THE STOCHASTIC DOMINANCE EFFICIENCY OF THE MEAN-VARIANCE FRONTIER

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A Thesis
in
The John Molson School of Business

Presented in Partial Fulfillment of the Requirements for the Degree of Master of Science in Administration at Concordia University
Montreal, Quebec, Canada

April 2006

Your file Votre référence
ISBN: 0-494-14379-7
Our file Notre référence
ISBN: 0-494-14379-7

ABSTRACT

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This paper examines the second-degree stochastic dominance (SSD) efficiency of the portfolios on the mean-variance (EV) frontier. By applying Post's linear programming tests to our weekly and monthly data of a sample of U.S. equity funds, we find that the higher portion of the EV frontier is SSD-efficient, while the lower portion is SSD-inefficient. The quadratic utility test confirms that the top of the EV frontier is SSD-efficient where the SSD-efficient portfolios have higher-means and higher-variances.

Based on Perrakis' theoretical inequalities on the central moments derived from polynomial utility functions, we test SSD efficiency on the third central moment and find more SSD-undominated portfolios following the quadratic utility-efficient portfolios; thus extending the SSD-efficient portion on the EV frontier. Then, we maximize the polynomial utility functions on the third and the fourth central moments for SSD and on the fourth central moment for TSD without constraining them to lie on the EV efficient set. Hence, those new generated portfolios should be SSD- or TSD- efficient, but may or may not be on the EV frontier. Our empirical work shows that such optimal portfolios further extend the SSD-efficient portion of the EV frontier, but still lie on the EV frontier, whose upper part lies entirely within the SSD-efficient set.

Acknowledgements

Dr. S. Perrakis. Thank you so much for your supervision. You gave me a lot of valuable advice. I really appreciate your patience to let me handle this difficult topic step by step. I find I have learned a lot through working with you on this thesis. It is an unforgettable experience.

Dr. S. Betton. Many thanks for all your supports after I entered the M.Sc. program. I do appreciate your guidance and help to build up my knowledge of finance, especially in the field of corporate finance.

Dr. L. Switzer. Thanks a lot for sharing your precious ideas with me to further improve my thesis. The knowledge of financial methodology I learned from you is a great resource and tool for my future research.

I am especially grateful to my dear parents. Thank you so much for giving me all your love and encouragement, even though we are now far from each other. I hope one day you will be proud of your daughter.

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I. Introduction

The two-moment Mean-Variance (EV) approach developed by Markowitz (1952, 1959) assumes either that decision makers have quadratic utility functions, or that their utility functions are consistent with the von Neumann-Morgenstern expected utility maximization on the basis of a restricted class of two-parameter distributions. Unfortunately, the EV assumptions have been seriously criticized by extensive studies both in its theoretical and in its empirical aspects. Fama (1965), and Breen and Savage (1968) have shown that distributions of stock price changes are inconsistent with the assumption of normal probability functions; rather, they conform quite well to the four-parameter Stable Paretian distributions. Moreover, the quadratic utility functions imply that beyond some level of wealth (or return) the investors' marginal utility for wealth becomes negative as their risk aversion increases with wealth. In other words, investors would prefer less wealth to more wealth, which is unrealistic.[1]

An important development after the EV approach is the theory of Stochastic Dominance (SD) for the optimal portfolio selection. The SD theory has firstly been developed by four papers independently published by Hadar and Russell (1969), Hanoch and Levy (1969), Rothschild and Stiglitz (1970), and Whitmore (1970). In contrast to the EV approach, the SD approach is non-parametric in the sense that its criteria impose neither explicit specifications of decision makers' utility functions nor restrictions on functional forms of probability distributions. Hence, the SD approach accounts for the entire probability distributions and employs some general condition for decision makers' risk preferences.

[1] See Baumol (1963), Borch (1969), Breen (1968), Feldstein (1969), Hadar and Russell (1969), Hanoch and Levy (1969), Lintner (1965), Pratt (1964), and Quirk and Saposnik (1962).

