Stochastic Dominance Option Pricing: An Alternative Paradigm

Ioan Mihai Oancea

A Thesis

in

The John Molson School

of

Business

Presented in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy at Concordia University
Montreal, Quebec, Canada

April 2006

Your file Votre référence
ISBN: 978-0-494-16306-1
Our file Notre référence
ISBN: 978-0-494-16306-1

Abstract

Stochastic Dominance Option Pricing:

An Alternative Paradigm

Ioan Mihai Oancea, Ph.D.

Concordia University

This thesis examines the pricing of options under several models with market incompleteness. The theoretical approach relies on the absence of stochastically dominating portfolios containing the underlying asset, the option and the riskless bond. The stochastic dominance approach provides two bounds on the equilibrium pricing of options by risk-averse investors. The two bounds are discounted conditional expectations of the option payoff under two probability measures.

This research generalizes the previous stochastic dominance pricing results in discrete time to non-i.i.d. underlying asset return processes and to contingent claims with non-convex payoffs. The new results are then used to examine the stochastic dominance pricing bounds for several discrete and continuous time processes of the underlying asset.

The continuous time bounds are obtained by constructing a sequence of discrete approximations that converge weakly to a given continuous time process. The weak convergence property provides the convergence of the two option bounds, which are discounted expectations of the option payoff. In the case of a univariate diffusion process, the two option bounds converge to a common limit. The two bounds converge to distinct limits when the underlying asset follows a jump-diffusion mixture.

The non-iid stochastic dominance pricing results are then applied to the pricing of options for a GARCH specification of the underlying asset returns. The two stochastic dominance bounds are obtained both for conditional normal and non-normal returns. The impact of the model estimation error is examined by generating a return sample from a known model and computing the stochastic dominance bounds implied by several estimated models.

Acknowledgements

I would like to express my gratitude to my thesis supervisor Professor Stylianos Perrakis. His contribution to the success of my thesis starts with his introduction of stochastic dominance as an option pricing paradigm more than two decades ago. His dedication to this topic and his guidance helped me succeed in my research. I would also like to thank Professors Peter Christoffersen, Sergey Isaenko and Jin-Chuan Duan for their feedback and comments, which strengthened the contribution of my thesis. A special mention goes to Professors Geneviève Gauthier and Benjamin Croitoru, whose courses helped me master the mathematics and the economic theory underlying my research.

Contents

List of Figures

List of Tables

List of Notations

PDE	Partial differential equation
SDE	Stochastic differential equation
RNVR	Risk-neutral valuation relationship
LRNVR	Local risk-neutral valuation relationship
CRRA	Constant relative risk aversion
GARCH	Generalized autoregressive conditional heteroscedasticity
NGARCH	Nonlinear asymmetric GARCH
FSD	First degree stochastic dominance
SSD	Second degree stochastic dominance
$(\Omega, \mathcal{F}, P)$	Probability space
S_t	Asset price at time t
K	Option strike price
$g(S_T)$	Option payoff
$\hat{z}_i$	Expectation of the lowest i outcomes of the multinomial random variable z
$\bar{C}$	Upper option bound
$\underline{C}$	Lower option bound
U	Upper bound pricing distribution
L	Lower bound pricing distribution
$E^U[X]$	Expectation of X under probability measure U

Chapter 1

Review of the Literature

Option pricing has been one of the most productive fields of financial economics and no review can be fair to all the contributors of this field. As an introduction to the stochastic dominance approach to option pricing, which is the main topic of this thesis, this literature review will present the main approaches used in option pricing - the arbitrage and the equilibrium approach - stressing the aspects of these approaches that are shared with the stochastic dominance approach. This review will also present the weaknesses of the arbitrage and equilibrium approaches when the markets are incomplete. These weaknesses are addressed by the stochastic dominance approach, which is the topic of this thesis.

1.1 The arbitrage pricing approach

The first approach used in option pricing relied on the construction of a strategy that replicated the option payoff, by the continuous rebalancing of a portfolio invested in the underlying asset and a riskless bond. The absence of arbitrage opportunities requires that the replicating portfolio have the same value as the option. An equivalent method is to construct a riskless hedge by the continuous rebalancing of a portfolio containing the option and the underlying asset. In this case, the absence of arbitrage requires that the rate of return on the hedge equals the riskless interest rate. This continuous hedging argument underlies the partial differential equations obtained by Black and Scholes (1973) or Merton (1973), which express such equalities implied by the absence of arbitrage opportunities. The option pricing formula obtained in the two above mentioned papers is the solution of the partial differential equation with a terminal condition given by the option payoff at maturity.

A breakthrough in the arbitrage pricing approach was the risk neutral pricing established by Cox and Ross (1976) through an economic insight and later by Harrison and Kreps (1979) and Harrison and Pliska (1981) through a probabilistic interpretation. Cox and Ross (1976) argued that the option price can be obtained as a risk-adjusted expectation of the payoff. But, since the arbitrage approach yields a unique price that is independent of the investor's preferences, the option price can be obtained, without loss of generality, as the expected option payoff from the point of view of a risk-neutral investor. The risk-neutral pricing approach reduces the option pricing problem to the calculation of an expectation, and the pricing of options on a variety of stochastic processes becomes mathematically tractable.

The formal proofs of the risk-neutral pricing approach were obtained by Harrison and Kreps (1979) and Harrison and Pliska (1981). The main result of these two papers is a theorem that proves the equivalence of the absence of arbitrage opportunities with the existence of a probability measure under which the contingent claims can be priced as discounted expectations of their payoffs. Under such a probability measure, the discounted price processes of all assets in the market are martingales. The risk neutral pricing of a contingent claim starts by finding a probability measure under which the discounted price processes of all assets in the market are martingales.

In a discrete time economy, the martingale measure is the solution of a linear system of equations, which reflect the martingale property of the prices. In continuous time, a risk-neutral price process can be obtained by adjusting the actual process with risk premia associated to the sources of uncertainty in the asset price.

The findings of Harrison and Kreps (1979) and Harrison and Pliska (1981) show that the arbitrage methodology relies on two fundamental assumptions that cannot be relaxed easily in most applications: the dynamic completeness of the markets and the absence of trading costs. The former assumption provides that all contingent claim payoffs can be manufactured from assets available in the market. The latter assumption reflects the requirement that the options are replicated using a self-financing strategy. That is, no cash can enter or exit the strategies and all the proceeds from selling one asset must be reinvested in the strategy.

In principle, a contingent claim is attainable if the market contains a sufficiently large number of securities. Breeden and Litzenberger (1978) proved that a market can be completed with an infinite number of call options spanning all possible strike prices. In practice, it is desirable that the option be replicated using only the underlying asset and a riskless bond. In this case, the completeness of the market can be ascertained from the stochastic process of the underlying asset.

In discrete time, a stock following a binomial price process provides a complete market. In the binomial tree model of Cox, Ross, and Rubinstein (1979), the risk neutral probabilities are the solution of two equations with two unknowns. Conversely, a stock following a trinomial price process renders the market incomplete, as will be shown in the next chapter. In continuous time, a stock price described by a univariate diffusion provides a complete market, as shown by Harrison and Pliska (1981). Other processes, such as the stochastic volatility specification of Hull and White (1987) or the jump diffusion specification of Merton (1976) do not satisfy the market completeness requirement.

The empirical rejection of the Black and Scholes (1973) and other univariate diffusion models - for instance the constant elasticity of variance model of Cox and Ross (1976) or the compound option model of Geske (1979) - redirected the option pricing research to more complex specifications, such as the stochastic volatility models of Hull and White (1987), Johnson and Shanno (1987), Heston (1993) or the stochastic volatility jump-diffusion mixtures of Bates (1996) or Bakshi, Cao, and Chen (1997). In fact, starting from the paper of Heston (1993), an entire strand of the option pricing literature specialized in the pricing of options on various processes using a Fourier transform technique. Duffie, Pan, and Singleton (2000) applied this technique to a large class of multivariate affine models that nests the previous stochastic volatility and jump models.

All these studies adopted the arbitrage methodology for its tractability, but did not always address the market incompleteness issue. Hull and White (1987) assumed that the volatility risk is diversifiable, but this assumption was later refuted by the study of Lamoureux and Lastrapes (1993), who found evidence that the volatility risk was priced. The pursuant papers adopted risk neutral models that included a price of volatility risk. In such models, the infinite number of possible risk neutral processes was reduced to a single process with parameters implied by the cross-section of option prices, under the assumption that the risk neutral model is correctly specified.

This approach is hard to support empirically, since the change of probability measure from the actual return distribution to the risk-neutral requires some relationships to be satisfied between the process parameters under the two probability measures. The studies of Bates (1996) and Bakshi, Cao, and Chen (1997) could not reconcile the parameters of the risk-neutral process estimated from the cross-section of option prices with the actual process parameters estimated from the time series of the returns. Pan (2002) performed a joint estimation of a jump-diffusion from the time series of the S&P500 index and the cross-section of options. She found that the price of the jump risk is much higher than the price of diffusion risk, which is hard to reconcile with the preferences of rational investors. Chernov (2003) used several asset classes - the S&P500 index, individual equities, T-bills and gold futures - to estimate the pricing kernel, which must be common to all assets to avoid arbitrage. The asset return dynamics was described by a multivariate stochastic volatility diffusion, with assets exposed both to systematic and specific risk. The estimated kernel is not consistent with a time-separable utility function.

Related to the estimation of a risk-neutral stochastic process from option prices is the nonparametric estimation of the entire risk neutral probability distribution from the cross-section of option prices. Rubinstein (1994) found the closest distribution to a lognormal prior, which prices the options correctly. Aït-Sahalia and Lo (1998) fit a kernel regression to option prices and obtained the risk neutral probability density as the second derivative of the option price with respect to the strike price - a relation established by Breeden and Litzenberger (1978). See the monograph of Jackwerth (2004) for a survey of these methods. Still, these risk neutral probabilities could not be reconciled with the actual returns distribution, unless the risk aversion had weird patterns. The studies of Aït-Sahalia and Lo (2000) and Jackwerth (2000) documented such findings.

1.2 The equilibrium pricing approach

Equilibrium pricing has been an alternative to the arbitrage approach from the very beginning of the option pricing literature. Black and Scholes (1973) provided an alternative derivation of their equation as an application of the capital asset pricing model in a continuous time economy.

The power of an equilibrium methodology was revealed in the studies of Rubinstein (1976) and Brennan (1979), who obtained the Black-Scholes formula in a two-period economy, without intermediate trading between the purchase and the maturity of the option. Their result was based on the valuation of the option payoff by rational investors and relied on assumptions on the representative investor's preferences and the distribution of the asset price at the expiration of the option in the second period. The law of one price provides that the value of the asset at time zero is the expectation of its second period payoff, multiplied by a random quantity called pricing kernel, state price or stochastic discount factor. This quantity is the marginal rate of substitution between a unit of first period consumption and a unit of random second period consumption and depends on the investor's utility function.

Rubinstein (1976) and Brennan (1979) found that, under certain combinations of assumptions on the asset return distribution and the investor's utility, the equilibrium option price is identical to the arbitrage price obtained under a risk neutral approach. Rubinstein (1976) obtained the Black-Scholes pricing formula under the joint assumptions of lognormal returns and constant proportional risk aversion (CPRA) preferences of the representative investor. Brennan (1979) found a larger class of distributions and pricing kernels that satisfy these so-called risk neutral valuation relationships (RNVR). Such relationships make the equilibrium pricing very tractable. Although an assumption is made on the functional form of the utility, the option pricing formula is preference free, since the risk aversion parameter vanishes from this formula.

The equilibrium pricing approach was extended to a multiperiod setting in the paper of Amin and Ng (1993). They applied this approach to the pricing of options on assets that feature stochastic volatility or jumps, whether or not a part of the stochastic volatility or jump risk can be diversified. Their results relied on the joint lognormality of the returns and consumption processes on the one hand, and the CPRA preferences of the representative investor on the other hand.

A similar equilibrium argument underlies all the GARCH option pricing models in the literature. Duan (1995) found that, under CPRA preferences and conditional lognormal returns, a so-called local risk neutral valuation relationship (LRNVR) provides the existence of a risk-neutral probability measure under which the options can be priced as discounted expectations of their payoffs. An important property of GARCH models is their convergence to stochastic volatility models, proved by Nelson (1990) and Duan (1997). Duan, Ritchken, and Sun (2005) developed a class of GARCH models that converge to various jump-diffusions, with jumps both in returns and in volatility. All these contributions make the GARCH models very attractive for option pricing, since they are much easier to estimate.

The equilibrium approach can also be used in the pricing of options in continuous time. In this case, the tractability is obtained by matching the stochastic process of the pricing kernel with the asset price process. For instance, the Fourier transform technique applied by Duffie, Pan, and Singleton (2000) to the pricing of options on affine processes can be applied in an equilibrium framework if the pricing kernel has itself an affine specification.

Unless they assume a CRRA utility and normal returns, equilibrium models require some information about the parameters of the utility function, such as the risk aversion parameter. This renders the equilibrium methodology problematic, as there exists no reliable way to estimate the pricing kernel. Unless other options are used in the estimation procedure, the utility function has to be estimated from consumption measurements. The inconsistencies between utility functions obtained in this way and asset prices have been documented by the equity premium puzzle literature.

The few cases in which a (local) risk neutral valuation relationship holds are too restrictive.

1.3 Option pricing in incomplete markets

The equilibrium methodology can give a unique option price under market incompleteness, if the absence of arbitrage assumption is complemented with assumptions about the investor preferences. Another strand of the literature focused on the derivation of price intervals for the contingent claims. Merton (1973) used absence of arbitrage arguments to obtain several important results regarding the pricing of options, such as inequalities that have to be satisfied by the option prices. These option pricing bounds are not very useful, since they are very wide.

Perrakis and Ryan (1984) and Perrakis (1986) obtained tighter bounds by finding conditions under which two portfolios containing the underlying asset, the option and the riskless bond stochastically dominate each other. This criterion yields two bounds on the prices at which all risk averse investors would agree to buy or sell the option. Their results were derived within the equilibrium framework of Rubinstein (1976) and Brennan (1979), using the extra assumption of a non-increasing ordering of the stochastic discount factor as a function of the price change. This assumption relaxes the standard CRRA or CARA assumptions of the equilibrium framework, while making no assumption on the distribution of the price process.

The results of Perrakis and Ryan (1984) and Perrakis (1986) were dependent on the choice of portfolios used in the stochastic dominance assessment. In the second paper, a better portfolio choice resulted in tighter bounds. Levy (1985) used a general stochastic dominance result that applies to all the possible portfolios containing the stock, the bond and the option. These bounds are the tightest possible bounds implied by a risk aversion assumption, though Perrakis (1986) obtained a tighter upper bound in the case when the next period asset price is strictly positive.

Ritchken (1985) expressed the arbitrage and equilibrium option pricing problems as linear programming problems. He obtained the Merton (1973) bounds as the solution of the arbitrage problem and the Levy (1985) as the solution of the equilibrium problem. The linear programming approach was extended to multiperiod models by Ritchken and Kuo (1988) and to the n-th degree stochastic dominance and decreasing absolute risk aversion (DARA) preferences by Ritchken and Kuo (1989). The upper bound remains the same as the second degree stochastic dominance bound. Mathur and Ritchken (1999) found the lower pricing bound under DARA and decreasing relative risk aversion (DRRA) preferences in a single period economy. The linear programming approach is the most tractable manner to obtain stochastic dominance bounds and will be presented in detail in the next chapter.

When the number of trading period increases, the option pricing bounds become tighter. Perrakis (1988) and Perrakis (1993) studied the convergence of the option bounds as the number of trading periods increases to infinity. When the underlying asset follows a Black-Scholes diffusion, both option bounds converge to the Black-Scholes price. The bounds converge to distinct values in the case of jump-diffusions.

As an alternative to these risk-aversion bounds, other studies obtain risky arbitrage option bounds. The risky arbitrage methodology rules out investment opportunities that are very attractive to the representative investor. Cochrane and SaaRequejo (2000) measured the attractiveness of an investment by its Sharpe ratio, while Bernardo and Ledoit (2001) defined a gain-loss ratio as a measure of the investment attractiveness. Another approach pursued by Carr and Madan (2001) was to define a set of probability measures and floors that had to be exceeded by the expected payoff, in order for an investment to be considered acceptable. The latter approach has been applied in a single period setting. None of these approaches provides a clear cut criterion for the attractiveness of an investment. In contrast, stochastic dominance is an economic criterion that can be verified. The violation of the stochastic dominance upper (lower) bound would permit an investor to increase utility by writing (purchasing) the option.

Chapter 2

The Stochastic Dominance Approach

In an economy with incomplete markets, options are not redundant securities and the replication argument used by the arbitrage pricing methodology fails to provide a unique price. The purpose of this chapter is to present an alternative class of models that were designed to handle market incompleteness and were more recently used in the pricing of options under transaction costs. These models rely on the absence of stochastically dominating strategies involving the available assets.

Stochastic dominance provides an ordering for risky assets. It is said that risky asset A dominates asset B in the sense of First degree stochastic dominance (FSD) if all investors with increasing utility functions prefer A to B. Asset A dominates B in the sense of Second degree stochastic dominance (SSD) if all risk averse investors prefer A to B.[1]

Perrakis and Ryan (1984) introduced this class of option pricing model as an extension of the equilibrium framework of Rubinstein (1976) and Brennan (1979) under very weak preference assumptions. Their option bounds relied on a result obtained by Perrakis and Ryan (1984), which provided conditions for the stochastic

[1]For a discussion of stochastic dominance, see for instance Chapter 2 of Huang and Litzenberger (1988).

dominance between the underlying asset on the one hand and a portfolio containing the option and the riskless asset on the other hand.[2] This result can be applied recursively to provide option bounds in a multiperiod economy.

Levy (1985) used a more general stochastic dominance criterion, which provided the tightest possible bounds in a single period economy, but could not be extended to a multiperiod setting.

A breakthrough in the stochastic dominance pricing of options was obtained by Ritchken (1985), who expressed the option pricing problem as a linear program with constraints given by the structure of preferences. This approach was extended to a multiperiod setting by Ritchken and Kuo (1988) and to more complex preferences by Ritchken and Kuo (1989). The linear programming formulation has been the workhorse of all the further applications of the stochastic dominance methodology in a market incompleteness setting.

An important application of the stochastic dominance pricing approach is the pricing of options in the presence of transaction costs.[3] This has been done done successfully using the portfolio construction approach of Perrakis and Ryan (1984), but none of the alternative methods could provide a solution to this problem so far.

[2]Perrakis and Ryan (1984) considered the following portfolios in their analysis

Portfolio A: one share of the underlying asset at price S_0.

Portfolio B: one call option at price C_0 and $S - C_0$ invested in the riskless asset.

Portfolio C: $\frac{S_0}{C_0}$ call options at price C_0 each.

Portfolios B and C can be obtained by varying the weights in a portfolio containing the riskless asset and the option. Perrakis (1986) analyzed the absence of stochastic dominance between the latter portfolio and the underlying asset

[3]See Constantinides and Perrakis (2002), which shows that the Perrakis-Ryan upper bound remains essentially unchanged if trading costs are introduced in trading the underlying asset. This paper also provides a tight lower bound for European put options under transaction costs. An equally tight upper bound holds for American call options and a tight lower bound holds for American put options. See Constantinides and Perrakis (2006).

2.1 Stochastic dominance option pricing in a two-period economy

2.1.1 First degree stochastic dominance

Ritchken (1985) studied the pricing of an option in a two-period economy, when the underlying asset price is a random variable with a multinomial distribution. The underlying asset has the price S_0 at time 0 and will take a random value S_T, which will be revealed in the second period at time T. The n possible states of the economy in the second period are described by the asset price outcomes $s_1 \leq s_2 \leq \ldots \leq s_n$, which are known at time 0. There is a riskless bond with price B_0 at $t = 0$, which will take the same value of one unit of consumption in any of the second period states.

The pricing problem consists in finding the price at time $t = 0$ of a contingent claim that pays $C_T = g(S_T)$ when it expires in the second period. The payoff of a call option with exercise price K is $g(S_T) = max(S_T - K, 0)$, while the payoff of a put option with the same exercise price is $g'(S_T) = max(K - S_T, 0)$. Denote $c_j = g(s_j)$ the amount paid by the claim in state j. Using the state price methodology introduced by Arrow (1964), each state in the second period can be assigned a fictitious security called state contingent claim, which pays one unit of consumption if the state is realized and zero otherwise. Denote $d_1, d_2, \ldots d_n$ the time zero prices of these claims, also called state prices. The law of one price implies that the underlying asset, the bond and the option have respectively the prices

$$\begin{aligned} S_0 &= \sum_{j=1}^{n} s_j d_j, \\ B_0 &= \sum_{j=1}^{n} d_j, \\ C_0 &= \sum_{j=1}^{n} c_j d_j. \end{aligned} \tag{2.1}$$

If $n = 2$, the two state prices can be derived uniquely from the stock and bond pricing equations. The unique option price can then be obtained by substituting this solution into the option pricing equation. This complete market example is a single period version of the binomial option pricing model derived by Cox, Ross, and Rubinstein (1979).

