

INDEPENDENT SETS IN GRAPH PRODUCTS VIA HARMONIC

ANALYSIS

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A Thesis

in

The Department

of

Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
for the Degree of Master of Science (Mathematics) at
Concordia University
Montréal, Québec, Canada

June 2005

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ISBN: 0-494-10211-X
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ISBN: 0-494-10211-X

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Abstract

Independent sets in graph products via Harmonic analysis

Mahya Ghandehari

In this thesis we study the independent sets of K_r^n , the weak product of n complete graphs on r vertices, which are close to be of maximum size. We review the previously known results. For constant r and arbitrary n , it was known that every such independent set is close to some independent set of maximum size. We prove that this statement holds for arbitrary r and n . The proof involves some techniques from Fourier analysis of Boolean functions on \mathbb{Z}_r^n . In fact we show that when most of the 2-norm weight of the Fourier expansion of a Boolean function on \mathbb{Z}_r^n is concentrated on the first two levels, then the function can be approximated by a Boolean function that depends only on one coordinate. A stronger analogue of this has been proven by Jean Bourgain for \mathbb{Z}_2^n . We present an expanded version of his proof in this thesis.

Acknowledgements

First and foremost, I would like to express my sincere gratitude to my supervisor, Benoit Larose, who has supported me with his guidance, patience and encouragement. I have greatly benefited from his valuable discussions and advises.

Many thanks to Vašek Chvátal and Galia Dafni for their helpful comments and suggestions on the earlier version of this thesis. Also, I would like to thank the staff and the faculty of the Department of Mathematics and Statistics at Concordia University for providing such a nice academic environment.

I would like to thank Hamed Hatami for being a nice collaborator and a supportive husband at the same time! He has always supported me with love, patience and encouragement. The results of Chapter 4 is a joint work with him.

My two years of study at Concordia University have been one of the most wonderful periods of my life. I am very grateful to my friends Aryan Bayani, Mona Mehrandish and Amir Memartoluie who made my life in Montréal so much fun.

Finally, I would like to thank my family: my parents Mahboobeh and Mostafa and my sisters Mahsa, Maryam and Mahta who always supported me with love and encouraged me to learn. This thesis is dedicated to my family.

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Chapter 1

Introduction

In the paper “Graph products, Fourier analysis and spectral techniques” [1] Alon *et al.* considered the following interesting combinatorial problem: “Assume that at a given road junction there are n three-position switches that control the red-yellow-green position of the traffic light. You are told that whenever you change the position of all the switches then the color of the light changes. Prove that in fact the light is controlled by only one of the switches.”

The above problem can be viewed as the problem of finding all the possible types of maximum independent sets in the product of complete graphs, which is studied in the present thesis. In fact, the configuration space of the switches described above can be simulated by the weak product of n copies of K_3 , the complete graph of size 3. Let us begin with some definitions.

Definition 1.0.1 *Recall the following notations:*

- *Given a graph G , the vertex set of G is denoted by $V(G)$, and the set of all edges of G is denote by $E(G)$.*
- *For a graph G , define $|G|$ to be the number of vertices of G , i.e.*

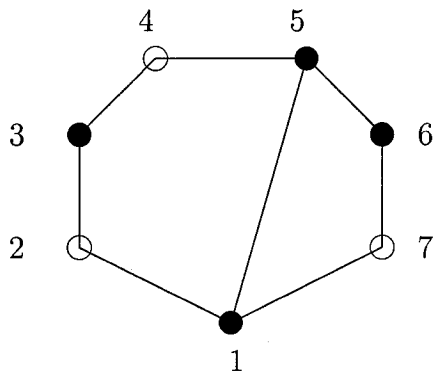
$$|G| = |V(G)|.$$

- For any two vertices u and v of G , by $u \sim v$ we mean that u and v are adjacent in G .

Definition 1.0.2 A set I is an independent set of the graph G if:

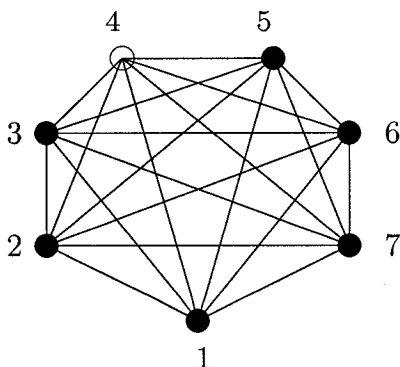
- $I \subseteq V(G)$.
- If $x, y \in I$ then $x \not\sim y$, i.e. no edge of G connects two vertices in I .

Example 1 The set $I = \{2, 4, 7\}$ forms an independent set of the graph G shown below.



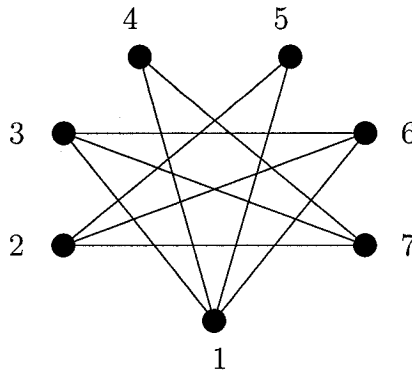
A graph G is a *complete graph* if any two of its vertices are adjacent. A complete graph on n vertices is denoted by K_n . In a complete graph any independent set has size at most one.

Example 2 The set $I = \{4\}$ is an independent set of K_7 .



A *maximum independent set* of a graph G is an independent set which has the maximum size. A *maximal independent set* is an independent set which is not a proper subset of any other independent set. Note that we study maximum independent sets in this thesis, and maximal independent sets will not be considered.

Example 3 In the following graph,



- $I_1 = \{1, 2\}$ is a maximal independent set, but not maximum.
- $I_2 = \{2, 3, 4\}$ is a maximum independent set.

Definition 1.0.3 The weak product of G and H , denoted by $G \times H$, is defined as follows.

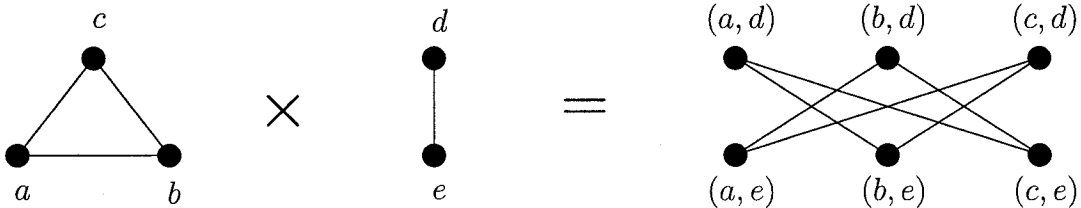
- The Cartesian product of $V(G)$ and $V(H)$ forms the vertex set of $G \times H$:

$$V(G \times H) = V(G) \times V(H).$$

- Let (g_1, h_1) and (g_2, h_2) be two vertices of $G \times H$. Then

$$(g_1, h_1) \sim (g_2, h_2) \text{ iff } g_1 \sim g_2 \text{ in } G \ \& \ h_1 \sim h_2 \text{ in } H.$$

Example 4 Consider the weak product of K_3 and K_2 :



In this thesis we study the n -fold product of complete graphs on $r > 2$ vertices,

$$G = K_r^n = \underbrace{K_r \times K_r \times \dots \times K_r}_n.$$

Let $r > 2$ be an integer. We identify $V(K_r)$ with \mathbb{Z}_r . It is then obvious that any two vertices i and j of K_r are adjacent iff $i \neq j$. Now we can identify the vertices of G with the elements of \mathbb{Z}_r^n in the natural way. By the definition of weak product, for any two vertices (x_1, \dots, x_n) and (y_1, \dots, y_n) of G ,

$$(x_1, \dots, x_n) \sim (y_1, \dots, y_n) \text{ iff } x_1 \neq y_1, \dots, x_n \neq y_n.$$

Let $0 \leq i \leq r - 1$ and $1 \leq j \leq n$ be two integers, and I_{ij} be the set of all vertices of G which have i in the j -th coordinate:

$$I_{ij} = \{v \in \mathbb{Z}_r^n : v_j = i\}.$$

Clearly, I_{ij} is an example of a large independent set. Calculate the ratio of this independent set, we have:

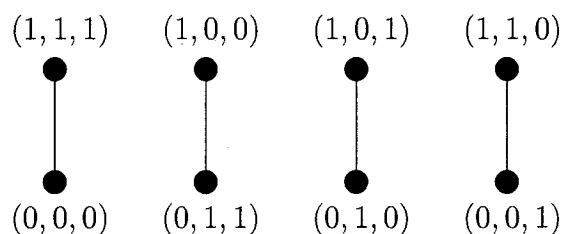
$$\frac{|I_{ij}|}{|G|} = \frac{r^{n-1}}{r^n} = \frac{1}{r}.$$

It is easy to show that this is the maximum ratio that an independent set can attain, and I_{ij} is a maximum independent set of G (See [1]). In fact, for $r > 2$, these sets are the only maximum independent sets of G , as Greenwell and Lovász showed in [26].

Theorem 1.0.4 (See [1]) *Let $G = K_r^n$, and assume $r \geq 3$. Let I be an independent set with $|I| = |G|/r$. Then there exists a coordinate $i \in \{1, \dots, n\}$ and $k \in \{0, \dots, r - 1\}$ such that*

$$I = \{v : v_i = k\}.$$

Remark. Theorem 1.0.4 is not necessarily true for $r = 2$. Note that for any n , the graph $G = K_2^n$ is the union of 2^{n-1} disjoint edges. As an example, the graph $G = K_2^3$ is presented below.



The set $I = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is obviously a maximum independent set in G , but the elements of I have no coordinate in common. It is easy to see that $G = K_2^3$ has 16 maximum independent sets (choose exactly one vertex from each edge). However, it has only 8 maximum independent sets of the form $I = \{v : v_i = k\}$, for $i = 1, 2, 3$ and $k = 1, 2$.

Remark. Using Theorem 1.0.4, it is not hard to see that for $r \geq 3$ and $n \geq 3$, one can find some independent set in K_r^n which does not extend to a maximum one. To see an example, let $I = \{(1, 1, 1), (2, 2, 1), (1, 2, 2)\}$. Obviously, I is an independent set in K_3^3 which does not extend to a maximum independent set.

Recently, Alon *et. al.* [1] proved a stability version of Theorem 1.0.4. They showed that for constant $r \geq 3$ and arbitrary n , if I is an independent set of size close to the maximum size in the graph K_r^n , then I is close to some maximum independent set. This is stated formally in Theorem 3.0.7. We generalize their result to arbitrary r and n in Theorem 4.1.1.

The proof of Theorem 4.1.1 as well as Theorem 3.0.7 is based on Fourier analysis on the group \mathbb{Z}_r^n . Fourier analysis has shown to be very useful in the study of some combinatorial structures. In recent years, this method has made significant contributions in combinatorics and computer science. For more examples, one can refer to an essay by Jean Bourgain [13] in a recent commemorative volume on “What can Combinatorics and Harmonic Analysis contribute to each other?”. Timothy Gowers’ solution [25] to the Erdős-Turan problem of arithmetic progressions, to which Endre Szemerédi had given a combinatorial solution before [45, 46], is an important instance that Bourgain mentions in his essay.

These methods are also playing an increasing role in the study of Boolean functions which began with Kahn, Kalai, and Linial's paper "The Influence of Variables on Boolean Functions" [29]. This topic is significantly important in theoretical computer science as well as economics (e.g., social choice) and statistical physics (e.g., percolation, spin glasses). One can refer to [1, 2, 3, 5, 8, 14, 20, 23, 29, 34, 36, 38, 48, 39] to see some examples.

Friedgut's [20] proof for the long lasting open problem of the existence of a sharp threshold for certain properties of random graphs is also based on Fourier analysis.

Recently many interesting nonembeddability results have been proven based on Fourier analysis on $\{0, 1\}^n$ and \mathbb{R}^n . Indeed, the proofs of results in [17, 43, 41, 33, 37, 32] all have Fourier analytic components.

Finally we should mention its significant role in the theory of lower bounds in approximation algorithms, especially in the near optimal inapproximability results of Johan Håstad [27].

Perhaps one reason for the effectiveness of Fourier methods in combinatorics is a general philosophy mentioned by Bourgain in [14] which claims that if f defines a property of "high complexity", then the support of its Fourier expansion has to be "spread out". Taking f to be the characteristic function of a combinatorial set enables us to use this rule of thumb to study combinatorial objects. Lemma 4.2.1, which we prove in Chapter 4, supports this philosophy.

This thesis is organized as follows. In Chapter 2 we review the basics of Fourier analysis on Abelian groups. Some useful inequalities in L_p spaces have been stated. We finish the chapter by a brief discussion on hyper-contractive inequalities. Chapter 3 reviews the proof in [1] for the stability version of Theorem 1.0.4 (Theorem 3.0.7). Chapter 4 contains our new results which generalize Theorem 3.0.7. Chapter 5 provides an expanded version of a paper of Jean Bourgain [14] where he proved a very strong analogue of Lemma 4.2.1 for \mathbb{Z}_2^n . Finally Chapter 6 contains concluding

remarks and a discussion about open problems and some ideas about future work.

Chapter 2

Background

In this chapter we describe the necessary background. We also introduce some notations and provide some tools for the following chapters.

Sections 2.1 and 2.2 review the basics of Fourier analysis on Abelian groups. Section 2.3 contains some useful inequalities in L_p spaces. Section 2.4 is devoted to a brief discussion on hyper-contractive inequalities which play a crucial role in the proofs presented in this thesis.

2.1 Fourier analysis on \mathbb{Z}_r^n

Let $r > 2$ and $G = \{0, 1, \dots, r-1\}^n = \mathbb{Z}_r^n$. For any $S \in G$, let $|S| = |\{i : S_i \neq 0\}|$ denote the number of nonzero coordinates of S . Let $\bar{0} = (0, 0, \dots, 0)$, and for each $1 \leq i \leq n$ let $e_i = (0, \dots, 1, \dots, 0)$ be the unit vector with 1 at the i -th coordinate and zero everywhere else. For $0 \leq k \leq n$, we define the “ k -th level” of $\{0, \dots, r-1\}^n$ to be the set of all S such that $|S| = k$.