In the SD theory, the first-degree stochastic dominance (FSD) criterion, originally developed by Quirk and Saposnik (1962), requires only that the first derivative of the utility function be positive throughout or monotone increasing; therefore, it assumes only non-satiation and allows for risk preference, risk indifference, or risk aversion. The second-degree stochastic dominance (SSD) criterion, developed by Hadar and Russell (1969), eliminates risk preference by adding the restriction that the second derivative of the utility function be everywhere non-positive. In other words, SSD adds the assumption of global risk-aversion; thus, utility is everywhere concave. The third-degree stochastic dominance (TSD), developed by Whitmore (1970), adds the requirement that the third derivative of the utility function be everywhere non-negative. Hence, TSD imposes skewness preference in addition to non-satiation and risk aversion.

Despite the theoretical appeal of the SD criteria, the two-moment EV approach has been more influential in empirical portfolio analysis. The attractiveness of the EV model lies in its ability to test and build efficient diversification strategies. Further, since only two parameters – mean and variance – are required for each portfolio, EV analysis is rather simple and inexpensive. By contrast, the practical application of the SD criteria demands a sophisticated technology. Specifically, once the probability functions are specified, a relatively large number of means and variances need to be calculated and ordered with a high degree of efficiency with the aid of a computer. However, with the increase in observations and samples, the burden of computations required for the implementation of various SD criteria makes such empirical work prohibitive. More importantly, the SD approach is restricted to pairwise comparison of a finite number of choice alternatives, which cannot be applied to problems with full diversification possibilities, although it is computationally efficient for simple crossing algorithms to check the difference of the empirical distribution functions (EDFs) of choice alternatives when we apply SD criteria to empirical data.[2] Because of these limitations of the SD application, decision makers still prefer the traditional EV approach when they believe that the EV efficiency and the SD efficiency do not make much difference for the optimal portfolio selection.

Recently, Post (2003) has developed the tractable Linear Programming (LP) tests for the SD efficiency of a given portfolio relative to all possible portfolios created from a set of assets. The development of the LP tests has provided us with an efficient and effective tool to present our empirical study in this paper. In particular, as the assumptions of the EV analysis suffer from serious flaws, the portfolios on the EV frontier (or the EV efficient portfolios) are not necessarily efficient based on the stochastic dominance criteria. Since FSD is a weak condition to impose on the investors' utilities and Post's LP tests are developed especially for the SSD efficiency, we focus our empirical work on the SSD efficiency of the EV frontier.

We use some of Perrakis' unpublished results to develop theoretical inequalities on the central moments of the return distributions of a pair of portfolios, which are necessary conditions for one portfolio to dominate the other in the second- or third-degree. The new inequalities involve the first three central moments for SSD and the first four central moments for both SSD and TSD. Hence, the central moment inequalities provide us with additional rules to test the stochastic dominance efficiency for the portfolios on the EV frontier. Then, we extend these developments to a new test based on the utility maximization introduced by Kroll, Levy, and Markowitz (1984). In such

[2] See Levy (1992) Appendix A.

programs, the investors' expected utilities drawn from a given set of utility functions are maximized without the constraint that the mean and the variance have to be on the EV efficient set. Therefore, the utility maximization includes all investment possibilities allowing for various investors' expected utilities and return distributions. The new generated optimal portfolios by such maximizing programs should be SSD- or TSD-efficient, but may or may not be on the EV frontier.

The remaining of the paper is arranged in the following structure: in section II, we provide the literature review with regard to the development of the SD theory and the research focus on the SD efficiency on the EV frontier; for the paper's integrity, in section III, we very briefly introduce the concepts and computation of the EV frontier based on Markowitz (1952, 1959); in section IV, we illustrate the theory of stochastic dominance in the first-, second-, and third-degree; in section V, we describe the LP tests by Post (2003) which are also the main tools for our empirical study to examine the SSD efficiency for the portfolios on the EV frontier; in section VI, we start discussing the polynomial utilities, based on which we introduce the central moment inequalities developed by Perrakis for further testing the SSD efficiency on the second and the third moments; in section VII, we extend our study to apply the utility maximizing programs on the third and the fourth moments; in section VIII, we offer our empirical results; in our final section IX, we give the conclusion and future research recommendations.