If $n \geq 3$, the stock and the bond pricing equation are not sufficient to determine the state prices. This is an obvious example of market incompleteness, in which the arbitrage methodology is not able to provide an option price. We use this setting to introduce the linear programming approach of Ritchken (1985). That study assumed that the underlying asset and the riskless bond are correctly priced. The option can take any value in an interval obtained by solving two linear programs:

$$\min(\max)\ C_0 = \sum_{j=1}^{n} c_j d_j \tag{2.2}$$

subject to:

$$S_0 = \sum_{j=1}^{n} s_j d_j$$

$$B_0 = \sum_{j=1}^{n} d_j$$

$$d_j \geq 0, \forall j = 1, \ldots n$$

The objective of the two linear programs is the option price, while the first two constraints are the arbitrage pricing equations of the underlying asset and the bond. The last set of constraints requires that all state prices be positive, a condition implied by the absence of arbitrage.

While linear programming had been used previously in option pricing applications - see for instance Garman (1976a), who derived the arbitrage bounds of Merton (1973) in this manner - the formulation of Ritchken (1985) can be easily enriched with information regarding preferences. Moreover, in the case of contingent claims with a convex payoff function, such as call and put options, the problems admits a closed form solution, which is presented here. Figure 2.1 depicts the payoff as a function of the stock return $z_k = \frac{s_k}{S_0}$. The convexity of the option payoff as a function of the asset price implies that the coordinates $(z_j, c_j), j = 1, \ldots n$ form a convex polygon. By writing the bond pricing equation as $\sum_{j=1}^{n} d_j / B_0 = 1$, we notice that the stock and the option prices are convex combinations of the discounted payoffs $B_0 c_j$ and respectively $B_0 s_j$. The feasible solutions must therefore lie inside the polygon determined by the option payoffs. The stock pricing equation, which

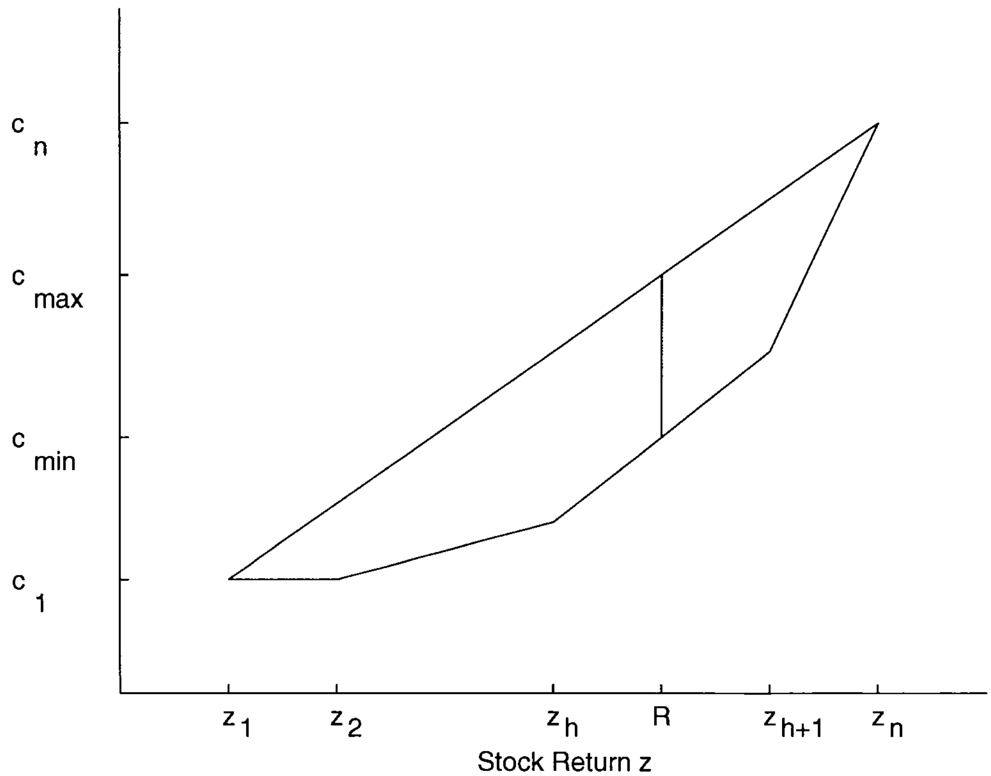

Figure 2.1: Geometric solution to the option bounds problem

can be rewritten as $\sum_{j=0}^{n} z_j d_j / B_0 = R$, further restricts the feasible solutions to the vertical line $z = R$. The solutions of the maximization, respectively minimization problems are the two intersections of the vertical line with the polygon. By inspecting Figure 2.1 and using the equality $B_0 = 1/R$, the upper and lower option bounds are

$$\begin{aligned} C_{max} &= \frac{1}{R}\left[\frac{z_n - R}{z_n - z_1} c_1 + \frac{R - z_1}{z_n - z_1} c_n\right] \\ C_{min} &= \frac{1}{R}\left[\frac{z_{h+1} - R}{z_{h+1} - z_h} c_h + \frac{R - z_h}{z_{h+1} - z_h} c_{h+1}\right] \end{aligned} \tag{2.3}$$

where the index h is obtained from the condition $z_h \leq R < z_{h+1}$.

There are two interesting limit cases that can be analyzed by inspecting the formulas and the geometric interpretation of the problem. The first is the case of the upper bound when the underlying asset price can reach zero, $z_1 = 0$. The option price is zero in this situation and the point (z_1, c_1) is the origin. This is the case with many return distributions, such as the lognormal. The upper bound is in this case

$$C_{max} = \frac{1}{z_n} c_n$$

The second interesting case is the lower bound for a return distribution when $z_h = R$ for some return index h. This latter case is useful in the extension of the results to continuous returns distributions In this case,

$$C_{min} = \frac{1}{R} c_h$$

The analysis of the linear program solutions that underlie the two option bounds provides more insight to the stochastic dominance option pricing. The FSD problem 2.2 consists in finding two risk neutral probability distributions under which the two bounds are attained. From the inspection of equations 2.3, the upper bound is reached under a binomial distribution of the returns, with outcomes equal to the extreme stock returns. The lower bound is attained under the binomial distribution of the two return outcomes that nest the riskless interest rate. The upper and lower bound distributions are the highest, respectively lowest return volatility distributions among all possible risk neutral distributions.

2.1.2 Second degree stochastic dominance

The previous section introduced the linear programming methodology of Ritchken (1985) in a first degree stochastic dominance setting. The linear programs have been adapted to other classes of preferences by modifying the constraints. Ritchken (1985) derived option bounds for risk-averse preferences, while Ritchken and Kuo (1989) studied the general case of n -th degree stochastic dominance, as well as the pricing of options under decreasing absolute risk aversion. While the latter case introduces nonlinear constraints, all the other applications are extensions of the FSD problem and can be solved in a similar manner. The second degree stochastic dominance case relies on the assumptions of the equilibrium pricing framework of Rubinstein (1976) and Brennan (1979):

Assumption 1: law of one price

Assumption 2: nonsatiation

Assumption 3: perfect competitive, Pareto efficient markets

Assumption 4: rational time-additive tastes, with a concave utility functions

Assumption 5: weak aggregation

The weak aggregation property assures that the state prices are identical to the state prices of an economy with a single utility maximizing investor who holds the aggregate consumption in the first period and in each second period state. Let the probabilities $p_1, p_2, \ldots p_n$ denote the beliefs of this investor about the second period values $s_1 \leq s_2 \leq \cdots \leq s_n$ of the underlying asset. The state prices are then $d_j = p_j m_j, \forall j = 1, \ldots, n$, where m_j, called stochastic discount factor, represents the marginal rate of substitution between consumption in state j and first period consumption[4].

A problem of the equilibrium models is the presence of consumption as an argument of the utility functions. Consumption cannot be measured in a reliable manner, and a set of beliefs on future consumption would not be a realistic assumption.

[4]See the discussion in the Chapter 5 of Huang and Litzenberger (1988)

Equilibrium option pricing models remove consumption from the pricing equation by making assumptions on the relation between the underlying asset price and the aggregate consumption. Most equilibrium models, starting from those of Rubinstein (1976) and Brennan (1979), assume the joint lognormality of the aggregate consumption and the underlying asset price. For mathematical tractability, these models also make assumptions on the functional form of the utility function, which usually belongs to the constant proportional risk aversion class. The counterpart of these assumptions in the stochastic dominance framework is the following milder assumption introduced by Perrakis and Ryan (1984):

Assumption 6: non-increasing ordering of the stochastic discount factor with respect to the underlying asset.

This is a basic assumption of the stochastic dominance approach, the monotonicity of the state contingent discount factors with respect to the stock returns. This assumption is rigorously justified when there exists at least one investor in the economy who holds only the stock, the option and the riskless asset. This assumption implies that the underlying asset has a positive consumption beta. However, the analysis can be repeated for negative beta assets. More importantly, the stochastic dominance approach does not make any distributional assumption on the return process, and can therefore accommodate any return distribution observed empirically. There are also no restrictions on preferences, except risk aversion.

The new preference assumptions transform the option bounds problem to the pair of linear programs

$$\min(\max)\ C_0 = \sum_{j=1}^{n} c_j m_j p_j \tag{2.4}$$

subject to:

$$S_0 = \sum_{j=1}^{n} s_j m_j p_j$$

$$B_0 = \sum_{j=1}^{n} m_j p_j$$

$$m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$$

The objective and the pricing constraints are identical to the FSD problem described by the linear program (2.2). The last set of constraints describes the risk averse preferences.

The SSD problem 2.4 optimizes the stochastic discount factors under which the two option bounds are attained. From a probabilistic point of view, the stochastic discount factors are, up to a constant of $1/R$, the Radon-Nikodym derivatives (likelihood ratios) that transform the beliefs of the representative investor into the risk-neutral distributions used to price the two option bounds. Let us assume that we found the stochastic discount factors m_j^U and $m_j^L, j = 1, \ldots, n$ under which the upper and respectively lower option bounds are attained. We can denote

$$U_j = \frac{m_j^U p_j}{B_0} \quad \text{and} \quad L_j = \frac{m_j^L p_j}{B_0} \tag{2.5}$$

It is clear from the bond pricing equation that $\sum_{j=1}^{n} U_j = 1$ and $\sum_{j=1}^{n} L_j = 1$, so U_j and $L_j, j = 1, \ldots, n$ define two probability measures U and L. With this new notation, we can write the two option bounds as

$$C_{max} = \frac{1}{R} E^U \left[g(S_T) \right],$$
$$C_{min} = \frac{1}{R} E^L \left[g(S_T) \right]. \tag{2.6}$$

Note also that we have

$$E^U \left[\frac{S_T}{S_0} \right] = E^L \left[\frac{S_T}{S_0} \right] = R \tag{2.7}$$

Equations (2.6) express the two option pricing bounds as discounted expectations of the option payoff under the probability measures U and L, while (2.7) shows that U and L are risk-neutral probability measures. The two risk neutral probability measures can be obtained from the solutions of the linear program (2.4), using equations (2.5).

Equations (2.6) express the stochastic dominance bounds of any contingent claim price. For call and put options, Ritchken (1985) obtained closed form solutions of the linear programs (2.4). By introducing the substitutions

$$\hat{s}_j = \frac{\sum_{i=1}^{j} s_i p_i}{\sum_{i=1}^{j} p_i} \qquad \hat{c}_j = \frac{\sum_{i=1}^{j} c_i p_i}{\sum_{i=1}^{j} p_i}, \tag{2.8}$$

$$y_j = \left(\sum_{i=1}^{j} p_i \right) x_j \quad \text{where } x_j \geq 0, \forall j = 1, \ldots n \quad \text{such that } \sum_{i=1}^{j} x_i = m_j, \tag{2.9}$$

the linear program becomes

$$\min(\max)\ C_0 = \sum_{j=1}^{n} \hat{c}_j y_j$$

subject to:

$$S_0 = \sum_{j=1}^{n} \hat{s}_j y_j$$

$$B_0 = \sum_{j=1}^{n} y_j$$

$$y_j > 0, \forall j = 1, \dots n$$

which is similar to 2.2, if we substitute y_j for q_j. Using the solution of the FSD problem and denoting $\hat{z}_j = \hat{s}_j / S_0$, the two option bounds under risk aversion are

$$\begin{aligned} C_{max} &= \frac{1}{R}\left[\frac{\hat{z}_n - R}{\hat{z}_n - z_1} c_1 + \frac{R - z_1}{\hat{z}_n - z_1}\hat{c}_n\right] \\ C_{min} &= \frac{1}{R}\left[\frac{\hat{z}_{h+1} - R}{\hat{z}_{h+1} - \hat{z}_h}\hat{c}_h + \frac{R - \hat{z}_h}{\hat{z}_{h+1} - \hat{z}_h}\hat{c}_{h+1}\right] \end{aligned} \tag{2.10}$$

The upper bound given by (2.10) is the same as the one obtained by Perrakis (1986) by a portfolio construction methodology. The solutions of the SSD problem can therefore be expressed in terms of risk neutral probabilities as shown in the studies of Ritchken and Kuo (1988), Perrakis (1988) and Perrakis (1993).

We can obtain the two risk neutral probabilities from the inspection of the SSD bounds equations 2.10. The option upper bound is a function of all the returns outcomes and the corresponding risk neutral distribution is a multinomial distribution with the outcome probabilities

$$\begin{aligned} U_1 &= \frac{\hat{z}_n - R}{\hat{z}_n - z_1} + \frac{R - z_1}{\hat{z}_n - z_1} p_1 \\ U_j &= \frac{R - z_1}{\hat{z}_n - z_1} p_j, \qquad j = 2, \ldots, n \end{aligned} \tag{2.11}$$

The lower bound is obtained from the first $h+1$ outcomes of the returns and the corresponding risk neutral distribution is a multinomial distribution with $h+1$ outcomes

$$\begin{aligned} L_j &= \frac{\hat{z}_{h+1} - R}{\hat{z}_{h+1} - \hat{z}_h} \frac{p_j}{\sum_{i=1}^{h} p_i} + \frac{R - \hat{z}_h}{\hat{z}_{h+1} - \hat{z}_h} \frac{p_j}{\sum_{i=1}^{h+1} p_i}, \qquad j = 1, \ldots, h \\ L_{h+1} &= \frac{R - \hat{z}_h}{\hat{z}_{h+1} - \hat{z}_h} \frac{p_{h+1}}{\sum_{i=1}^{h+1} p_i} \end{aligned} \tag{2.12}$$

An interesting case which simplifies the analysis of the lower bound when the return has a continuous distribution is the case when $\hat{z}_h = R$. In this case, the lower bound distribution becomes

$$L_j = \frac{p_j}{\sum_{i=1}^{h} p_i}, \qquad j = 1, \ldots, h,$$

a distribution obtained by truncating the actual return distribution and keeping only the outcomes $z_j | \hat{z}_j \leq R$.

We can now express the two bound distributions for an underlying asset with a continuous return distribution. The two distributions are risk neutral transformations of the actual return distribution. The upper bound distribution is a mixture of the whole return distribution and a distribution concentrated in the minimum return outcome, with weights that imply risk neutrality. The lower bound distribution is obtained from a truncation of the actual return distribution in such a manner that the resulting distribution is risk-neutral.

$$U(z) = \frac{R - z_{min}}{E^P[z] - z_{min}} P(z) + \frac{E^P[z] - R}{E^P[z] - z_{min}} \mathbf{1}_{z=z_{min}} \tag{2.13}$$
$$L(z) = \frac{P(z)\,\mathbf{1}_{z \le z*}}{P(z \le z^*)}, \text{ where } z^* \text{ solves } E^L(z) = R$$

If the asset return can be zero with a positive probability, $z_{min} = 0$ and the upper bound becomes

$$C_{max} = \frac{1}{E^P[z]} E^P[g(S_T)] \tag{2.14}$$

In practice, this upper bound can be obtained by replacing the riskless rate with the mean return in the risk-neutral option pricing formula.

If the underlying asset has a negative consumption beta then the last constraint in 2.4 is replaced by $0 \le m_1 \le \ldots \le m_n$. While this case is not very common in practice, it can arise in multiperiod models when the returns are not independent and identically distributed (iid) or when the underlying asset is not an equity. We provide here briefly the expressions that form the counterparts of (2.10) and (2.13) in this case. We define the conditional expectations

$$\bar{z}_j = \frac{\sum_{i=j}^{n} z_i p_i}{\sum_{i=j}^{n} p_i} = E\left[z_T | z_T \ge z_j\right] \tag{2.15}$$

Then instead of (2.10) we get

$$C'_{max} = \frac{1}{R}\left[\frac{\bar{z}_{h+1} - R}{\bar{z}_{h+1} - \bar{z}_h}\bar{c}_h + \frac{R - \bar{z}_h}{\bar{z}_{h+1} - \bar{z}_h}\bar{c}_{h+1}\right] \tag{2.16}$$
$$C'_{min} = \frac{1}{R}\left[\frac{\bar{z}_n - R}{\bar{z}_n - \bar{z}_1}\bar{c}_1 + \frac{R - \bar{z}_1}{\bar{z}_n - \bar{z}_1}\bar{c}_n\right]$$

Here again the two states $\bar{z}_h$ and $\bar{z}_{h+1}$ are defined from the relation $\bar{z}_h \leq R \leq \bar{z}_{h+1}$. For a continuous return distribution $P(z)$ the risk neutral distributions $U(z)$ and $L(z)$ of the upper and lower bounds respectively become, instead of (2.13)

$$\begin{aligned} U'(z) &= \frac{z_{\max} - R}{z_{\max} - E(z)} P(z) + \frac{R - E(z)}{z_{\max} - E(z)} \mathbf{1}_{z=z_{min}} \\ L'(z) &= \frac{P(z)\,\mathbf{1}_{z \geq z*}}{P(z \geq z^*)}, \text{ where } z^* \text{ solves } E^L(z) = R \end{aligned} \tag{2.17}$$

2.2 Stochastic dominance bounds in multiperiod economies

The linear programming problems used by Ritchken (1985) to derive the two option pricing bounds rely on general equilibrium and arbitrage arguments and can be used to price any contingent claim in an incomplete market with risk averse investors. The extension of the approach to the pricing of contingent claims in a multiperiod economy reduces to the recursive solving of the two optimization problems.

Under very mild assumptions call and put option prices are convex functions of the underlying asset price. When this property holds, the structure of the single period problem is preserved in each step of the multiperiod problem. Ritchken and Kuo (1988) used this idea to extend the linear programming approach to multiperiod economies where investors can revise their portfolio holdings during the life of the option. Perrakis (1988) and Perrakis (1993) formulated the option bounds in the spirit of the risk-neutral pricing approach presented in the previous section.

All these papers focused on the pricing of options on a process with iid multinomial returns. While such processes can be general enough to approximate the unconditional asset distribution at the expiration of the option, they cannot reproduce some features observed in the markets, such as a time-varying volatility. The two option bounds are obtained by dynamic programming and may be sensitive to the local properties of the asset price process. The reformulation of the multiperiod pricing bounds presented here is general enough to accommodate more recent specifications of the underlying asset distribution with non-iid returns, such as stochastic volatility, jump diffusion mixtures or GARCH processes.

The generalization of the multiperiod results to non-iid returns can be obtained by using the discrete time market model of Harrison and Pliska (1981). The market is specified by the probability space $(\Omega, \mathcal{F}, P)$. Each event in the sample space Ω represents a possible path of the underlying asset price (and possibly other variables such as volatility) and $\mathcal{F}$ is the σ - algebra generated by all the subsets of Ω. The probability measure P represents the beliefs of an investor regarding the time path of the asset price. The information structure of this market is represented by the filtration $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_T\}$ with $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_T = \mathcal{F}$.

Assuming the investors are risk averse, the two option pricing bounds can be obtained recursively by solving the following linear programs, for $t = T-1, \ldots, 0$

$$\min(\max)\ C_t = \sum_{j=1}^{n} c_{j,t+1} m_{j,t+1} p_{j,t+1} \tag{2.18}$$

subject to:

$$S_t = \sum_{j=1}^{n} s_{j,t+1} m_{j,t+1} p_{j,t+1}$$

$$B_t = \sum_{j=1}^{n} m_{j,t+1} p_{j,t+1}$$

$$m_{1,t+1} \geq m_{2,t+1} \geq \cdots \geq m_{n,t+1} \geq 0$$

where $p_{j,t+1} = \text{Prob}\left(S_{t+1} = s_j \,|\mathcal{F}_t\right)$ is the conditional probability of state j at time $t+$

1 given the information available at time t and $s_{j,t+1}$, $c_{j,t+1}$ and $m_{j,t+1}$ are respectively the underlying asset, the option price and the stochastic discount factor in that state at $t+1$.