Obviously G is an Abelian group. However, we can also think of G as a probability space equipped with the uniform (product) measure μ , i.e. $\mu(S) = \frac{1}{|G|}$ for every $S \in G$.

Let $f, g : G \rightarrow \mathbb{C}$. We then define:

- $\int_G f(x)dx = \sum_{x \in G} f(x)\mu(x) = \frac{1}{|G|} \sum_{x \in G} f(x)$.
- $\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)}$.
- $\|f\|_p = \left(\int_G |f(x)|^p dx\right)^{\frac{1}{p}}$.

Let us now find an orthonormal basis for the space of all complex-valued functions on G . For any $S \in G$, we define $u_S : G \rightarrow \mathbb{C}$ to be

$$u_S(T) = e^{\frac{2\pi i \langle S, T \rangle}{r}},$$

where $T \in G$ and $\langle S, T \rangle = \sum_{i=1}^n S_i T_i \pmod{r}$.

The functions u_S are called characters of G . The following properties of characters are well-known and easy to prove.

- The set of characters forms a group under the operation of pointwise multiplication.
 - The mapping $S \mapsto u_S$ is a homomorphism from G to the group of characters.
- Since for any $R \in G$, we have:

$$(u_S u_T)(R) = u_S(R)u_T(R) = e^{\frac{2\pi i \langle S, R \rangle}{r}} e^{\frac{2\pi i \langle T, R \rangle}{r}} = e^{\frac{2\pi i \langle S+T, R \rangle}{r}} = u_{S+T}(R),$$

and

$$u_{-T}(R) = e^{\frac{2\pi i \langle -T, R \rangle}{r}} = e^{\frac{-2\pi i \langle T, R \rangle}{r}} = (u_T^{-1})(R) = \overline{u_T}(R).$$

- Recall that the sum of the roots of unity is 0. So we have:
 1. $\sum_{T \in G} u_S(T) = 0$, for $S \neq \bar{0}$.
 2. $\frac{1}{|G|} \sum_{T \in G} u_{\bar{0}}(T) = 1$, since $u_{\bar{0}} \equiv 1$.
- The set of characters is an orthonormal basis:
 1. $\langle u_S, u_T \rangle = 0$ if $S \neq T$.

$$2. \langle u_S, u_S \rangle = 1.$$

- $u_S(T)$ is a function of $\{T_i : S_i \neq 0\}$.

As stated above, the set of all functions u_S forms an orthonormal basis for the space of all functions $f : G \rightarrow \mathbb{C}$. Therefore any such f has a unique expansion of the form $f = \sum_{S \in G} \widehat{f}(S) u_S$, where

$$\widehat{f}(S) = \langle f, u_S \rangle = \frac{1}{|G|} \sum_{T \in G} f(T) \overline{u_S(T)}.$$

From orthogonality it can be easily seen that

$$\|f\|_2^2 = \sum_{S \in G} |\widehat{f}(S)|^2, \quad (2.1)$$

and

$$\langle f, g \rangle = \sum_{S \in G} \widehat{f}(S) \overline{\widehat{g}(S)},$$

where the first equality is referred to as Parseval's identity. Let $f^{>k} = \sum_{|S|>k} \widehat{f}(S) u_S$ (similarly $f^{<k} = \sum_{|S|<k} \widehat{f}(S) u_S$) and $f^{=k} = \sum_{|S|=k} \widehat{f}(S) u_S$. We occasionally refer to $f^{=k}$ as the k -th *level* of Fourier expansion of f . Note that for any function f , $\widehat{f}(\bar{0})$ is the expectation of f , and $\|f^{\geq 1}\|_2^2$ is the variance of f .

2.2 Walsh expansion

Denote $\{1, 2, \dots, n\}$ by $[n]$.

Definition 2.2.1 (Biased Walsh-Products). Let $0 < p < 1$. For every $i \in [n]$, we define the i -th p -biased Rademacher function $r_i : \mathcal{P}([n]) \rightarrow \mathbb{R}$ by

$$r_i(x) = \begin{cases} \sqrt{\frac{p}{1-p}} & i \notin x \\ -\sqrt{\frac{1-p}{p}} & i \in x \end{cases}$$

For every set $\emptyset \neq S \subseteq [n]$, its corresponding p -biased Walsh-product is defined by

$$r_S = \prod_{i \in S} r_i.$$

Also let $r_\emptyset = 1$.

Note that every element of $\mathcal{P}([n])$ can be identified with a vector in \mathbb{Z}_2^n in the natural way. So we can restate Definition 2.2.1 as the following. Let $0 < p < 1$. For every $i \in [n]$, we define the i -th p -biased Rademacher function $r_i : \mathbb{Z}_2^n \rightarrow \mathbb{R}$ by

$$r_i(X) = \begin{cases} \sqrt{\frac{p}{1-p}} & X_i = 0 \\ -\sqrt{\frac{1-p}{p}} & X_i = 1 \end{cases}$$

for every $X = (X_1, \dots, X_n) \in \mathbb{Z}_2^n$.

The corresponding p -biased Walsh-product of S , $S = (S_1, \dots, S_n) \in \mathbb{Z}_2^n$, is now defined by

$$r_S = \prod_{S_i=1} r_i.$$

Let μ be the p -biased measure on \mathbb{Z}_2^n , which is defined as:

$$\mu(T) = p^{|T|}(1-p)^{n-|T|},$$

for any $T \in \mathbb{Z}_2^n$, i.e. every coordinate of T is 1 with probability p , and is 0 with probability $1-p$. We also define the integration with respect to the measure μ :

$$\int_{\mathbb{Z}_2^n} f d\mu(S) = \sum_{S \in \mathbb{Z}_2^n} f(S)\mu(S).$$

It is easy to see that the set of the p -biased Walsh-products forms an orthonormal basis for $L^2(\mathbb{Z}_2^n, d\mu)$, the L^2 -space of all real valued functions on \mathbb{Z}_2^n with

$$\|f\|_2^2 = \int_{\mathbb{Z}_2^n} |f|^2 d\mu = \sum_{S \in \mathbb{Z}_2^n} |f(S)|^2 \mu(S).$$

Therefore any function $f : \mathbb{Z}_2^n \rightarrow \mathbb{R}$ can be written as a linear combination $f = \sum_{S \in \mathbb{Z}_2^n} \widehat{f}(S)r_S$, called the Walsh expansion of f , where

$$\widehat{f}(S) = \langle f, r_S \rangle = \int_{\mathbb{Z}_2^n} f(x)r_S(x)d\mu(x).$$

If $p = \frac{1}{2}$, using the above argument, we get an orthonormal basis for the space of all functions $f : \mathbb{Z}_2^n \rightarrow \mathbb{R}$ with the uniform measure.

2.3 Some inequalities in L_p spaces

In this section we list a few elementary inequalities in L_p spaces.

Theorem 2.3.1 (Hölder) *If p and q are nonnegative extended real numbers (i.e. $p, q \in \mathbb{R}^{\geq 0} \cup \{\infty\}$) such that $\frac{1}{p} + \frac{1}{q} = 1$, and if $f \in L_p$ and $g \in L_q$, then $fg \in L_1$ and*

$$\int |fg| \leq \|f\|_p \|g\|_q.$$

Corollary 2.3.2 *Let $p, q, r \geq 1$ and $0 < \theta < 1$ be such that $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}$. Then for each $h \in L_p \cap L_r$ we have*

$$\|h\|_q \leq \|h\|_p^\theta \|h\|_r^{1-\theta}.$$

Proof. Let $p_1 = \frac{p}{q\theta}$ and $q_1 = \frac{r}{q(1-\theta)}$, so we have $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Using Hölder inequality for $f = h^{q\theta}$ and $g = h^{q(1-\theta)}$ we have:

$$\int (|h|^{q\theta})(|h|^{q(1-\theta)}) \leq \left(\int (|h|^{q\theta})^{\frac{p}{q\theta}} \right)^{\frac{q\theta}{p}} \left(\int (|h|^{q(1-\theta)})^{\frac{r}{q(1-\theta)}} \right)^{\frac{q(1-\theta)}{r}}.$$

Hence

$$\int |h|^q \leq \left(\int |h|^p \right)^{q\theta} \left(\int |h|^r \right)^{q(1-\theta)}.$$

Therefore we have:

$$\left(\int |h|^q \right)^{1/q} \leq \left(\int |h|^p \right)^{\theta/p} \left(\int |h|^r \right)^{\frac{1-\theta}{r}}.$$

■

The next theorem is also known as generalized Minkowski inequality. One can refer to [28] to see the proof.

Theorem 2.3.3 *(See [28]) Let $(M_1, \mathcal{M}_1, \mu_1)$ and $(M_2, \mathcal{M}_2, \mu_2)$ be two σ -finite measure spaces. If f is a measurable function on $M_1 \times M_2$ and $0 < q \leq r \leq \infty$, then*

$$\| \|f\|_{L_q(M_1, d\mu_1)} \| \|f\|_{L_r(M_2, d\mu_2)} \leq \| \|f\|_{L_r(M_2, d\mu_2)} \| \|f\|_{L_q(M_1, d\mu_1)}.$$

2.4 Hyper-contractive inequalities

Let \mathbb{Z}_2^n be the probability space equipped with the p -biased measure. Recall that the Bonami-Beckner operator T_ρ , $0 \leq \rho \leq 1$, is a linear operator on the space of functions $\mathbb{Z}_2^n \rightarrow \mathbb{R}$ defined by

$$T_\rho[f](x) = \mathbf{E}[f(y)],$$

where y is a ρ -correlated copy of x , i.e. at each point x , $T_\rho[f](x)$ is the expected value of f when a $(1 - \rho)$ -fraction of the coordinates in x are randomly re-assigned (we say that a $(1 - \rho)$ -noise is applied to x). This is equivalent to saying that $T_\rho[f](x)$, at each point x , is the expected value of f when each coordinate is chosen with the probability $1 - \rho$, and the chosen coordinate is randomly re-assigned.

Now let $f : \mathbb{Z}_2^n \rightarrow \mathbb{R}$ be a function, and consider its Walsh expansion $f = \sum_{S \in \mathbb{Z}_2^n} \hat{f}(S) r_S$. Obviously T_ρ is a linear operator since expectation is linear. Therefore,

$$T_\rho[f] = T_\rho\left[\sum_{S \in \mathbb{Z}_2^n} \hat{f}(S) r_S\right] = \sum_{S \in \mathbb{Z}_2^n} \hat{f}(S) T_\rho[r_S].$$

Hence it only remains to evaluate T_ρ on the biased Walsh-products, as shown in the following lemma.

Lemma 2.4.1 $T_\rho[r_S] = \rho^{|S|} r_S$

Proof. Let us first consider the case where $|S| = 1$. Without loss of generality we can assume that $r_S = r_1$. Let $X \in \mathbb{Z}_2^n$ such that $X_1 = 0$. Thus $r_1(X) = \sqrt{\frac{p}{1-p}}$. Now using the definition of T_ρ , we get:

$$T_\rho[r_S] = \rho \sqrt{\frac{p}{1-p}} + (1 - \rho)p \left(-\sqrt{\frac{1-p}{p}}\right) + (1 - \rho)(1 - p) \sqrt{\frac{p}{1-p}} = \rho \sqrt{\frac{p}{1-p}}.$$

Using the same argument for $X \in \mathbb{Z}_2^n$ with $X_1 = 1$, we conclude that $T_\rho[r_S] = \rho r_S$ when $|S| = 1$. It is also easy to see that $T_\rho[r_S] = \prod_{S_i=1} T_\rho[r_i]$ since r_i and r_j are independent when $i \neq j$. Therefore

$$T_\rho[r_S] = \prod_{S_i=1} \rho r_i = \rho^{|S|} r_S.$$

■

Recall that $f : \mathbb{Z}_2^n \rightarrow \mathbb{R}$ is a function with the Walsh expansion $f = \sum_{S \in \mathbb{Z}_2^n} \widehat{f}(S) r_S$. Now using Lemma 2.4.1, we have:

$$T_\rho[f] = \sum_{S \in \mathbb{Z}_2^n} \rho^{|S|} \widehat{f}(S) r_S. \quad (2.2)$$

Using Lemma 5 of [12], it can be easily proved that this operator is contractive with respect to any p -norm, $p \geq 1$. However, the Bonami-Beckner inequality is powerful since it shows that this operator remains contractive from L_p to L_q for certain values of p and q with $q > p$. This is the reason it is often referred to as a hyper-contractive inequality. The inequality was originally proved by Bonami in 1970 [9] and then independently by Beckner in 1973 [6]. It was first used to analyze discrete problems in a paper by Kahn, Kalai and Linial [29] where they considered the influence of variables on Boolean functions. The inequality is also very important in the study of combinatorics of $\{0, 1\}^n$ [15, 16, 20], percolation and random graphs [47, 22, 8, 21] and many other applications [7, 2, 44, 4, 19, 31, 42].

Theorem 2.4.2 (Bonami-Beckner) *Let \mathbb{Z}_2^n be a measure space endowed with the uniform measure. Let $f : \mathbb{Z}_2^n \rightarrow \mathbb{R}$ and $q \geq p \geq 1$. Then*

$$\|T_\rho f\|_q \leq \|f\|_p \text{ for all } 0 \leq \rho \leq \frac{(p-1)^{1/2}}{(q-1)^{1/2}}.$$

The dual version of Theorem 2.4.2 can be formulated as Theorem 2.4.3. The first part of this theorem has been presented in Lemma 2.1 of [1].