II. Literature Review

A. Development of the Stochastic Dominance Theory and Application

Since the publication of the four original papers[3], there has been a proliferation of papers on the SD theory published in a wide variety of finance and economics journals, books, and conferences. In the survey of the literatures about stochastic dominance, Bawa (1982) has listed about 400 publications, working papers, and books in his bibliography including basic concepts of decision-making under uncertainty and various applications in different scientific fields. The survey contains an exhaustive listing of papers that are either basic contributions to the SD subject or primarily concerned with applications of the SD concepts. It also contains selective listing of papers from finance, economics, mathematics, mathematical physics, mathematical psychology, operation research, and statistics literature to illustrate the wide applicability of SD concepts. Bawa states that in finance and economics, the foundation of SD is the mainstream von Neumann-Morgenstern expected utility paradigm. Its essence is to provide an admissible set of choices under restrictions on individual decision maker's utility functions that follow from prevalent and appealing models of economic behavior. With alternative choices equivalently characterized by probability distributions, SD can be viewed as inequalities involving functions of the probability distributions that induce partial orderings of the set of probability distributions.

In finance literature, the SD approach is applied to compare the performance of alternative investment portfolios regarding their observed rates of returns. Therefore, the empirical application of SD as well as its algorithmic development has typically focused

[3] These four original papers refer to Hadar and Russell (1969), Hanoch and Levy (1969), Rothschild and Stiglitz (1970), and Whitmore (1970).

on discrete empirical distributions where each observed state of nature occurs with equal probability. Bawa et al. (1979) notice that although SD rules are theoretically sound, they are hard to implement because they require comparisons of probability distributions over their entire ranges. In this sense, they develop an algorithm that should remedy this situation. It exploits the theoretical results and computational techniques to efficiently determine the SD admissible set of alternatives, which contains the optimal choices for all decision makers whose preferences satisfy reasonable economic criteria. Further, they indicate that in the portfolio choice problem which has an infinite choice set, the algorithm can provide reasonable approximations to the true set of optimal choices via the use of a suitably fine enough grid on the space of portfolios.

Applying Fishburn's (1974) conditions for convex stochastic dominance (CSD), Bawa et al. (1985) provide exact LP algorithms for assigning discrete return distributions into the FSD and SSD optimal sets. For TSD, a super-convex stochastic dominance approach is proposed which allows classification of choice elements into super-dominated, mixed, and super-optimal sets. In the first application of CSD rules for identifying the elements of the optimal set from the full choice set, the optimal sets were found to be significantly smaller than the previously identified admissible sets. The CSD procedure is a useful tool for limiting the set of alternatives that must be considered by an individual having utility functions in the classes examined. Finally, the applicability of super-convex SD for continuous distributions defined over a bounded interval is shown. The difficulties in identifying the elements of the super-dominated set for distributions defined over the entire real line are demonstrated in the determination of the dominated choices for a set of normally distributed mutual fund returns previously examined by

Meyer (1979) using the mean-variance approach. Specifically, they find that the dominated set determined by Meyer is too large.

However, an important reason why SD has not seen the proliferation that one might expect based on its theoretical attractiveness is that until recently, SD efficiency can only be tested pairwise. This restriction limits the scope of SD tests, because investors generally can diversify between a set of assets and they can effectively face infinitely many choice alternatives. That is probably the most important reason why the two-moment EV model is still extensively used in the empirical portfolio analysis, despite the theoretical superiority of the SD criteria that take into account the entire distribution. As Levy (1992) concludes in his paper, "Ironically, the main drawback of the SD framework is found in the area of finance where it is most intensively used, namely, in choosing the efficient diversification strategies. This is because as yet there is no way to find the SD efficient set of diversification strategies as prevailed by the M-V framework. Therefore, the next important contribution in this area will probably be in this direction."