The optimization problem for $t = T-1$ is the single period problem solved in the previous section. Assume that we have solved the optimization problem for $T-1, \ldots, t+1$. We can apply the single period solution to express the two bounds of the option price at time t.

$$\begin{aligned}
\bar{C}_t &= \frac{1}{R} E^U \left[C_{t+1} \left| \mathcal{F}_t \right. \right] \\
\underline{C}_t &= \frac{1}{R} E^L \left[C_{t+1} \left| \mathcal{F}_t \right. \right]
\end{aligned}$$

where the probability measures U and L are the multiperiod counterparts of the risk neutral probabilities defined in equations (2.11) and (2.12) for the single period case.

The application of the upper bound equation implies the recursion.

$$\begin{aligned}
C_t &\leq \frac{1}{R} E^U \left[C_{t+1} \left| \mathcal{F}_t \right. \right] \\
&\leq \frac{1}{R} E^U \left[\bar{C}_{t+1} \left| \mathcal{F}_t \right. \right] \\
&= \frac{1}{R} E^U \left[\frac{1}{R} E^U \left[C_{t+2} \left| \mathcal{F}_{t+1} \right. \right] \left| \mathcal{F}_t \right. \right] \\
&= \frac{1}{R^2} E^U \left[C_{t+2} \left| \mathcal{F}_t \right. \right]
\end{aligned} \tag{2.19}$$

The last equality of (2.19) is obtained by applying the law of iterated expectations. By repeatedly applying this recursion from $T-1$ to 0, the claim price C_0 must satisfy the inequality

$$C_0 \leq \frac{1}{R^T} E^U \left[g(S_T) \left| \mathcal{F}_0 \right. \right] \tag{2.20}$$

It can be shown in a similar recursive manner that the option price C_t must satisfy the inequality

$$C_0 \geq \frac{1}{R^T} E^L \left[g(S_T) \left| \mathcal{F}_0 \right. \right] \tag{2.21}$$

Equations 2.20 and 2.21 show that the option pricing bounds under risk aversion can be computed in the same manner as any other option price, by using the risk neutral pricing framework of Cox and Ross (1976) and Harrison and Kreps (1979). This interpretation of the option bounds is very useful since it extends the risk neutral pricing used in the arbitrage approach to the pricing of the two bounds under risk aversion. Most arbitrage pricing models are expressions or algorithms that compute the expectation of the payoff under a risk neutral distribution. These models extend easily to the two risk neutral distributions involved in computing the two option bounds, assuming we can obtain these distributions.

An application of the stochastic dominance approach in a multiperiod setting is the pricing of American options. We can use the same recursive method to obtain the pricing bounds for an American option on an asset that pays a dividend d_t equal to $d_t = \gamma S_t$

$$\begin{aligned}
\bar{C}_{A,T} = \underline{C}_{A,t} &= g(S_T(1+\gamma)), \\
\bar{C}_{A,t} &= \frac{1}{R} E^U[max(g(S_{t+1}(1+\gamma)), \bar{C}_{A,t+1}) | \mathcal{F}_t], \\
\underline{C}_{A,t} &= \frac{1}{R} E^L[max(g(S_{t+1}(1+\gamma)), \underline{C}_{A,t+1}) | \mathcal{F}_t].
\end{aligned}$$

The two bounds can be used to obtain the maximum buying price, and respectively the minimum writing price for an American option. However, they cannot be used to decide whether or not to exercise an option if the exercise value lies between the bounds.

2.2.1 Call and put option bounds

In general, the probability measures U and L can be obtained recursively by solving the linear programs (2.18) for $t = T-1, \ldots, 0$. If the option price C_{t+1} is a convex function of the underlying asset price S_{t+1}, the closed form solution obtained in the two-period case extends to the time t step of the problems (2.18). Using the solutions of the two period problem, it is convenient to express the probabilities U and L as conditional probabilities of the return between t and $t+1$ given the information available at time t.

$$\begin{aligned} U\left(z_{t+1}\left|\mathcal{F}_t\right.\right) &= \frac{R - z_{t+1,\min}}{E^P\left[z_{t+1}\left|\mathcal{F}_t\right.\right] - z_{t+1,\min}} P\left(z_{t+1}\left|\mathcal{F}_t\right.\right) \\ &+ \frac{E^P\left[z_{t+1}\left|\mathcal{F}_t\right.\right] - R}{E^P\left[z_{t+1}\left|\mathcal{F}_t\right.\right] - z_{t+1,\min}} \mathbf{1}_{z_{t+1}=z_{t+1,\min}} \end{aligned} \tag{2.22}$$

$$\begin{aligned} L\left(z_{t+1}\left|\mathcal{F}_t\right.\right) &= \frac{P\left(z_{t+1}\left|\mathcal{F}_t\right.\right) \mathbf{1}_{z_{t+1}\leq z^*_{t+1}}}{P\left(z_{t+1} \leq {z^*}_{t+1}\left|\mathcal{F}_t\right.\right)}, \\ &\text{where } z^*_{t+1} \text{ solves } E^L\left(z_{t+1}\left|\mathcal{F}_t\right.\right) = R \end{aligned} \tag{2.23}$$

The extension of the closed form results of the two-period problem relies on the convexity of the option payoff as a function of the underlying asset price. Merton (1973) proved that this property holds for call and put options if the distribution of the underlying asset returns is independent of the stock price. Bergman, Grundy, and Wiener (1996) showed that the price convexity holds whenever the underlying asset returns follow a univariate diffusion. This extends the validity of the closed form bounds defined by the probabilities (2.22) and (2.23) to some asset specifications with a return distribution that depends on the underlying asset price, for instance the constant elasticity of variance model of Cox and Ross (1976).

Most processes used in the modeling of asset prices assume the Markov property. If the information set $\mathcal{F}_t$ is defined by the evolution of the asset price $S_0, \ldots, S_t$, the Markov property provides that the conditional probabilities involved in (2.22) and (2.23) depend only on the time t value of the underlying asset price S_t. In subsequent chapters we examine the multiperiod option bounds in cases where the Markov property ensures that convexity holds, as well as the computation of stochastic dominance bounds whenever convexity is violated.

Chapter 3

Stochastic dominance option pricing in continuous time

The previous chapter examined the stochastic dominance approach to option pricing in discrete time. While asset prices are discrete by the nature of the trading process, many option pricing results have been obtained in a continuous time paradigm. The main reason is the mathematical tractability of continuous time models, in which option prices can be obtained by solving a partial differential equation. The constant volatility diffusion model of Black and Scholes (1973) is the best known continuous time model. Since then, a large number models relaxed the constant volatility assumption. Early models assumed univariate price processes, such as the constant elasticity of variance model of Cox and Ross (1976) or the jump diffusion mixture of Merton (1976). More complex models rely on bivariate processes, using price and volatility as state variables. Such stochastic volatility models were proposed by Garman (1976b), Hull and White (1987) and many others. Heston (1993) obtained a closed form for the characteristic function of the risk-neutral probability of a stochastic volatility diffusion. Bates (1996) and Bakshi, Cao, and Chen (1997) extended this technique to jump-diffusion mixtures, while the affine jump diffusions of Duffie, Pan, and Singleton (2000) can accommodate stochastic volatility of the volatility or jumps in volatility.

All these models have been priced in the arbitrage framework introduced by Black and Scholes (1973) and Merton (1973), by applying the risk neutral pricing methodology of Cox and Ross (1976) and Harrison and Kreps (1979). Under the risk neutral pricing approach, the price of any contingent claim is the discounted expectation of the payoff under a so-called risk-neutral probability measure. In continuous time models, the risk-neutral probability measure is obtained by applying Girsanov's theorem, such that the discounted underlying asset price follows a martingale under the new probability measure.

Unfortunately, the martingale probability measure is unique only when the underlying asset follows a univariate diffusion. When the underlying asset price process incorporates stochastic volatility or jumps, the market is incomplete and extra assumptions are needed to price the option.

This chapter derives stochastic dominance pricing bounds for options on diffusion and jump-diffusion processes. The bounds are obtained by analyzing the limit behavior of the discrete time stochastic dominance bounds. In discrete time, the two option bounds are discounted expectations of the option payoff under two probability measures U and L. Using the two risk neutral distributions, the two bounds for the price of a an option with payoff $g(S_T)$ can be written as

$$C_{max} = \frac{1}{R^T} E^U \left[g(S_T) \,|\mathcal{F}_0 \right] \tag{3.1}$$

$$C_{min} = \frac{1}{R^T} E^L \left[g(S_T) \,|\mathcal{F}_0 \right] \tag{3.2}$$

When $g(S_T)$ is a convex function of the underlying asset price, for instance in the case of call and put options, the two probability measures can be obtained in closed form from the actual return distribution.

Perrakis (1988) examined the convergence of the option bounds for the special case of a stock return following a trinomial distribution. It was shown that when that distribution tended to a diffusion process the limit of both upper and lower bounds was the Black-Scholes option price. The convergence criteria used in that study were the ones provided by Merton (1982) for iid returns following a general multinomial process. Since the bounds are available in closed form in such a case, it suffices to show that the limiting form of the multiperiod convolutions of the distributions $U(z)$ and $L(z)$ is a risk neutral diffusion with the same constant volatility as the initial process.

This line of approach is, unfortunately, not available when the underlying stock returns are not iid. Although the Merton (1982) criteria for the convergence to a diffusion of the multinomial discretization of the underlying stochastic process are still valid, they are not very useful in characterizing the limiting process. Further, the option bounds themselves are available only as recursive expressions of time-varying distributions, whose limiting form is not easy to ascertain under general conditions.

For this reason we shall examine the behavior of the bounds by adopting a more general approach to convergence analysis. In analyzing the convergence of stochastic processes it is important to use the right type of convergence. We adopt the notion of weak convergence. Given a sequence of random variables Z^n on the probability probability spaces $(\Omega^n, \mathcal{F}^n, P^n)$, and the random variable Z on the probability space $(\Omega, \mathcal{F}, P)$, Z^n is said to converge in distribution to Z and P^n is said to converge weakly to P when the following limit exists for any real value z.

$$\lim_{n \to \infty} P^n(Z^n \leq z) = P(Z \leq z) \tag{3.3}$$

The weak convergence property holds if and only if for any continuous bounded function g, we have

$$\lim_{n \to \infty} E^{P^n}\left[g(Z^n)\right] = E^P\left[g(Z)\right], \tag{3.4}$$

Since option prices are expectations of the option payoff, equation (3.4) underlies the use of discrete time models such as binomial trees to approximate options on continuous time processes.

In our stochastic dominance setting, if we can prove the convergence of the sequences of probability measures U^n and L^n to the limits U and L, the weak convergence property provides that the option prices on the continuous time process X_t (which contains, but is not necessarily limited to the underlying asset price S_t), are bounded by the discounted expectations of the payoff under the two probability measures U and L. A unique option price results if both U^n and L^n converge to the same limit.

We adopt a three-step procedure for the analysis, where we use the superscript h to denote a discrete process with sampling interval $\Delta t = h$:

- Find a sequence of stochastic processes X_t^h that converge weakly to the given continuous time process X_t as $h \to 0$.
- For each approximating stochastic process X^h, obtain the two risk-neutral probability measures U^h and L^h under which the stochastic dominance bounds are attained
- Analyze the convergence of the two probability measures as $h \to 0$.

In the following sections, we apply this methodology to examine the convergence of the option bounds when the underlying asset follows a diffusion and a jump-diffusion mixture. Apart from stochastic volatility and GARCH processes, which are examined in the next chapter, these two cases cover most continuous time option pricing models studied in the literature.

3.1 Option Bounds in Continuous Time: the Diffusion Case

We consider the general case of a multivariate diffusion process

$$dX_t = \mu(X_t)dt + \Sigma(X_t)dW, \tag{3.5}$$

where X_t is a d -dimensional vector of state variables known at time t including the underlying asset S_t. The drift coefficient vector $\mu(X_t)$ and the diffusion coefficient matrix $\Sigma(X_t)$ are unspecified continuous functions of X_t in $\mathbf{R}^d$ and respectively $\mathbf{R}^{dxd}$. Since they are continuous, the functions $\mu(X_t)$ and $\Sigma(X_t)$ satisfy a Lipschitz condition that provides the existence of a unique strong solution of the stochastic differential equation (3.5).

We assume without loss of generality that Σ is a lower triangular matrix. W_t is a d -dimensional Brownian motion with independent components. In the traditional Black-Scholes model $X_t \equiv S_t$ and both the mean and variance are linear functions with constant coefficients, $\mu(S) = \mu S$ and $\sigma(S) = \sigma S$. In bivariate models, the first component of X is the underlying asset price, while the second component is usually a measure of the return volatility.

In the first step of the convergence analysis, we seek a sequence of discrete time Markovian stochastic process over the interval $[0, T]$ to option expiration that converges to (3.5) as the length h of the elementary time period tends to zero. There are several ways to verify the weak convergence of Markov processes[1]. For instance, a necessary and sufficient condition for the convergence to a multivariate diffusion is the Lindeberg condition, which was used by Merton (1982) to develop criteria for the convergence of multinomial processes and by Nelson (1990) to prove the convergence of some ARCH type processes to diffusions. Let X_t^h denote a family of discrete multidimensional stochastic processes. The Lindeberg condition stipulates that for any fixed $\delta > 0$ we must have

$$\lim_{h \to 0} \frac{1}{h} \int_{||y-x||>\delta} P^h(x, dy) = 0 \tag{3.6}$$

where $P^h(x, dy)$ is the transition probability from $X_t^h = x$ to $X_{t+h}^h = y$ during the time interval h and $||\cdot||$ denotes the Euclidean norm in $\mathbb{R}^d$. Intuitively, the Lindeberg requires that X_t^h does not change very much when the time interval h tends to zero.

When the Lindeberg condition is satisfied, the following limits exist

$$\lim_{h \to 0} \frac{1}{h} \int_{||y-x||>\delta} (y_i - x_i) P^h(x, dy) = \mu_i(x) \tag{3.7}$$

$$\lim_{h \to 0} \frac{1}{h} \int_{||y-x||>\delta} (y_i - x_i)(y_j - x_j) P^h(x, dy) = \sigma_{ij}(x) \tag{3.8}$$

The conditions (3.6), (3.7) and (3.8) are equivalent to the weak convergence of the discrete process X_t^h to a diffusion process with the generator

$$\mathcal{A} = \frac{1}{2} \sum_{i=1,j=1}^{d} \sigma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \mu_i \frac{\partial}{\partial x_i} \tag{3.9}$$

The generator of the diffusion X_t is, by definition, the operator

[1] For more on weak convergence for Markov processes see Ethier and Kurz (1986), or Strook and Varadhan (1979).

$$\lim_{h\to 0} \frac{Eu(X_{t+h}) - u(X_t)}{h} = \mathcal{A}u$$

for any bounded, real valued function u. An alternative representation of this diffusion is the d -dimensional stochastic differential equation (3.5).

We now construct a sequence of discrete processes that converges to the diffusion (3.5). Inspired by the Euler approximation used in Monte Carlo simulations, we show that the following process is a valid discretization of (3.5).

$$X_{t+h} - X_t = \mu(X_t)h + \Sigma(X_t)\varepsilon_{t+h}\sqrt{h} \tag{3.10}$$

In (3.10), the elements of the return innovation ε_{t+h} are independent random variables drawn from a bounded continuous distribution of mean zero and variance one, $\varepsilon_{t+\Delta t} \sim D(0,1)$ and $\varepsilon_{\min} \leq \varepsilon_{t+\Delta t} \leq \varepsilon_{\max}$, but otherwise unrestricted. We express the convergence through the following result proved in the appendix.

Lemma 1. The discrete process described by equation (3.10) converges weakly to the diffusion (3.5).

Having proved the lemma, we can construct a process that converges weakly to a given diffusion. We first examine the convergence of the bounds when the underlying asset converges weakly to the univariate diffusion defined by the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu(S_t)dt + \sigma(S_t)dW. \tag{3.11}$$

We pick the following discrete approximation

$$(S_{t+h} - S_t)/S_t \equiv z_{t+h} = \mu(S_t)h + \sigma(S_t)\varepsilon\sqrt{h} \tag{3.12}$$

where the innovations have a uniform distribution with mean zero and variance one,

$$f(\varepsilon) = \frac{1}{2a}\mathbf{1}_{-a\leq\varepsilon\leq a}, \tag{3.13}$$

where $a = \sqrt{3}$.

In the second step of the convergence analysis, we construct the two bound probabilities U^h and L^h, corresponding to the sampling interval h. Since the convexity of the option price holds, the two bound distributions are available in the closed form defined by equations (2.22) and (2.23). The two option bounds are defined by the discrete processes

$$z^U_{t+h} = rh + \sigma(S_t)\varepsilon^U\sqrt{h}, \tag{3.14}$$

$$z^L_{t+h} = rh + \sigma(S_t)\varepsilon^L\sqrt{h}, \tag{3.15}$$

where the innovations of the bound processes are defined by the laws

$$\begin{aligned} f^U(\varepsilon^U) &= \left(1 - \frac{\mu - r}{\sigma a}\sqrt{h}\right)\frac{1}{2a}\mathbf{1}_{-a+\frac{(\mu-r)}{\sigma}\sqrt{h}\leq\varepsilon^U\leq a+\frac{(\mu-r)}{\sigma}\sqrt{h}} && (3.16)\\ &+ \frac{\mu - r}{\sigma a}\sqrt{h}\mathbf{1}_{\varepsilon=-a+\frac{(\mu-r)}{\sigma}\sqrt{h}}, \\ f^L(\varepsilon^L) &= \frac{1}{2\left(a - \frac{(\mu-r)}{\sigma}\sqrt{h}\right)}\mathbf{1}_{-a+\frac{(\mu-r)}{\sigma}\sqrt{h}\leq\varepsilon^L\leq a-\frac{(\mu-r)}{\sigma}\sqrt{h}}. && (3.17) \end{aligned}$$

It can be verified by elementary algebra that $E^U[\varepsilon^U] = E^L[\varepsilon^L] = 0$ and that $\lim_{h\to 0} Var^U[\varepsilon^U] = \lim_{h\to 0} Var^L[\varepsilon^L] = 1$.

By applying the lemma to the discrete bound processes z^U and z^L, we prove the weak convergence of both bound processes to the same risk-neutral pricing process. We summarize this result in the following two propositions[2]:

[2]Since (2.22) and (2.23) hold only whenever $\mu(S_t) > r$, we need to show that the convergence of the bounds as in Propositions 1 and 2 is also preserved in the case $\mu(S_t) \leq r$. The demonstration

Proposition 1. When the underlying asset follows a continuous time process described by (3.11) the stochastic dominance upper bound of a European call or put option converges to the discounted expectation of the terminal payoff of an option on an asset whose dynamics are described by the process

$$\frac{dS}{S} = rdt + \sigma(X_t)dW \tag{3.18}$$

where r is the (continuous time) riskless rate of interest.

Proposition 2. Under the conditions of Proposition 1 the stochastic dominance lower bound of a European call or put option converges to the same limit as the upper bound.

This result shows that, in a complete market setting where the underlying asset follows a univariate diffusion process, the two stochastic dominance option pricing bounds converge to the arbitrage option price.

Propositions 1 and 2 were shown to hold for a uniform distribution of the discrete innovations. In fact they hold under any return innovation that satisfies Lemma 1. In the appendix we prove the convergence of the two bounds to a common limit under such general conditions.

Note that the univariate Ito process is the only type of asset dynamics, corresponding to dynamically complete markets, for which options can be priced by arbitrage considerations alone. The stochastic volatility and mixed jump-diffusion models need additional assumptions beyond arbitrage in order to complete the market.

Figure 3.1 illustrates the convergence of the two option bounds for an at-the money call option with strike price $K = 100$ and maturity $T = 0.25$ years on a

is straightforward and will be omitted.