Theorem 2.4.3 *Let $f : \mathbb{Z}_2^n \rightarrow \mathbb{R}$ be a function that is a linear combination of the $\{u_T : |T| \leq k\}$. Then for $p \geq 2$,*

$$\|f\|_p \leq (\sqrt{p-1})^k \|f\|_2;$$

and for $1 \leq p \leq 2$,

$$\|f\|_p \geq (\sqrt{p-1})^k \|f\|_2.$$

Proof. First assume that $p \geq 2$. By Theorem 2.4.2, assuming $\rho = \sqrt{\frac{1}{p-1}}$ we have $\|T_\rho[f]\|_p \leq \|f\|_2$. Note that for every $T \in \mathbb{Z}_2^n$,

$$u_T(S) = e^{\frac{2\pi i}{2}\langle S, T \rangle} = (-1)^{\langle S, T \rangle} = r_T(S),$$

holds for every $S \in \mathbb{Z}_2^n$. Therefore $u_T = r_T$ in the case of the uniform measure on \mathbb{Z}_2^n . Thus since f is a linear combination of the $\{u_T : |T| \leq k\}$ and $\rho \leq 1$, by (2.2) we have $\|T_\rho[f]\|_p \geq \rho^k \|f\|_p$. So we have:

$$\rho^k \|f\|_p \leq \|f\|_2$$

which yields the claim.

For the case $1 \leq p \leq 2$, the same argument with $\rho = \sqrt{p-1}$ implies the claim. ■

Theorem 2.4.3 tells us that if a function f is a linear combination of the $\{u_T : |T| \leq k\}$, then all the p -norms ($p \geq 1$) of f are equivalent up to constants depending only on p and k and not on the function. This can even be generalized to $0 < p < 1$. However these should not be called norms since they do not satisfy the triangle inequality.

Theorem 2.4.4 *Let $f : \mathbb{Z}_2^n \rightarrow \mathbb{R}^{\geq 0}$ be a function that is a linear combination of the $\{u_T : |T| \leq k\}$. Then for $p \leq 1$,*

$$\|f\|_p \geq (3-p)^{-\frac{k(4-p)}{2p}} \|f\|_2.$$

Proof. Fix some $p_1 > 1$. By applying the Hölder inequality (Lemma 2.3.1) to $f^2 = f^{\frac{p}{p_1}} f^{2-\frac{p}{p_1}}$, we have

$$\int f^2 \leq \left(\int |f|^p \right)^{\frac{1}{p_1}} \left(\int |f|^{\left(\frac{2p_1-p}{p_1-1}\right)} \right)^{\frac{p_1-1}{p_1}}. \quad (2.3)$$

Since $\frac{2p_1-p}{p_1-1} > 2$, by Theorem 2.4.3

$$\left(\int |f|^{\left(\frac{2p_1-p}{p_1-1}\right)} \right)^{\frac{p_1-1}{2p_1-p}} \leq \left(\frac{2p_1-p}{p_1-1} - 1 \right)^{\frac{k}{2}} \|f\|_2. \quad (2.4)$$

Combining (2.3) and (2.4) we get

$$\left(\frac{p_1 - p + 1}{p_1 - 1}\right)^{-\frac{k(2p_1 - p)}{2p_1}} \|f\|_2^{2 - \frac{2p_1 - p}{p_1}} \leq \left(\int |f|^p\right)^{\frac{1}{p_1}},$$

or

$$\left(\frac{p_1 - p + 1}{p_1 - 1}\right)^{-\frac{k(2p_1 - p)}{2p}} \|f\|_2 \leq \|f\|_p.$$

Substituting $p_1 = 2$ implies the result. \blacksquare

The Bonami-Beckner inequality admits a reversed form which was first proved by Christer Borell [10] in 1982. The reversed form studies the case where $q \leq p \leq 1$, while the norms in the original inequality are all at least 1. We refer the reader to [40] for a proof of Theorem 2.4.5.

Theorem 2.4.5 (Reverse Bonami-Beckner) *Let $f : \mathbb{Z}_2^n \rightarrow \mathbb{R}^{\geq 0}$ and $q \leq p \leq 1$.*

Then

$$\|T_\rho f\|_q \geq \|f\|_p \text{ for all } 0 \leq \rho \leq \frac{(1-p)^{1/2}}{(1-q)^{1/2}}.$$

There are different generalizations of Theorem 2.4.2. We finish this chapter with stating a version dealing with functions on \mathbb{Z}_r^n , which has been proved in [1].

Theorem 2.4.6 [1] *For every $r \geq 2$ there exists $C > 0$ such that the following holds.*

Let $G = \mathbb{Z}_r^n$ be a probability space equipped with the uniform measure, and $f : G \rightarrow \mathbb{C}$ be a function whose Fourier expansion is concentrated on the first $k + 1$ levels, that is, f is a linear combination of the $\{u_T : |T| \leq k\}$. Then

$$\|f\|_4 \leq C^k \|f\|_2.$$

Chapter 3

Stability of independent sets

In this chapter we present some results concerning the maximum or nearly maximum independent sets of $G = K_r^n$. The results and proofs presented here are from a paper by Alon et. al. [1].

As stated in Theorem 1.0.4, the sets $I_{ij} = \{x \in \mathbb{Z}_r^n : x_i = j\}$, for $1 \leq i \leq n$ and $0 \leq j \leq r - 1$, are the only maximum independent sets of K_r^n for $r \geq 3$. A stability version of Theorem 1.0.4 has been proved in [1]:

Theorem 3.0.7 [1] *For every $r \geq 3$, there exists a constant $M = M(r)$ such that for any $\epsilon > 0$ the following is true. Let $G = K_r^n$ and J be an independent set of G such that $\frac{|J|}{|G|} = \frac{1}{r}(1 - \epsilon)$. Then there exists an independent set I with $\frac{|I|}{|G|} = \frac{1}{r}$ such that $\frac{|J \Delta I|}{|G|} < \frac{M\epsilon}{r}$.*

In Theorem 3.0.7, “ Δ ” denotes the symmetric difference. Theorem 3.0.7 asserts that any independent set J of G which is close to be of maximum size is close to some set I_{ij} , i.e. J is close to being determined by one vertex.

The proof of Theorem 3.0.7 uses some Fourier analysis methods. We present this proof in Section 3.1. There are a few lemmas regarding Boolean functions on \mathbb{Z}_r^n which are the key lemmas in the proof of the theorem. In Section 3.2 we review the proofs of these lemmas.

3.1 Proof of Theorem 3.0.7

Let $r \geq 3$ be an integer and $G = K_r^n$. We identify the vertices of G with the elements of \mathbb{Z}_r^n as shown in Chapter 1. Throughout this chapter, we think of \mathbb{Z}_r^n as a probability space with the uniform product measure, denoted by μ , i.e.

$$\mu(S) = \frac{1}{|G|} \text{ for every } S \in G.$$

Let I be an independent set in G . Let f be the characteristic function of I , i.e. f is a Boolean function on \mathbb{Z}_r^n such that

$$f(x) = \begin{cases} 1 & x \in I \\ 0 & x \notin I \end{cases}$$

The function f has the following properties:

- $\alpha = \frac{|I|}{|G|} = \int_G f(x) dx = \widehat{f}(\bar{0})$.
- $\alpha = \|f\|_2^2 = \sum_{S \in \mathbb{Z}_r^n} |\widehat{f}(S)|^2$.
- $\sum_{S \in \mathbb{Z}_r^n} |\widehat{f}(S)|^2 \left(\frac{-1}{r-1}\right)^{|S|} = 0$.

Where α is defined to be the ratio of I , $\alpha = \mu(I) = \frac{|I|}{|G|}$. The first equality is obvious. The second one can be easily shown using Parseval's identity. We will later present the proof of the third equality in Lemma 3.1.1. Assuming these facts, we get:

$$\sum_{S \neq \bar{0}} |\widehat{f}(S)|^2 = \alpha - \alpha^2. \quad (3.1)$$

$$\sum_{S \neq \bar{0}} |\widehat{f}(S)|^2 \left(\frac{-1}{r-1}\right)^{|S|} = -\alpha^2. \quad (3.2)$$

Now we get valuable information on the Fourier expansion of f using (3.1) and (3.2). We consider the following distribution to interpret this information. Let T be a random variable which takes values in $G \setminus \bar{0}$ with

$$\Pr[T = S] = \frac{|\widehat{f}(S)|^2}{\alpha - \alpha^2}.$$

Let

$$X = X(T) = \left(\frac{-1}{r-1} \right)^{|T|}.$$

Calculating $\mathbf{E}(X)$ and using (3.2), we have

$$\mathbf{E}(X) = \sum_{S \neq \emptyset} \left(\frac{-1}{r-1} \right)^{|S|} \frac{|\widehat{f}(S)|^2}{\alpha - \alpha^2} = \frac{-\alpha^2}{\alpha - \alpha^2} = \frac{-\alpha}{1 - \alpha}.$$

It is important to note that for all T , $X(T) \geq \frac{-1}{r-1}$. Also equality $X(T) = \frac{-1}{r-1}$ holds if and only if $|T| = 1$. We now break our analysis into three cases: $\alpha > 1/r$, $\alpha = 1/r$ and $\alpha = \frac{1-\epsilon}{r}$. A simple argument shows that the first case cannot happen. One can then use Lemmas 3.2.1 and 3.2.2, discussed in Section 3.2, to analyze the second and third cases.

Case 1:

Let $\alpha > 1/r$. Since $\frac{-x}{1-x}$ is a decreasing function, we have $\mathbf{E}(X) = \frac{-\alpha}{1-\alpha} < \frac{-1}{r-1}$. However, note that $\mathbf{E}(X) \geq \frac{-1}{r-1}$ since $X(T) \geq \frac{-1}{r-1}$ for all T . Therefore the assumption $\alpha > 1/r$ yields a contradiction. Hence if I is an independent set then $\mu(I) \leq 1/r$.

Case 2:

Let $\alpha = 1/r$. Then $\mathbf{E}(X) = \frac{-\alpha}{1-\alpha} = \frac{-1}{r-1}$. Moreover, since $X(T) \geq \frac{-1}{r-1}$ for all T , $\mathbf{E}(X) = \frac{-1}{r-1}$ implies that $X \equiv \frac{-1}{r-1}$. Now recalling $X(T) = \frac{-1}{r-1}$ iff $|T| = 1$, we conclude that for all S of size bigger than one, the following holds:

$$\Pr[T = S] = \frac{|\widehat{f}(S)|^2}{\alpha - \alpha^2} = 0.$$

Hence $\widehat{f}(S) = 0$ for all S of size bigger than one. Therefore f has all its Fourier expansion concentrated on the first two levels, which implies that f is constant or depends only on one coordinate (see Lemma 3.2.1). Obviously f is not a constant function, because it is the characteristic function of an independent set. Therefore it depends only on one coordinate, and it must be the characteristic function of I_{ij} for

some i and j . Thus we have just given another proof for Theorem 1.0.4.

Case 3:

Finally we consider the case $\alpha = \frac{1-\epsilon}{r}$, where α is slightly less than $1/r$. We then have

$$\mathbf{E}(X) = \frac{-1 + \epsilon}{r - 1 + \epsilon} > \frac{-1}{r - 1}.$$

So $\mathbf{E}(X)$ is very close to $\frac{-1}{r-1}$, its minimal value, from which we will conclude that most of the Fourier expansion of f is concentrated on the first two levels.

Recall that for all S of size bigger than one, we have

$$X(S) \geq \frac{-1}{(r-1)^3} > \frac{-1}{r-1}. \quad (3.3)$$

Now let $Y = Y(T) = X(T) + \frac{1}{r-1}$. It is easy to verify the following properties of Y :

1. $Y \geq 0$.
2. $\mathbf{E}(Y) = \mathbf{E}(X) + \frac{1}{r-1} = \frac{-1+\epsilon}{r-1+\epsilon} + \frac{1}{r-1} = \frac{\epsilon r}{(r-1)(\epsilon+r-1)}$.
3. $Y(T) > 0$ iff $X(T) > \frac{-1}{r-1}$ iff $|T| > 1$.
4. If $Y > 0$ then $Y \geq \frac{-1}{(r-1)^3} + \frac{1}{r-1} = \frac{r(r-2)}{(r-1)^3}$.

Now by Markov's inequality, we get

$$\Pr[Y > 0] \leq \mathbf{E}(Y) \frac{(r-1)^3}{r(r-2)}. \quad (3.4)$$

Using (3.4) and the second property of Y together with $r \geq 3$, we have

$$\Pr[Y > 0] \leq \frac{\epsilon}{\epsilon + r - 1} \cdot \frac{(r-1)^2}{r-2} \leq 2\epsilon. \quad (3.5)$$

The conclusion of (3.5) is that for each $\epsilon > 0$ and for every independent set I with $\alpha = \mu(I) = \frac{1-\epsilon}{r}$ we have

$$\sum_{|S|>1} |\widehat{f}(S)|^2 = (\alpha - \alpha^2) \sum_{|S|>1} \Pr[T = S] = (\alpha - \alpha^2) \Pr[Y > 0] \leq \frac{1}{r} (2\epsilon) \leq \frac{2\epsilon}{r},$$

where f is the characteristic function of I .

We have just shown that most of the Fourier expansion of f is concentrated on the first two levels. Now we deduce Theorem 3.0.7 from Lemma 3.2.2, which says that there exists some Boolean function g that depends on at most one coordinate such that

$$\|f - g\|_2^2 < \frac{(2K)(2\epsilon)}{(\alpha - \alpha^2 - \frac{2\epsilon}{r})(r)} < \frac{M\epsilon}{r},$$

where M is a function of r . Note that we assume ϵ to satisfy $\|f\|_2^2 - \frac{(2K)(2\epsilon)}{(\alpha - \alpha^2 - \frac{2\epsilon}{r})(r)} > 0$. This assumption disallows the uninteresting cases, since if ϵ is not small enough then $g \equiv 0$ is also a possibility.

We now continue with the proof of Lemma 3.1.1, which has been used in proving Theorem 3.0.7.