However, in recent years, many studies have tried to find computationally tractable empirical tests for SD efficiency with full diversification possibilities. The first breakthrough was made by Kuosmanen (2001). He has analytically characterized the sets of time series vectors that dominate a given evaluated portfolio by FSD, SSD, and TSD, respectively. Interestingly, these sets have a relatively simple polyhedral structure. Based on these insights, he has proposed tests of SD efficiency. The major innovation is that these tests account for diversification. In particular, he has formulated the FSD efficiency test as a 0-1 mixed integer linear programming problem, while the SSD and the TSD tests take the form of the standard LP problems. The generalizations to the higher order SD criteria follow in a straightforward manner. Therefore, he argues that such computationally tractable SD efficiency tests allow for diversified portfolios and significantly enhance the power of the SD criteria as well as extend their empirical applicability to areas where diversification plays an important role.

However, as Post (2003) states, the number of model variables increases progressively with the number of observations. The computational complexity of LP problems (as measured by the number of arithmetic operations, run time, and working memory requirements) also increases significantly with the number of model variables. Thus, the approach developed by Kuosmanen (2001) is computationally complicated for samples that are sufficiently large to allow for powerful analysis. Next, Kuosmanen (2004) has developed a series of operational tests for portfolio efficiency that are based on the general SD criteria and account for infinite numbers of diversification strategies. The key idea in this paper is to preserve the cross-sectional dependence of asset returns when forming portfolios. Instead of arranging data in the form of empirical distribution functions, he has re-expressed the SD criteria in T-dimensional Euclidean space spanned by return vectors representing rates of return in T different states of nature. Then, he derives explicit analytical characterizations for the FSD and SSD dominating sets as subsets of this T-dimensional state-space. Using these results, he further derives operational SD efficiency measures and test statistics that can be computed using standard mathematical programming algorithms and readily available software packages. Finally, the SD tests and efficiency measures are illustrated by an empirical application that analyzes industrial diversification of the market portfolio.

Post (2003), using the straightforward LP tests, has developed empirical tests for the SSD efficiency of a given portfolio with respect to all possible portfolios constructed from a set of assets. Specifically, the LP tests include the primal and the dual tests. The primal test checks whether we can construct a piecewise-linear utility function that rationalizes the evaluated portfolio, while the dual test checks whether we can construct a benchmark portfolio that outperforms the evaluated portfolio in terms of the ordered mean differences (OMDs) introduced by Bowden (2000). Then, he applies bootstrapping techniques and asymptotic distribution theory that can approximate the sampling properties of the test results and allow for statistical inference. His approach presents a very impressive start towards statistical inference in the SD framework with full diversification possibilities. As for the empirical application, Post has analyzed whether the Fama and French market portfolio is SSD efficient relative to all possible portfolios constructed from 25 Fama and French benchmark portfolios and the one-month U.S. Treasury bill (a riskless asset). The market portfolio is the value-weighted average of all NYSE, AMEX, and Nasdaq stocks. The benchmark portfolios are the intersections of five portfolios formed on market capitalization (size) and five portfolios formed on book-to-market equity ratio (B/M). Monthly returns from July 1963 to October 2001 (460 months) are used in his tests. The main results suggest that the market portfolio is SSD-inefficient.

The next paper of Post is to deal with the dual test which he has developed in Post (2003) for the SSD efficiency. To illustrate the dual SSD test, Post (2005) applies the test to analyze the effect of short-selling restrictions on the profitability of momentum investment strategies. The main finding is that with looser restrictions on short sales, the market portfolio is highly and significantly SSD inefficient, and that as an indicator of the statistical significance, the bootstrap p-value rejects the null hypothesis of the SSD efficiency. However, when imposing tighter restrictions on short sales, Post finds that the profitability of momentum strategies falls quickly. In fact, if short sales are excluded, the dual test and statistics suggest that no significant momentum effect remains.

Post and Vliet (2006) have also analyzed the SSD efficiency of the stock market portfolio. They first extend Post's (2003) empirical test for SSD efficiency in which they derive the asymptotic sampling distribution of the SSD test statistic under the true null of efficiency rather than the restrictive null of equal means that was used earlier. This extension is intended to avoid rejection of efficiency in cases where the market portfolio is efficient but the assets have substantially different means. Further, they derive a LP test for EV efficiency that can be compared directly with the SSD test, which allows for attributing differences between the two tests to omit moments exclusively. They find that the value-weighted CRSP all-share index is SSD-efficient relative to common benchmark portfolios formed on size, value, and momentum. However, the market portfolio is significantly EV-inefficient relative to value and momentum, consistent with the existing evidence on these puzzles. They also find that the SSD criterion is especially successful in rationalizing EV inefficiencies that occur in the 1970s and the early 1980s. This indicates that the asset pricing puzzles that occur in the EV framework can be explained by omitted return moments during this period.