Black-Scholes diffusion with the annual return mean μ ranging between 0.05 and 0.07, and annual volatility $\sigma = 0.1$. The riskless interest rate is $r = 0.03$. The diffusion process was approximated by a trinomial tree constructed according to the algorithm of Kamrad and Ritchken (1991) with 300 periods. The two option bounds have been computed as discounted expectations of the payoffs under the risk-neutral probabilities obtained by applying the closed formulas (2.22) and (2.23) to subtrees of the 300-period trinomial tree. For instance, the 150-period tree is a recombining tree obtained by self-replicating a 2-period trinomial tree. The risk-neutral probabilities are the 150-period convolutions of the elementary bound probabilities computed on the terminal distribution of the 2-period trinomial tree (which is a 5-nomial tree). The convolution is an application of the Fast Fourier Transform and is a convenient way to obtain the closed forms of the terminal distribution of the underlying asset and the bounds, when the single-period returns are i.i.d.

3.2 Mixed Jump-Diffusion Processes

Jump-diffusion processes characterize the dynamics of the underlying asset price distribution whenever there are discontinuous jumps in the time path of the stock price caused by the sudden and unexpected arrival of important information. Such jumps have long been recognized as an important source of market incompleteness. Their presence makes the valuation of options solely by arbitrage methods infeasible, except in a binomial model.[3] As for the stochastic dominance approach, it was shown that the two bounds converge to two different option values at the limit of continuous trading even in the case of a very simple three-state jump process (up, down, and stay the same).[4]

[3]See Cox and Rubinstein (1985, pp. 365-368).
[4]See Proposition 6 in Perrakis (1988).

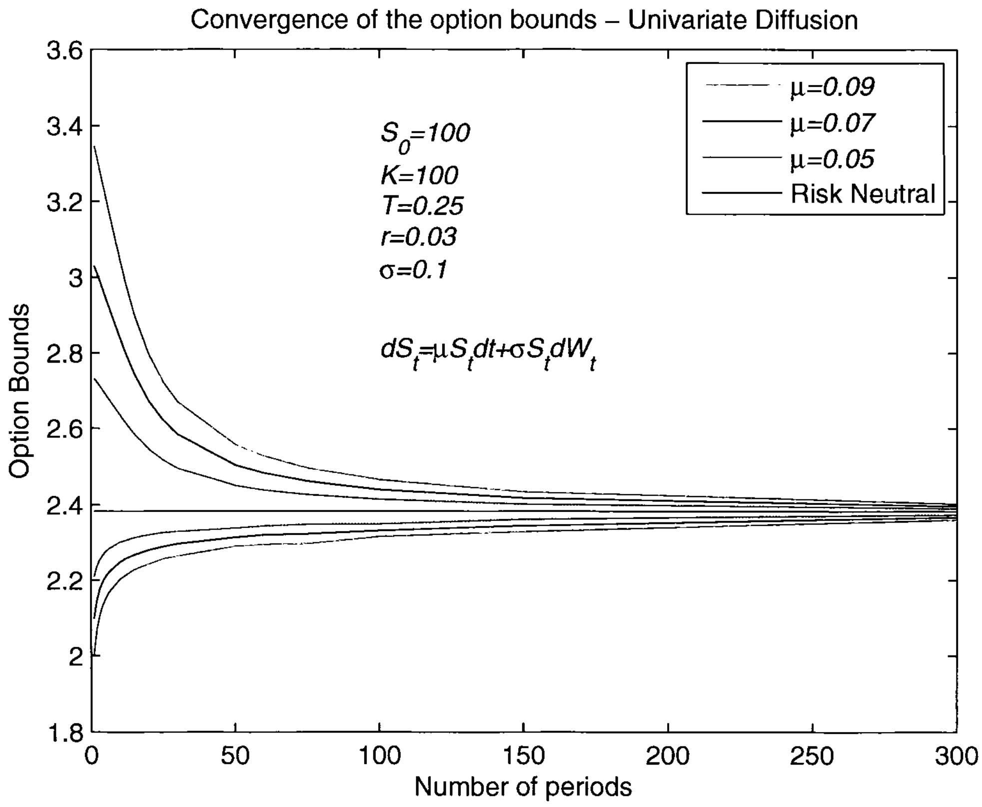

Figure 3.1: Convergence of the option bounds when the underlying process follows a diffusion

When there are jumps in the underlying asset price distribution it is not possible to replicate the option with a portfolio comprising the riskless asset and the underlying asset. The pricing of the option requires extra assumptions regarding the jump risk. The most common assumption, originally introduced by Merton (1976), is that the jump risk is diversifiable. In such a case the market will not pay a risk premium over the riskless rate for bearing the jump risk and risk neutral pricing applies by assuming that the jump probabilities are risk neutral. With such an assumption a closed-form expression for the option price was provided by Merton for the case where the amplitude of the jump size follows a lognormal distribution. Alternative approaches for valuing options in jump-diffusion cases have been provided by Amin and Ng (1993), Amin (1993) and Bates (1991, 1996).[5]

In this section we examine the stochastic dominance approach to option pricing in the case of underlying assets whose returns follow jump-diffusion processes. As with the general diffusion case, we first provide a discretization of the continuous time process that converges at the limit to the given jump-diffusion process. The option bounds are derived by the stochastic dominance approach from such a discretization by applying the risk neutral transformations (2.22) and (2.23) to the discrete one-period distribution. The two transformed distributions are then shown to converge at the continuous time limit to two different option prices. These prices correspond to two different risk neutral jump-diffusion processes, each one of which prices options in a manner similar to the Merton (1976) assumption of diversifiable jump risk. We provide two partial differential equations (pde) satisfied by the upper and lower bounds respectively.[6] Last but not least, we show that the two bounds contain all

[5]See also Bakshi, Cao and Chen (1997), who added jump components to a stochastic volatility model. More recently Duffie, Pan and Singleton (2000) have introduced option pricing models for underlying assets that contain jumps in both asset returns and their stochastic volatility.

[6]Of these two pde's only the one corresponding to the upper bound yields a closed-form solution under a lognormal distribution of the amplitude of the jumps. Closed form solutions also arise

the jump-diffusion option prices that have appeared in earlier studies, including the Merton (1976) price. We assume that the underlying asset returns follow the process

$$\frac{dS_t}{S_t} = (\mu_t - \lambda\mu_J)dt + \sigma_t dW_t + J_t dN_t \tag{3.19}$$

where the last term is a jump component added to the diffusion. Although our results apply to the more general case where both μ_t and σ_t are functions of S_t, we shall assume in what follows that $\mu_t = \mu, \sigma_t = \sigma, \lambda_t = \lambda$ and $J_t = J$, in line with earlier studies; we shall also assume that $\mu > r$. The variable J represents the logarithm of the jump size. It is a random variable with density function $f_J(J)$, with mean μ_J and variance σ_J. N is a Poisson counting process with intensity λ. In most of the literature it is assumed that the jumps are normally distributed.

The first step in deriving the bounds on this process is to find a discrete approximation that converges weakly to (3.19). It will be shown that the following process is such an approximation.

$$z_{t,t+h} = (\mu - \lambda\mu_J)h + \sigma\varepsilon\sqrt{h} + J\Delta N \tag{3.20}$$

where ε is a random variable with a given distribution, with a bounded continuous distribution D(0,1). We can pick any density function with these properties, for instance the uniform density function (3.13). The transition probability of the returns process can be characterized as a mixture of a diffusion and a jump, with

under simple discrete distributions, like the trinomial used in Perrakis (1993). For the lower bound, and for all other jump amplitude distributions, both option bounds can be obtained either through numerical methods or through their characteristic functions following the approach of Heston (1993) and Bates (1996).

corresponding probabilities $1-\lambda h$ and λh:

$$z_{t,t+\Delta t} = \begin{cases} z_D = (\mu - \lambda\mu_J)h + \sigma\varepsilon\sqrt{h} & \text{with probability} \quad 1-\lambda h \\ J & \text{with probability} \quad \lambda h \end{cases}$$

It can be easily seen that this process does not satisfy the Lindeberg condition, since

$$\lim_{h\to 0}\frac{1}{h}P^h(x,dy) = \lambda\int_{|z_{t,t+h}|>\delta} f_J(J)dJ + \lim_{h\to 0}\frac{1}{h}\int_{|z_{t,t+h}|>\delta}(1-\lambda h)f(\varepsilon)d\varepsilon$$

As shown in the proof of Lemma 1 for the diffusion case, the second integrand is zero for h sufficiently low. However, the first integrand is strictly positive for any h, implying that the process does not converge to a diffusion in continuous time. The following result, proved in the appendix, shows that (3.20) is a valid discrete time representation of (3.19).

Lemma 2. The discrete process described by (3.20) converges weakly to the jump-diffusion process (3.19).

Next we examine the limiting behavior of the stochastic dominance bounds derived from (3.20). We assume, without loss of generality, that the variable J takes both positive and negative values, or that $J_{\min} < 0 < J_{\max}$, implying that the jump amplitude takes values both above and below 1. For the option upper bound we apply (2.22) to the discrete time process defined by (3.20). For such a process we note that as h decreases, there exists an $\bar{h}$, such that for any $h \leq \bar{h}$, the minimum outcome of the jump component is less than the minimum outcome of the diffusion component, $J_{\min} < \mu_t h + \sigma_t \varepsilon_{\min}\sqrt{h}$. Consequently, for any $h \leq \bar{h}$, the minimum outcome of the returns distribution is $J_{\min}$, which is the value that we substitute for

$z_{\min}$ in (2.22). With such a substitution we have now the following result, proved in the appendix.

Proposition 3. When the underlying asset follows a jump-diffusion process described by (3.19), the upper option bound is the discounted expected payoff of an option on an asset whose dynamics is described by the jump-diffusion process

$$\frac{dS_t}{S_t} = \left[r - (\lambda + \lambda_U)\mu_J^U\right] dt + \sigma dW + J^U dN_t \tag{3.21}$$

where r is the riskless interest rate,

$$\lambda_U = -\frac{\mu - r}{J_{\min}},$$

J^U is a jump with the distribution

$$f_J^U(J) = \frac{\lambda}{\lambda + \lambda_U} f_J(J) + \frac{\lambda_U}{\lambda + \lambda_U} \mathbf{1}_{J=J_{\min}} \tag{3.22}$$

and μ_J^U is the mean of the jumps under U

$$\mu_J^U = \frac{\lambda}{\lambda + \lambda_U}\mu_J + \frac{\lambda_U}{\lambda + \lambda_U} J_{\min}$$

Given now Proposition 3, we can then use the results derived by Merton (1976) for options on assets following jump-diffusion processes with the jump risk fully diversifiable.[7] Applying Merton's approach to (3.21) we find that the upper bound $\overline{C}(S_t, t)$ on claim prices for the jump-diffusion process (3.19) must satisfy the following pde,

[7]Remark that in the stochastic dominance approach, we *do not* assume that the jump risk is diversifiable.

with terminal condition $C(S_T, T) = f(S_T)$:

$$\left[r - (\lambda + \lambda_U)\mu_J^U\right] S\frac{\partial \overline{C}}{\partial S} - \frac{\partial \overline{C}}{\partial T} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \overline{C}}{\partial S^2} + \lambda_U E^U[\overline{C}(SJ^U) - \overline{C}(S)] - r\overline{C} = 0 \quad (3.23)$$

An important special case is when the lower limit of the jump amplitude is equal to 0, in which case $J_{\min} = -\infty$. In such a case r is replaced by μ in Merton's pde, which becomes

$$[\mu - \lambda\mu_J]\, S\frac{\partial \overline{C}}{\partial S} - \frac{\partial \overline{C}}{\partial T} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \overline{C}}{\partial S^2} + \lambda E[\overline{C}(SJ) - \overline{C}(S)] - \mu\overline{C} = 0 \qquad (3.24)$$

If (3.24) holds and we assume, in addition, that the amplitude of the jumps has a lognormal distribution with $J \sim N(\mu_J, \sigma_J)$, the distribution of the asset price given that k jumps occurred is conditionally normal, with mean and variance

$$\mu_k = \mu - k\lambda\mu_J + \frac{k}{T}\ln(1 + \mu_J)$$
$$\sigma_k^2 = \sigma^2 + \frac{k}{T}\sigma_J^2$$

Hence, if k jumps occurred, the option price would be a Black-Scholes expression with μ_k replacing the riskless rate r, or $BS(S, X, T, \mu_k, \sigma_k)$. Integrating (3.24) would then yield the following upper bound, which can be obtained directly from Merton (1976) by replacing r by μ.

$$\overline{C} = \sum_{k=0}^{\infty} \exp[-\lambda(1 + \mu_J)t]\frac{[\lambda(1 + \mu_J)t]^n}{n!} BS(S, X, T, \mu_k, \sigma_k)$$

When the jump distribution is not normal, the conditional asset distribution given k jumps is the convolution of a normal and k jump distributions. The upper bound cannot be obtained in closed form, but it is possible to obtain the characteristic function of the bound distribution. Similar approaches can be applied to the integration of equation (3.23), which holds whenever $0 > J_{\min} > -\infty$. Closed form solutions can also be found whenever the amplitude of the jumps is fixed as, for instance, when there is only an up and a down jump of a fixed size.[8]

Next we examine the option lower bound for the jump-diffusion process given by (3.19) and its discretization (3.20). We apply now (2.23) to the process (3.20) and we prove in the appendix the following result.

Proposition 4. When the underlying asset follows a jump-diffusion process described by (3.19), the lower option bound is the discounted expected payoff of an option on an asset whose dynamics are described by the jump-diffusion process

$$\frac{dS_t}{S_t} = \left[r - \lambda \mu_J^L\right] dt + \sigma_t dW_t + J^L dN_t \tag{3.25}$$

where J^L is a jump with the truncated distribution $J|J \leq \bar{J}$ The mean of the jump and the value of $\bar{J}$ can be obtained by solving the equations

$$\mu - \lambda \mu_J + \lambda \mu_J^L = r \tag{3.26}$$

$$\mu_J^L = E(J|J \leq \bar{J})$$

Observe that (3.26) always has a solution since $\mu > r$ by assumption. From the discretization (3.20) it is also clear that as $h \to 0$ all the outcomes of the diffusion component will be lower than $\bar{J}$. Therefore, the limiting distribution will include the whole diffusion component and a truncated jump component. The maximum

[8]See, for instance, Perrakis (1993).

jump outcome in this truncated distribution is obtained from the condition that the distribution is risk neutral, which is expressed in (3.26). As with the upper bound, we can apply the Merton (1976) approach to derive the pde satisfied by the option lower bound, which is given by

$$\left[r - \lambda \mu_J^L\right] S \frac{\partial \underline{C}}{\partial S} - \frac{\partial \underline{C}}{\partial T} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \underline{C}}{\partial S^2} + \lambda E^L[\underline{C}(SJ^L) - \underline{C}(S)] - R\underline{C} = 0 \qquad (3.27)$$

with terminal condition $\underline{C_T} = C(S_T, T) = g(S_T)$. The solution of (3.27) can be obtained in closed form only when the jump amplitudes are fixed, since even when the jumps are normally distributed, the lower bound jump distribution is truncated.

Figure 3.2 illustrates the convergence of the two option bounds for an at-the money call option with strike price $K = 100$ and maturity $T = 0.25$ years on a jump-diffusion process described by the model of Merton (1976) with the annual return mean μ ranging between 0.05 and 0.09, and annual volatility of the diffusion component $\sigma = 0.1$. The jumps have the intensity $\lambda = 0.3$ and are normally distributed with mean $\mu_J = -0.05$ and volatility $\sigma_J = 0.07$. The riskless interest rate is $r = 0.03$. The jump-diffusion process was approximated by a 300-period tree built according to the method introduced by Amin (1993). The bounds were computed by taking the discounted expectation of the payoff under risk-neutral probabilities obtained by applying the closed formulas (2.22) and (2.23) to subtrees, in the same manner as in the diffusion case. The risk-neutral price is the the Merton (1976) price for this process.

The plot shows a spread of less than 10%. It is important to note that this range of allowable prices in the stochastic dominance approach is the exact counterpart of the inability of the "traditional" arbitrage-based approaches to produce a single option price for jump diffusion processes. Indeed, the exact option prices under jump

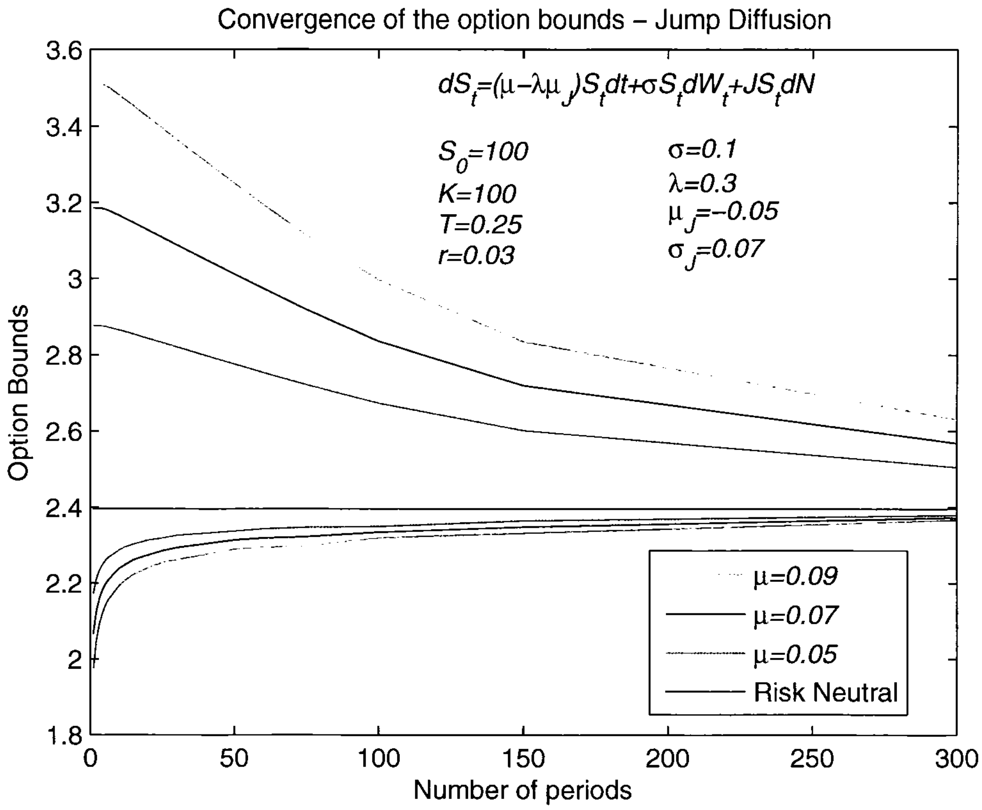

Figure 3.2: Convergence of the option bounds when the underlying process follows a jump-diffusion

diffusion derived in the well-known studies of Bates (1991), Amin and Ng (1993) and Amin (1993) are all functions of the risk aversion parameter of the CPRA utility function of consumption used in the derivations; see, for instance, equation (27) of Amin and Ng (1993), or equation (33) of Amin (1993). Further, the assumed monotonicity of the state-contingent discount factors of the stochastic dominance approach in an elementary discrete time period also holds in the combination of jump diffusion asset dynamics and CPRA utility of consumption used in the more traditional approaches. The stochastic dominance option bounds are, therefore, a more general approach to option pricing than general equilibrium based on specific forms of the utility function.

3.3 Multivariate diffusions

We now discuss the case when the underlying asset follows a multivariate diffusion, as in the stochastic volatility models of Garman (1976b), Hull and White (1987), Heston (1993) and many others. We specify the state vector X_t as the pair (S_t, Y_t), where S_t is the underlying asset price and Y_t incorporates other state variables that measure the asset volatility.

In principle, we can apply the lemma and construct a discretization of these models according to equation 3.10. However, the multiperiod optimization problem (2.18) is not easily tractable with this type of discretization. The main difficulty is that the next option price C_{t+h} is now a function of both the stock price S_{t+h} and the volatility Y_{t+h}. In the univariate case, the convexity of the option with respect to the stock return holds in most cases except a few perverse counterexamples[9]. This allows us to apply the closed forms of the two bound probabilities and express the bounds as expectations under these probabilities. In the multivariate case, we can often ascertain the convexity of the option price with respect to the stock price if the volatility is fixed[10]. However, the closed formulas (2.22) and (2.23) are a result of the convexity of $C_{t+h}(S_{t+h})$, rather than the convexity of $C_t(S_t)$, given the information available at time t. Since the next period states occur across various volatilities, neither the criteria of Bergman, Grundy, and Wiener (1996), nor that of Merton(73) provide the required convexity.

