_X(t)}{\bar{F}_X(t)}$, for $t > 0$. Obvious applications in risk theory are to model random phenomena with seasonal effects; car accidents, hurricanes. Some characterization properties of the NHP process with periodic failure rate are derived in Chukova et al. (1993) and Dimitrov et al. (1997). Applications of these models to environmental evolution under a periodic behavior are also considered by Dimitrov et al. (1998), for an overall population growth model in a periodic random environment, in Dimitrov and Khalil (1992), and by Dimitrov et al. (1996) for a random process in an environment with a double–periodic structure.

Garrido et al. (1996) and Dimitrov et al. (2000) exploit the corresponding properties in a risk model, where the claim intensity rates are modeled by a NHP process with (single) periodic intensity. Some properties of such processes are considered and possible forms of the intensity function over a period, like the beta–shape, are proposed. Morales (2004) further explores the single periodic NHP model by defining a Gaussian intensity with which he considers the problem of ruin through a simulation study. While Schmidli (2003) considers a NHP process to price catastrophe PCS options.

Furthermore, Lu (2001) and Garrido and Lu (2004) consider a model with a double periodic intensity rate, where periodicity does not repeat the exact same pattern in each short–term period, rather its peak intensity varies over a longer period. This model reflects periodic environments like those forming hurricanes, in alternating El Niño/La Niña years. Parametric forms of the doubly periodic intensity function, like the double–beta and the sine–beta, are proposed in Chapter 2.

There are surprisingly few results for the compound NHP process in the risk theory literature. A compound NHP process with periodic claim intensity rate, called periodic risk model, is considered by Dimitrov et al. (2000). Similar models are also considered by Dassios and Embrechts (1989), and by Asmussen and Rolski (1991, 1994) and Rolski et al. (1999). Dassios and Embrechts (1989) discusses a class of risk models with periodic claim arrival intensity by using the theory of piecewise–deterministic Markov processes, together with some standard martingale techniques. The other authors derive two–sided bounds and an asymptotic formula for the ruin probability by using an average arrival rate risk model.

Besides the reliability area, from the literature we can see also the application of NHP processes in different fields. A NHP process with a periodic intensity function is used in Parisi and Lund (2000) to model the annual cycle of hurricane arrival times. They study the annual arrival cycle and return period properties of Atlantic Basin hurricanes landfalling. Lu and Garrido (2004a) propose in Chapter 3 to use a NHP model with double periodicity to fit a large data set: the Atlantic hurricanes affecting the United States, 1899 through 2000. The main differences between Parisi and Lund's model and the model proposed in Lu and Garrido (2004a), in considering the seasonal effects on the hurricane arrival times, is that the latter consider global climatological effects (El Niño, La Niña) through a double periodic model.

To have NHP models fit real data more closely, one needs to estimate or approximate their intensity function. If this intensity function is given in a parametric form, like the double–beta ones proposed in Chapter 2, the powerful maximum likelihood method may be used to estimate the unknown parameters. Helmers and Zitikis (1999) present a computationally simple algorithm for estimating the intensity function of a Poisson process with exponential quadratic and cyclic of fixed frequency trends.

Helmers et al. (2003) construct a consistent kernel–type nonparametric estimator of the intensity function of a cyclic Poisson process when the period is unknown, while Helmers et al. (2005) further investigate its statistical properties.

As for the Cox risk model, an early reference to it is Ammeter (1948). In his model, the intensity λ_k over time intervals $[(k-1)\Sigma, k\Sigma)$ of (fixed) length Σ, for $k \in \mathbb{N}^+$, forms an i.i.d. sequence $\{\lambda_k;\, k \geq 0\}$. This model is generalized by Björk and Grandell (1988), who consider the intensity $\lambda(t) = L_i$ if $\Sigma_{i-1} \leq t < \Sigma_i$, where $\Sigma_i = \sigma_1 + \cdots + \sigma_i$ (denoting $\Sigma_0 = 0$) and (L_i, σ_i) is a sequence of i.i.d. random vectors. Ammeter's model is revisited by Grandell (1995) and more properties of the model are derived.

Ruin probabilities have been studied for these Cox models with a piecewise constant intensity. Lundberg inequalities hold, provided some assumptions are fulfilled. Björk and Grandell (1988) derived by a "martingale approach" a general (infinite–time) Lundberg inequality when the occurrence of the claims is described by a Cox process. They applied their general result to the "independent jump intensity" and "Markov renewal intensity" cases and got fairly explicit results. Grandell (1991) derived finite–time Lundberg inequalities in the case of Markovian intensities. Embrechts et al. (1993) extend the finite–time results for Markovian intensities to non–Markovian intensities within the classes mentioned. However, these may not be practical due to the difficulty in estimating the Lundberg coefficient and evaluating some constants within the inequalities.

Asmussen (1989) proposes a Cox risk model, called a Markov–modulated Poisson process, whose intensity process $\{\lambda(t);\, t \geq 0\}$ is given by $\lambda(t) = \lambda_{J(t)}$. Here the process $\{J(t);\, t \geq 0\}$ models the random environment of an insurance business and is assumed to be an irreducible continuous time Markov chain, with finite state space. This is a first attempt to define a risk model using an environmental Markov chain $\{J(t);\, t \geq 0\}$. The Markov–modulated risk model was first introduced by Janssen (1980) and also treated in Janssen and Reinhard (1985) and Reinhard (1984). Models of this type have also been investigated, e.g., by Rolski (1981), Asmussen et al. (1995), Bäuerle (1996) and Snoussi (2002). A survey of methods for statistical estimation of the parameters of Markov–modulated Poisson processes is given in Rydén (1994).

Reinhard (1984) considers the probability of ruin in a class of Markov–modulated risk models, in which the claim frequencies, amounts and premiums are influenced by an external Markovian environment process. Bäuerle (1996) has been interested in the expected ruin time of the same model. Wu (1999) develops generalized bounds, while Schmidli (1997) studies the estimation of the Lundberg coefficient, for the probability of ruin under a Markov–modulated risk model.

Snoussi (2002) studies the severity of ruin for the Markov–modulated risk models. An explicit formula is derived for the severity of ruin in the particular case where one has two possible states for the environmental process and where the amounts of claims are exponentially distributed.

Dickson and Waters (1992) present algorithms to calculate the probability and severity of ruin in both finite and infinite time for a discrete time risk model. The severity of ruin in the discrete risk model was also considered by Reinhard (1997). More recently, Reinhard and Snoussi (2002, 2004) extend the results in a discrete semi–Markov risk model. Other references are Dickson et al. (1995) and Gerber (1988).

Reinhard and Snoussi (2001, 2002) have also discussed the probability of ruin and the distribution of the surplus prior to ruin in a discrete semi–Markov risk model, respectively, where it is assumed that the claims are influenced by a Markov chain.

Furthermore, a Cox risk process with a piecewise constant intensity is considered by Schmidli (1996), where the sequence of successive levels of the intensity forms a Markov chain. Jasiulewicz (2001) considers the probability of ruin under the influence of a premium rate which varies with the level of free reserves, while Wu and Wei (2004) investigates the same problem but the premium rate varies according to the intensity of claims, in a Markovian environment.

There are very few results in the risk theory literature regarding Cox processes with other than piecewise constant intensities. Recently, Schmidli (2003) suggests a NHP model with doubly stochastic occurrences for the PCS catastrophes index, based on individual indices for PCS options, where the intensity is of the form $\Lambda\lambda(t)$, with Λ is stochastic and $\lambda(t)$ is a deterministic function.

Some natural phenomena evolve in a seasonal environment subject to random fluctuations which, in turn, affect insurance claims. For example, tropical storms and hurricanes periodically affect the coastal US states along the Atlantic and the Gulf of Mexico. The claim intensity then forms a specific pattern for each year which can be modeled by a periodic function. Speculation exists regarding the significance and potential effects of the El Niño phenomenon on hurricane frequency and the strength attained by tropical cyclones during alternating El Niño/La Niña years. These are random effects that, in some sense, affect the risk propensity or the peak level of the seasonal intensity, which can be modeled by a stochastic process.

Lu and Garrido (2004b) propose a Cox model that accounts for both, the seasonal variations and the random fluctuations in the claims intensity. An intensity process given by $\lambda(t) = \lambda_S(t)\mathbf{q}(t)$, for $t \geq 0$, is considered, where $\lambda_S(t)$ is the short–term intensity function with periodicity and $\mathbf{q}(t)$ is a stochastic (level) process. A Markov chain with finite states, corresponding to different levels, is chosen for the level process, yielding a so called regime–switching process. By properly choosing the Lundberg coefficient, Lundberg–type upper bounds for finite time ruin probabilities in a two–state case are derived. This two–state Cox process model with periodicity mirrors the underlying risk on the claim under, say "normal" or "extraordinary" conditions, and is akin to the regime–switching models used in finance.

In this thesis, we consider the generalizations and modifications of the classical Poisson model, both in "size fluctuations" and "risk fluctuations" in the claims intensity, as well as their application. The NHP processes with double periodicity and the periodic regime–switching Poisson models under Markovian environments are introduced. Their properties on risk theory and statistical inferences are discussed. The ruin probability and the severity of ruin are studied for Markov–modulated Poisson risk models.

Chapter 1 first reviews the literature on the NHP processes including the claim counting processes with periodicity, the compound claim processes with periodicity and the ruin models under a periodic environment. Then it reviews the literature on the Cox models with different intensity processes, especially the results on ruin probabilities and the severity of ruin in the Markov–modulated risk models.

Chapter 2 proposes NHP processes with periodic claim intensity rate as the claim counting process of risk theory. We introduce a doubly periodic Poisson model with short and long term trends, illustrated by a double–beta intensity function. Here periodicity does not repeat the exact same short term pattern every year, but lets its peak intensity vary over a longer period, reflecting periodic environments like those forming hurricanes, in alternating El Niño/La Niña years. The properties of the model are derived.

Chapter 3 gives the statistical inference on NHP models with double periodicity introduced in Chapter 2. Beta–type intensity functions are presented as illustrations. The likelihood function and the maximum likelihood estimates, as well as the estimated standard deviations of the model parameters, are derived. Finally, an application of the model to the dataset of Atlantic Hurricanes Affecting the United States (1899–2000) is discussed in detail.

Chapter 4 studies a Cox risk model that accounts for both, seasonal variations and random fluctuations in the claims intensity. More precisely, we define an intensity process, governed by a periodic function with a random peak level. The periodic intensity function follows a deterministic pattern in each short–term period, and is illustrated by a beta–type function. A finite state Markov chain defines the level process, yielding a periodic regime–switching Poisson process. The properties of the corresponding claim counting process are discussed. By properly defining the

Lundberg coefficient, Lundberg–type bounds for finite time ruin probabilities in a two–state case are derived.

Chapter 5 gives the statistical analysis for the periodic regime–switching Poisson models. The likelihood function and the maximum likelihood estimates of the model parameters are derived. An application of the model to the dataset of Atlantic Hurricanes Affecting the United States (1899-2000) is discussed.

Chapter 6 considers a Markov-modulated risk model in which the claim inter–arrivals, claim sizes and premiums are influenced by an external Markovian environment process. A system of Laplace transforms of non–ruin probabilities, as well as the probabilities of the severity of ruin, given the initial environment state, are established from a system of integro–differential equations. In the two–state model, explicit formulas for non–ruin probabilities and probabilities of the severity of ruin are obtained when the initial reserve is zero or when both claim size distributions are from the rational family. Examples with exponentially distributed claim sizes, as well as Erlang and mixture of exponentials, are given.

Chapter 1

Review of the literature

1.1 Non-homogeneous Poisson Model

1.1.1 NHP processes with periodicity

Homogeneous Poisson processes are commonly used in risk theory to model claim frequency. These sometimes give a crude representation since their claim intensity rate λ is constant. Time–dependent non–homogeneous Poisson (NHP) processes are proposed to generalize these models, as their intensity rate $\lambda(t)$ is a function of time. The NHP claim counting process has independent but not necessarily stationary increments. Due to this additional mathematical difficulty, the actuarial literature on NHP processes is rather limited, compared to that on the classical homogeneous Poisson process.

First we review the definition of the NHP process and study its properties, especially under periodicity.

Let λ be a non–negative (measurable and locally integrable) deterministic function. Consider the number of claims in the time interval $[s,\, t)$, denoted $N_{[s,\, t)}$ for $0 \leq s < t$ (simply $N(t)$ when $s = 0$). A NHP process is defined as follows [see for example De Vylder (1996)].

Definition 1.1 [**NHP process**] A counting process $\{N(t);\ t \geq 0\}$ is said to be non–homogeneous Poisson (NHP) with intensity function λ, where $\lambda(t) \geq 0$ for $t \geq 0$, if it satisfies:

(a) $N(t) = 0$ at $t = 0$,

(b) $\{N(t);\ t \geq 0\}$ has independent increments,

(c) $P\{N_{[t,\,t+h)} = 1\} = \lambda(t)h + o(h)$, for all $t,\ h \geq 0$,

(d) $P\{N_{[t,\,t+h)} \geq 2\} = o(h)$, for all $t,\ h \geq 0$.

The function Λ defined by

$$\Lambda(t) = \int_0^t \lambda(v)dv\,, \qquad t \geq 0\,, \tag{1.1}$$

is called the cumulative hazard function or the cumulative intensity function of the process.

Consider the claims count, $N_{[\tau,\,\tau+t)}$, in an interval of the form $[\tau,\ \tau + t)$, where $\tau, t \geq 0$. The time parameter τ, called the initial age of the process, marks the beginning of the time observation period when claims start to be counted. It is well known [De Vylder (1996)] that for a NHP process the probability of n claims occurring in a time interval of duration t starting at time τ is given by

$$P\{N_{[\tau,\,\tau+t)} = n\} = \frac{e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}[\Lambda(\tau + t) - \Lambda(\tau)]^n}{n!}\,, \qquad n \in \mathbb{N}\,. \tag{1.2}$$

That is, for a NHP process with intensity function λ, $N_{[\tau,\,\tau+t)}$ has a Poisson distribution with mean $\Lambda(\tau + t) - \Lambda(\tau) = \int_\tau^{\tau+t} \lambda(v)dv$.

A NHP process reduces to the classical homogeneous Poisson process when its intensity function does not depend on time, i.e. $\lambda(t) = \lambda$ for all $t \geq 0$, and therefore $\Lambda(t) = \lambda t$ is linear.

Beard et al. (1984) and Daykin et al. (1994) claim that the risk process is often subject to continual changes in risk propensity. This is true for both, the long–term, systematic, slow–changing trends, as well as the short–term random variations that affect the number of claims. The model to be employed then must consider a suitably defined time–dependent function or a stochastic process $\{\lambda(t);\, t \geq 0\}$, instead of a constant Poisson parameter λ.

In practice, many natural phenomena evolve in a periodic environment or under seasonal conditions. In turn, these events generate insurance claims. For example, weather factors are known to affect automobile or fire insurance claims, while seasonal snow storms in the north and hurricanes or floods in the south affect property insurance. A periodic time–dependent intensity rate is a reasonable model for the claim frequency in such situations. With some types of intensity function, it can also be tractable, even for the corresponding aggregate claim process.

We correspondingly consider the case where the risk process evolves in a periodic environment, as when the claim arrival rate may depend on the seasons. Then the intensity function of a NHP claim counting process $\{N(t);\, t \geq 0\}$ is a periodic function, say with a period of $c \in \mathbb{N}^+$ years. Consequently $t - \lfloor \frac{t}{c} \rfloor c \in [0, c)$ is the time of the season, where $\lfloor t \rfloor$ is the integer part of $t \geq 0$.

In fact, NHP processes are frequently used as reliability maintenance models under a minimal repair policy. The similarities between failure rate functions and NHP intensities helped explore different applications of NHP processes. Chukova et al. (1993) and Dimitrov et al. (1997) study NHP processes generated by distributions with (single) periodic failure rates and derive some characterization properties. Applications of these models to environmental evolution under a periodic behavior are also considered by Dimitrov and Khalil (1992), by Dimitrov et al. (1998) for an overall population growth model in a periodic random environment, and by Dimitrov et al. (1996) for a random process in an environment with a double–periodic structure.

We list here the following definitions and properties by Chukova et al. (1993) and Dimitrov et al. (1997) for the NHP process $\{N(t);\, t \geq 0\}$ with periodic failure rate.

Definition 1.2 **[Failure rate function]** The failure rate function is defined as

$$\lambda_Y(t) = \frac{f_Y(t)}{\bar{F}_Y(t)}, \qquad t \geq 0,$$

where f_Y and $\bar{F}_Y = 1 - F_Y$ are the underlying probability density function (pdf) and the tail of the cumulative distribution function (cdf) of the random variable (r.v.) Y, respectively.

Definition 1.3 **[Periodic failure rate function]** A r.v. Y has a Periodic Failure Rate Function (PFRF) of period $c > 0$ if and only if the following holds:

$$\lambda_Y(kc + t) = \lambda_Y(t), \qquad t \geq 0, \; k \in \mathbb{N}^+.$$

Lemma 1.1 If the NHP process $\{N(t); t \geq 0\}$ has periodic failure rate function $\lambda_Y(t)$ with period $c > 0$, then for any fixed integer $k \geq 0$, its cumulative hazard function Λ has the property

$$\Lambda(kc + t) = \Lambda(kc) + \Lambda(t) = k\Lambda(c) + \Lambda(t), \qquad t \geq 0. \tag{1.3}$$

Proof. See Chukova et al. (1993). □

Remark 1.1 (1.3) is called "almost linearity", a property of the cumulative hazard function Λ.

Lemma 1.2 Under the conditions of Lemma 1.1, for all $t \geq 0$ and any fixed integer $k \geq 0$, we have

(a) $P\{N_{[kc,\, kc+t)} = n\} = P\{N(t) = n\}$, for $n \in \mathbb{N}$.

(b) The random variables $N_{[kc,\, kc+t)}$ and $N(kc)$ are mutually independent.

Proof. See Chukova et al. (1993). □

Theorem 1.1 A NHP process $\{N(t); t \geq 0\}$ has periodic failure rate $\lambda(t)$ of period $c > 0$, if and only if the following two properties hold:

(a) For a fixed constant c and an arbitrary $t \geq 0$ it is true that

$$P\{N_{[c,\,c+t)} = n\} = P\{N(t) = n\}\,, \qquad n \in \mathbb{N}\,.$$

(b) The random variables $N(c)$ and $N_{[c,\,c+t)}$ are mutually independent, for any $t \geq 0$.

Proof. See Chukova et al. (1993). □

Corollary 1.1 If $\{N(t);\, t \geq 0\}$ is a NHP process with periodic failure rate function of period $c > 0$, then when $t > c$ the random variable $N(t)$ has the following decomposition:

$$N(t) = M_1 + M_2 + \cdots + M_{\lfloor \frac{t}{c} \rfloor} + N(t - \lfloor \frac{t}{c} \rfloor c)\,, \qquad \text{if } t > c\,,$$

where $\{M_i\}_{i \geq 1}$ are i.i.d. Poisson random variables of parameter $\Lambda(c) = \int_0^c \lambda(v)dv$, independent of the component $N(t - \lfloor \frac{t}{c} \rfloor c)$, itself a Poisson process of hazard function $\Lambda(y) = \int_0^y \lambda(v)dv$, for $y = t - \lfloor \frac{t}{c} \rfloor c \in [0,\, c)$.

Proof. See Chukova et al. (1993). □

Corollary 1.1 gives a clear and practical decomposition for the use of NHP processes with periodic failure rates. For time instant t that are integer multiples of c, i.e. $t = nc$, the process $N(t)$ is equivalent to the sum of n identically distributed independent Poisson processes. This property is not equivalent to the infinite divisibility of the time–homogeneous Poisson process but is analogous to it.

The probability distribution of $N(t)$ with a periodic failure rate is discussed in Dimitrov et al. (1997). The random variables under consideration can be either the number of events within a given time interval or the waiting time until some event occurs. This is also one of the main probability properties of the NHP process with periodic failure rate or periodic intensity function. Let Y be a non–negative r.v. with cdf F_Y and corresponding pdf f_Y, if it exists.

Definition 1.4 [Hazard distribution function] The hazard distribution function (HDF) $\Lambda_Y(t;\, y)$ is defined as the conditional cdf for the remaining lifetime of Y, given that $Y \geq y$, i.e., for any $y > 0$,

$$\Lambda_Y(t;\, y) = P\{Y - y < t \mid Y \geq y\} = \frac{F_Y(y+t) - F_Y(y)}{\bar{F}_Y(y)}, \qquad t > 0.$$

Definition 1.5 [Periodic hazard distribution function] A r.v. Y has a periodic hazard distribution function (PHDF) of period $c > 0$ if its HDF $\Lambda_Y(t;\, y)$ is periodic with respect to y, i.e., when $c > 0$

$$\Lambda_Y(t;\, y + c) = \Lambda_Y(t;\, y), \qquad y, t \geq 0.$$

Theorem 1.2 A r.v. Y has a PHDF of period $c > 0$ if and only if $F_Y(y)$ has the form

$$F_Y(y) = 1 - \alpha^{\lfloor \frac{y}{c} \rfloor}[1 - (1-\alpha)F_Z(y - \lfloor \frac{y}{c} \rfloor c)], \qquad y \geq 0,$$

where $\alpha \in (0, 1)$ and F_Z is a cdf with support on the interval $[0, 1)$.

Proof. See Dimitrov et al. (1997). □

Corollary 1.2 A r.v. Y with probability density f_Y has a PHDF of period $c > 0$ if and only if it has PFRF of period c and its pdf is of the form

$$f_Y(y) = (1-\alpha)\alpha^{\lfloor \frac{y}{c} \rfloor} f_Z(y - \lfloor \frac{y}{c} \rfloor c), \qquad y \geq 0,$$

where α and Z are specified as in Theorem 1.2, and

$$\alpha = \exp\left\{ -\int_0^c \lambda_Y(u)du \right\}, \qquad f_Z(z) = \frac{\lambda_Y(z)}{1-\alpha} e^{-\int_0^z \lambda_Y(v)dv}, \qquad z \in [0,\, c).$$

Proof. See Dimitrov et al. (1997). □

Chukova et al. (1993), Garrido et al. (1996) and Dimitrov et al. (2000) exploit the corresponding properties in a risk model, where the claim intensity rates are modeled by a NHP process with (single) periodic intensity. A family of beta intensity functions is proposed to model claim counting processes on a single period of time (short–term), for instance a year. Their motivation lies in the great flexibility of shape of the beta function in modeling different practical situations where periodicity occurs. Some properties of such processes, illustrated by a beta–shape periodic intensity function, are discussed and listed below.

Let

$$\lambda_1(t) = \lambda^* t^{p-1}(1-t)^{q-1}, \qquad \lambda^*, p, q > 0,\ t \in [0,1), \tag{1.4}$$

be the intensity of occurrence of insurance events at time $t \in [0,1)$, where λ^* is the level of the intensity. Then $\lambda_1(t - \lfloor t \rfloor)$, for $t \geq 0$, is a periodic function with a period of 1 year. Further assume that λ_1 is the claim intensity of one policyholder, then for K similar contracts in the portfolio at the beginning of the year, the corresponding intensity function is

$$\lambda(t) = K\lambda_1(t - \lfloor t \rfloor), \qquad t \geq 0, \tag{1.5}$$

which is also a periodic function of period 1.

Under this beta–shape periodic intensity assumption, explicit formulas are obtained for the distribution of the number of claims and total claims within a period or within an arbitrary time interval. Some characteristic properties of this corresponding claim counting process are also derived as follows.

(a) $N_{[\tau,\tau+t)}$ has a Poisson distribution with parameter $\Lambda(\tau+t)-\Lambda(\tau) = \int_\tau^{\tau+t} \lambda(v)dv$. Hence the probability of total number of claims within the time interval $[\tau,\ \tau+t)$ is:

$$P\{N_{[\tau,\tau+t)} = n\} = \frac{[\Lambda(\tau+t) - \Lambda(\tau)]^n}{n!} e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}, \qquad n \in \mathbb{N},$$

where $\lambda(t)$ is given in (1.5), while the probability of total number of claims within a policy year $[0,\ 1)$ is:

$$P\{N(1) = n\} = \frac{[K\,\lambda^*\,B(p,\,q)]^n}{n!} e^{-[K\,\lambda^*\,B(p,q)]}, \qquad n \in \mathbb{N},$$

where

$$B(p,\,q) = \int_0^1 v^{p-1}(1-v)^{q-1}dv = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad p, q > 0, \tag{1.6}$$

is the (complete) beta function.

(b) The moment generating function (mgf) of $N_{[\tau,\tau+t)}$ is obtained as

$$M_{N_{[\tau,\,\tau+t)}}(s) = E\big[e^{s\,N_{[\tau,\,\tau+t)}}\big] = e^{[\Lambda(\tau+t)-\Lambda(\tau)](e^s-1)}\,, \qquad s \in \mathbb{R},$$

and similarly the mgf of $N(1) = N_{[0,1)}$ is

$$M_{N(1)}(s) = E\big[e^{s\,N(1)}\big] = e^{\Lambda(1)(e^s-1)} = e^{K\,\lambda^*\,B(p,q)(e^s-1)}\,, \qquad s \in \mathbb{R}\,.$$

(c) The expected number of claims during the year coincides with its variance and is expressed by

$$E\big[N(1)\big] = V\big[N(1)\big] = K\,\lambda^*\,B(p,\,q)\,,$$

while the expected number of claims during any period $[\tau,\,\tau+t)$ within the year is

$$E\big[N_{[\tau,\tau+t)}\big] = K\,\lambda^*\big[B(p,\,q;\,\tau+t) - B(p,\,q;\,\tau)\big]\,, \qquad \tau \in [0,\,1),\ \tau+t \in (0,\,1]\,,$$

where

$$B(p,\,q;\,t) = \begin{cases} 0\,, & \text{if } t \le 0 \\ \int_0^t v^{p-1}\,(1-v)^{q-1}\,dv\,, & \text{if } t \in (0,\,1) \\ B(p,\,q)\,, & \text{if } t \ge 1 \end{cases}\,, \tag{1.7}$$

is the usual incomplete beta function.

(d) The probability to "survive" the time interval $[\tau,\,\tau+t)$ without a claim is

$$P\{N_{[\tau,\tau+t)} = 0\} = e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}\,, \qquad t \ge 0\,,$$

while for a one–year period it is

$$\alpha = e^{-\Lambda(1)} = e^{-K\,\lambda^*\,B(p,q)}\,. \tag{1.8}$$

(e) The distribution of the waiting time T_1 for the first claim is obtained directly by

$$\begin{aligned} F_{T_1}(t) &= P\{T_1 \le t\} = 1 - P\{N(t) = 0\}, \\ &= 1 - e^{-\Lambda(t)} = 1 - \alpha^{\lfloor t \rfloor} e^{-K\,\lambda^*\,B(p,q;\,t-\lfloor t \rfloor)}, \qquad t \ge 0, \end{aligned}$$

where α is given in (1.8), or alternatively by Theorem 1.2,

$$\begin{aligned} F_{T_1}(t) &= 1 - \alpha^{\lfloor t \rfloor}\left[1 - (1-\alpha) F_Z(t - \lfloor t \rfloor)\right] \\ &= 1 - \alpha^{\lfloor t \rfloor}\left[1 - (1-\alpha)\frac{F_{T_1}(t - \lfloor t \rfloor)}{F_{T_1}(1)}\right] \\ &= 1 - \alpha^{\lfloor t \rfloor} P\{N_{t - \lfloor t \rfloor} = 0\} \\ &= 1 - \alpha^{\lfloor t \rfloor} e^{-K\lambda^* B(p,q;t-\lfloor t \rfloor)}, \qquad t \geq 0, \end{aligned}$$

where F_Z is the cdf with support on the interval $[0,1)$ and clearly $F_{T_1}(1) = 1 - \alpha$. Thus, the pdf of T_1 is

$$f_{T_1}(t) = \alpha^{\lfloor t \rfloor} e^{-K\lambda^* B(p,q;t-\lfloor t \rfloor)} \lambda(t - \lfloor t \rfloor), \qquad t \geq 0,$$

and the expectation of T_1 is derived as

$$\begin{aligned} E[T_1] &= \frac{\alpha + K\lambda^* \int_0^1 t^p (1-t)^{q-1} e^{-K\lambda^* B(p,q;t)} dt}{1-\alpha} \\ &= \frac{\int_0^1 e^{-K\lambda^* B(p,q;t)} dt}{1-\alpha}. \end{aligned}$$

Berg and Haberman (1994) use a non–homogeneous Markov birth process, of which the NHP is a special case, to predict trends such as the time to the next claim or the expected total number of claims in a year in life insurance claim occurences.

1.1.2 Compound NHP processes with periodic intensity

Consider a NHP claim counting process $\{N(t);\, t \geq 0\}$. Then the corresponding aggregate claims process

$$S(t) = \begin{cases} \sum_{n=1}^{N(t)} X_n & \text{if } N(t) > 0 \\ 0 & \text{if } N(t) = 0 \end{cases}, \qquad t \geq 0,$$

is called a compound NHP process and is denoted as $S(t) \sim$ C.NHP.$[\Lambda;\, F_X]$, for $t \geq 0$. The X_n are i.i.d. claim severities, with common cdf F_X and finite mean μ, and are (mutually) independent of the Poisson r.v. $N(t)$ which has parameter $\Lambda(t)$.

Chukova et al. (1993), Garrido et al. (1996) and Dimitrov et al. (2000) proposed a compound NHP claim process with a periodic intensity function of arbitrary period $c > 0$. Let $\{X_n\}_{n\geq 1}$ be the sequence of recorded claim severities. Assume these severities are i.i.d. random variables with cdf F_X, independent of time. Then the aggregate claims accumulated over the time interval $[\tau, \tau + t)$ are given by

$$S_{[\tau,\tau+t)} = \sum_{n=1}^{N_{[\tau,\tau+t)}} X_n, \qquad t, \tau \geq 0, \tag{1.9}$$

where $N_{[\tau,\tau+t)}$ is supposed to be a NHP process with periodic intensity function λ and $S_{[\tau,\tau+t)} = 0$ if $N_{[\tau,\tau+t)} = 0$.

Theorem 1.3 The aggregate claim process $\{S(t);\, t \geq 0\}$ driven by a NHP process with periodic intensity rate of period $c = 1$ can be decomposed into

$$S(t) = S_1 + S_2 + \cdots + S_{\lfloor t \rfloor} + S(t - \lfloor t \rfloor), \qquad t \geq 0, \tag{1.10}$$

where $\{S_n;\, n \geq 1\}$ are i.i.d. random variables distributed as the compound Poisson sum $\sum_{n=1}^{N_1} X_n$ and $N(1)$ is a Poisson random variable of parameter $\Lambda(1) = \int_0^1 \lambda(v)dv$. The last term $S(t - \lfloor t \rfloor)$ in (1.10) is also a compound Poisson term, independent of the other S_n, but with a Poisson parameter $\Lambda(t - \lfloor t \rfloor) = \int_0^{t-\lfloor t \rfloor} \lambda(v)dv$.

Proof. See Chukova et al. (1993). □

Lu (2001) discusses the following general decomposition of the compound NHP process $S_{[\tau,\tau+t)}$ when the initial age of the process $\tau > 0$, given by (1.9), with periodic intensity function λ of period $c = 1$.

Theorem 1.4 Suppose that the intensity function λ is periodic of period 1, then the hazard function during the time period $[\tau,\, \tau + t)$, i.e. $\Lambda(\tau + t) - \Lambda(\tau)$, and the compound NHP process $S_{[\tau,\tau+t)}$ have the following representations:

(i) If t is an integer, independent of the initial age value τ, then

$$\Lambda(\tau + t) - \Lambda(\tau) = \Lambda(1)t\,, \qquad t \in \mathbb{N}^+$$

and $S_{[\tau,\tau+t)}$ can be decomposed as

$$S_{[\tau,\tau+t)} = S_1 + S_2 + \cdots + S_{\lfloor t \rfloor}, \qquad t \in \mathbb{N}^+ , \tag{1.11}$$

where all S_i's are i.i.d. random variables distributed as

$$S(1) = \sum_{n=1}^{N(1)} X_n ,$$

and $N(1)$ is a Poisson r.v. with parameter $\Lambda(1) = \int_0^1 \lambda(v)dv$.

(ii) If τ is an integer but t is not, then

$$\Lambda(\tau+t) - \Lambda(\tau) = \Lambda(1)\lfloor t \rfloor + \Lambda(t - \lfloor t \rfloor) , \qquad t > 0 ,$$

and the claim counting process $\{N_{[\tau,\tau+t)};\, t \geq 0\}$ is equivalent to the process which has the same time period but starts from $\tau = 0$, i.e. $\{N(t);\, t \geq 0\}$. Thus $S_{[\tau,\tau+t)}$ can be decomposed as

$$S_{[\tau,\tau+t)} = S_1 + S_2 + \cdots + S_{\lfloor t \rfloor} + S(t - \lfloor t \rfloor), \qquad t > 0 ,$$

where the last term is also a compound Poisson r.v., with parameter $\Lambda(t - \lfloor t \rfloor)$ for $t - \lfloor t \rfloor \in [0,\, 1)$, independent of other aggregate claims S_i's.

(iii) If neither τ nor t are integers, then

$$\Lambda(\tau+t) - \Lambda(\tau) = \begin{cases} \Lambda(1)\lfloor t \rfloor + \Lambda(\tau - \lfloor \tau \rfloor + t - \lfloor t \rfloor) - \Lambda(\tau - \lfloor \tau \rfloor) \\ \qquad\qquad\qquad\qquad \text{if } \lfloor \tau \rfloor + 1 - \tau \geq t - \lfloor t \rfloor \\ \Lambda(1)(\lfloor t \rfloor + 1) + \Lambda(\tau - \lfloor \tau \rfloor + t - \lfloor t \rfloor - 1) - \Lambda(\tau - \lfloor \tau \rfloor) \\ \qquad\qquad\qquad\qquad \text{otherwise} \end{cases} ,$$

and $S_{[\tau\,\tau+t)}$ may contain one or two incomplete terms.

When $\lfloor \tau \rfloor + 1 - \tau \geq t - \lfloor t \rfloor$, only one incomplete term appears as if aggregating claims on the time interval $[\tau - \lfloor \tau \rfloor,\, \tau - \lfloor \tau \rfloor + t - \lfloor t \rfloor) \subset [0,\, 1)$. In this case, $S_{[\tau,\tau+t)}$ can be decomposed as

$$S_{[\tau,\tau+t)} = S_1 + S_2 + \cdots + S_{\lfloor t \rfloor} + S_{[\tau - \lfloor \tau \rfloor, \tau - \lfloor \tau \rfloor + t - \lfloor t \rfloor)} ,$$

where the last term is a Poisson r.v. with parameter $\Lambda(\tau - \lfloor \tau \rfloor + t - \lfloor t \rfloor) - \Lambda(\tau - \lfloor \tau \rfloor)$, independent of other Poisson r.v.'s with parameter $\Lambda(1)$.

Otherwise, two incomplete terms appear in the decomposition form of $S_{[\tau,\tau+t)}$. One is equivalent to the accumulated claims from time $\tau - \lfloor \tau \rfloor$ to the end of the year, while the other is equal to accumulate claims from the beginning of the year up to $t - \lfloor t \rfloor - (\lfloor \tau \rfloor + 1 - \tau)$. Consequently, $S_{[\tau,\tau+t)}$ can be decomposed as

$$S_{[\tau,\tau+t)} = S_1 + S_2 + \cdots + S_{\lfloor t \rfloor} + S(t - \lfloor t \rfloor - (\lfloor \tau \rfloor + 1 - \tau)) + S_{[\tau - \lfloor \tau \rfloor, 1)} ,$$

where both incomplete terms are mutually independent Poisson r.v.'s, with parameters $\Lambda(t - \lfloor t \rfloor - (\lfloor \tau \rfloor + 1 - \tau))$ and $\Lambda(1) - \Lambda(\tau - \lfloor \tau \rfloor)$ respectively, independent of other aggregate claims as well.

Some other characteristic properties related to the aggregate claim process described in Theorem 1.3 are also derived as follows.

(a) The mgf of the aggregate claim process, $S_{[\tau,\tau+t)}$, is obtained as

$$M_{S_{[\tau,\,\tau+t)}}(s) = E\left[e^{s\,S_{[\tau,\,\tau+t)}}\right] = e^{[\Lambda(\tau+t)-\Lambda(\tau)][M_X(s)-1]}, \qquad s < a_X ,$$

where $M_X(s)$ is the mgf of X such that it is finite on $(-\infty, a_X)$.

(b) The expected total number of claims within time period $[0,\, t)$, and the corresponding variance are expressed as

$$E\left[S(t)\right] = \{\lfloor t \rfloor \Lambda(1) + \Lambda(t - \lfloor t \rfloor)\}E[X],$$
$$V\left[S(t)\right] = \{\lfloor t \rfloor \Lambda(1) + \Lambda(t - \lfloor t \rfloor)\}V[X] + \{\lfloor t \rfloor \Lambda(1) + \Lambda(t - \lfloor t \rfloor)\}\left(E[X]\right)^2 .$$

1.1.3 Ruin models for NHP processes

This section discusses the ruin problem for a general compound NHP process, with known intensity function λ and premium rate c.

Consider the surplus process, over the time interval $[\tau,\, \tau + t)$, for a compound NHP model with initial value of u at time τ. It is denoted by $U_{[\tau,\,\tau+t)}$ (or $U(t)$ if $\tau = 0$) and is given by

$$U_{[\tau,\,\tau+t)} = u + c\,t - \sum_{n=1}^{N_{[\tau,\,\tau+t)}} X_n, \qquad u,\, \tau,\, t \geq 0, \tag{1.12}$$

where $N_{[\tau,\,\tau+t)}$ is the corresponding NHP claim counting process, with intensity function λ, and claim severities $\{X_n;\, n \geq 1\}$ are i.i.d. with common cumulative distribution function F_X, independent of time t, and finite mean μ.

Let

$$T_\tau = \inf\left\{t \geq 0 \;\middle|\; U_{[\tau,\,\tau+t)} < 0\right\}, \qquad u,\, \tau \geq 0\,,$$

denote the time to ruin for the above surplus process $\{U_{[\tau,\,\tau+t)};\, t \geq 0\}$, with initial surplus u at initial age τ, and T denote the time to ruin when $\tau = 0$. Define the probability that ruin occurs before time $\tau + t$ as

$$\begin{aligned} \Psi_\tau(u,t) &= P\big\{T_\tau \leq t\big\}, \qquad u,\, t \geq 0, \\ &= P\big\{U_{[\tau,\,\tau+s)} < 0\,, \quad \text{for some } 0 < s \leq t\,\big\}\,. \end{aligned} \tag{1.13}$$

Like the parallel definition for the classical risk model, this is a finite time ruin problem and it depends not only on the initial surplus u and finite time period t, but also on the initial age τ.

The ultimate ruin probability with initial surplus u and initial age τ is then

$$\Psi_\tau(u) = \lim_{t\to\infty} \Psi_\tau(u,t), \qquad u,\, \tau \geq 0\,. \tag{1.14}$$

We denote $\Psi(u)$ to be the ultimate ruin probability with initial surplus u when $\tau = 0$ for simplicity.

Referring to Grandell (1991), the problem of calculating the ruin probability for the NHP processes in the following special case can be reduced to the ruin problem in the classical model.

The cumulative intensity function of the NHP process, Λ, given by

$$\Lambda(t) = \int_0^t \lambda(v)dv\,, \qquad t \geq 0\,, \tag{1.15}$$

is also called the intensity measure of the process. Assume that $\Lambda(t) < \infty$ for each $t < \infty$. As Λ is a continuous non–decreasing function with $\Lambda(0) = 0$, then Λ can be seen as the distribution function corresponding to the measure λ.

Define the inverse Λ^{-1} of Λ by

$$\Lambda^{-1}(t) = \sup\left\{s \,\middle|\, \Lambda(s) \leq t\right\}, \qquad t < \Lambda(\infty)\,. \tag{1.16}$$

Λ^{-1} is always right–continuous. Since Λ is continuous, Λ^{-1} is strictly increasing and

$$\Lambda \circ \Lambda^{-1}(t) = \Lambda[\Lambda^{-1}(t)] = t, \qquad \text{for } t < \Lambda(\infty).$$

The time scale defined by Λ^{-1} is generally called the operational time scale.

Definition 1.6 [Standard Poisson process] A Poisson process $\{\tilde{N}(t);\, t \geq 0\}$ with $\lambda = 1$ is called a standard Poisson process.

Let $\{N(t);\, t \geq 0\}$ be a NHP process with intensity measure Λ such that $\Lambda(\infty) = \infty$. Then it is easily seen that the point process $\tilde{N}(t)$ defined by $N[\Lambda^{-1}(t)]$ is a standard Poisson process.

It is natural to assume that the intensity function $\lambda(s)$ is proportional to the number of policyholders at time s. When the premium is determined individually for each policyholder it is also natural to assume that the aggregate (portfolio) risk premium is proportional to the number of policyholders. If the relative safety loading θ is constant we get the premium rate $c(t) = (1 + \theta)\,\mu\,\lambda(t)$ and the corresponding income process is given by

$$R(t) = (1 + \theta)\,\mu\,\Lambda(t) - \sum_{n=1}^{N(t)} X_n\,, \qquad t \geq 0\,,$$

where $N(t)$ is a NHP process with intensity measure Λ such that $\Lambda(\infty) = \infty$ and μ is the mean value of the i.i.d. random variables X_n.

Consider now the process $\{\tilde{U}(t);\, t \geq 0\}$ defined by

$$\tilde{U}(t) = U \circ \Lambda^{-1}(t) = (1+\theta)\mu\, t - \sum_{n=1}^{\tilde{N}(t)} X_n\,, \qquad t \geq 0\,.$$

This is a classical surplus process with $\lambda = 1$ and $u = 0$. Now rewrite (1.14) as

$$\Psi(u) = P\{\inf_{t\geq 0} U(t) < -u\}\,, \qquad u \geq 0\,,$$

which is the ultimate ruin probability with initial value u in the classical model. If Λ is increasing, or if $\lambda(t) > 0$, Λ^{-1} is continuous and it is obvious that $\inf_{t\geq 0} U_t = \inf_{t\geq 0} \tilde{U}_t$ and the problem of calculating the ruin probability is reduced to the classical case.

Now, we consider the ruin model under periodicity. In practical situations, not only the claim arrival rate can vary with the time of the year, but the claim size distribution may also depend on the seasons. Asmussen and Rolski (1994) proposes a risk process with the claim intensity, the claim size distribution and the premium rate at time t being periodic functions of t. By using an average arrival rate risk model, some properties of this periodic risk model, like two–sided bounds and asymptotic formulas, are derived. Here the premium rate is assumed to be constant and is equal to c for simplicity.

We refer to Asmussen and Rolski (1994) and Rolski et al. (1999) for the following average arrival rate risk model and its related two–sided bounds and asymptotic formula for the ruin probability $\Psi(u)$.

Let $\{F_t;\, t \geq 0\}$ be a family of distribution functions on $\mathbb{R}^+$ such that the mapping $t \to \int_0^\infty g(x)dF_t(x)$ is measurable and periodic, with period 1, for all integrable functions g.

Assume that claims arrive according to the NHP process $\{N(t);\, t \geq 0\}$, with periodic intensity function λ of period 1, so that $t - \lfloor t \rfloor$ is the time of the season. Furthermore, if a claim arrives at time t, then the claim size distribution is F_t, independent of everything else. Denote the corresponding mgf by

$$M_F(s;\, t) = \int_0^\infty e^{s\,x} dF_t(x)\,, \qquad s < a_{F_t}\,,$$

where $a_{F_t} > 0$ and $\lim_{s\uparrow a_{F_t}} M_F(s;\, t) = +\infty$.

Denote the average arrival rate by

$$\bar{\lambda} = \int_0^1 \lambda(v)dv = \Lambda(1), \tag{1.17}$$

while

$$F_X^0(x) = \frac{\int_0^1 \lambda(v)F_v(x)dv}{\bar{\lambda}}, \qquad x \geq 0, \tag{1.18}$$

is the distribution function of a typical claim size; a weighted average of distributions F_t for the different values of t over the year. Its mean is given by

$$\mu_X^0 = \int_0^\infty x\, dF_X^0(x)\,,$$

while the corresponding mgf

$$M_{F_X^0}(s) = \frac{\int_0^1 \lambda(v)M_F(s;\, v)dv}{\bar{\lambda}}, \qquad s < \inf_{0\leq t\leq 1}\{a_{F_t}\}\,.$$

Thus, the claim counting process $\{N(t);\, t \geq 0\}$ is a special homogeneous Poisson process with average arrival rate $\bar{\lambda}$. Its corresponding aggregate claims process $\{S(t);\, t \geq 0\}$, where $S(t) = \sum_{n=1}^{N(t)} X_n$ with $S(t) = 0$ if $N(t) = 0$, is also a special compound Poisson process with common cdf F_X^0. Furthermore, we call its corresponding surplus process $\{U(t);\, t \geq 0\}$, given by

$$U(t) = u + c\,t - \sum_{n=0}^{N(t)} X_n, \qquad t \geq 0\,, \tag{1.19}$$

an average arrival rate risk model, where $u \geq 0$ is the initial surplus and the constant premium rate $c > 0$ satisfies $c = (1+\theta)\,\bar{\lambda}\,\mu_X^0$ for the relative security loading $\theta > 0$.

Now consider the income process $\{R(t);\, t \geq 0\}$, where $R(t) = U(t) - u$, instead of the surplus. First, the Laplace–Stieltjes transform of $R(t)$, denoted as $M_{R(t)}(-s) = E\big[e^{-s\,R(t)}\big]$, is given in the following lemma.

Lemma 1.3 For $s < \inf_{0\leq t\leq 1}\{a_{F_t}\}$,

$$M_{R(t)}(-s) = e^{-c\,s\,t + \int_0^t \lambda(v)[M_F(s;\, v) - 1]dv}, \qquad t \geq 0 \tag{1.20}$$

where c is the constant premium rate.

Proof. See Rolski et al. (1999). □

For $a_X^0 = \inf_{0\le t\le 1}\{a_{F_t}\}$, define

$$\rho^*(s) = \bar{\lambda}\,[M_{F_X^0}(s) - 1] - c\,s. \tag{1.21}$$

Let $\gamma > 0$ be the solution of $\rho^*(s) = 0$, that is, $\int_0^1 \lambda(v)[M_F(\gamma;\, v) - 1]dv = c\,\gamma$. Then this γ is called the adjustment coefficient for the average arrival rate risk model. In this section, assume that such a γ exists in $[0,\, a_X^0)$, since $\rho^*(0) = 0$ and the derivative of $\rho^*(s)$ at zero is $\bar{\lambda}\,\mu_X^0 - c < 0$. The convexity of ρ^* ensures that

$$(\rho^*)'(\gamma) = \bar{\lambda}\int_0^\infty x\,e^{\gamma x}\,dF_X^0(x) - c > 0.$$

Let $x_v = \sup\{y \ge 0 : F_v(y) < 1\}$, then the following two–sided bound can be given for $\Psi(u)$, in the periodic Poisson model, with a time–dependent claim size distribution.

Theorem 1.5 For the ruin probability $\Psi(u)$ in the periodic Poisson model, where the claim size distribution is F_t time–dependent, the following inequalities hold:

$$a_-\,e^{-\gamma u} \le \Psi(u) \le a_+\,e^{-\gamma u}, \qquad u \ge 0, \tag{1.22}$$

where

$$a_- = \inf_{0\le v\le 1} M_{R(v)}(-\gamma)\,\alpha_-(v), \quad a_+ = \sup_{0\le v\le 1} M_{R(v)}(-\gamma)\,\alpha_+(v),$$
$$\alpha_-(v) = \inf_{0\le x\le x_v} \alpha(x,\, v), \quad \alpha_+(v) = \sup_{0\le x\le x_v} \alpha(x,\, v),$$

and

$$\alpha(x,\, v) = \frac{\bar{F}_v(x)}{\int_x^\infty e^{\gamma(y-x)}dF_v(y)}, \qquad x > 0\,.$$

Proof. See Rolski et al. (1999). □

Lu (2001) further considers some special cases of the above periodic Poisson model, where the claim size distribution is no longer time–dependent, i.e. $F_t = F_X$ for all $t > 0$, for which the two–sided bounds can be evaluated in practice. Thus, the severity mgf is $M_X(s) = \int_0^\infty e^{sx}dF_X(x)$, and the Laplace–Stieltjes transform (1.20) takes the form $M_{R(t)}(-s) = e^{-c\,s\,t+[M_X(s)-1]\Lambda(t)}$, where $\Lambda(t) = \int_0^t \lambda(v)dv$. Moreover, in this case, the average arrival rate risk model can be described as having the following features

$$\bar{\lambda} = \int_0^1 \lambda(v)dv, \quad F_X^0(x) = F_X(x), \qquad x \geq 0,$$

$$\mu_X^0 = \int_0^\infty x\,dF_X(x) = \mu \quad \text{and} \quad M_{F_X^0}(s) = M_X(s), \qquad s < a_X.$$

Here (1.21) becomes

$$\rho^*(s) = \bar{\lambda}\,[M_X(s) - 1] - c\,s, \qquad s < a_X, \tag{1.23}$$

which is exactly the same as in the classical case. Still assume that there exists a $\gamma > 0$ such that $\rho^*(\gamma) = 0$. Then we obtain two–sided bounds for ruin probabilities $\Psi(u)$ in this special periodic Poisson model, below. Let $x_0 = \sup\{y \geq 0 : \ F_X(y) < 1\}$.

Corollary 1.3 For the ruin probability $\Psi(u)$ in the periodic Poisson model with common claim size distribution, the following inequalities hold:

$$a_-^*\,e^{-\gamma\,u} \leq \Psi(u) \leq a_+^*\,e^{-\gamma\,u}, \qquad u \geq 0, \tag{1.24}$$

where

$$a_-^* = \inf_{0\leq x\leq x_0} \alpha(x) \inf_{0\leq v\leq 1} e^{-c\gamma\,[v-\frac{\Lambda(v)}{\bar{\lambda}}]}, \quad a_+^* = \sup_{0\leq x\leq x_0} \alpha(x) \sup_{0\leq v\leq 1} e^{-c\gamma\,[v-\frac{\Lambda(v)}{\bar{\lambda}}]}, \tag{1.25}$$

and

$$\alpha(x) = \frac{\bar{F}_X(x)}{\int_x^\infty e^{\gamma\,(y-x)}\,dF_X(y)}, \qquad x > 0.$$

Proof. Under the model assumptions, γ is the solution of (1.23), that is,

$$\rho^*(\gamma) = \bar{\lambda}\,[M_X(\gamma) - 1] - c\,\gamma = 0, \qquad \gamma < a_X.$$

Hence, the Laplace–Stieltjes transform $M_{R(v)}(-\gamma)$ is obtained as

$$\begin{aligned} M_{R(v)}(-\gamma) &= e^{-c\,\gamma\,v+[M_X(\gamma)-1]\,\Lambda(v)}, \qquad v \geq 0,\ \gamma < a_X, \\ &= e^{-c\,\gamma\,(v-\frac{\Lambda(v)}{\bar{\lambda}})}. \end{aligned}$$

Then (1.24) holds by Theorem 1.5. □

Corollary 1.4 Assume there is λ_{max}, such that

$$\lambda(t) \leq \lambda_{max}\,, \qquad \text{for } 0 \leq t \leq 1. \tag{1.26}$$

Then under the conditions of Corollary 1.3, the following inequalities hold:

$$a_-^{**}\, e^{-\gamma u} \leq \Psi(u) \leq a_+^{**}\, e^{-\gamma u}, \qquad u \geq 0, \tag{1.27}$$

where

$$a_-^{**} = \alpha_-^*\, e^{-c\gamma}, \quad a_+^{**} = \alpha_+^*\, e^{c\gamma \frac{\lambda_{max}}{\lambda}},$$

and α_-^*, α_+^* are given in (1.25).

Proof. The relation (1.26) implies that

$$\Lambda(v) = \int_0^v \lambda(s)\, ds \leq \lambda_{max} v \leq \lambda_{max}, \qquad \text{for } 0 \leq v \leq 1,$$

then

$$M_{R(v)}(-\gamma) \leq e^{c\gamma \frac{\Lambda(v)}{\lambda}} \leq e^{c\gamma \frac{\lambda_{max}}{\lambda}}, \qquad M_{R(v)}(-\gamma) \geq e^{-c\gamma (1 - \frac{\Lambda(v)}{\lambda})} \geq e^{-c\gamma}.$$

Hence, (1.27) holds. □

Corollary 1.5 Let $\lambda(t)$ take a specific parametric form from the beta family:

$$\lambda(t) = \lambda^*\, t^{p-1}\, (1-t)^{q-1}, \qquad 0 \leq t \leq 1,\ p,\ q > 1,$$

and $g(v) = v - \frac{\Lambda(v)}{\lambda} = v - \frac{B(p,q;v)}{B(p,q)}$, for $0 \leq v \leq 1$.

Assume that v_{max} exists such that $\sup_{0 \leq v \leq 1} g(v) = g(v_{max})$, Then

$$a_-^\star\, e^{-\gamma u} \leq \Psi(u) \leq a_+^\star\, e^{-\gamma u}, \qquad u \geq 0, \tag{1.28}$$

where

$$a_-^\star = \alpha_-^*\, e^{-c\gamma\, [v_{max} - \frac{B(p,q;\, v_{max})}{B(p,\, q)}]}, \qquad a_+^\star = \alpha_+^*\, e^{\frac{c\gamma}{B(p,\, q)} \frac{(p-1)^{p-1} (q-1)^{q-1}}{(p+q-2)^{p+q-2}}},$$

and α_-^*, α_+^* are given in (1.25).

Proof. Since in this case,

$$a_-^* = \alpha_-^* \inf_{0 \le v \le 1} e^{-c\gamma g(v)} \ge \alpha_-^* e^{-c\gamma g(v_{max})} = \alpha_-^* e^{-c\gamma[v_{max} - \frac{B(p,q;v_{max})}{B(p,q)}]}$$

and $\bar{\lambda} = \lambda^* B(p, q)$, $\lambda_{max} = \lambda^* \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}$, the result holds. □

Remark 1.2 When $\lambda(t) = \lambda$, $t \ge 0$, many sharper bounds have been derived for $\Psi(u)$ in this special case (the classical risk model). For example, see the lower bound of De Vylder and Goovaerts (1984), the upper bound of Dickson (1994) or the two–sided bounds of Cai and Garrido (1999a, 1999b).

Asmussen et al. (1994) also mentions that under some additional conditions, there is a Cramér–Lundberg type approximation for $\Psi(u)$ in the form of

$$\Psi(u) \sim c e^{-\gamma u}, \qquad u \to \infty.$$

Dassios and Embrechts (1989) discusses a class of risk models which allow for periodicity in the claim arrival intensity λ by using the theory of piecewise–deterministic Markov processes together with some standard martingale techniques.

Consider the surplus process $U(t) = u + ct - S(t)$, where $S(t) = \sum_{n=0}^{N(t)} X_n$ is the aggregate claims process. Assume that F_t is claim severity distribution at time t with support on $[0, \infty)$ and for $t \ge 0$ its corresponding mgf $M_F(s; t) = \int_0^\infty e^{sx} dF_t(x)$ exists for s in a suitable domain D containing 0 in its interior D^0. On D^0, also assume that $M_F(s; t)$ is twice differentiable with respect to s. Furthermore, $\forall s \in D$, $M_F(s; t)$ should be Riemann integrable. Again, $T = \inf\{t > 0 \mid U(t) \le 0\}$ is the time to ruin, $T = \infty$ if for all $t > 0$, $U(t) > 0$. The main result for this model is as follows.

Theorem 1.6 Assume that the claim severity distribution is independent of t, i.e., $F_t = F_X$, and that the claim arrival process is a NHP process with intensity parameter

$$\lambda(t) = \bar{\lambda} + \lambda_0 \sin[\frac{2\pi}{\Delta}(t + t_0)], \qquad t \ge 0$$

where $\bar{\lambda}$ is to be interpreted as the average claim arrival rate, λ_0 is the amplitude of the periodic component, Δ the period and t_0 the phase. If we denote by $\Psi(u; \Delta) =$

$P\{T < \infty \mid U(0) = u\}$ in this model, it then follows that

$$\Psi(u;\,\Delta) = \frac{e^{-s_0\,u + \frac{c\,s_0\lambda_0\Delta}{2\pi\,\lambda}\cos\left(\frac{2\pi t_0}{\Delta}\right)}}{E\left[e^{-s_0\,U(T) + \frac{c\,s_0\lambda_0\Delta}{2\pi\,\lambda}\cos\left[\frac{2\pi}{\Delta}(T+t_0)\right]}\;\Big|\; T < \infty\right]}, \qquad u \geq 0, \tag{1.29}$$

where s_0 satisfies $c\,s_0 = \bar{\lambda}\,[M_X(s_0) - 1]$.

Proof. See Dassios and Embrechts (1989). □

In general, it is difficult to calculate the denominator in (1.29), even in the exponential case where T and $U(T)$ are independent. However, inequalities can be obtained readily. For instance with probability one $|\cos\frac{2\pi}{\Delta}(T + t_0)| < 1$ so that

$$E\big[e^{-s_0 U(T)} \mid T < \infty\big]\;\Psi(u;\,\Delta) < e^{-s_0\,u + \frac{c\,s_0\lambda_0\Delta}{2\pi\lambda}\left[\cos\left(\frac{2\pi t_0}{\Delta}\right)+1\right]}, \qquad u \geq 0,$$

and a lower bound is

$$E\big[e^{-s_0 U(T)} \mid T < \infty\big]\;\Psi(u;\,\Delta) > e^{-s_0\,u + \frac{c\,s_0\lambda_0\,\Delta}{2\pi\lambda}\left[\cos\left(\frac{2\pi t_0}{\Delta}\right)-1\right]}, \qquad u \geq 0.$$

As special cases, for $t_0 = 0$

$$\Psi(u;\,\Delta) > \frac{e^{-s_0 u}}{E[e^{-s_0 U(T)} \mid T < \infty]}, \qquad u \geq 0\,,$$

and for $t_0 = \frac{\Delta}{2}$

$$\Psi(u;\,\Delta) < \frac{e^{-s_0 u}}{E[e^{-s_0 U(T)} \mid T < \infty]}, \qquad u \geq 0\,.$$

Morales (2004) defines a periodic NHP model with Gaussian intensity and considers the problem of ruin through a simulation approach.