Lemma 3.1.1 [1] *Let $I \subset \mathbb{Z}_r^n$ be an independent set in G , and let $f : \mathbb{Z}_r^n \rightarrow \{0, 1\}$ be its characteristic function, i.e. $f(x) = 1$ iff $x \in I$. Then,*

$$\sum_{S \in \mathbb{Z}_r^n} |\widehat{f}(S)|^2 \left(\frac{-1}{r-1} \right)^{|S|} = 0.$$

Proof. First define $D = (\mathbb{Z}_r \setminus \{0\})^n$ and $d = |D|$. It is then obvious that two vertices $u, v \in \mathbb{Z}_r^n$ are adjacent iff $u - v \in D$. Now define

- $f_\tau(x) = f(x + \tau)$ for any $\tau \in D$.
- $A(f) = \frac{1}{d} \sum_{\tau \in D} f_\tau$.

Note that $A(f)$ acts as an averaging operator which replaces $f(x)$ by the average of f on the neighbors of x . We now need to prove the following claims:

Claim 3.1.2 $\langle f, A(f) \rangle = 0$.

Proof. Let $x \in \mathbb{Z}_r^n$. Recall that for any $\tau \in D$, x and $x + \tau$ are adjacent in G . So for every $x \in \mathbb{Z}_r^n$, since I is an independent set, we have:

$$f_\tau(x)f(x) = 0.$$

Therefore

$$\langle f, f_\tau \rangle = 0. \quad (3.6)$$

Recalling the definition of $A(f)$, we have:

$$\langle f, A(f) \rangle = \langle f, \frac{1}{d} \sum_{\tau \in D} f_\tau \rangle = \frac{1}{d} \sum_{\tau \in D} \langle f, f_\tau \rangle = 0.$$

■

Now we compute the Fourier coefficients of f_τ and $A(f)$ in terms of the Fourier coefficients of f .

Claim 3.1.3 $\widehat{f}_\tau(S) = \widehat{f}(S)u_\tau(S)$.

Proof. Using the definition of the Fourier expansion of f_τ , we have:

$$\begin{aligned} \widehat{f}_\tau(S) &= \int f_\tau(x) \overline{u_S(x)} dx = \int f(x + \tau) \overline{u_S(x)} dx \\ &= \int f(x) \overline{u_S(x - \tau)} dx = \int f(x) \overline{u_S(x) u_S(-\tau)} dx \\ &= \int f(x) \overline{u_S(x)} u_S(\tau) dx = \widehat{f}(S) u_\tau(S). \end{aligned}$$

■

Claim 3.1.4 $\widehat{A(f)}(S) = \widehat{f}(S) \left(\frac{-1}{r-1}\right)^{|S|}$.

Proof. Let $\omega = e^{2\pi i/r}$. Recall that $D = (\mathbb{Z}_r \setminus \{0\})^n$ and $d = |D| = (r-1)^n$. By Claim 3.1.3,

$$\widehat{A(f)}(S) = \left(\frac{1}{d} \sum_{\tau \in D} f_\tau\right)(S) = \frac{1}{d} \sum_{\tau \in D} \widehat{f}_\tau(S) = \frac{1}{d} \widehat{f}(S) \sum_{\tau \in D} u_\tau(S).$$

We also have

$$\sum_{\tau \in D} u_\tau(S) = \prod_{j=1}^n \sum_{k=1}^{r-1} \omega^{kS_j} = \prod_{j:S_j=0} (r-1) \prod_{j:S_j \neq 0} (-1),$$

since the sum of the roots of unity is 0. Finally we get:

$$\widehat{A(f)}(S) = \frac{1}{(r-1)^n} \widehat{f}(S) (r-1)^{n-|S|} (-1)^{|S|} = \widehat{f}(S) \left(\frac{-1}{r-1}\right)^{|S|},$$

which completes the proof of Claim 3.1.4. \blacksquare

Back to the proof of Lemma 3.1.1, we can now use Claim 3.1.4 and Claim 3.1.2, and we have

$$0 = \langle f, A(f) \rangle = \sum_{S \in \mathbb{Z}_r^n} \widehat{f}(S) \overline{\widehat{A(f)}(S)} = \sum_{S \in \mathbb{Z}_r^n} |\widehat{f}(S)|^2 \left(\frac{-1}{r-1} \right)^{|S|}.$$

\blacksquare

3.2 Proofs of Lemmas 3.2.1 and 3.2.2

In this section we prove some lemmas which have been used in the proof of Theorem 3.0.7 in Section 3.1.

Lemma 3.2.1 [1] *Let $f : \mathbb{Z}_r^n \rightarrow \{0, 1\}$ be such that all its Fourier expansion is concentrated on the first two levels:*

$$|S| > 1 \Rightarrow \widehat{f}(S) = 0.$$

Then f is either constant or depends on precisely one coordinate.

Proof. The Fourier expansion of f is concentrated on the first two levels, therefore f is of the form $f = \widehat{f}(\bar{0}) + \sum_{T:|T|=1} \widehat{f}(T)$. Denoting $a_0 = \widehat{f}(\bar{0})$, the function f can be written as

$$f(S) = a_0 + \sum_{j=1}^n \sum_{k=1}^{r-1} a_{j,k} e^{\frac{2\pi i s_j k}{r}}.$$

Moreover $f = f^2$, since f is a Boolean function. Let us now compare the coefficients of $e^{\frac{2\pi i s_{j_1} k}{r}} e^{\frac{2\pi i s_{j_2} l}{r}}$ in f and f^2 . If $j_1 \neq j_2$ then the coefficients of $e^{\frac{2\pi i s_{j_1} k}{r}} e^{\frac{2\pi i s_{j_2} l}{r}}$ in f and f^2 are 0 and $2a_{j_1,k}a_{j_2,l}$ respectively, and by the uniqueness of the Fourier expansion, we get

$$a_{j_1,k}a_{j_2,l} = 0 \text{ whenever } j_1 \neq j_2.$$

It is now obvious that there exists one coordinate, say j_i , such that for all k and j , $j \neq j_i$, we have $a_{j,k} = 0$. Therefore f is of the form

$$f(S) = a_0 + \sum_{k=1}^{r-1} a_k e^{\frac{2\pi i S_1 k}{r}}.$$

■

Lemma 3.2.2 [1] *Let $r \geq 2$ and let $C = C(r)$ be the constant from Theorem 2.4.6. Let $K = 2 + 32C^8$. Then for any $\epsilon > 0$ the following holds. Let $f : \mathbb{Z}_r^n \rightarrow \{0, 1\}$ be a function such that $\Pr[f = 1] = \alpha$ and furthermore assume that*

$$\sum_{|S|>1} |\widehat{f}(S)|^2 = \epsilon.$$

Then there exists a Boolean function $g : \mathbb{Z}_r^n \rightarrow \{0, 1\}$ which depends on at most one coordinate such that

$$\|f - g\|_2^2 \leq \frac{2K}{\alpha - \alpha^2 - \epsilon} \epsilon.$$

Remark. Note that f is a Boolean function, and we assume that \mathbb{Z}_r^n is a probability space endowed with the uniform measure. Therefore we have

$$\alpha = \Pr[f = 1] = \int_{\mathbb{Z}_r^n} f(S) dS = \widehat{f}(\bar{0}),$$

and

$$\alpha = \int_{\mathbb{Z}_r^n} f(S) dS = \int_{\mathbb{Z}_r^n} |f(S)|^2 dS = \|f\|_2^2.$$

Hence

$$\alpha - \alpha^2 - \epsilon = \|f\|_2^2 - |\widehat{f}(\bar{0})|^2 - \sum_{|S|>1} |\widehat{f}(S)|^2 = \sum_{|S|=1} |\widehat{f}(S)|^2,$$

which implies that $\epsilon \leq \alpha - \alpha^2$.

Remark. In the above theorem, we can assume that $\epsilon \leq \frac{1}{4C^8}$, where C is the constant from Theorem 2.4.6. Note that if $\epsilon > \frac{1}{4C^8}$ then taking $K = 2 + 32C^8$ leads

to a trivial $K\epsilon$ -approximation of f . This is because $K\epsilon > 1$ and for any function $g : \mathbb{Z}_r^n \rightarrow \{0, 1\}$, obviously $\|f - g\|_2^2 \leq 1$.

Proof of Lemma 3.2.2. Let $f_S = \sum_{|T| \leq 1} \widehat{f}(T)u_T$ and $f_L = \sum_{|T| > 1} \widehat{f}(T)u_T$, where S represents small and L stands for large.

The lemma is trivial when f_S is assumed to be a Boolean function. Indeed, if f_S is Boolean then it depends on at most one coordinate by Lemma 3.2.1. Hence we can take g to be f_S and we are done. To prove the lemma in the general case, we will first show that f_S is close to being Boolean. We can then easily conclude the desired result.

Define $h = f_S^2 - f_S$. It is obvious that $h = 0$ iff f_S is Boolean. Thus we can consider h as a measure that determines how far f_S is from being Boolean. In order to show that f_S is close to being Boolean, we will prove that the 2-norm of h is small, as stated in the following lemma.

Lemma 3.2.3 [1] *Let C be the constant from Theorem 2.4.6 and let $\lambda = 32C^8$. Then*

$$\mathbf{E}(|h|^2) \leq \lambda\epsilon,$$

where $\sum_{|S| > 1} |\widehat{f}(S)|^2 = \epsilon$ and $\epsilon \leq \frac{1}{4C^8}$.

We postpone the proof of Lemma 3.2.3, and present its important corollary first.

Corollary 3.2.4 [1] *There exists $1 \leq j \leq n$ such that for the function g' , defined as*

$$g'(x) = \widehat{f}(\bar{0}) + \sum_{k=1}^{r-1} \widehat{f}(ke_j) e^{\frac{2\pi i k x_j}{r}}, \quad (3.7)$$

it is true that $\|g' - f\|_2^2 \leq \epsilon \left(1 + \frac{\lambda}{2(\alpha - \alpha^2 - \epsilon)}\right)$.

Assuming Corollary 3.2.4, Lemma 3.2.2 can be easily proven. Indeed, let $\text{maj}_j[f](y) : \mathbb{Z}_r^n \rightarrow \{0, 1\}$ be the function taking values of 0 or 1 according to the majority value of

$f(x)$ over all $\{x \in \mathbb{Z}_r^n : x_j = y_j\}$. It is easy to see that if $g : \mathbb{Z}_r^n \rightarrow \{0, 1\}$ is a Boolean function which depends only on the j -th coordinate, then

$$\|f - \text{maj}_j[f]\|_2^2 \leq \|f - g\|_2^2. \quad (3.8)$$

Let g'' be the function obtained by rounding g' to the nearest number in $\{0, 1\}$. Hence

$$\|f - g''\|_2^2 \leq 4\|f - g'\|_2^2. \quad (3.9)$$

Now using (3.8) and (3.9), we get:

$$\|f - \text{maj}_j[f]\|_2^2 \leq 4\|f - g'\|_2^2 \leq \frac{2K}{\alpha - \alpha^2 - \epsilon} \epsilon,$$

which completes the proof of Lemma 3.2.2. ■

Now it only remains to prove Corollary 3.2.4 and Lemma 3.2.3. We will first present the proof of the corollary assuming Lemma 3.2.3. We then finish the section with the review of the proof of this lemma.

Proof of Corollary 3.2.4. Let $g_0 = \widehat{f}(\overline{0})$. For $1 \leq i \leq n$, let $g_i = \sum_{k=1}^{r-1} \widehat{f}(ke_i)$. Recall that $f = g_0 + g_1 + g_2 + \dots + g_n$. Denote $a_i = \|g_i\|_2^2 = \sum_{k=1}^{r-1} |\widehat{f}(ke_i)|^2$. Without loss of generality we can assume that $\|g_1\|_2^2 \geq \|g_2\|_2^2 \geq \dots \geq \|g_n\|_2^2$.

Let us first compute $\widehat{h}(S)$ on the second level. Let $i \neq j$. Then, since $\widehat{f}(S)$ is 0 on the second level, we have:

$$\widehat{h}(k_1e_i + k_2e_j) = \widehat{f}_S^2(k_1e_i + k_2e_j) - \widehat{f}_S(k_1e_i + k_2e_j) = 2\widehat{f}(k_1e_i)\widehat{f}(k_2e_j). \quad (3.10)$$

Since $\mathbf{E}(|h|^2) = \|h\|_2^2 = \sum_{T \in \mathbb{Z}_r^n} |\widehat{h}(T)|^2$, we can get a lower bound for $\mathbf{E}(|h|^2)$ by summing only over T 's of the second level (i.e. $T = k_1e_i + k_2e_j$ for $i, j \in [n]$ with

$i \neq j$). Therefore using (3.10), we have

$$\begin{aligned}
\sum_{i < j} a_i a_j &= \sum_{i < j} \left[\sum_{k=1}^{r-1} |\widehat{f}(ke_i)|^2 \right] \left[\sum_{k=1}^{r-1} |\widehat{f}(ke_j)|^2 \right] \\
&= \sum_{i < j} \left[\sum_{k=1}^{r-1} \sum_{k'=1}^{r-1} |\widehat{f}(ke_i)|^2 |\widehat{f}(k'e_j)|^2 \right] \\
&= \sum_{i < j} \sum_{k=1}^{r-1} \sum_{k'=1}^{r-1} \frac{|\widehat{h}(ke_i + k'e_j)|^2}{4} \\
&\leq \frac{\mathbf{E}(|h|^2)}{4}.
\end{aligned} \tag{3.11}$$

We now use Lemma 3.2.3 together with (3.11), and we get:

$$\sum_{i < j} a_i a_j \leq \frac{\lambda}{4} \epsilon. \tag{3.12}$$

Hence,

$$\left(\sum_i a_i \right)^2 \leq \sum_i a_i^2 + \frac{\lambda}{2} \epsilon \leq a_1 \sum_i a_i + \frac{\lambda}{2} \epsilon. \tag{3.13}$$

According to the remark followed by Lemma 3.2.2, we also have $\sum_{i=1}^n a_i = \alpha - \alpha^2 - \epsilon$. Now by dividing both sides of (3.13) by $\sum_i a_i = \alpha - \alpha^2 - \epsilon$, we get:

$$a_1 \geq \alpha - \alpha^2 - \epsilon \left(1 + \frac{\lambda}{2(\alpha - \alpha^2 - \epsilon)} \right),$$

which means that $\|g'\|_2^2 = a_1 + \alpha^2 \geq \alpha - \epsilon \left(1 + \frac{\lambda}{2(\alpha - \alpha^2 - \epsilon)} \right)$. Finally, using the definition of g' , we have:

$$\begin{aligned}
\|f - g'\|_2^2 &= \|f\|_2^2 - \|g'\|_2^2 \\
&= \alpha - \|g'\|_2^2 \\
&\leq \left(1 + \frac{\lambda}{2(\alpha - \alpha^2 - \epsilon)} \right) \epsilon,
\end{aligned}$$

which implies the corollary. ■

Proof of Lemma 3.2.3. We first show that h is smaller than $O(\sqrt{\epsilon})$ on a relatively big subset of the domain. We then use a hyper-contractive argument to show that it cannot be much larger on the rest, which implies the lemma.