We can address this difficulty from two points of view. Early stochastic volatility models, such as that of Hull and White (1987), assumed that the volatility risk

[9]See for instance Bergman, Grundy, and Wiener (1996). However, their example is a non-Markovian process

[10]Bergman, Grundy, and Wiener (1996) examined the convexity of the option price with respect to the underlying asset for a bivariate diffusion. Most stochastic volatility models satisfy their convexity condition.

was diversifiable. Under this assumption, the result expressed by Propositions 1 and 2 still holds. Assume, without loss of generality, a stochastic volatility process described by the bivariate SDE

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu_S(S_t, Y_t)dt + \sigma_{SS}(S_t, Y_t)dW_S \\ dY_t &= \mu_Y(S_t, Y_t)dt + \sigma_{SY}(S_t, Y_t)dW_S + \sigma_{YY}(S_t, Y_t)dW_Y\end{aligned}$$

This form of the SDE is more tractable from a stochastic dominance perspective due to the univariate nature of the return equation. When the volatility is diversifiable, the argument of Hull and White (1987) applies to the two stochastic dominance option bounds. The argument that the volatility risk is diversifiable implies that C_t is the value of a portfolio of options that are contingent on the next volatility realization Y_{t+h}, and that the weights in this portfolio are proportional with the actual probabilities of Y_{t+h}. Assuming that we solved the linear programs (2.18) up to $t+h$, the following equation holds for the upper bound

$$\begin{aligned}C_t &\leq e^{-rh} E^Y \left[E^U \left[C_{t+h} \left| S_t, Y_{t+h} \right. \right] \right] \\ &\leq e^{-r(T-t)} E^{Y_{t,T}} \left[E^U \left[C_T \left| S_t, Y_{t,T} \right. \right] \right]\end{aligned}$$

where U denotes the univariate upper bound probability, $Y_{t,T}$ denote all the possible volatility paths to maturity and $E^Y, E^{Y_{t,T}}$ are expectations taken with respect to the volatility. Each of the volatility paths determines a univariate diffusion, for which Propositions 1 and 2 apply. The two bounds will converge to the discounted expectation of the payoff under the risk-neutral bivariate diffusion

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sigma_{SS}(S_t, Y_t)dW_S \\ dY_t &= \mu_Y(S_t, Y_t)dt + \sigma_{SY}(S_t, Y_t)dW_S + \sigma_{YY}(S_t, Y_t)dW_Y \end{aligned} \tag{3.28}$$

The fact that the volatility equations do not change under the risk-neutral probability measure can be expressed by a zero price of volatility risk. The common limit of the two option pricing bounds is a risk-neutral option price obtained under this price of risk assumption.

In fact, the pricing of volatility risk is an artifact of risk-neutral pricing in multivariate diffusion models. The change of probability measure is an application of Girsanov's theorem. The martingale pricing principle implies that the drift of the return must equal the riskless interest rate under the risk-neutral probability measure, but says nothing about how the other state variables are affected. That is, the martingale pricing does not say how the change of measure affects μ_Y in equation (3.28). The assumption that the volatility risk is diversifiable implies that μ_Y stays the same. When volatility is priced, the so-called "price of the volatility risk" provides a relation between μ_Y and μ'_Y, the volatility drifts under the two probability measures.

Implementations of stochastic volatility models handle this ambiguity by fitting a risk-neutral model of the returns. Equilibrium models resolve this issue by adding extra assumptions on preferences. In a paper that examines the convergence of GARCH processes to their diffusion limits, Duan (1996b) obtains a price of volatility risk that is implied by the convergence of the GARCH process under the equilibrium probability measure.

In principle, the linear programming approach can incorporate a discount factor that is contingent on volatility. The main problem with this approach is that, while the decreasing stochastic discount factor as a function of consumption is an expression of risk-aversion, we don't have an economic argument that tells us how the volatility states should be discounted.

However, stochastic dominance provides a more natural way to price contingent claims. Consider for instance the following discretization of the Hull and White (1987) model:

$$\begin{aligned} \ln \frac{S_{t+h}}{S_t} &= \mu h + V_{t+h}\varepsilon_1\sqrt{h}, \quad \varepsilon_1 = \pm 1 \\ \ln \frac{V_{t+h}}{V_t} &= mh + \sigma_V \varepsilon_2 \sqrt{h}, \quad \varepsilon_2 = \pm 1 \end{aligned} \tag{3.29}$$

The states at $t+h$ are

$\ln(S_{t+h}/S_t)$	$\ln(V_{t+h}/V_t)$	Probability
$z_1 = \mu h + V_1\sqrt{h}$	$V_1 = mh + \sigma_V\sqrt{h}$	1/4
$z_2 = \mu h + V_2\sqrt{h}$	$V_2 = mh - \sigma_V\sqrt{h}$	1/4
$z_3 = \mu h - V_2\sqrt{h}$	$V_2 = mh - \sigma_V\sqrt{h}$	1/4
$z_4 = \mu h - V_1\sqrt{h}$	$V_1 = mh + \sigma_V\sqrt{h}$	1/4

It can be verified that this discrete process converges weakly to the bivariate diffusion of Hull and White (1987). This discretization scheme can accommodate stochastic volatility processes with correlated return and volatility by changing the probabilities of the four states. The four values of the return S_{t+h}/S_t are ordered, and we can obtain the stochastic dominance bounds by recursively solving the two univariate linear programs (2.18). The two solutions of the linear programs can be expressed as two risk-neutral probability measures U and L, which change both the price and volatility processes. In general, the convexity of $C_{t+h}(z)$ does not hold and equations (2.22) and (2.23) may not apply. However, the probabilities obtained by solving (2.18) have a finite support and the lemma still applies, so the limit bound processes are diffusions. The parameters of the limit diffusions depend on the solution of the two dynamic programs and are not available in closed form. Conversely, in the arbitrage framework a single parameter - the price of volatility risk - determines the transformation between the actual and the risk-neutral process.

The discretization (3.29) can illustrate the stochastic dominance pricing for a bivariate diffusion of the underlying asset, but is of little practical use, since it uses a non-recombining tree. We can obtain the stochastic dominance bounds for a wide range of stochastic volatility processes using the lattice proposed by Leisen (2000). This bivariate tree can approximate stochastic volatility processes described by the following specification:

$$\begin{aligned} dS_t &= \nu(V_t)S_t dt + \psi(V_t)S_t dW_t^1 \\ dV_t &= \kappa(V_t - \bar{V})dt + \varphi(V_t)dW_t^2 \end{aligned}$$

with $corr(dW_t^1, dW_t^2) = \rho$. The discretization uses two volatility states and eight return states. Figure 3.3 shows a possible snapshot from this lattice. For a given price and volatility, the next period states are obtained by finding the closest points on the grid, such that the moments of the discrete process satisfy equations (3.7) and (3.8) and the resulting state probabilities are positive. Please refer to Leisen (2000) for the details of this discretization.

Another possible discretization is a GARCH approximation of the stochastic volatility process. Nelson (1990) and Duan (1997) proved the convergence of several GARCH processes to bivariate diffusions, including some popular stochastic volatility models. GARCH models have a single source of randomness, which perturbs both the price and the volatility. The application of the stochastic dominance approach

$$
\begin{array}{ccc}
 & & S_{t+1}^{uuu}, v_{t+1}^{u} \\
S_{t+1}^{duu}, v_{t+1}^{d} & & \\
 & & S_{t+1}^{uud}, v_{t+1}^{u} \\
S_{t+1}^{dud}, v_{t+1}^{d} & & \\
 & S_t, v_t & \\
S_{t+1}^{ddu}, v_{t+1}^{d} & & S_{t+1}^{udu}, v_{t+1}^{u} \\
S_{t+1}^{ddd}, v_{t+1}^{d} & & \\
 & & S_{t+1}^{udd}, v_{t+1}^{u}
\end{array}
$$

Figure 3.3: Prices and volatilities in Leisen's stochastic volatility trees

to the pricing of options on GARCH processes will be discussed in detail in the next chapter.

Figure 3.4 illustrates the convergence of the two option bounds for an at-the money call option with strike price $K = 100$ and maturity $T = 0.25$ years on a mean-reverting stochastic volatility process following the specification of Heston (1993). The process has an annual mean return $\mu = 0.05$ and an initial annual volatility of $\sqrt{V_0} = 0.1$. The mean reversion coefficient for the variance equation is $\kappa = 2$, while the long-run return variance is $\theta = 0.01$ and the volatility of the volatility is $\sigma = 0.1$. The two Brownian motions are negatively correlated, with $\rho = -0.5$. The riskless interest rate is $r = 0.03$. The bounds were computed by the Monte Carlo simulation method detailed in the next chapter, using the analytical option price as a control variate. The horizontal line depicts this price, which was computed under the assumption that the market price of volatility risk is zero. The plot shows a spread of less than 5%, which persists as the number of periods increases. The slow convergence of the upper bound can be attributed to the fact that the bounds were computed by discretising distinct realizations of the simulated processes. Thus, the simulation used to compute the 50-period bounds was not the same as the simulation used to compute the 300-period bounds. Moreover, the complexity of

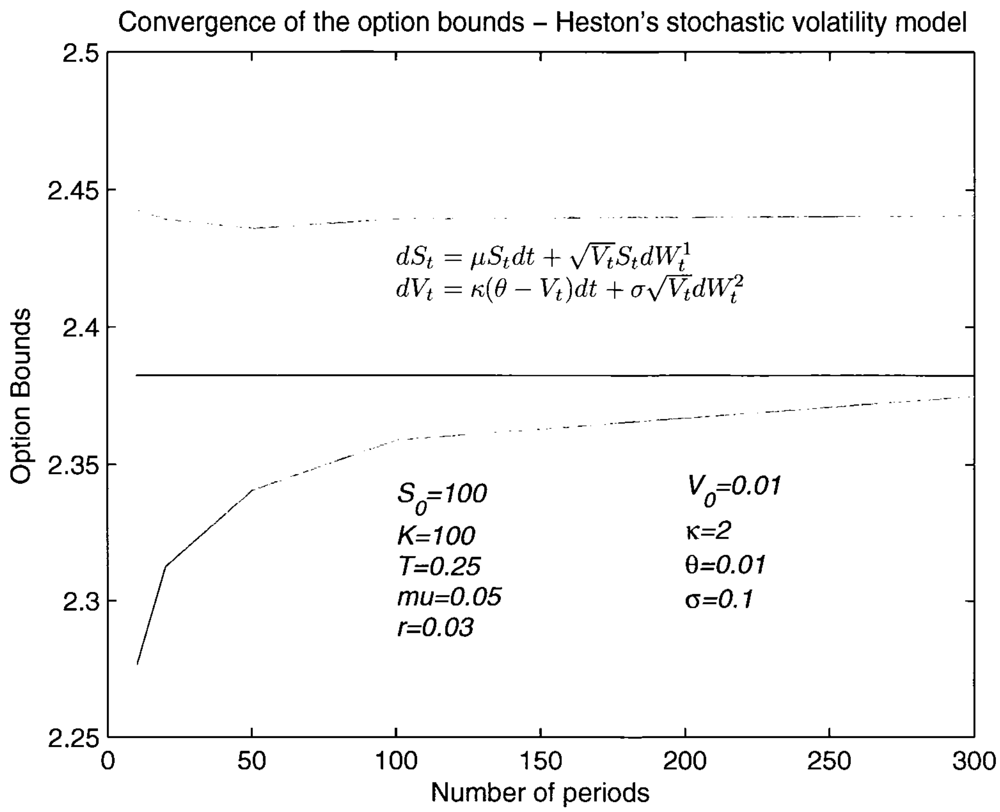

Figure 3.4: Convergence of the option bounds when the underlying process follows a stochastic volatility model

the bivariate process limits the number of states that can be spanned by a Monte Carlo simulation. Conversely, in the case of the bounds depicted in Figures 3.1 and 3.2, all the discretizations were sampled from the same process, which spanned all the possible realizations of the multinomial process for the given number of periods. Better results might be obtained with other discretizations of the stochastic volatility process. The discretization scheme used to approximate a continuous time process is an important implementation aspect of the stochastic dominance bounds and will be studied in future research.

In principle, this spread can be mapped into different values of the price of volatility risk by fitting risk-neutral option prices to the two option bounds. However, such models would be misspecified. When applying a risk neutral pricing approach, the price of risk is assumed to have a very simple functional form. This functional form does not matter when the risk-neutral model is estimated from option prices. But when the actual and risk neutral models are put side by side, for instance in Bates (1996), the parameters of the two models can hardly be reconciled only by a price of risk parameter. Likewise, since the two limit processes are the solutions of a dynamic program, their functional form is not available in closed form and cannot be obtained by transforming the actual process via the price of volatility risk.

Chapter 4

Preference-free option pricing under GARCH

One of the most documented empirical finding about asset returns is their departure from lognormality. In option pricing models, this departure is best described by the smile or smirk pattern obtained by plotting the implied volatilities of the options on the same asset against their strike price. This finding contrasts with the Black and Scholes (1973), assumption of lognormal returns, which would imply a constant volatility, irrespective of the strike price.

The first models that departed from the constant volatility model of Black and Scholes (1973) were more general univariate diffusions, such as the constant elasticity of variance specification of Cox and Ross (1976) or the displaced diffusion model of Rubinstein (1983). These models were soon replaced by bivariate diffusions, which specify a dynamic behavior of the asset volatility that is independent of the strike price. Such models have been studied by Garman (1976b), Hull and White (1987), Stein and Stein (1991) or Heston (1993). The more recent models of Bates (1996), Bakshi, Cao, and Chen (1997) or Duffie, Pan, and Singleton (2000) incorporated jumps in such multivariate specifications.

Stochastic volatility models have been developed with mathematical tractability in mind. To obtain a closed formula, the volatility, jumps and perhaps the pricing kernel must be affine functions of two or more state variables, of which only the asset price is observable. The difficulty of estimating these variables adds to the misspecification caused by the affine model assumption.

An alternative specification of the volatility randomness was introduced in the econometric literature by Engle (1982) and Bollerslev (1986). The (generalized) autoregressive conditional heteroscedasticity models or (G)ARCH are discrete time models that capture the clustering of the volatility of the financial returns. In all these models, volatility is a function of past values of the volatility and the return innovations. There is a very rich literature that specializes in the estimation and forecasting of the volatility using this class of models. See Bollerslev, Chou, and Kroner (1992) for an in-depth discussion.

The success of GARCH models in describing the asset returns made this class of models a good candidate for use in option pricing. However, any option pricing application of a GARCH model must address the market incompleteness created by the infinite number of possible outcomes of the innovation in any discrete time period. Duan (1995) addressed market incompleteness by using a multiperiod version of the equilibrium framework of Rubinstein (1976) and Brennan (1979). He established that, for certain types of preferences and return distributions, the option price can be computed as if the return process were risk neutral during any discrete time period. This set of conditions was called a local risk-neutral valuation relationship (LRNVR). Most GARCH specifications assume a conditional normal distribution of the log return innovations. In this case, a LRNVR is provided by constant proportional risk aversion (CPRA) preferences. Any extension of the GARCH model to more general distributions of the return innovation, such as the jump models of Duan, Ritchken, and Sun (2005), or the closed form GARCH model with a normal inverse Gaussian distribution developed by Christoffersen and Jacobs (2005b) requires a dedicated functional form of the pricing kernel, in order that a LRNVR holds[1].

This very strict preference requirement is a serious drawback of the GARCH models in option pricing. One could use the now traditional approach of estimating the model parameters from the cross-section of option prices. However, Christoffersen and Jacobs (2004) found discrepancies between the parameters of the risk-neutral process implied by the cross-section of options and the actual process estimated from the time series of the returns for several GARCH specifications. This finding implies that the pricing kernel used to price the options might be misspecified.

Besides the abundance of parametric specifications aiming at the description of various properties of the financial returns, a more accurate model that could capture the skewness and the fat tails of the conditional return distribution was desirable. Engle and Gonzalez-Rivera (1991) introduced a class of semiparametric ARCH models, where the return and volatility were described by a standard ARCH type specification, but the return innovation had a general distribution, specified by a nonparametric model. Option pricing applications for this class of models have been developed by Duan (2002) and Barone-Adesi, Engle, and Mancini (2004). To address market incompleteness, the former study prices the options under the risk neutral distribution that satisfies a minimum entropy criterion, while the latter estimates the two models implied by the actual and risk-neutral distributions without restricting the pricing kernel implied by the two distributions. Interestingly, the state price density implied by the two terminal distributions of the underlying asset is decreasing

[1] As an alternative to the preference-based LRNVR approach, Christoffersen and Jacobs (2005a) derived an arbitrage-based risk neutral probability measure for the pricing of options under GARCH.

for all values of the underlying asset. This property implied by risk aversion is not featured by the pricing kernels obtained through other empirical procedures.

In this chapter I derive preference free option pricing models for GARCH specifications of the underlying asset price. The models are an application of the stochastic dominance pricing methodology established by Perrakis and Ryan (1984), Ritchken (1985) and others. The approach consists in finding upper and lower option pricing bounds under which no stochastically dominating strategies are available in a market containing the underlying asset, the riskless bond and the option. Compared to the RNVR approaches, the stochastic dominance approach requires only that investors are risk-averse. Moreover, the approach does not require any assumptions regarding the return distribution. The stochastic dominance methodology works for any specification of the GARCH dynamics, including nonparametric models of the return innovations. Thus, model estimation by GARCH and stochastic dominance pricing provide a robust option pricing methodology that applies to any process, without requiring assumptions that cannot be validated empirically.

The details of the stochastic dominance approach have been presented in Chapter II. The two option bounds are obtained as discounted expectations of the option payoff under two risk-neutral probability measures. This relates the stochastic dominance option pricing with the arbitrage approach used in the continuous time models and with the RNVR approach used in the pricing of options on processes described by GARCH models.

Unlike in the previous applications, where the convexity of the option price with respect to the asset return resulted in closed forms of the two risk-neutral distributions, the convexity cannot be guaranteed for GARCH processes. However, the two bounds can be obtained by dynamic programming.

The next section presents the GARCH option pricing under the equilibrium and stochastic dominance approaches. Section 4.2 presents a numerical method used to price the stochastic dominance bounds for non-i.i.d. return processes. The last two sections present some results. Section 4.3 examines the stochastic dominance pricing bounds for conditionally normal and non-normal distributions of the return innovations. The implications of choosing a wrong model are presented in Section 4.4.

4.1 Option pricing under GARCH

4.1.1 The equilibrium approach

The literature on GARCH models for asset returns is very rich, with specifications that target various types of behavior of asset prices. This abundance of models is a consequence of the flexibility of the GARCH type models, which differ only by the formula used to update the volatility. Most option pricing applications have been developed around the NGARCH specification of Engle and Ng (1993) and Duan (1996a), but the extension to any other specification, such as the GJR-GARCH of Lawrence R. Glosten (1993) or the EGARCH model of Nelson (1991) is straightforward. For the sake of clarity, I use the NGARCH model throughout this chapter.

The NGARCH model assumes the following price and volatility dynamics under the probability measure P, which reflects the representative investor's beliefs

$$\begin{aligned} \ln \frac{S_{t+1}}{S_t} &= r + \lambda\sqrt{h_{t+1}} - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\epsilon_{t+1} \\ h_{t+1} &= \beta_0 + \beta_1 h_t + \beta_2 h_t \left(\epsilon_t - \gamma\right)^2 \qquad (4.1) \\ \epsilon_{t+1}|\mathcal{F}_t &\sim \mathcal{D}(0,1) \end{aligned}$$

In the log return equation, S_t is the asset price at time t, r is the riskless interest rate, λ is the price of risk[2] and h_{t+1} is the conditional variance of the random component. The second equation specifies the return variance, which differentiates among the GARCH specifications. In order that the volatility be positive the coefficients β_0, β_1 and β_2 must be positive. The positive parameter γ captures the negative correlation between the return and volatility. The log return and variance are driven by the same random component. In the specification of Duan (1996a), the conditional distribution of the innovation $\epsilon_{t+1}|\mathcal{F}_t$ is standard normal. More recent models depart from this assumption.

Since the return described by equation (4.1) can take an infinite number of values at any moment in time, the market containing a riskless bond and an asset following this price dynamic is obviously incomplete. The absence of arbitrage opportunities does not provide a unique option price. In his GARCH option pricing model, Duan (1995) applied the equilibrium approach of Rubinstein (1976) and Brennan (1979). He found that, for lognormal returns and a power utility of the representative investor, there exists a probability measure Q under which the option can be priced by assuming risk-neutrality.