1.2 Cox processes

The classical risk model is not realistic in some practical situations. Two main modifications are made here. First, a NHP process is used to model "size fluctuations" in the claim intensity of a risk subject to seasonality. Then, a Cox process, also called doubly stochastic Poisson process and a natural extension of the NHP process, is used to characterize the underlying "risk fluctuations" in the claims intensity.

This section first reviews the definition of the Cox process and the Cox models with different intensity processes considered in the risk theory literature. Then we give a brief review of the main results on a special Cox process, the Markov–modulated Poisson process.

1.2.1 Cox models and their intensity processes

Referring to Grandell (1991), we list the following definitions.

Definition 1.7 **[Random measure]** A stochastic process $\Lambda = \{\Lambda(t); t \geq 0\}$ with P–a.s. $\Lambda(0) = 0$, $\Lambda(t) < \infty$ for each $t < \infty$ and non–decreasing realizations is called here a random measure.

Definition 1.8 **[Cox process]** Let a random measure Λ and a standard Poisson process $\tilde{N}$ be independent of each other. The point process $N = \tilde{N} \circ \Lambda$ is called a Cox process (or a"doubly stochastic Poisson process").

Remark 1.3 Definition 1.8 is one of several equivalent definitions. Strictly speaking we only require that N and $\tilde{N} \circ \Lambda$ are equal in distribution. Further, we ought to show that the mapping $(\tilde{N}, \Lambda) \to \tilde{N} \circ \Lambda$ is measurable. For these questions see Grandell (1976).

Assume now that the random measure Λ has the following representation

$$\Lambda(t) = \int_0^t \lambda(v)dv, \qquad t \geq 0, \tag{1.30}$$

where $\lambda = \{\lambda(t); t \geq 0\}$ is called an intensity process. Obviously $\lambda(t) \geq 0$, P–a.s. Often it is more natural to define a Cox process by specifying the intensity λ rather than by specifying the measure Λ. In all cases to be considered $\lambda(\cdot)$ has right–continuous and Riemann integrable realizations. Then, by Grandell (1976), the mapping $\lambda \to \Lambda$ defined by (1.30) is measurable and the corresponding Cox process is well–defined.

Cox processes are very natural models to treat the underlying risk fluctuations in risk theory. Assuming that claims occur according to a Cox process N, intuitively means that conditioning upon the realization $\lambda_1(t)$ of a non–negative intensity process $\lambda = \{\lambda(t);\, t \geq 0\}$, N is a NHP process with this intensity function $\lambda_1(t)$.

An early reference to a Cox risk model is Ammeter (1948). In his model, the intensity process λ takes a piece–wise constant form

$$\lambda(t) = \lambda_k, \qquad (k-1)\,\Sigma \leq t < k\,\Sigma, \qquad k = 1, 2, \ldots,$$

where $\Sigma > 0$ is a (fixed) constant and $\{\lambda_k;\, k \geq 1\}$ is a sequence of non–negative, i.i.d. random variables. Ammeter's model is revisited by Grandell (1995) and more properties of the model are discovered.

Björk and Grandell (1988) further consider a Cox process, with an independent jump intensity, which is a generalization of the Ammeter process. An independent jump intensity is a process where the jump times form a renewal process and the value of the intensity between two successive jumps depends only on the distance between these two jumps. Let Σ_k, $k = 1, 2, \ldots$, denote the epoch of the k-th jump of the intensity process and let $\Sigma_0 = 0$. Define Λ_k and σ_k by

$$\Lambda_k = \lambda_{\Sigma_{k-1}}, \qquad k = 1, 2, \ldots,$$

$$\sigma_k = \Sigma_k - \Sigma_{k-1}, \qquad k = 1, 2, \ldots,$$

where $\{(\Lambda_k, \sigma_k);\, k \geq 1\}$ are assumed to be independent and identically distributed random vectors. The Cox process obtained by letting

$$\lambda(t) = \Lambda_k, \qquad \Sigma_{k-1} \leq t < \Sigma_k, \qquad k = 1, 2, \ldots,$$

is called an Björk–Grandell (B–G) process, where $\Sigma_k = \sigma_1 + \cdots + \sigma_k$ $(\Sigma_0 = 0)$. By using a martingale approach, extensions of the classical Lundberg (exponential) inequalities for the ruin probabilities of the model are obtained.

One of the characteristics of these B–G processes is that the random intensities Λ_k's, over their corresponding periods, are distributed independently with a common distribution function. It can be concluded that the fluctuations in intensity depend probably only on the changing of certain characteristics, but not on seasonal conditions.

The Markov–modulated risk process was first introduced by Janssen (1980) and also treated in Janssen and Rienhard (1985) and Rienhard (1984). The definition of this model using an environmental Markov chain $\{J(t);\, t \geq 0\}$ goes back to Asmussen (1989), who proposes a Cox risk model with intensity process $\{\lambda(t);\, t \geq 0\}$, given by

$$\lambda(t) = \lambda_{J(t)}, \qquad t \geq 0,$$

where the process $\{J(t);\, t \geq 0\}$ models the random environment in which an insurance business is assumed to be an irreducible continuous time Markov chain, with finite state space $\{1, 2, \ldots, m\}$. Furthermore, let the claim counting process $\{N(t);\, t \geq 0\}$ and the sequence of claim severities $\{X_i;\, i \geq 1\}$ be dependent via $\{J(t);\, t \geq 0\}$, that is, $\{N(t);\, t \geq 0\}$ and r.v.'s $\{X_i;\, i \geq 1\}$ are assumed to be conditionally independent given $\{J(t);\, t \geq 0\}$. The claim size distribution is also assumed to depend on time via $\{J(t);\, t \geq 0\}$. In contrast to the Björk–Grandell model, in Asmussen's model the sequence of successive intensities is not independent. Also, it can be shown that a Markov–modulated Poisson process has stationary increments if the environment process $\{J(t);\, t \geq 0\}$ has a stationary initial distribution.

A variety of methods, including approximations, simulation and numerical methods are studied in Asmussen (1989) to assess the values of ruin probabilities for the Markov–modulated risk model. Comparisons between the ruin function in the time–stationary Markov–modulated model and the ruin function in the corresponding average intensity compound Poisson model are given in Rolski (1981) and in Asmussen et al. (1995).

Schmidli (1996) introduces a Cox risk process with a piecewise constant intensity, including both the Björk–Grandell and Asmussen models as special cases. In his model, the intensity process $\{\lambda(t);\, t \geq 0\}$ is of the form

$$\lambda(t) = \lambda_k, \qquad \Sigma_{k-1} \leq t < \Sigma_k, \qquad k = 1, 2, \ldots,$$

where $\Sigma_k = \sigma_1 + \cdots + \sigma_k$ ($\Sigma_0 = 0$) and $\{(\lambda_k, \sigma_k);\, k \geq 1\}$ is a sequence of random vectors. The sequence $\{\lambda_k;\, k \geq 1\}$ of successive levels of the intensity is assumed to be a Markov chain with a stationary distribution, and the duration Σ_k of the level λ_k is assumed to be dependent only through λ_k. In the small–claim case a Lundberg inequality for the ruin probability is obtained via a martingale approach. A Cramèr–Lundberg approximation is also derived.

There are very few results in the risk theory literature regarding Cox processes with other than piecewise constant intensities. However, some authors propose Cox models by considering both, seasonal variations and stochastic fluctuations in the claims intensity.

Beard et al. (1984) and Daykin et al. (1994) suggest the intensity process λ as a composition of some factors:

$$\lambda(t) = \lambda_g(t)[1 + d(t)]\mathbf{q}(t), \qquad t \geq 0,$$

where $\lambda_g(t)$ is the trend adjusted expected number of claims in year t, $d(t)$ introduces deviations from the normal trend and $\mathbf{q}(t)$ indicates short–term variations in the risk propensity. A convenient formula for $d(t)$ is given by a sine–curve $d(t) = \alpha_d \sin(\omega t + \nu)$ corresponding to experience under a periodic environment, where α_d is the amplitude and $\omega = \frac{2\pi}{\Delta}$ the frequency, Δ being the length of the postulated cycle. The parameter ν determines the phase of the cycle. These authors do not discuss further the use of such intensity functions.

Recently, Schmidli (2003) considered a NHP model with doubly stochastic occurrences for the PCS catastrophes index, based on individual indices for PCS options, where the intensity is of the form

$$\lambda(t) = \lambda^* g(t), \qquad t \geq 0,$$

where λ^* is stochastic and $g(t)$ is a given function.

1.2.2 Markov–modulated Poisson processes

This section lists the main results on ruin probabilities and the severity of ruin in Markov–modulated risk models. Reinhard (1984) considers a class of semi–Markov risk models in which the claim frequencies and claim amounts are influenced by an external Markovian environment process. Later in Asmussen (1989), the concept of Markov–modulation has been introduced for the same model described by Reinhard (1984) in the sense that the rate of the Poisson arrival process and the distribution of the claim sizes are not fixed in time but depend on the state of an underlying Markov jump process.

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space and all the random variables defined below are on this space. Following Reinhard (1984) and Snoussi (2002), we introduce a Markov–modulated risk model involving in a Markovian environment process.

Consider a risk model in continuous time. Denote by $\{I(t);\, t \geq 0\}$ the external environment process, which influences the frequency of claims, the distribution of claims, and the rate of premiums. As pointed out by Asmussen (1989), in health insurance, sojourns of $\{I(t);\, t \geq 0\}$ could be certain types of epidemics or, in automobile insurance, these could be weather types (for example, icy, foggy,...). Suppose that $\{I(t);\, t \geq 0\}$ is a homogeneous, irreducible and recurrent Markov process with finite state space $J = \{1, 2, \ldots, m\}$. Denote by $A = (\alpha_{ij})_{i,j=1}^{m}$, with $\alpha_{ii} := -\alpha_i$, the intensity matrix of $\{I(t);\, t \geq 0\}$. The transition probability matrix of the embedded Markov chain is then given by

$$P = (p_{ij})_{i,j=1}^{m}, \qquad p_{ij} = \begin{cases} 0, & i = j, \\ \frac{\alpha_{ij}}{\alpha_i}, & i \neq j, \end{cases} \qquad i,\, j \in J\,. \tag{1.31}$$

Further assume that at time t claims occur according to a Poisson process with constant intensity rate $\lambda_i \in \mathbb{R}^+$, when $I(t) = i$ and the corresponding claim amounts have distribution $F_i(x)$, with density function $f_i(x)$ and finite mean μ_i $(i \in J)$. Moreover, we assume that premiums are received continuously at a positive constant rate

c_i during any time interval when the environment process remains in state i. Denote by W_n and X_n, respectively, the arrival time and the amount of the n-th claim, and by $T_n = W_n - W_{n-1}$ the inter–arrival time of the $(n-1)$-th claim and the n-th claim, with $W_0 = X_0 = T_0 = 0$.

Let J_n be the state of the process $\{I(t);\, t \geq 0\}$ at the arrival of the n-th claim, i.e.,

$$J_n = I(W_n)\,, \qquad n \in \mathbb{N}\,.$$

Reinhard (1984) shows the following result.

Theorem 1.7 Assume that the Markov chain $\{J_n;\, n \geq 0\}$ is irreducible and aperiodic (thus ergodic, as $m < \infty$), then its unique stationary probability distribution $\pi = (\pi_1, \ldots, \pi_m)$ is given by

$$\pi_i = \frac{\frac{\lambda_i \eta_i}{\alpha_i}}{\sum_{k=1}^{m} \frac{\lambda_k \eta_k}{\alpha_k}}\,, \qquad i \in J\,, \tag{1.32}$$

where $\eta = (\eta_1, \ldots, \eta_m)$ is the unique stationary probability distribution of the embedded Markov chain of process I, with transition probabilities given by (1.31).

Proof. See Reinhard (1984). □

Suppose that the sequences of random variables $\{X_n;\, n \in \mathbb{N}\}$ and $\{T_n;\, n \in \mathbb{N}\}$ are conditionally independent given $\{I(t);\, t \geq 0\}$.

Now define $N(t) = \sup\{n \in \mathbb{N} \,|\, \sum_{i=1}^{n} T_i \leq t\}$ as the number of claims that have occured before time t. The counting process $\{N(t);\, t \geq 0\}$ is called a Markov–modulated Poisson process, which is a special case of Cox processes. It also can be seen as a Poisson process with parameters modified by the transitions of the environment process. The corresponding surplus process $\{U(t);\, t \geq 0\}$ is given by

$$U(t) = u + C(t) - \sum_{n=1}^{N(t)} X_n\,, \qquad t \geq 0\,, \tag{1.33}$$

where $C(t)$ denotes the aggregate premium received during interval $(0, t]$ and $u\,(\geq 0)$ is the initial reserve. Let Z_n be the time at which the n-th transition of the environment process occurs and I_n be the state of the environment after its n-th transition. Assume that the company managing the risk receives premiums at a constant rate $c_i > 0$ during any time interval the environment process remains in state i. The premium income process is thus characterized by a vector $(c_1, \ldots, , c_m)$ with positive entries. Then we have

$$C(t) = \sum_{k=1}^{N_e(t)} c_{I_{k-1}}(Z_k - Z_{k-1}) + c_{I_{N_e(t)}}(t - T_{N_e(t)}), \qquad t \geq 0, \tag{1.34}$$

where $N_e(t) = \sup\{n \in \mathbb{N} : Z_n \leq t\}$.

Define

$$T = \inf\{t > 0 \mid U(t) < 0\},$$

to be the time of ruin and define ultimate ruin probabilities, given that the initial environment state is i and the initial reserve is u, by

$$\Psi_i(u) = P\{T < \infty \mid U(0) = u,\ I(0) = i\}, \qquad i \in J,\ u \geq 0,$$

and the probability of ultimate ruin in the stationary case by

$$\Psi(u) = \sum_{i=1}^{m} \pi_i \Psi_i(u), \qquad u \geq 0.$$

Their corresponding ultimate survival probabilities, or non–ruin probabilities, are defined, for $u \geq 0$, by $\bar{\Psi}_i(u) = 1 - \Psi_i(u)$, $i \in J$, and $\bar{\Psi}(u) = 1 - \Psi(u)$, respectively.

It is well known from the theory of random walks (see Reinhard (1984)) that

Theorem 1.8 For $u \geq 0$, $i \in J$, $\Psi_i(u) > 0$ and $\Psi_i(\infty) = 0$ if

$$d = \sum_{i=1}^{m} \pi_i (\frac{c_i}{\lambda_i} - \mu_i) > 0, \tag{1.35}$$

where π_i is given by (1.32), while $\Psi_i(u) = 1$ if $d \leq 0$.

We suppose therefore that $d > 0$.

Reinhard (1984) derives a system of integro–differential equations about the non–ruin probabilities, $\bar{\Psi}_i(u)$, for $i = 1, 2, \ldots, m$:

$$c_i \bar{\Psi}_i'(u) = (\lambda_i + \alpha_i)\bar{\Psi}_i(u) - \lambda_i \int_{0-}^{u} \bar{\Psi}_i(u-x)dF_i(x) - \alpha_i \sum_{k=1}^{m} p_{ik} \bar{\Psi}_k(u), \qquad u \geq 0, \tag{1.36}$$

which has a unique solution such that $\bar{\Psi}_i(\infty) = 1$, for $i \in J$.

Integrating (1.36) from 0 to t, we have

$$c_i\bar{\Psi}_i(t) = c_i\bar{\Psi}_i(0) + \lambda_i \int_0^t \bar{\Psi}_i(t-y)[1-F_i(y)]dy$$
$$+ \alpha_i \int_0^t \left[\bar{\Psi}_i(u) - \sum_{k=1}^m p_{ik}\bar{\Psi}_k(u)\right] du\,, \qquad i \in J,\ t \geq,, \tag{1.37}$$

which is a system of Volterra integral equations, but no longer the renewal type, for $m > 1$.

Letting t go to ∞ in (1.37) gives

$$\bar{\Psi}_i(0) \;=\; 1 - \frac{\lambda_i\mu_i}{c_i} - \frac{\alpha_i}{c_i}\int_0^\infty \left[\bar{\Psi}_i(u) - \sum_{k=1}^m p_{ik}\bar{\Psi}_k(u)\right] du\,, \qquad i \in J\,, \tag{1.38}$$

which does not give an explicit value for the probabilities $\bar{\Psi}_i(0)$ as in the classical case ($m = 1$).

However, when the claim severity distributions are exponential,

$$F_i(x) = 1 - e^{-\frac{x}{\mu_i}}\,, \qquad x \geq 0\,,$$

a further differentiation of both sides of (1.36) shows that the non–ruin probabilities are solution of the differential system:

$$\bar{\Psi}_i''(u) = \left(\frac{\lambda_i + \alpha_i}{c_i} - \frac{1}{\mu_i}\right)\bar{\Psi}_i'(u) - \frac{\alpha_i}{c_i}\sum_{k=1}^m p_{ik}\bar{\Psi}_k'(u) + \frac{\alpha_i}{c_i\mu_i}\bar{\Psi}_i(u)$$
$$- \frac{\alpha_i}{c_i\mu_i}\sum_{k=1}^m p_{ik}\bar{\Psi}_k(u)\,, \qquad i \in J,\ u \geq 0\,, \tag{1.39}$$

with boundary conditions

$$\bar{\Psi}_i(\infty) = 1\,, \qquad \bar{\Psi}_i'(0) = \frac{\lambda_i + \alpha_i}{c_i}\bar{\Psi}_i(0) - \frac{\alpha_i}{c_i}\sum_{k=1}^m p_{ik}\bar{\Psi}_k(0)\,, \qquad i \in J\,.$$

For instance, assume that $m = 2$, $p_{12} = p_{21} = 1$ and $p_{11} = p_{22} = 0$, thus there are two possible states for the environment, the sojourn times in each state being exponentially distributed.

Define

$$\rho_i = \frac{1}{\mu_i} - \frac{\lambda_i}{c_i}\,, \qquad i = 1, 2\,, \tag{1.40}$$

and assume without loss of generality that $\rho_1 \geq \rho_2$. The condition $d > 0$ is then equivalent to

$$\frac{\alpha_2}{c_2\,\mu_2}\,\rho_1 + \frac{\alpha_1}{c_1\,\mu_1}\,\rho_2 > 0\,. \tag{1.41}$$

The following result holds for the non–ruin probabilities when the claim severity distributions are exponential.

Theorem 1.9 If $m = 2$, $p_{12} = p_{21} = 1$, $d > 0$ and if the claim severity distributions are exponential, the non–ruin probabilities are given by

$$\begin{cases} \bar{\Psi}_1(u) = 1 + A_1 e^{k_1 u} + A_2 e^{k_2 u}\,, \\ \bar{\Psi}_2(u) = 1 - D(k_1) A_1 e^{k_1 u} - D(k_2) A_2 e^{k_2 u}\,, \end{cases} \qquad u \geq 0\,,$$

where k_1 and k_2 are the two negative roots of the characteristic equation

$$\begin{aligned} P(k) = k^3 &+ \left(\rho_1 + \rho_2 - \frac{\alpha_1}{c_1} - \frac{\alpha_2}{c_2}\right) k^2 \\ &+ \left[\left(\rho_1 - \frac{\alpha_1}{c_1}\right)\left(\rho_2 - \frac{\alpha_2}{c_2}\right) - \frac{\alpha_2}{c_2\mu_2} - \frac{\alpha_1}{c_1\mu_1} - \frac{\alpha_1\alpha_2}{c_1c_2}\right] k \\ &- \left(\frac{\alpha_2}{c_2\mu_2}\rho_1 + \frac{\alpha_1}{c_1\mu_1}\rho_2\right) = 0\,, \end{aligned} \tag{1.42}$$

with ρ_i given in (1.40), while the constants $D(k_i)$ are given by

$$D(k_i) = \frac{c_1\mu_1 k_i^2 + (c_1 - \lambda_1\mu_1 - \alpha_1\mu_1)k_i - \alpha_1}{\alpha_1\mu_1 k_i + \alpha_1}\,, \qquad i = 1, 2\,, \tag{1.43}$$

and A_1 and A_2 are solutions of the following linear system:

$$\begin{cases} \frac{1}{\mu_1 k_1 + 1} A_1 + \frac{1}{\mu_1 k_2 + 1} A_2 = -1 \\ \frac{D(k_1)}{\mu_2 k_1 + 1} A_1 + \frac{D(k_2)}{\mu_2 k_2 + 1} A_2 = 1 \end{cases}\,. \tag{1.44}$$

Proof. See Reinhard (1984). □

Remark 1.4 When $\lambda_1 = \lambda_2 = \lambda$, $\mu_1 = \mu_2 = \mu$, $c_1 = c_2 = c$ and if α_1 and α_2 are arbitrary positive numbers, then $k_2 = -\rho$ and k_1 is the negative root of

$$k^2 + \left(\rho - \frac{\alpha_1 + \alpha_2}{c}\right) k - \frac{\alpha_1 + \alpha_2}{c\mu} = 0\,.$$

We obtain then $D(k_2) = -1$, $D(k_1) = \frac{\alpha_2}{\alpha_1}$ and the solution of (1.44) is $A_1 = 0$ $A_2 = -\frac{\lambda\mu}{c}$. As expected the non–ruin probabilities $\bar{\Psi}_1(u)$ and $\bar{\Psi}_2(u)$ are in this case identical and equal to the ruin probabilities obtained for the classical Poisson model with exponentially distributed claim amounts:

$$\bar{\Psi}_1(u) = \bar{\Psi}_2(u) = 1 - \frac{\lambda\mu}{c} e^{-\rho u}, \qquad u \geq 0. \tag{1.45}$$

Now we consider the severity of ruin in a Markov–modulated model. The ruin probability is not always a satisfactory decision criteria. For instance it has been pointed out by Gerber et al. (1987) that it can be "very crude stability criterion" in some cases. The severity of ruin is then introduced to fill these deficiencies. The practical interest of this concept is, in particular, the possibility of bringing an additional element of information on ruin.

The severity of ruin was first studied by Gerber et al. (1987) for the classical continuous time risk model. They considered the probability, $G(y;\, u)$, that ruin occurs given initial surplus u and that the deficit at the time of ruin is less than y, with corresponding density function $g(y;\, u)$. General equations for G were derived and explicit solutions were obtained for certain claim severity distributions. This work was extended by Dufresne and Gerber (1988), who found explicit results for $G(y;\, u)$ when the claim severity distribution is a translation of a combination of exponential distributions. Dufresne (1989) also extended the results to a model where a diffusion process is added to the compound Poisson process. The generalization can be interpreted as allowing for some uncertainty, both in the premium income and in the claim amounts. Dickson (1989) showed that approximate values of $G(y;\, u)$ could be calculated by a recursive method by defining a relationship between survival probabilities and the density of $G(y;\, u)$. Snoussi (2002) studies the severity of ruin problem in a Markov–modulated model where the intensity and the premium can fluctuate according to a Markovian environment.

Considerable works dealing with the severity of ruin in a discrete risk model can also be found in the literature. Dickson and Waters (1992) present algorithms to calculate the probability and severity of ruin in both finite and infinite time for a discrete time risk model. The severity of ruin in the discrete risk model was also considered by Reinhard (1997). More recently, Reinhard and Snoussi (2002, 2004) extend the results in a discrete semi–Markov risk model. For other references see Dickson et al. (1995) and Gerber (1988).

Now for Markov–modulated risk models defined above, we define the severity of the ruin by

$$\Psi_i(y;\, u) = P\{T < \infty,\; U(T) < -y \,|\, U(0) = u, I(0) = i\}, \qquad i \in J,\; u, y \geq 0, \tag{1.46}$$

which represents the probability that ruin occurs and that at the time of ruin the surplus takes a value less than $-y$, or the deficit at the time of ruin is great than y, given an initial surplus u and an initial environment $i \in J$.

Note that by setting $y = 0$ in (1.46), we have $\Psi_i(0;\, u) = \Psi_i(u)$, and by the relationship

$$\begin{aligned} G_i(y;\, u) &= P\{T < \infty,\; U(T) \geq -y \mid U(0) = u,\; I(0) = i\} \\ &= \Psi_i(u) - \Psi_i(y;\, u), \qquad i \in J,\; u, y \geq 0, \end{aligned}$$

we are able to computer the probabilities $G_i(y;\, u)$. This generalizes the similar probability (also a distribution with respect to y) $G(y;\, u)$ introduced by Gerber et al. (1987) in the classical risk model.

Snoussi (2002) derives a differential system for $\Psi_i(y;\, u)$.

Theorem 1.10 For all $i \in J$, the probabilities of the severity of ruin satisfy the following system of differential equations

$$\begin{aligned} c_i \frac{\partial}{\partial u} \Psi_i(y;\, u) = (\lambda_i + \alpha_i)\Psi_i(y;\, u) - \lambda_i \left[\int_{0-}^{u} \Psi_i(y;\, u - x) dF_i(x) + 1 - F_i(u + y) \right] \\ - \alpha_i \sum_{k=1}^{m} p_{ik} \Psi_k(y;\, u), \qquad i \in J,\; u,\; y \geq 0. \end{aligned} \tag{1.47}$$

Proof. See Snoussi (2002). $\square$

It can be shown that (1.47) has a unique solution such that $\Psi_i(y;\, \infty) = 0$, for $i \in J,\ y \in \mathbb{R}^+$.

By integrating (1.36) from 0 to t, with respect to u, it yields the following corollary.

Corollary 1.6 For $i \in J$,

$$\begin{aligned} c_i\Psi_i(y;\, t) = c_i\Psi_i(y;\, 0) &+ \lambda_i \int_0^t \Psi_i(y;\, t-u)\big[1 - F_i(u)\big]du \\ &- \lambda_i \int_0^t \big[1 - F_i(u+y)\big]du \\ &+ \alpha_i \int_0^t \Big[\Psi_i(y;\, u) - \sum_{k=1}^m p_{ik}\Psi_k(y;\, u)\Big]du\,, \qquad t,\, y \geq 0\,. \end{aligned} \tag{1.48}$$

Remark 1.5 For $m = 1$, (1.48) is the well known renewal equation for the severity of ruin in the classical risk model and it is essentially formula (5.19) in Dufresne (1989), i.e.,

$$\begin{aligned} \Psi(y;\, u) = \Psi(y;\, 0) &+ \frac{\lambda}{c}\int_0^u \Psi(y;\, u-x)\big[1 - F(x)\big]dx \\ &- \frac{\lambda}{c}\int_0^u \big[1 - F(x+y)\big]dx\,, \qquad u,\, y \geq 0\,. \end{aligned}$$

For $m > 1$, (1.48) is a system of Volterra integral equations, no longer the renewal type equations.

Letting t go to ∞ in (1.48) gives:

Corollary 1.7 For $y \in \mathbb{R}^+$, $i \in J$,

$$\Psi_i(y;\, 0) = \frac{\lambda_i}{c_i}\int_y^\infty \big[1 - F_i(u)\big]du - \frac{\alpha_i}{c_i}\int_0^\infty \Big[\Psi_i(y;\, u) - \sum_{k=1}^m p_{ik}\Psi_k(y;\, u)\Big]du\,. \tag{1.49}$$

Remark 1.6 In the case $m = 1$, (1.49) gives the following result derived by Dufresne (1989),

$$\Psi(y;\, 0) = \frac{\lambda}{c}\int_y^\infty \big[1 - F(u)\big]du\,, \qquad y \geq 0\,,$$

which is equivalent to the result obtained by Bowers et al. (1986) (Theorem 12.2), see also Gerber et al. (1987).

Again, we consider the case where $I(t)$ is a two-state Markov process (only two environment states). Snoussi (2002) shows that an explicit formula for $\Psi_i(y;\, u)$ can be given under the additional assumption that the claim amounts distribution are exponential with finite mean μ_i, that is,

$$F_i(x) = 1 - e^{-\frac{x}{\mu_i}}\,, \qquad x \geq 0\,.$$

Further suppose that the premium income is constant and independent from the environment process (i.e. $c_i = c$, $i = 1, 2$). In this case (1.47) is reduced to

$$\begin{aligned} c\frac{\partial}{\partial u}\Psi_i(y;\, u) &= (\lambda_i + \alpha_i)\Psi_i(y;\, u) - \frac{\lambda_i}{\mu_i}\, e^{-\frac{u}{\mu_i}} \left[\int_0^u \Psi_i(y;\, z)\, e^{\frac{z}{\mu_i}}\, dz\right] \\ &\quad - \lambda_i\, e^{-\frac{u+y}{\mu_i}} - \alpha_i\, \Psi_{\vartheta(i)}(y;\, u)\,, \qquad i = 1, 2,\ u,\, y \geq 0\,, \end{aligned} \tag{1.50}$$

where $\vartheta(1) = 2$, $\vartheta(2) = 1$. Differentiation of (1.50) w.r.t. u leads to the following partial differential equation system:

$$\begin{aligned} c\,\frac{\partial^2}{\partial u^2}\Psi_i(y;\, u) &= \left(\lambda_i + \alpha_i - \frac{c}{\mu_i}\right)\frac{\partial}{\partial u}\Psi_i(y;\, u) + \frac{\alpha_i}{\mu_i}\Psi_i(y;\, u) - \alpha_i\frac{\partial}{\partial u}\Psi_{\vartheta(i)}(y;\, u) \\ &\quad - \frac{\alpha_i}{\mu_i}\Psi_{\vartheta(i)}(y;\, u)\,, \qquad i = 1, 2\,,\ u,\, y \geq 0\,, \end{aligned} \tag{1.51}$$

with boundary conditions, for $i = 1, 2,\ y \geq 0$,

$$\begin{cases} \Psi_i(y;\, \infty) = 0\,, \\ c\frac{\partial}{\partial u}\Psi_i(y;\, u)\Big|_{u=0} = (\lambda_i + \alpha_i)\Psi_i(y;\, 0) - \lambda_i[1 - F_i(y)] - \alpha_i\, \Psi_{\vartheta(i)}(y;\, 0)\,, \end{cases}$$

The following result holds for the probabilities of severity of ruin, when the claim severity distributions are exponential.

Theorem 1.11 If $\frac{\alpha_2}{c_2\,\mu_2}\,\rho_1 + \frac{\alpha_1}{c_1\,\mu_1}\,\rho_2 > 0$ and if the claim severity distributions are exponential, the probabilities of severity of ruin are given by

$$\begin{cases} \Psi_1(y;\, u) = B_1 e^{k_1\, u} + B_2 e^{k_2\, u}\,, \\ \Psi_2(y;\, u) = -D(k_1)\, B_1 e^{k_1\, u} - D(k_2)\, B_2 e^{k_2\, u}\,, \end{cases} \qquad u,\, y \geq 0\,,$$

where k_1 and k_2 are the two negative roots of the characteristic equation (1.42) with $c_1 = c_2 = c$ and $\rho_i = \frac{1}{\mu_i} - \frac{\lambda_i}{c}$, while the constants $D(k_i)$ are given by

$$D(k_i) = \frac{c\,\mu_1\,k_i^2 + (c_1 - \lambda_1\,\mu_1 - \alpha_1\,\mu_1)k_i - \alpha_1}{\alpha_1\mu_1\,k_i + \alpha_1}\,, \qquad i = 1, 2\,,$$

and B_1 and B_2 are solutions of the following linear system:

$$\begin{cases} \frac{1}{\mu_1 k_1 + 1} B_1 + \frac{1}{\mu_1 k_2 + 1} B_2 = e^{-\frac{y}{\mu_1}}\,, \\ -\frac{D(k_1)}{\mu_2 k_1 + 1} B_1 - \frac{D(k_2)}{\mu_2 k_2 + 1} B_2 = e^{-\frac{y}{\mu_2}}\,, \end{cases} \qquad y \geq 0\,. \tag{1.52}$$

Proof. See Snoussi (2002). □

Note that when $y = 0$, B_i in (1.52) is equal to $-A_i$ in (1.44) for $i = 1, 2$.

Remark 1.7 When $\lambda_1 = \lambda_2 = \lambda$, $\mu_1 = \mu_2 = \mu$, then $k_2 = -\rho$ and k_1 is the negative root of

$$k^2 + \left(\rho - \frac{\alpha_1 + \alpha_2}{c}\right) k - \frac{\alpha_1 + \alpha_2}{c\mu} = 0\,,$$

while $D(k_2) = -1$, $D(k_1) = \frac{\alpha_2}{\alpha_1}$ and the solution of (1.52) is $A_1 = 0$ $A_2 = \frac{\lambda\mu}{c} e^{-\frac{y}{\mu}}$. We obtain the probabilities of severity of ruin for the classical Poisson model with exponentially distributed claim amounts:

$$\Psi_1(y;\, u) = \Psi_2(y;\, u) = \frac{\lambda\mu}{c} e^{-\rho u - \frac{y}{\mu}}\,, \qquad u,\, y \geq 0\,. \tag{1.53}$$

Chapter 2

NHP processes with double periodicity

2.1 A doubly periodic intensity model

NHP processes are considered a more realistic alternative than the classical Poisson process to model the frequency of claims in risk theory. The NHP time–dependent intensity function is appropriate to describe the fluctuations of risks, subject to seasonality in their claims intensity.

Many natural phenomena evolve in a periodic environment or under seasonal conditions. In turn, these events generate insurance claims. For example, weather factors are known to affect automobile or fire insurance claims, while seasonal snow storms in the north and hurricanes or floods in the south affect property insurance. A periodic time–dependent intensity rate is a reasonable model for the claim frequency in such situations.

A more general case is when the periodic environment does not repeat itself exactly from year to year, but the short term peak changes over a relatively long period, with different levels in each year. This defines a double periodic environment, especially appropriate to model natural catastrophes, such as hurricanes, which have a peak season in the middle of the year, but with an intensity level also depending on long term climatological effects like La Niña or El Niño. In this sense, Garrido and Lu (2004) introduce a corresponding Poisson process model with double periodicity. Parametric forms of the doubly periodic intensity function, like the double–beta and the sine–beta, are proposed.

NHP models with single or double periodic intensity are described in Section 2.1.1, illustrated by beta–type functions. A double-beta periodic intensity model is introduced in Section 2.1.2. Its related characteristics are also discussed.

2.1.1 Some simple periodic intensity models

Assume that the short–term period is of length 1 (year). Let λ_1 be a beta–type function, with parameters $p_1, q_1 \geq 1$, defined on [0, 1], such that $\lambda_1(t_1^*) = 1$, where $t_1^* \in [0,1]$ is the mode of the function. That is

$$\lambda_1(t) = \begin{cases} \frac{\left(\frac{t-m_1}{D}\right)^{p_1-1}\left(1-\frac{t-m_1}{D}\right)^{q_1-1}}{\alpha_1^*}, & 0 \leq m_1 \leq t \leq m_2 \leq 1 \\ 0, & \text{otherwise} \end{cases}, \tag{2.1}$$

where $D = m_2 - m_1$ is the fraction of the year, over which λ_1 is non–zero, and

$$\alpha_1^* = \left(\frac{t_1^* - m_1}{D}\right)^{p_1-1}\left(1 - \frac{t_1^* - m_1}{D}\right)^{q_1-1}, \tag{2.2}$$

is a scale factor. Here

$$t_1^* = m_1 + D\,\frac{p_1 - 1}{p_1 + q_1 - 2}, \tag{2.3}$$

is the mode of λ_1, so that at the mode, $\lambda_1(t_1^*) = 1$ is the peak level.

Then the single periodic beta intensity function is given by

$$\lambda(t) = \lambda_0^*\,\lambda_1\big(t - \lfloor t \rfloor\big), \qquad t \geq 0, \tag{2.4}$$

where $\lambda_0^* > 0$ is the (constant) peak level for this intensity and λ_1 is given in (2.1).

The corresponding cumulative intensity function $\Lambda(t)$, for $t \geq 0$, is

$$\begin{aligned} \Lambda(t) &= \int_0^t \lambda(v)dv = \int_0^t \lambda_0^*\,\lambda_1\big(v - \lfloor v \rfloor\big)dv \\ &= \int_{m_1}^t \frac{\lambda_0^*}{\alpha_1^*}\left(\frac{v - \lfloor v \rfloor - m_1}{D}\right)^{p_1-1}\left(1 - \frac{v - \lfloor v \rfloor - m_1}{D}\right)^{q_1-1} dv. \end{aligned} \tag{2.5}$$

Let $s = \frac{v - \lfloor v \rfloor - m_1}{D}$ in (2.5), then we get

$$\begin{aligned}\Lambda(t) &= \lfloor t \rfloor \int_0^1 \frac{\lambda_0^*}{\alpha_1^*} s^{p_1 - 1} (1-s)^{q_1 - 1} D\, ds + \int_0^{\frac{t - \lfloor t \rfloor - m_1}{D}} \frac{\lambda_0^*}{\alpha_1^*} s^{p_1 - 1} (1-s)^{q_1 - 1} D\, ds \\ &= \frac{\lambda_0^* D}{\alpha_1^*} \left[\lfloor t \rfloor B(p_1,\, q_1) + B\left(p_1,\, q_1; \frac{t - \lfloor t \rfloor - m_1}{D}\right)\right], \qquad t \geq 0, \qquad (2.6)\end{aligned}$$

where $B(p_1,\, q_1)$ is the beta function at $p_1,\, q_1 > 1$, and $B(p,\, q;\, t)$ is the usual incomplete beta function, given by (1.6) and (1.7), respectively.

Following Corollary 1.1, the NHP process $\{N(t);\, t \geq 0\}$ with intensity function given in (2.4) can be decomposed as

$$N(t) = M_1 + M_2 + \cdots + M_{\lfloor t \rfloor} + N(t - \lfloor t \rfloor), \qquad t > 0,$$

where $\{M_i;\, i \geq 1\}$ are i.i.d. Poisson random variables distributed as $N(1)$, with mean $\Lambda(1)$, representing counts for complete years. These M_i are independent of $N(t - \lfloor t \rfloor)$, the latter being a Poisson r.v. with mean $\Lambda(t - \lfloor t \rfloor)$, for $t - \lfloor t \rfloor \in [0,\, 1)$, representing the count in the final incomplete year. Here $\Lambda(1)$ and $\Lambda(t - \lfloor t \rfloor)$ are derived from (2.6), respectively as:

$$\begin{aligned}\Lambda(1) &= \frac{\lambda_0^* D}{\alpha_1^*} B(p_1,\, q_1), \\ \Lambda(t - \lfloor t \rfloor) &= \frac{\lambda_0^* D}{\alpha_1^*} B\left(p_1,\, q_1; \frac{t - \lfloor t \rfloor - m_1}{D}\right), \qquad t \geq 0.\end{aligned}$$

An alternative simple form for λ_1, which can result in a better fit with real data, is the generalized 3–parameter beta function [denoted $G3B(p_1,\, q_1,\, \epsilon)$, see Johnson et al. (1995), Chapter 25], given by

$$\lambda_1(t) = \begin{cases} \frac{\left(\frac{t - m_1}{D}\right)^{p_1 - 1} \left(1 - \frac{t - m_1}{D}\right)^{q_1 - 1}}{\alpha_1^* \left[1 - (1-\epsilon)\left(\frac{t - m_1}{D}\right)\right]^{p_1 + q_1}}, & 0 \leq m_1 \leq t \leq m_2 \leq 1 \\ 0, & \text{otherwise} \end{cases}, \qquad (2.7)$$

where $p_1, q_1 \geq 1$ as before, the third parameter $\epsilon > 0$, the fraction of the year $D = m_2 - m_1$ remains the same and

$$\alpha_1^* = \frac{\left(\frac{t_1^* - m_1}{D}\right)^{p_1 - 1} \left(1 - \frac{t_1^* - m_1}{D}\right)^{q_1 - 1}}{\left[1 - (1-\epsilon)\frac{t_1^* - m_1}{D}\right]^{p_1 + q_1}}, \qquad (2.8)$$

is again a scale factor. The mode of this new intensity function λ_1 in (2.7) now becomes

$$t_1^* = m_1 + D\,\frac{3 - p_1 - (1+q_1)\epsilon + \sqrt{[1 + p_1 + (1+q_1)\epsilon]^2 - 8(p_1+q_1)\epsilon}}{4(1-\epsilon)}, \tag{2.9}$$

such that $\lambda_1(t_1^*) = 1$. Note that as $\epsilon \to 1$, then (2.9) tends to the 2–parameter form in (2.3).

Then for the intensity function, given by (2.4), the corresponding cumulative intensity function Λ can be derived as follows:

$$\begin{aligned}\Lambda(t) &= \int_0^t \lambda(v)\,dv = \int_0^t \lambda_0^*\,\lambda_1(v - \lfloor v \rfloor)\,dv \\ &= \int_{m_1}^t \lambda_0^*\,\frac{\left(\frac{v-\lfloor v\rfloor - m_1}{D}\right)^{p_1-1}\left(1 - \frac{v-\lfloor v\rfloor - m_1}{D}\right)^{q_1-1}}{\alpha_1^*\left[1-(1-\epsilon)\left(\frac{v-\lfloor v\rfloor - m_1}{D}\right)\right]^{p_1+q_1}}\,dv, \qquad t \geq 0. \end{aligned} \tag{2.10}$$

Let $u = \frac{v-\lfloor v\rfloor - m_1}{D}$ in (2.10), then we have

$$\Lambda(t) = \lfloor t\rfloor \int_0^1 \frac{\lambda_0^*\,u^{p_1-1}\,(1-u)^{q_1-1}}{\alpha_1^*\,[1-(1-\epsilon)u]^{p_1+q_1}}\,D\,du + \int_0^{\frac{t-\lfloor t\rfloor-m_1}{D}} \frac{\lambda_0^*\,u^{p_1-1}\,(1-u)^{q_1-1}}{\alpha_1^*\,[1-(1-\epsilon)u]^{p_1+q_1}}\,D\,du.$$

Now setting $s = \frac{1-u}{1-(1-\epsilon)u}$ in the above integrals, then $ds = \frac{-\epsilon\,du}{[1-(1-\epsilon)u]^2}$ and it follows that

$$\begin{aligned}\Lambda(t) &= \lfloor t\rfloor \int_0^1 \frac{\lambda_0^*\,s^{q_1-1}\,(1-s)^{p_1-1}}{\alpha_1^*\,\epsilon^{p_1}}\,D\,ds + \int_{\frac{1-\frac{t-\lfloor t\rfloor-m_1}{D}}{1-(1-\epsilon)\frac{t-\lfloor t\rfloor-m_1}{D}}}^1 \frac{\lambda_0^*\,s^{q_1-1}\,(1-s)^{p_1-1}}{\alpha_1^*\,\epsilon^{p_1}}\,D\,ds \\ &= \frac{\lambda_0^* D}{\alpha_1^*\,\epsilon^{p_1}}\left[\lfloor t\rfloor B(p_1,\,q_1) + B\left(p_1,\,q_1;\,\frac{\epsilon\left\{\frac{t-\lfloor t\rfloor-m_1}{D}\right\}}{1-(1-\epsilon)\left\{\frac{t-\lfloor t\rfloor-m_1}{D}\right\}}\right)\right], \qquad t \geq 0. \end{aligned} \tag{2.11}$$

More generally, now assume that the (constant) peak values or the levels of the short–term intensity function vary periodically with period c (an integer number of years), we have the following double periodic intensity model:

$$\lambda(t) = \begin{cases} \lambda_0^*\lambda_1(t - \lfloor t\rfloor) & \text{if } 0 \leq t - \lfloor \frac{t}{c}\rfloor c < 1 \\ \lambda_1^*\lambda_1(t - \lfloor t\rfloor) & \text{if } 1 \leq t - \lfloor \frac{t}{c}\rfloor c < 2 \\ \vdots & \vdots \\ \lambda_{c-1}^*\lambda_1(t - \lfloor t\rfloor) & \text{if } c-1 \leq t - \lfloor \frac{t}{c}\rfloor c < c \end{cases}, \tag{2.12}$$

where $\lambda_0^*, \ldots, \lambda_{c-1}^*$ are all positive levels. If, as above, the short–term intensity is the beta–shape function given in (2.1), then the resulting cumulative intensity function $\Lambda(t)$ can be derived as follows:

$$
\begin{aligned}
\Lambda(t) &= \int_0^t \lambda(v)dv \\
&= \lfloor\frac{t}{c}\rfloor \sum_{j=0}^{c-1} \frac{\lambda_j^*}{\alpha_1^*} \int_{j+m_1}^{j+m_1+D} \left(\frac{v-j-m_1}{D}\right)^{p_1-1} \left(1-\frac{v-j-m_1}{D}\right)^{q_1-1} dv \\
&\quad + \sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor -1} \frac{\lambda_j^*}{\alpha_1^*} \int_{j+m_1}^{j+m_1+D} \left(\frac{v-j-m_1}{D}\right)^{p_1-1} \left(1-\frac{v-j-m_1}{D}\right)^{q_1-1} dv \\
&\quad + \frac{\lambda^*_{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor}}{\alpha_1^*} \int_{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor + m_1}^{t-\lfloor\frac{t}{c}\rfloor c} \left(\frac{v-\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor - m_1}{D}\right)^{p_1-1} \\
&\qquad \left(1-\frac{v-\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor - m_1}{D}\right)^{q_1-1} dv\,. \qquad (2.13)
\end{aligned}
$$

Letting $s = \frac{v-j-m_1}{D}$ in the first two integrals and $s = \frac{v-\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor - m_1}{D}$ in the last integral in (2.13) gives

$$
\begin{aligned}
\Lambda(t) &= \lfloor\frac{t}{c}\rfloor \sum_{j=0}^{c-1} \frac{\lambda_j^*}{\alpha_1^*} \int_0^1 s^{p_1-1}(1-s)^{q_1-1} D\, ds \\
&\quad + \sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor -1} \frac{\lambda_j^*}{\alpha_1^*} \int_0^1 s^{p_1-1}(1-s)^{q_1-1} D\, ds \\
&\quad + \frac{\lambda^*_{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor}}{\alpha_1^*} \int_0^{\frac{t-\lfloor\frac{t}{c}\rfloor c - \lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor - m_1}{D}} s^{p_1-1}(1-s)^{q_1-1} D\, ds \\
&= \lfloor\frac{t}{c}\rfloor D\, B(p_1,\, q_1) \sum_{j=0}^{c-1} \frac{\lambda_j^*}{\alpha_1^*} + D\, B(p_1,\, q_1) \sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor -1} \frac{\lambda_j^*}{\alpha_1^*} \\
&\quad + D\, B\left(p_1,\, q_1;\, \frac{t-\lfloor t\rfloor - m_1}{D}\right) \frac{\lambda^*_{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor}}{\alpha_1^*}\,, \qquad t \geq 0\,.
\end{aligned}
$$

The corresponding NHP process $\{N(t);\, t \geq 0\}$ with double periodic intensity can be decomposed as

$$
N(t) = M_1 + \cdots + M_{\lfloor\frac{t}{c}\rfloor} + N^*(\frac{t-\lfloor t\rfloor - m_1}{D})\,, \qquad t \geq 0\,, \qquad (2.14)
$$

where

$$N^*(\frac{t-\lfloor t\rfloor-m_1}{D}) = \sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1} N^{(j)}(c) + N^{(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor)}(\frac{t-\lfloor t\rfloor-m_1}{D}), \tag{2.15}$$

and the $\{M_i;\, i \geq 1\}$ are i.i.d. Poisson distributed with mean $D\,B(p_1,\, q_1)\sum_{j=0}^{c-1}\frac{\lambda_j^*}{\alpha_1^*}$, while $N^{(j)}(c)$ is Poisson with mean $D\,B(p_1,\, q_1)\frac{\lambda_j^*}{\alpha_1^*}$, for $j = 0, 1, \ldots, \lfloor t - \lfloor\frac{t}{c}\rfloor c\rfloor - 1$, respectively. Furthermore, $N^{(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor)}(\frac{t-\lfloor t\rfloor-m_1}{D})$ is also Poisson random variable with mean $D\,B\big(p_1,\, q_1;\, \frac{t-\lfloor t\rfloor-m_1}{D}\big)\frac{\lambda^*_{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor}}{\alpha_1^*}$. All these random variables are mutually independent.

2.1.2 A double–beta periodic intensity model

One way to reduce the number of free parameters, in the previous model in (2.12), is to assume a parametric form also for the long–term changes in intensity. Here this is reasonable if it can be assumed that the short–term peak intensity values are affected periodically by some smoothly varying conditions, like the surface water temperatures in the alternating cycles of the El Niño/La Niña phenomenon.

To illustrate a NHP process with periodic intensity, consider the 155 hurricanes recorded by the National Hurricane Center along the US Atlantic coastline (Texas to Maine), from 1899 through 1992, and reported by Neumann et al. (1993). If this data set is augmented by the 12 additional hurricanes occurred from 1993 to 2000, more recently reported by Landreneau (2001), we obtain a total of 167 observations. In each case we have the time (month) that the hurricane hit the US coastline, allowing us to draw a frequency histogram in Figure 2.1.

Figure 2.1 also gives the 3-parameter beta intensity described above, appropriately fitted to these annual hurricane frequencies (see Section 3.3.2 for the statistical estimation). The constant intensity of a classical homogeneous Poisson process is also given for comparison. Clearly the classical model gives a crude representation of hurricane frequencies.

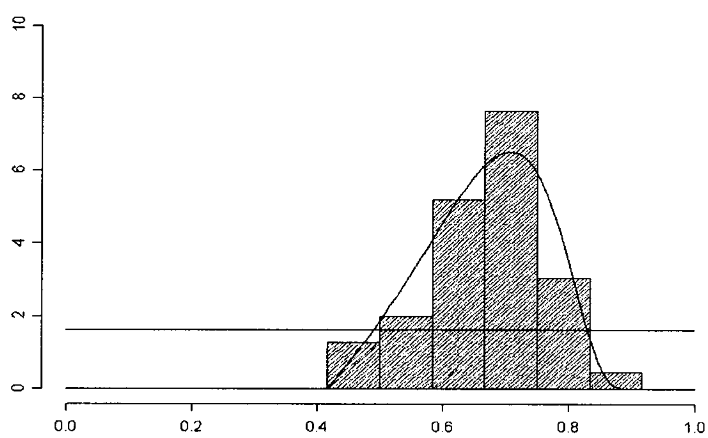

Figure 2.1: Histogram and 3-parameter beta fitted hurricane intensities $\lambda(t)$ over a 1-year cycle

Although the beta periodic claim intensity seems to provide a better fit to hurricane frequencies, climatological studies suggest that the claim intensity does not repeat the exact same short term pattern every year. Rather, it slightly varies from year to year, as in alternating El Niño–La Niña cycles. This motivates our study of the doubly periodic NHP process presented in this section. Here the seasonality repeats a similar short term pattern every year, letting the peak intensity vary over a longer periodic cycle.

More specifically, here we assume that the peak beta values, $\lambda_0^*, \ldots, \lambda_{c-1}^*$ in the short–term intensities, follow another continuous function of period c (an integer number of years), called the long–term intensity function. For instance, a beta function λ_c, is also proposed for the long–term intensity:

$$\begin{aligned}\lambda_c(t) &= a + \frac{b-a}{\alpha_c^*}\left(\frac{t-m_c}{c} - \lfloor\frac{t-m_c}{c}\rfloor\right)^{p_c-1} \\ &\quad \left[1 - \left(\frac{t-m_c}{c} - \lfloor\frac{t-m_c}{c}\rfloor\right)\right]^{q_c-1}, \qquad t \geq 0, \qquad (2.16)\end{aligned}$$

where

$$\alpha_c^* = \left(\frac{t_c^* - m_c}{c}\right)^{p_c-1}\left(1 - \frac{t_c^* - m_c}{c}\right)^{q_c-1}, \qquad (2.17)$$

is again a scale factor, so that a and b are, respectively, the minimum and maximum amplitude of the peak values. Here m_c is the starting point of the complete cycle of the long–term beta function and

$$t_c^* = m_c + c\left(\frac{p_c - 1}{p_c + q_c - 2}\right) \tag{2.18}$$

denotes the mode of this long–term λ_c.

Then the double–beta intensity function is given by the product

$$\lambda(t) = \lambda_c\big(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor + t_1^*\big)\lambda_1\big(t - \lfloor t \rfloor\big)\,, \qquad \text{for } t \geq 0\,, \tag{2.19}$$

where λ_1 and λ_c are given in (2.1) and (2.16), respectively. Then the NHP process with intensity function (2.19) is called a NHP process with double–beta periodicity.

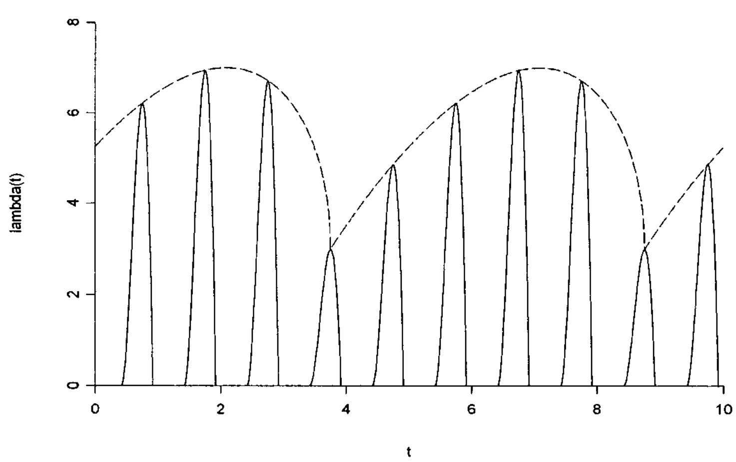

Figure 2.2: Double–beta intensity function $\lambda(t)$

The solid line in Figure 2.2 illustrates the shape of $\lambda(t)$ in (2.19), when $p_1 = 3$, $q_1 = 2$, $m_1 = \frac{5}{12}$, $D = \frac{6}{12}$, $c = 5$, $p_c = 2$, $q_c = 1\frac{2}{3}$, $m_c = 3.75$, $a = 3$ and $b = 7$. The peak values of the short–term beta λ_1 fall on the dotted line, plotting the long–term beta λ_c. It serves to explain the fluctuations in the peak values of λ_1, the short–term beta periodicity.

By Corollary 1.1, we can also obtain in the double–beta periodic case an explicit expression for the cumulative hazard function Λ, defined in general by (1.1). The corresponding claim counting process $\{N(t);\ t \geq 0\}$ is also decomposed in i.i.d. components.

Theorem 2.1 Assume that the intensity function λ is given by (2.19), then

(a) The cumulative hazard function Λ has the almost linear property, given by

$$\Lambda(t) = \lfloor \frac{t}{c} \rfloor D\,B(p_1,\,q_1) \sum_{j=0}^{c-1} \frac{\lambda_c(j+t_1^*)}{\alpha_1^*} + D\,B(p_1,\,q_1) \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor -1} \frac{\lambda_c(j+t_1^*)}{\alpha_1^*}$$
$$+ D\,B\big(p_1,\,q_1;\,\frac{t - \lfloor t \rfloor - m_1}{D}\big) \frac{\lambda_c(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor + t_1^*)}{\alpha_1^*}, \qquad t \geq 0, \qquad (2.20)$$

where $\lambda_c(t)$ is given by (2.16).

(b) For any $t \geq 0$, the random variable N_t is decomposed as the independent sum of $\lfloor \frac{t}{c} \rfloor$ i.i.d. Poisson variables M_i, for the complete periods, and a different Poisson variable for the incomplete period:

$$N(t) = M_1 + \cdots + M_{\lfloor \frac{t}{c} \rfloor} + N^*(\frac{t - \lfloor t \rfloor - m_1}{D}), \qquad t \geq 0, \qquad (2.21)$$

where

$$M_i = \sum_{j=0}^{c-1} N_{[ic+j,\,ic+j+1)} = \sum_{j=0}^{c-1} N^{(j)}(1), \qquad i = 0, 1, \ldots, \lfloor \frac{t}{c} \rfloor - 1 \qquad (2.22)$$

and

$$N^*(\frac{t - \lfloor t \rfloor - m_1}{D}) = \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor -1} N^{(j)}(1) + N^{(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor)}(\frac{t - \lfloor t \rfloor - m_1}{D}). \qquad (2.23)$$

The M_i are i.i.d. Poisson with mean $D\,B(p_1,\,q_1) \sum_{j=0}^{c-1} \frac{\lambda_c(j+t_1^*)}{\alpha_1^*}$, independent of $N^{(j)}(c)$, for $j = 0, 1, \ldots, \lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - 1$, and $N^{(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor)}(\frac{t - \lfloor t \rfloor - m_1}{D})$, which are all Poisson random variables with mean $D\,B(p_1,\,q_1) \frac{\lambda_c(j+t_1^*)}{\alpha_1^*}$, where $j = 0, 1, 2, \ldots, \lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - 1$, and $D\,B\left(p_1,\,q_1;\,\frac{t - \lfloor t \rfloor - m_1}{D}\right) \frac{\lambda_c(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor + t_1^*)}{\alpha_1^*}$, respectively.