Let $k = 2C^4$. Define set $Z = \{x \in \mathbb{Z}_r^n : |f_L(x)| \leq k\sqrt{\epsilon}\}$, and let $\bar{Z} = \mathbb{Z}_r^n - Z$ be the complement of Z . Then,

- $\Pr_x[x \in Z] \geq 1 - 1/k^2$. Since if $\Pr_x[x \in Z] < 1 - 1/k^2$ then $\Pr_x[x \in \bar{Z}] > 1/k^2$, which implies that

$$\|f_L\|_2^2 = \mathbf{E}(|f_L|^2) \geq \Pr_x[x \in \bar{Z}]k^2\epsilon > \frac{\epsilon k^2}{k^2}.$$

But this is a contradiction, since $\|f_L\|_2^2 \leq \epsilon$.

- For the values of x , $x \in Z$, $h(x)$ is bounded by $2k\sqrt{\epsilon}$. To see this, recall that $f_S = f - f_L$ and that $f^2 - f = 0$. Therefore,

$$h = (f_S)^2 - f_S = (f - f_L)^2 - (f - f_L) = f_L^2 + f_L(1 - 2f).$$

Now, for every $x \in Z$,

$$\begin{aligned} |h(x)| &\leq |f_L^2(x)| + |f_L(x)||1 - 2f(x)| \\ &\leq |f_L(x)|^2 + |f_L(x)| \\ &\leq 2|f_L(x)| \\ &\leq 2k\sqrt{\epsilon}. \end{aligned} \tag{3.14}$$

Here we have used our assumption that $\epsilon \leq \frac{1}{4C^8}$ which is equivalent to saying that $k\sqrt{\epsilon} \leq 1$.

According to the bound obtained above, $\mathbf{E}(|h|^2)$ can be much larger than ϵ only if this expectation is built mostly from the values of x in \bar{Z} , where h gets values much larger than $\sqrt{\epsilon}$. However, using the hyper-contractive argument (2.4.6), we show that this is impossible. This is “intuitively because having only low Fourier frequencies means $|h|$ is fairly “concentrated” around its expectation”, as stated in [1].

Denote $X = \mathbf{E}(|h|^2)$ and $Y = \mathbf{E}(|h|^4)$. Let also $p = \Pr_x[x \notin Z] \leq \frac{1}{k^2}$. Since in the probability spaces p -norm increases as p increases, for every function h , $X \leq \sqrt{Y}$

holds. Now recall that $h = (f_S)^2 - f_S$. Therefore the Fourier support of h is on the first three levels, i.e. h is a linear combination of the $\{u_T : |T| \leq 2\}$. So Theorem 2.4.6 implies that $\sqrt{Y} \leq C^4 X$. Thus,

$$\begin{aligned}
X &= \mathbf{E}(|h|^2) \\
&= (1-p)\mathbf{E}(|h(x)|^2|x \in Z) + p\mathbf{E}(|h(x)|^2|x \notin Z) \\
&\leq (1-p)4k^2\epsilon + p\sqrt{\mathbf{E}(|h(x)|^4|x \notin Z)} \\
&\leq 4k^2\epsilon + p\sqrt{\frac{Y}{p}} \leq 4k^2\epsilon + \sqrt{p}C^4X \leq 4k^2\epsilon + \frac{1}{2}X, \tag{3.15}
\end{aligned}$$

where we used $p \leq \frac{1}{k^2} = \frac{1}{4C^8}$ for the last inequality. Thus, $X \leq 8k^2\epsilon = 32C^8\epsilon$. ■

Chapter 4

Large independent sets

In this chapter we present our new results [24] which contain a stronger version of Theorem 3.0.7 (Theorem 4.1.1).

4.1 Introduction

Theorem 3.0.7 asserts that any independent set which is close to being of maximum size is close to being determined by one coordinate. We will later show that the best function $M(r)$ that can be obtained from the proof of [1] is $\Omega(r^6)$. When r is a constant, for every constant $\delta > 0$ one can choose ϵ to be a sufficiently small constant so that $\frac{|J\Delta I|}{|G|} < \frac{\delta}{r}$. But when r tends to infinity, to obtain any nontrivial result from Theorem 3.0.7, ϵ must be less than $\frac{1}{M(r)} = O(r^{-6})$ which is not a constant. The main result of this chapter is to show that in Theorem 3.0.7, M does not need to be a function of r . Note that this major improvement makes Theorem 3.0.7 as powerful for large values of r as for constant r . We formalize this in the following theorem.

Theorem 4.1.1 *Let $G = K_r^n$, $r \geq 20$ and $\epsilon < 10^{-9}$. Suppose that J is an independent set of G such that $\frac{|J|}{|G|} = \frac{1}{r}(1 - \epsilon)$. Then there exists an independent set I with $\frac{|I|}{|G|} = \frac{1}{r}$ such that $\frac{|J\Delta I|}{|G|} < \frac{40\epsilon}{r}$.*

Remark. Note that for $\epsilon \geq 10^{-9}$, we have the trivial bound $\frac{|J \Delta I|}{|G|} \leq \frac{2 \times 10^9 \epsilon}{r}$, where I is an arbitrary independent set. We also assumed that $r \geq 20$, for technical reasons. However, one can use Theorem 3.0.7 when $r < 20$, as $M(r)$ is a constant for those values of r .

Let I be a maximum independent set of $G = K_r^n$, and J be an independent set of G such that $J \not\subseteq I$. So there exists some $(a_1, \dots, a_n) \in \mathbb{Z}_r^n$ which belongs to J and does not belong to I . Since I is a maximum independent set, by Theorem 1.0.4, there exist $i \in \{1, \dots, n\}$ and $k \in \{0, \dots, r-1\}$ such that $I = \{v : v_i = k\}$. So we conclude that $a_i \neq k$. Now taking all the elements of the form $(b_1, \dots, k, \dots, b_n)$, which have k in the i -th coordinate and satisfy $b_j \neq a_j$ for every $j \neq i$, it is then obvious that all these elements belong to $I \setminus J$. Therefore:

$$\frac{|I \setminus J|}{|G|} \geq \frac{(r-1)^{n-1}}{r^n}.$$

So we obtain the following as a corollary of Theorem 4.1.1.

Corollary 4.1.2 *Let $G = K_r^n$, $r \geq 20$ and $\epsilon < c$ where $c = \min(10^{-9}, (1 - \frac{1}{r})^{n-1})/40$. Let J be an independent set of G such that $\frac{|J|}{|G|} = \frac{1}{r}(1 - \epsilon)$. Then there exists an independent set I with $\frac{|I|}{|G|} = \frac{1}{r}$ such that $J \subseteq I$.*

Note that if in Corollary 4.1.2, $r > c'n$ for some constant c' , then one can take c to be a constant that does not depend on n .

The proof of Theorem 4.1.1 is based on Fourier analysis on the group \mathbb{Z}_r^n (as was Theorem 3.0.7). In order to prove Theorem 4.1.1 we show that a Boolean function which has most of its 2-norm weight concentrated on the first two levels of its Fourier expansion is close to being determined by one coordinate. Thus Lemma 4.2.1 which formulates this, might be of independent interest in the direction of extending results of [23, 14, 34] from \mathbb{Z}_2^n to \mathbb{Z}_r^n .

The following version of Bennett's inequality which can be easily obtained from the one stated in [11] will be used in the proof of Lemma 4.2.1 below.

Theorem 4.1.3 (Bennett's Inequality) *Let X_1, \dots, X_n be independent real-valued random variables with zero mean, and assume that $X_i \leq c$ with probability one. Let*

$$\sigma^2 = \sum_{i=1}^n \mathbf{Var}[X_i].$$

Then for any $t > 0$,

$$\Pr\left[\sum X_i \geq t\right] \leq e^{-\frac{\sigma^2}{c^2} h\left(\frac{tc}{\sigma^2}\right)},$$

where $h(u) = (1 + u) \ln(1 + u) - u$ for $u \geq 0$.

4.2 Main results

Before starting the proof of Theorem 4.1.1, we show that the function $M(r)$ in [1] is of order $\Omega(r^6)$. According to the proof given in [1], $M(r) = \Theta(r^2\delta^{-8})$ where δ is such that for every $f : \mathbb{Z}_r^n \rightarrow \mathbb{C}$, $\|T_\delta f\|_4 \leq \|f\|_2$ holds and T_δ is the Bonami-Beckner operator defined as $T_\delta f = \sum \delta^{|S|} \widehat{f}(S) u_S$. Consider the one dimensional case, $n = 1$. Let $f(k) = \sum_{j=1}^{r-1} e^{\frac{2\pi i(k \times j)}{r}}$. Then we have $\|T_\delta f\|_4^4 = \delta^4(r-1)(r^2 - 3r + 3)$ and $\|f\|_2^2 = r-1$, which implies that $\delta^4(r-1)(r^2 - 3r + 3) \leq r-1$. So $\delta^{-8} \geq (r^2 - 3r + 3)^2$.

In [14, 23, 34] results of the following type have been proven. Let f be a Boolean function on \mathbb{Z}_2^n and $f^{>k}$ is sufficiently small for some constant k , then f is close to being determined by a few number of coordinates. The following result which is the key lemma in the proof of Theorem 4.1.1 is a result of this type for \mathbb{Z}_r^n .

Lemma 4.2.1 *Let $f : \mathbb{Z}_r^n \rightarrow \mathbb{C}$ be a Boolean function such that $\|f^{=1}\|_2^2 \leq \frac{1}{r}$ and $\|f^{>1}\|_2^2 \leq \epsilon$, where $\epsilon < \frac{1}{10^8 r}$ and $r \geq 20$. Then denoting by $1 \leq i_0 \leq n$ the index such that $\sum_{j=1}^{r-1} |\widehat{f}(je_{i_0})|^2$ is maximized, we have*

$$\left\| f - \left(\widehat{f}(\overline{0}) + \sum_{j=1}^{r-1} \widehat{f}(je_{i_0}) u_{je_{i_0}} \right) \right\|_2^2 < 5\epsilon.$$

Remark. Lemma 4.2.1 shows that f is close to a function which depends only on the i_0 -th coordinate. We do not know if the condition $\|f^{\neq 1}\|_2^2 \leq \frac{1}{r}$ is a weakness of our proof or it is essential. The condition $\epsilon < \frac{1}{10^8 r}$ is not a major weakness, since for $\epsilon \geq \frac{1}{10^8 r}$, we have the trivial bound of $(10^8 + 1)\epsilon$.

We postpone the proof of Lemma 4.2.1 until Section 4.2.1. We now give the proof of Theorem 4.1.1, assuming Lemma 4.2.1.

Proof of Theorem 4.1.1. Let J be an independent set of G such that $\frac{|J|}{|G|} = \frac{1}{r}(1-\epsilon)$. Let f be the characteristic function of J . Then according to the proof of Theorem 3.0.7, we have

$$\|f^{>1}\|_2^2 = \sum_{|S|>1} |\widehat{f}(S)|^2 \leq \frac{2\epsilon}{r}.$$

Since

$$\|f^{\neq 1}\|_2^2 \leq \|f\|_2^2 = \mu(J) \leq \frac{1}{r},$$

by Lemma 4.2.1, there exists a function $g : \mathbb{Z}_r^n \rightarrow \mathbb{C}$ which depends on one coordinate and $\|f - g\|_2^2 \leq \frac{10\epsilon}{r}$. By rounding g to the nearest of 0 or 1, we get a Boolean function g_1 which depends on one coordinate, and since f is Boolean

$$\|f - g_1\|_2^2 \leq 4\|f - g\|_2^2 \leq \frac{40\epsilon}{r}.$$

■

4.2.1 Proof of Lemma 4.2.1

The proof of Lemma 4.2.1 shares similar ideas with the proof of Theorem 8 in [34]; however, dealing with (complex) Fourier expansions on \mathbb{Z}_r^n instead of (real) generalized Walsh expansions on \mathbb{Z}_2^n required new ideas.

For every complex number z , let $d(z, \{0, 1\}) = \min(|z|, |z - 1|)$ denote its distance from the nearest element in $\{0, 1\}$. For any function f , denote $\widehat{f}(S)u_S$ by F_S for convenience. For $1 \leq i \leq n$, let $g_i = \sum_{j=1}^{r-1} F_{j e_i}$, and define $g_0 = \widehat{f}(\bar{0})$. For $0 \leq i \leq n$

let $a_i = \|g_i\|_2$. Without loss of generality assume that $a_1 \geq a_2 \geq \dots \geq a_n$. To obtain

$$\|f - (g_0 + g_1)\|_2^2 = \sum_{i=2}^n a_i^2 + \|f^{>1}\|_2^2 \leq 5\epsilon,$$

we will first show that a_2 is small (Claim 4.2.2). This will allow us to apply a concentration theorem and conclude that $\sum_{i=2}^n a_i^2$ is very small (Claim 4.2.3).