The price and volatility dynamics under Q are described by the following GARCH

[2]An exception is the model of Heston and Nandi (2000), which uses a risk premium that is proportional with the return variance, rather than the standard deviation.

$$\ln \frac{S_{t+1}}{S_t} = r + \left(\lambda - \frac{1}{2}\right) h_{t+1} + \sqrt{h_{t+1}}\epsilon_{t+1}$$

However, the option pricing principles are the same.

model

$$
\begin{aligned}
\ln \frac{S_{t+1}}{S_t} &= r - \frac{1}{2} h_{t+1} + \sqrt{h_{t+1}} \epsilon^*_{t+1}, \\
h_{t+1} &= \beta_0 + \beta_1 h_t + \beta_2 h_t \left(\epsilon^*_t - \gamma *\right)^2, \\
\epsilon^*_{t+1} | \mathcal{F}_t &\sim \mathcal{N}(0,1)
\end{aligned} \tag{4.2}
$$

where $\epsilon^*_{t+1}|\mathcal{F}_t$ is a standardized normal random variable under Q and $\gamma^* = \gamma + \lambda$. The price at time t of a European contingent claim with maturity T and payoff described by the function $g(S_T)$ is then

$$
c_t = e^{-r(T-t)} E^Q[g(S_T)|\mathcal{F}_t], \tag{4.3}
$$

the conditional expectation of the payoff under Q, given the information available at time t.

The risk neutral valuation approach is very tractable, but unfortunately relies on strong preference assumptions. Under the conditional lognormality assumption used in early GARCH models, any power utility function would provide a LRNVR. More recent GARCH models depart from the conditional lognormality assumption in the attempt to provide a better description of the fat tails observed in daily returns. The departure from the conditional lognormality assumption requires a pricing kernel that is tailored to the functional form of the return distribution, in order that a LRNVR holds. For instance, in the GARCH model with jumps of Duan, Ritchken, and Sun (2005), the pricing kernel and the return process must have a jump component with the same intensity. In the model with inverse-gaussian innovations by Christoffersen and Jacobs (2005b), the LRNVR is provided by a power utility function with parameters related to the parameters of the underlying innovation distribution. The pricing approach proposed in the next section can accommodate any conditional return distribution without making other preference assumptions than risk aversion.

4.1.2 The stochastic dominance approach

The stochastic dominance option pricing approach introduced by Perrakis and Ryan (1984) extended the equilibrium framework of Rubinstein (1976) and Brennan (1979) under very weak preference assumptions. The approach relies on the absence of stochastically dominating portfolios containing the underlying asset, the option and the riskless asset. This condition provides two bounds on admissible option prices.

The stochastic dominance pricing approach has been presented in detail in Chapter II. The pricing problem consists in finding the price at time $t = 0$ of a contingent claim on the stock with the return dynamics described by the GARCH model 4.1. The claim pays $C_T = g(S_T)$ when it expires. The payoff of a call option with exercise price K is $g(S_T) = max(S_T - K, 0)$, while the payoff of a put option with the same exercise price is $max(K - S_T, 0)$.

In the formulation of Ritchken (1985), the option bounds problem reduces to a pair of linear programs that minimize, respectively maximize the option price c_t, provided that the underlying asset and the riskless bond are correctly priced by a risk-averse representative investor.

$$\min_{m_{t+1}}(\max) c_t = E[m_{t+1} c_{t+1} | \mathcal{F}_t]$$

subject to

$$S_t = E[m_{t+1} S_{t+1} | \mathcal{F}_t]$$
$$\frac{1}{R} = E[m_{t+1} | \mathcal{F}_t]$$
$$m_{t+1} \text{ non-increasing function of } S_{t+1}$$

where m_{t+1} is the stochastic discount factor used to price assets at time t. We include the assumption that investors are risk-averse by constraining the stochastic discount factor to be a non-increasing function of S_{t+1}. This monotonicity condition holds for assets with a positive consumption beta. In the case of the GARCH process described by equations (4.1), this condition is provided by the positive risk premium factor λ.

The solutions of the two linear programs are two stochastic discount factors $\bar{m}_{t+1}$ and $\underline{m}_{t+1}$ under which the option bounds are attained. As shown in Chapter II, the two bounds can also be expressed as discounted conditional expectations of the payoff under two probability measures U and L.

$$\bar{c}_t = \frac{1}{R^{T-t}} E^U[g(S_T) | \mathcal{F}_t]$$
$$\underline{c}_t = \frac{1}{R^{T-t}} E^L[g(S_T) | \mathcal{F}_t]$$

This formulation of the option bounds is useful in proving convergence results such as the diffusion limits obtained in Chapter III. It is also useful in the implementation of the two bounds. However, the two probability measures U and L are available in closed form only when the next period option price is a convex function of next period return.

In the case of a GARCH process, we can invoke the criterion of Merton (1973), that the option price is a convex function of the underlying asset price when the return distribution is independent of the price. That is, the option price $C_t(S_t)$ is a convex function if the terminal return distribution $f(S_T/S_t|\mathcal{F}_t)$ is independent of S_t. Unfortunately, we cannot invoke this criterion to verify the convexity of the next period option price C_{t+1} with respect to the asset return S_{t+1}/S_t. The GARCH dynamics affect both price and volatility, and next period's option prices will be realized in different volatility states.

An intuitive argument against convexity is that, for a strictly increasing set of next period prices driven by increasing ϵ_t, the next period volatility will have a minimum at $\epsilon_{t+1} = \gamma$. The option price change between states with ϵ_{t+1} slightly below γ, will have an increasing component as a result of increasing volatility and a decreasing component caused by the decreasing price. With typical values of the leverage parameter γ around one, this option price behavior corresponds to a one standard deviation price increase and should be captured by the return model.

4.2 A Markov chain approximation

The implementation of the linear programs (2.18) requires a discrete representation of the conditional return distribution. Monte Carlo simulation has been the traditional approach to the pricing of options on assets following a GARCH process. However, its application to the stochastic dominance pricing problem (2.18) is not straightforward. The main difficulty of applying Monte Carlo simulation is that, in finding the two option bounds, we need to maximize, respectively minimize the *conditional* expectation of the next period option price. However, each path in the Monte Carlo simulation provides a distinct value of the price and volatility at each moment of time. Since Monte Carlo computes expectations by taking the mean of a sample, we need a way to produce such samples.

An alternative approach is the Markov chain approximation of Duan and Simonato (2001). In this approach, the return process is approximated by a homogeneous Markov chain. The ingredients of the Markov chain approximation are a discretization of the state space and a matrix containing the probabilities of transitions between the states. In the case of a Black-Scholes diffusion, the states are defined by log price intervals. In a Markov chain with n states, the process is in state i whenever the log price belongs to the interval $[L_i, L_{i+1}[$, with $L_1 = -\infty$ and $L_{n+1} = \infty$. In the Markov chain approximation of a GARCH process, the states are defined by log price-volatility pairs. Both the Black-Scholes and the GARCH process have a univariate random component ϵ_{t+1}, which is assumed to be normally distributed. The transition probabilities are obtained in closed form from the distribution of this random component. For instance, in the case of the Black-Scholes diffusion, with drift μ and diffusion coefficient σ, the transition probability from state i to state j is

$$\begin{aligned}
&\operatorname{Prob}\left(L_j \leq L(t+1) \leq L_{j+1} | L(t) = (L_i + L_{i+1})/2\right) \\
&\qquad = N\left[\frac{L_{j+1} - (L_i + L_{i+1})/2 - \mu}{\sigma}\right] - N\left[\frac{L_j - (L_i + L_{i+1})/2 - \mu}{\sigma}\right],
\end{aligned}$$

where $N[\cdot]$ is the cumulative normal distribution function. In a similar manner, Duan (2002) computes the transition matrix of a GARCH process with a general innovation term.

The risk-neutral pricing of European and American options reduces to a matrix multiplication. If Q denotes the risk-neutral transition probability, the price at time 0 of a European option with payoff $g(S_T)$ is $R^{-T}Q^T g(S)$, where the vector S contains the prices in all the states of the Markov chain.

This approach can be easily adapted to the stochastic dominance pricing of options on i.i.d. processes, where one could use equations (2.22) and (2.23) to derive the transition matrices of the two bound pricing probabilities. However, the possible convexity violations in the case of non-iid processes require the recursive solving of the linear programs (2.18) at each period of time, rather than computing the bound probabilities and taking the expectations by a matrix multiplication. A Markov chain approximation with 501 prices and 101 volatilities, used for instance in Duan (2002) would require the solving of 50601 linear programs in each discrete time period. However, the size of these problems is small in the case of a GARCH process.

The method adopted here is a combination of Monte Carlo simulation and Markov chain approximation. The insight of this method is Merton's finding that the option price is homogeneous in the underlying asset, if the return distribution is independent of the price. This property holds for many GARCH and stochastic volatility specifications.[3] If we condition the return distribution on the volatility, we can group all the paths that have the same volatility at time t and solve the linear programs (2.18) using $S_t^{'} = 1$, $S_{t+1}^{'} = S_{t+1}/S_t$ and $c_{t+1}^{'} = c_{t+1}/S_t$. From the solutions $\bar{c}_t^{'}$ and $\underline{c}_t^{'}$ of the two problems, we recover the option bounds at time t as $\bar{c}_t = S_t\bar{c}_t^{'}$ and $\underline{c}_t = S_t\underline{c}_t^{'}$.

We can condition on the volatility by approximating this state variable with a discrete variable that can take m values. In a Monte Carlo simulation, we can ac-

[3]Remark that the homogeneity of the option price C_t in the underlying asset price S_t holds for all paths that have a common volatility h_t. There is a similar convexity property of $C_t(S_t)$, but the closed form solutions of the bounds probabilities require the convexity of all paths of $C_{t+1}(S_{t+1})$ that start from h_t. The latter convexity property does not hold in general.

complish this by accomplish this by assigning the value $(V_i + V_{i+1})/2$ to any volatility belonging to the interval $[V_i, V_{i+1}[$. This is akin to a Markov chain approximation of the volatility, but not of the price. By sampling from a given return process and assigning volatilities in this manner, the transition probability is built into the Monte Carlo simulation.

The same state aggregation approach is used to compute the closed form pricing probabilities given by equations (2.22) and (2.23). Using this closed form, the bounds are computed much faster than by linear programming. However, the cost of this gain in speed is the bias of the bounds when the option price convexity is violated. The closed form upper bound is lower than the maximum objective of (2.18), while the closed form lower bound is higher than the minimum objective of this program. Consequently, the closed form bounds are tighter than the optimal bounds obtained from the linear programs.[4]

While there is no mention of this Monte Carlo technique in the mainstream option pricing literature, this feature is available in Monte Carlo simulation software. See for instance Meyer (2004). This technique was used in the previous chapter to price the two bounds of a stochastic volatility process.

4.3 The impact of the innovations distribution

The stochastic dominance option pricing approach can easily be applied to the available GARCH models without forced assumptions about the pricing kernel. In the tradition of the GARCH option pricing literature, the return process follows equations (4.1), which describe the NGARCH(1,1) of Engle and Ng (1993) and Duan (1996a). To illustrate the power of the stochastic dominance approach, two distri-

[4]Due to the numerical instability of the available linear programming software, all the results in this chapter report the closed form bounds

butions of the return innovation will be examined

- normal distribution
- Generalized Error Distribution (GED)

The latter distribution is widely used in the modeling of fat-tailed asset returns. Duan (1999) developed a risk-neutral pricing approach for GARCH processes with conditional returns described by a GED distribution. The GED distribution is standardized, such that the innovations have mean zero and variance one. The parameter of the GED distribution is $v = 1.6$, obtained by estimating a GARCH(1,1) model from S&P 500 returns during 2000-2005[5]. Figure 4.1 depicts the two distributions.

The underlying asset is assumed to follow a GARCH specification similar to that of Duan and Simonato (2001). Under the actual return distribution, the process follows equations (4.1) with the parameters $\beta_0 = 0.00001$, $\beta_1 = 0.8$, $\beta_2 = 0.1$, $\gamma = 0.49$ and $\lambda = 0.01$. The annualized riskless rate is $r = 0.05$. The initial stock price is $S_0 = 100$ and the initial return volatility is assumed to be equal to the stationary volatility $h_1 = \beta_0/(1 - \beta_1 - \beta_2/(1 + \gamma^2))$, which gives an annualized volatility of 0.18.

Table 4.1 presents the prices and Figure 4.2 depicts the volatility smiles for 3-month options. The stochastic dominance bounds were computed in closed form using equations (2.22) and (2.23), while the equilibrium prices were computed using the approach of Duan (1996a). It can be seen from the table that both normal bounds bracket the equilibrium prices. The tight fat-tailed bounds indicate larger biases of the closed form bounds from the solutions of the linear programs (2.18). These biases indicate violations of the convexity assumption underlying the closed form bound

[5]This parameter of the GED distribution is somewhat artificial. The estimation of a NGARCH(1,1) from the series of S&P 500 returns using 1-month LIBOR rates as riskless rates yielded normal residuals.

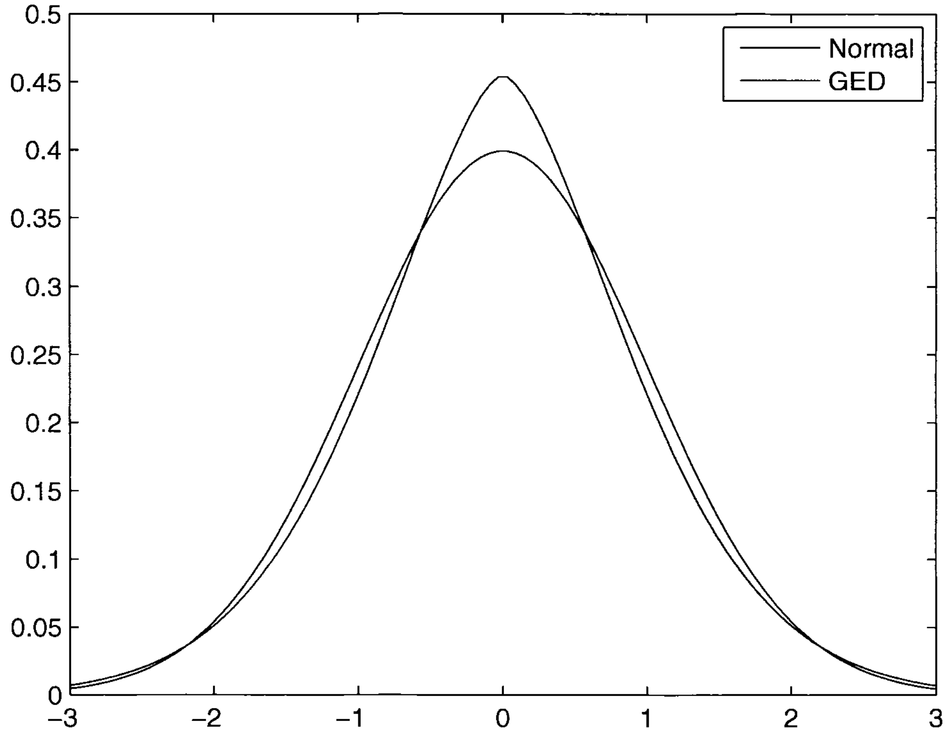

Figure 4.1: Probability densities of the GARCH error term

	Normal innovations			GED innovations	
Strike	Equilibrium	Upper	Lower	Upper	Lower
		T=1 month			
90	10.5057	10.5231	10.5041	10.6031	10.5928
95	5.8740	5.9072	5.8579	5.9696	5.9406
100	2.2743	2.3116	2.2378	2.3457	2.3002
105	0.5017	0.5214	0.4696	0.5301	0.4984
110	0.0597	0.0641	0.0491	0.0652	0.0559
		T=3 months			
90	11.6602	11.7016	11.6218	11.9319	11.8680
95	7.4951	7.5667	7.4300	7.7461	7.6512
100	4.1789	4.2620	4.0938	4.3822	4.2709
105	1.9568	2.0288	1.8718	2.0947	1.9925
110	0.7492	0.7962	0.6843	0.8250	0.7524

Table 4.1: GARCH option prices for normal and GED innovations. The parameters are $S_0 = 100$, $r = 0.05$ (annualized). The parameters of the GARCH process are $\beta_0 = 0.00001$, $\beta_1 = 0.8$, $\beta_2 = 0.1$, $\gamma = 0.49$, $\lambda = 0.01$. The initial return volatility is 0.18 (annualized). The GED innovations are standardized and the parameter of the GED distribution is $v = 1.6$

pricing probabilities (2.22) and (2.23). Another indication of such violations is that the equilibrium prices of "in the money" options under GED distributed conditional returns (not reported here) violate the lower pricing bound. An in-depth analysis of the implications of the convexity assumptions is left for future research.

4.4 Option bounds from estimated models

This final application examines the option bounds implications of the model estimation. We assume that we know the underlying asset process and we sample the returns from this process. We compute the option pricing bounds using the estimated models and compare them with the bounds derived from the true model.

Let us assume that the returns are sampled from an NGARCH process with constant mean returns and conditionally normal innovations:

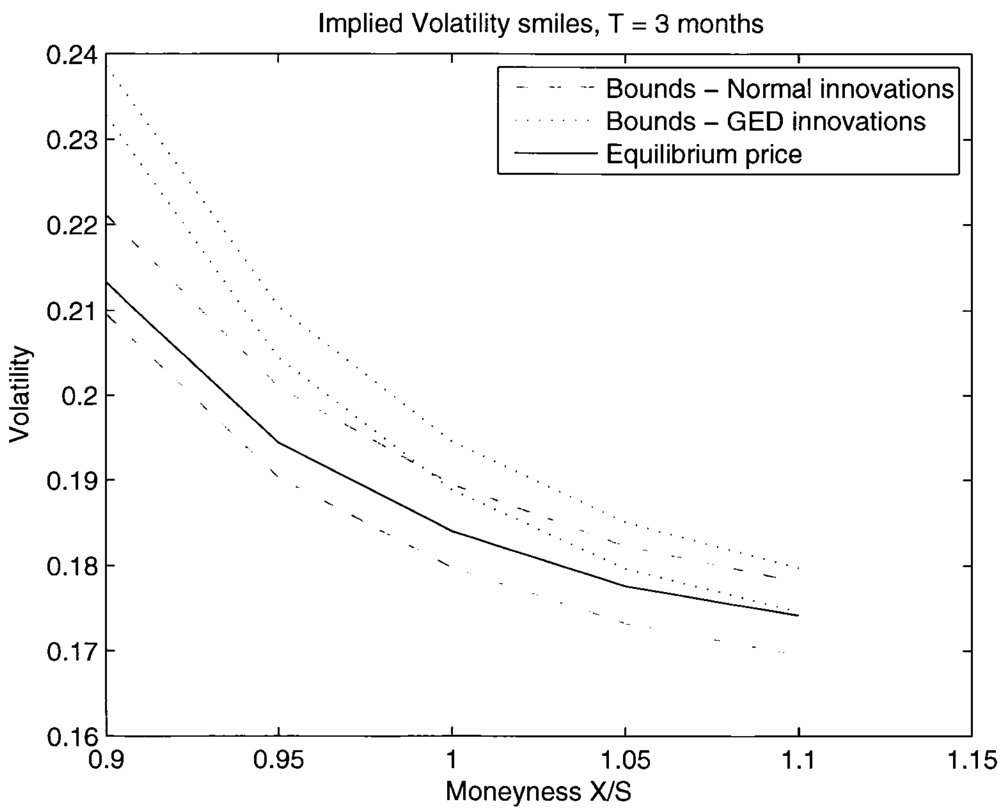

Figure 4.2: Implied volatility smiles for the option bounds and equilibrium price

$$
\begin{aligned}
\ln \frac{S_{t+1}}{S_t} &= \mu - \frac{1}{2} h_{t+1} + \sqrt{h_{t+1}} \epsilon_{t+1} \\
h_{t+1} &= \beta_0 + \beta_1 h_t + \beta_2 h_t \left(\epsilon_t - \gamma\right)^2 \\
\epsilon_{t+1} | \mathcal{F}_t &\sim \mathcal{N}(0,1)
\end{aligned}
\tag{4.4}
$$

Unlike the process described in the previous section, this specification does not contain a GARCH-in-mean term. While such a term may be helpful in modeling risk premia, the estimation of this term is not very reliable. In general this term is not statistically significant and its unconstrained estimation may result in negative values during periods of market decline.

With the exception of λ which is now zero, the process has the same parameters as the process examined in the previous section: $\beta_0 = 0.00001$, $\beta_1 = 0.8$, $\beta_2 = 0.1$ and $\gamma = 0.49$. The annualized riskless rate is $r = 0.05$ and the mean return is 0.09. The initial stock price is $S_0 = 100$ and the initial return volatility is assumed to be equal to the stationary volatility $h_1 = \beta_0/(1 - \beta_1 - \beta_2/(1 + \gamma^2))$, which gives an annualized volatility of 0.18.

The theoretical pricing bounds for options with 1 and 3 months to maturity and moneyness between 0.9 and 1.1 are given in the leftmost columns of Table 4.6. The first application examines the implications of the errors in the estimation of a model with the same structure as the data generating model. The second application examines the bounds pricing implications of a misspecified model.