More recently, Post and Versijp (2006) have developed SSD and TSD efficiency tests within the framework of Generalized Method of Moments (GMM). In contrast to Post's (2003) LP tests, the GMM tests consider all pricing errors rather than the maximum positive error only. Their application demonstrates that the mean-variance inefficiency of the CRSP all-share index may reflect left-tail risk not captured by variance. The low-beta stocks, which seem underpriced in mean-variance terms, typically have relatively low tail-betas.

Starting from the reward-risk model for portfolio selection introduced in De Giorgi (2005), De Giorgi and Post (2005) derive the reward-risk CAPM analogous to the classic mean-variance CAPM. In contrast to the mean-variance model, reward-risk portfolio selection arises from general axioms of investors' preferences including consistency with the second-order stochastic dominance. They establish that a necessary condition for the existence of market equilibrium in complete markets with risk averse investors is that investors' optimal allocations are co-monotonic. Then, they derive the pricing kernel as an explicitly given monotone decreasing function of market portfolio return, depending on the representative agent's risk perception through his probability distortion function. An empirical application suggests that the reward-risk CAPM better captures the cross-section of US stock returns than mean-variance CAPM does. Moreover, they find that the pricing kernel arising from the reward-risk analysis is similar to that obtained in the mean-semivariance equilibrium model.

Levy (1998) has developed the prospect stochastic dominance (PSD), which assumes a S-shaped utility function that is convex for losses and concave for gains. In addition, Levy and Levy (2002) have developed the Markowitz stochastic dominance (MSD), which assumes a reverse S-shaped utility function that is concave for losses and convex for gains. Levy and Post (2005) use the SSD, PSD, and MSD criteria to analyze investor behavior. Based on the assumptions of the M-V CAMP, they use a single-period, portfolio-oriented model of a frictionless and competitive capital market. They use various SD criteria that account for the possibility that investors exhibit local risk seeking behavior. They also develop a general LP test based upon Post (2003) for fitting SD efficiency criteria to empirical data and derive the asymptotic sampling distribution of the test results. After they analyze the SD efficiency classification of the value-weighted market portfolio relative to benchmark portfolios based on market capitalization, book-to-market equity ration, and momentum, their results suggest that reverse S-shaped utility functions, with risk aversion for losses and risk seeking for gains, can explain stock returns. Those results are also consistent with a reverse S-shaped pattern of subjective probability distortion, as in the cumulative prospect theory.

Empirically, co-skewness of asset returns seems to explain a substantial part of the cross-sectional variation of mean return not explained by beta. This finding is typically interpreted in terms of a risk averse representative investor with a cubic utility function. However, Post, Vilet, and Levy (2006) have questioned this interpretation in their paper. They show that the empirical tests fail to impose risk aversion and the implied utility function takes an inverse S-shape. Unfortunately, the first-order conditions are not sufficient to guarantee that the market portfolio is the global maximum for this utility function, and their results suggest that the market portfolio is more likely to represent the global minimum. In addition, co-skewness has minimal explanatory power if they impose risk aversion.

B. Research on the Stochastic Dominance Efficiency and the Mean-Variance Efficiency

Although the SD criteria have been shown theoretically superior to the EV approach, the question remains: If we apply the SD criteria to the portfolios that are on the EV frontier, do these EV efficient portfolios satisfy the SD efficiency at the same time? This question has been examined in several previous studies.