Proof. (a) By (2.19) and the periodicity of the intensity function λ,

$$\begin{aligned}
\Lambda(t) &= \lfloor\frac{t}{c}\rfloor \int_0^c \frac{\lambda_c(\lfloor v\rfloor + t_1^*)}{\alpha_1^*}\left(\frac{v-\lfloor v\rfloor - m_1}{D}\right)^{p_1-1}\left(1-\frac{v-\lfloor v\rfloor - m_1}{D}\right)^{q_1-1} dv \\
&+ \int_0^{t-\lfloor\frac{t}{c}\rfloor c} \frac{\lambda_c(\lfloor v\rfloor + t_1^*)}{\alpha_1^*}\left(\frac{v-\lfloor v\rfloor - m_1}{D}\right)^{p_1-1}\left(1-\frac{v-\lfloor v\rfloor - m_1}{D}\right)^{q_1-1} dv \\
&= \lfloor\frac{t}{c}\rfloor \sum_{j=0}^{c-1} \frac{\lambda_c(j + t_1^*)}{\alpha_1^*}\int_{j+m_1}^{j+m_1+D}\left(\frac{v-j - m_1}{D}\right)^{p_1-1}\left(1-\frac{v-j - m_1}{D}\right)^{q_1-1} dv \\
&+ \sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor -1} \frac{\lambda_c(j + t_1^*)}{\alpha_1^*}\int_{j+m_1}^{j+m_1+D}\left(\frac{v-j - m_1}{D}\right)^{p_1-1}\left(1-\frac{v-j - m_1}{D}\right)^{q_1-1} dv \\
&+ \frac{\lambda_c(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor + t_1^*)}{\alpha_1^*}\int_{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor + m_1}^{t-\lfloor\frac{t}{c}\rfloor c}\left(\frac{v-\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor - m_1}{D}\right)^{p_1-1} \\
&\qquad \left(1-\frac{v-\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor - m_1}{D}\right)^{q_1-1} dv\,. \qquad (2.24)
\end{aligned}$$

Letting $s = \frac{v-j-m_1}{D}$ in the first two integrals and $s = \frac{v-\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor - m_1}{D}$ in the last integral in (2.24) gives

$$\begin{aligned}
\Lambda(t) &= \lfloor\frac{t}{c}\rfloor \sum_{j=0}^{c-1} \frac{\lambda_c(j + t_1^*)}{\alpha_1^*}\int_0^1 s^{p_1-1}(1-s)^{q_1-1} D\, ds \\
&+ \sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor -1} \frac{\lambda_c(j + t_1^*)}{\alpha_1^*}\int_0^1 s^{p_1-1}(1-s)^{q_1-1} D\, ds \\
&+ \frac{\lambda_c(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor + t_1^*)}{\alpha_1^*}\int_0^{\frac{t-\lfloor\frac{t}{c}\rfloor c - \lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor - m_1}{D}} s^{p_1-1}(1-s)^{q_1-1} D\, ds\,.
\end{aligned}$$

Since $t-\lfloor\frac{t}{c}\rfloor c - \lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor = t - \lfloor t\rfloor$, then (2.20) follows by definition of the beta and incomplete beta functions.

(b) By Corollary 1.1, for $t \geq 0$, N_t can be decomposed as follows

$$N(t) = \sum_{i=0}^{\lfloor\frac{t}{c}\rfloor-1}\sum_{j=0}^{c-1} N_{[ic+j,\, ic+j+1)} + \sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor -1} N_{[\lfloor\frac{t}{c}\rfloor c+j,\, \lfloor\frac{t}{c}\rfloor c+j+1)} + N_{[\lfloor\frac{t}{c}\rfloor c+\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor,\, t)}\,, \quad (2.25)$$

where the first term sums over the complete period sub–sums, while the second summation in (2.25) accounts for the complete years included in the last (incomplete) period. Finally the last term represents the claim count for the last (incomplete) year of the last (incomplete) period.

By periodicity of the function λ and Lemma 1.2, it is clear that $N_{[ic+j,\, ic+j+1)}$, $i = 0, 1, \ldots, \lfloor \frac{t}{c} \rfloor$, the claim counts over the j-th year within the periods, are mutually independent and Poisson distributed r.v.'s with mean $D\,B(p_1,\, q_1)\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}$, denoted by $N^{(j)}(1)$, for $j = 0, 1, \ldots, c-1$.

Similarly, $N_{[\lfloor \frac{t}{c} \rfloor c + \lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor,\, t)} = N_{[t-\lfloor t \rfloor,\, t)}^{(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor)}$ is a Poisson random variable with mean $D\,B(p_1,\, q_1;\, \frac{t-\lfloor t \rfloor - m_1}{D})\frac{\lambda_c(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor + t_1^*)}{\alpha_1^*}$, like $N^{(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor)}(\frac{t-\lfloor t \rfloor - m_1}{D})$.

Now (2.25) becomes

$$N(t) = \sum_{i=1}^{\lfloor \frac{t}{c} \rfloor} \sum_{j=0}^{c-1} N^{(j)}(1) + \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - 1} N^{(j)}(1) + N^{(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor)}(\frac{t - \lfloor t \rfloor - m_1}{D}), \qquad t \geq 0. \tag{2.26}$$

Further denote by $M_i = \sum_{j=0}^{c-1} N^{(j)}(1)$, for $i = 0, 1, \ldots, \lfloor \frac{t}{c} \rfloor - 1$, the claim counts over the i-th period. By the additive property of NHP processes, we consequently get that M_i, given in (2.22), $i = 1, 2, \ldots, \lfloor \frac{t}{c} \rfloor$ are i.i.d. Poisson random variables distributed as $N(c)$, with mean $D\,B(p_1,\, q_1) \sum_{j=0}^{c-1} \frac{\lambda_c(j+t_1^*)}{\alpha_1^*}$.

Setting

$$\begin{aligned} N^*(\frac{t - \lfloor t \rfloor - m_1}{D}) &= \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - 1} N_{[\lfloor \frac{t}{c} \rfloor c + j,\, \lfloor \frac{t}{c} \rfloor c + j + 1)} + N_{[\lfloor \frac{t}{c} \rfloor c + \lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor,\, t)} \\ &= \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - 1} N^{(j)}(1) + N^{(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor)}(\frac{t - \lfloor t \rfloor - m_1}{D}), \end{aligned} \tag{2.27}$$

gives (2.23). Combining with (2.22), (2.26) leads to (2.21) and hence (b) holds. □

This decomposition property of periodic NHP models is particularly useful for statistical inference, as seen in the next chapter.

Now consider $N_{[\tau,\, \tau+t)}$, the number of claims in the time interval $[\tau,\, \tau + t)$. It is assumed to follow a NHP process with parameter $\lambda(t)$ given by (2.19). From Theorem 2.1, we have the following corollary.

Corollary 2.1 For a NHP process with intensity function λ given by (2.19), the following properties hold:

(a) The probability of n claims in the time interval $[\tau,\ \tau+t)$ is:

$$P\{N_{[\tau,\tau+t)}=n\}=\frac{[\Lambda(\tau+t)-\Lambda(\tau)]^n}{n!}e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}\,,\qquad n\in\mathbb{N}\,,\qquad (2.28)$$

where $\Lambda(\tau+t)-\Lambda(\tau)=\int_\tau^{\tau+t}\lambda(v)dv$ can be derived from (2.20).

(b) The mgf of $N_{[\tau,\tau+t)}$ is given by

$$M_{N_{[\tau,\,\tau+t)}}(s)=E\big[e^{sN_{[\tau,\,\tau+t)}}\big]=e^{[\Lambda(\tau+t)-\Lambda(\tau)](e^s-1)}\,,\qquad s\in\mathbb{R}\qquad (2.29)$$

and the expected number of claims over this time interval equals its variance and is given by

$$E\big[N_{[\tau,\tau+t)}\big]=V\big[N_{[\tau,\tau+t)}\big]=\Lambda(\tau+t)-\Lambda(\tau)\,,\qquad \tau,\,t\geq 0\,.$$

In particular, the mgf of the number of claims over one period of length c, with an initial age of τ equals

$$M_{N_{[\tau,\,\tau+c)}}(s)=e^{\Lambda(c)(e^s-1)}=e^{D\,B(p_1,q_1)(e^s-1)\sum_{j=0}^{c-1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}\,,\qquad s\in\mathbb{R},\qquad (2.30)$$

where $\lambda_c(j+t_1^*)$ can be derived from (2.16) for $j=0,1,\cdots,c-1$.

(c) The probability to survive the time interval $[\tau,\ \tau+t)$ without a claim is

$$P\{N_{[\tau,\tau+t)}=0\}=e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}\,,\qquad \tau,\,t\geq 0,$$

while for a c-year period this probability (that we will denote by δ) is

$$\delta=P\{N_{[\tau,\tau+c)}=0\}=e^{-\Lambda(c)}=e^{-D\,B(p_1,q_1)\sum_{j=0}^{c-1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}\,.\qquad (2.31)$$

(d) The cdf of the waiting time T_1 for the first claim in $[0,\ t)$ is

$$P\{T_1\leq t\}=1-\delta^{\lfloor\frac{t}{c}\rfloor}e^{-D\,B(p_1,q_1)\sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}$$
$$e^{-D\,B\left(p_1,q_1;\,\frac{t-\lfloor t\rfloor-m_1}{D}\right)\frac{\lambda_c(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor+t_1^*)}{\alpha_1^*}}\,,\qquad t\geq 0\,,\qquad (2.32)$$

where δ is given by (2.31). The corresponding pdf is

$$f_{T_1}(t)\ =\ \delta^{\lfloor\frac{t}{c}\rfloor}e^{-D\,B(p_1,q_1)\sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}e^{-D\,B\left(p_1,q_1;\,\frac{t-\lfloor t\rfloor-m_1}{D}\right)\frac{\lambda_c(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor+t_1^*)}{\alpha_1^*}}$$
$$\lambda_c\left(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor+t_1^*\right)\lambda_1\left(t-\lfloor t\rfloor\right)\,,\qquad t\geq 0,\qquad (2.33)$$

while the expectation of T_1 is given by

$$E[T_1] = m_1 + \frac{c\,\delta + \sum_{j=1}^{c-1} j\, e^{-DB(p_1,q_1)\sum_{l=0}^{j-1}\frac{\lambda_c(l+t_1^*)}{\alpha_1^*}}\left(1 - e^{-DB(p_1,q_1)\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}\right)}{1-\delta}$$
$$+ \frac{D\sum_{j=0}^{c-1} e^{-D\,B(p_1,q_1)\sum_{l=0}^{j-1}\frac{\lambda_c(l+t_1^*)}{\alpha_1^*}} \int_0^1 s\, d\left[-e^{-D\,B(p_1,q_1;s)\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}\right]}{1-\delta}. \tag{2.34}$$

(e) At time t, the excess–life until the next claim, $T_{N(t)+1} - t$ is distributed as

$$P\{T_{N(t)+1} - t \le t_1\} = 1 - e^{-[\Lambda(t+t_1)-\Lambda(t)]}, \qquad t,\, t_1 \ge 0. \tag{2.35}$$

Proof.

(a) By the well known property of NHP processes, $N_{[\tau,\tau+t)}$ has a Poisson distribution with mean $\Lambda(\tau+t) - \Lambda(\tau)$, and hence (a) holds.

(b) Since the probability generating function (pgf) of a Poisson random variable Z with parameter λ^* is

$$\mathrm{P}_Z(z) = e^{\lambda^*(z-1)}, \qquad 0 < z < 1,$$

the mgf of $N_{[\tau,\tau+t)}$ (2.29) follows immediately.

In particular, when the time period of the counting process N is of length c, with an initial age of τ, the periodicity of intensity function λ with period of c gives

$$\Lambda(\tau+c) - \Lambda(\tau) = \Lambda(c) = D\,B(p_1,\, q_1)\sum_{j=0}^{c-1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}, \qquad \tau \ge 0.$$

This yields (2.30).

(c) This property follows from (a) by letting $n = 0$ in (2.28).

(d) As $P\{T_1 \le t\} = 1 - P\{N(t) = 0\} = 1 - e^{-\Lambda(t)}$, and

$$f_{T_1}(t) = \frac{d}{dt}\left[P\{T_1 \le t\}\right] = e^{-\Lambda(t)}\lambda(t),$$

(2.32) and (2.33) follow by the expression of $\lambda(t)$ given by (2.20) in Theorem 2.1 and the definition of δ in (2.31).

By the definition of the expectation of T_1,

$$\begin{aligned} E[T_1] &= \int_0^\infty t e^{-\Lambda(t)}\lambda(t)\,dt = \sum_{k=0}^{\infty}\int_0^c (kc+u)\,e^{-\Lambda(kc+u)}\,\lambda(kc+u)\,du \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{c-1}\int_0^1 (kc+j+v)\,e^{-\Lambda(kc+j+v)}\,\lambda(kc+j+v)\,dv \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{c-1}\int_{m_1}^{m_1+D} (kc+j+v)\,\delta^k e^{-D\,B(p_1,q_1)\sum_{l=0}^{j-1}\frac{\lambda_c(l+t_1^*)}{\alpha_1^*}} \\ &\qquad\qquad e^{-D\,B(p_1,q_1;\,\frac{v-m_1}{D})\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}\,\lambda_c(j+t_1^*)\,\lambda_1(v)\,dv, \end{aligned} \tag{2.36}$$

where $\lambda_1(v)$ is given by (2.1).

Let $s = \frac{v-m_1}{D}$, then the summation due to the first term kc in $(kc+j+v)$ of (2.36) can be simplified as

$$\begin{aligned} &\sum_{k=0}^{\infty} kc\,\delta^k \sum_{j=0}^{c-1} e^{-D\,B(p_1,q_1)\sum_{l=0}^{j-1}\frac{\lambda_c(l+t_1^*)}{\alpha_1^*}} \\ &\qquad\qquad \int_0^1 e^{-D\,B(p_1,q_1;s)\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}\,\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}\,s^{p_1-1}(1-s)^{q_1-1}\,D\,ds \\ &= \frac{c\,\delta}{(1-\delta)^2}\sum_{j=0}^{c-1} e^{-D\,B(p_1,q_1)\sum_{l=0}^{j-1}\frac{\lambda_c(l+t_1^*)}{\alpha_1^*}} \int_0^1 d\left[-e^{-D\,B(p_1,q_1;s)\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}\right] \\ &= \frac{c\,\delta}{(1-\delta)^2}\sum_{j=0}^{c-1} e^{-D\,B(p_1,q_1)\sum_{l=0}^{j-1}\frac{\lambda_c(l+t_1^*)}{\alpha_1^*}}\left[1 - e^{-D\,B(p_1,q_1)\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}\right] \\ &= \frac{c\,\delta}{(1-\delta)^2}\left[1 - e^{-D\,B(p_1,q_1)\sum_{l=0}^{c-1}\frac{\lambda_c(l+t_1^*)}{\alpha_1^*}}\right] \\ &= \frac{c\,\delta}{(1-\delta)^2}(1-\delta) = \frac{c\,\delta}{1-\delta}. \end{aligned} \tag{2.37}$$

Similarly, the summation due to the second term j in $(kc+j+v)$ of (2.36) is

$$\sum_{k=0}^{\infty}\delta^k \sum_{j=0}^{c-1} j\, e^{-D\,B(p_1,q_1)\sum_{l=0}^{j-1}\frac{\lambda_c(l+t_1^*)}{\alpha_1^*}}$$
$$\int_0^1 e^{-D\,B(p_1,q_1;\,s)\,\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}\,\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}\,s^{p_1-1}\,(1-s)^{q_1-1}\,D\,ds$$
$$= \frac{1}{1-\delta}\sum_{j=1}^{c-1} j\, e^{-D\,B(p_1,q_1)\sum_{l=0}^{j-1}\frac{\lambda_c(l+t_1^*)}{\alpha_1^*}} \int_0^1 d\left[-e^{-D\,B(p_1,q_1;\,s)\,\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}\right]$$
$$= \frac{1}{1-\delta}\sum_{j=1}^{c-1} j\, e^{-D\,B(p_1,q_1)\sum_{l=0}^{j-1}\frac{\lambda_c(l+t_1^*)}{\alpha_1^*}} \left[1-e^{-D\,B(p_1,q_1)\,\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}\right], \qquad (2.38)$$

while the summation due to the last term v in $(kc+j+v)$ of (2.36) can be written as

$$\sum_{k=0}^{\infty}\delta^k \sum_{j=0}^{c-1} e^{-D\,B(p_1,q_1)\sum_{l=0}^{j-1}\frac{\lambda_c(l+t_1^*)}{\alpha_1^*}}$$
$$\int_0^1 (D\,s+m_1)\, e^{-D\,B(p_1,q_1;\,s)\,\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}\,\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}\,s^{p_1-1}\,(1-s)^{q_1-1}\,D\,ds$$
$$= \frac{1}{1-\delta}\sum_{j=0}^{c-1} m_1\, e^{-D\,B(p_1,q_1)\sum_{l=0}^{j-1}\frac{\lambda_c(l+t_1^*)}{\alpha_1^*}} \int_0^1 d\left[-e^{-D\,B(p_1,q_1;\,s)\,\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}\right]$$
$$+\frac{D}{1-\delta}\sum_{j=0}^{c-1} e^{-D\,B(p_1,q_1)\sum_{l=0}^{j-1}\frac{\lambda_c(l+t_1^*)}{\alpha_1^*}} \int_0^1 s\,d\left[-e^{-D\,B(p_1,q_1;\,s)\,\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}\right]$$
$$= m_1+\frac{D}{1-\delta}\sum_{j=0}^{c-1} e^{-D\,B(p_1,q_1)\sum_{l=0}^{j-1}\frac{\lambda_c(l+t_1^*)}{\alpha_1^*}} \int_0^1 s\,d\left[-e^{-D\,B(p_1,q_1;\,s)\,\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}\right]. \qquad (2.39)$$

Finally combining (2.37), (2.38) and (2.39) yields (2.34).

(e) The distribution of the excess–life until the next claim in (2.35) follows easily by the relationship:

$$P\{T_{N(t)+1}-t\le t_1\} = 1-P\{T_{N(t)+1}-t>t_1\} = 1-P\{N_{[t,\,t+t_1)}=0\}\,.$$

□

The flexibility of the beta family of intensity functions, which depends on the value of the shape parameters p and q, provides many possible forms of short and long term seasonal claim intensities.

Other shapes, like periodic trigonometric functions could also be considered to model the long term periodicity. For example,

$$\lambda_c(t) = a + b \sin 2\pi \left(\frac{t - m_c}{c} - \lfloor \frac{t - m_c}{c} \rfloor \right),$$

where $a \geq b$ and $a + b$, $a - b$ represent, respectively, the maximum and minimum amplitude of the peak values for the long term periodicity, while m_c is the starting point of the periodic sine function.

Figure 2.3 illustrates the shape of $\lambda(t)$ for $p_1 = q_1 = 2$, $m_1 = 0$, $d = 1$, $c = 4$, $m_c = \frac{3}{2}$ $a = \frac{5}{4}$ and $b = 1$. Here the short–term beta peak values vary according to the sine function (dotted line). The properties for the corresponding hazard function Λ and claim counting process $\{N(t);\, t \geq 0\}$ can be derived analogously.

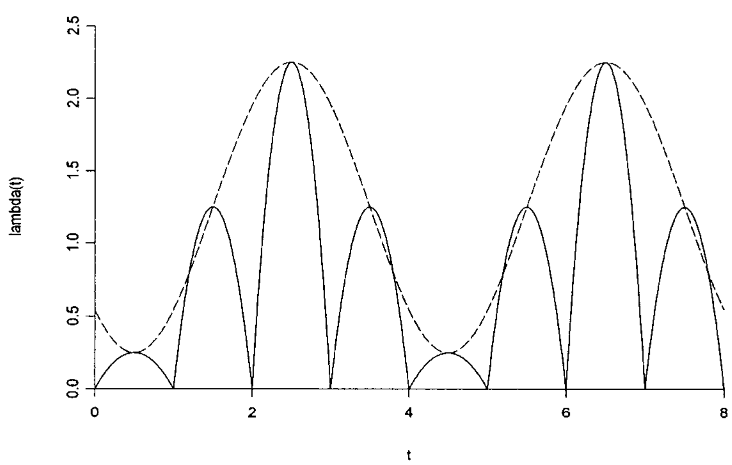

Figure 2.3: Sine–beta intensity function $\lambda(t)$

2.2 The Aggregate Claims Process

The decompositions of Theorem 2.1 for the NHP process can be extended to compound NHP sums.

Again consider a NHP claim counting process $\{N(t);\, t \geq 0\}$. Then the corresponding aggregate claims process

$$S(t) = \begin{cases} \sum_{n=1}^{N(t)} X_n & \text{if } N(t) > 0 \\ 0 & \text{if } N(t) = 0 \end{cases},$$

is called a compound NHP process and is denoted as $S(t) \sim$ C.P.$[\Lambda;\, F_X]$. The $\{X_n;\, n \geq 1\}$ are i.i.d. claim severities, with common cdf F_X and finite mean μ, independent of $N(t)$.

Consider the claim counting process $\{N_{[\tau,\tau+t)};\, t \geq 0\}$, for a fixed initial age τ and periodic intensity function λ. As in Theorem 2.1-(a), we have the following corollary for the corresponding cumulative hazard function of the process $\{N_{[\tau,\tau+t)};\, t \geq 0\}$.

Corollary 2.2 For a claim counting process $\{N_{[\tau,\tau+t)};\, t \geq 0\}$, with an initial age τ and periodic intensity function λ, given by (2.19), the cumulative hazard function Λ has the following structure:

$$\begin{aligned}
\Lambda(\tau + t) - \Lambda(\tau) \;=\; & D\,B\Big(p_1, q_1;\, \frac{\tau - \lfloor \tau \rfloor - m_1}{D},\, 1\Big) \frac{\lambda_c(\lfloor \tau - \lfloor \frac{\tau}{c} \rfloor c \rfloor + t_1^*)}{\alpha_1^*} \\
& + D\,B(p_1,\, q_1) \sum_{j=\lfloor \tau - \lfloor \frac{\tau}{c} \rfloor c \rfloor + 1}^{c-1} \frac{\lambda_c(j + t_1^*)}{\alpha_1^*} \\
& + \left(\lfloor \frac{t+\tau}{c} \rfloor - \lfloor \frac{\tau}{c} \rfloor - 1 \right) D\,B(p_1,\, q_1) \sum_{j=0}^{c-1} \frac{\lambda_c(j + t_1^*)}{\alpha_1^*} \\
& + D\,B(p_1,\, q_1) \sum_{j=0}^{\lfloor \tau + t - \lfloor \frac{\tau+t}{c} \rfloor c \rfloor - 1} \frac{\lambda_c(j + t_1^*)}{\alpha_1^*} \\
& + D\,B\Big(p_1,\, q_1;\, \frac{\tau + t - \lfloor \tau + t \rfloor - m_1}{D}\Big) \frac{\lambda_c(\lfloor \tau + t - \lfloor \frac{\tau+t}{c} \rfloor c \rfloor + t_1^*)}{\alpha_1^*}\,, \\
& \qquad \text{for } t,\, \tau \geq 0\,, \qquad (2.40)
\end{aligned}$$

where for any $p, q > 0$,

$$B(p,\, q;\, t,\, 1) = \begin{cases} B(p,\, q)\,, & \text{if } t \leq 0 \\ B(p,\, q) - B(p,\, q;\, t)\,, & \text{if } 0 < t < 1 \\ 0\,, & \text{if } t \geq 1 \end{cases}. \qquad (2.41)$$

Proof. By the definition of $\Lambda(t)$ and the periodicity of the function λ, we have

$$
\begin{aligned}
\Lambda(\tau+t)-\Lambda(\tau) &= \int_{\tau}^{\tau+t}\lambda(t)\,dt \\
&= \int_{\tau}^{\lfloor \tau-\lfloor \frac{\tau}{c}\rfloor c\rfloor+1}\lambda(t)\,dt+\int_{\lfloor \tau-\lfloor \frac{\tau}{c}\rfloor c\rfloor+1}^{c}\lambda(t)\,dt+\int_{0}^{\left(\lfloor\frac{t+\tau}{c}\rfloor-\lfloor\frac{\tau}{c}\rfloor-1\right)c}\lambda(t)\,dt \qquad (2.42)\\
&\quad+\int_{0}^{\lfloor \tau+t-\lfloor \frac{\tau+t}{c}\rfloor c\rfloor}\lambda(t)\,dt+\int_{\lfloor \tau+t-\lfloor \frac{\tau+t}{c}\rfloor c\rfloor}^{\tau+t}\lambda(t)\,dt \qquad (2.43)\\
&= \int_{\tau}^{\lfloor \tau-\lfloor \frac{\tau}{c}\rfloor c\rfloor+1}\lambda(t)\,dt+\sum_{j=\lfloor \tau-\lfloor \frac{\tau}{c}\rfloor c\rfloor+1}^{c-1}\int_{j}^{j+1}\lambda(t)\,dt+\sum_{j=0}^{\lfloor\frac{t+\tau}{c}\rfloor-\lfloor\frac{\tau}{c}\rfloor-2}\int_{j\,c}^{(j+1)\,c}\lambda(t)\,dt\\
&\quad+\sum_{j=0}^{\lfloor \tau+t-\lfloor \frac{\tau+t}{c}\rfloor c\rfloor-1}\int_{j}^{j+1}\lambda(t)\,dt+\int_{\lfloor \tau+t-\lfloor \frac{\tau+t}{c}\rfloor c\rfloor}^{\tau+t}\lambda(t)\,dt, \qquad (2.44)
\end{aligned}
$$

where the first and second integrals in (2.42) are the integrals over incomplete and complete years, respectively, in the first (incomplete) period, the third integral in (2.42) is over the complete cycles between the time period $[\tau,\ \tau+t)$, while the integrals in (2.43) are the corresponding integrals over complete years and the incomplete year, respectively, in the last (incomplete) period.

Now setting $s=\frac{t-m_1}{D}$, the first integral in (2.44) can be written as:

$$
\begin{aligned}
\int_{\tau}^{\lfloor \tau-\lfloor \frac{\tau}{c}\rfloor c\rfloor+1}\lambda(t)\,dt &= \lambda_c(\lfloor \tau-\lfloor\frac{\tau}{c}\rfloor c\rfloor+t_1^*)\int_{\tau-\lfloor\tau\rfloor}^{m_2}\lambda_1(t)\,dt\\
&= \frac{\lambda_c(\lfloor \tau-\lfloor\frac{\tau}{c}\rfloor c\rfloor+t_1^*)}{\alpha_1^*}\int_{\frac{\tau-\lfloor\tau\rfloor-m_1}{D}}^{1}s^{p_1-1}\,(1-s)^{q_1-1}\,D\,ds\\
&= D\,B\big(p_1,\,q_1;\,\frac{\tau-\lfloor\tau\rfloor-m_1}{D},\,1\big)\frac{\lambda_c(\lfloor \tau-\lfloor\frac{\tau}{c}\rfloor c\rfloor+t_1^*)}{\alpha_1^*},
\end{aligned}
$$

where $B(p,\,q;\,t,\,1)$, for $p,\,q>0$ and $t\in\mathbb{R}$, is given by (2.41).

Similarly, the last integral in (2.44) can be represented as incomplete beta functions:

$$
\begin{aligned}
\int_{\lfloor \tau+t-\lfloor \frac{\tau+t}{c}\rfloor c\rfloor}^{\tau+t}\lambda(t)\,dt &= \lambda_c(\lfloor \tau+t-\lfloor\frac{\tau+t}{c}\rfloor c\rfloor+t_1^*)\int_{m_1}^{\tau+t-\lfloor\tau+t\rfloor}\lambda_1(t)\,dt\\
&= \frac{\lambda_c(\lfloor \tau+t-\lfloor\frac{\tau+t}{c}\rfloor c\rfloor+t_1^*)}{\alpha_1^*}\int_{0}^{\frac{\tau+t-\lfloor\tau+t\rfloor-m_1}{D}}s^{p_1-1}\,(1-s)^{q_1-1}\,D\,ds\\
&= B\big(p_1,\,q_1;\,\frac{\tau+t-\lfloor\tau+t\rfloor-m_1}{D}\big)\frac{\lambda_c(\lfloor \tau+t-\lfloor\frac{\tau+t}{c}\rfloor c\rfloor+t_1^*)}{\alpha_1^*}.
\end{aligned}
$$

By the definition of the beta function, the two integrals over complete years in incomplete periods can be expressed as:

$$\begin{aligned}
\sum_{j=\lfloor \tau-\lfloor \frac{\tau}{c}\rfloor c\rfloor+1}^{c-1} \int_j^{j+1} \lambda(t)\, dt &= \sum_{j=\lfloor \tau-\lfloor \frac{\tau}{c}\rfloor c\rfloor+1}^{c-1} \lambda_c(j+t_1^*) \int_{m_1}^{m_2} \lambda_1(t)\, dt \\
&= \sum_{j=\lfloor \tau-\lfloor \frac{\tau}{c}\rfloor c\rfloor+1}^{c-1} \frac{\lambda_c(j+t_1^*)}{\alpha_1^*} \int_0^1 s^{p_1-1}\,(1-s)^{q_1-1}\, D\, ds \\
&= D\, B(p_1,\, q_1) \sum_{j=\lfloor \tau-\lfloor \frac{\tau}{c}\rfloor c\rfloor+1}^{c-1} \frac{\lambda_c(j+t_1^*)}{\alpha_1^*},
\end{aligned}$$

and respectively,

$$\begin{aligned}
\sum_{j=0}^{\lfloor \tau+t-\lfloor \frac{\tau+t}{c}\rfloor c\rfloor-1} \int_j^{j+1} \lambda(t)\, dt &= \sum_{j=0}^{\lfloor \tau+t-\lfloor \frac{\tau+t}{c}\rfloor c\rfloor-1} \lambda_c(j+t_1^*) \int_{m_1}^{m_2} \lambda_1(t)\, dt \\
&= \sum_{j=0}^{\lfloor \tau+t-\lfloor \frac{\tau+t}{c}\rfloor c\rfloor-1} \frac{\lambda_c(j+t_1^*)}{\alpha_1^*} \int_0^1 s^{p_1-1}\,(1-s)^{q_1-1}\, D\, ds \\
&= D\, B(p_1,\, q_1) \sum_{j=0}^{\lfloor \tau+t-\lfloor \frac{\tau+t}{c}\rfloor c\rfloor-1} \frac{\lambda_c(j+t_1^*)}{\alpha_1^*}.
\end{aligned}$$

Finally, since the intensity λ is a periodic function of period c, the integral over the complete cycles is given by

$$\begin{aligned}
\sum_{j=0}^{\lfloor \frac{t+\tau}{c}\rfloor-\lfloor \frac{\tau}{c}\rfloor-2} \int_{jc}^{(j+1)c} \lambda(t)\, dt &= \sum_{j=0}^{\lfloor \frac{t+\tau}{c}\rfloor-\lfloor \frac{\tau}{c}\rfloor-2} \sum_{k=0}^{c-1} \lambda_c(k+t_1^*) \int_{m_1}^{m_2} \lambda_1(t)\, dt \\
&= \left(\lfloor \frac{t+\tau}{c}\rfloor - \lfloor \frac{\tau}{c}\rfloor - 1\right) \sum_{j=0}^{c-1} \frac{\lambda_c(j+t_1^*)}{\alpha_1^*} \int_0^1 s^{p_1-1}\,(1-s)^{q_1-1}\, D\, ds \\
&= \left(\lfloor \frac{t+\tau}{c}\rfloor - \lfloor \frac{\tau}{c}\rfloor - 1\right) D\, B(p_1,\, q_1) \sum_{j=0}^{c-1} \frac{\lambda_c(j+t_1^*)}{\alpha_1^*}.
\end{aligned}$$

Combining all of the above integral expressions gives (2.40). □

The aggregate claims over $[\tau,\, \tau+t)$ are then given by $S_{[\tau,\tau+t)} = \sum_{n=1}^{N_{[\tau,\,\tau+t)}} X_n$, where $N_{[\tau,\tau+t)}$ is a NHP process with periodic intensity function λ as in (2.19) and $S_{[\tau,\tau+t)} = 0$ if $N_{[\tau,\tau+t)} = 0$. Theorem 2.1 implies the following decomposition result.

Corollary 2.3 For $t \geq 0$, independent of the initial age $\tau \geq 0$, then by Theorem 2.1 $S_{[\tau,\tau+t)}$ can be decomposed as independent sums of random variables:

$$
\begin{aligned}
S_{[\tau,\tau+t)} &= S^*_{[\tau,\lfloor\tau\rfloor+1)} + \sum_{j=\lfloor\tau-\lfloor\frac{\tau}{c}\rfloor c\rfloor+1}^{c-1} S^*_j + S_1 + \cdots + S_{\lfloor\frac{t+\tau}{c}\rfloor-\lfloor\frac{\tau}{c}\rfloor-1} \\
&+ \sum_{j=0}^{\lfloor\tau+t-\lfloor\frac{\tau+t}{c}\rfloor c\rfloor-1} S^*_j + S^*_{[\lfloor\tau+t\rfloor,\tau+t)},
\end{aligned}
\tag{2.45}
$$

where $S^*_{[\tau,\lfloor\tau\rfloor+1)}$, S^*_j, for $j = \lfloor\tau - \lfloor\frac{\tau}{c}\rfloor c\rfloor + 1, \ldots, c-1$, are compound Poisson sums, with parameters of $DB\left(p_1,\, q_1;\, \frac{\tau-\lfloor\tau\rfloor-m_1}{D},\, 1\right)\frac{\lambda_c(\lfloor\tau-\lfloor\frac{\tau}{c}\rfloor c\rfloor+t^*_1)}{\alpha^*_1}$, $DB(p_1,\, q_1)\frac{\lambda_c(j+t^*_1)}{\alpha^*_1}$, for $j = \lfloor\tau - \lfloor\frac{\tau}{c}\rfloor c\rfloor + 1, \ldots, c-1$, respectively, and S_i, $i = 1, \ldots, \lfloor\frac{t+\tau}{c}\rfloor - \lfloor\frac{\tau}{c}\rfloor - 1$, are i.i.d. random variables, distributed as $S(c) = \sum_{n=1}^{N(c)} X_n$, and $N(c)$ is a Poisson r.v. with parameter $\Lambda(c)$. While the terms S^*_j, for $j = 0, \ldots, \lfloor\tau + t - \lfloor\frac{\tau+t}{c}\rfloor c\rfloor - 1$, and $S^*_{[\lfloor\tau+t\rfloor,\tau+t)}$ are also the compound Poisson sums, with parameters of $DB(p_1,\, q_1)\frac{\lambda_c(j+t^*_1)}{\alpha^*_1}$, for $j = 0, \ldots, \lfloor\tau + t - \lfloor\frac{\tau+t}{c}\rfloor c\rfloor - 1$, and $DB(p_1,\, q_1;\, \frac{\tau+t-\lfloor\tau+t\rfloor-m_1}{D})\frac{\lambda_c(\lfloor\tau+t-\lfloor\frac{\tau+t}{c}\rfloor c\rfloor+t^*_1)}{\alpha^*_1}$, respectively. All these compound sums are mutually independent.

Proof. By Theorem 2.1-(b), the decomposition structure of the aggregate claims process (2.45) follows straightforwardly from the structure of its corresponding cumulative hazard function (2.40) given in Corollary 2.2.

Here $S^*_{[\tau,\lfloor\tau\rfloor+1)}$ and S^*_j, for $j = \lfloor\tau - \lfloor\frac{\tau}{c}\rfloor c\rfloor + 1, \ldots, c-1$, in the first sum of (2.45) represent claims for incomplete and complete years, in the first (incomplete) period. According to the corresponding Poisson counting process, these are compound Poisson sums, respectively, with parameters of $DB\left(p_1,\, q_1;\, \frac{\tau-\lfloor\tau\rfloor-m_1}{D},\, 1\right)\frac{\lambda_c(\lfloor\tau-\lfloor\frac{\tau}{c}\rfloor c\rfloor+t^*_1)}{\alpha^*_1}$, $DB(p_1,\, q_1)\frac{\lambda_c(j+t^*_1)}{\alpha^*_1}$, for $j = \lfloor\tau - \lfloor\frac{\tau}{c}\rfloor c\rfloor + 1, \ldots, c-1$.

The terms S_i, $i = 1, \ldots, \lfloor\frac{t+\tau}{c}\rfloor - \lfloor\frac{\tau}{c}\rfloor - 1$, in (2.45) represent claims for complete cycles. They are i.i.d. random variables, distributed as the compound Poisson sum $S(c) = \sum_{n=1}^{N(c)} X_n$, and $N(c)$ is a Poisson r.v. with parameter $\Lambda(c)$.

The terms S^*_j, for $j = 0, \ldots, \lfloor\tau + t - \lfloor\frac{\tau+t}{c}\rfloor c\rfloor - 1$, and $S^*_{[\lfloor\tau+t\rfloor,\tau+t)}$ in the last sum of (2.45) represent complete years and incomplete year in the last (incomplete) period.

They are also compound Poisson sums, respectively, with means of $D\,B(p_1,\,q_1)\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}$, for $j=0,\ldots,\lfloor\tau+t-\lfloor\frac{\tau+t}{c}\rfloor c\rfloor-1$, and $D\,B(p_1,\,q_1;\,\frac{\tau+t-\lfloor\tau+t\rfloor-m_1}{D})\frac{\lambda_c(\lfloor\tau+t-\lfloor\frac{\tau+t}{c}\rfloor c\rfloor+t_1^*)}{\alpha_1^*}$.

Finally by the property of the Poisson processes, all these compound sums are mutually independent. □

Moreover, the mgf of $S_{[\tau,\tau+t)}$ is obtained as

$$M_{S_{[\tau,\,\tau+t)}}(s)=E\big[e^{s\,S_{[\tau,\,\tau+t)}}\big]=e^{[\Lambda(\tau+t)-\Lambda(\tau)][M_X(s)-1]}\,,\qquad s<a_X\,,\tag{2.46}$$

where M_X is the mgf of the claims severity distribution such that it is finite on $(-\infty,a_X)$. Moments of $S_{[\tau,\tau+t)}$ are easily obtained from (2.46). For instance, the total initial premium is given by

$$E\big[S_{[\tau,\tau+t)}\big]=[\Lambda(\tau+t)-\Lambda(\tau)]E[X_1]\,.$$

Chapter 3

Statistical inference on NHP processes and an application

In this chapter, we are interested in statistical aspects of the proposed periodic NHP models. The decomposition property of the models, given in Theorem 2.1-(b), makes it easier to construct likelihood functions for NHP processes. Then the maximum likelihood estimation method can be used to estimate the parameters in the parametric intensity functions.

Statistical inferences on the model parameters, including the maximum likelihood estimators, the information matrix and the estimated standard deviations, are presented in Section 3.1 and 3.2, respectively. An application of the model to the dataset of Atlantic Hurricanes Affecting the United States (1899-2000) is discussed in detail in Section 3.3.

3.1 Maximum likelihood function for periodic intensities

The double–beta periodic intensity model in (2.19) is a parametric function with 6 parameters: p_1, q_1, p_c, q_c, a and b. It is possible to estimate these parameters from data using maximum likelihood estimation. Note that other model time–scale parameters as m_1, m_2, m_c and c can usually be set visually at values observed from empirical plots of the dataset.

Let d be the time scale in each short–term cycle; here $d = \frac{1}{12}$, a month in each year. Then, for the short–term intensity function in (2.1), denote by m_1 and m_2 two integer–multiples of d; here m_1 and m_2 correspond to two specific months in the year, marking the beginning and end of the hurricane season. Furthermore define J as

$$J = \frac{m_2 - m_1}{d} = \frac{D}{d} \,,$$

that is the total number of months in each year over which the intensity function is positive. This gives a convenient partition of each year cycle $[0, m_1)$, $[m_1, t_1)$, $[t_1, t_2)$, ..., $[t_J, m_2)$, $[m_2, 1]$, where

$$t_j = m_1 + j\,d \,, \qquad \text{for } j = 0, \ldots, J \,. \tag{3.1}$$

Under the double–beta intensity function given in (2.19), the contribution to the likelihood for the first year of the first cycle is:

$$\begin{aligned} L_{1,1} &= e^{-\int_0^{m_1} \lambda(v)\,dv} \prod_{j=1}^{J} \left[e^{-\int_{t_{j-1}}^{t_j} \lambda(v)\,dv} \left(\int_{t_{j-1}}^{t_j} \lambda(v)\,dv \right)^{n_{j,1}^{(1)}} \right] e^{-\int_{m_2}^{1} \lambda(v)\,dv} \qquad (3.2) \\ &= e^{-\int_0^1 \lambda(v)\,dv} \prod_{j=1}^{J} \left(\int_{t_{j-1}}^{t_j} \lambda(v)\,dv \right)^{n_{j,1}^{(1)}} \,, \end{aligned}$$

where $n_{j,1}^{(1)}$ is the number of events which occured within the j-th month $[t_{j-1}, t_j)$ of the first year of the first cycle, for $j = 1, \ldots, J$. The first and the last term in (3.2) represent the likelihood of having no events outside the time interval $[m_1, m_2]$.

In general, the contribution to the likelihood by the k-th year of the i-th cycle is similarly given by

$$L_{k,i} = e^{-\int_{k-1}^{k} \lambda(v)\,dv} \prod_{j=1}^{J} \left(\int_{(k-1)+t_{j-1}}^{(k-1)+t_j} \lambda(v)\,dv \right)^{n_{j,k}^{(i)}} \,, \qquad k = 1, \ldots, c, \; i = 1, \ldots, \lfloor \frac{t}{c} \rfloor \,,$$

where $n_{j,k}^{(i)}$ is the number of hurricanes within the j-th month of the k-th year of the i-th cycle.

Hence for the i-th cycle, $i = 1, \ldots, \lfloor \frac{t}{c} \rfloor$, the total contribution to the likelihood is given by

$$L_i = \prod_{k=1}^{c} L_{k,i} = e^{-\int_0^c \lambda(u)\,du} \prod_{k=1}^{c} \prod_{j=1}^{J} \left(\int_{(k-1)+t_{j-1}}^{(k-1)+t_j} \lambda(v)\,dv \right)^{n_{j,k}^{(i)}},$$

while the likelihood function for all $\lfloor \frac{t}{c} \rfloor$ complete cycles is

$$L_{comp} = e^{-\lfloor \frac{t}{c} \rfloor \int_0^c \lambda(v)\,dv} \prod_{k=1}^{c} \prod_{j=1}^{J} \left(\int_{(k-1)+t_{j-1}}^{(k-1)+t_j} \lambda(v)\,dv \right)^{\sum_{i=1}^{\lfloor \frac{t}{c} \rfloor} n_{j,k}^{(i)}}. \tag{3.3}$$

Finally, the contribution to the likelihood from the last incomplete cycle is composed of the contributions by complete years in the last cycle, the complete months in the last incomplete year and the last incomplete month. For simplicity, set $\tau_c = \lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor$ to be the number of years in the last incomplete cycle, we have

$$\begin{aligned} L_{incomp} = & \prod_{k=1}^{\tau_c} \left[e^{-\int_{k-1}^{k} \lambda(v)\,dv} \prod_{j=1}^{J} \left(\int_{(k-1)+t_{j-1}}^{(k-1)+t_j} \lambda(v)\,dv \right)^{n_{j,k}^{(\lfloor \frac{t}{c} \rfloor+1)}} \right] \\ & \times e^{-\int_{\lfloor \frac{t}{c} \rfloor c+\tau_c}^{\lfloor \frac{t}{c} \rfloor c+\tau_c+m_1} \lambda(v)\,dv} \prod_{j=1}^{J^*} \left[e^{-\int_{\tau_c+t_{j-1}}^{\tau_c+t_j} \lambda(v)\,dv} \left(\int_{\tau_c+t_{j-1}}^{\tau_c+t_j} \lambda(v)\,dv \right)^{n_{j,\tau_c+1}^{(\lfloor \frac{t}{c} \rfloor+1)}} \right] \\ & \times e^{-\int_{\lfloor \frac{t}{c} \rfloor c+\tau_c+t_{J^*}}^{t} \lambda(v)\,dv} \left(\int_{\tau_c+t_{J^*}}^{t-\lfloor \frac{t}{c} \rfloor c} \lambda(v)\,dv \right)^{n_{J^*+1,\tau_c+1}^{(\lfloor \frac{t}{c} \rfloor+1)}}, \end{aligned} \tag{3.4}$$

where

$$J^* = \left\lfloor \frac{t - \lfloor \frac{t}{c} \rfloor c - \lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - m_1}{d} \right\rfloor = \left\lfloor \frac{t - \lfloor \frac{t}{c} \rfloor c - \tau_c - m_1}{d} \right\rfloor$$

is the number of months in the last incomplete year (set to 0 when J^* is a negative integer).

Hence the full likelihood function is given by (3.3) and (3.4) to be

$$
\begin{aligned}
L &= L_{comp} \cdot L_{incomp} \\
&= e^{-\Lambda(t)} \prod_{k=1}^{\tau_c} \prod_{j=1}^{J} \left(\int_{(k-1)+t_{j-1}}^{(k-1)+t_j} \lambda(v)\, dv \right)^{\sum_{i=1}^{\lfloor \frac{t}{c} \rfloor + 1} n_{j,k}^{(i)}} \\
&\quad \times \prod_{k=\tau_c+1}^{c} \prod_{j=1}^{J} \left(\int_{(k-1)+t_{j-1}}^{(k-1)+t_j} \lambda(v)\, dv \right)^{\sum_{i=1}^{\lfloor \frac{t}{c} \rfloor} n_{j,k}^{(i)}} \\
&\quad \times \prod_{j=1}^{J^*} \left(\int_{\tau_c+t_{j-1}}^{\tau_c+t_j} \lambda(v)\, dv \right)^{n_{j,\tau_c+1}^{(\lfloor \frac{t}{c} \rfloor + 1)}} \left(\int_{\tau_c+t_{J^*}}^{t-\lfloor \frac{t}{c} \rfloor c} \lambda(v)\, dv \right)^{n_{J^*+1,\tau_c+1}^{(\lfloor \frac{t}{c} \rfloor + 1)}} . \qquad (3.5)
\end{aligned}
$$

Substituting λ for the double–beta periodic intensity function in (2.19), we have for $l = 0, \ldots, c-1$,

$$
\int_{l+t_{j-1}}^{l+t_j} \lambda(v)\, dv = \frac{\lambda_c(l+t_1^*)}{\alpha_1^*} \int_{l+t_{j-1}}^{l+t_j} \left(\frac{v-l-m_1}{D} \right)^{p_1-1} \left(1 - \frac{v-l-m_1}{D} \right)^{q_1-1} dv \, . \qquad (3.6)
$$

Setting $s = \frac{v-l-m_1}{D}$ in (3.6) and following the notation in (3.1) gives

$$
\begin{aligned}
\int_{l+t_{j-1}}^{l+t_j} \lambda(v)\, dv &= \frac{\lambda_c(l+t_1^*)}{\alpha_1^*} \int_{\frac{(j-1)d}{D}}^{\frac{jd}{D}} s^{p_1-1} (1-s)^{q_1-1} D\, ds \\
&= \frac{\lambda_c(l+t_1^*)}{\alpha_1^*} D \left[B(p_1, q_1; \frac{jd}{D}) - B(p_1, q_1; \frac{(j-1)d}{D}) \right] .
\end{aligned}
$$

Similarly, by setting $s = \frac{v-\tau_c-m_1}{D}$,

$$
\begin{aligned}
\int_{\tau_c+t_{J^*}}^{t-\lfloor \frac{t}{c} \rfloor c} \lambda(v)\, dv &= \frac{\lambda_c(\tau_c+t_1^*)}{\alpha_1^*} \int_{\tau_c+t_{J^*}}^{t-\lfloor \frac{t}{c} \rfloor c} \left(\frac{v-\tau_c-m_1}{D} \right)^{p_1-1} \\
&\qquad \left(1 - \frac{v-\tau_c-m_1}{D} \right)^{q_1-1} dv \\
&= \frac{\lambda_c(\tau_c+t_1^*)}{\alpha_1^*} \int_{\frac{J^*d}{D}}^{\frac{t-\lfloor \frac{t}{c} \rfloor c-\tau_c-m_1}{D}} s^{p_1-1} (1-s)^{q_1-1} D\, ds \\
&= \frac{\lambda_c(\tau_c+t_1^*)}{\alpha_1^*} D \left[B(p_1, q_1; \frac{t-\lfloor \frac{t}{c} \rfloor c-\tau_c-m_1}{D}) - B(p_1, q_1; \frac{J^*d}{D}) \right] .
\end{aligned}
$$

Then the integrals in (5.5) can be represented as incomplete beta functions, yielding:

$$
\begin{aligned}
L = e^{-\Lambda(t)} \prod_{k=1}^{\tau_c} \prod_{j=1}^{J} & \left\{ \frac{\lambda_c(k-1+t_1^*)}{\alpha_1^*} D \left[B(p_1, q_1; \frac{jd}{D}) - B(p_1, q_1; \frac{(j-1)d}{D}) \right] \right\}^{\sum_{i=1}^{\lfloor \frac{t}{c} \rfloor + 1} n_{j,k}^{(i)}} \\
\times \prod_{k=\tau_c+1}^{c} \prod_{j=1}^{J} & \left\{ \frac{\lambda_c(k-1+t_1^*)}{\alpha_1^*} D \left[B(p_1, q_1; \frac{jd}{D}) - B(p_1, q_1; \frac{(j-1)d}{D}) \right] \right\}^{\sum_{i=1}^{\lfloor \frac{t}{c} \rfloor} n_{j,k}^{(i)}} \\
\times \prod_{j=1}^{J^*} & \left\{ \frac{\lambda_c(\tau_c+t_1^*)}{\alpha_1^*} D \left[B(p_1, q_1; \frac{jd}{D}) - B(p_1, q_1; \frac{(j-1)d}{D}) \right] \right\}^{n_{j,\tau_c+1}^{(\lfloor \frac{t}{c} \rfloor + 1)}} \\
\times & \left\{ \frac{\lambda_c(\tau_c+t_1^*)}{\alpha_1^*} D \left[B(p_1, q_1; \frac{t - \lfloor \frac{t}{c} \rfloor c - \tau_c - m_1}{D}) - B(p_1, q_1; \frac{J^* d}{D}) \right] \right\}^{n_{J^*+1,\tau_c+1}^{(\lfloor \frac{t}{c} \rfloor + 1)}} ,
\end{aligned}
\tag{3.7}
$$

where the function λ_c is given in (2.16) and the cumulative intensity function $\Lambda(t)$ is given by (2.20).

Consequently, the log likelihood function is given by

$$
\begin{aligned}
l = & -\Lambda(t) \\
& + \sum_{k=1}^{\tau_c} \sum_{j=1}^{J} \left[\sum_{i=1}^{\lfloor \frac{t}{c} \rfloor + 1} n_{j,k}^{(i)} \right] \log \left[\frac{\lambda_c(k-1+t_1^*)}{\alpha_1^*} \right. \\
& \qquad \left. \times D \left(B(p_1, q_1, \frac{jd}{D}) - B(p_1, q_1, \frac{(j-1)d}{D}) \right) \right] \\
& + \sum_{k=\tau_c+1}^{c} \sum_{j=1}^{J} \left[\sum_{i=1}^{\lfloor \frac{t}{c} \rfloor} n_{j,k}^{(i)} \right] \log \left[\frac{\lambda_c(k-1+t_1^*)}{\alpha_1^*} \right. \\
& \qquad \left. \times D \left(B(p_1, q_1, \frac{jd}{D}) - B(p_1, q_1, \frac{(j-1)d}{D}) \right) \right] \\
& + \sum_{j=1}^{J^*} n_{j,\tau_c+1}^{(\lfloor \frac{t}{c} \rfloor + 1)} \log \left[\frac{\lambda_c(\tau_c+t_1^*)}{\alpha_1^*} D \left(B(p_1, q_1, \frac{jd}{D}) - B(p_1, q_1, \frac{(j-1)d}{D}) \right) \right] \\
& + n_{J^*+1,\tau_c+1}^{(\lfloor \frac{t}{c} \rfloor + 1)} \log \left[\frac{\lambda_c(\tau_c+t_1^*)}{\alpha_1^*} D \left(B(p_1, q_1, \frac{t - \lfloor \frac{t}{c} \rfloor c - \tau_c - m_1}{D}) \right. \right. \\
& \qquad \left. \left. - B(p_1, q_1, \frac{J^* d}{D}) \right) \right] .
\end{aligned}
\tag{3.8}
$$

Denote by $N = \sum_{i,j,k} n_{j,k}^{(i)}$ the total number of occurrences, for $1 \le i \le \lfloor \frac{t}{c} \rfloor + 1$,

$1 \le j \le J$ and $1 \le k \le c$. Further denote by

$$n_{\cdot,k}^{(\cdot)} = \sum_{j=1}^{J} \sum_{i=1}^{\lfloor \frac{t}{c} \rfloor} n_{j,k}^{(i)} , \qquad k = 1, 2, \ldots, c ,$$

the total number of occurrences in the k-th year of all complete cycles, while $n_{\cdot,k}^{(\lfloor \frac{t}{c} \rfloor + 1)}$ stands for the count in the k-th year of the last incomplete cycle. Similarly

$$n_{j,\cdot}^{(\cdot)} = \sum_{k=1}^{c} \sum_{i=1}^{\lfloor \frac{t}{c} \rfloor} n_{j,k}^{(i)} , \qquad j = 1, 2, \ldots, J ,$$

denotes the total number of occurrences in the j-th month of all complete cycles, while $n_{j,\cdot}^{(\lfloor \frac{t}{c} \rfloor + 1)}$ stands for the count in the j-th complete month of the last incomplete cycle. Consequently, the log likelihood function (3.8) can be written as

$$\begin{aligned} l &= -\Lambda(t) + N \log \frac{D}{\alpha_1^*} + \sum_{k=1}^{c} \left[n_{\cdot,k}^{(\cdot)} + n_{\cdot,k}^{(\lfloor \frac{t}{c} \rfloor + 1)} \right] \log \lambda_c(k - 1 + t_1^*) \\ &\quad + \sum_{j=1}^{J} \left[n_{j,\cdot}^{(\cdot)} + n_{j,\cdot}^{(\lfloor \frac{t}{c} \rfloor + 1)} \right] \log \left[B(p_1, q_1; \frac{jd}{D}) - B(p_1, q_1; \frac{(j-1)d}{D}) \right] \\ &\quad + n_{J^*+1,\tau_c+1}^{(\lfloor \frac{t}{c} \rfloor + 1)} \log \left[B(p_1, q_1; \frac{t - \lfloor \frac{t}{c} \rfloor c - \tau_c - m_1}{D}) - B(p_1, q_1; \frac{J^* d}{D}) \right] . \end{aligned} \quad (3.9)$$

The maximum likelihood estimators for p_1, q_1, p_c, q_c, a and b in the double–beta intensity function are obtained by maximizing l numerically.

Similarly, the maximum likelihood estimators for parameters in the model given in Section 2.1.1, with a double periodic intensity function (2.12) and a generalized 3–parameter beta short–term intensity function λ_1, can be derived as follows.

To simplify expressions, let t be an integer number here. Assume that the short–term intensity function for the k-th year of a cycle is of the generalized 3–parameter beta form in (2.7) with parameters $p_1^{(k)}$, $q_1^{(k)}$ and $\epsilon^{(k)}$ and λ_k^* is the peak value, where $k = 1, 2, \ldots, c$, that is,

$$\lambda(t) = \lambda_k^* \, \lambda_1\big(t - \lfloor t \rfloor\big) , \qquad k - 1 \le t - \lfloor \frac{t}{c} \rfloor c < k , \quad (3.10)$$

where function λ_1 is of the form

$$\lambda_1(t) = \frac{\left(\frac{t-m_1}{D}\right)^{p_1^{(k)}-1} \left(1 - \frac{t-m_1}{D}\right)^{q_1^{(k)}-1}}{\alpha_1^{(k)} \left[1 - (1 - \epsilon^{(k)}) \left(\frac{t-m_1}{D}\right)\right]^{p_1^{(k)}+q_1^{(k)}}} , \qquad 0 \le m_1 \le t \le m_2 \le 1 , \quad (3.11)$$

while $\alpha_1^{(k)}$ is the scale factor of the k-th year of each cycle.

For $1 \le k \le \tau_c$, the likelihood function for all the k-th years over the entire time period $[0,\ t)$ is given by

$$\begin{aligned}
L^{(k)} &= \prod_{i=1}^{\lfloor \frac{t}{c} \rfloor + 1} e^{-\lambda_k^* \int_{k-1}^{k} \lambda_1(v)\, dv} \prod_{j=1}^{J} \left(\lambda_k^* \int_{(k-1)+t_{j-1}}^{(k-1)+t_j} \lambda_1(v)\, dv \right)^{n_{j,k}^{(i)}} , \qquad k = 1, \ldots, \tau_c \\
&= e^{-(\lfloor \frac{t}{c} \rfloor + 1) \lambda_k^* \int_{k-1}^{k} \lambda_1(v)\, dv} \prod_{j=1}^{J} \left(\lambda_k^* \int_{(k-1)+t_{j-1}}^{(k-1)+t_j} \lambda_1(v)\, dv \right)^{\sum_{i=1}^{\lfloor \frac{t}{c} \rfloor + 1} n_{j,k}^{(i)}} \qquad (3.12) \\
&= e^{-(\lfloor \frac{t}{c} \rfloor + 1) \frac{\lambda_k^* D B(p_1^{(k)}, q_1^{(k)})}{\alpha_1^{(k)} [\epsilon^{(k)}]^{p_1^{(k)}}}} \left(\frac{\lambda_k^* D}{\alpha_1^{(k)} [\epsilon^{(k)}]^{p_1^{(k)}}} \right)^{\sum_{j=1}^{J} \left[n_{j,k}^{(\cdot)} + n_{j,k}^{(\lfloor \frac{t}{c} \rfloor + 1)} \right]} \\
&\quad \times \prod_{j=1}^{J} \left[B\left(p_1^{(k)}, q_1^{(k)}; \frac{\epsilon^{(k)} \frac{jd}{D}}{1 - (1 - \epsilon^{(k)}) \frac{jd}{D}} \right) \right. \\
&\qquad \left. - B\left(p_1^{(k)}, q_1^{(k)}; \frac{\epsilon^{(k)} \frac{(j-1)d}{D}}{1 - [1 - \epsilon^{(k)}] \frac{(j-1)d}{D}} \right) \right]^{n_{j,k}^{(\cdot)} + n_{j,k}^{(\lfloor \frac{t}{c} \rfloor + 1)}} ,
\end{aligned}$$

where $n_{j,k}^{(\cdot)} = \sum_{i=1}^{\lfloor \frac{t}{c} \rfloor} n_{j,k}^{(i)}$ is the total number of occurrences in the j-th month of the k-th year of all complete cycles. Similarly as the derivation of (2.11), the integrals in (3.12) are derived in terms of incomplete beta functions.