	β_0	β_1	β_2	θ
Min.	4.66E-06	0.650	0.043	0.083
1st Qu.	8.33E-06	0.769	0.082	0.406
Median	1.03E-05	0.797	0.098	0.503
Mean	1.05E-05	0.795	0.098	0.520
3rd Qu.	1.23E-05	0.823	0.112	0.616
Max.	2.54E-05	0.895	0.175	1.387
Std	3.04E-06	0.040	0.022	0.166

Table 4.2: Descriptive Statistics of the estimated NGARCH parameters
The models are estimated from 500 samples of 1500 returns each, that are randomly generated from the NGARCH process described by equation (4.4), with parameters $\mu = 0.09$, $\beta_0 = 0.00001$, $\beta_1 = 0.8$, $\beta_2 = 0.1$ and $\gamma = 0.49$.

4.4.1 Pricing with estimated parameters

In this section, we draw 500 samples of returns from the process (4.4) with the known parameters. Each sample has 1500 daily returns. We estimate the model parameters and compute the bounds using the point estimation of each sample. Table 4.2 presents the descriptive statistics of the estimated parameters and Table 4.3 presents the results. In the pricing of the bounds, the mean return was assumed to be the same as the theoretical mean. The top panel presents the theoretical bounds. The middle panel presents the descriptive statistics of the upper bounds and the number of violation cases in which both estimated bounds are below the theoretical lower bound. The bottom panel presents the descriptive statistics of the lower bounds and the number of violation cases in which both estimated bounds are above the theoretical upper bound. In most of the 81 violations of the theoretical bounds, the two bounds implied by the estimated model are above the theoretical upper bound. Most of these violations occur for deep in the money options.

Option Bounds, Actual and Estimated Processes					
Moneyness	0.9	0.95	1	1.05	1.1
	Actual Process Bounds				
Upper Bound	10.467	5.883	2.313	0.526	0.068
Lower Bound	10.394	5.769	2.169	0.432	0.042
	Estimated Processes, Upper Bound				
Min.	10.377	5.771	2.243	0.457	0.036
1st Qu.	10.450	5.853	2.290	0.512	0.059
Median	10.467	5.876	2.311	0.526	0.067
Mean	10.467	5.877	2.312	0.527	0.067
3rd Qu.	10.486	5.902	2.332	0.540	0.074
Max.	10.566	6.003	2.444	0.609	0.106
Std	0.027	0.037	0.032	0.023	0.011
Violation	5	0	0	0	2
	Estimated Processes, Lower Bound				
Min.	10.408	5.746	2.155	0.396	0.024
1st Qu.	10.436	5.795	2.192	0.441	0.040
Median	10.448	5.817	2.207	0.453	0.045
Mean	10.448	5.817	2.209	0.454	0.046
3rd Qu.	10.458	5.836	2.225	0.464	0.051
Max.	10.502	5.924	2.331	0.517	0.075
Std	0.016	0.030	0.026	0.018	0.008
Violation	53	13	1	0	7

Table 4.3: Option Bounds implied by estimated parameters

Option bounds implied by the estimation of 500 samples of 1500 returns drawn from the NGARCH process described by equation (4.4), with parameters $\mu = 0.09$, $\beta_0 = 0.00001$, $\beta_1 = 0.8$, $\beta_2 = 0.1$ and $\gamma = 0.49$. In the pricing of the bounds, the mean return was assumed to be the same as the theoretical mean. The top panel presents the theoretical bounds. The middle panel presents the descriptive statistics of the upper bounds and the number of violation cases in which both estimated bounds are below the theoretical lower bound. The bottom panel presents the descriptive statistics of the lower bounds and the number of violation cases in which both estimated bounds are above the theoretical upper bound.

4.4.2 Pricing with a misspecified model

In this section, we draw a sample of 1500 daily returns from the known process and estimate several models from this sample. Figures 4.3, 4.4 and 4.5 present the simulated price, the histogram and the normal probability plot of the log returns. Table 4.4 presents the descriptive statistics and two normality tests of the log returns. It can be seen from this table that both the Kolmogorov-Smirnov and the Shapiro-Wilk test reject the normality. The departure of the log returns from normality is also obvious in the histogram and the normal probability plot.

The following models are estimated from the simulated data and used to price the options:

- an NGARCH model with the same structure
- a linear GARCH(1,1) model with $\gamma = 0$
- a nonparametric i.i.d. return model.

The first two models belong to the same class as the model from which the sample was drawn. While the first model has the same specification as the actual model, the second model has a simpler GARCH specification, which does not capture the leverage effect. While the two models are parametric, they provide us with an estimate of the conditional return distribution. The third model attempts to capture the properties of the returns from the data, without knowing the stochastic process that generated the data. The calculation of the stochastic dominance bounds requires only the return distribution, which can be obtained in this way.

Table 4.5 presents the estimated parameters. The option bounds are presented in Table 4.6. It can be seen from the table that both GARCH models outperform the nonparametric model. This finding suggests the importance of modeling the

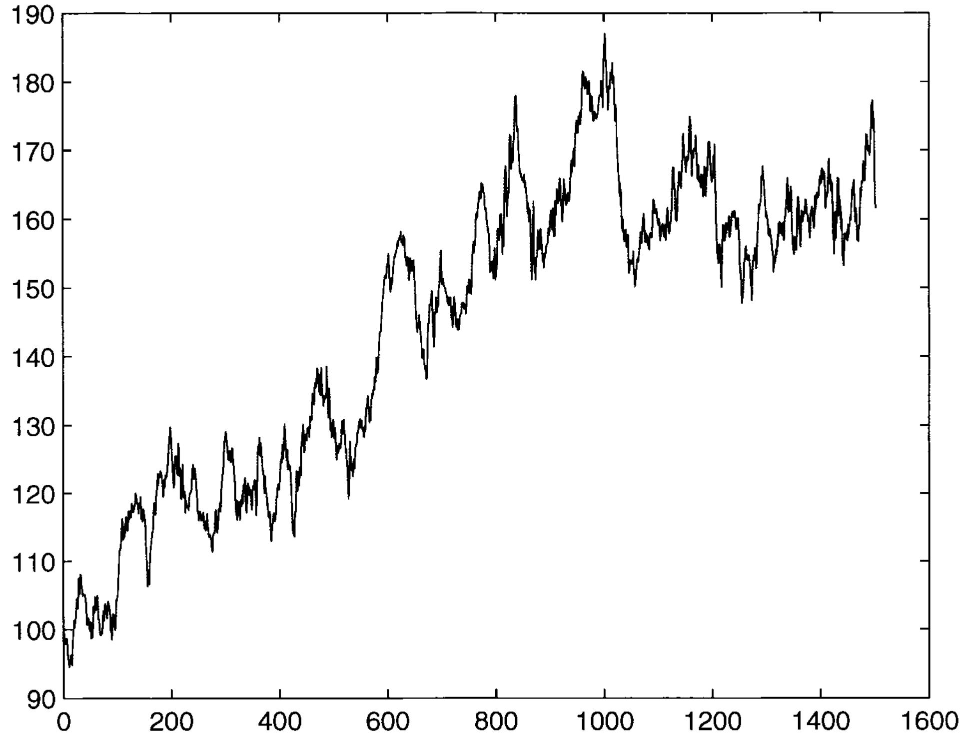

Figure 4.3: Simulated Price
The price was obtained by simulating the NGARCH process described by equation (4.4), with parameters $\mu = 0.09$, $\beta_0 = 0.00001$, $\beta_1 = 0.8$, $\beta_2 = 0.1$ and $\gamma = 0.49$.

Descriptive Statistics	
Min.	-0.05423
1st Qu.	-0.00719
Median	0.00015
Mean	0.00032
3rd Qu.	0.00758
Max.	0.04363
Normality tests	
Shapiro-Wilk	
W = 0.9971	p = 0.0079
Kolmogorov-Smirnov	
D = 0.4848	p < 2.2e-16

Table 4.4: Descriptive statistics of the log returns

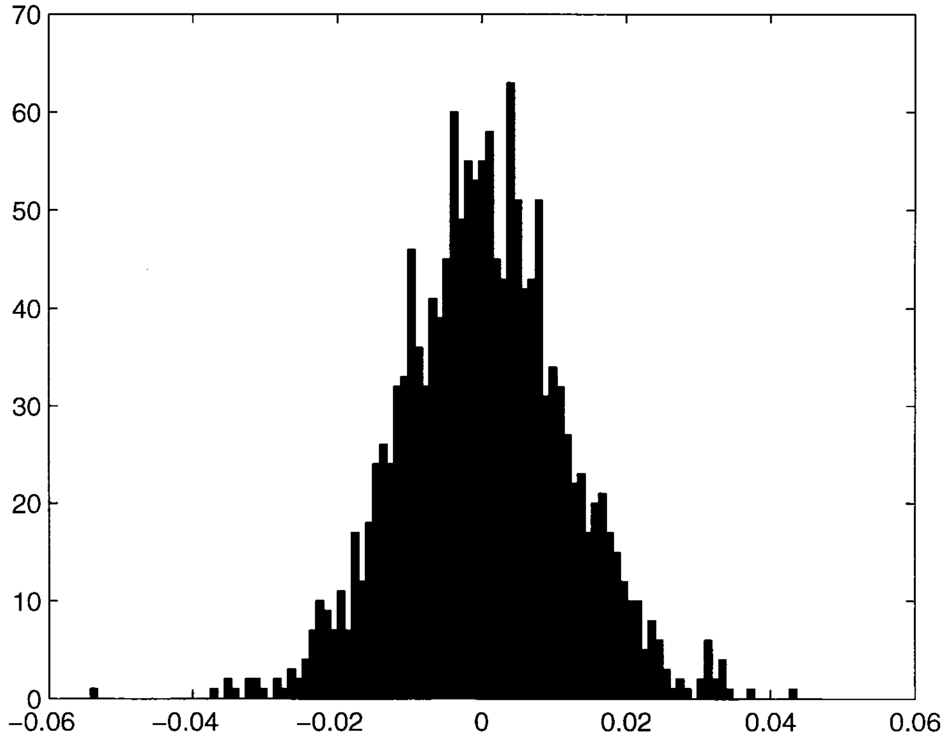

Figure 4.4: Histogram of the daily log returns
The returns are simulated from the NGARCH process described by equation (4.4), with parameters $\mu = 0.09$, $\beta_0 = 0.00001$, $\beta_1 = 0.8$, $\beta_2 = 0.1$ and $\gamma = 0.49$.

	NGARCH		GARCH	
	Coefficient	Std. Error	Coefficient	Std. Error
β_0	1.2E-05	3.95E-06	1.12E-05	3.74E-06
β_1	0.7705	0.04472321	0.768577	0.045313
β_2	0.1319	0.02337707	0.148774	0.026613
γ	0.2757	0.10611231		

Table 4.5: GARCH model parameter estimation
The models are estimated from returns that are randomly generated from the NGARCH process described by equation (4.4), with parameters $\mu = 0.09$, $\beta_0 = 0.00001$, $\beta_1 = 0.8$, $\beta_2 = 0.1$ and $\gamma = 0.49$.

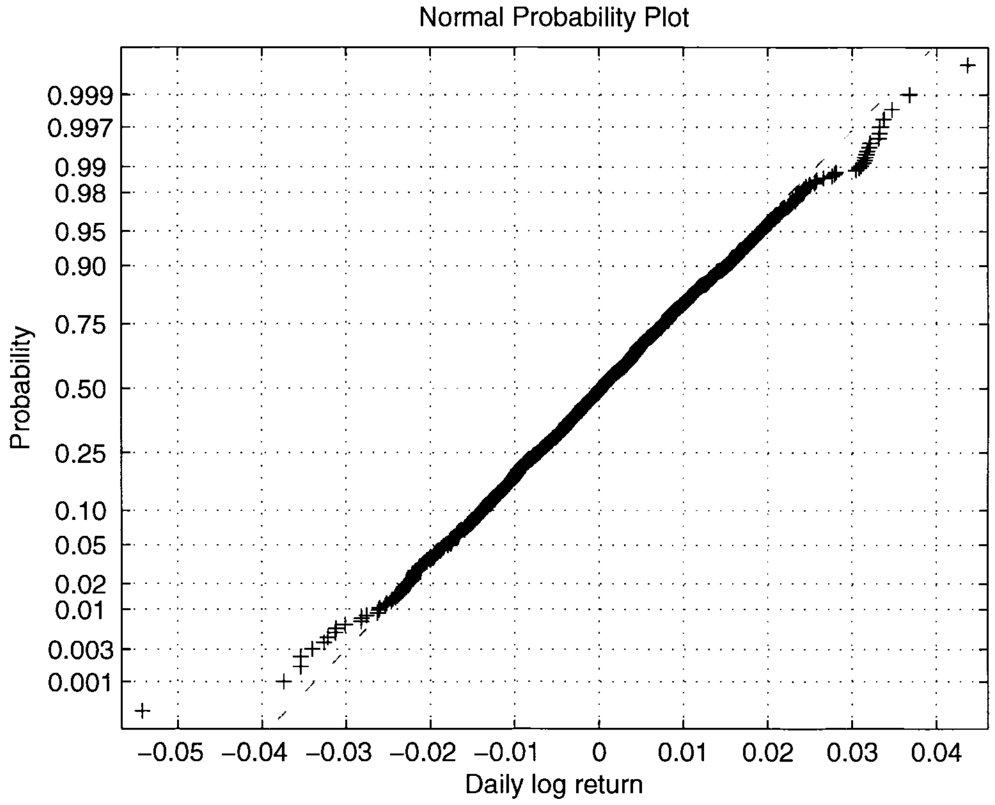

Figure 4.5: Normal Q-Q plot of the daily log returns

conditional rather than the unconditional returns for stochastic dominance pricing purposes. Whether a parametric or a non-parametric conditional return model performs better is a topic of future research.

A second finding is that short term options are priced better. This is probably caused by the numerical method, which uses the daily discretization period used in the estimation of the GARCH model. The possibility of a GARCH model to describe conditional non-normal returns can be used together with a longer sampling period to mitigate this effect.

In both applications, the theoretical mean return was used to price the bounds from the estimated models. In principle, the sample mean could have been used as an estimation of the mean return, but the poor predictive power of past returns is well known. The bounds implications of return models, such as factor models, will be examined in future research.

	Theoretical		NGARCH		GARCH		Nonparametric		BS
T=1 month									
Strike	Upper	Lower	Upper	Lower	Upper	Lower	Upper	Lower	
90	10.467	10.394	10.486	10.390	10.372	10.357	10.521	10.480	10.408
95	5.883	5.769	5.899	5.756	5.779	5.692	5.925	5.776	5.776
100	2.313	2.169	2.348	2.176	2.298	2.146	2.375	2.184	2.304
105	0.526	0.432	0.566	0.464	0.610	0.502	0.598	0.471	0.590
110	0.068	0.042	0.088	0.057	0.125	0.084	0.092	0.049	0.092
T=3 months									
Strike	Upper	Lower	Upper	Lower	Upper	Lower	Upper	Lower	
90	11.649	11.406	11.663	11.369	11.601	11.332	12.177	11.835	11.511
95	7.578	7.276	7.606	7.258	7.533	7.194	8.124	7.645	7.434
100	4.319	3.994	4.389	4.015	4.348	3.984	4.852	4.339	4.258
105	2.088	1.808	2.199	1.869	2.216	1.895	2.566	2.122	2.135
110	0.839	0.655	0.958	0.726	1.015	0.783	1.205	0.886	0.932

Table 4.6: Pricing with the wrong model

The theoretical bounds are priced assuming that the returns follow the NGARCH process described by equation (4.4), with parameters $\mu = 0.09$, $\beta_0 = 0.00001$, $\beta_1 = 0.8$, $\beta_2 = 0.1$ and $\gamma = 0.49$. The NGARCH and GARCH bounds are computed from the estimated parameters listed in Table 4.5. The nonparametric bounds are obtained from an i.i.d. model with the theoretical mean $\mu = 0.09$ and sample variance and $\sigma = 0.1818$ (annualized) and the distribution described by the histogram in Figure 4.4. The Black-Scholes prices assume the sample volatility. The annualized riskless interest rate is $r = 0.05$.

Chapter 5

Conclusions

This thesis presented a new approach to option pricing, the stochastic dominance approach. This approach derives two bounds on allowable option prices dependent on the entire distribution of underlying asset returns. The distribution can be of any type, but the contingent claim bounds have a closed form only for options with convex payoffs. We show that the two bounds are discounted payoff expectations under two risk neutral transformations of the original asset dynamics.

We then examined the convergence of the discrete time option bounds derived by stochastic dominance methods in a multiperiod context as trading becomes progressively more dense, under a variety of assumptions about the limiting distribution of the underlying asset returns. We found that this stochastic dominance approach nests virtually the entire set of option prices available in the literature under a variety of alternative methods, including arbitrage and general equilibrium. Specifically, they nest all the models where the distribution of the underlying asset depends on a single random factor, as well as the models in which this same distribution can be approximated by a discrete model with a single random factor.

The probabilistic interpretation of the two option bounds extends to options with non-convex payoffs, although the two bound distributions cannot be obtained in closed form. This extension is useful in deriving convergence results. A topic of future research will be the convergence of the two bounds for stochastic volatility and jump models approximated by discrete GARCH processes.

There are two major advantages of the stochastic dominance approach over alternative derivatives pricing methods. The first one is that it does produce useful results in the presence of market frictions such as transaction costs, in sharp contrast to the arbitrage approach. The second one is that it is not necessary to know the stochastic process governing the evolution of the price of the underlying asset in order to price the derivative, as long as an empirical distribution represented by a histogram of possible future values (or returns) of the asset is available. Such an empirical distribution is sufficient to derive the risk neutral U and L distributions that define the option bounds. The pricing implications stemming from the choice of the wrong model have been briefly examined for a GARCH specification of the returns. The results show the importance of modeling the conditional return distribution when the price is generated by a non-i.i.d. process. A thorough examination of this important topic will be the subject of future research.

Bibliography

Amin, Kaushik I., 1993, Jump Diffusion Option Valuation in Discrete Time, *Journal of Finance* 48, 1833–1863.

Amin, Kaushik I., and Victor K. Ng, 1993, Option Valuation with Systematic Stochastic Volatility, *Journal of Finance* 48, 881–910.

Arrow, Kenneth J., 1964, The Role of Securities in the Optimal Allocation of Risk-bearing, *The Review of Economic Studies* 31, 91–96.

Aït-Sahalia, Yacine, and Andrew Lo, 1998, Nonparametric Estimation of State-Price Densities Implicit in Financial Asset Prices, *Journal of Finance* 53, 499–547.

Aït-Sahalia, Yacine, and Andrew Lo, 2000, Nonparametric Risk Management and Implied Risk Aversion, *Journal of Econometrics* 94, 9–51.

Bakshi, Gurdip S., Charles Cao, and Zhiwu Chen, 1997, Empirical Performance of Alternative Option Pricing Models, *JF* 52, 2003–2049.

Barone-Adesi, Giovanni, Robert Engle, and Loriano Mancini, 2004, GARCH Options in Incomplete Markets, University of Lugano, Switzerland Working Paper.

Bates, David S., 1991, The Crash of '87 – Was It Expected? The Evidence from Options Markets, *Journal of Finance* 46, 1009–1044.

Bates, David S., 1996, Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options, *Review of Financial Studies* 9, 69–107.

Bergman, Yaacov Z., Bruce D. Grundy, and Zvi Wiener, 1996, General Properties of Option Prices, *Journal of Finance* 51, 1573–1610.

Bernardo, Antonio E., and Olivier Ledoit, 2001, Gain, Loss and Asset Pricing, *Journal of Political Economy* 108, 144–172.

Black, Fischer, and Myron Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637–659.

Bollerslev, Tim, 1986, Generalized autoregressive conditional heteroskedasticity, *Journal of Econometrics* 31, 307–327.

Bollerslev, Tim, Ray Y. Chou, and Kenneth F. Kroner, 1992, ARCH modeling in finance : A review of the theory and empirical evidence, *Journal of Econometrics* 52, 5–59.

Breeden, Douglas T., and Robert Litzenberger, 1978, Prices of State-Contingent Claims Implicit in Options Prices, *Journal of Business* 51, 621–651.

Brennan, Michael, 1979, The pricing of contingent claims in discrete-time models, *Journal of Finance* 34, 53–68.

Carr, Peter, and Dilip B. Madan, 2001, Pricing and Hedging in Incomplete Markets, *Journal of Financial Economics* 62, 131–167.

Chernov, Mikhail, 2003, Empirical reverse engineering of the pricing kernel, *Journal of Econometrics* 116, 329–364.

Christoffersen, Peter F., and Kris Jacobs, 2004, Which GARCH Model for Option Valuation?, *Management Science* 50, 1204–1221.