First note that

$$\|f^{=1}\|_2^2 = \sum_{i=1}^n a_i^2 \leq \frac{1}{r},$$

which implies that $a_2^2 \leq \frac{1}{2r}$. Now since $\|g_2\|_2^2 \leq \frac{1}{2r}$, for every $0 \leq x_2 \leq r-1$,

$$|g_2(x_2)| \leq \sqrt{1/2}. \quad (4.1)$$

Claim 4.2.2 $a_2^2 < 2000\epsilon$.

Proof. Consider an arbitrary assignment $\delta_1, \delta_3, \dots, \delta_n$ to x_1, x_3, \dots, x_n , and let

$$l = \widehat{f}(\overline{0}) + g_1(\delta_1) + \sum_{i=3}^n g_i(\delta_i).$$

Since for every $0 \leq x_2 \leq r-1$,

$$d(l, \{0, 1\}) \leq |g_2(x_2)| + d(l + g_2(x_2), \{0, 1\}),$$

we have

$$\|d(l, \{0, 1\})\|_2^2 \leq 2(\|g_2\|_2^2 + \|d(l + g_2, \{0, 1\})\|_2^2),$$

or equivalently

$$d(l, \{0, 1\})^2 \leq 2(a_2^2 + \|d(l + g_2, \{0, 1\})\|_2^2). \quad (4.2)$$

Note that

$$\|d(f^{\leq 1}, \{0, 1\})\|_2^2 \leq 2(\|d(f, \{0, 1\})\|_2^2 + \|f^{>1}\|_2^2) \leq 2\epsilon.$$

Therefore we can find an assignment $\delta_1, \delta_3, \dots, \delta_n$ such that

$$\|d(l + g_2, \{0, 1\})\|_2^2 \leq 2\epsilon. \quad (4.3)$$

By (4.2) for any such assignment, we have $d(l, \{0, 1\})^2 \leq \frac{1}{r} + 4\epsilon \leq 1/16$, which implies either $|l| \leq \frac{1}{4}$ or $|l - 1| \leq \frac{1}{4}$.

Define $\lambda = \frac{1 - \sqrt{\frac{1}{2} - \frac{1}{4}}}{\sqrt{\frac{1}{2} + \frac{1}{4}}}$. Now (4.1) implies that for any $0 \leq x_2 \leq r - 1$,

Case 1: If $|l| < \frac{1}{4}$, then $|(l + g_2(x_2)) - 1| \geq \lambda|l + g_2(x_2)|$.

Case 2: If $|l - 1| < \frac{1}{4}$, then $|l + g_2(x_2)| \geq \lambda|(l + g_2(x_2)) - 1|$.

Let $A = \{x_2 \in \mathbb{Z}_r : |l + g_2(x_2)| \leq |(l + g_2(x_2)) - 1|\}$ and denote its complement by \bar{A} . Representing $\|d(l + g_2, \{0, 1\})\|_2^2$ as a sum of two integrals over A and \bar{A} , and using (4.1), in Cases 1 and 2 one can show that

$$\|d(l + g_2, \{0, 1\})\|_2^2 \geq \lambda^2 \|g_2\|_2^2 > \frac{a_2^2}{1000}.$$

Note that the assumption $a_2^2 \geq 2000\epsilon$ will imply $\|d(l + g_2, \{0, 1\})\|_2^2 > 2\epsilon$ which contradicts (4.3). Thus $a_2^2 < 2000\epsilon$. \blacksquare

Claim 4.2.3 $\sum_{i=2}^n a_i^2 \leq 4\epsilon$.

Proof. Let $2 \leq m \leq n$ be the minimum index which satisfies

$$\sum_{i=m}^n a_i^2 \leq 10^4 \epsilon. \quad (4.4)$$

Denote $I = \{m, \dots, n\}$, and for every $y \in \mathbb{Z}_r^{m-1}$ let $f_{I[y]}^*$ be a function of \mathbb{Z}_r^{n-m+1} (with uniform measure μ') defined as

$$f_{I[y]}^*(x) = f^{\leq 1}(y \cup x).$$

Obviously

$$\int \|d(f_{I[y]}^*(x), \{0, 1\})\|_2^2 \mu'(dy) = \|d(f^{\leq 1}, \{0, 1\})\|_2^2 \leq 2\epsilon.$$

Hence for some y , $\|d(f_{I[y]}^*(x), \{0, 1\})\|_2^2 \leq 2\epsilon$. Let $b = \widehat{f}(\bar{0}) + \sum_{i=1}^{m-1} g_i(y_i)$. Then

$$f_{I[y]}^*(x) = b + \sum_{i=m}^n g_i(x_i).$$

Applying Lemma 4.2.4 below to $f_{I[y]}^*$ for $\epsilon' = 2\epsilon$ shows that $\sum_{i=m}^n a_i^2 \leq 4\epsilon$. This will imply that $m = 2$, as $a_2^2 < 2000\epsilon$ and m was the minimum index satisfying (4.4), which completes the proof. \blacksquare

Lemma 4.2.4 *Let $f : \mathbb{Z}_r^n \rightarrow \mathbb{C}$ be a function satisfying $f^{>1} \equiv 0$. Let $\|d(f, \{0, 1\})\|_2^2 \leq \epsilon'$, and suppose that $\|f^{-1}\|_2^2 < 10^4\epsilon'$ and $\epsilon' < \frac{2}{108r}$. Then we have*

$$\|f^{-1}\|_2^2 < 2\epsilon'.$$

Proof. Suppose that $f = b + \sum_{i=1}^n g_i$, where $b = \widehat{f}(\overline{0})$ and $g_i = \sum_{j=1}^{r-1} F_{je_i}$. We have

$$\|d(b, \{0, 1\})\|_2^2 \leq 2(\|d(f, \{0, 1\})\|_2^2 + \|f - b\|_2^2) \leq 20002\epsilon'.$$

Without loss of generality assume that $d(b, 1) \leq \sqrt{20002\epsilon'}$ which implies that

$$\operatorname{Re}(b) > 2/3. \quad (4.5)$$

We have

$$\|f - 1\|_2^2 - \|d(f, \{0, 1\})\|_2^2 = \int (|f - 1|^2 - |f|^2)\zeta dx,$$

where

$$\zeta(x) = \begin{cases} 1 & \operatorname{Re}(f(x)) < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

So

$$\|f - 1\|_2^2 - \|d(f, \{0, 1\})\|_2^2 = \int (1 - 2\operatorname{Re}(f))\zeta dx. \quad (4.6)$$

The next step is to show that (4.6) is less than ϵ' . Note that $\operatorname{Re}(f) = \operatorname{Re}(b) + \sum_{i=1}^n \operatorname{Re}(g_i)$, and $\int \operatorname{Re}(g_i) = 0$. Moreover

$$\int \operatorname{Re}(g_i)^2 = \|\operatorname{Re}(g_i)\|_2^2 \leq \|g_i\|_2^2.$$

So

$$\|\operatorname{Re}(g_i)\|_2^2 \leq \sum \|g_i\|_2^2 \leq 10^4\epsilon'$$

from which follows that for every x ,

$$|\operatorname{Re}(g_i(x))| \leq \sqrt{10^4 r \epsilon'} \leq \sqrt{2} \times 10^{-2} \doteq c.$$

Applying Theorem 4.1.3 with $X_i = -\operatorname{Re}(g_i)$, we get

$$\Pr\left[\sum \operatorname{Re}(g_i) \leq -t\right] \leq e^{-\frac{10^4 \epsilon'}{c^2} h(10^{-4} t c / \epsilon')}. \quad (4.7)$$

Note that $h(u) \geq u \ln(\frac{u}{e})$, for $u \geq e$; which implies that for $t \geq \frac{1}{6} \geq 10^4 e \epsilon' / c$,

$$\Pr\left[\sum \operatorname{Re}(g_i) \leq -t\right] \leq e^{-\frac{t}{c} \ln(10^{-4} t c / \epsilon')}. \quad (4.8)$$

Now

$$(4.6) = \int_{t=0}^{\infty} \Pr[1 - 2\operatorname{Re}(f) > t] = \int_{t=0}^{\infty} \Pr\left[\operatorname{Re}(b) + \sum \operatorname{Re}(g_i) < \frac{1-t}{2}\right].$$

Substituting (4.5) we get

$$(4.6) \leq \int_{t=0}^{\infty} \Pr\left[\sum \operatorname{Re}(g_i) < \frac{1-t}{2} - \frac{2}{3}\right] \leq 2 \int_{t=1/6}^{\infty} \Pr\left[\sum \operatorname{Re}(g_i) < -t\right].$$

Now by (4.8)

$$\begin{aligned} (4.6) &\leq 2 \int_{t=1/6}^{\infty} e^{\frac{t}{c} \ln(10^4 e \epsilon' / t c)} \leq 2 \int_{t=1/6}^{\infty} \left(\frac{1 - \ln(10^4 e \epsilon' / t c)}{c}\right) e^{\frac{t}{c} \ln(10^4 e \epsilon' / t c)} = \\ &= 2e^{\frac{1}{6c} \ln(6 \times 10^4 e \epsilon' / c)} < \epsilon', \end{aligned} \quad (4.9)$$

because $\epsilon' \leq 10^{-8}$. Finally by (4.9)

$$\|f^{-1}\|_2^2 \leq \|f - 1\|_2^2 \leq \|d(f, \{0, 1\})\|_2^2 + \epsilon' \leq 2\epsilon'.$$

■

Chapter 5

Fourier spectrum of Boolean functions

In this chapter we provide an expanded version of a paper of Jean Bourgain [14] where he proved a very strong analogue of Lemma 4.2.1 for \mathbb{Z}_2^n .

5.1 Fourier spectrum of Boolean functions

As Jean Bourgain says “There is a general philosophy which claims that if f defines a property of “high complexity”, then $\text{supp} \hat{f}$, the support of the Fourier expansion, has to be “spread out”.” The result shown in this chapter is one more illustration of this phenomenon. For a real function f on \mathbb{Z}_2^N let $f = \sum \hat{f}(S)r_S$ be its Walsh expansion. Now if f is not essentially determined by a few variables, then the tail distribution of \hat{f} satisfies a lower bound, i.e. for every $\epsilon > 0$ there exists c_ϵ such that

$$\sum_{|S|>k} |\hat{f}(S)|^2 \gg c_\epsilon k^{-\frac{1}{2}-\epsilon}$$

holds for all fixed k . A precise formulation of the above statement appears in Theorem 5.1.1. This estimate expresses to what extent \hat{f} may be fully concentrated on coefficients $\hat{f}(S)$ of low weight $|S|$. It is worth to note that Theorem 5.1.1 is basically

sharp, as shown in Corollary 5.1.4.

For any subset $A \subseteq \mathbb{Z}_2^N$, let χ_A denote the characteristic function of A . We also write some inequalities in the form of “ $a \lesssim b$ ” by which we mean that there exists some c such that $a \leq bc$ holds.

The main result of this chapter is the following.

Theorem 5.1.1 *Let $f = \chi_A$, $A \subseteq \mathbb{Z}_2^N$. Let $k > 0$ be an integer and $\gamma > 0$ a fixed constant. Assume*

$$\sum \{|\widehat{f}(S)|^2 : |\widehat{f}(S)| < \gamma 4^{-k^2}\} > \gamma^2. \quad (5.1)$$

Then for every $\epsilon > 0$, there exists the lower bound $c_\epsilon k^{-\frac{1}{2}-\epsilon}$ such that

$$\sum_{|S|>k} |\widehat{f}(S)|^2 \geq c_\epsilon k^{-\frac{1}{2}-\epsilon}. \quad (5.2)$$

Remark. Suppose f is a function that satisfies (5.1), i.e. the summation of all relatively small Fourier coefficients of f is large. Therefore f must have a large number of nonzero Fourier coefficients, which means that f is of high complexity.

Proof. We may assume that

$$\sum_{|S|>k} |\widehat{f}(S)|^2 < \frac{1}{100} \gamma^2, \quad (5.3)$$

since if $\sum_{|S|>k} |\widehat{f}(S)|^2 \geq \frac{1}{100} \gamma^2$, it is enough to take $c_\epsilon = \frac{1}{100} \gamma^2$ and we are done. Fix $0 < \kappa < 1$ and define

$$I_0 = \{i \in [1, N] : \sum_{i \in S, |S| \leq k} |\widehat{f}(S)|^2 > \kappa\}.$$

Then we have

$$\kappa |I_0| \leq \sum_{i=1}^N \sum_{i \in S, |S| \leq k} |\widehat{f}(S)|^2,$$

and since each $|\widehat{f}(S)|^2$ contributes at most k times in the sum we get:

$$\sum_{i=1}^N \sum_{i \in S, |S| \leq k} |\widehat{f}(S)|^2 \leq k \sum |\widehat{f}(S)|^2 = k \|f\|_2^2.$$

Recall that f is a Boolean function, which implies that $\|f\|_2^2 \leq 1$. So we get

$$\kappa|I_0| \leq \sum_{i=1}^N \sum_{i \in S, |S| \leq k} |\widehat{f}(S)|^2 \leq k,$$

and therefore

$$|I_0| < \kappa^{-1}k. \quad (5.4)$$

Thus

$$\sum \left\{ |\widehat{f}(S)|^2 : S \subseteq I_0, |S| \leq k, |\widehat{f}(S)| < \gamma 4^{-k^2} \right\} < (\kappa^{-1}k)^k \gamma^2 16^{-k^2} < \frac{\gamma^2}{100}, \quad (5.5)$$

if we assume

$$(\kappa^{-1}k)^k 16^{-k^2} < \frac{1}{100}. \quad (5.6)$$

Denote

$$I'_0 = [1, N] \setminus I_0.$$

It follows from (5.1), (5.3), and (5.5) that

$$\sum_{S \cap I'_0 \neq \emptyset, |S| \leq k} |\widehat{f}(S)|^2 > \gamma^2 - \frac{1}{100}\gamma^2 - \frac{1}{100}\gamma^2 > \frac{1}{2}\gamma^2. \quad (5.7)$$

Define for $t \geq 0$

$$\rho_t = \sum_{2^t \leq |S \cap I'_0| \leq 2^{t+1}} |\widehat{f}(S)|^2, \quad (5.8)$$

so that (5.7) implies that

$$\sum_{0 \leq t \leq \log k} \rho_t > \frac{\gamma^2}{2} \quad (5.9)$$

(where $\log k = \log_2 k$).