Accordingly, the log–likelihood function is given by

$$\begin{aligned}
l^{(k)} &= -\left(\lfloor \frac{t}{c} \rfloor + 1 \right) \frac{\lambda_k^* D B(p_1^{(k)}, q_1^{(k)})}{\alpha_1^{(k)} [\epsilon^{(k)}]^{p_1^{(k)}}} + [n_{\cdot,k}^{(\cdot)} + n_{\cdot,k}^{(\lfloor \frac{t}{c} \rfloor + 1)}] \log \left(\frac{\lambda_k^* D}{\alpha_1^{(k)} [\epsilon^{(k)}]^{p_1^{(k)}}} \right) \\
&\quad + \sum_{j=1}^{J} [n_{j,k}^{(\cdot)} + n_{j,k}^{(\lfloor \frac{t}{c} \rfloor + 1)}] \log \left[B\left(p_1^{(k)}, q_1^{(k)}; \frac{\epsilon^{(k)} \frac{jd}{D}}{1 - (1 - \epsilon^{(k)}) \frac{jd}{D}} \right) \right. \\
&\qquad \left. - B\left(p_1^{(k)}, q_1^{(k)}; \frac{\epsilon^{(k)} \frac{(j-1)d}{D}}{1 - [1 - \epsilon^{(k)}] \frac{(j-1)d}{D}} \right) \right] , \qquad k = 1, \ldots, \tau_c . \qquad (3.13)
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
l^{(k)} &= -\lfloor\frac{t}{c}\rfloor \frac{\lambda_k^* D B(p_1^{(k)}, q_1^{(k)})}{\alpha_1^{(k)}[\epsilon^{(k)}]^{p_1^{(k)}}} + n_{\cdot,k}^{(\cdot)} \log\left(\frac{\lambda_k^* D}{\alpha_1^{(k)}[\epsilon^{(k)}]^{p_1^{(k)}}}\right) \\
&\quad + \sum_{j=1}^{J} n_{j,k}^{(\cdot)} \log\left[B\left(p_1^{(k)}, q_1^{(k)}; \frac{\epsilon^{(k)}\frac{jd}{D}}{1-(1-\epsilon^{(k)})\frac{jd}{D}}\right)\right. \\
&\quad \left. - B\left(p_1^{(k)}, q_1^{(k)}; \frac{\epsilon^{(k)}\frac{(j-1)d}{D}}{1-(1-\epsilon^{(k)})\frac{(j-1)d}{D}}\right)\right], \qquad k = \tau_c + 1, \ldots, c\,. \quad (3.14)
\end{aligned}
$$

The maximum likelihood estimators for $p_1^{(k)}$, $q_1^{(k)}$, $\epsilon^{(k)}$ and λ_k^*, for $k = 1, 2, \ldots, c$, in this double periodic intensity function are obtained by maximizing (3.13) and (3.14) numerically.

3.2 Information matrix and estimated standard deviations

It is well-known that the maximum likelihood estimators have the smallest variance among estimators that have asymptotic normal distributions. Further, an explicit formula for the asymptotic variance of the estimators can be derived.

The variance–covariance matrix is obtained by inverting the matrix whose rs–element is

$$
\mathbf{I}(\theta)_{rs} = -E\left[\frac{\partial^2}{\partial\theta_s \partial\theta_r} l(\theta)\right], \qquad (3.15)
$$

where θ is the vector of MLE's in the model, and it has an asymptotic multivariate normal distribution. This matrix is often called the information matrix. The estimated standard deviations of the MLE's are then obtained by taking the square root of the variances on the main diagonal.

Here we give a detailed calculation of the estimated standard deviations of the MLE's for the parameters in our double–beta intensity model. Assume that the parameter vector $\theta = (p_1,\, p_c,\, a,\, b)^T$, and t_1^*, t_c^*, m_1, m_2 and m_c can be observed from the data. Other parameters, like q_1 and q_c, can be evaluated from (2.3) and (2.18), respectively, once θ is known.

For simplicity, we assume that t is an integer in the following derivation. In this case, the log–likelihood function is given by

$$\begin{aligned} l &= -\Lambda(t) + N \log \frac{D}{\alpha_1^*} + \sum_{k=1}^{c} n_{\cdot,k}^{(\cdot)} \log \lambda_c(k-1+t_1^*) \\ &\quad + \sum_{j=1}^{J} n_{j,\cdot}^{(\cdot)} \log \left[B(p_1, q_1; \frac{jd}{D}) - B(p_1, q_1; \frac{(j-1)d}{D}) \right], \end{aligned} \tag{3.16}$$

where the cumulative intensity function Λ has the form

$$\Lambda(t) = \lfloor \frac{t}{c} \rfloor D\, B(p_1, q_1) \sum_{j=0}^{c-1} \frac{\lambda_c(j+t_1^*)}{\alpha_1^*} + D\, B(p_1, q_1) \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - 1} \frac{\lambda_c(j+t_1^*)}{\alpha_1^*}, \tag{3.17}$$

and $\lambda_c(t)$ is given by (2.16).

The first partial derivatives of (3.16) w.r.t. p_1, p_c, a and b are:

$$\begin{aligned} \frac{\partial l}{\partial p_1} &= -\frac{\partial \Lambda(t)}{\partial p_1} - N \frac{\frac{\partial \alpha_1^*}{\partial p_1}}{\alpha_1^*} + \sum_{j=1}^{J} n_{j,\cdot}^{(\cdot)} \left[\frac{B'_{p_1}(p_1, q_1; \frac{jd}{D}) - B'_{p_1}(p_1, q_1; \frac{(j-1)d}{D})}{B(p_1, q_1; \frac{jd}{D}) - B(p_1, q_1; \frac{(j-1)d}{D})} \right], \\ \frac{\partial l}{\partial p_c} &= -\frac{\partial \Lambda(t)}{\partial p_c} + \sum_{k=1}^{c} n_{\cdot,k}^{(\cdot)} \frac{\frac{\partial \lambda_c(k-1+t_1^*)}{\partial p_c}}{\lambda_c(k-1+t_1^*)}, \\ \frac{\partial l}{\partial a} &= -\frac{\partial \Lambda(t)}{\partial a} + \sum_{k=1}^{c} n_{\cdot,k}^{(\cdot)} \frac{\frac{\partial \lambda_c(k-1+t_1^*)}{\partial a}}{\lambda_c(k-1+t_1^*)}, \\ \frac{\partial l}{\partial b} &= -\frac{\partial \Lambda(t)}{\partial b} + \sum_{k=1}^{c} n_{\cdot,k}^{(\cdot)} \frac{\frac{\partial \lambda_c(k-1+t_1^*)}{\partial b}}{\lambda_c(k-1+t_1^*)}, \end{aligned}$$

where

$$\frac{\frac{\partial \alpha_1^*}{\partial p_1}}{\alpha_1^*} = \nu^* := \ln\left(\frac{t_1^* - m_1}{D}\right) + \left(\frac{q_1 - 1}{p_1 - 1}\right) \ln\left(1 - \frac{t_1^* - m_1}{D}\right),$$

$$B'_{p_1}(p_1, q_1; t) = \int_0^t v^{p_1 - 1} (1-v)^{q_1 - 1} \Big[\ln(v) + \left(\frac{q_1 - 1}{p_1 - 1}\right) \ln(1-v) \Big] dv, \qquad t \in (0,\ 1).$$

Note here $\frac{q_1-1}{p_1-1} = \frac{1}{\frac{(t_1^* - m_1)}{D}} - 1$, which is a constant. Accordingly, ν^* is also a constant.

While the first partial derivatives of (3.17) w.r.t. p_1, p_c, a and b are given by

$$\begin{aligned}
\frac{\partial\Lambda(t)}{\partial p_1} &= \Lambda(t)\left[\frac{B'_{p_1}(p_1,q_1)}{B(p_1,q_1)} - \nu^*\right],\\
\frac{\partial\Lambda(t)}{\partial p_c} &= \frac{D\,B(p_1,q_1)}{\alpha_1^*}\left[\lfloor\frac{t}{c}\rfloor\sum_{j=0}^{c-1}\frac{\partial\lambda_c(j+t_1^*)}{\partial p_c} + \sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1}\frac{\partial\lambda_c(j+t_1^*)}{\partial p_c}\right],\\
\frac{\partial\Lambda(t)}{\partial a} &= \frac{D\,B(p_1,q_1)}{\alpha_1^*}\left[\lfloor\frac{t}{c}\rfloor\sum_{j=0}^{c-1}\frac{\partial\lambda_c(j+t_1^*)}{\partial a} + \sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1}\frac{\partial\lambda_c(j+t_1^*)}{\partial a}\right],\\
\frac{\partial\Lambda(t)}{\partial b} &= \frac{D\,B(p_1,q_1)}{\alpha_1^*}\left[\lfloor\frac{t}{c}\rfloor\sum_{j=0}^{c-1}\frac{\partial\lambda_c(j+t_1^*)}{\partial b} + \sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1}\frac{\partial\lambda_c(j+t_1^*)}{\partial b}\right],
\end{aligned}$$

here $B'_{p_1}(p_1,q_1) = B'_{p_1}(p_1,q_1;1)$ and the first partial derivatives of $\lambda_c(t)$ w.r.t. p_c, a and b are given by

$$\begin{aligned}
\frac{\partial\lambda_c(j+t_1^*)}{\partial p_c} &= \left[\lambda_c(j+t_1^*) - a\right]\Bigg\{\ln\left(\frac{t-m_c}{c} - \lfloor\frac{t-m_c}{c}\rfloor\right) - \ln\left(\frac{t_c^*-m_c}{c}\right)\\
&\quad + \left(\frac{q_c-1}{p_c-1}\right)\left[\ln\left(1-\frac{t-m_c}{c} - \lfloor\frac{t-m_c}{c}\rfloor\right) - \ln\left(1-\frac{t_c^*-m_c}{c}\right)\right]\Bigg\},\\
\frac{\partial\lambda_c(j+t_1^*)}{\partial a} &= 1 - \frac{1}{\alpha_c^*}\left(\frac{t-m_c}{c} - \lfloor\frac{t-m_c}{c}\rfloor\right)^{p_c-1}\left(1-\frac{t-m_c}{c} - \lfloor\frac{t-m_c}{c}\rfloor\right)^{q_c-1},\\
\frac{\partial\lambda_c(j+t_1^*)}{\partial b} &= \frac{1}{\alpha_c^*}\left(\frac{t-m_c}{c} - \lfloor\frac{t-m_c}{c}\rfloor\right)^{p_c-1}\left(1-\frac{t-m_c}{c} - \lfloor\frac{t-m_c}{c}\rfloor\right)^{q_c-1}.
\end{aligned}$$

Note here $\frac{q_c-1}{p_c-1} = \frac{1}{\frac{(t_c^*-m_c)}{c}} - 1$, which is also a constant.

Now the second partial derivative of (3.16) w.r.t. p_1 is obtained as follows:

$$\frac{\partial^2 l}{\partial p_1^2} = -\frac{\partial^2\Lambda(t)}{\partial p_1^2} + \sum_{j=1}^{J} n_{j,\cdot}^{(\cdot)}\left[\frac{B'_{p_1}(p_1,q_1;\frac{jd}{D}) - B'_{p_1}(p_1,q_1;\frac{(j-1)d}{D})}{B(p_1,q_1;\frac{jd}{D}) - B(p_1,q_1;\frac{(j-1)d}{D})}\right]'_{p_1},$$

where

$$\begin{aligned}
\left[\frac{B'_{p_1}(p_1,q_1;\frac{jd}{D}) - B'_{p_1}(p_1,q_1;\frac{(j-1)d}{D})}{B(p_1,q_1;\frac{jd}{D}) - B(p_1,q_1;\frac{(j-1)d}{D})}\right]'_{p_1} &= \frac{B''_{p_1,p_1}(p_1,q_1;\frac{jd}{D}) - B''_{p_1,p_1}(p_1,q_1;\frac{(j-1)d}{D})}{B(p_1,q_1;\frac{jd}{D}) - B(p_1,q_1;\frac{(j-1)d}{D})}\\
&\quad - \left[\frac{B'_{p_1}(p_1,q_1;\frac{jd}{D}) - B'_{p_1}(p_1,q_1;\frac{(j-1)d}{D})}{B(p_1,q_1;\frac{jd}{D}) - B(p_1,q_1;\frac{(j-1)d}{D})}\right]^2,
\end{aligned}$$

here we define for $t \in (0,\ 1)$,

$$\begin{aligned}
B''_{p_1,p_1}(p_1,q_1;t) &= \int_0^t v^{p_1-1}(1-v)^{q_1-1}\Big[\ln^2(v) + 2\left(\frac{q_1-1}{p_1-1}\right)\ln(v)\ln(1-v)\\
&\quad + \left(\frac{q_1-1}{p_1-1}\right)^2\ln^2(1-v)\Big]\,dv,
\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial^2 \Lambda(t)}{\partial p_1^2} &= \frac{\partial \Lambda(t)}{\partial p_1}\left[\frac{B_{p_1}(p_1,q_1)}{B(p_1,q_1)} - \nu^*\right] + \Lambda(t)\left[\frac{B'_{p_1}(p_1,q_1)}{B(p_1,q_1)} - \nu^*\right]'_{p_1} \\
&= \frac{\partial \Lambda(t)}{\partial p_1}\left[\frac{B'_{p_1}(p_1,q_1)}{B(p_1,q_1)} - \nu^*\right] + \Lambda(t)\left[\frac{B''_{p_1,p_1}(p_1,q_1)}{B(p_1,q_1)} - \left(\frac{B''_{p_1}(p_1,q_1)}{B(p_1,q_1)}\right)^2\right],
\end{aligned}
$$

where $B''_{p_1,p_1}(p_1,q_1) = B''_{p_1,p_1}(p_1,q_1;1)$.

The second order mixed partial derivatives of (3.16) w.r.t. p_1 and other parameters p_c, a and b are:

$$
\begin{aligned}
\frac{\partial^2 l}{\partial p_1\,\partial p_c} &= -\frac{\partial^2 \Lambda(t)}{\partial p_1\,\partial p_c} = -\frac{\partial \Lambda(t)}{\partial p_c}\left[\frac{B'_{p_1}(p_1,q_1)}{B(p_1,q_1)} - \nu^*\right], \\
\frac{\partial^2 l}{\partial p_1\,\partial a} &= -\frac{\partial^2 \Lambda(t)}{\partial p_1\,\partial a} = -\frac{\partial \Lambda(t)}{\partial a}\left[\frac{B'_{p_1}(p_1,q_1)}{B(p_1,q_1)} - \nu^*\right], \\
\frac{\partial^2 l}{\partial p_1\,\partial b} &= -\frac{\partial^2 \Lambda(t)}{\partial p_1\,\partial b} = -\frac{\partial \Lambda(t)}{\partial b}\left[\frac{B'_{p_1}(p_1,q_1)}{B(p_1,q_1)} - \nu^*\right],
\end{aligned}
$$

while the second partial derivatives of (3.16) w.r.t. p_c, a and b, and other second order mixed partial derivatives of (3.16) are:

$$
\begin{aligned}
\frac{\partial^2 l}{\partial p_c^2} &= -\frac{\partial^2 \Lambda(t)}{\partial p_c^2} + \sum_{k=1}^{c} n_{\cdot,k}^{(\cdot)} \frac{\partial}{\partial p_c}\left[\frac{\frac{\partial \lambda_c(k-1+t_1^*)}{\partial p_c}}{\lambda_c(k-1+t_1^*)}\right] \\
&= -\frac{\partial^2 \Lambda(t)}{\partial p_c^2} + \sum_{k=1}^{c} n_{\cdot,k}^{(\cdot)} \left\{\frac{\frac{\partial^2 \lambda_c(k-1+t_1^*)}{\partial p_c^2}}{\lambda_c(k-1+t_1^*)} - \left[\frac{\frac{\partial \lambda_c(k-1+t_1^*)}{\partial p_c}}{\lambda_c(k-1+t_1^*)}\right]^2\right\}, \\
\frac{\partial^2 l}{\partial p_c\,\partial a} &= -\frac{\partial^2 \Lambda(t)}{\partial p_c\,\partial a} + \sum_{k=1}^{c} n_{\cdot,k}^{(\cdot)} \frac{\partial}{\partial a}\left[\frac{\frac{\partial \lambda_c(k-1+t_1^*)}{\partial p_c}}{\lambda_c(k-1+t_1^*)}\right] \\
&= -\frac{\partial^2 \Lambda(t)}{\partial p_c\,\partial a} + \sum_{k=1}^{c} n_{\cdot,k}^{(\cdot)} \left\{\frac{\frac{\partial^2 \lambda_c(k-1+t_1^*)}{\partial p_c\,\partial a}}{\lambda_c(k-1+t_1^*)} - \frac{\frac{\partial \lambda_c(k-1+t_1^*)}{\partial p_c}\,\frac{\partial \lambda_c(k-1+t_1^*)}{\partial a}}{\left[\lambda_c(k-1+t_1^*)\right]^2}\right\}, \\
\frac{\partial^2 l}{\partial p_c\,\partial b} &= -\frac{\partial^2 \Lambda(t)}{\partial p_c\,\partial b} + \sum_{k=1}^{c} n_{\cdot,k}^{(\cdot)} \frac{\partial}{\partial b}\left[\frac{\frac{\partial \lambda_c(k-1+t_1^*)}{\partial p_c}}{\lambda_c(k-1+t_1^*)}\right] \\
&= -\frac{\partial^2 \Lambda(t)}{\partial p_c\,\partial b} + \sum_{k=1}^{c} n_{\cdot,k}^{(\cdot)} \left\{\frac{\frac{\partial^2 \lambda_c(k-1+t_1^*)}{\partial p_c\,\partial b}}{\lambda_c(k-1+t_1^*)} - \frac{\frac{\partial \lambda_c(k-1+t_1^*)}{\partial p_c}\,\frac{\partial \lambda_c(k-1+t_1^*)}{\partial b}}{\left[\lambda_c(k-1+t_1^*)\right]^2}\right\}, \\
\frac{\partial^2 l}{\partial a^2} &= -\sum_{k=1}^{c} n_{\cdot,k}^{(\cdot)} \left[\frac{\frac{\partial \lambda_c(k-1+t_1^*)}{\partial a}}{\lambda_c(k-1+t_1^*)}\right]^2,
\end{aligned}
$$

$$\frac{\partial^2 l}{\partial a\,\partial b} = -\sum_{k=1}^{c} n_{\cdot,k}^{(\cdot)} \left[\frac{\frac{\partial \lambda_c(k-1+t_1^*)}{\partial a} \frac{\partial \lambda_c(k-1+t_1^*)}{\partial b}}{\lambda_c(k-1+t_1^*)} \right]^2 ,$$

$$\frac{\partial^2 l}{\partial b^2} = -\sum_{k=1}^{c} n_{\cdot,k}^{(\cdot)} \left[\frac{\frac{\partial \lambda_c(k-1+t_1^*)}{\partial b}}{\lambda_c(k-1+t_1^*)} \right]^2 ,$$

where

$$\frac{\partial^2 \Lambda(t)}{\partial p_c^2} = \frac{D\,B(p_1,q_1)}{\alpha_1^*} \left[\lfloor \frac{t}{c} \rfloor \sum_{j=0}^{c-1} \frac{\partial^2 \lambda_c(j+t_1^*)}{\partial p_c^2} + \sum_{j=0}^{\lfloor t-\lfloor \frac{t}{c} \rfloor c \rfloor -1} \frac{\partial^2 \lambda_c(j+t_1^*)}{\partial p_c^2} \right] ,$$

$$\frac{\partial^2 \Lambda(t)}{\partial p_c\,\partial a} = \frac{D\,B(p_1,q_1)}{\alpha_1^*} \left[\lfloor \frac{t}{c} \rfloor \sum_{j=0}^{c-1} \frac{\partial^2 \lambda_c(j+t_1^*)}{\partial p_c\,\partial a} + \sum_{j=0}^{\lfloor t-\lfloor \frac{t}{c} \rfloor c \rfloor -1} \frac{\partial^2 \lambda_c(j+t_1^*)}{\partial p_c\,\partial a} \right] ,$$

$$\frac{\partial^2 \Lambda(t)}{\partial p_c\,\partial b} = \frac{D\,B(p_1,q_1)}{\alpha_1^*} \left[\lfloor \frac{t}{c} \rfloor \sum_{j=0}^{c-1} \frac{\partial^2 \lambda_c(j+t_1^*)}{\partial p_c\,\partial b} + \sum_{j=0}^{\lfloor t-\lfloor \frac{t}{c} \rfloor c \rfloor -1} \frac{\partial^2 \lambda_c(j+t_1^*)}{\partial p_c\,\partial b} \right] ,$$

while

$$\begin{aligned}
\frac{\partial^2 \lambda_c(j+t_1^*)}{\partial p_c^2} &= \frac{\partial \lambda_c(j+t_1^*)}{\partial p_c} \Big\{ \ln\big(\frac{t-m_c}{c} - \lfloor \frac{t-m_c}{c} \rfloor\big) - \ln\big(\frac{t_c^*-m_c}{c}\big) \\
&\quad + \Big(\frac{q_1-1}{p_1-1}\Big) \Big[\ln\big(1-\frac{t-m_c}{c} - \lfloor \frac{t-m_c}{c} \rfloor\big) - \ln\Big(1-\frac{t_c^*-m_c}{c}\Big) \Big] \Big\} , \\
\frac{\partial^2 \lambda_c(j+t_1^*)}{\partial p_c\,\partial a} &= \Big[\frac{\partial \lambda_c(j+t_1^*)}{\partial a} - 1 \Big] \Big\{ \ln\big(\frac{t-m_c}{c} - \lfloor \frac{t-m_c}{c} \rfloor\big) - \ln\big(\frac{t_c^*-m_c}{c}\big) \\
&\quad + \Big(\frac{q_1-1}{p_1-1}\Big) \Big[\ln\big(1-\frac{t-m_c}{c} - \lfloor \frac{t-m_c}{c} \rfloor\big) - \ln\Big(1-\frac{t_c^*-m_c}{c}\Big) \Big] \Big\} , \\
\frac{\partial^2 \lambda_c(j+t_1^*)}{\partial p_c \partial b} &= \frac{\partial \lambda_c(j+t_1^*)}{\partial b} \Big\{ \ln\big(\frac{t-m_c}{c} - \lfloor \frac{t-m_c}{c} \rfloor\big) - \ln\big(\frac{t_c^*-m_c}{c}\big) \\
&\quad + \Big(\frac{q_1-1}{p_1-1}\Big) \Big[\ln\big(1-\frac{t-m_c}{c} - \lfloor \frac{t-m_c}{c} \rfloor\big) - \ln\Big(1-\frac{t_c^*-m_c}{c}\Big) \Big] \Big\} .
\end{aligned}$$

Now (3.15), the elements of the information matrix $\mathbf{I}(\theta)$, can be calculated with the following formulas:

$$\begin{aligned}
-E\left[\frac{\partial^2 l}{\partial p_1^2}\right] &= \frac{\partial^2 \Lambda(t)}{\partial p_1^2} - \sum_{j=1}^{J} E\big[n_{j,\cdot}^{(\cdot)}\big] \left[\frac{B'_{p_1}(p_1,q_1;\frac{jd}{D}) - B'_{p_1}(p_1,q_1;\frac{(j-1)d}{D})}{B(p_1,q_1;\frac{jd}{D}) - B(p_1,q_1;\frac{(j-1)d}{D})} \right]'_{p_1} , \\
-E\left[\frac{\partial^2 l}{\partial p_1\,\partial p_c}\right] &= \frac{\partial \Lambda(t)}{\partial p_c} \left[\frac{B'_{p_1}(p_1,q_1)}{B(p_1,q_1)} - \nu* \right] , \\
-E\left[\frac{\partial^2 l}{\partial p_1\,\partial a}\right] &= \frac{\partial \Lambda(t)}{\partial a} \left[\frac{B'_{p_1}(p_1,q_1)}{B(p_1,q_1)} - \nu* \right] ,
\end{aligned}$$

$$-E\left[\frac{\partial^2 l}{\partial p_1\,\partial b}\right] = \frac{\partial\Lambda(t)}{\partial b}\left[\frac{B'_{p_1}(p_1,q_1)}{B(p_1,q_1)} - \nu*\right],$$

$$-E\left[\frac{\partial^2 l}{\partial p_c^2}\right] = \frac{\partial^2\Lambda(t)}{\partial p_c^2} - \sum_{k=1}^{c} E\big[n^{(\cdot)}_{\cdot,k}\big]\left\{\frac{\frac{\partial^2\lambda_c(k-1+t_1^*)}{\partial p_c^2}}{\lambda_c(k-1+t_1^*)} - \left[\frac{\frac{\partial\lambda_c(k-1+t_1^*)}{\partial p_c}}{\lambda_c(k-1+t_1^*)}\right]^2\right\},$$

$$-E\left[\frac{\partial^2 l}{\partial p_c\,\partial a}\right] = \frac{\partial^2\Lambda(t)}{\partial p_c\,\partial a} - \sum_{k=1}^{c} E\big[n^{(\cdot)}_{\cdot,k}\big]\left\{\frac{\frac{\partial^2\lambda_c(k-1+t_1^*)}{\partial p_c\,\partial a}}{\lambda_c(k-1+t_1^*)} - \frac{\frac{\partial\lambda_c(k-1+t_1^*)}{\partial p_c}\frac{\partial\lambda_c(k-1+t_1^*)}{\partial a}}{\big[\lambda_c(k-1+t_1^*)\big]^2}\right\},$$

$$-E\left[\frac{\partial^2 l}{\partial p_c\,\partial b}\right] = \frac{\partial^2\Lambda(t)}{\partial p_c\,\partial b} - \sum_{k=1}^{c} E\big[n^{(\cdot)}_{\cdot,k}\big]\left\{\frac{\frac{\partial^2\lambda_c(k-1+t_1^*)}{\partial p_c\,\partial b}}{\lambda_c(k-1+t_1^*)} - \frac{\frac{\partial\lambda_c(k-1+t_1^*)}{\partial p_c}\frac{\partial\lambda_c(k-1+t_1^*)}{\partial b}}{\big[\lambda_c(k-1+t_1^*)\big]^2}\right\},$$

$$-E\left[\frac{\partial^2 l}{\partial a^2}\right] = \sum_{k=1}^{c} E\big[n^{(\cdot)}_{\cdot,k}\big]\left[\frac{\frac{\partial\lambda_c(k-1+t_1^*)}{\partial a}}{\lambda_c(k-1+t_1^*)}\right]^2,$$

$$-E\left[\frac{\partial^2 l}{\partial a\,\partial b}\right] = \sum_{k=1}^{c} E\big[n^{(\cdot)}_{\cdot,k}\big]\left[\frac{\frac{\partial\lambda_c(k-1+t_1^*)}{\partial a}\frac{\partial\lambda_c(k-1+t_1^*)}{\partial b}}{\lambda_c(k-1+t_1^*)}\right]^2,$$

$$-E\left[\frac{\partial^2 l}{\partial b^2}\right] = \sum_{k=1}^{c} E\big[n^{(\cdot)}_{\cdot,k}\big]\left[\frac{\frac{\partial\lambda_c(k-1+t_1^*)}{\partial b}}{\lambda_c(k-1+t_1^*)}\right]^2,$$

where the expectations $E\big[n^{(\cdot)}_{j,\cdot}\big]$ and $E\big[n^{(\cdot)}_{\cdot,k}\big]$ are given by

$$E\big[n^{(\cdot)}_{j,\cdot}\big] = D\left[B(p_1,q_1;\frac{jd}{D}) - B(p_1,q_1;\frac{(j-1)d}{D})\right]$$
$$\left[\lfloor\frac{t}{c}\rfloor\sum_{j=0}^{c-1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*} + \sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}\right], \qquad j=1,2,\ldots,6,$$

$$E\big[n^{(\cdot)}_{\cdot,k}\big] = \begin{cases}\left(\lfloor\frac{t}{c}\rfloor+1\right)D\,B(p_1,q_1)\,\frac{\lambda_c(k-1+t_1^*)}{\alpha_1^*}, & 1\le k\le\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor\\ \lfloor\frac{t}{c}\rfloor D\,B(p_1,q_1)\,\frac{\lambda_c(k-1+t_1^*)}{\alpha_1^*}, & \lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor<k\le c\end{cases}.$$

Then the variance–covariance matrix is obtained by inverting the matrix $\mathbf{I}(\theta)$ and the estimated standard deviations of the MLE's are obtained by taking the square root of the elements on the main diagonal.

An analog of this method can be applied to find the information matrix and to calculate the estimated standard deviations of the MLE's for the parameters in other double periodic intensity models.

3.3 An application to the hurricanes dataset

Tropical storms and hurricanes periodically affect every coastal US state along the Atlantic and the Gulf of Mexico, from Texas to Maine, year after year. According to Cole and Pfaff (1997), much speculation exists regarding the significance of the El Niño effect. This is a phenomenon generating abnormally warm surface water temperatures off the coasts of Ecuador and Peru, affecting global climate in the short-term, including weather patterns across North America. Particular attention has been directed toward the potential effects of the El Niño phenomenon on hurricane frequency and the strength attained by tropical cyclones during El Niño years, in comparison to non–El Niño years (called La Niña). These can be seen as long–term climatological and periodical effects on North American weather.

Parisi and Lund (2000) study the annual arrival cycle and return period properties of landfalling Atlantic Basin hurricanes. A NHP process with a periodic intensity function is used to model the annual cycle of hurricane arrival times. The data used in their study contains all Atlantic Basin hurricanes that have made a landfall in the contiguous United States during the years 1935–98, inclusive. Kernel methods are used to estimate the intensity function and the standard normal kernel function is selected.

In this section, apart from considering the seasonal effects on the hurricane arrival times, we also consider global climatological and periodical effects and try to model the occurrence times of Atlantic hurricanes using a double periodic NHP process. A double beta–type intensity function is used in this parametric model and the Atlantic hurricanes affecting the United States 1899–2000 dataset [Neumann et al. (1993) and Landreneau (2001)] is used to estimate the parameters in the model. By contrast to the method proposed by Parisi and Lund (2000), a parametric statistical inference approach is used here to estimate the intensity function. Maximum likelihood estimators, as well as estimated standard deviations, of model parameters for this dataset are obtained. The fit of different models to the hurricane data and the goodness–of–fit assessment are discussed.

3.3.1 The hurricane dataset

The data used for our study comes from Neumann et al. (1993), which reports 155 hurricanes that crossed or passed immediately adjacent to the Unites States coastline (Texas to Maine), 1899 through 1992. Landreneau (2001) contains 12 additional hurricanes for the years 1993 through 2000 and is obtained from the National Hurricane Center Web site. Henceforth we call this combined dataset "the hurricanes data". Thus, over the 102–year period 1899 though 2000, a total of 167 category 1 though 5 hurricanes crossed the Atlantic United States coastline at one or more points.

The average annual number is 1.64 over the whole period, which means an average of one to two hurricane landfalls per year. The years with a maximum number of 6 hurricanes were 1916 and 1985, while 19 out of the 102 years had no hurricanes. It can be observed that the hurricane season starts in June and ends in November over those years. Furthermore, the hurricane season peak period lasts from mid-August through October, with September having had the most major hurricanes (38.9% of all hurricanes). Figure 3.1 shows the annual distribution of those 167 Atlantic hurricanes, while Table 3.1 gives their monthly distribution.

Table 3.1: Monthly distribution of the hurricanes data

Month	Number of occurrences	Proportion (%)
June	11	6.6
July	17	10.2
August	44	26.3
September	65	38.9
October	26	15.6
November	4	2.4
	167	100.0

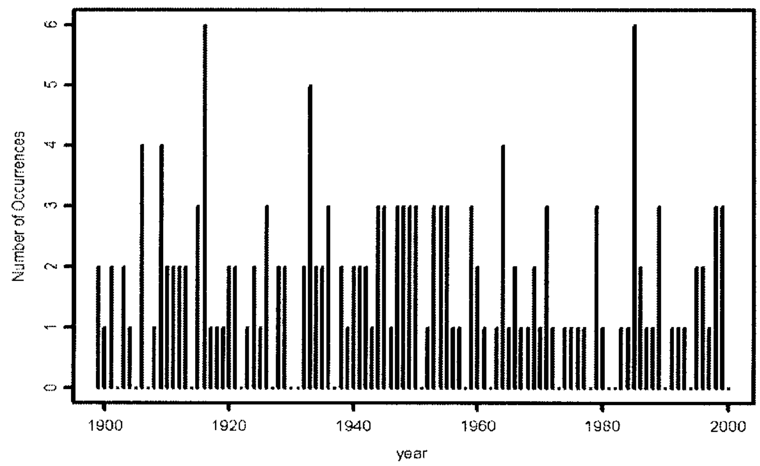

Figure 3.1: US Atlantic Hurricanes (1899-2000) Annual Counts

3.3.2 Models and statistical inference

As outlined in Section 3.3.1, the illustrative dataset used here comprises 167 hurricanes that made a landfall somewhere on the Atlantic United States coastline, over the 102–year period 1899 through 2000. These exhibit clear seasonal patterns. First, all hurricanes happened between the months of June to November. September generated more major hurricanes than any other month. On average, there were 1 to 2 hurricane landfalls per year over the whole period. A short–term (annual) periodic model thus seems appropriate.

First consider a NHP model with single periodicity. Let the generalized 3–parameter beta function described in (2.7) be the intensity of each year for this model. Then the intensity function is given by

$$\lambda(t) = \lambda_0^* \frac{\left(\frac{t-\lfloor t\rfloor-m_1}{D}\right)^{p_1-1}\left(1-\frac{t-\lfloor t\rfloor-m_1}{D}\right)^{q_1-1}}{\alpha_1^*\left[1-(1-\epsilon)\left(\frac{t-\lfloor t\rfloor-m_1}{D}\right)\right]^{p_1+q_1}}, \qquad m_1 \le t-\lfloor t\rfloor \le m_2\,, \tag{3.18}$$

where we set $m_1 = \frac{5}{12}$, $m_2 = \frac{11}{12}$ and $t_1^* = \frac{8.5}{12}$. These choices are motivated by the fact that no hurricanes ever happened outside the months of June to November and September has the largest number of hurricane occurrences. Hence $D = m_2 - m_1 = \frac{6}{12}$, is the fraction of the year with positive hurricane intensity, while α_1^* is given by (2.8).

Note that for this hurricanes dataset, $t = 102$ is an integer and $J = 6$, represents the total number of months in each year over which the intensity function is positive, i.e., $D = \frac{6}{12}$.

The parameters in (3.18) can be estimated by using the method described in Section 3.1. The log–likelihood function of this model is given by

$$l = -\,(102)\,\frac{\lambda_0^*\,D\,B(p_1,q_1)}{\alpha_1\,\epsilon^{p_1}} + N\,\log\left(\frac{\lambda_0^*\,D}{\alpha_1\,\epsilon^{p_1}}\right)$$
$$+\sum_{j=1}^{6}\left[n_{j,\cdot}^{(\cdot)} + n_{j,\cdot}^{(21)}\right]\,\log\left[B\Big(p_1,q_1;\,\frac{\epsilon^{\frac{jd}{D}}}{1-(1-\epsilon)^{\frac{jd}{D}}}\Big) - B\Big(p_1,q_1;\,\frac{\epsilon^{\frac{(j-1)d}{D}}}{1-[1-\epsilon]^{\frac{(j-1)d}{D}}}\Big)\right], \tag{3.19}$$

where $D = \frac{6}{12}$, $\frac{d}{D} = \frac{1}{6}$, $N = 167$, while $n_{j,\cdot}^{(\cdot)}$ and $n_{j,\cdot}^{(\lfloor\frac{t}{c}\rfloor+1)}$ for $j = 1,\ldots,6$, are given in Table 3.2.

Table 3.2: Empirical counts used in parameter estimation

j (month)	1	2	3	4	5	6	Total
$n_{j,\cdot}^{(\cdot)}$	11	17	43	64	25	4	164
$n_{j,\cdot}^{(21)}$			1	1	1		3
k (year)	1	2	3	4	5		
$n_{\cdot,k}^{(\cdot)}$	41	36	38	18	31		164
$n_{\cdot,k}^{(21)}$	3	0					3

Figure 3.2 gives the generalized 3–parameter beta intensity, that was fitted to these annual hurricane frequencies. The parameter MLE's here are $\hat{p}_1 = 1.9198$, $\hat{q}_1 = 11.3050$ and $\hat{\epsilon} = 0.1349$ for the beta function, and $\hat{\lambda}_0^* = 6.5145$ for the peak intensity. These were obtained by maximizing the log likelihood (3.19) with the Excel solver.

The constant intensity $\lambda_1(t) = \hat{\lambda} = 1.64$, the homogeneous Poisson process MLE, is also graphed on Figure 3.2 for comparison. Graphically it is clear that the classical

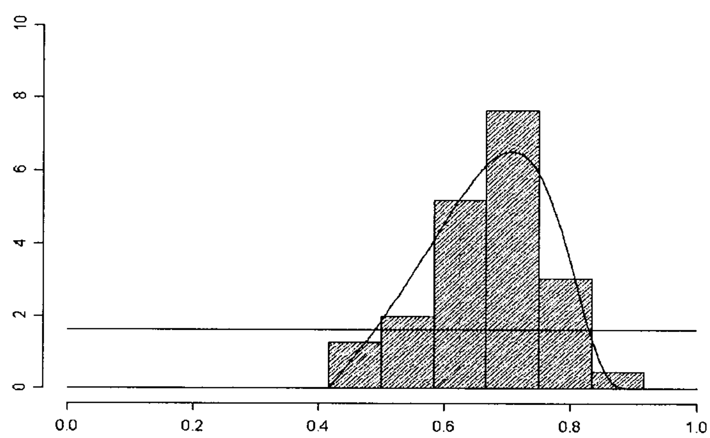

Figure 3.2: Histogram and fitted hurricane intensity $\lambda(t)$ for 1–year cycles

model gives here a crude representation of hurricane frequencies (this hypothesis is tested more formally in Section 3.3.3).

Climatological studies suggest that the hurricane intensity does not repeat the exact same short–term pattern every year. Rather, it slightly varies from year to year, as in alternating El Niño/La Niña cycles. In general, the weaker tropical cyclones and relatively fewer hurricanes, characteristic of an El Niño's influence on Atlantic hurricanes. The studies also show that there is lack of tropical cyclones in the Atlantic in the year following El Niño [see Cole and Pfaff (1997)]. Some actuaries also believe that El Niño/La Niña cycles in the Pacific affect tropical storm systems in the Atlantic.

Our hurricane data also exhibits some long–term periodicity, under the influence of the global El Niño/La Niña phenomenon. The 5–year cycle in Figure 3.3 shows how the 3rd and 4th years of the cycle have lower occurrences of hurricanes, the 4th year being the lowest. This is followed by a peak lasting for a period nearly three years long.

This motivates our assumptions of the doubly periodic NHP process presented in Section 2.1. Here the seasonality of the Atlantic hurricane repeats a similar short–

term pattern every year meanwhile the peak intensity, affected by the El Niño phenomenon, varies over a longer periodic cycle.

Climatologists observed that the typical El Niño cycle occurs within a two to seven year period. From a graphical analysis of the dataset, we conclude that a long–term period $c = 5$ years and a short–term period of one year reasonably describe the Atlantic hurricanes. Then the double–beta intensity function is given by

$$\lambda(t) = \lambda_c\big(\lfloor t - 5\lfloor\frac{t}{5}\rfloor\rfloor + t_1^*\big)\lambda_1\big(t - \lfloor t\rfloor\big)\ , \qquad t \geq 0\ , \tag{3.20}$$

where λ_1 and λ_c are given by the 2–parameter beta functions in (2.1) and (2.16), respectively, with $m_1 = \frac{5}{12}$, $m_2 = \frac{11}{12}$, $D = \frac{6}{12}$ and $t_1^* = \frac{8.5}{12}$. Furthermore, set the starting point of the long–term cycle to $m_c = t_1^* + 3$, the short–term peak of the year have lowest occurrences of hurricanes within a cycle. Similarly, let the mode of the long-term beta function to be $t_c^* = t_1^* + 1$.

The log-likelihood function of this model is given by (3.9), with $t = 102$, $\tau_c = 2$ and $J^* = 0$, that is,

$$\begin{aligned} l &= -\Lambda(102) + N\log\frac{D}{\alpha_1^*} + \sum_{k=1}^{5}[n_{\cdot,k}^{(\cdot)} + n_{\cdot,k}^{(21)}]\log\lambda_c(k - 1 + t_1^*) \\ &\quad + \sum_{j=1}^{6}[n_{j,\cdot}^{(\cdot)} + n_{j,\cdot}^{(21)}]\log\left[B(p_1, q_1; \frac{jd}{D}) - B(p_1, q_1; \frac{(j-1)d}{D})\right]\ , \end{aligned} \tag{3.21}$$

where $D = \frac{6}{12}$, $\frac{d}{D} = \frac{1}{6}$ and $N = 167$, while $\Lambda(t)$ is given by (2.20), i.e.,

$$\Lambda(102) = (20)\,\frac{D\,B(p_1, q_1)}{\alpha_1^*}\sum_{j=0}^{4}\lambda_c(j + t_1^*) + \frac{D\,B(p_1, q_1)}{\alpha_1^*}\sum_{j=0}^{1}\lambda_c(j + t_1^*)\,.$$

Here $\lambda_c(t)$ is given by (2.16). Meanwhile, $n_{\cdot,k}^{(\cdot)}$ and $n_{\cdot,k}^{(21)}$, for $k = 1, \ldots, 5$ and $n_{j,\cdot}^{(\cdot)}$ and $n_{j,\cdot}^{(21)}$, for $j = 1, \ldots, 6$ are given in Table 3.2.

The numerical maximization of (3.21) is simple and accesible to actuaries on any PC. Using the solver in Excel, we obtain the following MLE's: $\hat{p}_1 = 3.0145$, $\hat{q}_1 = 2.4389$, $\hat{p}_c = 1.5463$, $\hat{q}_c = 1.3642$, $\hat{a} = 3.2354$ and $\hat{b} = 6.9634$, where $\hat{q}_1$ and $\hat{q}_c$ are obtained from (2.3) and (2.18), respectively. Figure 3.3 compares the observed and expected monthly average number of hurricanes over the 5–year cycle for the 1899–2000 hurricane dataset.

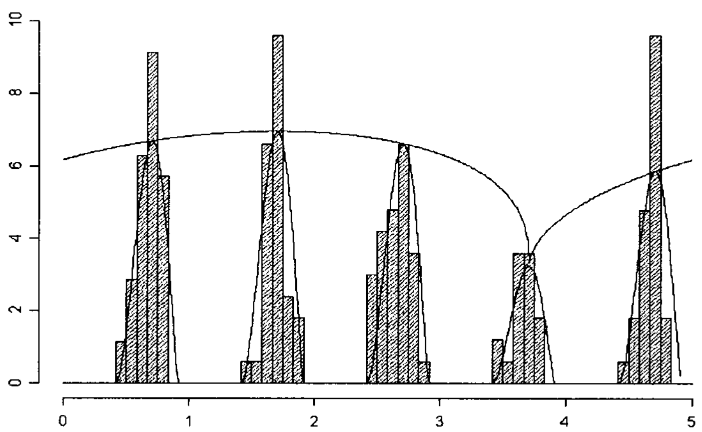

Figure 3.3: Hurricanes data and 5–year double–beta intensities

Now by the formulas derived in Section 3.2, the corresponding information matrix and its inverse are given by

$$\mathbf{I}(\theta) = \begin{pmatrix} 12.76374 & 2.64269 & -1.39801 & -3.487777 \\ 2.64269 & 3.14166 & -0.53981 & -1.94906 \\ -1.39801 & -0.53981 & 1.80932 & 0.32309 \\ -3.48777 & -1.94906 & 0.32309 & 2.75323 \end{pmatrix},$$

$$\mathbf{I}^{-1}(\theta) = \begin{pmatrix} 0.12831 & 0.00000 & 0.07162 & 0.15414 \\ 0.00000 & 0.58574 & 0.10286 & 0.40258 \\ 0.07162 & 0.10286 & 0.62256 & 0.09049 \\ 0.15414 & 0.40258 & 0.09049 & 0.83285 \end{pmatrix}.$$

Hence the estimated standard deviations for MLE's $\hat{p}_1$, $\hat{p}_c$, $\hat{a}$ and $\hat{b}$ are 0.3582, 0.7653, 0.7890 and 0.9126, respectively. The corresponding approximate 95% confidence intervals for the true parameter values are constructed in Table 3.3. These would be 1.96 standard deviations either side of the estimates.

Table 3.3: Asymptotic 95% confidence intervals for MLE's

Parameter	$\hat{p}_1$	$\hat{p}_c$	$\hat{a}$	$\hat{b}$
CI	3.0145 ± 0.7021	1.5463 ± 1.5001	3.2354 ± 1.5465	6.9634 ± 1.7887

Climatology suggests that the levels for the long–term cycle are governed by some underlying smoothly changing function, represented by the second beta function. The fit for each short–term cycle seems quite good, supporting our periodic theory. But the model does not adequately explains the short–term peaks over the long–term cycle. The El Niño/La Niña is a global phenomenon, perhaps too complex to capture with such a simple 4–parameter model.

Depending on the intended use of the model, the fit can be improved by the introduction of additional parameters. For instance when a generalized 3–parameter beta intensity is used for the short–term cycle, while the long–term beta function is kept at 2–parameters, the log–likelihood function becomes:

$$\begin{aligned} l &= -\Lambda(102) + N \log \frac{D}{\alpha_1^* \, \epsilon^{p_1}} + \sum_{k=1}^{5} [n_{\cdot,k}^{(\cdot)} + n_{\cdot,k}^{(21)}] \log \lambda_c(k - 1 + t_1^*) \\ &\quad + \sum_{j=1}^{6} [n_{j,\cdot}^{(\cdot)} + n_{j,\cdot}^{(21)}] \log \Bigg[B\Big(p_1, q_1; \frac{\epsilon \frac{jd}{D}}{1 - (1-\epsilon)\frac{jd}{D}}\Big) \\ &\qquad - B\Big(p_1, q_1; \frac{\epsilon \frac{(j-1)d}{D}}{1 - [1-\epsilon]\frac{(j-1)d}{D}}\Big) \Bigg] , \end{aligned} \tag{3.22}$$

where the cumulative hazard function Λ is given by

$$\Lambda(102) = (20) \frac{D\, B(p_1, q_1)}{\epsilon^{p_1} \, \alpha_1^*} \sum_{j=0}^{4} \lambda_c(j + t_1^*) + \frac{D\, B(p_1, q_1)}{\epsilon^{p_1} \, \alpha_1^*} \sum_{j=0}^{1} \lambda_c(j + t_1^*) \, ,$$

with $D = \frac{6}{12}$ and $\lambda_c(t)$ is given by (2.16).

By maximizing (3.22), the following MLE's are obtained: $\hat{p}_1 = 1.8946$, $\hat{q}_1 = 12.3899$, $\hat{\epsilon} = 0.1205$, $\hat{p}_c = 1.5639$, $\hat{q}_c = 1.3921$, $\hat{a} = 3.5868$ and $\hat{b} = 7.7307$. It is clear from Figure 3.4 that the fit is improved (also see the corresponding test in Section 3.3.3), although not perfect, at the cost of introducing only one additional parameter.

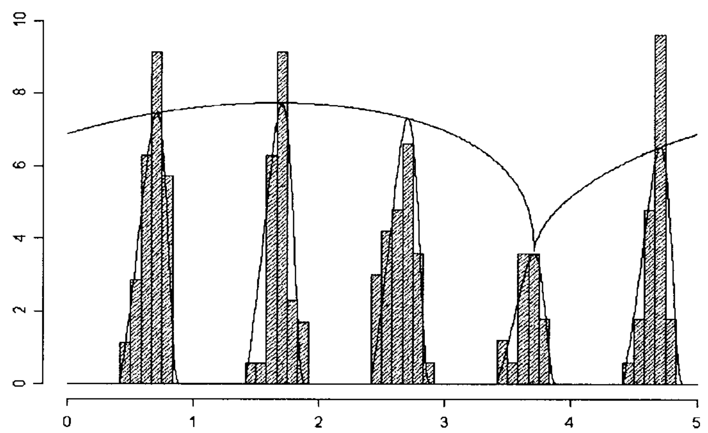

Figure 3.4: Hurricanes data and 5–year generalized double–beta intensities

If fit is more important than simplicity of the model or smoothness, the number of parameters can be further increased by letting the short–term cycle peak values be free parameters. Figure 3.5 gives the histogram and fitted beta intensities, as in (2.12), for monthly hurricane frequencies over a 5–year long–term cycle. Here the generalized 3–parameter beta function in (2.7) was used as the short–term intensity.

The log–likelihood functions used here are given by (3.13) and (3.14), i.e.,

$$\begin{aligned} l^{(k)} &= -(21)\,\frac{\lambda_k^* D B(p_1^{(k)}, q_1^{(k)})}{\alpha_1^{(k)}[\epsilon^{(k)}]^{p_1^{(k)}}} + [n_{\cdot,k}^{(\cdot)} + n_{\cdot,k}^{(21)}]\log\left(\frac{\lambda_k^* D}{\alpha_1^{(k)}[\epsilon^{(k)}]^{p_1^{(k)}}}\right) \\ &\quad + \sum_{j=1}^{6}[n_{j,k}^{(\cdot)} + n_{j,k}^{(21)}]\log\left[B\Big(p_1^{(k)}, q_1^{(k)}; \frac{\epsilon^{(k)}\frac{jd}{D}}{1-(1-\epsilon^{(k)})\frac{jd}{D}}\Big)\right. \\ &\quad \left. - B\Big(p_1^{(k)}, q_1^{(k)}; \frac{\epsilon^{(k)}\frac{(j-1)d}{D}}{1-[1-\epsilon^{(k)}]\frac{(j-1)d}{D}}\Big)\right], \qquad k = 1, 2, \qquad (3.23) \end{aligned}$$

and

$$\begin{aligned} l^{(k)} &= -(20)\,\frac{\lambda_k^* D B(p_1^{(k)}, q_1^{(k)})}{\alpha_1^{(k)}[\epsilon^{(k)}]^{p_1^{(k)}}} + n_{\cdot,k}^{(\cdot)}\log\left(\frac{\lambda_k^* D}{\alpha_1^{(k)}[\epsilon^{(k)}]^{p_1^{(k)}}}\right) \\ &\quad + \sum_{j=1}^{6} n_{j,k}^{(\cdot)}\log\left[B\Big(p_1^{(k)}, q_1^{(k)}; \frac{\epsilon^{(k)}\frac{jd}{D}}{1-(1-\epsilon^{(k)})\frac{jd}{D}}\Big)\right. \\ &\quad \left. - B\Big(p_1^{(k)}, q_1^{(k)}; \frac{\epsilon^{(k)}\frac{(j-1)d}{D}}{1-(1-\epsilon^{(k)})\frac{(j-1)d}{D}}\Big)\right], \qquad k = 3, 4, 5, \qquad (3.24) \end{aligned}$$

where quantities $n_{j,k}^{(\cdot)}$ and $n_{j,k}^{(21)}$, for $j = 1, \ldots, 6$ and $k = 1, \ldots, 5$ are shown in Table A.1 of Appendix A.1.

The MLE's, given in Table 3.4, were derived from (3.23) and (3.24). The fit improvement is substantial for each short–term intensity in the 5–year cycle. Yet, the model now fails to explain through a functional relation how hurricane intensities vary from El Niño to La Niña years. A possible remedy may be the use of random effects on certain years of the cycle. This will be discussed later in Chapter 5 by considering a regime–switching double–periodic Cox process.

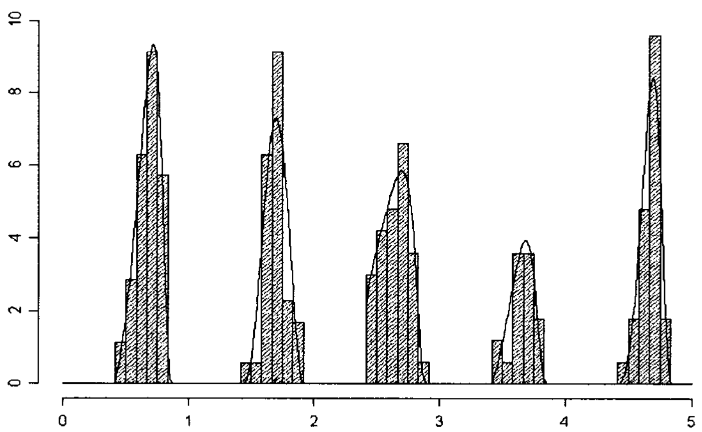

Figure 3.5: Hurricanes data 5-year short-term generalized beta fit

Table 3.4: MLE's of 3–parameter beta intensities for the hurricanes data

Year	$p_1^{(k)}$	$q_1^{(k)}$	$\epsilon^{(k)}$	λ_k^*
1	2.0087	150.0076	0.0097	9.3381
2	4.7926	3.0123	1.3227	7.2847
3	1.1459	11.9872	0.0820	5.8373
4	2.0586	121.7060	0.0150	3.9563
5	3.0769	155.4399	0.0165	8.4431

3.3.3 Goodness-of-fit and comments

Figures 3.1 and 3.2 provide graphical evidence that annual, respectively monthly, hurricane counts show a periodic behaviour.

More formally, we can test the alternate hypothesis of a constant hurricane intensity, $\lambda_1(t) = \hat{\lambda} = 1.637254902$, resulting in a Poisson number of hurricanes per year.

The Chi-square goodness-of-fit test is the most commonly used generic test. The test statistic is

$$X^2 = \sum_{j=1}^{k} \frac{(Ej - O_j)^2}{E_j},$$

where E_j is the number of expected observations (assuming that the hypothesized model is true and the parameters have their estimated values) and O_j is the number of observations in the j-th interval. The null hypothesis is rejected if X^2 exceeds $\chi^2_{d;\alpha}$, where $d = k - r - 1$, is the number of degrees of freedom, r is the number of parameters, and α is the significance level.

Table 3.5 reports the Poisson expected and observed number of years with 0, 1, 2, 3 and 4 or more hurricanes (the last observations were grouped to be representative). The Chi-square test statistic $X^2 = 1.81 < \chi^2_{3;0.05} = 7.81$, does not reject the homogeneous Poisson assumption. Still, it is clear from Table 3.5 that the fit is poor in the tail of the distribution.

The Poisson model with constant intensity predicts well the expected number of years with lower hurricane frequencies (e.g. $n = 0$, 1 or 2 hurricanes per year), but gives a poorer prediction of the number of years with higher frequencies ($n = 3$ and $n \geq 4$). The fit in the tail is usually very important in insurance applications.

Furthermore, the homogeneous Poisson model fails to recognize the short–term seasonal and long–term cyclical patterns that the hurricanes data exhibit in Figure 3.4. A more appropriate statistical inference here is to test the significance of the additional parameters in our double–beta periodic models.

Table 3.5: Chi-square goodness–of–fit test for the homogeneous Poisson model

Counts	Observed	Expected	Chi-square
0	19	19.84	0.04
1	34	32.48	0.07
2	25	26.59	0.10
3	18	14.51	0.84
4+	6	8.57	0.77
Total	102	102.00	1.81

Since the classical Poisson model is a special case of the double–beta periodic model with 4 parameters, in Figure 3.3, we can use a likelihood ratio test for the homogeneous Poisson hypothesis, against the alternative of a full 4–parameters model. The test statistic is

$$r = 2\left[\log L(\theta_A) - \log L(\theta_0)\right],$$

where θ_0 is the value of the parameters (within the null hypothesis) that maximizes the likelihood function, while θ_A is the MLE where the parameters vary over all possible values from the alternative hypothesis.

The null hypothesis is rejected if $r > \chi^2_{d;\,\alpha}$, where d, the number of degrees of freedom, equals to the number of free parameters in the model from alternative hypothesis less the one from the null hypothesis, and α is the significance level.

Now here the test statistic $r = 2(499.645 - 345.407) = 308.476 > \chi^2_{3;0.05} = 7.81$, is significant, supporting the full model hypothesis.

Similarly, in testing for the extra parameter in the complete model used for Figure 3.4, with a generalized 3–parameter beta function for the short–term intensity, the statistic $r = 2(345.407 - 335.936) = 18.942 > \chi^2_{1;0.05} = 3.84$ is also significant. This complete double–beta periodic model with 5 parameters explains the observed periodicity more adequately than the above 1 or 4–parameter reduced models.

The other assumption that should be tested is that of dependence on time. The hurricane counts observed here are not assumed to be mutually dependent (auto-correlated), but rather dependent on the time (season) of occurrence. Once a cycle completes, every 5 years, then this dependence on time gets reset. Subsequent 5–year cycles are thus independent, as in the decomposition in (2.14). The autocorrelation function plot in the lower part of Figure 3.6 shows the absence of serial correlations, in these 5-year cycle counts, since all values lie within the horizontal dashed lines for lags great than 0.

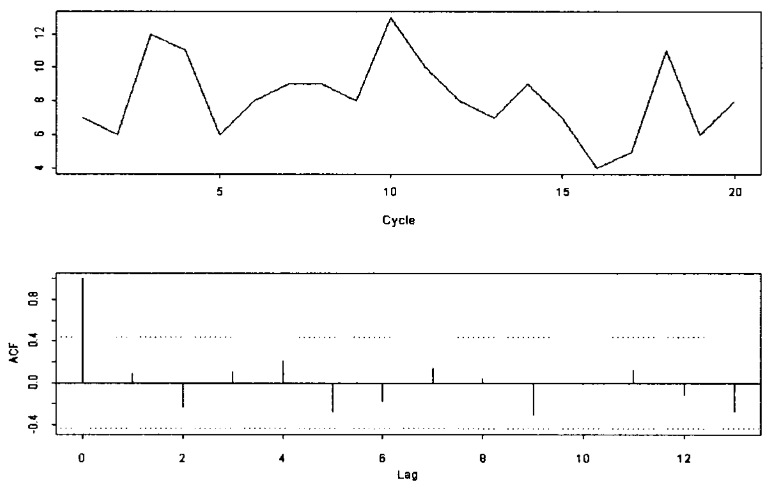

Figure 3.6: Hurricanes data autocorrelations, 5-year cycle counts

In conclusion, it appears that NHP risk models are more realistic in practice than classical Poisson processes, as their intensity rate is a function of time. This is clearly the case for hurricane landfalls.