Porter and Gaumnitz (1972) have presented the results of several empirical studies of the similarities and differences between the EV and the SD efficiency. Their main finding is that the differences between the EV and the SSD efficiency are not great, and that the most significant difference between the EV results and the SSD and the TSD results is the tendency that the SD rules eliminate the EV efficient portfolios in the low-mean and low-variance range. Interestingly, this result implies that the less risk averse an investor is, the more indifferent he or she would be regarding a choice between EV and SSD as efficiency criteria. This implication follows from the fact that as the degree of risk aversion decreases, the investor moves up to the range of higher-mean and higher-variance where the EV and SSD-efficient sets become similar. On the other hand, the highly risk-averse investor is most likely to suffer from the use of the EV model. The reason is that the EV efficient portfolios on the lower tier of the EV frontier are excluded from the SSD-efficient set. Thus, when risk aversion is strong, the SSD and the TSD criteria are more consistent with the maximization of expected utility than is the EV rule.

Then, Porter (1973) has further developed his study to examine the SD-EV conflict from an empirical point of view. His findings confirm the conclusion in Porter and Gaumnitz (1972) that the differences are almost non-existent in the range of high-mean and high-variance of return, while most of the portfolios that are EV but not SSD- or TSD-efficient occur in the range of low-mean and low-variance. In this paper, Porter has also used different types of historical data – monthly, quarterly, semiannually, and annually – in order to figure out the most appropriate proxy for future data. The results suggest a use of monthly data; however, the significance of the differences in the choice of data should be evaluated more fully.

After a series of articles[4] in which Porter and his associates have conducted empirical comparisons of the EV and the SD portfolio choice criteria, Perrakis and Zerbinis (1978) have offered an exact theoretical justification of those empirical results of the aforementioned studies. They show that, for all cases of practical interest, a portion of the EV frontier is a subset of the SSD-efficient set. First, they define a necessary and sufficient condition for the corresponding portfolio to be quadratic utility-efficient. It is shown that the EV efficient portfolios which meet the quadratic utility criterion on the EV frontier are also SSD-efficient. In this sense, the theoretical proof confirms the empirical conclusion that the high-mean and high-variance portfolios on the upper range of EV frontier are both EV- and SSD-efficient. However, it does not follow necessarily that the remaining portion of the EV frontier is SSD-inefficient. The reason lies that the class of quadratic utilities (that is, the range of values of the parameter that guarantees a non-decreasing and concave utility for all random returns, and then find expected utility-maximizing portfolios for various values of the parameter within the range) can also be done for other classes of utility functions, for example, the cubic utility functions. Since the possibility exists that other classes of utility functions may also be satisfied by the

[4] See Porter (1973), Porter and Bey (1974), Porter and Carey (1974), and Porter and Gaumnitz (1972).

efficient portfolios on the EV frontier, the low-mean and low-variance EV efficient portfolios can be SSD-efficient or -inefficient.

Finally, Kroll, Levy, and Markowitz (1984) have developed the so called "direct utility maximization" program which allows for various expected utility functions for a case with an infinite number of alternative probability distributions. The optimal portfolios generated by the direct utility maximization are not necessarily consistent with the utility maximization required of the EV efficient set. Then, they compare the empirical results from the EV utility maximization with those from the direct utility maximization, finding that the mean and the variance of both maximization methods are very much the same and that the direct maximization portfolios are almost on the EV frontier. This result implies that the quadratic utility function assumed on the EV criterion is a good proxy for the optimal portfolio selection. They further argue that additional moments besides the mean and the variance will not improve the optimal portfolio selection substantially.

III. Theory and Computation of the Mean-Variance (EV) Frontier

Harry Markowitz first published his Nobel Prize winning mean-variance portfolio selection theory in 1952, and later, in 1959, published his book *Portfolio Selection*[5], one of the most important books published in the history of financial economics. It rests firmly at the root of the next half-century of research. His model is precisely step one of portfolio management: the identification of the efficient set of portfolios, or, as it is often called, the efficient frontier of risky assets.

The principal idea behind the Markowitz portfolio selection theory is that the "risk" of a portfolio can be adequately represented by the variance of its return. Given that, in identifying the frontier set of risky portfolios we must find, for any risk level, the portfolio with the highest expected return. Alternatively, the frontier is the set of portfolios that minimizes the variance for any target expected return.