Next, fix a subset

$$I_1 \subseteq I'_0.$$

Write the variable $x \in \mathbb{Z}_2^N$ as $x = (x_1, x_2)$ where x_1 is the part with coordinates in I_1 .

For a fixed x_2 write $f_{x_2}(x_1)$ for $f(x_1, x_2)$ and write also $F_T(x_2)$ for $\widehat{f_{x_2}}(T)$. Thus,

$$f_{x_2}(x_1) = f(x_1, x_2) = \sum_{T \subseteq I_1} F_T(x_2) r_T(x_1).$$

Fix $0 < \delta < 1$ and $\{\xi_i\}_{i \in I_1}$ independent $\{0, 1\}$ -valued selectors of mean $1 - \delta$, i.e. ξ_i 's are independent $\{0, 1\}$ -valued random variables which take 0 with probability δ . Define

$$I(\omega) = \{i \in I_1 : \xi_i(\omega) = 1\}.$$

Fix x_2 . Since f_{x_2} is $\{0, 1\}$ -valued,

$$2 \int |f_{x_2} - \mathbf{E}_{I(\omega)}[f_{x_2}]|^2 dx_1 = \int |f_{x_2} - \mathbf{E}_{I(\omega)}[f_{x_2}]| dx_1. \quad (5.10)$$

where

$$\mathbf{E}_{I(\omega)}[f_{x_2}] = \sum_{T \subseteq I(\omega)} F_T(x_2) r_T,$$

and

$$f_{x_2} - \mathbf{E}_{I(\omega)}[f_{x_2}] = \sum_{T \subseteq I_1, T \not\subseteq I(\omega)} F_T(x_2) r_T. \quad (5.11)$$

Fix $1 < p < 2$. Then by applying the Bonami-Beckner inequality (Theorem 2.4.3) on the function $\sum_{i \in I_1 \setminus I(\omega)} F_{\{i\}}(x_2) r_{\{i\}}$ for fixed x_2 , we have

$$(p-1)^{\frac{1}{2}} \left(\sum_{i \in I_1 \setminus I(\omega)} |F_{\{i\}}(x_2)|^2 \right)^{1/2} \leq \left\| \sum_{i \in I_1 \setminus I(\omega)} F_{\{i\}}(x_2) r_{\{i\}} \right\|_p. \quad (5.12)$$

According to Proposition 6 of [12], the orthogonal projection of a function to the first k levels of its Walsh expansion is bounded in p -norm by C_p^k . Therefore

$$\left\| \sum_{i \in I_1 \setminus I(\omega)} F_{\{i\}}(x_2) r_{\{i\}} \right\|_p \lesssim \|f_{x_2} - \mathbf{E}_{I(\omega)}[f_{x_2}]\|_p. \quad (5.13)$$

Using Lemma 2.3.2 for $\frac{1}{p} = \frac{2/p'}{2} + \frac{1-2/p'}{1}$ we get

$$\|f_{x_2} - \mathbf{E}_{I(\omega)}[f_{x_2}]\|_p \leq \|f_{x_2} - \mathbf{E}_{I(\omega)}[f_{x_2}]\|_1^{1-2/p'} \|f_{x_2} - \mathbf{E}_{I(\omega)}[f_{x_2}]\|_2^{2/p'}, \quad (5.14)$$

and by (5.10) we have

$$\|f_{x_2} - \mathbf{E}_{I(\omega)}[f_{x_2}]\|_1^{1-2/p'} = 2 \|f_{x_2} - \mathbf{E}_{I(\omega)}[f_{x_2}]\|_2^{2(1-2/p')}. \quad (5.15)$$

Now by (5.12), (5.13), (5.14) and (5.15) we get

$$(p-1)^{\frac{1}{2}} \left(\sum_{i \in I_1 \setminus I(\omega)} |F_{\{i\}}(x_2)|^2 \right)^{1/2} \leq 2 \|f_{x_2} - \mathbf{E}_{I(\omega)}[f_{x_2}]\|_2^{2/p} = 2 \left[\sum_{T \subseteq I_1, T \not\subseteq I(\omega)} |F_T(x_2)|^2 \right]^{1/p},$$

or

$$\left[\sum_{i \in I_1} [1 - \xi_i(\omega)] |F_{\{i\}}(x_2)|^2 \right]^{p/2} \lesssim (p-1)^{-p/2} \sum_{T \subseteq I_1} [1 - \prod_{i \in T} \xi_i(\omega)] |F_T(x_2)|^2. \quad (5.16)$$

Note that $|x + y| \leq |x| + |y|$. Raising both sides to the power of r , $0 \leq r \leq 1$, we get

$$|x + y|^r \leq (|x| + |y|)^r \leq |x|^r + |y|^r,$$

where the last inequality holds since $r \leq 1$. Thus the left hand side of (5.16) is at

least

$$\begin{aligned} & \delta^{p/2} \left[\sum_{i \in I_1} |F_{\{i\}}(x_2)|^2 \right]^{p/2} - \left| \sum_{i \in I_1} (\xi_i(\omega) - (1 - \delta)) |F_{\{i\}}(x_2)|^2 \right|^{p/2} \\ & \geq \delta^{p/2} \left[\sum_{i \in I_1} |F_{\{i\}}(x_2)|^2 \right] - \left| \sum_{i \in I_1} (\xi_i(\omega) - (1 - \delta)) |F_{\{i\}}(x_2)|^2 \right|^{p/2}, \end{aligned} \quad (5.17)$$

because $\sum_{i \in I_1} |F_{\{i\}}(x_2)|^2 \leq \|f\|_2^2 \leq 1$ and $p/2 < 1$. Thus

$$\begin{aligned} \delta^{p/2} \sum_{i \in I_1} |F_{\{i\}}(x_2)|^2 & \lesssim (p-1)^{-p/2} \left[\sum_{T \subseteq I_1} [1 - \prod_{i \in T} \xi_i(\omega)] |F_T(x_2)|^2 \right] \\ & \quad + \left| \sum_{i \in I_1} (\xi_i(\omega) - (1 - \delta)) |F_{\{i\}}(x_2)|^2 \right|^{p/2}. \end{aligned} \quad (5.18)$$

Integrating (5.18) in x_2 and ω , and using the generalized Minkowski inequality (Lemma 2.3.3) we get

$$\begin{aligned} & \int \int \left| \sum_{i \in I_1} (\xi_i(\omega) - (1 - \delta)) |F_{\{i\}}(x_2)|^2 \right|^{p/2} d\omega dx_2 \\ & \leq \left[\int \int \left| \sum_{i \in I_1} (\xi_i(\omega) - (1 - \delta)) |F_{\{i\}}(x_2)|^2 \right| d\omega dx_2 \right]^{p/2}. \end{aligned}$$

To bound the last term, we use the fact that if X_1, X_2, \dots, X_n are independent random variables with mean 0, then $\mathbf{E}(|X_1 + \dots + X_n|) \leq 2\mathbf{E}\sqrt{X_1^2 + X_2^2 + \dots + X_n^2}$. It is not hard to prove this statement using some symmetrization arguments (see [32]).

We can now conclude that

$$\int \left| \sum_{i \in I_1} (\xi_i(\omega) - (1 - \delta)) |F_{\{i\}}(x_2)|^2 \right| d\omega \leq 2 \int \left(\sum_{i \in I_1} |F_{\{i\}}(x_2)|^4 (\xi_i(\omega) - (1 - \delta))^2 \right)^{1/2} d\omega.$$

Thus

$$\begin{aligned} & \int \int \left| \sum_{i \in I_1} (\xi_i(\omega) - (1 - \delta)) |F_{\{i\}}(x_2)|^2 \right|^{p/2} d\omega dx_2 \lesssim \\ & \lesssim \left(\int \int \left(\sum_{i \in I_1} (1 - \xi_i(\omega)) |F_{\{i\}}(x_2)|^4 \right)^{1/2} d\omega dx_2 \right)^{p/2}. \end{aligned} \quad (5.19)$$

Since

$$\begin{aligned} |F_{\{i\}}(x_2)| &= \left| \sum_{S \cap I_1 = \{i\}} \widehat{f}(S) r_S(x) \right| \\ &\leq \left| \sum_{|S| \leq k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right| + \left| \sum_{|S| > k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right|, \end{aligned}$$

it follows that

$$\begin{aligned} |F_{\{i\}}(x_2)|^4 &\leq \left(\left| \sum_{|S| \leq k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right| + \left| \sum_{|S| > k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right| \right)^4 \\ &\leq 16 \left| \sum_{|S| \leq k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right|^4 + 16 \left| \sum_{|S| > k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right|^4. \end{aligned}$$

Therefore we have

$$\begin{aligned} & \left(\sum_{i \in I_1} |F_{\{i\}}(x_2)|^4 (1 - \xi_i(\omega)) \right)^{1/2} \lesssim \\ & \left(\sum_{i \in I_1} \left| \sum_{|S| \leq k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right|^4 (1 - \xi_i(\omega)) + \sum_{i \in I_1} \left| \sum_{|S| > k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right|^4 (1 - \xi_i(\omega)) \right)^{1/2} \leq \\ & \left(\sum_{i \in I_1} \left| \sum_{|S| \leq k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right|^4 (1 - \xi_i(\omega)) \right)^{1/2} \\ & + \\ & \left(\sum_{i \in I_1} \left| \sum_{|S| > k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right|^4 (1 - \xi_i(\omega)) \right)^{1/2} \leq \end{aligned}$$

$$\left(\sum_{i \in I_1} \left| \sum_{|S| \leq k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right|^4 \right)^{1/2} + \sum_{i \in I_1} \left| \sum_{|S| > k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right|^2 (1 - \xi_i(\omega)).$$

Hence

$$\left(\sum_{i \in I_1} |F_{\{i\}}(x_2)|^4 (1 - \xi_i(\omega)) \right)^{1/2} \lesssim \left(\sum_{i \in I_1} \left| \sum_{|S| \leq k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right|^4 \right)^{1/2} + \sum_{i \in I_1} \left| \sum_{|S| > k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right|^2 (1 - \xi_i(\omega)).$$

Integrating both sides in x_2 and ω , using Jensen's inequality we get

$$\begin{aligned} & \iint \left(\sum_{i \in I_1} |F_{\{i\}}(x_2)|^4 (1 - \xi_i(\omega)) \right)^{1/2} d\omega dx_2 \\ & \lesssim \left(\iint \sum_{i \in I_1} \left| \sum_{|S| \leq k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right|^4 d\omega dx_2 \right)^{1/2} \\ & + \sum_{i \in I_1} \iint \left| \sum_{|S| > k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right|^2 (1 - \xi_i(\omega)) d\omega dx_2. \end{aligned} \quad (5.20)$$

Applying the Bonami-Beckner inequality (Theorem 2.4.3) on (5.20) we obtain

$$\begin{aligned} \sum_{i \in I_1} \int \left| \sum_{|S| \leq k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right|^4 dx_2 &= \sum_{i \in I_1} \left\| \sum_{|S| \leq k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right\|_4^4 \\ &\leq 3^{2k} \sum_{i \in I_1} \left\| \sum_{|S| \leq k, S \cap I_1 = \{i\}} \widehat{f}(S) r_S \right\|_2^4. \end{aligned}$$

Recall that $\int (1 - \xi_i(\omega)) d\omega = \delta$. Therefore, from (5.20) we have:

$$\begin{aligned} & \iint \left(\sum_{i \in I_1} |F_{\{i\}}(x_2)|^4 (1 - \xi_i(\omega)) \right)^{1/2} d\omega dx_2 \\ & \leq 3^k \left[\sum_{i \in I_1} \left(\sum_{|S| \leq k, S \cap I_1 = \{i\}} |\widehat{f}(S)|^2 \right)^2 \right]^{1/2} + \delta \sum_{|S| > k} |\widehat{f}(S)|^2 \\ & \leq 3^k \max_{i \in I_1} \left(\sum_{|S| \leq k, i \in S} |\widehat{f}(S)|^2 \right)^{1/2} + \delta \sum_{|S| > k} |\widehat{f}(S)|^2 \end{aligned} \quad (5.21)$$

Note that in (5.21) since f is a Boolean function, $\sum_{i \in I_1} \sum_{|S| \leq k, S \cap I_1 = \{i\}} |\widehat{f}(S)|^2 \leq 1$, which implies the last inequality. We now use the fact that $I_1 \cap I_0 = \emptyset$, so by definition of I_0 we have

$$\int \int \left(\sum_{i \in I_1} |F_{\{i\}}(x_2)|^4 (1 - \xi_i(\omega)) \right)^{1/2} d\omega dx_2 < 3^k \kappa^{1/2} + \delta \sum_{|S| > k} |\widehat{f}(S)|^2. \quad (5.22)$$

Now integrate both sides of (5.18) in x_2 and ω , then substitute (5.22) and (5.19).