In general, NHP processes with a periodic claim intensity can be useful in modeling risk processes that evolve in a periodic environment. The proposed double–beta periodic claim intensity not only generalizes the classical risk model, but it can also give a more realistic representation than (single) periodic models with only short–term periodic intensity functions.

The flexibility in shape of the beta function and the explicit results obtained for the risk process, as well as the tractability of the statistical estimation of model parameters, should make these double–beta periodic models easy to use in practice. We hope that the illustration of the hurricane dataset serves to show that NHP risk models can also be tractable if properly parameterized.

Chapter 4

Regime-switching periodic NHP processes

As we have seen in the previous chapters, the classical risk model is not realistic in some practical situations, such as modelling hurricane data. Two main modifications can be made [see Grandell (1991)]. First, a NHP process is used to model "size fluctuations" in the claim intensity of a risk subject to seasonality. Then, a Cox process, also called doubly stochastic Poisson process and a natural extension of the NHP process, is used to characterize the underlying "risk fluctuations" in the claims intensity.

In this chapter we propose a Cox model that accounts for both, the seasonal variations and the random fluctuations in the claims intensity. Beard et al. (1984) and Daykin et al. (1994) suggest an intensity process λ as a composition of some factors, such as the normal trend, deviations from it and the short–term variations in risk propensity. Here we simply consider an intensity process with the following structure

$$\lambda(t) = \lambda_S(t)\,\mathbf{q}(t)\,, \qquad t \geq 0\,, \tag{4.1}$$

where $\lambda_S(t)$ is the short–term intensity function and $\{\mathbf{q}(t);\, t \geq 0\}$ is a stochastic (level) process. The periodicity of the short–term intensity function is also considered, which takes into account those insurance claims affected by a periodic environment, like hurricanes or seasonal storms. A Markov chain with finite state space, corresponding to the different (low, high, $\cdots$) levels, is chosen for the level process, yielding a so called regime–switching process. Under this intensity process, properties of the claim counting process and its corresponding risk process are studied in detail. By properly choosing the Lundberg coefficient, Lundberg–type upper bounds for finite time ruin probabilities in a two-state case are derived.

The model is defined in Section 4.1. Section 4.2 discusses the properties of the claim counting process. This gives a precise description of the model characteristics, such as the probabilities of recording k claims during the time interval $[0, t)$, for $t \geq 0$ and $k \in \mathbb{N}$, and the expectation of the integrated intensities in (4.1). In Section 4.3 we derive Lundberg–type upper bounds for finite time ruin probabilities and illustrate the results by some examples.

4.1 A Cox model with a regime–switching periodic intensity

Consider an intensity process $\{\lambda(t);\, t \geq 0\}$ governed by a deterministic pattern in each short–term period, say a year, and a random effect on its peak level, that is the amplitude of the pattern. This fixed intensity pattern can be seen as the short-term periodicity, like in the NHP process. Assume we have m different risk levels; λ_0, λ_1, λ_2, ..., which represent the risk under "low season", "median season", "high season", etc., risk conditions, respectively. In practice, such conditions can be slippery roads, foggy days, stormy weather, years affected by the El Niño phenomenon and so on.

Furthermore, assume that the intensity level is modulated by an irreducible discrete time Markov process, $\kappa = \{\kappa_n;\, n \geq 0\}$, with finite state space $J = \{0, 1, \ldots, m-$

1} and the transition probability matrix P, given by

$$P = \begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0,m-1} \\ p_{10} & p_{11} & \cdots & p_{1,m-1} \\ \cdots & \cdots & & \\ p_{m-1,0} & p_{m-1,1} & \cdots & p_{m-1,m-1} \end{pmatrix}. \tag{4.2}$$

Without loss of generality, we assume that the short–term period is 1. Let β be a function defined on $[0, 1]$, such that $\beta(t^*) = 1$, where $t^* \in [0, 1]$ is the mode of the function. Consider the intensity process λ, given by

$$\lambda(t) = \lambda_{\kappa_{\lfloor t \rfloor}}\, \beta(t - \lfloor t \rfloor)\,, \qquad t \geq 0\,. \tag{4.3}$$

This gives $\lambda(n + t^*) = \lambda_{\kappa_n}\, \beta(t^*) = \lambda_{\kappa_n}$ for $n \in \mathbb{N}$, that is, the peak of the function $\lambda(t)$ within the $(n + 1)$-th year [i.e. $t \in [n, n + 1)$] is λ_{κ_n}, which changes according to the Markov chain κ. As such, we call λ_{κ_n} the intensity level for year $n + 1$, and correspondingly call $\{\lambda_{\kappa_n};\, n \geq 0\}$ the intensity level process.

In the sequel, we illustrate the annual common intensity pattern as a beta–type function with parameters $p \geq 1$ and $q \geq 1$, given by

$$\beta(t) = \begin{cases} \frac{\left(\frac{t-m_1}{D}\right)^{p-1}\left(1-\frac{t-m_1}{D}\right)^{q-1}}{\alpha^*}, & 0 \leq m_1 \leq t \leq m_2 \leq 1 \\ 0, & \text{otherwise} \end{cases}, \tag{4.4}$$

where $D = m_2 - m_1$ is the fraction of the year for which the beta intensity function is not equal to zero, while α^* is a scale factor, given by

$$\alpha^* = \left(\frac{t^* - m_1}{D}\right)^{p-1} \left(1 - \frac{t^* - m_1}{D}\right)^{q-1}, \tag{4.5}$$

and

$$t^* = m_1 + D\,\frac{p-1}{p+q-2}, \tag{4.6}$$

is the mode of $\beta(t)$, $t \in [0, 1]$, so that at the mode $\beta(t^*) = 1$ is the peak level [see Figure 4.1]. if $t \leq 0$).

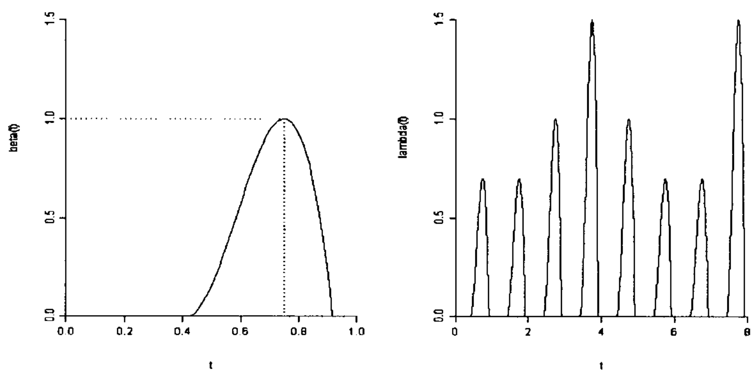

Figure 4.1: $\beta(t)$ and one realization of intensity process λ.

Figure 4.1 illustrates function $\beta(t)$, when $p = 3$ and $q = 2$, as well as a realization of the intensity process λ, when $p = 3$, $q = 2$, $\lambda_0 = 0.75$, $\lambda_1 = 1$, $\lambda_2 = 1.5$.

Consider a special Cox process, the claim counting process $\{N(t);\, t \geq 0\}$ with an intensity process as in (4.3). Due to the periodicity of the function $\beta(t - \lfloor t \rfloor)$, for $t \geq 0$, and the transitions, from year to year, within finite m levels $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$, we call this risk model a regime–switching periodic non–homogeneous Poisson (NHP) process.

Let $\{N_i(t);\, t \geq 0\}$, for $i = 0, 1, \ldots, m-1$, (with $N_i(0) = 0$) denote a claim counting NHP process with intensity function $\lambda_i\, \beta(t - \lfloor t \rfloor)$ over the time interval $[0, t)$. That is $N_i(t)$ is Poisson distributed with mean

$$\lambda_i \int_0^t \beta(v - \lfloor v \rfloor) dv = \frac{\lambda_i\, D}{\alpha^*} \left[\lfloor t \rfloor B(p, q) + B(p, q, \frac{t - \lfloor t \rfloor - m_1}{D}) \right], \qquad t \geq 0,$$

where $B(p, q)$ and $B(p, q;\, t)$ are given by (1.6) and (1.7), respectively.

Then the process $\{N(t);\, t \geq 0\}$ can be represented as

$$N(t) = \sum_{i=0}^{m-1} \sum_{j=1}^{Y_i(\lfloor t \rfloor)} N_{i,j} + N_{\kappa_{\lfloor t \rfloor}}(t) - N_{\kappa_{\lfloor t \rfloor}}(\lfloor t \rfloor), \qquad t \geq 0, \tag{4.7}$$

where $Y_i(\lfloor t \rfloor) = \sum_{n=0}^{\lfloor t \rfloor - 1} \mathbb{I}(\kappa_n = i)$ denotes the number of years in $[0, \lfloor t \rfloor)$ that κ spends in state i, for $i = 0, 1, \ldots, m-1$, while $N_{i,j}$ are i.i.d. random variables distributed as $N_i(1)$. This implies that, the conditional expected number of claims in the time interval $[0, t)$, given the environment, is:

$$
\begin{aligned}
&E\big[N(t) \mid \kappa_0, \kappa_1, \ldots, \kappa_{\lfloor t \rfloor}\big] \\
&\quad = \sum_{n=0}^{\lfloor t \rfloor - 1} \int_0^1 \lambda_{\kappa_n} \beta(v)\, dv + \lambda_{\kappa_{\lfloor t \rfloor}} \int_0^{t - \lfloor t \rfloor} \beta(v)\, dv \\
&\quad = L\big(\lfloor t \rfloor\big) \frac{D\, B(p,q)}{\alpha^*} + \frac{\lambda_{\kappa_{\lfloor t \rfloor}} D}{\alpha^*} B\big(p, q; \frac{t - \lfloor t \rfloor - m_1}{D}\big), \qquad t \geq 0,
\end{aligned}
$$

where

$$
L\big(\lfloor t \rfloor\big) = \sum_{i=0}^{m-1} Y_i\big(\lfloor t \rfloor\big)\, \lambda_i\,, \qquad t \geq 0, \tag{4.8}
$$

denotes the sum of the corresponding $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$ values that occurred in $[0, \lfloor t \rfloor)$. Hence, we have

$$
E[N(t)] \leq \max_{0 \leq i \leq m-1} \{\lambda_i\} \frac{D}{\alpha^*} \left[\lfloor t \rfloor B(p,q) + B\big(p, q, \frac{t - \lfloor t \rfloor - m_1}{D}\big) \right].
$$

The corresponding compound NHP process $\{S(t);\, t \geq 0\}$ is given by

$$
S(t) = \sum_{j=1}^{N(t)} X_j\,, \qquad t \geq 0, \tag{4.9}
$$

where the X_j's are the claim sizes with distribution function F_X, expected claim size $\mu = \int_0^\infty v\, dF_X(v)$ and moment generating function $\hat{m}_X(s) = \int_0^\infty e^{sv}\, dF_X(v)$, for $s < \alpha_X$. These claim severities are assumed independent of the Markov environment process κ and hence of the claim counting process $\{N(t);\, t \geq 0\}$. As in (4.7) process $\{S(t);\, t \geq 0\}$ can also be represented as

$$
S(t) \;=\; \sum_{i=0}^{m-1} \sum_{j=1}^{Y_i(\lfloor t \rfloor)} S_{i,j} + S_{\kappa_{\lfloor t \rfloor}}(t) - S_{\kappa_{\lfloor t \rfloor}}\big(\lfloor t \rfloor\big)\,, \qquad t \geq 0,
$$

where $S_{i,j}$ are i.i.d. random variables distributed as $S_i(1) = \sum_{n=1}^{N_i(t)} X_{i,n}$ and $S_i(t) = \sum_{n=1}^{N_i(t)} X_{i,n}$.

Now consider the continuous–time surplus process $\{U(t);\, t \geq 0\}$, given by

$$
U(t) = u + c\,t - S(t)\,, \qquad t \geq 0, \tag{4.10}
$$

where u is the initial capital value and c is the constant premium rate. The aggregate claim process $\{S(t);\, t \geq 0\}$ is given in (4.9) and the claim counting process $\{N(t);\, t \geq 0\}$ is the regime-switching periodic NHP process in (4.7).

Since the Markov environment process κ is assumed irreducible, it has a stationary initial distribution, denoted by $\pi = (\pi_0, \pi_1, \ldots, \pi_{m-1})$. Then by the law of large numbers for irreducible Markov processes, we have the following theorem for this model.

Theorem 4.1 Assume that the Markov process $\{\kappa_n;\, n \geq 0\}$ is irreducible. Then

$$\lim_{t\to\infty} \frac{R(t)}{t} = c - \mu \sum_{i=0}^{m-1} \pi_i\, \lambda_i\, \frac{D\, B(p,q)}{\alpha^*}\,. \tag{4.11}$$

Proof. Without loss of generality, we can assume that $u = 0$. Observe that $\frac{R(t)}{t}$ can be rewritten as

$$\begin{aligned} \frac{R(t)}{t} &= \sum_{i=0}^{m-1} \frac{Y_i(\lfloor t \rfloor)}{t} \left\{ \frac{1}{Y_i(\lfloor t \rfloor)} \sum_{k=0}^{Y_i(\lfloor t \rfloor)-1} \left(c - \sum_{j=1}^{N_i(1)} X_j \right) \right\} \\ &\quad + \frac{c\,(t - \lfloor t \rfloor) - \sum_{j=1}^{N_{\kappa_{\lfloor t \rfloor}}(t-\lfloor t \rfloor)} X_j}{Y_{\kappa_{\lfloor t \rfloor}}(\lfloor t \rfloor)}\,, \end{aligned} \tag{4.12}$$

when the $Y_i(\lfloor t \rfloor)$ are not equal to zero (the result is trivial when one of them is zero). Since $\{\kappa_n;\, n \geq 0\}$ is irreducible, $Y_i(\lfloor t \rfloor)$ tends to infinity as $t \to \infty$ and

$$\lim_{t\to\infty} \frac{Y_i(\lfloor t \rfloor)}{t} = \pi_i\,, \qquad i = 0, 1, \ldots, m-1\,.$$

It is not difficult to see that the second term in (4.12) tends to zero as $t \to \infty$ and $\frac{1}{Y_i(\lfloor t \rfloor)} \sum_{k=0}^{Y_i(\lfloor t \rfloor)-1} (c - \sum_{j=1}^{N_i(1)} X_j)$ in the first summation tends to $c - \lambda_i \frac{D\, B(p,q)}{\alpha^*}\, \mu$. Then the result in (4.11) follows. □

The limit result in (4.11) implies that ruin occurs almost surely if the process has a negative drift, that is $c \leq \mu \sum_{i=0}^{m-1} \pi_i\, \lambda_i\, \frac{D\, B(p,q)}{\alpha^*}$. Therefore we assume that the net profit condition

$$c > \mu \sum_{i=0}^{m-1} \pi_i\, \lambda_i\, \frac{D\, B(p,q)}{\alpha^*}\,, \tag{4.13}$$

holds in the sequel.

4.2 Properties of the regime–switching periodic process

For the regime–switching periodic NHP process defined above, the random measure Λ in this Cox process, given the realization of the environment process κ up to time $\lfloor t \rfloor$, is:

$$\Lambda(t) = \int_0^t \lambda(v)\, dv = L\big(\lfloor t \rfloor\big) \frac{D\, B(p,q)}{\alpha^*} + \frac{\lambda_{\kappa_{\lfloor t \rfloor}}\, D}{\alpha^*}\, B(p,q;\, \frac{t - \lfloor t \rfloor - m_1}{D}), \qquad t \geq 0, \tag{4.14}$$

where $L\big(\lfloor t \rfloor\big)$ is given in (4.8). Then the conditional probability that the number of claims be k in the time interval $[0, t)$ is obtained as:

$$P\{N(t) = k \mid \kappa_0, \kappa_1, \ldots, \kappa_{\lfloor t \rfloor}\} = \frac{[\Lambda(t)]^k}{k!}\, e^{-\Lambda(t)}, \qquad k \in \mathbb{N},$$

where $\Lambda(t)$ is given in (4.14).

In order to calculate $P\{N(t) = k\}$, we need to know how many times the levels $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$ appear in the sequence $\{\lambda_{\kappa_0}, \lambda_{\kappa_1}, \ldots, \lambda_{\kappa_{\lfloor t \rfloor}}\}$. This is equivalent to finding how many times $0, 1, \ldots, m-1$ appear in the corresponding sequence $\{\kappa_0, \kappa_1, \ldots, \kappa_{\lfloor t \rfloor}\}$. Denote $Y_i(n)$ to be the number of times that successive n-length sequences of the time–homogeneous $\{0, 1, \ldots, m-1\}$-valued Markov process κ are in state i, for $i = 0, 1, \ldots, m-1$.

Many papers discuss formulas or recursions for the distribution of success runs of several lengths in a finite state Markov chain [for example, see Han and Aki (1998)]. From these, it is not difficult to extend the results to the m–state case and further derive the distribution of the number of successes, $Y_i(n)$, which takes values in $\{0, 1, \ldots, n\}$, for $i = 0, 1, \ldots, m-1$, and can be obtained as follows.

Let $p_i(n;\, y_0, y_1, \ldots, y_{m-1})$ denote the conditional probability of a total number y_j of visits to state j out of an $(n+1)$-length sequence, for $j = 0, 1, \ldots, m-1$, respectively, given that the sequence starts from state i, $i = 0, 1, \ldots, m-1$. Note here $0 \leq y_j \leq n$, $j = 0, 1, \ldots, m-1$, and $y_0 + y_1 + \cdots + y_{m-1} = n$. For convenience, define $p_i(n;\, y_0, y_1, \ldots, y_{m-1}) = 0$ for either $y_j < 0$, $n \geq 0$ and $i, j = 0, 1, \ldots, m-1$. It is also clear that $p_i(n;\, y_0, y_1, \ldots, y_{m-1}) = 0$ when $y_0 + y_1 + \cdots + y_{m-1} > n$. We have the following lemma for probabilities $p_i(n;\, y_0, y_1, \ldots, y_{m-1})$.

Lemma 4.1 The recursive formulas for $E_i(n; y_0, y_1, \ldots, y_{m-1})$ are given by

$$\begin{aligned}
p_i(0;\, 0, \ldots, 0) &= 1\,, \qquad i = 0, 1, \ldots, m-1, \\
p_i(n;\, y_0, y_1, \ldots, y_{m-1}) &= \sum_{j=0}^{m-1} p_{ij}\, p_j(n-1;\, y_0, \ldots, y_j - 1, \ldots, y_{m-1}) \\
& \quad 0 \leq y_j \leq n\,, \quad j = 0, 1, \ldots, m-1\,, \quad y_0 + y_1 + \cdots + y_{m-1} = n\,, \\
& \quad i = 0, 1, \ldots, m-1\,, \quad n \geq 1\,.
\end{aligned} \tag{4.15}$$

Denote by $P(n;\, y_0, y_1, \ldots, y_{m-1})$, the probability of $Y_j(n) = y_j$, for $j = 0, 1, \ldots, m-1$ in an n–length sequence of the $\{0, 1, \ldots, m-1\}$–valued irreducible Markov chain κ. The following lemma gives the formulas for the probabilities $P_i(n; y_0, y_1, \ldots, y_{m-1})$.

Lemma 4.2 Let $\{\kappa_n;\, n \geq 0\}$ be the $\{0, 1, \ldots, m-1\}$–valued irreducible Markov chain with initial distribution $(\pi_0, \pi_1, \ldots, \pi_{m-1})$. Then the probabilities of $Y_j(n) = y_j$, for $j = 0, 1, \ldots, m-1$, in an n–length sequence of the Markov chain κ are given by

$$\begin{aligned}
P(0;\, 0, 0, \ldots, 0) &= 1\,, \\
P(n;\, y_0, y_1, \ldots, y_{m-1}) &= \sum_{j=0}^{m-1} \pi_j\, p_j(n-1;\, y_0, \ldots, y_j - 1, \ldots, y_{m-1}) \\
& \quad 0 \leq y_j \leq n\,, \quad j = 0, 1, \ldots, m-1\,, \quad y_0 + y_1 + \cdots + y_{m-1} = n\,, \\
& \quad i = 0, 1, \ldots, m-1\,, \quad n \geq 1\,.
\end{aligned} \tag{4.16}$$

where $p_j(n-1;\, y_0, \ldots, y_j - 1, \ldots, y_{m-1})$, $j = 0, 1, \ldots, m-1$, can be calculated recursively from (4.15).

Proof. Assuming that this n-length sequence starts at κ_0, the law of the total probabilities gives:

$$
\begin{aligned}
&P(n;\, y_0, y_1, \ldots, y_{m-1}) \\
&\quad = \sum_{j=0}^{m-1} P\{Y_0(n) = y_0, \ldots, Y_{m-1}(n) = y_{m-1} \mid \kappa_0 = j\}\, P\{\kappa_0 = j\} \\
&\quad = \sum_{j=0}^{m-1} \pi_i\, P\{Y_0(n-1) = y_0, \ldots, Y_j(n-1) = y_j - 1, \ldots, \\
&\qquad\qquad\qquad\qquad Y_{m-1}(n-1) = y_{m-1} \mid \kappa_0 = j\} \\
&\quad = \sum_{j=0}^{m-1} \pi_j\, p_j(n-1;\, y_0, \ldots, y_j - 1, \ldots, y_{m-1}),
\end{aligned}
$$

where $(\pi_0, \pi_1, \ldots, \pi_{m-1})$ is the initial distribution of Markov chain κ. □

Example 4.1 In a 3–length sequence, the probability that there are 3 visits of state 0 and no visits of states 2 and 3 is given by

$$P(3;\, 3,0,0) = \pi_0\, p_0(2;\, 2,0,0) = \pi_0\, p_{00}\, p_0(1;\, 1,0,0) = \pi_0\, (p_{00})^2,$$

since first it has to start from state 0 and then stays at 1 for the next two steps, while the probability that there is 1 visit of state 0, 2 visits of state 1 and no visit to state 3 is

$$
\begin{aligned}
P(3;\, 1,2,0) &= \pi_0\, p_0(2;\, 0,2,0) + \pi_1\, p_1(2;\, 1,1,0) \\
&= \pi_0\, [p_{01}\, p_1(1;\, 0,1,0) + \pi_1\, [p_{10}\, p_0(1;\, 0,1,0) + p_{11}\, p_1(1;\, 1,0,0)] \\
&= \pi_0\, p_{01}\, p_{11} + \pi_1\, [p_{10}\, p_{01} + p_{11}\, p_{10}],
\end{aligned}
$$

since if the sequence starts at 0, it must go to 1 twice in the next two steps, and if the sequence starts at 1, it goes to 0 for the next step and goes back to 1 for the last step, or goes to 1 for the next step and goes back to 0 at the last step.

We introduce the following notation for abbreviations. Let $\Lambda(n;\, y_0, y_1, \ldots, y_{m-1})$ be the random measure under a realization of $y_0, \ldots, y_{m-1}$ periods at levels $\lambda_0, \ldots, \lambda_{m-1}$, respectively, in the sequence $\{\lambda_{\kappa_0}, \lambda_{\kappa_1}, \ldots, \lambda_{\kappa_{n-1}}\}$. That is

$$\Lambda(n;\, y_0, y_1, \ldots, y_{m-1}) = \left(y_0\, \lambda_0 + \cdots + y_{m-1}\, \lambda_{m-1}\right) \frac{D\, B(p,q)}{\alpha^*}, \tag{4.17}$$

where $0 \leq y_j \leq n,\ j = 0, 1, \ldots, m-1$ and $y_0 + y_1 + \cdots + y_{m-1} = n$. Then we have the following theorem for the probabilities $P\{N(t) = k\}$.

Theorem 4.2 Let $\kappa = \{\kappa_n;\ n \geq 0\}$ be a $\{0, 1, \ldots, m-1\}$-valued irreducible Markov chain with (4.2) as transition probabilities and $(\pi_0, \pi_1, \ldots, \pi_{m-1})$ as initial distribution. For the counting process $\{N(t);\ t \geq 0\}$, given by (4.7), the probabilities that there be k claim occurrences during the time interval $[0, t)$, for $t \geq 0$ and $k \in \mathbb{N}$, is given by

$$P\{N(t) = k\} = \sum_{\substack{0 \leq y_j \leq \lfloor t \rfloor \\ y_0 + \cdots + y_{m-1} = \lfloor t \rfloor}} P(\lfloor t \rfloor;\ y_0, \ldots, y_{m-1}) \left\{ \sum_{i=0}^{m-1} \left(\sum_{j=0}^{m-1} \pi_j\, p_{ji} \right) e^{-\left[\Lambda(\lfloor t \rfloor;\ y_0, \ldots, y_{m-1}) + \frac{\lambda_i\, D\, B(p, q, \frac{t - \lfloor t \rfloor - m_1}{D})}{\alpha^*}\right]} \frac{1}{k!} \left[\Lambda(\lfloor t \rfloor;\ y_0, \ldots, y_{m-1}) + \frac{\lambda_i\, D\, B(p, q, \frac{t - \lfloor t \rfloor - m_1}{D})}{\alpha^*}\right]^k \right\}, \quad (4.18)$$

where $P(\lfloor t \rfloor;\ y_0, \ldots, y_{m-1})$ and $\Lambda(\lfloor t \rfloor;\ y_0, \ldots, y_{m-1})$ can be obtained from (4.16) and (4.17), respectively.

Proof. By the law of the total probabilities, it is easily seen that

$$\begin{aligned} P\{N(t) = k\} &= P\{N(\lfloor t \rfloor) + [N(t) - N(\lfloor t \rfloor)] = k\} \\ &= \sum_{l=0}^{k} P\{N(\lfloor t \rfloor) = l\} P\{N(t) - N(\lfloor t \rfloor) = k - l\} \\ &= \sum_{l=0}^{k} P\{N(\lfloor t \rfloor) = l\} \left[\sum_{j=0}^{m-1} P\{N(t) - N(\lfloor t \rfloor) = k - l \mid \kappa_{\lfloor t \rfloor - 1} = j\} P\{\kappa_{\lfloor t \rfloor - 1} = j\} \right]. \end{aligned}$$

Furthermore, since

$$\begin{aligned} P\{N(\lfloor t \rfloor) = l\} &= \sum_{\substack{0 \leq y_j \leq \lfloor t \rfloor \\ y_0 + \cdots + y_{m-1} = \lfloor t \rfloor}} P\{N(\lfloor t \rfloor) = l \mid Y_0(\lfloor t \rfloor) = y_0, \ldots, Y_{m-1}(\lfloor t \rfloor) = y_{m-1}\} \\ &\qquad P\{Y_0(\lfloor t \rfloor) = y_0, \ldots, Y_{m-1}(\lfloor t \rfloor) = y_{m-1}\} \\ &= \sum_{\substack{0 \leq y_j \leq \lfloor t \rfloor \\ y_0 + \cdots + y_{m-1} = \lfloor t \rfloor}} \frac{1}{l!} e^{-\Lambda(\lfloor t \rfloor;\ y_0, \ldots, y_{m-1})} \left[\Lambda(\lfloor t \rfloor;\ y_0, \ldots, y_{m-1})\right]^l P(\lfloor t \rfloor;\ y_0, \ldots, y_{m-1}), \end{aligned}$$

and

$$P\{N(t) - N(\lfloor t \rfloor) = k - l\}$$
$$= \sum_{j=0}^{m-1} P\{N(t) - N(\lfloor t \rfloor) = k - l \mid \kappa_{\lfloor t \rfloor - 1} = j\} P\{\kappa_{\lfloor t \rfloor - 1} = j\}$$
$$= \sum_{j=0}^{m-1} \pi_j \left[\sum_{i=0}^{m-1} P\{N(t) - N(\lfloor t \rfloor) = k - l \mid \kappa_{\lfloor t \rfloor - 1} = j, \kappa_{\lfloor t \rfloor} = i\} \right.$$
$$\left. P\{\kappa_{\lfloor t \rfloor} = i \mid \kappa_{\lfloor t \rfloor - 1} = j\} \right]$$
$$= \sum_{i=0}^{m-1} \left[\sum_{j=0}^{m-1} \pi_j p_{ji} \right] \frac{1}{(k-l)!} e^{-\frac{\lambda_i D B(p,q,\frac{t - \lfloor t \rfloor - m_1}{D})}{\alpha^*}} \left[\frac{\lambda_i D B(p,q,\frac{t - \lfloor t \rfloor - m_1}{D})}{\alpha^*} \right]^{k-l},$$

we now can write

$$\begin{aligned} P\{N(t) = k\} &= \sum_{\substack{0 \le y_j \le \lfloor t \rfloor \\ y_0 + \cdots + y_{m-1} = \lfloor t \rfloor}} P(\lfloor t \rfloor; y_0, \ldots, y_{m-1}) \\ &\quad \left\{ \sum_{i=0}^{m-1} \left(\sum_{j=0}^{m-1} \pi_j p_{ji} \right) e^{-\left[\Lambda(\lfloor t \rfloor; y_0, \ldots, y_{m-1}) + \frac{\lambda_i D B(p,q,\frac{t - \lfloor t \rfloor - m_1}{D})}{\alpha^*}\right]} \right. \\ &\quad \left. \sum_{l=0}^{k} \frac{[\Lambda(\lfloor t \rfloor; y_0, \ldots, y_{m-1})]^l}{l!\,(k-l)!} \left[\frac{\lambda_i D B(p,q,\frac{t - \lfloor t \rfloor - m_1}{D})}{\alpha^*} \right]^{k-l} \right\} \\ &= \sum_{\substack{0 \le y_j \le \lfloor t \rfloor \\ y_0 + \cdots + y_{m-1} = \lfloor t \rfloor}} P(\lfloor t \rfloor; y_0, \ldots, y_{m-1}) \\ &\quad \left\{ \sum_{i=0}^{m-1} \left(\sum_{j=0}^{m-1} \pi_j p_{ji} \right) e^{-\left[\Lambda(\lfloor t \rfloor; y_0, \ldots, y_{m-1}) + \frac{\lambda_i D B(p,q,\frac{t - \lfloor t \rfloor - m_1}{D})}{\alpha^*}\right]} \right. \\ &\quad \left. \frac{1}{k!} \left[\Lambda(\lfloor t \rfloor; y_0, \ldots, y_{m-1}) + \frac{\lambda_i D B(p,q,\frac{t - \lfloor t \rfloor - m_1}{D})}{\alpha^*} \right]^k \right\}, \end{aligned}$$

which completes the proof. □

Remark 4.1 Note here that (4.18) can be re–written as

$$P\{N(t) = k\} = E\big[P\{N(t) = k \,|\, \kappa_0, \kappa_1, \ldots, \kappa_{\lfloor t \rfloor}\}\big] = E\left[\frac{\Lambda(t)^k}{k!} e^{-\Lambda(t)} \right],$$

where $\Lambda(t)$ is given by (4.14). It means that this regime–switching periodic NHP process can also be interpreted as a mixed Poisson process.

The random measure $\Lambda(t)$ of this special Cox process is given by (4.14). Its expectation, $E[\Lambda(t)]$, and its mgf, $M_{\Lambda(t)}(s)$, are given in the following corollary.

Corollary 4.1 The random measure $\Lambda(t)$ in (4.14) has the following expectation

$$\begin{aligned} E\big[\Lambda(t)\big] &= \frac{D\,B(p,q)}{\alpha^*} \sum_{\substack{0 \le y_i \le \lfloor t \rfloor \\ y_0+\cdots+y_{m-1}=\lfloor t \rfloor}} P(\lfloor t \rfloor;\, y_0,\ldots,y_{m-1}) \left(\sum_{i=0}^{m-1} y_i\,\lambda_i\right) \\ &\quad + \frac{D\,B(p,q,\frac{t-\lfloor t \rfloor - m_1}{D})}{\alpha^*} \sum_{i=0}^{m-1} \lambda_i \left(\sum_{j=0}^{m-1} \pi_j\, p_{ji}\right), \qquad t \ge 0, \end{aligned} \tag{4.19}$$

while its mgf is

$$\begin{aligned} M_{\Lambda(t)}(s) &= \sum_{\substack{0 \le y_j \le \lfloor t \rfloor \\ y_0+\cdots+y_{m-1}=\lfloor t \rfloor}} P(\lfloor t \rfloor;\, y_0,\ldots,y_{m-1}) \sum_{i=0}^{m-1} \left\{ \left(\sum_{j=0}^{m-1} \pi_j\, p_{ji}\right) \right. \\ &\qquad \left. e^{s\left[\Lambda(\lfloor t \rfloor;\, y_0,\ldots,y_{m-1}) + \frac{\lambda_i\, D\, B(p,q,\frac{t-\lfloor t \rfloor - m_1}{D})}{\alpha^*}\right]} \right\}, \qquad s < a_\Lambda, \end{aligned} \tag{4.20}$$

where a_Λ is a number such that $\lim_{s \uparrow a_\Lambda} M_{\Lambda(t)}(s) = +\infty$, and $P(\lfloor t \rfloor;\, y_0,\ldots,y_{m-1})$ can be obtained from (4.16).

Proof. Taking expectations in (4.14) directly gives

$$\begin{aligned} E[\Lambda(t)] &= \frac{D\,B(p,q)}{\alpha^*} E\big[L(\lfloor t \rfloor)\big] + E\left[\lambda_{\kappa_{\lfloor t \rfloor}}\right] \frac{D\,B(p,q,\frac{t-\lfloor t \rfloor - m_1}{D})}{\alpha^*} \\ &= \frac{D\,B(p,q)}{\alpha^*} \sum_{i=0}^{m-1} \lambda_i\, E\big[Y_i(\lfloor t \rfloor)\big] + \frac{D\,B(p,q,\frac{t-\lfloor t \rfloor - m_1}{D})}{\alpha^*} E\big[\lambda_{\kappa_{\lfloor t \rfloor}}\big], \end{aligned}$$

where $E\big[Y_i(\lfloor t \rfloor)\big] = \sum_{y_i=0}^{\lfloor t \rfloor} y_i\, P\big\{Y_i(\lfloor t \rfloor) = y_i\big\}$ and for $0 \le y_i \le \lfloor t \rfloor$,

$$P\big\{Y_i(\lfloor t \rfloor) = y_i\big\} = \sum_{\substack{0 \le z_j \le \lfloor t \rfloor - y_i \\ \sum_{j \ne i} z_j = \lfloor t \rfloor - y_i}} P\big(\lfloor t \rfloor;\, z_0,\ldots,z_{i-1},y_i,z_{i+1},\ldots,z_{m-1}\big).$$

Since the following equations

$$\begin{aligned} &\sum_{\substack{0 \le y_j \le \lfloor t \rfloor \\ y_0+\cdots+y_{m-1}=\lfloor t \rfloor}} y_i\, P(\lfloor t \rfloor;\, y_0,\ldots,y_{m-1}) \\ &= \sum_{y_i=0}^{m-1} \sum_{\substack{0 \le z_j \le \lfloor t \rfloor - y_i \\ \sum_{j \ne i} z_j = \lfloor t \rfloor - y_i}} y_i\, P\big(\lfloor t \rfloor;\, z_0,\ldots,z_{i-1},y_i,z_{i+1},\ldots,z_{m-1}\big), \qquad i = 0,\ldots,m-1, \end{aligned}$$

and $E\big[\lambda_{\kappa_{\lfloor t\rfloor}}\big] = \sum_{i=0}^{m-1} \lambda_i \left(\sum_{j=0}^{m-1} \pi_j\, p_{ji}\right)$, (4.19) follows immediately.

It is not difficult to see that (4.19) is equivalent to

$$\begin{aligned} E[\Lambda(t)] &= \sum_{\substack{0\le y_j\le \lfloor t\rfloor \\ y_0+\cdots+y_{m-1}=\lfloor t\rfloor}} P(\lfloor t\rfloor;\, y_0,\ldots,y_{m-1}) \sum_{i=0}^{m-1} \Bigg\{ \left(\sum_{j=0}^{m-1} \pi_j\, p_{ji}\right) \\ &\qquad \left[\Lambda(\lfloor t\rfloor;\, y_0,\ldots,y_{m-1}) + \frac{\lambda_i\, D\, B(p,q,\frac{t-\lfloor t\rfloor - m_1}{D})}{\alpha^*}\right] \Bigg\}, \end{aligned}$$

then when $t \geq 0$ and $s < a_\Lambda$, the mgf of $\Lambda(t)$, $M_{\Lambda(t)}(s) = E\big[e^{s\Lambda(t)}\big]$, (4.20) follows. □

Remark 4.2 It is interesting to see that (4.20) can be rewritten as

$$\begin{aligned} M_{\Lambda(t)}(s) &= \sum_{\substack{0\le y_j\le \lfloor t\rfloor \\ y_0+\cdots+y_{m-1}=\lfloor t\rfloor}} P(\lfloor t\rfloor;\, y_0,\ldots,y_{m-1}) e^{s\,\Lambda(\lfloor t\rfloor;\, y_0,\ldots,y_{m-1})} \\ &\qquad \sum_{i=0}^{m-1} \Bigg\{ \left(\sum_{j=0}^{m-1} \pi_j\, p_{ji}\right) e^{s\,\frac{\lambda_i\, D\, B(p,q,\frac{t-\lfloor t\rfloor-m_1}{D})}{\alpha^*}} \Bigg\} \\ &= M_{\Lambda(\lfloor t\rfloor)}(s)\, M_{\Lambda(t-\lfloor t\rfloor)}(s)\,, \qquad s < a_\Lambda\,, \end{aligned}$$

showing that $\Lambda(t) = \Lambda(\lfloor t\rfloor) + \Lambda(t - \lfloor t\rfloor)$ and that these are independent.

Theorem 4.2 and the above results on $\Lambda(t)$ lead to the following corollary for the pgf of $N(t)$.

Corollary 4.2 For $t \geq 0$,

$$\mathrm{P}_{N(t)}(s) = M_{\Lambda(t)}(s-1)\,, \qquad |\, s \,| < 1\,, \tag{4.21}$$

where $M_{\Lambda(t)}(s-1)$ can be derived from (4.20) in Corollary 4.1, and hence

$$E\Big[N(t)[N(t)-1]\cdots[N(t)-r+1]\Big] = E\big[\Lambda(t)^r\big]\,, \qquad r \in \mathbb{N}^+\,. \tag{4.22}$$

In particular,

$$E\big[N(t)\big] = E\big[\Lambda(t)\big] \quad \text{and} \quad V\big[N(t)\big] = V\big[\Lambda(t)\big] + E\big[\Lambda(t)\big]\,, \tag{4.23}$$

implying that the index of dispersion is given by

$$I_{N(t)} = \frac{V\big[N(t)\big]}{E\big[N(t)\big]} = 1 + I_{\Lambda(t)}\,. \tag{4.24}$$

Proof. Applying Fubini's Theorem, by (4.18), we have

$$\begin{aligned}
\mathrm{P}_{N(t)}(s) &= E\big[s^{N(t)}\big] = \sum_{k=0}^{\infty} P\{N(t)=k\}\, s^k, \qquad |\, s \,|< 1, \\
&= \sum_{k=0}^{\infty} \sum_{\substack{0\le y_j \le \lfloor t \rfloor \\ y_0+\cdots+y_{m-1}=\lfloor t \rfloor}} P(\lfloor t \rfloor;\, y_0,\ldots,y_{m-1}) \\
&\qquad \left\{ \sum_{i=0}^{m-1} \left(\sum_{j=0}^{m-1} \pi_j\, p_{ji} \right) e^{-\left[\Lambda(\lfloor t \rfloor;\, y_0,\ldots,y_{m-1}) + \frac{\lambda_i\, D\, B(p,q,\frac{t-\lfloor t \rfloor - m_1}{D})}{\alpha^*}\right]} \right. \\
&\qquad \left. \frac{1}{k!}\left[\Lambda(\lfloor t \rfloor;\, y_0,\ldots,y_{m-1}) + \frac{\lambda_i\, D\, B(p,q,\frac{t-\lfloor t \rfloor - m_1}{D})}{\alpha^*}\right]^k \right\} s^k \\
&= \sum_{\substack{0\le y_j \le \lfloor t \rfloor \\ y_0+\cdots+y_{m-1}=\lfloor t \rfloor}} P(\lfloor t \rfloor;\, y_0,\ldots,y_{m-1}) \\
&\qquad \sum_{i=0}^{m-1} \left(\sum_{j=0}^{m-1} \pi_j\, p_{ji} \right) e^{-\left[\Lambda(\lfloor t \rfloor;\, y_0,\ldots,y_{m-1}) + \frac{\lambda_i\, D\, B(p,q,\frac{t-\lfloor t \rfloor - m_1}{D})}{\alpha^*}\right]} \\
&\qquad \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ s \left[\Lambda(\lfloor t \rfloor;\, y_0,\ldots,y_{m-1}) + \frac{\lambda_i\, D\, B(p,q,\frac{t-\lfloor t \rfloor - m_1}{D})}{\alpha^*}\right]\right\}^k \\
&= \sum_{\substack{0\le y_j \le \lfloor t \rfloor \\ y_0+\cdots+y_{m-1}=\lfloor t \rfloor}} P(\lfloor t \rfloor;\, y_0,\ldots,y_{m-1}) \\
&\qquad \sum_{i=0}^{m-1} \left(\sum_{j=0}^{m-1} \pi_j\, p_{ji} \right) e^{(s-1)\left[\Lambda(\lfloor t \rfloor;\, y_0,\ldots,y_{m-1}) + \frac{\lambda_i\, D\, B(p,q,\frac{t-\lfloor t \rfloor - m_1}{D})}{\alpha^*}\right]} \\
&= E\big[e^{(s-1)\Lambda(t)}\big] = M_{\Lambda(t)}(s-1),
\end{aligned}$$

which gives (4.21). Furthermore, taking the r-th derivative of $\mathrm{P}_{N(t)}(s)$ with respect to $s \in (0,1)$, $\mathrm{P}^{(r)}_{N(t)}(s)$, and its limit as $s \uparrow 1$, yields the successive factorial moments of $N(t)$ in (4.22) (that these be finite or not):

$$\begin{aligned}
E\Big[N(t)[N(t)-1]\cdots[N(t)-r+1]\Big] &= \mathrm{P}^{(r)}_{N(t)}(1) = \lim_{s\uparrow 1} \mathrm{P}^{(r)}_{N(t)}(s) \\
&= \lim_{s\uparrow 1} E\big[\Lambda(t)^r e^{(s-1)\Lambda(t)}\big] = E\big[\Lambda(t)^r\big].
\end{aligned}$$

Finally, (4.23) and (4.24) can easily be evaluated from (4.22). □

4.3 A Lundberg upper bound for finite time ruin probabilities

This last section discusses the ruin problem for our special Cox process. The income process, over the time interval $[0, t)$, with initial value $R(0) = 0$ and a constant premium rate c, is given as

$$R(t) = c\,t - S(t) = c\,t - \sum_{j=1}^{N(t)} X_j\,, \qquad t \geq 0\,, \tag{4.25}$$

where the claim counting process $\{N(t);\, t \geq 0\}$ is the regime–switching periodic NHP process driven by a $\{0, 1\}$–valued Markov chain κ and $S(t)$ is as in (4.9). For simplicity, we assume here that $m_1 = 0$, $m_2 = 1$, thus $D = 1$, for the beta–type function (4.4) used in the intensity process λ given by (4.25). Further assume that the mgf $M_X(s) = \int_0^\infty e^{sx}\, dF_X(x)$ is twice differentiable on an interval $[0, a_X)$, where $a_X > 0$ and $\lim_{s \uparrow a_X} M_X(s) = +\infty$.

Assuming the Laplace–Stieltjes transform of $R(t)$, $M_{R(t)}(-s) = E[e^{-s\,R(t)}]$, exists, it is given by

$$M_{R(t)}(-s) = e^{\Lambda(t)\left[M_X(s)-1\right]-s\,c\,t}\,, \qquad s > a_{R(t)}\,,\, t \geq 0\,. \tag{4.26}$$

Similarly, for $i = 0, 1$, let

$$\begin{aligned} M_{R_i(t)}(-s) &= E\big[e^{-s\,R_i(t)}\big] = E\big[e^{-s\,(c\,t-\sum_{j=1}^{N_i(t)} X_j)}\big] \\ &= e^{\lambda_i \frac{\lfloor t\rfloor B(p,q)+B(p,q;\,t-\lfloor t\rfloor)}{\alpha^*}\left[M_X(s)-1\right]-s\,c\,t}\,, \qquad s > a_{R_i(t)}\,,\ t \geq 0\,. \end{aligned} \tag{4.27}$$

Let the time to ruin be defined in the usual way:

$$T = \inf\big\{t \geq 0 \mid u + R(t) < 0\big\}\,, \qquad u \geq 0\,.$$

The ultimate ruin probability $\Psi(u)$ is then given by:

$$\Psi(u) = P\{T < \infty\}\,, \qquad u \geq 0\,.$$

Using the martingale approach to Cox models discussed in Grandell (1991) we can prove the following result.

Theorem 4.3 The following Lundberg–type upper bound holds for the finite time ruin probability in model (4.25):

$$P\{T \leq t_0\} \leq e^{-su} E\left[\sup_{0\leq t\leq t_o} M_{R(t)}(-s)\right], \qquad 0 \leq t_0 < \infty, \tag{4.28}$$

where $L(s;\, t)$ is given by (4.26).

A tighter upper bound can also be obtained for $0 \leq t_0 < \infty$, as:

$$P\{T \leq t_0\} \leq e^{-su} E\left[\sup_{0\leq t\leq t_o} M_{R(t)}(-s)\right] \sup_{y\geq 0}\left\{\frac{e^{sy}\,\bar{F}_X(y)}{\int_y^\infty e^{sx}\,dF_X(x)}\right\}, \tag{4.29}$$

where $\bar{F}_X = 1 - F_X$ is the tail of the distribution function of X.

Proof. Consider the martingale approach to Cox models discussed in Grandell (1991). Let $\mathbf{F}$ be a suitable filtration, M be a positive $\mathbf{F}$-martingale (or a positive $\mathbf{F}$-supermartingale) and T be an $\mathbf{F}$-stopping time. Choose $t_0 < \infty$ and consider $t_0 \wedge T$, a bounded $\mathbf{F}$-stopping time.

By the optional stopping theorem, we have that

$$M(0) \geq E^{\mathcal{F}_0}\big[M(t_0 \wedge T)\big] \geq E^{\mathcal{F}_0}\big[M(T)\,\big|\, T \leq t_0\big]\, P^{\mathcal{F}_0}\{T \leq t_0\},$$

and therefore

$$P^{\mathcal{F}_0}\{T \leq t_0\} \leq \frac{M(0)}{E^{\mathcal{F}_0}\big[M(T)\,\big|\,T \leq t_0\big]}, \qquad t_0 < \infty.$$

Let the risk process R be adapted to $\mathbf{F}$, that is $\mathcal{F}_t \supseteq \mathcal{F}_t^R$ for all $t \geq 0$. Then the ultimate ruin probability $\Psi(u)$ is seen to be:

$$\Psi(u) = P\{T < \infty\} = E\big[P^{\mathcal{F}_0}\{T < \infty\}\big], \qquad u \geq 0.$$

Now consider N to be a Cox process with intensity process $\{\lambda(t);\, t \geq 0\}$ and random intensity measure Λ, given by $\Lambda(t) = \int_0^t \lambda(v)\,dv$. A suitable filtration $\mathbf{F}$ is defined as $\mathcal{F}_t = \mathcal{F}_\infty^\Lambda \vee \mathcal{F}_t^R$ and thus $\mathcal{F}_0 = \mathcal{F}_\infty^\Lambda$. Consider the following choice of process M:

$$M(t) = \frac{e^{-s\,[u+R(t)]}}{M_{R(t)}(-s)} = \frac{e^{-s\,[u+R(t)]}}{e^{\Lambda(t)\,[M_X(s)-1]-s\,c\,t}}, \qquad t \geq 0,$$

where $R(t)$ is given in (4.25).

It can be shown that M is an $\mathbf{F}$-martingale where the filtration is given by $\mathcal{F}_t = \mathcal{F}_\infty^\Lambda \vee \mathcal{F}_t^R$. A lower bound is obtained when $0 \le t_0 < \infty$ as

$$\begin{aligned} E^{\mathcal{F}_0}\big[M(T)\,\big|\,T \le t_0\big] &\ge E^{\mathcal{F}_0}\big[e^{-\Lambda(T)\,[M_X(s)-1]+s\,c\,T}\,\big|\,T \le t_0\big] \\ &\ge \inf_{0\le t\le t_o} e^{-\Lambda(t)\,[M_X(s)-1]+s\,c\,t}\,. \end{aligned} \tag{4.30}$$

More precisely,

$$\begin{aligned} &E^{\mathcal{F}_0}\big[M(T)\,\big|\,T \le t_0\big] \\ &\quad = E^{\mathcal{F}_0}\big[e^{-s\,[u+R(T)]}\,e^{-\Lambda(T)\,[M_X(s)-1]+s\,c\,T}\,\big|\,T \le t_0\big] \\ &\quad \ge \inf_{0\le t\le t_o}\big\{e^{-\Lambda(t)\,[M_X(s)-1]+s\,c\,t}\big\}\,E^{\mathcal{F}_0}\big[e^{-s\,[u+R(T)]}\,\big|\,T \le t_0\big] \\ &\quad \ge \inf_{0\le t\le t_o}\big\{e^{-\Lambda(t)\,[M_X(s)-1]+s\,c\,t}\big\}\,\inf_{y\ge 0}\left\{\frac{\int_y^\infty e^{-s(y-x)}\,dF_X(x)}{1-F_X(y)}\right\}. \end{aligned} \tag{4.31}$$

Then we get, from (4.30), that

$$P^{\mathcal{F}_0}\{T \le t_0\} \le \frac{M(0)}{E^{\mathcal{F}_0}\big[M(T)\,\big|\,T \le t_0\big]} \le e^{-s\,u} \sup_{0\le t\le t_o} M_{R(t)}(-s)\,. \tag{4.32}$$

Taking expectations proves (4.28). Using (4.31) in (4.32) yields (4.29). □

The upper bound given in (4.29) is difficult to use in practice. To derive a corresponding useful bound for our regime–switching periodic NHP model, first define the average risk level, given by

$$\bar{\lambda} = \pi_0\,\lambda_0 + \pi_1\,\lambda_1\,, \tag{4.33}$$

and consider, for $s \ge 0$, the equation

$$\theta(s) = \frac{\bar{\lambda}\,B(p,q)}{\alpha^*}\big[M_X(s) - 1\big] - s\,c = 0\,. \tag{4.34}$$

The solution, $\gamma > 0$, to (4.34) satisfies:

$$\frac{\bar{\lambda}\,B(p,q)}{\alpha^*}\big[M_X(\gamma) - 1\big] = \gamma\,c\,. \tag{4.35}$$

Here γ is an adjustment coefficient for the average risk level $\bar{\lambda}$ in (4.33), where λ_1, the peak intensity under "high risk" years, is assumed larger than that in the "low season" (i.e. $\lambda_0 < \lambda_1$). It follows from (4.35) that

$$\frac{\lambda_i\, B(p,q)}{\alpha^*}\big[M_X(\gamma)-1\big] = \frac{\lambda_i}{\bar{\lambda}}\,\gamma\, c\,, \qquad i=0,1\,. \tag{4.36}$$

The existence and unicity of γ in $[0, a_X)$ is guaranteed because $\theta(0)=0$ and $\theta'(0) = \frac{\bar{\lambda}\, B(p,q)}{\alpha^*}\,\mu - c < 0$, provided that the net profit condition (4.13) holds, and hence the convexity of $\theta(s)$ ensures that $\theta'(\gamma) > 0$.

Assume that t_0 is an integer. Then with probabilities $P(t_0;\, t_0-y, y)$ (here $Y_1(t_0) = y$ implies that $Y_0(t_0) = t_0 - y$), given by (4.16) when $m=2$, $\Lambda(t_0)$ takes the following realizations:

$$\Lambda(t_0;\, t_0-y, y) = \big[(t_0-y)\,\lambda_0 + y\,\lambda_1\big]\,\frac{B(p,q)}{\alpha^*}\,, \qquad 0 \le y \le t_0\,,\ t_0 \in \mathbb{N}\,.$$

When $0 \le t \le t_0$, we have two possibilities for $\Lambda(t)$, depending on the value of $\lambda_{\kappa_{\lfloor t \rfloor}}$. One is

$$\Lambda(t) = \big[(\lfloor t\rfloor - z)\,\lambda_0 + z\,\lambda_1\big]\,\frac{B(p,q)}{\alpha^*} + \lambda_0\,\frac{B(p,q;\, t-\lfloor t\rfloor)}{\alpha^*}\,, \qquad 0 \le t \le t_0\,, \tag{4.37}$$

where $0 \le z \le \min\{\lfloor t\rfloor, y\}$ and $\lfloor t\rfloor - z + 1 \le t_0 - y$, or equivalently, $z \in C(t+1, y) = \big[\max\{0, \lfloor t\rfloor + 1 - (t_0 - y)\}, \min\{\lfloor t\rfloor, y\}\big]$. While the other is

$$\Lambda(t) = \big[(\lfloor t\rfloor - z)\,\lambda_0 + z\,\lambda_1\big]\,\frac{B(p,q)}{\alpha^*} + \lambda_1\,\frac{B(p,q;\, t-\lfloor t\rfloor)}{\alpha^*}\,, \qquad 0 \le t \le t_0\,, \tag{4.38}$$

where similarly, $0 \le z \le \min\{\lfloor t\rfloor, y-1\}$ and $\lfloor t\rfloor - z \le t_0 - y$, or equivalently, $z \in C(t, y-1) = \big[\max\{0, \lfloor t\rfloor - (t_0-y)\}, \min\{\lfloor t\rfloor, y-1\}\big]$.

When $\Lambda(t)$ is given by (4.37), then (4.35) and (4.36) imply that:

$$\begin{aligned}
\Lambda(t)\,\big[M_X(\gamma)-1\big] - \gamma\, c\, t \;&=\; (\lfloor t\rfloor - z)\,\Big[\frac{\lambda_0\, B(p,q)}{\alpha^*}\,(M_X(\gamma)-1) - \gamma\, c\Big] \\
&\quad + z\,\Big[\frac{\lambda_1\, B(p,q)}{\alpha^*}\,(M_X(\gamma)-1) - \gamma\, c\Big] \\
&\quad + \frac{\lambda_0\, B(p,q;\, t-\lfloor t\rfloor)}{\alpha^*}\,(M_X(\gamma)-1) - \gamma\, c(t-\lfloor t\rfloor) \\
&=\; -\lfloor t\rfloor\Big(\frac{\bar{\lambda}-\lambda_0}{\bar{\lambda}}\Big)\,\gamma\, c + z\,\Big(\frac{\lambda_1-\lambda_0}{\bar{\lambda}}\Big)\,\gamma\, c \\
&\quad + \frac{\lambda_0\, B(p,q;\, t-\lfloor t\rfloor)}{\alpha^*}\,\big[M_X(\gamma)-1\big] - \gamma\, c(t-\lfloor t\rfloor).
\end{aligned}$$

In turn

$$
\begin{aligned}
\sup_{0\le t\le t_0} M_{R(t)}(-\gamma) &= \sup_{0\le t\le t_0} e^{\Lambda(t)\left[M_X(\gamma)-1\right]-\gamma c t} \\
&\le \sup_{\substack{0\le t\le t_0\\ z\in C(t+1,y)}} e^{z\left(\frac{\lambda_1-\lambda_0}{\lambda}\right)\gamma c+\frac{\lambda_0\, B(p,q;\, t-\lfloor t\rfloor)}{\alpha^*}\left[M_X(\gamma)-1\right]-\gamma c(t-\lfloor t\rfloor)} \\
&= \max_{\substack{0\le t\le t_0\\ z\in C(t+1,y)}} e^{z\left(\frac{\lambda_1-\lambda_0}{\lambda}\right)\gamma c} \max_{0\le v<1} M_{R_0(v)}(-\gamma) \\
&\le e^{y\left(\frac{\lambda_1-\lambda_0}{\lambda}\right)\gamma c} \max_{0\le v<1} M_{R_0(v)}(-\gamma)\,. \qquad (4.39)
\end{aligned}
$$

Similarly, when $\Lambda(t)$ is given by (4.38), then

$$
\sup_{0\le t\le t_0} M_{R(t)}(-\gamma) \le e^{(y-1)\left(\frac{\lambda_1-\lambda_0}{\lambda}\right)\gamma c} \max_{0\le v<1} M_{R_1(v)}(-\gamma) \le e^{y\left(\frac{\lambda_1-\lambda_0}{\lambda}\right)\gamma c} \max_{0\le v<1} M_{R_1(v)}(-\gamma)\,,
$$

which has a similar form as (4.39). Taking expectations gives

$$
E\left[\sup_{0\le t\le t_o} M_{R(t)}(-\gamma)\right] \le \left[\sum_{y=0}^{t_0} P(t_0;\, t_0-y,y)\, e^{y\left(\frac{\lambda_1-\lambda_0}{\lambda}\right)\gamma c}\right] \max_{\substack{0\le v<1\\ i=0,1}} M_{R_i(v)}(-\gamma)\,.
$$

Finally, a Lundberg–type upper bound for the finite time ruin probability in (4.29), for $t_0 \in \mathbb{N}$, is given in the following corollary:

Corollary 4.3 For the finite time ruin probability $P\{T \le t_0\}$ in the regime–stitching periodic NHP model driven by a $\{0,1\}$–valued Markov chain κ, where t_0 is assumed to be an integer value, the following inequality holds:

$$
\begin{aligned}
P\{T\le t_0\} \;&\le\; e^{-\gamma u}\left[\sum_{y=0}^{t_0} P(t_0;\, t_0-y,y)\, e^{y\left(\frac{\lambda_1-\lambda_0}{\lambda}\right)\gamma c}\right] \\
&\qquad \max_{\substack{0\le v<1\\ i=0,1}} M_{R_i(v)}(-\gamma) \sup_{y\ge 0}\left\{\frac{e^{\gamma y}\,\bar{F}_X(y)}{\int_y^\infty e^{\gamma x}\, dF_X(x)}\right\}\,, \qquad t_0\in\mathbb{N}\,, \qquad (4.40)
\end{aligned}
$$

where γ satisfies (4.35) and $P(t_0;\, t_0-y,y)$ is given in (4.16) when $m=2$.