Markowitz (1952, 1959) defines mean (E) and variance (V) of a specific portfolio as follows:

$$E = \sum_{i=1}^{N} \lambda_i \mu_i$$

$$V = \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \lambda_j \sigma_{ij} \qquad (1)$$

where the λ's are the weights put on different securities, and should satisfy $\sum_{i=1}^{N} \lambda_i = 1$. μ_i is the expected return and σ_i is the standard deviation of security *i*.

[5] We use the EV theory from Markowitz, H., 1959(1st edition) and 1991(2nd edition), *Portfolio Selection*, John Wiley & Sons, New York.

Markowitz also shows how to compute the EV efficient set. Specifically,

$$\lambda = \begin{bmatrix} \lambda_1 \\ \cdot \\ \cdot \\ \lambda_n \end{bmatrix} \text{ is a portfolio.} \tag{2}$$

A portfolio is legitimate if it satisfies constraints:

$$\begin{aligned} A\lambda &= b, \\ \lambda &\geq 0 \end{aligned} \tag{3}$$

where A is an m by n matrix and b is an m component column vector. The second constraint implies that no short sales are allowed, a restriction that we shall adopt in our study but that is not part of the Markowitz theory. μ_j is the expected return on the jth security; σ_{jk} is the covariance between the jth and kth securities:

$$\mu = \begin{bmatrix} \mu_1 \\ \cdot \\ \cdot \\ \mu_n \end{bmatrix},$$

$$C = \begin{bmatrix} \sigma_{11} & \cdot & \cdot & \sigma_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \sigma_{n1} & \cdot & \cdot & \sigma_{nn} \end{bmatrix} \tag{4}$$

The covariance matrix C is symmetric and positive semi-definite.

The expected return E of a portfolio is

$$E = \mu'\lambda. \tag{5}$$

The variance of return V of a portfolio is

$$V = \lambda' C \lambda. \tag{6}$$

An E, V combination (E^0, V^0) is obtainable if there is a legitimate portfolio λ^0 with

$$E^0 = \mu'\lambda^0 \text{ and } V^0 = \lambda^0{}'C\lambda^0. \tag{7}$$

An E, V combination (E^0, V^0) is efficient if:

1. E^0, V^0 is obtainable, and
2. no obtainable combination E^1, V^1 exists, such that either $E^1 > E^0$ and $V^1 \leq V^0$ or else $E^1 \geq E^0$ and $V^1 < V^0$.

A portfolio λ is efficient if it is legitimate and if its E, V combination is efficient. Therefore, our problem is to find:

1. the set of all efficient E, V combination, and
2. a legitimate portfolio for each efficient E, V combination.

In sum, the EV rule states that the investor would (or should) want to select one of those portfolios which give rise to the (E, V) combinations with minimum V for given E or more and maximum E for given V or less.

IV. Theory of Stochastic Dominance (SD)

In this section, we are going to introduce the theory of stochastic dominance in the first-, second-, and third-degree developed by Levy (1998).

A. First-degree Stochastic Dominance (FSD)

Suppose that the investor wishes to rank two investments whose cumulative distributions are F and G. We denote these two investments by F and G, respectively. The FSD rule is a criterion that tells us whether one investment dominates another investment where the only available information is that $U \in U_1$, namely that $U' \geq 0$ and, to avoid the trivial case of U' coinciding with the horizontal axis, there is a range where $U' > 0$. Actually, this is the weakest assumption on preference because we assume only that decision makers like more money rather than less money which conforms with the monotonicity axiom. Hence, we assume that U is a continuous non-decreasing function which implies that it is differentiable apart from a set of points whose measure is zero.

Theorem 1: Let F and G be the cumulative distributions of two distinct investments. Then F dominates G by FSD (which we denote by FD_1G, where D_1 denotes dominance by the first order and the subscript 1 indicates that we assume only one piece of information on U, namely that U is non-decreasing) for all $U \in U_1$ if and only if $F(x) \leq G(x)$ for all values x, and there is at least some x_0 for which a strong inequality holds. As FSD relates to $U \in U_1$, it can be summarized as follows:

$F(x) \leq G(x)$ for all x with a strong inequality for at least one x_0

$\Leftrightarrow$ $E_F U(x) \geq E_G U(x)$ for all $U \in U_1$ with a strong inequality for at least one $U_0 \in U_1$.