Therefore we get:

$$\begin{aligned} \int \int \delta^{p/2} \sum_{i \in I_1} |F_{\{i\}}(x_2)|^2 dx_2 d\omega &\lesssim \int \int (p-1)^{-p/2} \left[\sum_{T \subseteq I_1} [1 - \prod_{i \in T} \xi_i(\omega)] |F_T(x_2)|^2 \right] dx_2 d\omega \\ &+ \left(3^k \kappa^{1/2} + \delta \sum_{|S| > k} |\widehat{f}(S)|^2 \right)^{p/2}. \end{aligned}$$

We also know that $\frac{p}{2} < 1$. Thus,

$$\begin{aligned} \delta^{p/2} \int \int \sum_{i \in I_1} |F_{\{i\}}(x_2)|^2 dx_2 d\omega &\lesssim \\ (p-1)^{-p/2} \left[\sum_{T \subseteq I_1} \int [1 - \prod_{i \in T} \xi_i(\omega)] d\omega \right] &\left[\int |F_T(x_2)|^2 dx_2 \right] \\ &+ \\ (3^k \kappa^{1/2})^{p/2} + \delta^{p/2} \left(\sum_{|S| > k} |\widehat{f}(S)|^2 \right)^{p/2}. \end{aligned}$$

Recalling $|F_{\{i\}}(x_2)| = |\sum_{S \cap I_1 = \{i\}} \widehat{f}(S) r_S(x_2)|$, we now use Parseval's equality to write:

$$\begin{aligned} \delta^{p/2} \sum_{|S \cap I_1| = 1} |\widehat{f}(S)|^2 &\lesssim (p-1)^{-p/2} \sum_S [1 - (1-\delta)^{|S \cap I_1|}] |\widehat{f}(S)|^2 \\ &+ \delta^{p/2} \left(\sum_{|S| > k} |\widehat{f}(S)|^2 \right)^{p/2} + (3^k \kappa^{1/2})^{p/2}. \end{aligned} \quad (5.23)$$

Recall that $(1-\delta)^n \geq 1 - \delta^n$. Now we can estimate

$$\begin{aligned} 1 - (1-\delta)^{|S \cap I_1|} &\leq \delta |S \cap I_1| && \text{if } |S \cap I_0| \leq k, \\ &< 1 && \text{otherwise.} \end{aligned}$$

Thus

$$\begin{aligned}
\delta^{p/2} \sum_{|S \cap I_1|=1} |\widehat{f}(S)|^2 &\lesssim (p-1)^{-p/2} \delta \sum_{|S \cap I'_0| \leq k} |S \cap I_1| |\widehat{f}(S)|^2 \\
&+ (p-1)^{-p/2} \sum_{|S| > k} |\widehat{f}(S)|^2 \\
&+ \delta^{p/2} \left(\sum_{|S| > k} |\widehat{f}(S)|^2 \right)^{p/2} + (3^k \kappa^{1/2})^{p/2}.
\end{aligned} \tag{5.24}$$

We will now specify the set $I_1 \subseteq I'_0$. Fix $0 \leq t_0 \leq \log k$ and let $I_1 = I_{\omega'}$ be a random subset of I'_0 , choosing each element of I'_0 independently with probability $10^{-3}2^{-t_0}$. This ensures that if $2^{t_0} \leq |S \cap I'_0| < 2^{t_0+1}$, then

$$\Pr[|S \cap I_1| = 1] > 10^{-4}. \tag{5.25}$$

Also

$$\mathbf{E}_{\omega'}[|S \cap I_1|] = 10^{-3}2^{-t_0}|S \cap I'_0|. \tag{5.26}$$

Now recalling the definition (5.8) of ρ_t and using (5.25) we have:

$$\begin{aligned}
\rho_{t_0} &= \sum_{2^{t_0} \leq |S \cap I'_0| < 2^{t_0+1}} |\widehat{f}(S)|^2 \\
&\leq 10^4 \sum_{2^{t_0} \leq |S \cap I'_0| < 2^{t_0+1}} |\widehat{f}(S)|^2 \Pr[|S \cap I_1| = 1] \\
&\leq 10^4 \sum_S |\widehat{f}(S)|^2 \Pr[|S \cap I_1| = 1] \\
&= 10^4 \mathbf{E}_{\omega'} \left[\sum_{|S \cap I_1|=1} |\widehat{f}(S)|^2 \right],
\end{aligned} \tag{5.27}$$

and using (5.26) we have:

$$\begin{aligned}
\mathbf{E}_{\omega'} \left[\sum_{|S \cap I'_0| \leq k} |S \cap I_1| |\widehat{f}(S)|^2 \right] &= \sum_{|S \cap I'_0| \leq k} |\widehat{f}(S)|^2 \mathbf{E}_{\omega'}[|S \cap I_1|] \\
&= \sum_{|S \cap I'_0| \leq k} |\widehat{f}(S)|^2 10^{-3}2^{-t_0} [|S \cap I'_0|] \\
&\leq \sum_{t \leq \log k} \sum_{2^t \leq |S \cap I'_0| < 2^{t+1}} |\widehat{f}(S)|^2 10^{-3}2^{t-t_0+1} \\
&= 2 \times 10^{-3} \sum_{t \leq \log k} 2^{t-t_0} \rho_t.
\end{aligned} \tag{5.28}$$

Applying $\mathbf{E}_{\omega'}$ to (5.24) and using (5.27) and (5.28), we get

$$\begin{aligned} \delta^{p/2} \rho_{t_0} &\lesssim (p-1)^{-p/2} \delta \left(\sum_{t \leq \log k} 2^{t-t_0} \rho_t \right) + (p-1)^{-p/2} \sum_{|S| > k} |\widehat{f}(S)|^2 \\ &\quad + \delta^{p/2} \left(\sum_{|S| > k} |\widehat{f}(S)|^2 \right)^{p/2} + (3^k \kappa^{1/2})^{p/2}. \end{aligned} \quad (5.29)$$

In order to have the left term in (5.29) larger than the first term on the right, take

$$\delta \sim (p-1)^{p/(2-p)} \left(\frac{2^{t_0} \rho_{t_0}}{\sum_{t \leq \log k} 2^t \rho_t} \right)^{2/(2-p)}. \quad (5.30)$$

Taking

$$\kappa = 10^{-k} \quad (5.31)$$

to make the last term in (5.29) negligible, Condition (5.6) is satisfied. Recall that $a + b \leq 2 \max\{a, b\} \lesssim \max\{a, b\}$, so (5.29) and (5.30) imply

$$\sum_{|S| > k} |\widehat{f}(S)|^2 \gtrsim \min \left\{ (p-1)^{p/(2-p)} \left(\frac{2^{t_0} \rho_{t_0}}{\sum_{t \leq \log k} 2^t \rho_t} \right)^{p/(2-p)}, \rho_{t_0}, \rho_{t_0}^{2/p} \right\}. \quad (5.32)$$

Here $0 \leq t_0 \leq \log k$ and $1 < p < 2$ are arbitrary.

We distinguish two cases.

Case 1:

$$\sum_{t \leq \log k} 2^t \rho_t < \sqrt{k}. \quad (5.33)$$

Recalling (5.9), we may take $0 \leq t_0 \leq \log k$ such that

$$\rho_{t_0} \geq \frac{\gamma^2}{2} \left(\frac{1}{\log k} \right),$$

or equivalently

$$\rho_{t_0} \gtrsim 1/\log k. \quad (5.34)$$

Take in (5.32)

$$p = 1 + \epsilon_1 / \log k. \quad (5.35)$$

From (5.33), (5.34) and (5.35),

$$(5.32) \gtrsim \min((\log k)^{-2}k^{-1/2}, (\log k)^{-2}) \gtrsim (\log k)^{-2}k^{-1/2-2\epsilon} \gtrsim k^{-1/2-\epsilon}, \quad (5.36)$$

for appropriate choice of ϵ_1 .

Case 2:

$$\sum_{t \leq \log k} 2^t \rho_t \geq \sqrt{k}.$$

Choose t_0 s.t.

$$2^{t_0} \rho_{t_0} > \frac{1}{\log k} \sum_{t \leq \log k} 2^t \rho_t > \frac{\sqrt{k}}{\log k}, \quad (5.37)$$

hence

$$\rho_{t_0} > (\log k)^{-1}k^{-1/2}. \quad (5.38)$$

Now let p tend to 2 in (5.32). We get for every ϵ

$$(5.32) \gtrsim \min((\log k)^{-2/(2-p)-1}k^{-1/2}, (\log k)^{-2}k^{-1/p}) > c_\epsilon k^{-1/2-\epsilon}. \quad (5.39)$$

Thus (5.2) follows from (5.36) and (5.39). ■

Corollary 5.1.2 *Let $f = \chi_A$, $A \subseteq \{-1, 1\}^N$ satisfying*

$$|A|(1 - |A|) > \frac{1}{10}. \quad (5.40)$$

Let k be an integer and assume

$$\max_{|S| \leq k} |\widehat{f}(S)|^2 < 4^{-k^2-1}. \quad (5.41)$$

Then for every $\epsilon > 0$

$$\sum_{|S| > k} |\widehat{f}(S)|^2 \gtrsim k^{-1/2-\epsilon}. \quad (5.42)$$

Proof. From (5.40) we have

$$\frac{1 - \sqrt{\frac{3}{5}}}{2} \leq |A| \leq \frac{1 + \sqrt{\frac{3}{5}}}{2}.$$

Now suppose (5.42) does not hold. So for some $\epsilon > 0$ we have

$$\sum_{|S|>k} |\widehat{f}(S)|^2 \leq k^{-1/2-\epsilon},$$

which implies that

$$\sum \{|\widehat{f}(S)|^2 : |\widehat{f}(S)| < 4^{-k^2-1}\} \geq \sum_{|S|\leq k} |\widehat{f}(S)|^2 \geq \frac{1 - \sqrt{\frac{3}{5}}}{2} - k^{-1/2-\epsilon} > \frac{1}{16}.$$

Using Theorem 5.1.1 for $\gamma = \frac{1}{4}$, we obtain $\sum_{|S|>k} |\widehat{f}(S)|^2 \gtrsim k^{-1/2-\epsilon}$ for every $\epsilon > 0$. ■

Suppose that a function f does not satisfy (5.2). Since there are at most $\frac{4^{2k^2}}{\gamma^2}$ values for S such that $|\widehat{f}(S)| > \gamma 4^{-k^2}$, Theorem 5.1.1 implies that there exists a function g which depends on at most $\frac{4^{2k^2}}{\gamma^2}$ coordinates, and $\|f - g\|_2^2 \leq \gamma^2$.

A closer look at the proof of Theorem 5.1.1 will lead to a better result. By (5.31) and (5.4) we have $|I_0| \leq k10^k$. The analysis of Case 1 and Case 2 in the proof of Theorem 5.1.1 shows that if f does not satisfy (5.2), then (5.9) is not satisfied. This together with (5.3) imply

$$\|f - \sum_{S \subseteq I_0} \widehat{f}(S)r_S\|_2^2 \leq \gamma^2.$$

Note that $g = \sum_{S \subseteq I_0} \widehat{f}(S)r_S$ depends only on the variables in I_0 . By rounding g to the nearest element in $\{0, 1\}$ we obtain the following corollary.

Corollary 5.1.3 *Let $f : \mathbb{Z}_2^N \rightarrow \{0, 1\}$ be a Boolean function, and let k be a positive integer and $\epsilon, \gamma > 0$ any fixed constants. Then there is a constant $c_{\epsilon, \gamma}$, such that if $\sum_{|S|>k} |\widehat{f}(S)|^2 < c_{\epsilon, \gamma} k^{-1/2-\epsilon}$, then there exists a Boolean function $h : \mathbb{Z}_2^N \rightarrow \{0, 1\}$ which depends on $k10^k$ variables, and for which $\|f - h\|_2^2 \leq \gamma$.*

Theorem 5.1.1 turns out to be basically sharp.

Theorem 5.1.4 *There is a function f which satisfies the following two conditions:*

$$\sum \{|\widehat{f}(S)|^2 : |\widehat{f}(S)| < \gamma 4^{-k^2}\} > \gamma^2, \quad (5.43)$$

and

$$\sum_{|S|>k} |\widehat{f}(S)|^2 \sim k^{-1/2}. \quad (5.44)$$

Proof. Take f to be the “majority function” which we define as the $\{0, 1\}$ -valued function

$$f(\epsilon) = \chi_{[\epsilon_1 + \epsilon_2 + \dots + \epsilon_N > N/2]}$$

on \mathbb{Z}_2^N . It is known (see [30]) that in this case

$$|\widehat{f}(S)|^2 \sim \binom{N}{k}^{-1} k^{-3/2} \quad \text{for } |S| = k > 0.$$

Hence

$$\sum_{|S|=k} |\widehat{f}(S)|^2 \sim k^{-3/2},$$

and

$$\sum_{|S|>k} |\widehat{f}(S)|^2 \sim k^{-1/2}.$$

■

Chapter 6

Concluding remarks

In this thesis we studied the independent sets of K_r^n which are close to be of maximum size. We reviewed the proof of the previously known result, Theorem 3.0.7. We improved that result in Theorem 4.1.1. The proof involves some techniques from Fourier analysis of Boolean functions on \mathbb{Z}_r^n .

In order to prove Theorem 4.1.1, we studied Boolean functions on \mathbb{Z}_r^n for which most of the 2-norm weight of the Fourier expansion is concentrated on the first two levels. Lemma 4.2.1 asserts that every such function can be approximated by a Boolean function that depends only on one coordinate. One possible generalization of this lemma would be to show that a Boolean function on \mathbb{Z}_r^n whose Fourier expansion is concentrated on the first l levels for some constant l , can be approximated by a Boolean function that depends on $k(l)$ coordinates, for some function $k(l)$. Analogues of this for \mathbb{Z}_2^n have been proven in [14] and [34].

Consider a graph G whose vertices are the elements of the symmetric group S_n , and two vertices π and π' are adjacent if $\pi(i) \neq \pi'(i)$ for every $1 \leq i \leq n$. For every $1 \leq i, j \leq n$ the set S_{ij} of the vertices π satisfying $\pi(i) = j$ forms an independent set of size $(n-1)!$. Recently Cameron and Ku [18] have proved that these sets are the only maximum independent sets of this graph. Similar results have been proven for

generalizations of this graph in [35]. Cameron and Ku made the following conjecture:

Conjecture 6.0.5 [18] *There is a constant c such that every independent set of size at least $c(n-1)!$ is a subset of an independent set of size $(n-1)!$.*

One might notice the similarity of Conjecture 6.0.5 and Corollary 4.1.2 for $r = n$. Despite this similarity we are not aware of any possible way to apply the techniques used in this paper to the problem. Since S_n is not Abelian, the methods of the present thesis fail to apply directly to this problem. So an answer to Conjecture 6.0.5 or its analogues for the graphs studied in [35] (which do not even have a group structure) might lead to new techniques.

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