Remark 4.3 Obviously, the simpler bound for $P\{T \le t_0\}$ given by (4.28) can also be derived here:

$$
P\{T\le t_0\} \le e^{-\gamma u}\left[\sum_{y=0}^{t_0} P(t_0;\, t_0-y,y)\, e^{y\left(\frac{\lambda_1-\lambda_0}{\lambda}\right)\gamma c}\right] \max_{\substack{0\le v<1\\ i=0,1}} M_{R_i(v)}(-\gamma)\,, \qquad (4.41)
$$

but (4.40) is tighter than (4.41), as shown in the following examples.

Example 4.2 Consider claim sizes that are exponentially distributed with mean μ. Their mgf $M_X(s) = \frac{1}{1-\mu s}$, for $s < a_X = \frac{1}{\mu}$. The adjustment coefficient for parameter λ_0, is then given by

$$\gamma = \frac{c - \lambda_0\, B(p,q)\,\mu}{c\,\mu\,\alpha^*} = \frac{1}{\mu} - \frac{\bar{\lambda}\, B(p,q)}{c\,\alpha^*}\,, \tag{4.42}$$

which is the positive solution to equation (4.35). The corresponding $M_{R_i(v)}(-\gamma)$, given in (4.27), takes the form

$$M_{R_i(v)}(-\gamma) = e^{\left(\frac{\lambda_i\, B(p,q;\,v)}{\lambda\, B(p,q)} - v\right)\gamma c}\,, \qquad 0 \le v < 1\,,\ i = 0,1\,. \tag{4.43}$$

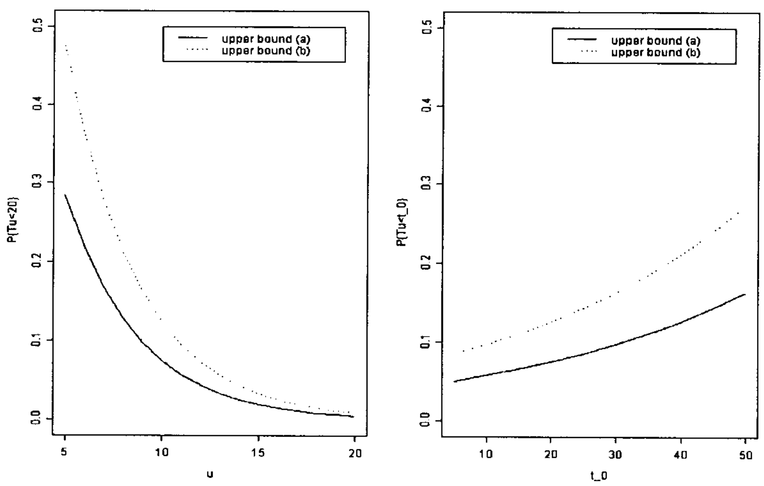

Figure 4.2: Upper bounds for exponential claims vs u ($t_0 = 20$) and t_0 ($u = 10$).

Figure 4.2 illustrates the upper bounds in this exponential case, as a function of u (left graph), when $t_0 = 20$, and as a function of t_0 (right graph), when $u = 10$. The other parameters are chosen to be $\lambda_0 = 1$, $\lambda_1 = 1.2$, $p = 3$, $q = 2$, $p_{01} = 0.25$, $p_{10} = 0.5$, $c = 1.5$, $\mu = 1.5$ and $\gamma = 0.267$, which is obtained from (4.42). Clearly, the upper bounds (a), given by (4.40) are sharper than those in (b), given by (4.41).

Example 4.3 Consider the case of inverse Gaussian distributed claims, with mean μ, variance $\mu\,\beta$ and density function

$$f_X(x) = \frac{\mu}{\sqrt{2\pi\beta x^3}}\, e^{-\frac{(x-\mu)^2}{2\beta x}}\,, \qquad x > 0\,.$$

Their mgf $M_X(s) = e^{\frac{\mu}{\beta}(1-\sqrt{1-2\beta s})}$ exists for $s < \frac{1}{2\beta}$. The adjustment coefficient γ with respect to parameter $\bar{\lambda}$ is the positive solution to the equation

$$\frac{\bar{\lambda}\, B(p,q)}{\alpha^*}\left[e^{\frac{\mu}{\beta}\left(1-\sqrt{1-2\beta\gamma}\right)} - 1\right] = \gamma\, c\,, \tag{4.44}$$

and $M_{R_i(v)}(-\gamma)$, for $i = 0, 1$, is of the same form as in (4.43).

Figure 4.3 illustrates the upper bounds in this inverse Gaussian case, again as a function of u (left graph), when $t_0 = 20$, and as a function of t_0 (right graph), when $u = 10$. The other parameters are chosen as for Figure 4.2 and $\beta = \frac{8}{3}$, which gives a variance of 4. Here $\gamma = 0.155$ is obtained from (4.44). Again the upper bounds in (a), given by (4.40) are sharper than those in (b), given by (4.41).

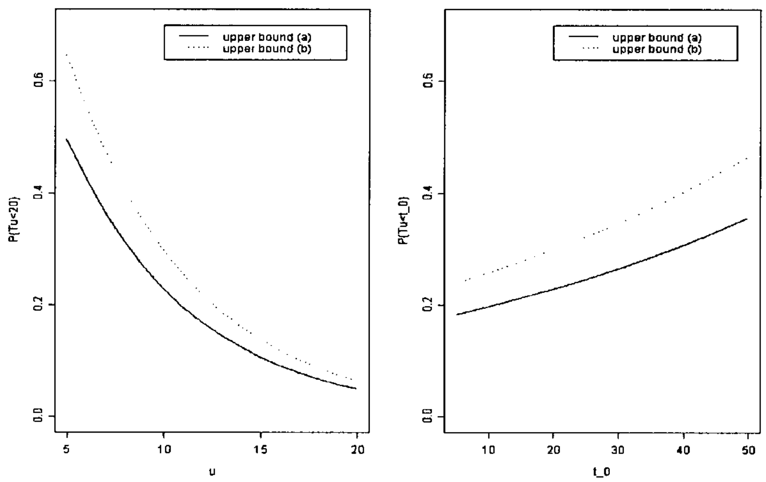

Figure 4.3: Upper bounds for inverse Gaussian claims vs t_0 ($u = 10$) and u ($t_0 = 20$).

In summary, the regime–switching periodic NHP processes discussed here can be useful in modeling risk processes under periodic and random environments. A beta–type short–term intensity function is proposed with a finite state Markov process to model the peak level in the intensity of this Cox risk process. This generalizes the periodic NHP model. It can also provide more realistic descriptions than Cox models with piecewise constant intensities.

The flexible shape of the beta function and the explicit results obtained for the Cox risk process should make these regime–switching period NHP models more practical than Cox processes with piecewise constant intensities, or than the usual NHP process. However, this work can be extended to other reasonable short–term intensity functions. Furthermore, statistical methods to estimate from real dataset the beta parameters and level parameters of the model are discussed and illustrated in next chapter.

Chapter 5

Statistical inference for regime–switching NHP models and an application

We have discussed the statistical inference on NHP models with short–term and long–term periodicity in Chapter 3. These models reflect periodic environments affected by seasonal conditions. Chapter 4 introduces a regime–switching periodic Poisson model that accounts for both, seasonal variations and random fluctuations in the claim intensity. Like the periodic NHP process, the intensity function follows a deterministic pattern in each short–term period, but its peak level is not governed by another long–term periodic function, but rather by a random Markov chain.

In this chapter, we study the statistical inference for periodic regime–switching Poisson models. The likelihood function and the maximum likelihood estimates of the model parameters are derived in Section 5.1. An application of the model to the dataset of Atlantic Tropical Storms and Hurricanes Affecting the United States is discussed in Section 5.2. Two classifications of the hurricane seasons corresponding to different level processes are discussed.

5.1 Maximum likelihood estimation of the intensity process

For the regime–switching Cox counting process $\{N(t);\, t \geq 0\}$ in (4.7), the intensity process $\{\lambda(t);\, t \geq 0\}$, given by (4.3), that is,

$$\lambda(t) = \lambda_{\kappa_{\lfloor t \rfloor}}\, \beta(t - \lfloor t \rfloor)\,, \qquad t \geq 0\,,$$

is of a parametric form. Here we assume that the short–term annual intensity function β is of beta–type with parameters p and q, given by (4.4). Other parameters for this intensity process λ are $\lambda_0, \ldots, \lambda_{m-1}$ for the level process κ, and $(p_{ij})_{i,j=0}^{m-1}$ for the transition probabilities. It is possible to estimate these parameters from data using maximum likelihood estimation (MLE). Note that other model parameters as m_1 and m_2 can usually be set graphically at values observed from the dataset.

Let $m^{(i,j)}$ be the total number of complete years at which the intensity level changes from λ_i to λ_j, then the empirical estimation of the transition probabilities is given by

$$\hat{p}_{i,j} = \frac{m^{(i,j)}}{\sum_{i=0}^{m-1} m^{(i,j)}}\,, \qquad i, j = 0, \ldots, m-1\,. \tag{5.1}$$

Let d be the time scale in each year cycle; here $d = \frac{1}{12}$ corresponding to a month. Also, for the annual intensity function in (4.4), let m_1 and m_2 be two integer–multiples of d; here m_1 and m_2 correspond to two specific months in the year, marking the beginning and end of the hurricane season. Further for simplicity, assume that t is also an integer–multiple of d.

As in the periodic NHP case, define J as

$$J = \frac{m_2 - m_1}{d} = \frac{D}{d}\,,$$

that is the total number of months in each year over which the intensity function is positive. This gives a convenient partition of each year cycle $[0,\, m_1)$, $[m_1,\, t_1)$, $[t_1,\, t_2)$, ..., $[t_J,\, m_2)$, $[m_2,\, 1]$, where

$$t_l = m_1 + l\, d\,, \qquad \text{for } l = 0, \ldots, J\,.$$

Furthermore, denote by $n_{l,1}$, the number of events occured within the l-th month $[t_{l-1}, t_l)$ of the first year, and by $n_{l,k}^{(i,j)}$, the number of events occured within the l-th month $[t_{l-1}, t_l)$ of the k-th year in which the intensity level changes from λ_i to λ_j, where $l = 1, \ldots, J$, $m = 1, \ldots, m^{(i,j)}$ and $i, j = 0, \ldots, m-1$.

Suppose now that the level process $\{\lambda_{\kappa_n}; n \geq 0\}$ starts from level λ_{ς_0}, i.e., $\lambda_{\kappa_0} = \lambda_{\varsigma_0}$ where ς_0 is given, taking a value in $\{0, \ldots, m-1\}$. The contribution to the likelihood for the first year is:

$$
\begin{aligned}
L_1 &= \prod_{l=1}^{J} \left[e^{-\int_{t_{l-1}}^{t_l} \lambda_{\varsigma_0} \beta(v)\, dv} \left(\int_{t_{l-1}}^{t_l} \lambda_{\varsigma_0} \beta(v)\, dv \right)^{n_{l,1}} \right] \\
&= e^{-\int_{m_1}^{m_2} \lambda_{\varsigma_0} \beta(v)\, dv} \prod_{l=1}^{J} \left(\int_{t_{l-1}}^{t_l} \lambda_{\varsigma_0} \beta(v)\, dv \right)^{n_{l,1}}, \qquad (5.2)
\end{aligned}
$$

which counts for the likelihood of having $n_{l,1}$ events within the l-th month $[t_{l-1}, t_l)$ of the first year, for $l = 1, \ldots, J$.

In general, the contribution to the likelihood from years in which the intensity levels change from λ_i to λ_j, for $i, j = 0, \ldots, m-1$, is similarly given by

$$
\begin{aligned}
L^{(i,j)} &= (\hat{p}_{ij})^{m^{(i,j)}} \prod_{k=1}^{m^{(i,j)}} \prod_{l=1}^{J} e^{-\int_{t_{l-1}}^{t_l} \lambda_j \beta(v)\, dv} \left(\int_{t_{l-1}}^{t_l} \lambda_j \beta(v)\, dv \right)^{n_{l,k}^{(i,j)}}, \\
&= (\hat{p}_{ij})^{m^{(i,j)}} e^{-m^{(i,j)} \int_{m_1}^{m_2} \lambda_j \beta(v)\, dv} \prod_{l=1}^{J} \left(\int_{t_{l-1}}^{t_l} \lambda_j \beta(v)\, dv \right)^{\sum_{k=1}^{m^{(i,j)}} n_{l,k}^{(i,j)}}, \qquad (5.3)
\end{aligned}
$$

where the first term in (5.3) represents the likelihood contribution of all transition probabilities, for $m^{(i,j)}$ years with intensity levels changing from λ_i to λ_j, and where $\sum_{k=1}^{m^{(i,j)}} n_{l,k}^{(i,j)}$ counts the corresponding total number of events occured within the l-th month of these years.

Further, consider the likelihood from the last (incomplete) year. Suppose that the intensity level for the last complete year is $\lambda_{\lfloor t \rfloor - 1} = \lambda_{\varsigma_1}$ which changes to level $\lambda_{\lfloor t \rfloor} = \lambda_{\varsigma_2}$ for the last (incomplete) year, where ς_1 and ς_2 are from the set $\{0, 1, \ldots, m-1\}$. Then the contribution to the likelihood from the last incomplete year is the total contributions from the complete months in the last incomplete year, given by

$$L^{(\varsigma_1,\varsigma_2)} = \hat{p}_{\varsigma_1,\varsigma_2} \prod_{l=1}^{J^*} e^{-\int_{t_{l-1}}^{t_l} \lambda_{\varsigma_2} \beta(v)\, dv} \left(\int_{t_{l-1}}^{t_l} \lambda_{\varsigma_2} \beta(v)\, dv \right)^{n_{l,\lfloor t \rfloor+1}} , \qquad (5.4)$$

where

$$J^* = \left\lfloor \frac{t - -\lfloor t \rfloor - m_1}{d} \right\rfloor$$

is the number of (complete) months in the last incomplete year, while $n_{l,\lfloor t \rfloor+1}$ is the number of events occured in the l-th (complete) month ($l = 1, \ldots, J^*$) of the last (incomplete) year.

Hence the full likelihood function is given by (5.2), (5.3) and (5.4) to be

$$\begin{aligned} L &= L_1 \left(\prod_{i,j=0}^{m-1} L^{(i,j)} \right) L^{(\varsigma_1,\varsigma_2)} \\ &= \left[\hat{p}_{\varsigma_1,\varsigma_2} \prod_{i,j=0}^{m-1} (\hat{p}_{ij})^{m^{(i,j)}} \right] e^{-\left[\lambda_{\varsigma_0} + \sum_{j=0}^{m-1} \lambda_j \left(\sum_{i=0}^{m-1} m^{(i,j)} \right)\right] \int_{m_1}^{m_2} \beta(v)\, dv - \lambda_{\varsigma_2} \int_{t_0}^{t_{J^*}} \beta(v)\, dv} \\ & \prod_{l=1}^{J} \left\{ \left(\int_{t_{l-1}}^{t_l} \lambda_{\varsigma_0} \beta(v)\, dv \right)^{n_{l,1}} \prod_{j=0}^{m-1} \left(\int_{t_{l-1}}^{t_l} \lambda_j \beta(v)\, dv \right)^{\sum_{i=0}^{m-1} \sum_{k=1}^{m^{(i,j)}} n_{l,k}^{(i,j)}} \right\} \\ & \prod_{l=1}^{J^*} \left(\int_{t_{l-1}}^{t_l} \lambda_{\varsigma_2} \beta(v)\, dv \right)^{n_{l,\lfloor t \rfloor+1}} . \end{aligned}$$

Substituting β for the beta intensity function in (4.4), by similar derivations to those in Section 3.1, the integrals above can be represented as complete and incomplete beta functions, yielding:

$$\begin{aligned} L = e^{-\Lambda(t)} & \left[\hat{p}_{\varsigma_1,\varsigma_2} \prod_{i,j=0}^{m-1} (\hat{p}_{ij})^{m^{(i,j)}} \right] \prod_{l=1}^{J} \left\{ \frac{\lambda_{\varsigma_0} D \left[B(p,\, q;\, \frac{ld}{D}) - B(p,\, q;\, \frac{(l-1)d}{D}) \right]}{\alpha^*} \right\}^{n_{l,1}} \\ & \prod_{l=1}^{J} \prod_{j=0}^{m-1} \left\{ \frac{\lambda_j D \left[B(p,\, q;\, \frac{ld}{D}) - B(p,\, q;\, \frac{(l-1)d}{D}) \right]}{\alpha^*} \right\}^{\sum_{i=0}^{m-1} \sum_{k=1}^{m^{(i,j)}} n_{l,k}^{(i,j)}} \\ & \prod_{l=1}^{J^*} \left\{ \frac{\lambda_{\varsigma_2} D \left[B(p,\, q;\, \frac{ld}{D}) - B(p,\, q;\, \frac{(l-1)d}{D}) \right]}{\alpha^*} \right\}^{n_{l,\lfloor t \rfloor+1}} , \end{aligned} \qquad (5.5)$$

where the empirical random measure $\Lambda(t)$ is given by

$$\begin{aligned}\Lambda(t) &= \left[\lambda_{\varsigma_0} + \sum_{j=0}^{m-1} \lambda_j \left(\sum_{i=0}^{m-1} m^{(i,j)}\right)\right] \frac{D\,B(p,\,q)}{\alpha^*} \\ &\quad + \lambda_{\varsigma_2} \sum_{j=1}^{J^*} \frac{D\big[B(p,\,q;\,\frac{ld}{D}) - B(p,\,q;\,\frac{(l-1)d}{D})\big]}{\alpha^*}.\end{aligned}$$

Further denote by $n_1 = \sum_{l=1}^{J} n_{l,1}$ and $n_{\lfloor t\rfloor+1} = \sum_{l=1}^{J} n_{l,\lfloor t\rfloor+1}$, the total number of events occured during the first year and the last (incomplete) year, respectively, and by $n^{(j)} = \sum_{i=0}^{m-1}\sum_{l=1}^{J}\sum_{k=1}^{m^{(i,j)}} n_{l,k}^{(i,j)}$, the total number of events occured during the years at which the intensity levels change to λ_j, where $j = 0, 1, \ldots, m-1$. Then define

$$N_l = n_{l,1} + \sum_{i,j=0}^{m-1}\sum_{k=1}^{m^{(i,j)}} n_{l,k}^{(i,j)} + n_{l,\lfloor t\rfloor+1}, \qquad l = 1, \ldots, J,$$

to be the total number of events occured within the l-th month $[t_{l-1}, t_l)$ for the given dataset, where for $l = J^* + 1, \ldots, J$, $n_{l,\lfloor t\rfloor+1} = 0$. Clearly then

$$N = \sum_{l=1}^{J} N_l$$

is the total number of occurrences on time interval $[0,\,t)$.

Consequently, the corresponding log likelihood function is given by

$$\begin{aligned}l &= -\Lambda(t) + \log \hat{p}_{\varsigma_1,\varsigma_2} + \sum_{i,j=0}^{m-1} m^{(i,j)} \log \hat{p}_{ij} + N \log\left(\frac{D}{\alpha^*}\right) \\ &\quad + n_1 \log \lambda_{\varsigma_0} + \sum_{j=0}^{m-1} n^{(j)} \log \lambda_j + n_{\lfloor t\rfloor+1} \log \lambda_{\varsigma_2} \\ &\quad + \sum_{j=1}^{J} N_l \log\left[B(p,\,q;\,\frac{j\,d}{D}) - B(p,\,q;\,\frac{(j-1)\,d}{D})\right].\end{aligned} \tag{5.6}$$

The maximum likelihood estimators for p, q, and $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$ for the regime–switching Cox model with a beta–type intensity process are then obtained by maximizing (5.6) numerically.

5.2 An application to tropical storms and hurricanes dataset

By considering global climatological and periodical effects on North American weather, we model the frequencies of Atlantic hurricanes using the NHP processes with double periodicity in Section 3.3. The dataset Atlantic Tropical Storms and Hurricanes Affecting the United States: 1899–2002 [see Landreneau (2003)] is used to fit the periodic NHP counting model. With a double–beta intensity function, the model explains generally the short–term seasonal pattern and long–term peak variation affected by El Niño/La Niña phenomenon. In this section, we consider possible random effects on the short–term peaks and then use periodic regime–switching NHP (Cox) processes discussed in Chapter 4 to model this Atlantic tropical storms and hurricanes dataset.

5.2.1 Model and fit under the count classification

As we have seen in Section 3.3.1, the North Atlantic hurricane season officially runs form June to November. During this period the average number of systems reaching tropical storm (maximum sustained winds between 39–73 mph), hurricane (maximum sustained winds of at least 74 mph) and major (or intense) hurricane status (maximum sustained winds exceeding 110 mph, categories 3-5 on the Saffir–Simpson scale) are approximately ten, six and two, respectively. However, the vast majority of tropical storms and hurricanes typically occur during the August–October period, with September having had the most major ones. It is considered the peak of the hurricane season. The monthly distribution of total 168 Atlantic hurricanes (1899–2002) is given in Table 3.1. Hence, for the regime–switching NHP model, we still choose the beta–type function as the short–term (annual) intensity.

Suppose that the regime–switching NHP model has a level process $\{\lambda_{\kappa_n}; n \geq 0\}$ which reflects the long–term (between year) fluctuations, where λ_{κ_n} is the peak value of the deterministic short–term intensity function of year $n+1$ and $\{\kappa_n; n \geq 0\}$ is assumed a Markov chain, with finite state space. To apply precisely this regime–switching NHP model, we need to classify the hurricane seasons (years) into m states. An intuitively classification is based on the number of hurricanes occurred during the year, mirroring naturally the intensity of the hurricane seasons, and is call the count classification. We categorize the hurricane season in below–normal, near–normal or above–normal if the number of hurricanes occurred during the year are less, equal or greater than 2. Hence the level process $\{\lambda_{\kappa_n}; n \geq 0\}$ is assumed to be a Markov chain, with state space $\{0, 1, 2\}$. That is, the corresponding levels λ_0, λ_1 and λ_2 represent the levels for below–, near– and above–normal years, respectively.

We assume that the matrix of transition probabilities of this Markov chain κ is

$$P = \begin{pmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{pmatrix},$$

such that $p_{i0} + p_{i1} + p_{i2} = 1$, for all $i = 0, 1, 2$.

The empirical estimation as well as the MLE of the transition probabilities $\hat{p}_{ij}$ are obtained by (5.1) and given in Table 5.1.

Table 5.1: Empirical counts and MLE's of the transition probabilities (under the count classification)

$m^{(0,0)}$	$m^{(0,1)}$	$m^{(0,2)}$	$\hat{p}_{00}$	$\hat{p}_{01}$	$\hat{p}_{02}$
30	11	14	0.5455	0.2	0.2545
$m^{(1,0)}$	$m^{(1,1)}$	$m^{(1,2)}$	$\hat{p}_{10}$	$\hat{p}_{11}$	$\hat{p}_{12}$
14	9	2	0.56	0.36	0.08
$m^{(2,0)}$	$m^{(2,1)}$	$m^{(2,2)}$	$\hat{p}_{20}$	$\hat{p}_{21}$	$\hat{p}_{22}$
11	4	8	0.4783	0.1739	0.3478

The beta–type short–term intensity function is of the form in (4.4) with parameters p and q. Then the intensity process λ is given by

$$\lambda(t) = \lambda_{\kappa_{\lfloor t \rfloor}} \frac{\left(\frac{t-m_1}{D}\right)^{p-1}\left(1-\frac{t-m_1}{D}\right)^{q-1}}{\alpha^*}, \qquad m_1 \le t - \lfloor t \rfloor \le m_2, \tag{5.7}$$

where m_1 and m_2 are set to be $\frac{5}{12}$ and $\frac{11}{12}$, respectively, and $\lambda_{\kappa_{\lfloor t \rfloor}}$ takes value over the set $\{\lambda_0, \lambda_1, \lambda_2\}$ according to the Markov chain $\{\kappa_n;\ n \ge 0\}$. Hence $D = m_2 - m_1 = \frac{6}{12}$ is the fraction of the year over which the intensity is positive, while α^* is given by (4.5).

For this dataset, $t = 104$ is an integer and $J = 6$ is the total number of months with positive intensity. The parameters in (5.7) can be estimated by using the method described in Section 5.1. As the year 1899 had two hurricanes, it is a near–normal year, the log–likelihood given in (5.6) is then

$$\begin{aligned} l \;=\; & -\Lambda(104) + \sum_{i,j=0}^{2} m^{(i,j)} \log \hat{p}_{ij} + N \log\left(\frac{D}{\alpha^*}\right) + n_1 \log \lambda_1 \\ & + \sum_{j=0}^{2} n^{(j)} \log \lambda_j + \sum_{j=1}^{J} N_l \log\left[B(p,\,q;\,\frac{j\,d}{D}) - B(p,\,q;\,\frac{(j-1)\,d}{D})\right], \end{aligned} \tag{5.8}$$

where $d = \frac{1}{12}$, while $\Lambda(104)$ is

$$\Lambda(104) = \left[\lambda_1 + \sum_{j=0}^{2} \lambda_j \left(\sum_{i=0}^{2} m^{(i,j)}\right)\right] \frac{D\,B(p,\,q)}{\alpha^*}.$$

The empirical counts used for estimating the parameters of Atlantic Hurricanes Affecting United States (1899–2002) are given in Table 5.2, while Table 5.3 gives the MLE's of the parameters, which were obtained by maximizing the log–likelihood (5.8) with the Excel solver.

Table 5.2: Empirical counts (hurricanes) used for estimating the parameters (under the count classification)

Dataset	N_1	N_2	N_3	N_4	N_5	N_6	N	n_1	$n^{(0)}$	$n^{(1)}$	$n^{(2)}$
Hurricane	11	17	44	65	27	4	168	2	35	48	83

Table 5.3: MLE's of parameters for the hurricanes dataset (1899–2002) (under the count classification)

Dataset	$\hat{p}$	$\hat{q}$	$\hat{\lambda}_0$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	log-likelihood
Hurricane	3.1348	3.0795	2.4322	7.6442	13.2180	-408.86

Figure 5.1 compares the observed and expected monthly average number of Atlantic hurricanes for below–, near– and above–normal years under the count classification, respectively. Here the intensity with a beta–type function as well as a 3–level process was fitted to these annual hurricane frequencies of 1899–2002 dataset. The fit seems quite good for the seasonal hurricane characteristics of each short–term (year) cycle, as well as the level classifications. As we can see from this graph, the estimated level parameters distinguish significantly between below–, near– and above–normal seasons, providing a feasible and clarified criterion for insurance businesses.

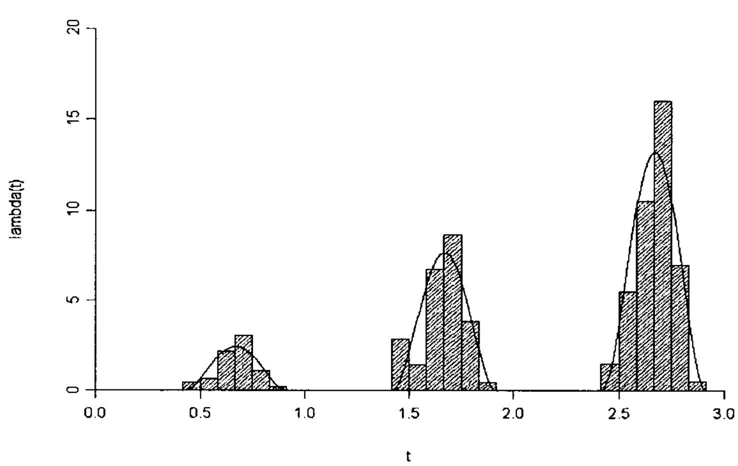

Figure 5.1: Hurricanes data and 3–level beta intensities

However, in climatology there is an official risk classification based on the Accumulated Cyclone Energy (ACE) index for the Atlantic hurricane season. We call this classification the ACE classification. Next section gives the necessary background information on ACE classification, while Section 5.2.3 studies the statistical inference for the Atlantic tropical storms and hurricanes (1950–2001) dataset under this ACE classification.

5.2.2 Background information on ACE classification

The National Oceanic and Atmospheric Administration's (NOAA) outlook usually predicts the probability that the Atlantic hurricane season for a specific year will be above–normal, near–normal or below–normal. For example, NOAA's updated outlook for the 2004 Atlantic hurricane season, issued in August 2004, indicates a 45% probability of an above–normal season, a 45% probability of a near–normal season, and only a 10% chance of a below–normal one (see NOAA: August 2004 update to Atlantic hurricane season outlook).

The phrase "total seasonal activity" refers to the collective intensity and duration of Atlantic tropical storms and hurricanes occurring during a given season. The measure of total seasonal activity used by NOAA in US is called the Accumulated Cyclone Energy (ACE) index. The ACE index is a wind energy index, defined as the sum of the squares of the estimated 6–hourly maximum sustained wind speed (knots) for all named systems while they are at least of tropical storm strength.

NOAA uses the ACE index, in combination with the numbers of named storms, hurricanes, and major hurricanes, to categorize North Atlantic hurricane seasons as being above–normal, near–normal and below–normal. The 1950–2003 mean value of the ACE index is 97.3, and the median value is 88.

The definitions of above–normal, near–normal and below–normal Atlantic hurricane seasons are given in the following (see Background information: the North Atlantic hurricane season).

Above–normal season: An ACE index value well above $103 \times 104\ kt^2$ (corresponding to 117% of the median ACE value or 110% of the mean), or an ACE value slightly above $103 \times 104\ kt^2$ combined with at least two of the following three parameters being above the long–term average: number of tropical storms, hurricanes, and major hurricanes.

Near–normal season: An ACE index value in the range 66 – 103 × 104 kt^2 (corresponding to 75%–117% of the median or 71%–110% of the mean), or an ACE index value slightly above 103 × 104 kt^2, but with less than two of the following three parameters being above the long–term average: numbers of tropical storms, hurricanes, and major hurricanes.

Below–normal season: An ACE index value below 66 × 104 kt^2, corresponding to below 75% of the median or 71% of the mean.

The NOAA's ACE index values as percents of the long–term median of 88 (1950–2003) are shown in Figure 5.2. The upper dotted line represents the critical value (117%) between above– and near–normal seasons, while the lower dotted line represents the critical value (75%) between near– and below–normal seasons.

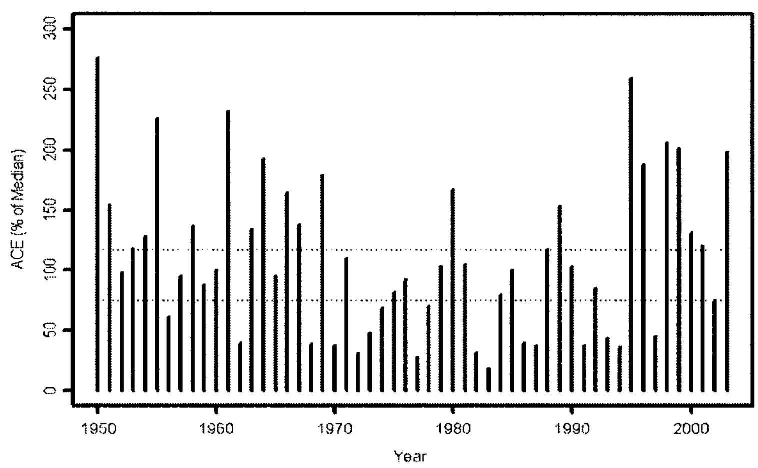

Figure 5.2: ACE values as percents of the long–term median (1950–2003)

5.2.3 Model and fit under the ACE classification

As described in Section 5.2.2, NOAA classifies North Atlantic hurricane seasons (years) to be above–normal, near–normal and below–normal, according to the ACE index combining with the numbers of named tropical storms, hurricanes, and major hurricanes. The Atlantic Hurricane Seasons dataset shows the statistics associated with above–, near– and below–normal seasons for the period of 1950–2003 [see Atlantic hurricane season statistics (1950–2003) Table A.2 in appendix]. Based on this fact, the natural environment presumably makes transitions between these 3 states according to some process.

The dataset Atlantic Tropical Storms and Hurricanes Affecting the United States: 1899–2002 in Landreneau (2003) is used to fit the periodic NHP counting model given in Chapter 4. Combing this with the dataset Atlantic Hurricane Seasons, we have the new dataset with the ACE index for Atlantic hurricane seasons and the Atlantic tropical storms and hurricanes affecting the United States for the time period 1950–2002. Therefore it is possible to fit this dataset to our periodic regime–switching NHP counting model under the ACE classification.

Over the 53–year period 1950 though 2002, a total of 76 category 1 though 5 hurricanes and 117 tropical storms have affected the Atlantic United States. Table 5.4 shows the monthly distribution of those 76 Atlantic hurricanes (H) and tropical storms (TS). Note that two of the tropical storms observed in February 1952 and May 1959, respectively, are not officially considered here. Apart from these two, all other Atlantic hurricanes and tropical storms happened between the months of June and November, and within those months, September had the largest portion. The average annual number of Atlantic hurricanes and tropical storms affecting United States are 1.43 and 2.2, respectively. As in the NHP periodic case, a short–term (annual) periodic function as part of intensity is appropriate.

Now the level process $\{\lambda_{\kappa_n}; n \geq 0\}$ in a regime–switching NHP model is assumed a Markov chain, with the state space $\{0, 1, 2\}$, where levels λ_0, λ_1 and λ_2 correspond to the levels for below–, near– and above–normal years, respectively.

The empirical estimation as well as the MLE of the transition probabilities $\hat{p}_{ij}$, under the ACE classification, are obtained from (5.1) and given in Table 5.5.

We also assume that the short–term intensity is a beta–type function given by

Table 5.4: Monthly distribution of the hurricanes and tropical storms (1950–2002 dataset)

Month	Number of H	Proportion (%)	Number of TS	Proportion (%)
June	4	5.3	14.5	12.6
July	6	7.9	11.5	10.0
August	21	27.6	27	23.5
September	32	42.1	44	38.3
October	12	15.8	13.5	11.7
November	1	1.3	4.5	3.9
	76	100.0	115	100.0

Table 5.5: Empirical counts and MLE's of the transition probabilities (under the ACE classification)

$m^{(0,0)}$	$m^{(0,1)}$	$m^{(0,2)}$	$\hat{p}_{00}$	$\hat{p}_{01}$	$\hat{p}_{02}$
6	7	4	0.3529	0.4118	0.2353
$m^{(1,0)}$	$m^{(1,1)}$	$m^{(1,2)}$	$\hat{p}_{10}$	$\hat{p}_{11}$	$\hat{p}_{12}$
7	3	6	0.4375	0.1875	0.3750
$m^{(2,0)}$	$m^{(2,1)}$	$m^{(2,2)}$	$\hat{p}_{20}$	$\hat{p}_{21}$	$\hat{p}_{22}$
5	6	8	0.2632	0.3158	0.4211

(4.4) and the intensity process λ is then given by (5.7), where the starting and ending hurricane season parameters, m_1 and m_2, are set to be $\frac{5}{12}$ and $\frac{11}{12}$, respectively. Again $D = m_2 - m_1 = \frac{6}{12}$ is the duration within a year for which the intensity is positive.

For this dataset, $t = 53$ is an integer and $J = 6$ is the total number of months with positive intensity. Parameters p, q, λ_0, λ_1 and λ_2 in (5.7) can be estimated by using the method presented in Section 5.1. As the year 1950 was an above–normal season, the log–likelihood given in (5.6) here is

$$\begin{aligned} l &= -\Lambda(53) + \sum_{i,j=0}^{2} m^{(i,j)} \log \hat{p}_{ij} + N \log\left(\frac{D}{\alpha^*}\right) + n_1 \log \lambda_2 \\ &\quad + \sum_{j=0}^{2} n^{(j)} \log \lambda_j + \sum_{j=1}^{J} N_l \log\left[B(p,q;\frac{j\,d}{D}) - B(p,q;\frac{(j-1)\,d}{D})\right], \end{aligned} \tag{5.9}$$

where $d = \frac{1}{12}$, represents the length of a month in a year, and the empirical random measure $\Lambda(53)$ is given by

$$\Lambda(53) = \left[\lambda_2 + \sum_{j=0}^{2} \lambda_j \left(\sum_{i=0}^{2} m^{(i,j)}\right)\right] \frac{D\,B(p,q)}{\alpha^*}.$$

The empirical counts used for estimating the parameters of Atlantic Hurricanes and Tropical Storms Affecting United States (1950–2002) are given in Table 5.6, while Table 5.7 gives the MLE's of the parameters, which were obtained by maximizing the log–likelihood (5.9) with the Excel solver. The corresponding log–likelihood values are also given in Table 5.7.

Table 5.6: Empirical counts used for estimating the parameters

(under the ACE classification)

Dataset	N_1	N_2	N_3	N_4	N_5	N_6	N	n_1	$n^{(0)}$	$n^{(1)}$	$n^{(2)}$
Hurricane	4	6	21	32	12	1	76	3	14	26	33
Tropical Storm	14	12	28	43	14	4	115	4	29	35	47

Table 5.7: MLE's of parameters for the hurricanes and tropical storms dataset

(under the ACE classification)

Dataset	$\hat{p}$	$\hat{q}$	$\hat{\lambda}_0$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	log–likelihood
Hurricane	3.8707	3.6883	3.3033	6.9015	8.0470	-212.06
Tropical Storm	2.1891	2.4343	5.2569	7.1376	8.7583	-269.36

Figure 5.3 and Figure 5.4 compare the observed and expected monthly average number of Atlantic hurricanes and tropical storms, under the ACE classification,

for below–, near– and above–normal years, respectively. Here the intensity with a beta–type function as well as a 3–level process was fitted respectively to these annual hurricane and tropical storms frequencies of 1950–2002 dataset. Overall, the fit was good for the seasonal characteristics of each short–term (year) cycle, but some levels were under–estimated.

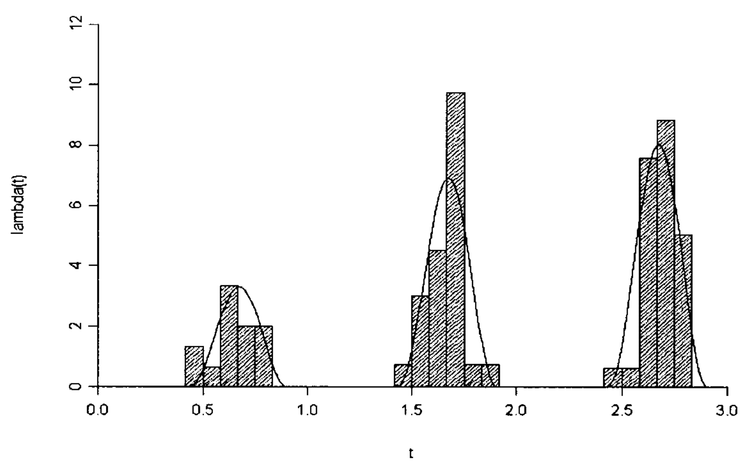

Figure 5.3: Hurricanes data and 3–level beta intensities

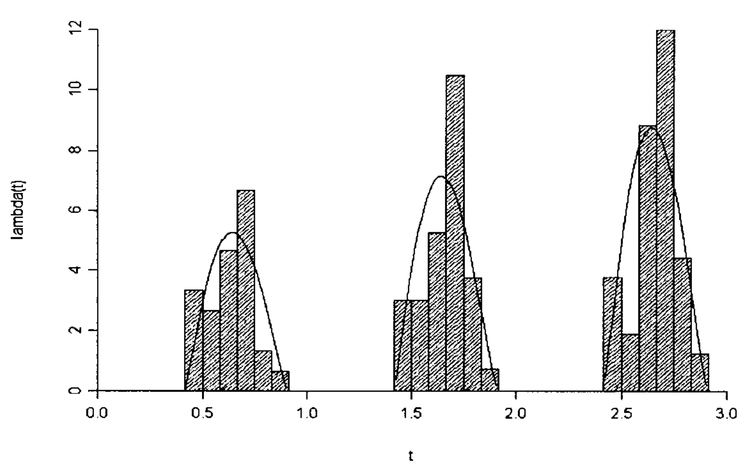

Figure 5.4: Tropical Storms data and 3–level beta intensities

The reality is that, under this ACE classification, the observed average number of hurricanes in September for near–normal seasons is significantly larger than the one for above–normal seasons, showing a somewhat inaccurate criterion for level classification. Nevertheless, the ACE classification is an official and climatological method to categorize the Atlantic hurricane seasons by NOAA. The intensity of Atlantic hurricane seasons are affected by some crucial climate phenomenon such as Atlantic multi–decadal signal, Atlantic sea surface temperatures (SSTs) and the El Niño/Southern Oscillation (ENSO).

Atlantic hurricane seasons exhibit prolonged periods lasting decades of generally above–normal or below–normal activity. The atmospheric and oceanic conditions controlling these very long–period fluctuations in hurricane activity are referred to as the Atlantic multi–decadal signal. These multi–decadal fluctuations in hurricane activity result almost entirely from variations in the number of hurricanes and major hurricanes. ENSO is another climate phenomenon that can significantly impact seasonal Atlantic hurricane activity, with El Niño acting to reduce activity while La Niña increases it. As such, it is difficult to model perfectly such a complex phenomena with a simple parametric processes. The fit under our count classification seems more natural and acceptable than the result under ACE classification for Atlantic hurricane seasons.

Perhaps the fit could be improved by choosing other intensity functions than the beta or by increasing the number of parameters, such as the 3–parameter generalized beta function. Goodness–of–fit tests can also be considered here, as in the NHP model case. These will be studied in the subsequent work. Further investigation could also be carried out through different directions, for examples, by considering the possible covariants, like wind speeds and water temperatures in the intensity or the non–homogeneous time–dependent level processes.

By the properties derived in Section 4.2, with the MLE's of the parameters in the model, it is possible to calculate some quantities for the claim counting process $\{N(t);\, t \geq 0\}$, such as the probabilities that there be k claim occurrences during the

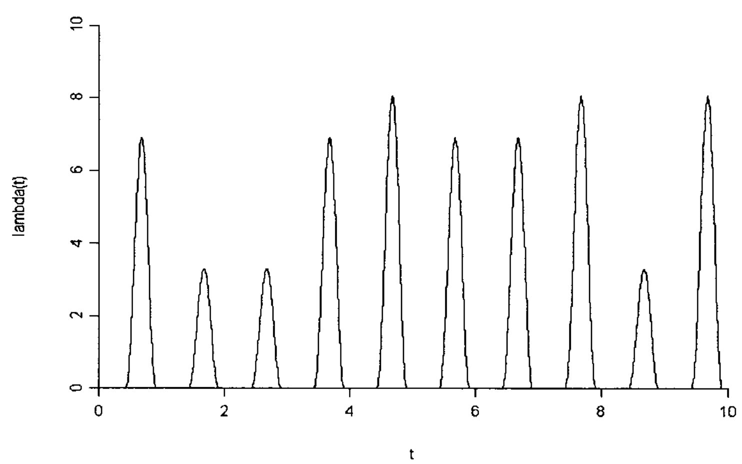

Figure 5.5: Simulated intensities for the next 10 Atlantic hurricane seasons

time interval $[0, t)$, $P\{N(t) = k\}$, for $t \geq 0$ and $k \in \mathbb{N}$, given in (4.18), and the expected number of claims occured in $[0, t)$, $E[N(t)] = E[\Lambda(t)]$, for $t \geq 0$, in (4.19).

Finally, we simply simulate future intensities for our regime–switching model, using the MLE's of the model parameters in Tables 5.1 and 5.3 obtained from the dataset under the count classification. Figure 5.5 shows simulated intensities for the Atlantic hurricanes affecting the United States for the next 10 seasons.

We conclude that these periodic regime–switching NHP (Cox) processes are practical and valuable in modeling counting processes under periodic and random environments. They give an alternative method to study the Atlantic tropical storms and hurricanes dataset. Furthermore, it seems that these models are more practical than Cox processes with piecewise constant intensities. It also provides an efficient method to take into account random fluctuations on the levels of the short–term intensity rather than the deterministic patterns discussed in our pervious NHP processes.

As in the NHP model case, the flexible shape of beta functions and the feasible statistical estimation of model parameters make these periodic regime–switching NHP models simple to use in practice. We hope that this work can contribute to show their usefulness.

Chapter 6

Markov–modulated Poisson risk models

Chapters 2 and 3 discuss time–dependent NHP processes that generalize the classical homogeneous Poisson risk model, as their intensity rate is a function of time rather than a constant. Chapters 4 and 5 introduce and study a special Cox process called the regime–switching NHP process. This further generalizes the NHP process, in the sense that the time–dependent intensity for regime–switching NHP process fluctuates under a Markov environment. This chapter considers a process that has Cox as well as a class of semi–Markov properties, called Markov–modulated Poisson process. It takes the form of a finite number of different homogeneous Poisson processes according to an external Markov process. This can be seen as an alternative generalization of classical Poisson risk models without specific emphasis on the seasonally periodic intensities.

One objective of ruin theory is to obtain exact formulas or approximations of ruin probabilities in various kinds risk models. In this chapter, we are interested in the ruin probability and the severity of ruin in a Markov-modulated risk model, as described in Section 1.2.2, where claim intensities, claim amounts and premiums vary according to a Markovian environment. The severity of ruin is considered as an additional element of information on ruin.

Reinhard (1984) considers a class of semi–Markov risk models in which the claim frequencies and claim amounts are influenced by an external Markovian environment process. Asmussen (1989) conceptualizes this model as the Markov–modulated Poisson risk model. In Section 1.2.2, we review the main results derived by Reinhard (1984). A system of integro–differential equations for the non–ruin probability is given by (1.36). In a particular case with two possible states for the environment, and when the claim severity distributions are exponential, the solution to this system is derived. Lu and Li (2004) generilize these results to allowing the claim severity distributions from a much wider class, the rational family.

Snoussi (2002) studies the severity of ruin in a Markov–modulated risk model where the claim intensity, claim amounts and the premium fluctuate according to a Markovian environment. In his paper, a differential system (1.47) for the distribution of the severity of ruin is established. An explicit formula for the severity of ruin is derived in the particular case where the environment process has two possible states and where the claim severities are exponentially distributed. These results are given in Theorem 1.11.

The description of the Markov-modulated model, where we assume that the claim intensities, claim severities and the premiums vary according to a Markovian environment, is given in Section 1.2.2. A system of Laplace transforms is established in Section 6.1 and Section 6.2, respectively, for the non–ruin probabilities and the distribution of the severity of ruin, given an initial environment state. These are obtained from the corresponding systems of Volterra integral equations. In the two–state model, explicit formulas for non–ruin probabilities and the distribution of the severity of ruin are then derived, when the initial reserve is zero or when both claim severity distributions belong to the rational family.

6.1 Ruin probabilities

Now consider the Markov–modulated Poisson counting process $\{N(t);\, t \geq 0\}$ with an external environment process $\{I(t);\, t \geq 0\}$, assumed to be a homogeneous, irreducible and recurrent Markov chain with finite state space $J = \{1, 2, \ldots, m\}$.

Under the assumptions given in Section 1.2.2, at time t claims occur according to a Poisson process with constant intensity rate $\lambda_i \in \mathbb{R}^+$, when $I(t) = i \in J$, and the corresponding claim severity distribution is $F_i(x)$, with density function $f_i(x)$ and finite mean μ_i. Moreover, premiums are received continuously at a positive constant rate c_i over any time interval in which the environment process I remains in state i. The corresponding surplus process $\{U(t);\, t \geq 0\}$ is given by (1.33):

$$U(t) = u + C(t) - \sum_{n=1}^{N(t)} X_n, \qquad t \geq 0, \tag{6.1}$$

where $u \geq 0$ is the initial surplus and $C(t)$ is the aggregate premium income process, given by (1.34).

Recall that

$$T = \inf\{t > 0 \mid U(t) < 0\}, \qquad (\infty, \text{ otherwise}),$$

is the time to ruin for this surplus process U defined by (6.1) and $\Psi_i(u)$ is the probability of ultimate ruin , when the initial environment state is i and the initial reserve is u. It is given by

$$\Psi_i(u) = P\{T < \infty \mid U(0) = u,\ I(0) = i\}, \qquad i \in J\,,\ u \geq 0\,,$$

while $\Psi(u) = \sum_{i=1}^{m} \pi_i\, \Psi_i(u)$, for $u \geq 0$, is the probability of ultimate ruin in the stationary case and π_i is given by (1.32) in Theorem 1.7.

Moreover, $\bar{\Psi}_i(u) = 1 - \Psi_i(u)$, $i \in J$, and $\bar{\Psi}(u) = 1 - \Psi(u)$ are the corresponding ultimate survival probabilities, or non–ruin probabilities, respectively.

6.1.1 Laplace transforms of $\bar{\Psi}_i(u)$

A system of Volterra integral equations for the non–ruin probabilities $\bar{\Psi}_i(u)$, for $i \in J = \{1, 2, \ldots, m\}$, is derived by Reinhard (1984) as follows:

$$c_i \bar{\Psi}_i(t) = c_i \bar{\Psi}_i(0) + \lambda_i \int_0^t \bar{\Psi}_i(t-y)\, \bar{F}_i(y) dy + \alpha_i \int_0^t \Big[\bar{\Psi}_i(u) - \sum_{k=1}^m p_{ik} \bar{\Psi}_k(u)\Big] du\,, \qquad t \geq 0\,, \tag{6.2}$$

where $\bar{F}_i$ is the survival function of the distribution function F_i. Equation (6.2) has a unique solution such that $\bar{\Psi}_i(\infty) = 1$, for $i \in J$, or equivalently,

$$\bar{\Psi}_i(0) = 1 - \frac{\lambda_i \mu_i}{c_i} - \frac{\alpha_i}{c_i} \int_0^\infty \Big[\bar{\Psi}_i(u) - \sum_{k=1}^m p_{ik} \bar{\Psi}_k(u)\Big] du\,, \qquad i \in J\,, \tag{6.3}$$

where the transition probabilities p_{ij}, for $i, j \in J$, are given by (1.31).

We now apply Laplace transforms to solve the system of equations (6.2). Let $\hat{\bar{\Psi}}_i(s)$ and $\hat{f}_i(s)$ be the Laplace transforms of $\bar{\Psi}_i$ and f_i, respectively, i.e.,

$$\hat{\bar{\Psi}}_i(s) = \int_0^\infty e^{-su}\, \bar{\Psi}_i(u) du\,, \qquad \hat{f}_i(s) = \int_0^\infty e^{-su}\, f_i(u) du\,, \qquad i \in J\,,\ s \in \mathbb{C}\,.$$

Taking Laplace transforms on both sides of equation (6.2) yields

$$c_i\, \hat{\bar{\Psi}}_i(s) = c_i\, \frac{\bar{\Psi}_i(0)}{s} + \lambda_i\, \hat{\bar{\Psi}}_i(s) \left[\frac{1 - \hat{f}_i(s)}{s}\right] + \alpha_i \left[\frac{\hat{\bar{\Psi}}_i(s)}{s} - \sum_{k=1}^m p_{ik} \frac{\hat{\bar{\Psi}}_k(s)}{s}\right] du\,,$$

which can be rewritten as

$$\left[s - \frac{\lambda_i + \alpha_i}{c_i} + \frac{\lambda_i}{c_i} \hat{f}_i(s)\right] \hat{\bar{\Psi}}_i(s) + \frac{\alpha_i}{c_i} \sum_{k=1}^m p_{ik}\, \hat{\bar{\Psi}}_k(s) = \bar{\Psi}_i(0)\,, \qquad i \in J\,,\ s \in \mathbb{C}\,,$$

or in a matrix form

$$A(s)\, \hat{\bar{\mathbf{\Psi}}}(s) = \bar{\mathbf{\Psi}}(0)\,, \qquad s \in \mathbb{C}\,, \tag{6.4}$$

where

$$A(s) = \begin{pmatrix} s - \frac{\lambda_1[1-\hat{f}_1(s)]+\alpha_1}{c_1} & & \\ & \ddots & \\ & & s - \frac{\lambda_m[1-\hat{f}_m(s)]+\alpha_m}{c_m} \end{pmatrix} + \begin{pmatrix} \frac{\alpha_i}{c_i} & & \\ & \ddots & \\ & & \frac{\alpha_m}{c_m} \end{pmatrix} P\,, \tag{6.5}$$

$\hat{\bar{\Psi}}(s) = \left(\hat{\bar{\Psi}}_1(s), \ldots, \hat{\bar{\Psi}}_m(s)\right)'$, $\bar{\Psi}(0) = \left(\bar{\Psi}_1(0), \ldots, \bar{\Psi}_m(0)\right)'$, and P is given by (1.31), with $p_{ii} = 0$, for $i \in J$.

Then the vector of Laplace transforms $\hat{\bar{\Psi}}(s)$ can be obtained as $\hat{\bar{\Psi}}(s) = [A(s)]^{-1}\bar{\Psi}(0)$, and the equation

$$|A(s)| = 0\,, \qquad s \in \mathbb{C}\,, \tag{6.6}$$

is called the characteristic equation of (6.4), where $|A(s)|$ is the determinant of the matrix $A(s)$.

Lemma 6.1 The characteristic equation in (6.6) is of the form

$$\prod_{j=1}^{m} S_j(s) + \sum_{j=1}^{m-2} \sum_{1 \le i_1 < i_2 < \cdots < i_j \le m} S_{i_1}(s) \cdots S_{i_j}(s) |P_{i_1,\ldots,i_j}| = -|P|\,, \tag{6.7}$$

where P is given by (1.31), with $p_{ii} = 0$, for $i \in J$, while

$$S_j(s) = \frac{s - \frac{\lambda_j[1-\hat{f}_j(s)]+\alpha_j}{c_j}}{\frac{\alpha_j}{c_j}} = \frac{c_j s - \left\{\lambda_j[1 - \hat{f}_j(s)] + \alpha_j\right\}}{\alpha_j}\,, \qquad j \in J\,,\; s \in \mathbb{C}\,, \tag{6.8}$$

and $|P_{i_1,\ldots,i_j}|$ is a $(m-j) \times (m-j)$ minor, which is the reduced determinant of $|P|$ formed by omitting the i_1th, $\ldots$, i_jth rows and column of matrix P.

Proof. Since $A(s)$ can be rewritten as

$$A(s) = \begin{pmatrix} \frac{\alpha_i}{c_i} & & \\ & \ddots & \\ & & \frac{\alpha_m}{c_m} \end{pmatrix} \left[\begin{pmatrix} S_1(s) & & \\ & \ddots & \\ & & S_m(s) \end{pmatrix} + P \right],$$

and $\alpha_i > 0$, for $i = 1, \ldots, m$, it is easy to see that equation (6.6) is equivalent to:

$$\left| \begin{pmatrix} S_1(s) & & \\ & \ddots & \\ & & S_m(s) \end{pmatrix} + P \right| = \begin{vmatrix} S_1(s) & p_{12} & \cdots & p_{1m} \\ p_{21} & S_2(s) & \cdots & p_{2m} \\ \cdots & \cdots & & \\ p_{m1} & p_{m2} & \cdots & S_m(s) \end{vmatrix} = 0\,. \tag{6.9}$$

Then, after some tedious algebraic manipulations, (6.7) follows from (6.9) by properties of the determinant and the method of induction. □

6.1.2 Characteristic equation of a two–state model

Now we consider the case when $m = 2$, that is $\{I(t);\, t \geq 0\}$ is a two–state Markov process, which reflects the random environmental effects due to "normal" vs. "abnormal", or "high season" vs. "low season" conditions. The unique stationary probability distribution π_i can be obtained from (1.32) as

$$\pi_i = \frac{\frac{\lambda_i}{\alpha_i}}{\frac{\lambda_1}{\alpha_1} + \frac{\lambda_2}{\alpha_2}}, \qquad i = 1, 2, \tag{6.10}$$

and the positive loading condition (1.35) becomes

$$d = \frac{\frac{\lambda_1}{\alpha_1}\left(\frac{c_1}{\lambda_1} - \mu_1\right) + \frac{\lambda_2}{\alpha_2}\left(\frac{c_2}{\lambda_2} - \mu_2\right)}{\frac{\lambda_1}{\alpha_1} + \frac{\lambda_2}{\alpha_2}} > 0. \tag{6.11}$$

In this case the matrix $A(s)$ in (6.5) takes the form

$$A(s) = \begin{pmatrix} s - \frac{\lambda_1+\alpha_1}{c_1} + \frac{\lambda_1}{c_1}\hat{f}_1(s) & \frac{\alpha_1}{c_1} \\ \frac{\alpha_2}{c_2} & s - \frac{\lambda_2+\alpha_2}{c_2} + \frac{\lambda_2}{c_2}\hat{f}_2(s) \end{pmatrix},$$

and the transition matrix P becomes

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.12}$$

Hence the characteristic equation (6.7) is of the form

$$S_1(s)\, S_2(s) = 1\,, \tag{6.13}$$

where $S_j(s)$, $j = 1, 2$, is given by (6.8), or equivalently,

$$Q(s) := \left[s - \frac{\lambda_1[1 - \hat{f}_1(s)] + \alpha_1}{c_1}\right]\left[s - \frac{\lambda_2[1 - \hat{f}_2(s)] + \alpha_2}{c_2}\right] = \frac{\alpha_1\, \alpha_2}{c_1\, c_2}\,. \tag{6.14}$$

Note that $s = 0$ is one root of equation (6.14). The following theorem shows that it also has one and only one positive real root, which plays a key role in deriving the non–ruin probabilities $\bar{\Psi}_i(u)$.

Theorem 6.1 The characteristic equation in (6.14) has exactly one positive real root, say ρ, on the right half complex plane.

Proof. For $\delta > 0$, define

$$Q_\delta(s) := \left[s - \frac{(\lambda_1 + \alpha_1 + \delta) - \lambda_1 \hat{f}_1(s)}{c_1}\right] \left[s - \frac{(\lambda_2 + \alpha_2 + \delta) - \lambda_2 \hat{f}_2(s)}{c_2}\right], \qquad s \in \mathbb{C}.$$

It is easy to check that equation $Q_\delta(s) = 0$ has two positive real roots. Noting that $Q_\delta(0) = \frac{(\alpha_1+\delta)(\alpha_2+\delta)}{c_1\, c_2} > \frac{\alpha_1\, \alpha_2}{c_1\, c_2}$ and $Q_\delta(+\infty) = +\infty$, we conclude that equation

$$Q_\delta(s) = \frac{\alpha_1\, \alpha_2}{c_1\, c_2} \tag{6.15}$$

has at least two positive real roots.