Christoffersen, Peter F., Redouane Elkamhi, and Kris Jacobs, 2005a, No-Arbitrage Valuation of Contingent Claims in Discrete Time, working paper, McGill University.

Christoffersen, Peter F., Steven Heston, and Kris Jacobs, 2005b, Option Valuation with Conditional Skewness, *Journal of Econometrics, forthcoming.*

Cochrane, John H., and Jésus SaaRequejo, 2000, Beyond Arbitrage: Good-Deal Asset Price Bounds in Incomplete Markets, *Journal of Political Economy* 107, 79–119.

Constantinides, G. M., and S. Perrakis, 2002, Stochastic Dominance Bounds on Derivative prices in a Multiperiod Economy with Proportional Transaction Costs, *Journal of Economic Dynamics and Control* 26, 1323–1352.

Constantinides, G. M., and S. Perrakis, 2006, Stochastic Dominance Bounds on American Option Prices in Markets with Frictions, *Review of Finance, forthcoming.*

Cox, John, and Stephen A. Ross, 1976, The Valuation of Options for Alternative Stochastic Processes, *Journal of Financial Economics* 3, 145–166.

Cox, John, Stephen A. Ross, and Mark Rubinstein, 1979, Option Pricing: A Simplified Approach, *Journal of Financial Economics* 7, 229–263.

Duan, Jin-Chuan, 1995, The GARCH Option Pricing Model, *Mathematical Finance* 5, 13–32.

Duan, Jin-Chuan, 1996a, Cracking the smile, *Risk* 9, 55–59.

Duan, Jin-Chuan, 1996b, A Unified Theory of Option Pricing under Stochastic Volatility- from GARCH to Diffusion, University of Toronto Working Paper.

Duan, Jin-Chuan, 1997, Augmented GARCH(p,q) process and its diffusion limit, *Journal of Econometrics* 79, 97–127.

Duan, Jin-Chuan, 1999, Conditionally Fat-Tailed Distributions and the Volatility Smile in Options, University of Toronto Working Paper.

Duan, Jin-Chuan, 2002, Nonparametric Option Pricing by Transformation, University of Toronto Working Paper.

Duan, Jin-Chuan, Peter H. Ritchken, and Z. Sun, 2005, Jump Starting GARCH: Pricing and Hedging Options with Jumps in Returns and Volatilities, University of Toronto Working Paper.

Duan, Jin-Chuan, and Jean-Guy Simonato, 2001, American option pricing under GARCH using a Markov chain approximation, *Journal of Economic Dynamics and Control* 25, 1689–1718.

Duffie, Darrell, Jun Pan, and Kenneth Singleton, 2000, Transform analysis and asset pricing for affine jump-diffusions, *Econometrica* 68, 1343–1376.

Engle, Robert, 1982, Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of U.K. Inflation, *Econometrica* 50, 987–1008.

Engle, Robert, and Victor K. Ng, 1993, Measuring and Testing of the Impact of News on Volatility, *Journal of Finance* 48, 1749–1778.

Engle, Robert F., and Gloria Gonzalez-Rivera, 1991, Semiparametric ARCH Models, *Journal of Business and Economic Statistics* 9, 345–359.

Garman, Mark B., 1976a, An algebra for evaluating hedge portfolios, *Journal of Financial Economics* 3, 403–427.

Garman, Mark B., 1976b, A General Theory of Asset Valuation Under Diffusion State Processes, University of California, Berkeley Working Paper.

Geske, Robert, 1979, The Valuation of Compound Options, *Journal of Financial Economics* 7, 63–81.

Harrison, John M., and David M. Kreps, 1979, Martingales and Arbitrage in Multiperiod Securities Markets, *Journal of Economic Theory* 20, 381–408.

Harrison, John M., and Stanley Pliska, 1981, Martingales and Stochastic Integrals in the Theory of Continuous Trading, *Stochastic Processes and their Applications* 11, 215–260.

Heston, Steven, 1993, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies* 6, 327–343.

Heston, Steven, and Saikat Nandi, 2000, A closed-form GARCH option valuation model, *Review of Financial Studies* 13, 585–625.

Huang, Chi-Fu, and Robert H. Litzenberger, 1988, *Foundations for Financial Economics.* (North-Holland New York).

Hull, John, and Alan White, 1987, The Pricing of Options on Assets with Stochastic Volatilities, *Journal of Finance* 42, 281–300.

Jackwerth, Jens Carsten, 2000, Recovering Risk Aversion from Option Prices and Realized Returns, *Review of Financial Studies* 13, 433–451.

Jackwerth, Jens Carsten, 2004, *Option-Implied Risk-Neutral Distributions and Risk Aversion.* (Research Foundation of AIMR Charlottesville, VA).

Johnson, Herb, and David Shanno, 1987, Option Pricing when the Variance is Changing, *Journal of Financial and Quantitative Analysis* 22, 143–151.

Kamrad, Bardia, and Peter H. Ritchken, 1991, Multinominal Approximating Models for Options with k State Variables, *Management Science* 37, 1640–1652.

Lamoureux, Christopher G., and William D. Lastrapes, 1993, Forecasting Stock-Return Variance: Toward an Understanding of Stochastic Implied Volatilities, *Review of Financial Studies* 6, 293–326.

Lawrence R. Glosten, Ravi Jaganathan, David E. Runkle, 1993, On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks, *Journal of Finance* 48, 1779–1801.

Leisen, Dietmar, 2000, Stock Evolution Under Stochastic Volatility: A Discrete Approach, *Journal of Derivatives* 8, 9–27.

Levy, Haim, 1985, Upper and Lower Bounds of Put and Call Option Value: Stochastic Dominance Approach, *Journal of Finance* 40, 1197–1217.

Mathur, Kamlesh, and Peter H. Ritchken, 1999, Minimum Option Prices Under Decreasing Absolute Risk Aversion, *Review of Derivatives Research* 3, 135–156.

Merton, Robert C., 1973, Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science* 4, 141–183.

Merton, Robert C., 1976, Option Pricing when the Underlying Stock Returns are Discontinuous, *Journal of Financial Economics* 3, 125–144.

Merton, Robert C., 1982, On the mathematics and economics assumptions of continuous-time models, in *Essays in Honor of Paul Cootner* (Prentice Hall, Englewood Cliffs, NJ).

Meyer, Michael J., 2004, Monte Carlo Simulation With Java And C++, Project Martingale, http://martingale.berlios.de/Martingale.html.

Nelson, Daniel B., 1990, ARCH models as diffusion approximations, *Journal of Econometrics* 45, 7–38.

Nelson, Daniel B., 1991, Conditional Heteroskedasticity in Asset Returns: A New Approach, *Econometrica* 59, 347–370.

Pan, Jun, 2002, The jump-risk premia implicit in options: evidence from an integrated time-series study, *Journal of Financial Economics* 63, 3–50.

Perrakis, Stylianos, 1986, Option Bounds in Discrete Time: Extensions and the Pricing of the American Put, *Journal of Business* 59, 119–141.

Perrakis, Stylianos, 1988, Preference-free option prices when the stock returns can go up, go down or stay the same, in Frank J. Fabozzi, eds.: *Advances in Futures and Options Research* (JAI Press, Greenwich, Conn).

Perrakis, Stylianos, 1993, Option pricing for multinomial stock returns in diffusion and mixed processes, *Revue Canadienne Des Sciences De L'Administration* 10, 68–82.

Perrakis, Stylianos, and Peter J. Ryan, 1984, Option Bounds in Discrete Time, *Journal of Finance* 39, 519–525.

Ritchken, Peter H., 1985, On Option Pricing Bounds, *Journal of Finance* 40, 1219–1233.

Ritchken, Peter H., and Shyanjaw Kuo, 1988, Option Bounds with Finite Revision Opportunities, *Journal of Finance* 43, 301–308.

Ritchken, Peter H., and Shyanjaw Kuo, 1989, Stochastic Dominance and Decreasing Absolute Risk Averse Option Pricing Bounds, *Management Science* 35, 51–59.

Rubinstein, Mark, 1976, The Valuation Of Uncertain Income Streams And The Pricing Of Options, *Bell Journal of Economics* 2, 407–425.

Rubinstein, Mark, 1983, Displaced Diffusion Option Pricing, *Journal of Finance* 38, 213–217.

Rubinstein, Mark, 1994, Implied Binomial Trees, *Journal of Finance* 49, 771–818.

Stein, Elias M., and Jeremy C. Stein, 1991, Stock Price Distributions with Stochastic Volatility: An Analytic Approach, *Review of Financial Studies* 4, 727–752.

Appendix

Proof of Lemma 1

The proof is similar to the one used by Merton (1982), the only difference being that ε_{t+h} is now a bounded continuous random variable rather than a multinomial discrete one. Denote $Q_t(\delta)$ the conditional probability that $|X_{t+h} - X_t| > \delta$, given the information available at time t. Since ε_{t+h} is bounded, define $\bar{\varepsilon} = \max|\varepsilon_{t+h}| = \max(|\varepsilon_{\min}|, |\varepsilon_{\max}|)$. For any $\delta t > 0$, define $h^*(\delta)$ as the solution of the equation $\delta = \mu h + \sigma\bar{\varepsilon}\sqrt{h}$. This equation admits a positive solution

$$\sqrt{h^*} = \frac{-\sigma\bar{\varepsilon} + \sqrt{\sigma^2\bar{\varepsilon}^2 + 4\mu\delta}}{\mu}.$$

For any $h < h^*(\delta)$ and for any possible X_{t+h}, we have

$$|X_{t+h} - X_t| = |\mu h + \sigma\varepsilon_{t+h}\sqrt{h}| < \mu h^* + \sigma\bar{\varepsilon}\sqrt{h^*} = \delta$$

so $Q_t(\delta) = \Pr(|X_{t+\Delta t} - X_t| > \delta \equiv 0$ whenever $h < h^*$ and hence $\lim\limits_{h\to 0}\frac{1}{h}Q_t(\delta) = 0$ The Lindeberg condition is thus satisfied. Equations (3.7) and (3.8) are satisfied by the construction of this discrete process, so the diffusion limit of (3.10) is (3.5), QED.

Proof of Proposition 1

We shall consider only the case $\mu > r$; the proof for the case $\mu \leq r$ is similar and is omitted. Under the upper bound probability given by (2.13), the return process becomes

$$z_{t+h}^{U} = \mu(X_t)h + \sigma(X_t)\sqrt{h} \begin{cases} \varepsilon_t & \text{with probability } 1-Q \\ \varepsilon_{\min} & \text{with probability } Q \end{cases}, \tag{1}$$

where Q is the following probability

$$\begin{aligned} Q &= \frac{E(z) - rh}{E(z) - \min(z_{t+h})} \\ &= \frac{\mu h - rh}{\mu h - (\mu h + \sigma \varepsilon_{\min}\sqrt{h})} = -\frac{\mu - r}{\sigma \varepsilon_{\min}}\sqrt{h}. \end{aligned}$$

The random component of the returns in (1) has a bounded continuous distribution, so the upper bound process satisfies the Lindeberg condition. This process has the mean

$$\begin{aligned} E_t^U\left[z_{t+h}\right] &= \mu h + Q\sigma\sqrt{h})\varepsilon_{\min} \\ &= rh. \end{aligned}$$

Its variance is

$$\begin{aligned} \mathrm{Var}_t^U\left[z_{t+h}\right] &= \sigma^2 h\left[(1-Q)\,\mathrm{Var}_t\left[z_{t+h}\right] + Q\varepsilon_{\min}^2\right] \\ &= \sigma^2 h\left[1 + \frac{\mu - r}{\sigma\varepsilon_{\min}}\sqrt{h} - \frac{\mu - r}{\sigma\varepsilon_{\min}}\sqrt{h}\varepsilon_{\min}^2\right] \\ &= \sigma^2 h + O(h^{3/2}) \end{aligned}$$

Consequently, the upper bound process converges weakly to the diffusion (3.18).

Proof of Proposition 2

As with the proof of Proposition 1, we shall consider only the case $\mu > r$. Under the probability distribution given by (2.13) for the lower bound the transformed returns process becomes

$$z_{t+h} = \mu(X_t)h + \sigma(X_t)\hat{\varepsilon}_t\sqrt{h},$$

where $\hat{\varepsilon}_t$ is a truncated random variable $\{\varepsilon_t|\varepsilon_t < \bar{\varepsilon}\}$, with $\bar{\varepsilon}$ found from the condition $E^L[z_{t+h}] = rh$. Since $\hat{\varepsilon}_t$ is truncated from a bounded continuous distribution the Lindeberg condition is satisfied. The risk neutrality of the lower bound distribution implies that

$$\mu h + \sigma\sqrt{h}E[\hat{\varepsilon}_t] = rh$$

and the mean of $\hat{\varepsilon}_t$ is

$$E[\hat{\varepsilon}_t] = -\frac{\mu - r}{\sigma}\sqrt{h} \tag{2}$$

Since this random variable is drawn from a distribution that is truncated from the distribution of ε_t we get

$$E[\hat{\varepsilon}_t] = \frac{1}{\Pr(\varepsilon_t < \bar{\varepsilon}_t)}\int_{\varepsilon_{\min}}^{\bar{\varepsilon}_t} \varepsilon_t df(\varepsilon_t).$$

We picked ε_t such that $E_t[\hat{\varepsilon}_t] = 0$ and we have

$$\int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon_t df(\varepsilon_t) = \int_{\varepsilon_{\min}}^{\bar{\varepsilon}_t} \varepsilon_t df(\varepsilon_t) + \int_{\bar{\varepsilon}_t}^{\varepsilon_{\max}} \varepsilon_t df(\varepsilon_t) = 0. \tag{3}$$

Then, from (2)-(3) we get

$$
\begin{aligned}
\frac{\mu - r}{\sigma}\sqrt{h} &= \frac{1}{\Pr(\varepsilon_t < \bar{\varepsilon}_t)} \int_{\bar{\varepsilon}_t}^{\varepsilon_{\max}} \varepsilon_t df(\varepsilon_t) \\
&\geq \frac{1}{1 - \Pr(\varepsilon_t > \bar{\varepsilon}_t)} \int_{\bar{\varepsilon}_t}^{\varepsilon_{\max}} \bar{\varepsilon}_t df(\varepsilon_t) \\
&= \frac{\bar{\varepsilon}_t \Pr(\varepsilon_t > \bar{\varepsilon}_t)}{1 - \Pr(\varepsilon_t > \bar{\varepsilon}_t)}.
\end{aligned}
$$

From the last inequality we get

$$
\Pr(\varepsilon_t > \bar{\varepsilon}_t) \leq \frac{\frac{\mu - R}{\sigma}\sqrt{h}}{\bar{\varepsilon}_t + \frac{\mu - R}{\sigma}\sqrt{h}} = O(\sqrt{h}).
$$

Since $\varepsilon_t \leq \varepsilon_{\max}$, we also have

$$
\int_{\bar{\varepsilon}_t}^{\varepsilon_{\max}} \varepsilon_t^2 df(\varepsilon_t) \leq \varepsilon_{\max}^2 \int_{\bar{\varepsilon}_t}^{\varepsilon_{\max}} df(\varepsilon_t) = O(\sqrt{h}).
$$

The last two result are used to compute the variance of $\hat{\varepsilon}_t$

$$
\begin{aligned}
Var\left[\hat{\varepsilon}_t\right] &= E\left[\hat{\varepsilon}_t^2\right] - \left(E\left[\hat{\varepsilon}_t\right]\right)^2 \\
&= \frac{1}{\Pr(\varepsilon_t < \bar{\varepsilon}_t)} \int_{\varepsilon_{\min}}^{\bar{\varepsilon}_t} \varepsilon_t^2 df(\varepsilon_t) - \left(\frac{\mu - r}{\sigma}\right)^2 h \\
&= \frac{1}{1 - \Pr(\varepsilon_t > \bar{\varepsilon}_t)} \left(1 - \int_{\bar{\varepsilon}_t}^{\varepsilon_{\max}} \varepsilon_t^2 df(\varepsilon_t)\right) - \left(\frac{\mu - r}{\sigma}\right)^2 h \\
&= 1 + O(\sqrt{h})
\end{aligned}
$$

It follows that $Var_t^L\left[z_{t+h}\right] = \sigma^2 h + O(h^{3/2})$. The diffusion limit is, therefore, the process described by equation (3.18), QED.

Proof of Lemma 2

As shown in the proof of Lemma 1, the first two terms of (3.20) converge to a diffusion. The generator of this diffusion is

$$\begin{aligned}\mathcal{A}u &= \lim_{h\to 0}\frac{E[u(z_{D,t+h},t+h)]-u(z_{D,t},t)}{h}\\ &= (\mu_t-\lambda\mu_J)S\frac{\partial u}{\partial S}+\frac{\partial u}{\partial t}+\frac{1}{2}\sigma_t^2 S^2\frac{\partial^2 u}{\partial S^2}.\end{aligned}$$

Denote $\mathcal{A}_J^h$ the generator of the discrete process described by (3.20). This generator converges to

$$\begin{aligned}\lim_{\Delta t\to 0}\mathcal{A}_J^h u &= \lim_{h\to 0}\frac{E[u(z_{t+h},t+h)]-u(z_t,t)}{h}\\ &= \lim_{h\to 0}(1-\lambda h)\frac{E[u(z_{D,t+h},t+h)]-u(z_{D,t},t)}{h}\\ &+\lambda h\frac{E[u(z_{J,t+h},t+h)]-u(z_{J,t},t)}{h}\\ &= (\mu_t-\lambda\mu_J)S\frac{\partial u}{\partial S}+\frac{\partial u}{\partial t}+\frac{1}{2}\sigma_t^2S^2\frac{\partial^2 u}{\partial S^2}+\lambda E[u(SJ)-u(S)],\end{aligned}$$

which is the generator of 3.19, QED.[1]

Proof of Proposition 3

As with Propositions 1 and 2, we consider the multiperiod discrete time bounds of Chapter 2, obtained by successive expectations under the risk-neutral upper bound distribution. We then seek the limit of this distribution as $h \to 0$. The mixing

[1]See for instance Merton (1992) for a discussion on the generators of diffusions and jump processes.

probability in (2.22) is given by

$$\frac{E(z)-rh}{E(z)-J_{\min}}=\frac{(\mu-r)h}{\mu h-J_{\min}}\sim\lambda_U h$$

where $\lambda_U = -\frac{\mu_t - r}{J_{\min}}$, since the expected return under the subjective probability distribution is

$$E(z_{t+h})=(1-\lambda h)(\mu-\lambda\mu_J)h+\lambda\mu_J h=\mu h+o(h)$$

Observe that λ_U is always positive since $J_{\min}<0$ and $E(z)>rh$. Hence, the discrete time upper bound process is

$$z_{t,t+h}=\begin{cases} z_D & \text{with probability} \quad (1-\lambda h)(1-\lambda_U h), \\ J & \text{with probability} \quad \lambda h(1-\lambda_U h), \\ J_{\min} & \text{with probability} \quad \lambda_U h. \end{cases}$$

By removing the terms in $o(h)$, the upper bound process becomes

$$z_{t,t+h}=\begin{cases} z_D & \text{with probability} \quad 1-(\lambda+\lambda_U)h \\ J^U & \text{with probability} \quad (\lambda+\lambda_U)h \end{cases},$$

where J^U is given by (3.22). This is a mixture of the diffusion component and a jump with intensity $\lambda+\lambda_U$. It can be readily verified that the upper bound process is risk neutral by construction. By Lemma 2, therefore, it converges weakly to a jump-diffusion process with the generator

$$\begin{aligned}\mathcal{A}^U f &= \left[r-(\lambda+\lambda_U)\mu_J^U\right] S\frac{\partial f}{\partial S}+\frac{\partial f}{\partial t} \\ &+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}+\lambda_U E^U[f(SJ^U)-f(S)].\end{aligned}$$

This process, however, corresponds to (3.21), QED.

Proof of Proposition 4

The proof is very similar to those of Lemma 2 and Proposition 3. We apply equation (2.22) and observe that, as with the upper bound, the lower bound distribution over $(t, t+h)$ is a mixture of the diffusion component and a jump of intensity λ and log-amplitude distribution J^L, the truncated distribution $\{J|J \leq \bar{J}\}$.

$$z_{t,t+h} = \begin{cases} z_D & \text{with probability} \quad 1-\lambda h \\ J^L & \text{with probability} \quad \lambda h \end{cases} .$$

By Lemma 2 this process converges weakly for $h \to 0$ to a jump-diffusion process with generator

$$\begin{aligned} \mathcal{A}^L f = & \left[r - \mu_J^L\right] S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} \\ & + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \lambda E^L[f(S+J^L) - f(S)]. \end{aligned}$$

which corresponds to (3.25), QED.