Furthermore, if s is on the half circle: $\{z \in \mathbb{C};\ |z| = r > 0 \text{ and } \Re(z) \geq 0\}$ of the complex plane, then $|Q_\delta(s)| > \frac{\alpha_1\, \alpha_2}{c_1\, c_2}$, for r is sufficiently large, while if s is on the imaginary axis $\{s \in \mathbb{C};\ \Re(s) = 0\}$, then $|Q_\delta(s)| \geq \frac{(\alpha_1+\delta)(\alpha_2+\delta)}{c_1\, c_2} > \frac{\alpha_1\, \alpha_2}{c_1\, c_2}$, which is the right side of equation (6.15). This implies on the boundary of the contour enclosed by the half circle and the imaginary axis, that $|Q_\delta(s)| > \frac{\alpha_1\, \alpha_2}{c_1\, c_2}$. We conclude, by Rouché's Theorem [see Szegö (1975)], that on the right half plane, the number of roots to equation (6.15) equals the number of roots of the equation $Q_\delta(s) = 0$. Moreover, by Rouché's Theorem, the latter only has exactly two roots with a positive real part on the right half complex plane.

It follows that equation (6.15) also only has exactly two roots with a positive real part, say, $\rho_1(\delta)$ and $\rho_2(\delta)$, on the right half complex plane.

Finally, as $\delta \to 0^+$, $\rho_j(\delta) \to \rho_j(0)$, $j = 1, 2$, where $\rho_j(0)$ are roots of equation (6.14). Moreover, the fact that $s = 0$ is a root of (6.14) shows that $\lim_{\delta\to 0^+} \rho_1(\delta) = 0$ and $\lim_{\delta\to 0^+} \rho_2(\delta) = \rho > 0$, the unique positive real root of (6.14). □

6.1.3 Formulas for $\bar{\Psi}_1(0)$ and $\bar{\Psi}_2(0)$

Now equation (6.4) has the form

$$\begin{pmatrix} s - \frac{\lambda_1+\alpha_1}{c_1} + \frac{\lambda_1}{c_1} \hat{f}_1(s) & \frac{\alpha_1}{c_1} \\ \frac{\alpha_2}{c_2} & s - \frac{\lambda_2+\alpha_2}{c_2} + \frac{\lambda_2}{c_2} \hat{f}_2(s) \end{pmatrix} \begin{pmatrix} \hat{\bar{\Psi}}_1(s) \\ \hat{\bar{\Psi}}_2(s) \end{pmatrix} = \begin{pmatrix} \bar{\Psi}_1(0) \\ \bar{\Psi}_2(0) \end{pmatrix},$$

or

$$\begin{cases} \hat{\bar{\Psi}}_1(s) = \dfrac{\bar{\Psi}_1(0)\left[s - \frac{\lambda_2+\alpha_2}{c_2} + \frac{\lambda_2}{c_2}\hat{f}_2(s)\right] - \bar{\Psi}_2(0)\frac{\alpha_1}{c_1}}{Q(s) - \frac{\alpha_1\alpha_2}{c_1c_2}} \\ \hat{\bar{\Psi}}_2(s) = \dfrac{\bar{\Psi}_2(0)\left[s - \frac{\lambda_1+\alpha_1}{c_1} + \frac{\lambda_1}{c_1}\hat{f}_1(s)\right] - \bar{\Psi}_1(0)\frac{\alpha_2}{c_2}}{Q(s) - \frac{\alpha_1\alpha_2}{c_1c_2}} \end{cases}, \tag{6.16}$$

where $Q(s)$ is given in (6.14).

Since $\hat{\bar{\Psi}}_1(s)$ and $\hat{\bar{\Psi}}_2(s)$ are finite for all s with $\Re(s) > 0$ and by Theorem 6.1 ρ is the unique positive real root of (6.14), that is,

$$Q(\rho) = \left[\rho - \frac{\lambda_1[1 - \hat{f}_1(\rho)] + \alpha_1}{c_1}\right]\left[\rho - \frac{\lambda_2[1 - \hat{f}_2(\rho)] + \alpha_2}{c_2}\right] = \frac{\alpha_1\,\alpha_2}{c_1\,c_2}, \tag{6.17}$$

we have that both the numerators in (6.16) are zero when $s = \rho$:

$$\bar{\Psi}_1(0)\left[\rho - \frac{\lambda_2 + \alpha_2}{c_2} + \frac{\lambda_2}{c_2}\hat{f}_2(\rho)\right] = \bar{\Psi}_2(0)\,\frac{\alpha_1}{c_1}, \tag{6.18}$$

or equivalently by equation (6.17),

$$\bar{\Psi}_2(0)\left[\rho - \frac{\lambda_1 + \alpha_1}{c_1} + \frac{\lambda_1}{c_1}\hat{f}_1(\rho)\right] = \bar{\Psi}_1(0)\,\frac{\alpha_2}{c_2}. \tag{6.19}$$

Applying (6.18) and (6.19) to numerators in (6.16), the latter can then be rewritten as

$$\begin{cases} \hat{\bar{\Psi}}_1(s) = \dfrac{\bar{\Psi}_1(0)\left[\left(s-\rho\right) + \frac{\lambda_2}{c_2}\left(\hat{f}_2(s) - \hat{f}_2(\rho)\right)\right]}{Q(s) - \frac{\alpha_1\alpha_2}{c_1c_2}}, \\ \hat{\bar{\Psi}}_2(s) = \dfrac{\bar{\Psi}_2(0)\left[\left(s-\rho\right) + \frac{\lambda_1}{c_1}\left(\hat{f}_1(s) - \hat{f}_1(\rho)\right)\right]}{Q(s) - \frac{\alpha_1\alpha_2}{c_1c_2}}. \end{cases} \tag{6.20}$$

On the other hand, equation (6.3) takes the form

$$\bar{\Psi}_1(0) = 1 - \frac{\lambda_1\mu_1}{c_1} - \frac{\alpha_1}{c_1}\int_0^\infty \left[\bar{\Psi}_1(u) - \bar{\Psi}_2(u)\right]du,$$
$$\bar{\Psi}_2(0) = 1 - \frac{\lambda_2\mu_2}{c_2} - \frac{\alpha_2}{c_2}\int_0^\infty \left[\bar{\Psi}_2(u) - \bar{\Psi}_1(u)\right]du,$$

which yields

$$\frac{\alpha_2}{c_2}\bar{\Psi}_1(0) + \frac{\alpha_1}{c_1}\bar{\Psi}_2(0) = \frac{\alpha_1}{c_1}\left(1 - \frac{\lambda_2\mu_2}{c_2}\right) + \frac{\alpha_2}{c_2}\left(1 - \frac{\lambda_1\mu_1}{c_1}\right). \tag{6.21}$$

Combining (6.18) or (6.19) with (6.21), we get the following theorem for $\bar{\Psi}_1(0)$ and $\bar{\Psi}_2(0)$, the non–ruin probabilities when the initial surplus is zero.

Theorem 6.2 For the risk model given by (6.1), with $m = 2$ and $d > 0$, the non–ruin probabilities when the initial surplus is zero are given by

$$\begin{cases} \bar{\Psi}_1(0) = \dfrac{\frac{\alpha_1}{c_1}\left(1-\frac{\lambda_2\mu_2}{c_2}\right)+\frac{\alpha_2}{c_2}\left(1-\frac{\lambda_1\mu_1}{c_1}\right)}{\rho-\frac{\lambda_2}{c_2}\left[1-\hat{f}_2(\rho)\right]}\,, \\ \bar{\Psi}_2(0) = \dfrac{\frac{\alpha_1}{c_1}\left(1-\frac{\lambda_2\mu_2}{c_2}\right)+\frac{\alpha_2}{c_2}\left(1-\frac{\lambda_1\mu_1}{c_1}\right)}{\rho-\frac{\lambda_1}{c_1}\left[1-\hat{f}_1(\rho)\right]}\,, \end{cases} \tag{6.22}$$

where ρ is the only positive real root to equation (6.14).

6.1.4 Explicit Results for $\bar{\Psi}_1(u)$ and $\bar{\Psi}_2(u)$

In this section, we derive closed–form expressions for the non–ruin probabilities when $m = 2$. The analysis of the roots of equation (6.14) allows for the analytical inversion of the Laplace transforms of $\bar{\Psi}_1(u)$ and $\bar{\Psi}_2(u)$, at least for certain types of claim size distributions.

Consider the case where the claim size distributions f_1 and f_2 are from the rational family. We first define this family of distributions.

Definition 6.1 **[Rational distribution on $\mathbb{R}^+$]** A probability distribution F on $\mathbb{R}^+$ is called a rational distribution, or is said to belong to the rational distribution family $\mathcal{R}_f^+$, if it admits a density function f, and the Laplace transform of this density is a rational function (ratio of two polynomials) as follows:

$$\hat{f}(s) = \frac{\prod_{i=1}^{n} q_i + s\,\beta(s)}{\prod_{i=1}^{n}(s+q_i)}\,, \qquad \text{for } \Re(s) > \max\{-\Re(q_i);\, i = 1,2,\ldots,n\}\,, \tag{6.23}$$

where $q_1, q_2, \cdots, q_n$ form pairs of conjugate complex numbers, with $\Re(q_i) > 0$, and $\beta(s) = \sum_{i=0}^{n-2} \beta_i\, s^i$ is a polynomial of degree $n-2$ or less.

Remark 6.1 When all $q_i > 0$, for $i = 1, 2, \ldots, n$ above, the probability distribution F is said to belong to the K_n family, $n \in \mathbb{N}^+$, which is a subclass of the family $\mathcal{R}_f^+$.

The class of K_n distribution includes, as special cases, Erlang, Coxian and some phase–type distributions, as well as mixtures of these. It is widely used in applied probability models [see Cohen (1982) and Tijms (1994)]. Li (2004) considers the evaluation of the Gerber–Shiu penalty function [see Gerber and Shiu (1998) for the definition] for a class of renewal risk process, in which the claim inter–arrival times are K_n distributed.

The rational family $\mathcal{R}_f^+$ is even a wider class of distributions, including the K_n family and distributions with damped sine and cosine functions as part of their densities.

Assume that two claim severity distributions f_1 and f_2 belong to $\mathcal{R}_f^+$, namely, their Laplace transforms are of the form:

$$\begin{cases} \hat{f}_1(s) &= \frac{p_{k-1}(s)}{p_k(s)}\,, \qquad \text{for } \Re(s) \in (a_{f_1}, \infty) \text{ and } k \in \mathbb{N}^+\,, \\ \hat{f}_2(s) &= \frac{q_{l-1}(s)}{q_l(s)}\,, \qquad \text{for } \Re(s) \in (a_{f_2}, \infty) \text{ and } l \in \mathbb{N}^+\,, \end{cases} \tag{6.24}$$

where $a_{f_i} = \inf\{s \in \mathbb{R} : E[e^{-s X_i}] < \infty\}$ and X_1, X_2 are the claim severity random variables, while p_k and q_l are polynomials of degrees k and l, respectively, with leading coefficient 1 and satisfying $p_{k-1}(0) = p_k(0)$ and $q_{l-1}(0) = q_l(0)$. Further, equations $p_k(s)$ and $q_l(s)$ have roots with only negative real parts.

It turns out that the equations in (6.20) can be transformed to rational expressions by multiplying both numerators and denominators by $p_k(s)q_l(s)$:

$$\begin{aligned} \hat{\bar{\Psi}}_1(s) &= \frac{\bar{\Psi}_1(0)(s-\rho)\, p_k(s) \left\{ q_l(s) + \frac{\lambda_2}{c_2\, q_l(\rho)} \left[\frac{q_{l-1}(s)\, q_l(\rho) - q_{l-1}(\rho)\, q_l(s)}{s-\rho} \right] \right\}}{p_k(s)\, q_l(s) [Q(s) - \frac{\alpha_1 \alpha_2}{c_1 c_2}]} \\ &= \frac{\bar{\Psi}_1(0)(s-\rho)\, p_k(s) \left\{ q_l(s) + \frac{\lambda_2}{c_2} \left(q_{l-1}[s,\rho] - \frac{q_{l-1}(\rho)}{q_l(\rho)}\, q_l[s,\rho] \right) \right\}}{p_k(s)\, q_l(s) [Q(s) - \frac{\alpha_1 \alpha_2}{c_1 c_2}]}, \end{aligned} \tag{6.25}$$

$$\begin{aligned} \hat{\bar{\Psi}}_2(s) &= \frac{\bar{\Psi}_2(0)(s-\rho)\, q_l(s) \left\{ p_k(s) + \frac{\lambda_1}{c_1\, p_k(\rho)} \left[\frac{p_{k-1}(s)\, p_k(\rho) - p_{k-1}(\rho)\, p_k(s)}{s-\rho} \right] \right\}}{p_k(s)\, q_l(s) [Q(s) - \frac{\alpha_1 \alpha_2}{c_1 c_2}]} \\ &= \frac{\bar{\Psi}_2(0)(s-\rho)\, q_l(s) \left\{ p_k(s) + \frac{\lambda_1}{c_1} \left(p_{k-1}[s,\rho] - \frac{p_{k-1}(\rho)}{p_k(\rho)}\, p_k[s,\rho] \right) \right\}}{p_k(s)\, q_l(s) [Q(s) - \frac{\alpha_1 \alpha_2}{c_1 c_2}]}, \end{aligned} \tag{6.26}$$

where $p_{k-1}[s,\rho] := \frac{p_{k-1}(s) - p_{k-1}(\rho)}{s-\rho}$, a polynomial of degree $k-2$, is the first order divided difference of $p_{k-1}(s)$ with respect to ρ, while $p_k[s,\rho]$, $q_{l-1}[s,\rho]$ and $q_l[s,\rho]$ have similar definitions. An introduction to divided differences can be found in Freeman (1960). It is now clear that both numerators of (6.25) and (6.26) are polynomials of degree $k+l+1$.

For simplicity, denote by $D_{k+l+2}(s)$ the common denominator of (6.25) and (6.26), which is clearly a polynomial of degree $k+l+2$ with the leading coefficient 1. Then equation $D_{k+l+2}(s)=0$, i.e.,

$$\left[\left(s-\frac{\lambda_1+\alpha_1}{c_1}\right)p_k(s)+\frac{\lambda_1}{c_1}\,p_{k-1}(s)\right]\left[\left(s-\frac{\lambda_2+\alpha_2}{c_2}\right)q_l(s)+\frac{\lambda_2}{c_2}\,q_{l-1}(s)\right] \\ -\frac{\alpha_1\,\alpha_2}{c_1\,c_2}\,p_k(s)\,q_l(s)=0\,, \tag{6.27}$$

has $k+l+2$ roots on the complex plane and all of which come in conjugate pairs. Noting that $s=0$ and $s=\rho$ are two roots, then we can rewrite $D_{k+l+2}(s)$ as:

$$D_{k+l+2}(s)=s\,(s-\rho)\prod_{i=1}^{k+l}(s+R_i)\,.$$

We remark that all R_i's have positive real parts, since, otherwise, they would also be roots to the characteristic equation (6.14), which is a contradiction to the conclusion in Theorem 6.1 that there is only one root on the right half complex plane.

Then (6.25) and (6.26) can be simplified to

$$\hat{\bar{\Psi}}_1(s)=\frac{\bar{\Psi}_1(0)\,p_k(s)\left\{q_l(s)+\frac{\lambda_2}{c_2}\left(q_{l-1}[s,\rho]-\frac{q_{l-1}(\rho)}{q_l(\rho)}\,q_l[s,\rho]\right)\right\}}{s\prod_{i=1}^{k+l}(s+R_i)}=\frac{\bar{\Psi}_1(0)\,g_{k+l}(s)}{s\prod_{i=1}^{k+l}(s+R_i)}\,,$$

$$\hat{\bar{\Psi}}_2(s)=\frac{\bar{\Psi}_2(0)\,q_l(s)\left\{p_k(s)+\frac{\lambda_1}{c_1}\left(p_{k-1}[s,\rho]-\frac{p_{k-1}(\rho)}{p_k(\rho)}\,p_k[s,\rho]\right)\right\}}{s\prod_{i=1}^{k+l}(s+R_i)}=\frac{\bar{\Psi}_2(0)\,h_{k+l}(s)}{s\prod_{i=1}^{k+l}(s+R_i)}\,,$$

where we define

$$g_{k+l}(s):=p_k(s)\left\{q_l(s)+\frac{\lambda_2}{c_2}\left(q_{l-1}[s,\rho]-\frac{q_{l-1}(\rho)}{q_l(\rho)}\,q_l[s,\rho]\right)\right\}\,, \tag{6.28}$$

$$h_{k+l}(s):=q_l(s)\left\{p_k(s)+\frac{\lambda_1}{c_1}\left(p_{k-1}[s,\rho]-\frac{p_{k-1}(\rho)}{p_k(\rho)}\,p_k[s,\rho]\right)\right\}\,. \tag{6.29}$$

Then if R_i, $i=1,2,\ldots,k+l$, are distinct real numbers, we obtain, by partial fractions, that

$$\hat{\bar{\Psi}}_1(s)=\bar{\Psi}_1(0)\left[\frac{g_0}{s}+\sum_{i=1}^{k+l}\frac{g_i}{s+R_i}\right],\qquad \hat{\bar{\Psi}}_2(s)=\bar{\Psi}_2(0)\left[\frac{h_0}{s}+\sum_{i=1}^{k+l}\frac{h_i}{s+R_i}\right],$$

and accordingly their inverse Laplace transforms are:

$$\bar{\Psi}_1(u) = \bar{\Psi}_1(0)\left[g_0 + \sum_{i=1}^{k+l} g_i\, e^{-R_i u}\right], \qquad \bar{\Psi}_2(u) = \bar{\Psi}_2(0)\left[h_0 + \sum_{i=1}^{k+l} h_i\, e^{-R_i u}\right], \quad (6.30)$$

where $g_0 = \frac{g_{k+l}(0)}{\prod_{i=1}^{k+l} R_i}$, $h_0 = \frac{h_{k+l}(0)}{\prod_{i=1}^{k+l} R_i}$, and

$$g_i = \frac{-g_{k+l}(-R_i)}{R_i \prod_{j=1,j\neq i}^{k+l}(R_j - R_i)}, \qquad h_i = \frac{-h_{k+l}(-R_i)}{R_i \prod_{j=1,j\neq i}^{k+l}(R_j - R_i)}, \qquad i = 1, 2, \ldots, k+l. \quad (6.31)$$

From (6.30), we immediately have that $g_0 = \frac{1}{\bar{\Psi}_1(0)}$, $h_0 = \frac{1}{\bar{\Psi}_2(0)}$, since $\lim_{u\to\infty} \bar{\Psi}_i(u) = 1$, for $i = 1, 2$, and $\sum_{i=0}^{k+l} g_i = \sum_{i=0}^{k+l} h_i = 1$.

We summarize the above derivations in the following theorem.

Theorem 6.3 For the risk model given by (1.33), with $m = 2$ and $d > 0$, if the claim severity distributions are in the rational family (6.23), the non–ruin probabilities are given by

$$\bar{\Psi}_1(u) = 1 + \bar{\Psi}_1(0)\sum_{i=1}^{k+l} g_i\, e^{-R_i u}, \qquad \bar{\Psi}_2(u) = 1 + \bar{\Psi}_2(0)\sum_{i=1}^{k+l} h_i\, e^{-R_i u}, \qquad (6.32)$$

where $-R_1, -R_2, \ldots, -R_{k+l}$, are distinct roots of equation (6.27), with negative real parts, and $\bar{\Psi}_1(0)$ and $\bar{\Psi}_2(0)$ are given by (6.22), while g_i, h_i are given by (6.31).

Remark 6.2 When some R_i's come in conjugate pairs of complex numbers, then the non–ruin probabilities may contain damped trigonometric functions. This is illustrated in the next section through examples.

6.1.5 Ruin probability examples

In this section, we illustrate the derivation of the non–ruin probabilities $\bar{\Psi}_1(u)$ and $\bar{\Psi}_2(u)$, given by (6.32), through some examples. In the first example, we assume that both claim severity distributions are exponential with mean μ_1 and μ_2, respectively.

Example 6.1 Let $f_i(x) = \frac{1}{\mu_i} e^{-\frac{x}{\mu_i}}$, for $x \geq 0$ and $i = 1, 2$. Then their Laplace transforms are $\hat{f}_i(s) = \frac{\frac{1}{\mu_i}}{s + \frac{1}{\mu_i}}$, for $i = 1, 2$, and hence, $p_0(s) = \frac{1}{\mu_1}$, $p_1(s) = s + \frac{1}{\mu_1}$, $q_0(s) = \frac{1}{\mu_2}$ and $q_1(s) = s + \frac{1}{\mu_2}$.

The explicit expressions of $\bar{\Psi}_1(0)$ and $\bar{\Psi}_2(0)$, given by (6.22), are now obtained as

$$\bar{\Psi}_1(0) = \frac{\frac{\alpha_1}{c_1}\left(1 - \frac{\lambda_2\,\mu_2}{c_2}\right) + \frac{\alpha_2}{c_2}\left(1 - \frac{\lambda_1\,\mu_1}{c_1}\right)}{\rho\big[1 - \frac{\lambda_2\,\mu_2}{c_2(\mu_2\,\rho+1)}\big]}, \tag{6.33}$$

$$\bar{\Psi}_2(0) = \frac{\frac{\alpha_1}{c_1}\left(1 - \frac{\lambda_2\,\mu_2}{c_2}\right) + \frac{\alpha_2}{c_2}\left(1 - \frac{\lambda_1\,\mu_1}{c_1}\right)}{\rho\big[1 - \frac{\lambda_1\,\mu_1}{c_1(\mu_1\,\rho+1)}\big]}, \tag{6.34}$$

where ρ is the positive root of equation:

$$Q(s) := \left[s - \frac{\lambda_1 + \alpha_1}{c_1} + \frac{\lambda_1}{c_1}\frac{\frac{1}{\mu_1}}{s + \frac{1}{\mu_1}}\right]\left[s - \frac{\lambda_2 + \alpha_2}{c_2} + \frac{\lambda_2}{c_2}\frac{\frac{1}{\mu_2}}{s + \frac{1}{\mu_2}}\right] = \frac{\alpha_1\,\alpha_2}{c_1\,c_2}, \tag{6.35}$$

which, once multiplied by $p_1(s)\,q_1(s) = (s + \frac{1}{\mu_1})(s + \frac{1}{\mu_2})$, is equivalent to

$$\begin{aligned} D_4(s) &= \left[\left(s - \frac{\lambda_1 + \alpha_1}{c_1}\right)\left(s + \frac{1}{\mu_1}\right) + \frac{\lambda_1}{c_1\mu_1}\right]\left[\left(s - \frac{\lambda_2 + \alpha_2}{c_2}\right)\left(s + \frac{1}{\mu_2}\right) + \frac{\lambda_2}{c_2\mu_2}\right] \\ &\quad - \frac{\alpha_1\,\alpha_2}{c_1\,c_2}\left(s + \frac{1}{\mu_1}\right)\left(s + \frac{1}{\mu_2}\right) \\ &= s^4 + \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} - \frac{\lambda_1 + \alpha_1}{c_1} - \frac{\lambda_2 + \alpha_2}{c_2}\right)s^3 \\ &\quad + \left[\left(\frac{1}{\mu_1} - \frac{\lambda_1 + \alpha_1}{c_1}\right)\left(\frac{1}{\mu_2} - \frac{\lambda_2 + \alpha_2}{c_2}\right) - \frac{\alpha_1}{c_1\mu_1} - \frac{\alpha_2}{c_2\mu_2} - \frac{\alpha_1\alpha_2}{c_1c_2}\right]s^2 \\ &\quad - \left[\frac{\alpha_1}{c_1\mu_1}\left(\frac{1}{\mu_2} - \frac{\lambda_2}{c_2}\right) + \frac{\alpha_2}{c_2\mu_2}\left(\frac{1}{\mu_1} - \frac{\lambda_1}{c_1}\right)\right]s = 0\,. \end{aligned} \tag{6.36}$$

It has exactly 4 roots, $s = 0$, $s = \rho$, $s = -R_1$ and $s = -R_2$. Note here that equation (6.36) is equivalent to equation (6.16) in Reinhard (1984). Now (6.28) and (6.29) are of the form

$$g_2(s) = \left(s + \frac{1}{\mu_1}\right)\left\{\left(s + \frac{1}{\mu_2}\right) - \frac{\lambda_2}{c_2(\mu_2\,\rho + 1)}\right\},$$

$$h_2(s) = \left(s + \frac{1}{\mu_2}\right)\left\{\left(s + \frac{1}{\mu_1}\right) - \frac{\lambda_1}{c_1(\mu_1\,\rho + 1)}\right\},$$

respectively. Further (6.31) gives:

$$g_1 = \frac{\left(R_1 - \frac{1}{\mu_1}\right)\left\{\left(\frac{1}{\mu_2} - R_1\right) - \frac{\lambda_2}{c_2(\mu_2\,\rho+1)}\right\}}{R_1(R_2 - R_1)}, \quad g_2 = \frac{\left(R_2 - \frac{1}{\mu_1}\right)\left\{\left(\frac{1}{\mu_2} - R_2\right) - \frac{\lambda_2}{c_2(\mu_2\,\rho+1)}\right\}}{R_2(R_1 - R_2)},$$

$$h_1 = \frac{\left(R_1 - \frac{1}{\mu_2}\right)\left\{\left(\frac{1}{\mu_1} - R_1\right) - \frac{\lambda_1}{c_1(\mu_1\,\rho+1)}\right\}}{R_1(R_2 - R_1)}, \quad h_2 = \frac{\left(R_2 - \frac{1}{\mu_2}\right)\left\{\left(\frac{1}{\mu_1} - R_2\right) - \frac{\lambda_1}{c_1(\mu_1\,\rho+1)}\right\}}{R_2(R_1 - R_2)}.$$

Therefore, the non–ruin probabilities $\bar{\Psi}_1(u)$ and $\bar{\Psi}_2(u)$ are given by

$$\begin{cases} \bar{\Psi}_1(u) = 1 + \bar{\Psi}_1(0)\big[g_1 e^{-R_1 u} + g_2 e^{-R_2 u}\big]\,, \\ \bar{\Psi}_2(u) = 1 + \bar{\Psi}_2(0)\big[h_1 e^{-R_1 u} + h_2 e^{-R_2 u}\big]\,, \end{cases} \qquad u \geq 0\,,$$

where $\bar{\Psi}_1(0)$ and $\bar{\Psi}_2(0)$ are given by (6.33) and (6.34), respectively. Then the ultimate non–ruin probability in the stationary case, $\bar{\Psi}(u)$, is given by

$$\bar{\Psi}(u) = 1 + \sum_{i=1}^{2} \big[\pi_1\, \bar{\Psi}_1(0)\, g_i + \pi_2\, \bar{\Psi}_2(0)\, h_i\big] e^{-R_i u}\,, \qquad u \geq 0\,,$$

where the stationary probability distribution π_i, $i = 1, 2$, is given by (6.10).

Remark 6.3 If we set the claim intensity rates, premium rates and claim severity distributions to be identical for the two Poisson processes in Example 6.1, that is, $\lambda_1 = \lambda_2 = \lambda$, $c_1 = c_2 = c$ and $\mu_1 = \mu_2 = \mu$, then it is easy to check that the characteristic equation (6.36) can be decomposed as

$$D_4(s) = s\,(s+\gamma)\left[s^2 + \Big(\gamma - \frac{\alpha_1 + \alpha_2}{c}\Big) s - \frac{\alpha_1 + \alpha_2}{c\,\mu}\right] = 0\,, \qquad (6.37)$$

where $\gamma = \frac{1}{\mu} - \frac{\lambda}{c}$. Note here $-R_1 = -\gamma$ is one root of equation (6.37), while ρ and $-R_2$ are the other two roots, corresponding to

$$s^2 + \Big(\gamma - \frac{\alpha_1 + \alpha_2}{c}\Big) s - \frac{\alpha_1 + \alpha_2}{c\,\mu} = 0\,.$$

We can also derive that $\bar{\Psi}_1(0)\, g_1 = \bar{\Psi}_2(0)\, h_1 = -\frac{\lambda \mu}{c}$ and $g_2 = h_2 = 0$. Hence

$$\bar{\Psi}_1(u) = \bar{\Psi}_2(u) = 1 - \frac{\lambda\,\mu}{c}\, e^{-\gamma u}, \qquad u \geq 0,$$

where the identical non–ruin probabilities $\bar{\Psi}_1(u)$ and $\bar{\Psi}_2(u)$, as expected, are the non–ruin probabilities obtained from the classical Poisson risk model with exponentially distributed claim sizes.

The next example gives numerical non–ruin probabilities when one claim severity class is distributed as Erlang(2) and the other as a mixture of two exponentials.

Example 6.2 Let $f_1(x) = \frac{x}{\beta^2} e^{-\frac{x}{\beta}}$ and $f_2(x) = \frac{\xi}{\beta_1} e^{-\frac{x}{\beta_1}} + \frac{1-\xi}{\beta_2} e^{-\frac{x}{\beta_2}}$, for $x \geq 0$ and $0 \leq \xi \leq 1$, with $\mu_1 = 2\beta$ and $\mu_2 = \xi\beta_1 + (1-\xi)\beta_2$. The Laplace transforms of f_1 and f_2 are given by

$$\hat{f}_1(s) = \frac{p_1(s)}{p_2(s)} = \frac{\frac{1}{\beta^2}}{(s+\frac{1}{\beta})^2}, \qquad \hat{f}_2(s) = \frac{q_1(s)}{q_2(s)} = \frac{\left(\frac{\xi}{\beta_1} + \frac{1-\xi}{\beta_2}\right) s + \frac{1}{\beta_1 \beta_2}}{\left(s+\frac{1}{\beta_1}\right)\left(s+\frac{1}{\beta_2}\right)}, \qquad s \in \mathbb{C}.$$

Then equation (6.27) becomes

$$\begin{aligned} D_6(s) = & \left[\left(s - \frac{\lambda_1 + \alpha_1}{c_1}\right)\left(s + \frac{1}{\beta}\right)^2 + \frac{\lambda_1}{c_1 \beta^2}\right] \\ & \times \left[\left(s - \frac{\lambda_2 + \alpha_2}{c_2}\right)\left(s + \frac{1}{\beta_1}\right)\left(s + \frac{1}{\beta_2}\right) + \frac{\lambda_2}{c_2}\left(\frac{\xi}{\beta_1} + \frac{1-\xi}{\beta_2}\right)s + \frac{\lambda_2}{c_2 \beta_1 \beta_2}\right] \\ & - \frac{\alpha_1 \alpha_2}{c_1 c_2}\left(s + \frac{1}{\beta}\right)^2\left(s + \frac{1}{\beta_1}\right)\left(s + \frac{1}{\beta_2}\right) = 0, \qquad s \in \mathbb{C}. \end{aligned} \tag{6.38}$$

It has exactly 6 roots: 0, ρ and $-R_i$, $i = 1, \ldots, 4$ that can be obtained numerically for chosen parameter values.

For example, let the intensity rates α_1 and α_2 of the external environmental process I be $\alpha_1 = \frac{1}{3}$ and $\alpha_2 = \frac{2}{3}$, respectively. Further letting $\lambda_1 = 0.5$, $\lambda_2 = 2$, $c_1 = 1$, $c_2 = 2$, $\beta = 1$, $\beta_1 = 0.5$, $\beta_2 = 2$, and $\xi = 0.8$, we get that $\mu_1 = 2$, $\mu_2 = 0.8$, and the stationary probabilities $\pi_1 = \frac{1}{3}$, $\pi_2 = \frac{2}{3}$. Solving equation (6.38) using mathematical softwares, we then obtain $\rho = 1.06554$, $R_1 = 0.07054$, $R_2 = 0.41863$, $R_3 = 1.45485 + 0.04652i$, $R_4 = 1.45485 - 0.04652i$. Note that in this case, the positive safety load factor, $\frac{d}{\pi_1 \mu 1 + \pi_2 \mu_2}$, is 11.11%.

Expressions (6.33) and (6.34) give that the non–ruin probabilities at $u = 0$ are $\bar{\Psi}_1(0) = 0.10235$ and $\bar{\Psi}_2(0) = 0.09765$, respectively. Inverting directly the Laplace transforms in (6.25) and (6.26), finally yields for $u \geq 0$:

$$\begin{cases} \bar{\Psi}_1(u) & = 1 - .00327\, e^{-.41863\, u} - .90331\, e^{-.07054\, u} + 0.00893\, e^{-1.45485\, u} \cos(.04652\, u) \\ & \qquad + .01390\, e^{-1.45485\, u} \sin(0.04652\, u), \qquad u \geq 0, \\ \bar{\Psi}_2(u) & = 1 + .00223\, e^{-.41863\, u} - .8810\, e^{-.07054\, u} - 0.02358\, e^{-1.45485\, u} \cos(.04652\, u) \\ & \qquad + .01197\, e^{-1.45485\, u} \sin(0.04652\, u), \qquad u \geq 0. \end{cases}$$

Finally, the probability of ultimate non–ruin, $\bar{\Psi}(u) = \pi_1 \bar{\Psi}_1(u) + \pi_2 \bar{\Psi}_2(u)$, in the stationary case is

$$\bar{\Psi}(u) = 1 + .00040\, e^{-.41863\, u} - .88844\, e^{-.07054\, u} - 0.01274\, e^{-1.45485\, u} \cos(.04652\, u) + .01261\, e^{-1.45485\, u} \sin(0.04652\, u)\,, \qquad u \geq 0\,.$$

These numerical inversions can be done easily and quickly using mathematical softwares, for example, Maple.

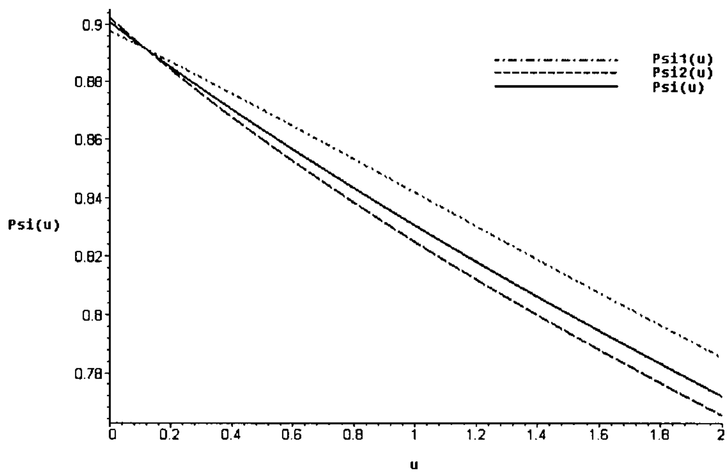

Figure 6.1: Probabilities of ultimate ruin for $0 \leq u \leq 2$

Figure 6.1 shows the probabilities of ultimate ruin, given that the initial environment state is i and the initial reserve is u, $\Psi_i(u)$, for $i = 1, 2$, and the probabilities of ultimate ruin in the stationary case, $\Psi(u)$, for different values of $u \in [0,\ 2]$. From this graph, we can see that, as expected, these ruin probabilities decrease as the initial surplus u increases. Moreover, when u is very small, $\Psi_1(u)$ (dotted line) is smaller than $\Psi_2(u)$ (dashed line) and the reverse situation occurs as u becomes larger, while the value of $\Psi(u)$ (solid line) is always between the values of $\Psi_1(u)$ and $\Psi_2(u)$. Figure 6.2 graphs these ruin probabilities again, but for values of the initial surplus $0 \leq u \leq 10$, showing how fast the ruin probabilities decrease when u increases.

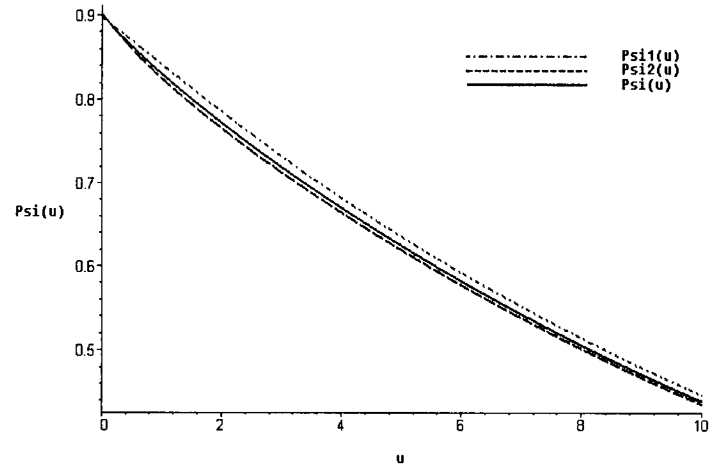

Figure 6.2: Probabilities of ultimate ruin for $0 \leq u \leq 10$

Reinhard (1984) studies the same problem for two–state models, however, there are two main differences with our work here: first, the Laplace transform approach is used here to solve the system of integro–differential equations; and second, the characteristic equation is fully discussed here, while Reinhard only considers it when claim severities are exponentially distributed. By these, explicit formulas for non–ruin probabilities when the initial surplus is zero or when both claim severity distributions are from the rational family are derived for the two-state model. The rational family of distributions is a wide–ranging family in practice and it includes, as the special cases, Erlang, Coxian, phase-type and K_n distributions, as well as mixture of these distributions.

6.2 The severity of ruin

In this section, we are interested in the severity of ruin for the Markov–modulated risk model given by (6.1). The severity of ruin is considered as an additional element of information on ruin. This problem was first explored by Gerber et al. (1987) for the classical continuous time risk model, and is studied, among others, by Dufresne and Gerber (1988), Dufresne (1989), Dickson (1989), Dickson and Waters (1992) and Snoussi (2002), who derives the explicit formulas of the probability of severity of ruin in a 2–state Markov–modulated model where the claim sizes are exponentially distributed.

For Markov–modulated risk models and definitions of the ultimate ruin probabilities defined in Section 6.1, we further define the probability of the severity of ruin by

$$\Psi_i(y;\, u) = P\{T < \infty,\, U(T) < -y \,|\, U(0) = u, I(0) = i\}\,, \qquad i \in J\,,\ u,\, y \geq 0\,, \tag{6.39}$$

which represents the probability that ruin occurs and that at the time of ruin the surplus takes a value less than $-y$, or eqivalently, that the deficit at the time of ruin be greater than y, given an initial surplus u and an initial environment $i \in J$.

Note that by setting $y = 0$ in (6.39), we have $\Psi_i(0;\, u) = \Psi_i(u)$, and by the relationship introduced by Gerber et al. (1987) for the classical risk model:

$$\begin{aligned} G_i(y;\, u) &= P\{T < \infty,\, U(T) \geq -y \,|\, U(0) = u,\, I(0) = i\}\,, \\ &= \Psi_i(u) - \Psi_i(y;\, u)\,, \qquad i \in J\,,\ u,\, y \geq 0\,, \end{aligned} \tag{6.40}$$

we are able to computer the probabilities $G_i(y;\, u)$ and the corresponding densities, defined by

$$g_i(y;\, u) = \frac{\partial}{\partial y} G_i(y;\, u)\,, \qquad i \in J\,,\ u,\, y \geq 0\,. \tag{6.41}$$

6.2.1 Laplace transforms of $\Psi_i(y;\, u)$

Snoussi (2002) derives a system of Volterra integral equations for the probabilities of the severity of ruin in Corollary 1.6:

$$c_i\,\Psi_i(y;\,t) = c_i\,\Psi_i(y;\,0) + \lambda_i \int_0^t \Psi_i(y;\,t-u)\,\bar{F}_i(u)\,du - \lambda_i \int_0^t \bar{F}_i(u+y)\,du$$
$$+\,\alpha_i \int_0^t \Big[\Psi_i(y;\,u) - \sum_{k=1}^m p_{ik}\,\Psi_k(y;\,u)\Big]\,du\,, \qquad i \in J\,,\ t,\, y \geq 0\,, \tag{6.42}$$

which has a unique solution such that $\Psi_i(y;\,\infty) = 0$, for $i \in J,\ y \in \mathbb{R}^+$, or equivalently, for $y \in \mathbb{R}^+$, $i \in J$,

$$\Psi_i(y;\,0) = \frac{\lambda_i}{c_i}\int_y^\infty \bar{F}_i(u)\,du - \frac{\alpha_i}{c_i}\int_0^\infty \Big[\Psi_i(y;\,u) - \sum_{k=1}^m p_{ik}\,\Psi_k(y;\,u)\Big]\,du\,, \tag{6.43}$$

where the transition probabilities p_{ij}, for $i, j = 1, 2, \ldots, m$, are given by (1.31).

We now apply Laplace transforms to solve the system of equations in (6.42). Let $\hat{\Psi}_i(s;\,y)$ be the Laplace transform of $\Psi_i(y;\,u)$ with respect to u, and respectively, $\hat{f}_i(s)$ be the Laplace transform of f_i, for $i \in J$, namely,

$$\hat{\Psi}_i(s;\,y) = \int_0^\infty e^{-s\,u}\,\Psi_i(y;\,u)du, \qquad s > a_{\Psi_i},\ y \geq 0\,,$$
$$\hat{f}_i(s) = \int_0^\infty e^{-s\,u}\,f_i(u)du, \qquad s > a_{f_i}\,.$$

As in Dickson and Hipp (2001), we define an operator T_r of a real-valued function f, with respect to a complex number r, to be

$$T_r f(y) = \int_y^\infty e^{-r(x-y)} f(x)dx\,, \qquad r \in \mathbb{C}\,,\ y \geq 0\,. \tag{6.44}$$

It is clear that the Laplace transform of f, $\hat{f}(s) = \int_0^\infty e^{-s\,u} f(u)du$, can be expressed as $T_s f(0)$, and for distinct complex numbers r_1 and r_2, we have the properties that

$$T_{r_1}T_{r_2}f(y) = T_{r_2}T_{r_1}f(y) = \frac{T_{r_1}f(y) - T_{r_2}f(y)}{r_2 - r_1}\,, \qquad r_1 \neq r_2 \in \mathbb{C}\,,\ y \geq 0\,. \tag{6.45}$$

By contrast, when $r_1 = r_2 = r \in \mathbb{C}$,

$$T_r T_r f(y) = \lim_{s\to r}\frac{T_r f(y) - T_s f(y)}{s - r} = \int_y^\infty e^{-r(x-y)}(x-y)f(x)dx\,, \qquad y \geq 0\,. \tag{6.46}$$

Further properties of the operator T_r can be found in Li and Garrido (2004).

Taking Laplace transforms on both sides of equation (6.42) yields,

$$c_i \, \hat{\Psi}_i(s;\, y) = c_i \, \frac{\Psi_i(y;\, 0)}{s} + \lambda_i \, \hat{\Psi}_i(s;\, y) \left[\frac{1 - \hat{f}_i(s)}{s} \right] - \lambda_i \int_0^\infty e^{-st} \left[\int_0^t \bar{F}_i(u+y)\, du \right] dt$$
$$+ \alpha_i \left[\frac{\hat{\Psi}_i(s;\, y)}{s} - \sum_{k=1}^{m} p_{ik} \frac{\hat{\Psi}_k(s;\, y)}{s} \right] du\,, \qquad i \in J\,. \tag{6.47}$$

By standard properties of the Laplace transform [see Oberhettinger and Badii (1973)], we have

$$\int_0^\infty e^{-st} \left[\int_0^t \bar{F}_i(u+y)\, du \right] dt$$
$$= \frac{1}{s} \int_0^\infty e^{-st} \bar{F}_i(t+y) dt = \frac{1}{s} \left[\bar{F}_i(t+y) \frac{e^{-st}}{(-s)} \Big|_0^\infty - \frac{1}{s} \int_0^\infty e^{-st} f_i(t+y) dt \right]$$
$$= \frac{1}{s} \left[\frac{\bar{F}_i(y)}{s} - \frac{1}{s} \int_y^\infty e^{-s(t-y)} f_i(t) dt \right] = \frac{T_0 f_i(y) - T_s f_i(y)}{s} = T_s T_0 f_i(y)\,.$$

Then equation (6.47) can be rewritten as

$$\left[s - \frac{\lambda_i + \alpha_i}{c_i} + \frac{\lambda_i}{c_i} \hat{f}_i(s) \right] \hat{\Psi}_i(s;\, y) + \frac{\alpha_i}{c_i} \sum_{k=1}^{m} p_{ik} \hat{\Psi}_k(s;\, y) = \Psi_i(y;\, 0) - \frac{\lambda_i}{c_i} T_s T_0 f_i(y)\,,$$

or in a matrix form

$$A(s)\, \hat{\mathbf{\Psi}}(s;\, y) = B(s;\, y), \qquad s > \max\{a_{\Psi_1}, \ldots, a_{\Psi_m}, a_{f_1}, \ldots, a_{f_m}\}, \tag{6.48}$$

where $A(s)$ has the same form as given in (6.5), $\mathbf{B}(s, y)$ is defined as

$$\mathbf{B}(s;\, y) = \left(\Psi_1(y;\, 0) - \frac{\lambda_1}{c_1} T_s T_0 f_1(y), \ldots, \Psi_m(y;\, 0) - \frac{\lambda_m}{c_m} T_s T_0 f_m(y) \right)',$$

and $\hat{\mathbf{\Psi}}(s;\, y) = \left(\hat{\Psi}_1(s;\, y), \ldots, \hat{\Psi}_m(s;\, y) \right)'$. The transition matrix P is given by (1.31), with $p_{ii} = 0$, for $i \in J$.

Then the vector of Laplace transforms $\hat{\mathbf{\Psi}}(s;\, y)$ can be solved as

$$\hat{\mathbf{\Psi}}(s;\, y) = [A(s)]^{-1} \mathbf{B}(s;\, y)\,, \qquad y \geq 0\,,$$

and equation (6.6), $|A(s)| = 0$, is again the characteristic equation of (6.48).

Now we consider that the external environmental process $\{I(t);\, t \geq 0\}$ is a two–state Markov chain, with transition matrix P given by (6.12) and the unique stationary probability distribution π given by (6.10). In this case, equation (6.14) is the characteristic equation of (6.48) and by Theorem 6.1, it has exactly one positive real root on the right half complex plane. We denote this unique positive real root by ρ.

The next section derives explicit expressions for the distribution of the severity of ruin when $m = 2$, given the initial surplus $u = 0$ and the initial environment state.

6.2.2 Formulas for $\Psi_1(y;\, 0)$ and $\Psi_2(y;\, 0)$

Now equation (6.48) has the form

$$\begin{pmatrix} s - \frac{\lambda_1+\alpha_1}{c_1} + \frac{\lambda_1}{c_1}\hat{f}_1(s) & \frac{\alpha_1}{c_1} \\ \frac{\alpha_2}{c_2} & s - \frac{\lambda_2+\alpha_2}{c_2} + \frac{\lambda_2}{c_2}\hat{f}_2(s) \end{pmatrix} \begin{pmatrix} \hat{\Psi}_1(s;\, y) \\ \hat{\Psi}_2(s;\, y) \end{pmatrix}$$
$$= \begin{pmatrix} \Psi_1(y;\, 0) - \frac{\lambda_1}{c_1} T_s T_0 f_1(y) \\ \Psi_2(y;\, 0) - \frac{\lambda_2}{c_2} T_s T_0 f_2(y) \end{pmatrix}, \qquad y \geq 0\,,$$

or for $s > a^* := \max\{a_{\Psi_1},\, a_{\Psi_2},\, a_{f_1},\, a_{f_1}\}$,

$$\begin{cases} \hat{\Psi}_1(s;\, y) = \dfrac{\left[\Psi_1(y;0) - \frac{\lambda_1}{c_1} T_s T_0 f_1(y)\right]\left[s - \frac{\lambda_2+\alpha_2}{c_2} + \frac{\lambda_2}{c_2}\hat{f}_2(s)\right] - \left[\Psi_2(y;0) - \frac{\lambda_2}{c_2} T_s T_0 f_2(y)\right]\frac{\alpha_1}{c_1}}{Q(s) - \frac{\alpha_1\alpha_2}{c_1 c_2}}\,, \\ \hat{\Psi}_2(s;\, y) = \dfrac{\left[\Psi_2(y;0) - \frac{\lambda_2}{c_2} T_s T_0 f_2(y)\right]\left[s - \frac{\lambda_1+\alpha_1}{c_1} + \frac{\lambda_1}{c_1}\hat{f}_1(s)\right] - \left[\Psi_1(y;0) - \frac{\lambda_1}{c_1} T_s T_0 f_1(y)\right]\frac{\alpha_2}{c_2}}{Q(s) - \frac{\alpha_1\alpha_2}{c_1 c_2}}\,, \end{cases} \quad y \geq 0\,, \tag{6.49}$$

where $Q(s)$ is given in (6.14).

Since $\hat{\Psi}_1(s;\, y)$ and $\hat{\Psi}_2(s;\, y)$ are finite for all s with $\Re(s) \geq 0$ and the fact that ρ satisfies (6.17) by Theorem 6.1, i.e., $Q(\rho) = \frac{\alpha_1\alpha_2}{c_1 c_2}$, we have that both the numerators in (6.49) are zero when $s = \rho$:

$$\left[\Psi_1(y;\, 0) - \frac{\lambda_1}{c_1} T_\rho T_0 f_1(y)\right]\left[\rho - \frac{\lambda_2 + \alpha_2}{c_2} + \frac{\lambda_2}{c_2}\hat{f}_2(\rho)\right]$$
$$= \left[\Psi_2(y;\, 0) - \frac{\lambda_2}{c_2} T_\rho T_0 f_2(y)\right]\frac{\alpha_1}{c_1}, \qquad y \geq 0\,, \tag{6.50}$$

or equivalently by equation (6.17),

$$\begin{aligned} &\left[\Psi_2(y;\,0) - \frac{\lambda_2}{c_2} T_\rho T_0 f_2(y)\right] \left[\rho - \frac{\lambda_1 + \alpha_1}{c_1} + \frac{\lambda_1}{c_1}\hat{f}_1(\rho)\right] \\ &\quad = \left[\Psi_1(y;\,0) - \frac{\lambda_1}{c_1} T_\rho T_0 f_1(y)\right] \frac{\alpha_2}{c_2}, \qquad y \geq 0. \end{aligned} \tag{6.51}$$

Moreover, by the following equations obtained from (6.43),

$$\Psi_1(y;\,0) = \frac{\lambda_1}{c_1} \int_y^\infty \bar{F}_1(u)\,du - \frac{\alpha_1}{c_1} \int_0^\infty \left[\Psi_1(y;\,u) - \Psi_2(y;\,u)\right] du\,,$$

$$\Psi_2(y;\,0) = \frac{\lambda_2}{c_2} \int_y^\infty \bar{F}_2(u)\,du - \frac{\alpha_2}{c_2} \int_0^\infty \left[\Psi_2(y;\,u) - \Psi_1(y;\,u)\right] du\,,$$

we get

$$\frac{\alpha_2}{c_2}\Psi_1(y;\,0) + \frac{\alpha_1}{c_1}\Psi_2(y;\,0) = \frac{\lambda_1\alpha_2}{c_1c_2} \int_y^\infty \bar{F}_1(u)du + \frac{\lambda_2\alpha_1}{c_1c_2} \int_y^\infty \bar{F}_2(u)du\,. \tag{6.52}$$

It is easy to verify that equation (6.52) is consistent with the fact that both numerators in (6.49) are zero when $s = 0$. Now solving equations (6.50) and (6.52) for $\Psi_1(y;\,0)$ and $\Psi_2(y;\,0)$ and using equation (6.51) yields the following theorem for the severity of ruin when the initial surplus is zero.

Theorem 6.4 For risk model given by (6.1), with $m = 2$ and $d > 0$, the probabilities of the severity of ruin when the initial reserve is zero are given by

$$\begin{cases} \Psi_1(y;\,0) = \frac{\lambda_1}{c_1} T_\rho T_0 f_1(y) + \dfrac{\frac{\lambda_2\alpha_1}{c_1c_2}\left[\int_y^\infty \bar{F}_2(u)du - T_\rho T_0 f_2(y)\right] + \frac{\lambda_1\alpha_2}{c_1c_2}\left[\int_y^\infty \bar{F}_1(u)du - T_\rho T_0 f_1(y)\right]}{\rho - \frac{\lambda_2}{c_2}\left[1 - \hat{f}_2(\rho)\right]}, \\ \Psi_2(y;\,0) = \frac{\lambda_2}{c_2} T_\rho T_0 f_2(y) + \dfrac{\frac{\lambda_2\alpha_1}{c_1c_2}\left[\int_y^\infty \bar{F}_2(u)du - T_\rho T_0 f_2(y)\right] + \frac{\lambda_1\alpha_2}{c_1c_2}\left[\int_y^\infty \bar{F}_1(u)du - T_\rho T_0 f_1(y)\right]}{\rho - \frac{\lambda_1}{c_1}\left[1 - \hat{f}_1(\rho)\right]}, \end{cases}$$

where $y \geq 0$, and ρ is the only positive real root to equation (6.14).

6.2.3 Explicit Results for $\Psi_1(y;\,u)$ and $\Psi_2(y;\,u)$

In this section we derive explicit expressions for the probabilities of the severity of ruin for a two–state model. By a discussion similar to that of Section 6.1.4 for the roots of equation (6.14), the Laplace transform of $\hat{\Psi}_1(s;\,y)$ and $\hat{\Psi}_2(s;\,y)$ can be inverted for the claim severity distributions in the rational family.

Let $n_1(s;\, y)$ and $n_2(s;\, y)$ be the numerators in (6.49), respectively. That is,

$$\begin{aligned} n_1(s;\, y) &= \Big[\Psi_1(y;\, 0) - \frac{\lambda_1}{c_1} T_s T_0 f_1(y)\Big]\Big[s - \frac{\lambda_2+\alpha_2}{c_2} + \frac{\lambda_2}{c_2}\hat{f}_2(s)\Big] \\ &\quad - \Big[\Psi_2(y;\, 0) - \frac{\lambda_2}{c_2} T_s T_0 f_2(y)\Big]\frac{\alpha_1}{c_1}\,, \end{aligned} \tag{6.53}$$

$$\begin{aligned} n_2(s;\, y) &= \Big[\Psi_2(y;\, 0) - \frac{\lambda_2}{c_2} T_s T_0 f_2(y)\Big]\Big[s - \frac{\lambda_1+\alpha_1}{c_1} + \frac{\lambda_1}{c_1}\hat{f}_1(s)\Big] \\ &\quad - \Big[\Psi_1(y;\, 0) - \frac{\lambda_1}{c_1} T_s T_0 f_1(y)\Big]\frac{\alpha_2}{c_2}\,. \end{aligned} \tag{6.54}$$

Since $s = 0$ is a root of $n_1(s;\, y)$ and $n_2(s;\, y)$, it follows that

$$\begin{aligned} n_1(s;\, y) &= n_1(s;\, y) - n_1(y;\, 0) \\ &= s\,\Psi_1(y;\, 0) + \Psi_1(y;\, 0)\frac{\lambda_2}{c_2}\big[T_s f_2(0) - T_0 f_2(0)\big] - \frac{\lambda_1}{c_1} s\, T_s T_0 f_1(y) \\ &\quad + \frac{\lambda_1(\lambda_2+\alpha_2)}{c_1\, c_2}\big[T_s T_0 f_1(y) - T_0 T_0 f_1(y)\big] \\ &\quad - \frac{\lambda_1\,\lambda_2}{c_1\, c_2}\big[T_s T_0 f_1(y)\, T_s f_2(0) - T_0 T_0 f_1(y)\, T_0 f_2(0)\big] \\ &\quad + \frac{\lambda_2\,\alpha_1}{c_1\, c_2}\big[T_s T_0 f_2(y) - T_0 T_0 f_2(y)\big]\,. \end{aligned} \tag{6.55}$$

By using of the operator defined in (6.44) and the property (6.45), we have

$$\begin{aligned} T_s f_2(0) - T_0 f_2(0) &= -s\, T_s T_0 f_2(0)\,, \\ T_s T_0 f_1(y) - T_0 T_0 f_1(y) &= -s\, T_s T_0 T_0 f_1(y)\,, \\ T_s T_0 f_1(y)\, T_s f_2(0) - T_0 T_0 f_1(y)\, T_0 f_2(0) &= -s\big[T_s T_0 f_1(y)\, T_s T_0 f_2(0) + T_s T_0 T_0 f_1(y)\big]\,, \\ T_s T_0 f_2(y) - T_0 T_0 f_2(y) &= -s\, T_s T_0 T_0 f_2(y)\,. \end{aligned}$$

After some manipulations, (6.55) can be rewritten as follows:

$$\begin{aligned} n_1(s;\, y) &= s\Big\{\Big[\Psi_1(y;\, 0) - \frac{\lambda_1}{c_1} T_s T_0 f_1(y)\Big]\Big[1 - \frac{\lambda_2}{c_2} T_s T_0 f_2(0)\Big] \\ &\qquad - \frac{\lambda_1\,\alpha_2}{c_1\, c_2} T_s T_0 T_0 f_1(y) - \frac{\lambda_2\,\alpha_1}{c_1\, c_2} T_s T_0 T_0 f_2(y)\Big\} \\ &= s\, n_1^*(s;\, y)\,, \end{aligned} \tag{6.56}$$

where by properties (6.45) and (6.46), $T_s T_0 T_0 f_i(y) = \frac{T_0 T_0 f_i(y) - T_s T_0 f_i(y)}{s}$, for $i = 1, 2$, and $T_0 T_0 f_i(y) = \int_y^\infty (x-y) f_i(x)dx = \int_y^\infty \bar{F}_i(x)dx$, while $n_1^*(s;\, y)$ has the following expression:

$$n_1^*(s;\,y) = \Big[\Psi_1(y;\,0) - \frac{\lambda_1}{c_1} T_s T_0 f_1(y)\Big]\Big[1 - \frac{\lambda_2}{c_2} T_s T_0 f_2(0)\Big] - \frac{\lambda_1\,\alpha_2}{c_1\,c_2} T_s T_0 T_0 f_1(y) - \frac{\lambda_2\,\alpha_1}{c_1\,c_2} T_s T_0 T_0 f_2(y)\,.$$

Furthermore, the fact that $s = \rho$ is also a root of $n_1(s;\,y)$ and therefore it must also be a root of $n_1^*(s;\,y)$, which implies that

$$\begin{aligned} n_1^*(s;\,y) &= n_1^*(s;\,y) - n_1^*(\rho;\,y) \\ &= -\Psi_1(y;\,0)\frac{\lambda_2}{c_2}\big[T_s T_0 f_2(0) - T_\rho T_0 f_2(0)\big] - \frac{\lambda_1}{c_1}\big[T_s T_0 f_1(y) - T_\rho T_0 f_1(y)\big] \\ &\quad + \frac{\lambda_1\,\lambda_2}{c_1\,c_2}\big[T_s T_0 f_1(y)\,T_s T_0 f_2(0) - T_\rho T_0 f_1(y)\,T_\rho T_0 f_2(0)\big] \\ &\quad - \frac{\lambda_1\,\alpha_2}{c_1\,c_2}\big[T_s T_0 T_0 f_1(y) - T_\rho T_0 T_0 f_1(y)\big] \\ &\quad - \frac{\lambda_2\,\alpha_1}{c_1\,c_2}\big[T_s T_0 T_0 f_2(y) - \rho\, T_\rho T_0 T_0 f_2(y)\big]\,. \end{aligned} \tag{6.57}$$

Now by the same technique used for deriving (6.56), $n_1^*(s;\,y)$ can be rewritten as

$$\begin{aligned} n_1^*(s;\,y) = (s-\rho)\Big\{ &\frac{\lambda_2}{c_2}\Big[\Psi_1(y;\,0) - \frac{\lambda_1}{c_1} T_\rho T_0 f_1(y)\Big] T_s T_\rho T_0 f_2(0) + \frac{\lambda_2\,\alpha_1}{c_1\,c_2} T_s T_\rho T_0 T_0 f_2(y) \\ &+ \frac{\lambda_1\,\alpha_2}{c_1\,c_2} T_s T_\rho T_0 T_0 f_1(y) + \frac{\lambda_1}{c_1}\Big[1 - \frac{\lambda_2}{c_2} T_s T_0 f_2(0)\Big] T_s T_\rho T_0 f_1(y)\Big\}\,, \end{aligned}$$

and hence $n_1(s;\,y) = s\,(s-\rho)\,m_1(s;\,y)$, where

$$\begin{aligned} m_1(s;\,y) = &\frac{\lambda_2}{c_2}\Big[\Psi_1(y;\,0) - \frac{\lambda_1}{c_1} T_\rho T_0 f_1(y)\Big] T_s T_\rho T_0 f_2(0) + \frac{\lambda_2 \alpha_1}{c_1\,c_2} T_s T_\rho T_0 T_0 f_2(y) \\ &+ \frac{\lambda_1 \alpha_2}{c_1\,c_2} T_s T_\rho T_0 T_0 f_1(y) + \frac{\lambda_1}{c_1}\Big[1 - \frac{\lambda_2}{c_2} T_s T_0 f_2(0)\Big] T_s T_\rho T_0 f_1(y)\,. \end{aligned} \tag{6.58}$$

Similarly, we can rewrite (6.54) as $n_2(s;\,y) = s\,(s-\rho)\,m_2(s;\,y)$, where

$$\begin{aligned} m_2(s;\,y) = &\frac{\lambda_1}{c_1}\Big[\Psi_2(y;\,0) - \frac{\lambda_2}{c_2} T_\rho T_0 f_2(y)\Big] T_s T_\rho T_0 f_1(0) + \frac{\lambda_1 \alpha_2}{c_1\,c_2} T_s T_\rho T_0 T_0 f_1(y) \\ &+ \frac{\lambda_2 \alpha_1}{c_1\,c_2} T_s T_\rho T_0 T_0 f_2(y) + \frac{\lambda_2}{c_2}\Big[1 - \frac{\lambda_1}{c_1} T_s T_0 f_1(0)\Big] T_s T_\rho T_0 f_2(y)\,. \end{aligned} \tag{6.59}$$

In (6.58) and (6.59), the composite operators are given by

$$T_s T_\rho T_0 f_i(y) = \frac{T_\rho T_0 f_i(y) - T_s T_0 f_i(y)}{s-\rho}, \qquad i = 1, 2\,,$$

$$T_s T_\rho T_0 T_0 f_i(y) = \frac{T_\rho T_0 T_0 f_i(y) - T_s T_0 T_0 f_i(y)}{s-\rho}, \qquad i = 1, 2\,,$$

while $\Psi_1(y;\,0)$, $\Psi_2(y;\,0)$ are given by (6.4) in Theorem (6.4).

Finally (6.49) is written as the following form:

$$\begin{cases} \hat{\Psi}_1(s;\,y) &= \frac{s\,(s-\rho)\,m_1(s;\,y)}{Q(s)-\frac{\alpha_1\,\alpha_2}{c_1\,c_2}}\,, \\ \hat{\Psi}_2(s;\,y) &= \frac{s\,(s-\rho)\,m_2(s;\,y)}{Q(s)-\frac{\alpha_1\,\alpha_2}{c_1\,c_2}}\,, \end{cases} \qquad s > a^*,\ y \geq 0\,, \tag{6.60}$$

where $m_1(s;\,y)$ and $m_2(s;\,y)$ are given by (6.58) and (6.59), respectively.

As in the ruin problem case, we now consider claim severity distributions f_1 and f_2 that are in the rational family and assume that their Laplace transforms are of the form in (6.24), where $p_k(s)$ and $q_l(s)$ have roots with negative real parts only.

Multiplying by $p_k(s)q_l(s)$, both numerators and denominators of equation (6.60) yields the following expressions which can be easily inverted:

$$\begin{cases} \hat{\Psi}_1(s;\,y) = \frac{s(s-\rho)m_1(s;\,y)p_k(s)q_l(s)}{\left[Q(s)-\frac{\alpha_1\alpha_2}{c_1c_2}\right]p_k(s)q_l(s)}, \\ \hat{\Psi}_2(s;\,y) = \frac{s(s-\rho)m_2(s;\,y)p_k(s)q_l(s)}{\left[Q(s)-\frac{\alpha_1\alpha_2}{c_1c_2}\right]p_k(s)q_l(s)}, \end{cases} \tag{6.61}$$

where

$$\begin{aligned} m_1(s;\,y)p_k(s)q_l(s) = {} & \frac{\lambda_2}{c_2}\left[\Psi_1(y;\,0) - \frac{\lambda_1}{c_1}T_\rho T_0 f_1(y)\right]\left[T_s T_\rho T_0 f_2(0) q_l(s)\right] p_k(s) \\ & + \left[\frac{\lambda_2\alpha_1}{c_1c_2}T_sT_\rho T_0T_0f_2(y) + \frac{\lambda_1\alpha_2}{c_1c_2}T_sT_\rho T_0T_0f_1(y)\right] p_k(s)q_l(s) \\ & + \frac{\lambda_1}{c_1}\left[q_l(s) - \frac{\lambda_2}{c_2}\left[T_sT_0f_2(0)\right]q_l(s)\right] p_k(s)T_sT_\rho T_0 f_1(y)\,, \\ m_2(s;\,y)p_k(s)q_l(s) = {} & \frac{\lambda_1}{c_1}\left[\Psi_2(y;\,0) - \frac{\lambda_2}{c_2}T_\rho T_0 f_2(y)\right]\left[T_s T_\rho T_0 f_1(0) p_k(s)\right] q_l(s) \\ & + \left[\frac{\lambda_2\alpha_1}{c_1c_2}T_sT_\rho T_0T_0f_2(y) + \frac{\lambda_1\alpha_2}{c_1c_2}T_sT_\rho T_0T_0f_1(y)\right] p_k(s)q_l(s) \\ & + \frac{\lambda_2}{c_2}\left[p_k(s) - \frac{\lambda_1}{c_1}\left[T_sT_0f_1(0)\right]p_k(s)\right] q_l(s)T_sT_\rho T_0 f_2(y)\,. \end{aligned}$$

Further denote by $p_k[s,\,\rho] := \frac{p_k(s)-p_k(\rho)}{s-\rho}$ the first order divided difference of $p_k(s)$, and by $p_k[s,\,\rho,\,0] := \frac{p_k[s,0]-p_k[\rho,0]}{s-\rho}$ the second order divided difference of $p_k(s)$. Note here polynomials $p_k[s,\,\rho]$ and $p_k[s,\,\rho,\,0]$ are of degrees $k-1$ and $k-2$, respectively, with leading coefficient 1. Then since $p_k(0) = p_{k-1}(0)$, we get that

$$\begin{aligned}[T_sT_0f_1(0)]p_k(s) &= \frac{T_0f_1(0) - T_sf_1(0)}{s}p_k(s) = \frac{1 - \frac{p_{k-1}(s)}{p_k(s)}}{s}p_k(s) \\ &= \frac{p_k(s) - p_{k-1}(s)}{s} = \frac{[p_k(s) - p_k(0)] - [p_{k-1}(s) - p_{k-1}(0)]}{s} \\ &= p_k[s,\, 0] - p_{k-1}[s,\, 0]\,,\end{aligned}$$

and similarly $\left[T_sT_0f_2(0)\right]q_l(s) = q_l[s,\, 0] - q_{l-1}[s,\, 0]$, and

$$\left[T_sT_\rho T_0f_1(0)\right]p_k(s) = \left[T_\rho T_0f_1(0)\right]p_k[s,\, \rho] - p_k[s,\, \rho,\, 0] + p_{k-1}[s,\, \rho,\, 0]\,,$$

$$\left[T_sT_\rho T_0f_2(0)\right]q_l(s) = \left[T_\rho T_0f_1(0)\right]q_l[s,\, \rho] - q_l[s,\, \rho,\, 0] + q_{l-1}[s,\, \rho,\, 0]\,.$$

It is clear that $[T_sT_0f_1(0)]p_k(s)$ and $T_sT_0f_2(0)q_l(s)$ are polynomials of degree $k-1$ and $l-1$, while $T_sT_\rho T_0f_1(0)p_k(s)$ and $T_sT_\rho T_0f_2(0)q_l(s)$ are polynomials of degree $k-2$ and $l-2$, respectively.

For simplicity, let $D_{k+l+2}(s)$ be the common denominator of (6.61), which is clearly a polynomial of degree $k+l+2$ with the leading coefficient 1. Then equation (6.27) has $k+l+2$ roots on the complex plane and all the complex roots are in conjugate pairs. By the same arguments given in Section 6.1,

$$D_{k+l+2}(s) = s\,(s-\rho)\prod_{i=1}^{k+l}(s+R_i)\,,$$

where all R_i's have a positive real part, for $i = 1, 2, \ldots, k+l$. Then (6.61) can be simplified to

$$\begin{aligned}\hat{\Psi}_1(s;\, y) = \frac{\lambda_2}{c_2}\left[\Psi_1(y;\, 0) - \frac{\lambda_1}{c_1}T_\rho T_0f_1(y)\right]\frac{g_1(s)}{\prod_{i=1}^{k+l}(s+R_i)} + \frac{\lambda_1}{c_1}\left[T_sT_\rho T_0f_1(y)\right]\frac{h_1(s)}{\prod_{i=1}^{k+l}(s+R_i)} \\ + \left[\frac{\lambda_2\alpha_1}{c_1c_2}T_sT_\rho T_0T_0f_2(y) + \frac{\lambda_1\alpha_2}{c_1c_2}T_sT_\rho T_0T_0f_1(y)\right]\frac{p_k(s)q_l(s)}{\prod_{i=1}^{k+l}(s+R_i)}\,, \qquad (6.62)\end{aligned}$$

$$\begin{aligned}\hat{\Psi}_2(s;\, y) = \frac{\lambda_1}{c_1}\left[\Psi_2(y;\, 0) - \frac{\lambda_2}{c_2}T_\rho T_0f_2(y)\right]\frac{g_2(s)}{\prod_{i=1}^{k+l}(s+R_i)} + \frac{\lambda_2}{c_2}\left[T_sT_\rho T_0f_2(y)\right]\frac{h_2(s)}{\prod_{i=1}^{k+l}(s+R_i)} \\ + \left[\frac{\lambda_1\alpha_2}{c_1c_2}T_sT_\rho T_0T_0f_1(y) + \frac{\lambda_2\alpha_1}{c_1c_2}T_sT_\rho T_0T_0f_2(y)\right]\frac{p_k(s)q_l(s)}{\prod_{i=1}^{k+l}(s+R_i)}\,, \qquad (6.63)\end{aligned}$$

where we define

$$g_1(s) = \Big\{ \big[T_sT_\rho T_0 f_2(0)\big] q_l(s) \Big\} p_k(s)\,, \quad h_1(s) = \Big\{ q_l(s) - \frac{\lambda_2}{c_2} \big[T_sT_0 f_2(0)\big] q_l(s) \Big\} p_k(s)\,,$$
$$g_2(s) = \Big\{ \big[T_sT_\rho T_0 f_1(0)\big] p_k(s) \Big\} q_l(s)\,, \quad h_2(s) = \Big\{ p_k(s) - \frac{\lambda_1}{c_1} \big[T_sT_0 f_1(0)\big] p_k(s) \Big\} q_l(s)\,,$$

in which $g_i(s)$ and $h_i(s)$ are polynomials of degree $k+l-1$ and $k+l$, respectively, with leading coefficient 1.

If R_i, $i = 1, 2, \ldots, k+l$, are distinct real numbers, then by decomposing the rational expressions (6.62) and (6.63) in partial fractions, we get

$$\begin{aligned}\hat{\Psi}_1(s;\,y) = \frac{\lambda_2}{c_2}\Big[\Psi_1(y;\,0) - \frac{\lambda_1}{c_1} T_\rho T_0 f_1(y)\Big] \sum_{i=1}^{k+l} \frac{g_{1i}}{s+R_i} + \frac{\lambda_1}{c_1} T_sT_\rho T_0 f_1(y)\Big[1 + \sum_{i=1}^{k+l} \frac{h_{1i}}{s+R_i}\Big] \\ + \left[\frac{\lambda_1\,\alpha_2}{c_1\,c_2} T_sT_\rho T_0T_0 f_1(y) + \frac{\lambda_2\,\alpha_1}{c_1\,c_2} T_sT_\rho T_0T_0 f_2(y)\right]\left[1 + \sum_{i=1}^{k+l} \frac{g_i}{s+R_i}\right],\end{aligned} \tag{6.64}$$

$$\begin{aligned}\hat{\Psi}_2(s;\,y) = \frac{\lambda_1}{c_1}\Big[\Psi_2(y;\,0) - \frac{\lambda_2}{c_2} T_\rho T_0 f_2(y)\Big] \sum_{i=1}^{k+l} \frac{g_{2i}}{s+R_i} + \frac{\lambda_2}{c_2} T_sT_\rho T_0 f_2(y)\Big[1 + \sum_{i=1}^{k+l} \frac{h_{2i}}{s+R_i}\Big] \\ + \left[\frac{\lambda_2\,\alpha_1}{c_1\,c_2} T_sT_\rho T_0T_0 f_2(y) + \frac{\lambda_1\,\alpha_2}{c_1\,c_2} T_sT_\rho T_0T_0 f_1(y)\right]\left[1 + \sum_{i=1}^{k+l} \frac{g_i}{s+R_i}\right],\end{aligned} \tag{6.65}$$

where

$$g_{ij} = \frac{g_i(-R_j)}{\prod\limits_{\nu=1,\nu\neq j}^{k+l} (R_\nu - R_j)}\,, \quad h_{ij} = \frac{h_i(-R_j)}{\prod\limits_{\nu=1,\nu\neq j}^{k+l} (R_\nu - R_j)}\,, \quad g_j = \frac{p_k(-R_j)q_l(-R_j)}{\prod\limits_{\nu=1,\nu\neq j}^{k+l} (R_\nu - R_j)}\,, \tag{6.66}$$

for $j = 1, 2, \ldots, k+l$, and $i = 1, 2$. By definition of the operator T_r in (6.44), we have that

$$T_sT_\rho T_0 f_i(y) = \int_y^\infty e^{-s(x-y)} T_\rho T_0 f_i(x)dx = \int_0^\infty e^{-s\,t} T_\rho T_0 f_i(y+t)dt\,,$$

and hence the Laplace inversion of $T_sT_\rho T_0 f_i(y)$ is derived as

$$\mathcal{L}^{-1}[T_sT_\rho T_0 f_i(y)] = T_\rho T_0 f_i(y+u) = \frac{1}{\rho}\int_u^\infty \big[1 - e^{-\rho(t-u)}\big] f_i(t+y)dt\,, \qquad u,\, y \geq 0\,.$$

Correspondingly $\mathcal{L}^{-1}[T_s T_\rho T_0 T_0 f_i(y)] = T_\rho T_0 T_0 f_i(y+u)$, for $u,\, y \geq 0$ and $i = 1, 2$. Thus inverting the Laplace transforms in (6.62) and (6.63) yields the following theorem for the distribution of the severity of ruin.

Theorem 6.5 For the risk model given by (1.33), with $m = 2$ and $d > 0$, if the claim severity distributions are in the rational family (6.23), the distributions of the severity of ruin are given, for $u, y \geq 0$, by

$$\begin{aligned}\Psi_1(y;\, u) = & \left[\Psi_1(y;\, 0) - \frac{\lambda_1}{c_1} T_\rho T_0 f_1(y)\right] \sum_{j=1}^{k+l} g_{1j}\, e^{-R_j\, u} + \frac{\lambda_1}{c_1} T_\rho T_0 f_1(u+y) \\ & + \frac{\lambda_1}{c_1} \sum_{j=1}^{k+l} h_{1j}\, e^{-R_j\, u} * T_\rho T_0 f_1(u+y) \\ & + \frac{\lambda_1\, \alpha_2}{c_1\, c_2} \left[T_\rho T_0 T_0 f_1(y+u) + \sum_{j=1}^{k+l} g_j\, e^{-R_j\, u} * T_\rho T_0 T_0 f_1(u+y)\right] \\ & + \frac{\lambda_2\, \alpha_1}{c_1\, c_2} \left[T_\rho T_0 T_0 f_2(y+u) + \sum_{j=1}^{k+l} g_j\, e^{-R_j\, u} * T_\rho T_0 T_0 f_2(u+y)\right], \end{aligned} \tag{6.67}$$

$$\begin{aligned}\Psi_2(y;\, u) = & \left[\Psi_2(y;\, 0) - \frac{\lambda_2}{c_2} T_\rho T_0 f_2(y)\right] \sum_{j=1}^{k+l} g_{2j}\, e^{-R_j\, u} + \frac{\lambda_2}{c_2} T_\rho T_0 f_2(u+y) \\ & + \frac{\lambda_2}{c_2} \sum_{j=1}^{k+l} h_{2j}\, e^{-R_j\, u} * T_\rho T_0 f_2(u+y) \\ & + \frac{\lambda_2\, \alpha_1}{c_1\, c_2} \left[T_\rho T_0 T_0 f_2(y+u) + \sum_{j=1}^{k+l} g_j\, e^{-R_j\, u} * T_\rho T_0 T_0 f_2(u+y)\right] \\ & + \frac{\lambda_1\, \alpha_2}{c_1\, c_2} \left[T_\rho T_0 T_0 f_1(y+u) + \sum_{j=1}^{k+l} g_j\, e^{-R_j\, u} * T_\rho T_0 T_0 f_1(u+y)\right], \end{aligned} \tag{6.68}$$

where $-R_1, -R_2, \ldots, -R_{k+l}$ are distinct real roots of equation (6.27), ρ is the unique positive real root of the characteristic equation (6.14), and $\Psi_1(y;\, 0)$ and $\Psi_2(y;\, 0)$ are given in Theorem 6.4, while the coefficients g_{ij}, h_{ij}, g_j are given by (6.66), for $j = 1, 2, \ldots, k+l$, and $i = 1, 2$.

In formulas (6.67) and (6.68), $e^{-R_j u} * T_\rho T_0 f_i(u+y)$ and $e^{-R_j\, u} * T_\rho T_0 T_0 f_i(u+y)$, for $i = 1, 2$ and $j = 1, 2, \ldots, k+l$, are convolutions that can be calculated by the following property of T_r [see Li (2004)]:

$$T_r[f * g(x)] = f * [T_r g(x)] + [T_r g(0)][T_r f(x)]\,, \qquad x \geq 0\,.$$

We remark that if there are pairs of complex roots to equation (6.27), the distributions of the severity of ruin involve trigonometric functions, as illustrated by the examples that follow.

6.2.4 Ruin severity examples

In this section, we illustrate the derivation of some distributions of the severity of ruin $\Psi_1(u;\, y)$ and $\Psi_2(u;\, y)$, given by (6.67) and (6.68). In the first example we assume that both claim severities are exponentially distributed with mean $\mu_1$and μ_2, respectively, as in Example 6.1 for the probability of ruin.

Example 6.3 Let $f_i(x) = \frac{1}{\mu_i} e^{-\frac{x}{\mu_i}}$, for $x \geq 0$ and $i = 1, 2$. Then their Laplace transforms take the form $\hat{f}_i(s) = \frac{\frac{1}{\mu_i}}{s+\frac{1}{\mu_i}}$, for $i = 1, 2$, where $p_0(s) = \frac{1}{\mu_1}$, $p_1(s) = s + \frac{1}{\mu_1}$, $q_0(s) = \frac{1}{\mu_2}$ and $q_1(s) = s + \frac{1}{\mu_2}$.

Now since in this case, $T_\rho T_0 f_i(y)$ can be expressed as

$$T_\rho T_0 f_i(y) = \frac{1}{\rho} \int_y^\infty \left[1 - e^{-\rho(x-y)}\right] \frac{1}{\mu_i} e^{-\frac{x}{\mu_i}} dx = \frac{e^{-\frac{y}{\mu_i}}}{\rho + \frac{1}{\mu_i}}\,, \qquad y \geq 0\,, i = 1, 2\,,$$

the explicit expressions $\Psi_1(y;\, 0)$ and $\Psi_2(y;\, 0)$, given in Theorem 6.4, are obtained as

$$\Psi_1(y;\, 0) = \frac{\lambda_1}{c_1(\rho + \frac{1}{\mu_1})} e^{-\frac{y}{\mu_1}} + \frac{\frac{\lambda_2 \alpha_1 \mu_2}{c_1 c_2 (\rho + \frac{1}{\mu_2})} e^{-\frac{y}{\mu_2}} + \frac{\lambda_1 \alpha_2 \mu_1}{c_1 c_2 (\rho + \frac{1}{\mu_1})} e^{-\frac{y}{\mu_1}}}{1 - \frac{\lambda_2}{c_2(\rho + \frac{1}{\mu_2})}}\,, \qquad y \geq 0\,, \tag{6.69}$$

$$\Psi_2(y;\, 0) = \frac{\lambda_2}{c_2(\rho + \frac{1}{\mu_2})} e^{-\frac{y}{\mu_2}} + \frac{\frac{\lambda_1 \alpha_2 \mu_1}{c_1 c_2 (\rho + \frac{1}{\mu_1})} e^{-\frac{y}{\mu_1}} + \frac{\lambda_2 \alpha_1 \mu_2}{c_1 c_2 (\rho + \frac{1}{\mu_2})} e^{-\frac{y}{\mu_2}}}{1 - \frac{\lambda_1}{c_1(\rho + \frac{1}{\mu_1})}}\,, \qquad y \geq 0\,, \tag{6.70}$$

where ρ is the positive real root of equation (6.35). Equation $D_4(s) = Q(s)\, p_1(s)\, q_1(s) =$

0 is given by (6.36), where $Q(s)$ is defined in (6.35). That is,

$$
\begin{aligned}
D_4(s) = s^4 &+ \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} - \frac{\lambda_1+\alpha_1}{c_1} - \frac{\lambda_2+\alpha_2}{c_2}\right) s^3 \\
+ & \left[\left(\frac{1}{\mu_1} - \frac{\lambda_1+\alpha_1}{c_1}\right)\left(\frac{1}{\mu_2} - \frac{\lambda_2+\alpha_2}{c_2}\right) - \frac{\alpha_1}{c_1\mu_1} - \frac{\alpha_2}{c_2\mu_2} - \frac{\alpha_1\alpha_2}{c_1c_2}\right] s^2 \\
- & \left[\frac{\alpha_1}{c_1\mu_1}\left(\frac{1}{\mu_2} - \frac{\lambda_2}{c_2}\right) + \frac{\alpha_2}{c_2\mu_2}\left(\frac{1}{\mu_1} - \frac{\lambda_1}{c_1}\right)\right] s = 0 \,, \qquad (6.71)
\end{aligned}
$$

which is equation (6.16) in Reinhard (1984) or equation (12) in Snoussi (2002). It has exactly 4 roots, $s=0$, $s=\rho$, $s=-R_1$ and $s=-R_2$. Now here $T_sT_0f_i(0) = \frac{1}{s+\frac{1}{\mu_i}}$, $T_sT_\rho T_0 f_i(0) = \frac{1}{(s+\frac{1}{\mu_i})(\rho+\frac{1}{\mu_i})}$, for $i=1,2$, and therefore,

$$
g_1(s) = \frac{s+\frac{1}{\mu_1}}{\rho+\frac{1}{\mu_2}}\,, \qquad h_1(s) = \big(s+\frac{1}{\mu_2}-\frac{\lambda_2}{c_2}\big)\big(s+\frac{1}{\mu_1}\big)\,,
$$

$$
g_2(s) = \frac{s+\frac{1}{\mu_2}}{\rho+\frac{1}{\mu_1}}\,, \qquad h_2(s) = \big(s+\frac{1}{\mu_1}-\frac{\lambda_1}{c_1}\big)\big(s+\frac{1}{\mu_2}\big)\,.
$$

Furthermore, the coefficients in (6.66) are here:

$$
g_{1j} = \frac{-R_j+\frac{1}{\mu_1}}{(-1)^j(R_1-R_2)(\rho+\frac{1}{\mu_2})}\,, \qquad g_{2j} = \frac{-R_j+\frac{1}{\mu_2}}{(-1)^j(R_1-R_2)(\rho+\frac{1}{\mu_1})}\,,
$$

$$
h_{1j} = \frac{(-R_j+\frac{1}{\mu_2}-\frac{\lambda_2}{c_2})(-R_j+\frac{1}{\mu_1})}{(-1)^j(R_1-R_2)}\,, \qquad h_{2j} = \frac{(-R_j+\frac{1}{\mu_1}-\frac{\lambda_1}{c_1})(-R_j+\frac{1}{\mu_2})}{(-1)^j(R_1-R_2)}\,,
$$

$$
g_j = \frac{(-R_j+\frac{1}{\mu_2})(-R_j+\frac{1}{\mu_1})}{(-1)^j(R_1-R_2)}\,, \qquad j=1,2.
$$

To get the distributions of the severity of ruin $\Psi_1(u;\, y)$ and $\Psi_2(u;\, y)$ given by (6.67) and (6.68), we still need expressions for the following terms:

$$
T_\rho T_0 f_i(u+y) = \frac{1}{\rho+\frac{1}{\mu_i}} e^{-\frac{y+u}{\mu_i}}\,, \qquad T_\rho T_0 T_0 f_i(u+y) = \frac{\mu_i}{\rho+\frac{1}{\mu_i}} e^{-\frac{y+u}{\mu_i}}\,,
$$

$$
e^{-R_j u} * T_\rho T_0 f_i(u+y) = \frac{e^{-\frac{y}{\mu_i}}}{(\rho+\frac{1}{\mu_i})(R_j-\frac{1}{\mu_i})}\big[e^{-\frac{u}{\mu_i}} - e^{-R_j u}\big]\,,
$$

$$
e^{-R_j u} * T_\rho T_0 T_0 f_i(u+y) = \frac{\mu_i\, e^{-\frac{y}{\mu_i}}}{(\rho+\frac{1}{\mu_i})(R_j-\frac{1}{\mu_i})}\big[e^{-\frac{u}{\mu_i}} - e^{-R_j u}\big]\,, \qquad j=1,2\,.
$$

These yield the following distributions of the severity of ruin:

$$\Psi_1(y;\,u) = \left[\Psi_1(y;\,0) - \frac{\lambda_1}{c_1(\rho+\frac{1}{\mu_1})}e^{-\frac{y}{\mu_1}}\right]\sum_{j=1}^{2} g_{1j}\,e^{-R_j\,u} + \frac{\lambda_1}{c_1(\rho+\frac{1}{\mu_1})}e^{-\frac{y+u}{\mu_1}}$$
$$+ \frac{\lambda_1\,e^{-\frac{y}{\mu_1}}}{c_1(\rho+\frac{1}{\mu_1})}\sum_{j=1}^{2}\left[h_{1j} + \frac{\alpha_2\,\mu_1}{c_2}\,g_j\right]\frac{e^{-\frac{u}{\mu_1}} - e^{-R_j\,u}}{(R_j - \frac{1}{\mu_1})} + \frac{\lambda_1\,\alpha_2\,\mu_1}{c_1\,c_2(\rho+\frac{1}{\mu_1})}e^{-\frac{y+u}{\mu_1}}$$
$$+ \frac{\lambda_2\,\alpha_1\,\mu_2}{c_1\,c_2(\rho+\frac{1}{\mu_2})}e^{-\frac{y+u}{\mu_2}} + \frac{\mu_2\,e^{-\frac{y}{\mu_2}}}{(\rho+\frac{1}{\mu_2})}\sum_{j=1}^{2}\frac{g_j\left[e^{-\frac{u}{\mu_2}} - e^{-R_j\,u}\right]}{(R_j - \frac{1}{\mu_2})}, \tag{6.72}$$

$$\Psi_2(y;\,u) = \left[\Psi_2(y;\,0) - \frac{\lambda_2}{c_2(\rho+\frac{1}{\mu_2})}e^{-\frac{y}{\mu_2}}\right]\sum_{j=1}^{2} g_{2j}\,e^{-R_j\,u} + \frac{\lambda_1}{c_2(\rho+\frac{1}{\mu_2})}e^{-\frac{y+u}{\mu_2}}$$
$$+ \frac{\lambda_2\,e^{-\frac{y}{\mu_2}}}{c_2(\rho+\frac{1}{\mu_2})}\sum_{j=1}^{2}\left[h_{2j} + \frac{\alpha_1\,\mu_2}{c_1}\,g_j\right]\frac{e^{-\frac{u}{\mu_2}} - e^{-R_j\,u}}{(R_j - \frac{1}{\mu_2})} + \frac{\lambda_2\,\alpha_1\,\mu_2}{c_1\,c_2(\rho+\frac{1}{\mu_2})}e^{-\frac{y+u}{\mu_2}}$$
$$+ \frac{\lambda_1\,\alpha_2\,\mu_1}{c_1\,c_2(\rho+\frac{1}{\mu_1})}e^{-\frac{y+u}{\mu_1}} + \frac{\mu_1\,e^{-\frac{y}{\mu_1}}}{(\rho+\frac{1}{\mu_1})}\sum_{j=1}^{2}\frac{g_j\left[e^{-\frac{u}{\mu_1}} - e^{-R_j\,u}\right]}{(R_j - \frac{1}{\mu_1})}, \tag{6.73}$$

where $\Psi_1(y;\,0)$ and $\Psi_1(y;\,0)$ are given by (6.69) and (6.70).

Remark 6.4 If we set $\lambda_1 = \lambda_2 = \lambda$, $c_1 = c_2 = c$ and $\mu_1 = \mu_2 = \mu$, and let $\gamma = \frac{1}{\mu} - \frac{\lambda}{c}$, then like in Example 6.1, $-R_1 = -\gamma$ is one root of equation (6.71), while ρ and $-R_2$ are the other two roots of:

$$s^2 + \left(\gamma - \frac{\alpha_1+\alpha_2}{c}\right)s - \frac{\alpha_1+\alpha_2}{c\mu} = 0\,.$$

Then (6.72) and (6.73) reduce to

$$\Psi_1(y;\,u) = \Psi_2(y;\,u) = \frac{\lambda\mu}{c}e^{-\gamma u - \frac{y}{\mu}}, \qquad u,\,y \geq 0,$$

where the identical distributions of the severity of ruin, $\Psi_1(u;\,y)$ and $\Psi_2(u;\,y)$ are, as expected, the same distributions obtained with the classical Poisson risk model with exponentially distributed claim severities [see Snoussi (2002)].

Example 6.4 Now assume that $\lambda_1 = \lambda_2 = 1$, $c_1 = c_2 = 2$ and that the two identical severity distributions are a mixture of two Erlang(2) distributions, i.e., the corresponding density is given by

$$f_1(x) = f_2(x) = A_1\,\frac{x}{\beta_1^2}e^{-\frac{x}{\beta_1}} + A_2\,\frac{x}{\beta_2^2}e^{-\frac{x}{\beta_2}}\,, \qquad x \geq 0\,, \tag{6.74}$$

where we set $A_1 = A_2 = \frac{1}{2}$, $\beta_1 = \frac{1}{3-\sqrt{3}}$ and $\beta_2 = \frac{1}{3+\sqrt{3}}$, implying that $\mu_1 = \mu_2 = A_1(2\beta_1) + A_2(2\beta_2) = 1$. These choices are as in Example 4 of Gerber et al. (1987) and reduce the Markov–modulated model to the classical Poisson one. The Laplace transform of (6.74) is given by

$$\hat{f}_1(s) = \hat{f}_2(s) = \frac{p_1(s)}{p_2(s)} = A_1 \frac{\frac{1}{\beta_1^2}}{(s+\frac{1}{\beta_1})^2} + A_2 \frac{\frac{1}{\beta_2^2}}{(s+\frac{1}{\beta_2})^2}\,.$$

Now in this case, $T_sT_0f_1(y) = T_sT_0f_2(y)$ is of the form:

$$\begin{aligned} T_sT_0f_1(y) = {} & \big(\frac{1}{2}\big)\frac{(\frac{y}{\beta_1}+1)e^{-\frac{y}{\beta_1}}}{s} + \big(\frac{1}{2}\big)\frac{(\frac{y}{\beta_2}+1)e^{-\frac{y}{\beta_2}}}{s} \\ & - \big(\frac{1}{2}\big)\frac{\big[y(s+\frac{1}{\beta_1})+1\big]e^{-\frac{y}{\beta_1}}}{\beta_1^2\, s\,(s+\frac{1}{\beta_1})^2} - \big(\frac{1}{2}\big)\frac{\big[y(s+\frac{1}{\beta_2})+1\big]e^{-\frac{y}{\beta_2}}}{\beta_2^2\, s\,(s+\frac{1}{\beta_2})^2}\,, \qquad y \geq 0\,, \end{aligned}$$

and equation (6.14) or equivalently equation (6.27) has 6 roots, which are, 0, 0.708, −0.506, −1.765, −3.544, −5.685. Here $\rho = 0.708$ is the only positive real root. The identical explicit expressions $\Psi_1(y;\,0) = \Psi_2(y;\,0) = \Psi(y;\,0)$, given in Theorem 6.4, reduce to:

$$\Psi(y;\,0) = (0.25\,y + 0.394)\,e^{-1.268\,y} + (0.25\,y + 0.106)\,e^{-4.732\,y}\,, \qquad y \geq 0\,, \qquad (6.75)$$

and the identical explicit expressions for the distribution of the severity of ruin, $\Psi_1(y;\,u) = \Psi_2(y;\,u) = \Psi(y;\,u)$, given by (6.67), yield:

$$\begin{aligned} \Psi(y;\,u) = {} & (.031 + .023\,y)\,e^{-1.268\,y-5.685\,u} + (-.079 - .069\,y)\,e^{-1.268\,y-3.544\,u} \\ & + (-.054 + .125\,y)\,e^{-1.268\,y-1.765\,u} + (.496 + .172\,y)\,e^{-1.268\,y-.506\,u} \\ & + (-.067 + .107\,y)\,e^{-4.732\,y-5.685\,u} + (.168 + .133\,y)\,e^{-4.732\,y-3.544\,u} \\ & + (-.016 - .021\,y)\,e^{-4.732\,y-1.765\,u} + (.02 + .031\,y)\,e^{-4.732\,y-.506\,u}\,, \\ & \qquad u,\, y \geq 0\,, \qquad (6.76) \end{aligned}$$

We obtain also the corresponding probability density $g(y;\,u)$, given by (6.41), which is consistent with that derived by Gerber et al. (1987) for the classical model:

$$
\begin{aligned}
g(y;\, u) = {} & (.016 + .029\, y)\, e^{-1.268\, y - 5.685\, u} + (-.031 - .088\, y)\, e^{-1.268\, y - 3.544\, u} \\
& + (-.193 + .158\, y)\, e^{-1.268\, y - 1.765\, u} + (.458 + .218\, y)\, e^{-1.268\, y - .506\, u} \\
& + (-.424 + .506\, y)\, e^{-4.732\, y - 5.685\, u} + (.663 + .629\, y)\, e^{-4.732\, y - 3.544\, u} \\
& + (-.054 - .099\, y)\, e^{-4.732\, y - 1.765\, u} + (.066 + .147\, y)\, e^{-4.732\, y - .506\, u}\,, \\
& \qquad u,\, y \geq 0\,. \qquad (6.77)
\end{aligned}
$$

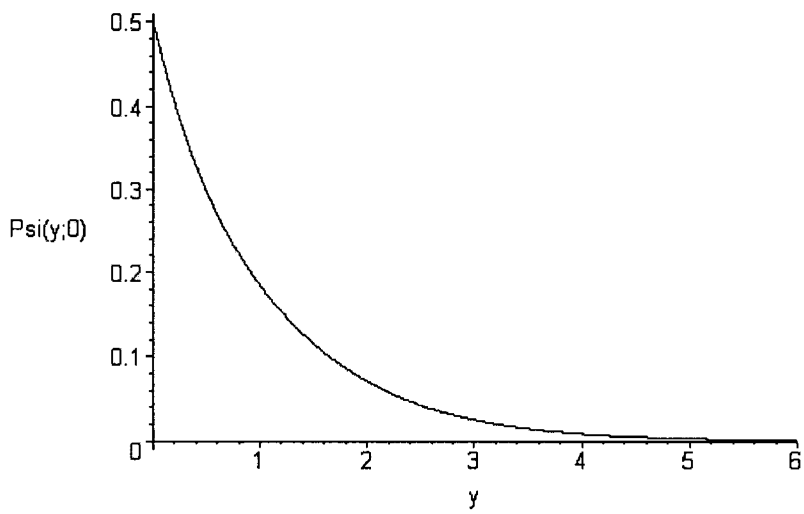

Figure 6.3: Distribution of the severity of ruin when $u = 0$ and $0 \leq y \leq 6$

Figure 6.3 shows the distribution of the severity of ruin $\Psi_i(y;\, 0)$, given by (6.75), when the initial surplus $u = 0$, for $0 \leq y \leq 6$. From this graph, we can see that this probability starts at 0.5 and obviously decreases as y increases. The curve is close to zero already when y reaches 5. Figure 6.4 gives the surface for severity of ruin probabilities $\Psi_i(y;\, u)$ in (6.77), when the initial surplus $0 \leq u \leq 5$ and $0 \leq y \leq 5$. As u and y simultaneously decrease, so does $\Psi_i(y;\, u)$, but the degree of decline with respect to the severity measure y reduces as u increases.

The next and last example gives the distribution of the severity of ruin when claim severities are Erlang(2) distributed with mean μ_1 for one class of claims while for the

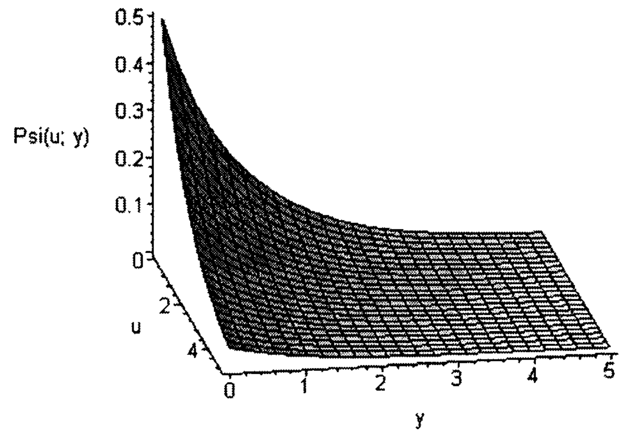

Figure 6.4: Probabilities of ultimate ruin when $0 \le u \le 10$

other class the claim severities are a mixture of exponentials with mean μ_2. This example continues the discussion in Example 6.2.

Example 6.5 Under the same assumptions as in Example 6.2, that is, $f_1(x) = \frac{x}{\beta^2} e^{-\frac{x}{\beta}}$ and $f_2(x) = \frac{\xi}{\beta_1} e^{-\frac{x}{\beta_1}} + \frac{1-\xi}{\beta_2} e^{-\frac{x}{\beta_2}}$, for $x \ge 0$ and $0 \le \xi \le 1$, with $\mu_1 = 2\beta$ and $\mu_2 = \xi\beta_1 + (1-\xi)\beta_2$, the operators $T_sT_0f_1(y)$ and $T_sT_0f_2(y)$ take the form:

$$T_sT_0f_1(y) = \frac{1}{s}\left[\frac{y}{\beta} + 1 - \frac{y\left(s+\frac{1}{\beta}\right)+1}{\beta^2\left(s+\frac{1}{\beta}\right)^2}\right] e^{-\frac{y}{\beta}}, \qquad y \ge 0,$$

$$T_sT_0f_2(y) = \frac{1}{s}\left[\xi\left(1 - \frac{1}{\beta_1 s + 1}\right)e^{-\frac{y}{\beta_1}} + (1-\xi)\left(1 - \frac{1}{\beta_2 s + 1}\right)e^{-\frac{y}{\beta_2}}\right], \qquad y \ge 0.$$

Again setting $\alpha_1 = \frac{1}{3}$ and $\alpha_2 = \frac{2}{3}$, $\lambda_1 = 0.5$ and $\lambda_2 = 2$, $c_1 = 1$ and $c_2 = 2$, $\beta = 1$, $\beta_1 = 0.5$ and $\beta_2 = 2$, and $\xi = 0.8$, gives $\mu_1 = 2$, $\mu_2 = 0.8$, and the stationary probabilities $\pi_1 = \frac{1}{3}$, $\pi_2 = \frac{2}{3}$. The roots to the equation (6.38) are 0, 1.066, -0.071, -0.419, $-1.455 + 0.047i$ and $-1.455 - 0.047i$, where $\rho = 1.066$ is the only positive real root.

Now $\Psi_1(y;\, 0)$ and $\Psi_2(y;\, 0)$, given in Theorem 6.4, are obtained as

$$\Psi_1(y;\, 0) = (.374\, y + .687)e^{-y} + .071\, e^{-2y} + .139\, e^{-.5y}, \qquad y \ge 0,$$

$$\Psi_2(y;\, 0) = (.126\, y + .313)e^{-y} + .329\, e^{-2y} + .261\, e^{-.5y}, \qquad y \ge 0,$$

which gives $\Psi_1(0;\,0) = .898$ and $\Psi_2(0;\,0) = .902$ and consequently $\bar{\Psi}_1(0) = 0.102$ and $\bar{\Psi}_2(0) = 0,098$. These values are consistent with those derived in Example 6.2. Furthermore, the expressions of $\Psi_1(y;\,u)$ and $\Psi_2(y;\,u)$, given by (6.67) and (6.68), give here:

$$\begin{aligned}
\Psi_1(y;\,u) = &\big[-.150\,e^{-.5\,y} - .018\,e^{-2\,y} + (.170 + .052\,y)e^{-y}\big]e^{-.419\,u}\\
&+ \big[.277\,e^{-.5\,y} + .057\,e^{-2\,y} + (.569 + .188\,y)e^{-y}\big]e^{-.071\,u}\\
&+ \big[.012\,\cos(.047\,u) + .069\,\sin(.047\,u)\big]e^{-.5\,y-1.455\,u}\\
&+ \big[(-0.052 + 0.134\,y)\cos(.047\,u) + 0.084\,\sin(.047\,u)\big]e^{-y-1.455\,u}\\
&+ \big[.032\,\cos(.047\,u) - .167\,\sin(.047\,u)\big]e^{-2y-1.455\,u}, \qquad u,\,y \geq 0,\\
\Psi_2(y;\,u) = &\big[.102\,e^{-.5\,y} + 0.012\,e^{-2\,y} + (-.116 - .036\,y)e^{-y}\big]e^{-.419\,u}\\
&+ \big[.270\,e^{-.5\,y} + .056\,e^{-2\,y} + (.555 + .183\,y)e^{-y}\big]e^{-.071\,u}\\
&+ \big[-.112\,\cos(.047\,u) + .008\,\sin(.047\,u)\big]e^{-.5\,y-1.455\,u}\\
&+ \big[(-0.126 - 0.022\,y)\cos(.047\,u)\big]e^{-y-1.455\,u}\\
&+ \big[(-0.097 + 0.214\,y)\,\sin(.047\,u)\big]e^{-y-1.455\,u}\\
&+ \big[.261\,\cos(.047\,u) + .077\,\sin(.047\,u)\big]e^{-2y-1.455\,u}, \qquad u,\,y \geq 0,
\end{aligned}$$

which yield the same expressions of the ruin probability $\Psi_1(u)$ and $\Psi_2(u)$ given in Example 6.2 by setting $y = 0$.

Figure 6.5 graph the surface formed by the severity of ruin probabilities $\Psi_i(y;\,u)$, given the initial environment state, $i = 1, 2$, for small initial surpluses u ($0 \leq u \leq 1$) and small y values ($0 \leq y \leq 1$). From this graph, we can see that $\Psi_1(y;\,u)$ and $\Psi_2(y;\,u)$ decrease as u and y increase, while $\Psi_1(y;\,u)$ is overall smaller than $\Psi_2(y;\,u)$. Figure 6.6 gives the same probabilities for relatively larger initial surpluses u ($0 \leq u \leq 10$) and larger y ($0 \leq y \leq 10$). Clearly the two probabilities $\Psi_1(y;\,u)$ and $\Psi_2(y;\,u)$ are very close and near zero, when u and y become bigger.

Figures 6.7 illustrates the severity of ruin probability in the stationary case, $\Psi(y;\,u) = \pi_1\,\Psi_1(y;\,u) + \pi_2\,\Psi_2(y;\,u)$, for fixed $y = 0, 1, \ldots, 5$ and different values

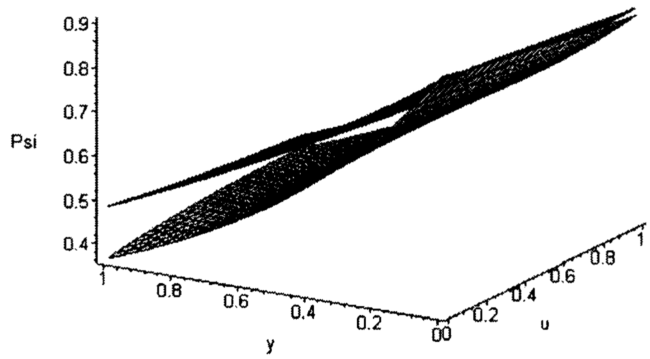

Figure 6.5: Severity of ruin probabilities $\Psi_1(y;\, u)$ and $\Psi_2(y;\, u)$ for small u and y

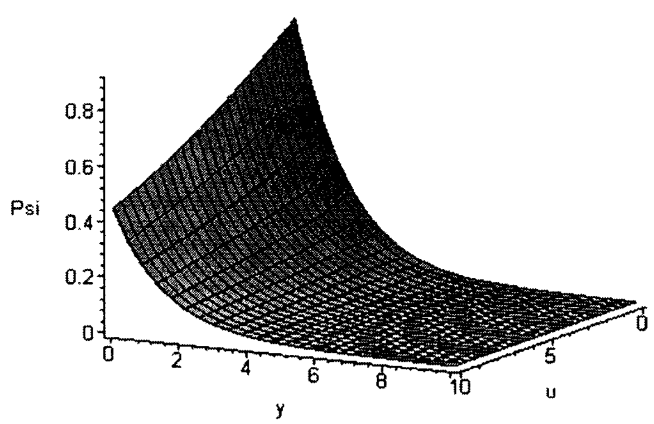

Figure 6.6: Severity of ruin probabilities $\Psi_1(y;\, u)$ and $\Psi_2(y;\, u)$ for $0 \leq u,\, y \leq 10$

of u ($0 \leq u \leq 10$). It shows that these probabilities decrease generally as the initial surplus u increases, while $\Psi(0;\ u)$ decreases faster than the others. Moreover, when y is large, $\Psi(y;\ u)$ is quite small and near constant, showing that increasing the initial surplus u does not affect much on the probability $\Psi(y;\ u)$ for large y.

Figure 6.8 gives these stationary severity of ruin probabilities for fixed $u = 0, 1, \ldots, 5$ and $0 \leq y \leq 2$. It can be observed that probabilities $\Psi(y;\ u)$, for $u = 1, 2, \ldots, 5$, decrease in parallel as y increases, while $\Psi(y;\ 0)$ drops faster, even becoming smaller than $\Psi(y;\ 1)$ when y is bigger than 0.8.

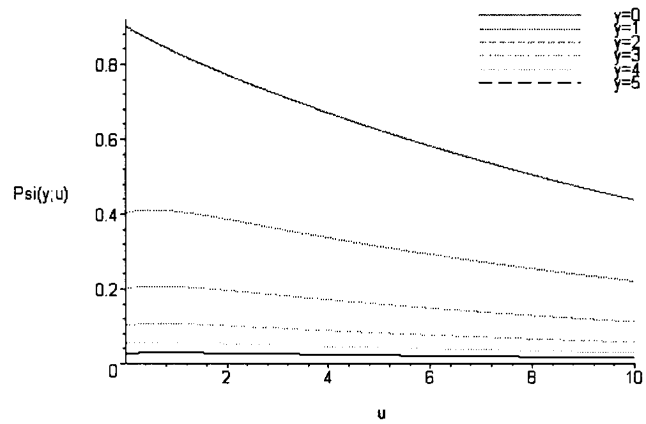

Figure 6.7: Stationary case severity of ruin probabilities for fixed y

In summary, this section gives the explicit formulas of the distribution of the severity of ruin in Markov–modulated models, where the claim intensities, the claim amounts and the premiums vary according to a Markovian environment. The same problem is studied by Snoussi (2002) for exponentially distributed claim severities. However a different approach is used here to generalize Snoussi's result to a much wider family of claim severity distributions.

We use Laplace transforms to solve the system of the Volterra–type integral equations and discuss the general characteristic equation in detail for a two–state model.

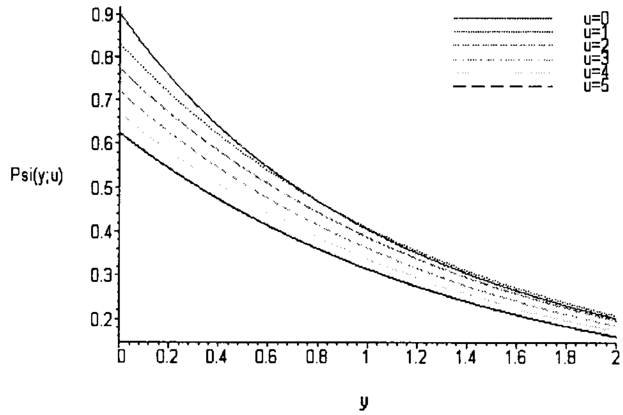

Figure 6.8: Stationary case severity of ruin probabilities for fixed u

The main techniques used here are similar to that used in Li (2004), Li and Garrido (2004) and Li and Lu (2004). With these, explicit formulas for the distribution of the severity of ruin, given an initial environmental condition (state) and the initial surplus is zero, are derived.

The expressions for the distributions of the severity of ruin are also obtained when both claim size distributions are in the rational family. Numerical inversions, as shown in the examples, can be done easily and quickly by using standard mathematical software.

Conclusions and further research

Two generalizations of the classical homogeneous Poisson risk model are considered in this thesis. A time–dependent NHP process is introduced to model claim frequency risks subject to seasonality, while a Markov–modulated Cox process is presented to characterize the underlying risk fluctuations affected by an environment external to the Poisson processes. Regime–switching NHP processes are further considered to account for both, the seasonal variations and the random fluctuations in the claims intensity.

Periodicity is considered for the model intensities of NHP processes and of regime–switching NHP processes. These yield models that are more attractive and practical, as many natural phenomena evolving in a periodic environment, or under seasonal conditions, can affect insurance claims.

NHP risk models are more realistic in practice, as their intensity rate is a function of time. This is clearly the case for hurricane landfalls. It provides an effective method for insurance companies to measure risk more accurately by using time dependent, rather than constant, intensity rates. Thanks to the flexibility of the beta function parametric form and its explicit expression for the cumulative intensity function, this allows for wide–ranging applications of the model with beta–type intensities.

The periodic regime–switching NHP (Cox) processes are practical and useful in modeling counting processes under periodic and random environments. A beta–type short–term intensity function, with a m–state Markov level process for the intensity of the Cox risk model, generalizes the periodic NHP model. It provides an efficient method to take into account random fluctuations on the levels of the periodic short–term intensity function, as an alternative to the deterministic patterns discussed in our NHP processes. It also provides a more realistic model compared to Cox models with piecewise constant intensities. Furthermore, the explicit results obtained for this Cox risk process make regime–switching periodic NHP models more practical and useful.

The Markov–modulated Poisson process, in which the claim intensities, claim severities and premiums vary according to an external Markovian environment, generalizes the homogeneous Poisson process. The study of the ruin problem, as well as the explicit results for the model, governed by a two–state Markov process with claim severity distributions from the rational family, make theoretical contributions in risk theory. The corresponding numerical evaluations of these probabilities make this model more practical for insurance businesses. Here the severity of ruin is considered as an additional element of information on ruin.

Moreover, statistical inferences are included for doubly periodic NHP and periodic regime–switching NHP claim counting processes, illustrated by beta–type intensities. The explicit expressions obtained for the likelihood functions and the feasible statistical estimation of the model parameters make these models simple to use in practice. The ensuing discussion on the evaluation of estimated parameters and the goodness–of–fit tests provides effective methods to perform model selection and assess model accuracy.

The dataset of the Atlantic hurricanes affecting the Unites States, from 1899 through 2000, is fitted by a doubly periodic NHP model and a periodic regime–switching Poisson model. This gives an application of the models to real data that is affected by a natural periodic environment, such as the short–term seasonality and long–term climatological fluctuations, or by a random environments.

Further, these models may be extended to other reasonable intensity functions and fitted to other datasets, for example, the dataset of Atlantic Tropical Storms Affecting the Unites States (1899-2002) or automobile accidents data reported in the Ontario Road Safety Annual Report. Besides, one may also consider the statistical aspects and applications of the Markov–modulated risk model to make it more tractable in practice.

For the regime–switching NHP model, one may further consider a periodic non–homogeneous Markov chain instead of our ordinary Markov chain, hence reflecting the long–term time–dependent, random and periodic natural environment. The interpretation of such a model within a fixed period, would be that the transition probabilities become time–dependent but are reset at the beginning of each new period.

Finally, for the two–state Markov–modulated Poisson model, we may consider the evaluation of the expected discounted penalty function defined by Gerber and Shiu (1998) and recently discussed for the classical risk model, the Sparre Andersen's risk model, as well as the diffusion perturbed these models. We may also consider the Markov–modulated diffusion perturbed Poisson model. These results would also be more attractive if they could be extended to the finite m–state case.

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Appendix A

Some tables

A.1 Emprical counts used in parameter estimation

Table A.1: Empirical counts used in the estimation for the double periodic model

j (month)	1	2	3	4	5	6	Total
$n_{j,1}^{(\cdot)}$	2	5	10	15	9	0	41
$n_{j,1}^{(21)}$			1	1	1		3
$n_{j,2}^{(\cdot)}$	1	1	11	16	4	3	36
$n_{j,2}^{(21)}$							0
$n_{j,3}^{(\cdot)}$	5	7	8	11	6	1	38
$n_{j,3}^{(21)}$							0
$n_{j,4}^{(\cdot)}$	2	1	6	6	3	0	18
$n_{j,4}^{(21)}$							0
$n_{j,5}^{(\cdot)}$	1	3	8	16	3	0	31
$n_{j,5}^{(21)}$							0

A.2 Atlantic hurricane season statistics (1950–2003)

Table A.2: Atlantic hurricane season statistics (1950–2003)

Above normal			Near normal			Below normal		
Year	TS,H,MH	ACE	Year	TS,H,MH	ACE	Year	TS,H,MH	ACE
1950	13,11,8	243	1952	7,6,3	87	1956	8,4,2	54
1951	10,8,5	137	1957	8,3,2	84	1962	5,3,1	36
1953	14,6,4	104	1959	11,7,2	78	1968	7,4,0	35
1954	11,8,2	113	1960	7,4,2	88	1970	10,5,2	34
1955	12,9,6	199	1963	9,7,2	118	1972	4,3,0	28
1958	10,7,5	121	1965	6,4,1	84	1973	7,4,1	43
1961	11,8,7	205	1967	8,6,1	122	1974	7,4,2	61
1964	12,6,6	170	1971	13,6,1	97	1977	6,5,1	25
1966	11,7,3	145	1975	8,6,3	73	1978	11,5,2	62
1969	17,12,5	158	1976	8,6,2	81	1982	5,2,1	29
1980	11,9,2	147	1979	8,5,2	91	1983	4,3,1	17
1988	12,5,3	103	1981	11,7,3	93	1986	6,4,0	36
1989	11,7,2	135	1984	12,5,1	71	1987	7,3,1	34
1995	19,11,5	228	1985	11,7,3	88	1991	8,4,2	34
1996	13,9,6	166	1990	14,8,1	91	1993	8,4,1	39
1998	14,10,3	182	1992	6,4,1	75	1994	7,3,0	32
1999	12,8,5	177				1997	7,3,1	40
2000	14,8,3	116				2000	12,4,2	66
2001	15,9,4	106						
2003	16,7,3	175						

Source: http://www.cpc.ncep.noaa.gov/products/outlooks/figure8.gif

In Table A.2, for each category the second column shows the seasonal total numbers of tropical storms (TS), hurricanes (H) and major hurricanes (MH), and the third column shows the ACE index for the entire basin.