Defining $G(x) - F(x) = I_1(x)$ (again the subscript 1 reminds us of dealing with first order stochastic dominance), then the condition for FSD of F over G if that $I_1(x) \geq 0$ for all x and $I_1(x_0) > 0$ for some x_0.

B. Second-degree Stochastic Dominance (SSD)

So far, the only assumption for the FSD rule is that $U \in U_1$, namely, $U' \geq 0$. There is much evidence that most, if not all, investors are probably risk averters. Therefore, an additional rule has been developed appropriate for all risk averters. In all the discussions below, we deal only with non-decreasing utility function, $U \in U_1$, and add the assumption of risk aversion, $U'' \leq 0$.

We define the set of all concave utility functions corresponding to risk aversion by U_2. Of course, $U_2 \subseteq U_1$ when U_1 corresponds to FSD. However, we have to keep in mind that although all investors would agree that $U \in U_1$, not all would agree that $U \in U_2$.[6] Nevertheless, there is much evidence that for virtually all decision makers, $U \in U_2$. The fact that the cost of capital of most firms is generally higher than the riskless interest rate is an indication that stockholders are risk averse and require a risk premium.

[6] There is some evidence of risk-seeking behavior by some people, as demonstrated by gambling and lotteries.

Theorem 2: Let F and G be two investments whose density functions are f(x) and g(x), respectively. Then F dominates G by SSD denoted by FD_2G (the subscript 2 indicates a second order stochastic dominance) for all risk averters if and only if:

$$I_2(x) \equiv \int_a^x [G(t) - F(t)]dt \geq 0 \tag{8}$$

For all $x \in [a,b]$ and there is at least one x_0 for which there a strict inequality. This theorem can also be stated as follows:

$\int_a^x [G(t) - F(t)]dt \geq 0$ for all x with at least one strict inequality for some x_0

$\Leftrightarrow$ $E_F U(x) - E_G U(x) \geq 0$ for all $U \in U_2$ with at least one $U_0 \in U_2$ for which there is a strict inequality.

C. Third-degree Stochastic Dominance (TSD)

TSD is accordance with the set of utility functions $U \in U_3$ where $U' \geq 0$, $U'' \leq 0$, and $U''' \geq 0$. The assumptions $U' \geq 0$ and $U'' \leq 0$ are easier to grasp: $U' \geq 0$ simply assumes that investors prefer more money to less money (which stems from the monotonicity axiom), and $U'' \leq 0$ assumes risk aversion: other things being equal, decision makers dislike uncertainty or risk. But what is the meaning of the assumption of $U''' \geq 0$?

Arrow and Pratt have defined the absolute risk aversion measure "in the small" as the risk premium $\pi(w)$ given by $\pi(w) = -U''(w)/U'(w)$, where w is the investor's wealth. It is claimed that investor's behavior reveals that $\frac{\partial \pi(w)}{\partial(w)} < 0$. This property is called the decreasing absolute risk aversion (DARA), which implies that $U''' \geq 0$.[7] In the TSD theory, U''' refers to the distribution skewness.

Theorem 3: Let F(x) and G(x) be the cumulative distributions of two investments under consideration whose density functions are f(x) and g(x), respectively. Then F dominates G by TSD if and only if the following two conditions hold:

$$I_3(x) = \int_a^x \int_a^z [G(t) - F(t)] dt dz \geq 0 \qquad (9a)$$

(for the sake of brevity, we denote the double integral by $I_3(x)$; hence, it is required that $I_3 \geq 0$).

$$E_F(x) \geq E_G(x) \text{ or } I_2(b) \geq 0 \qquad (9b)$$

and there is at least one strict inequality, namely:

$I_3(x) \geq 0$ and $I_2(b) > 0 \Leftrightarrow E_F U(x) \geq E_G U(x)$ for all $U \in U_3$.

To have a dominance we require that either $I_3(x_0) \geq 0$ for some x, or $I_2(b) > 0$, which guarantees that a strong inequality holds for some $U_0 \in U_3$ (recall that $U \in U_3$ if $U' \geq 0$, $U'' \leq 0$, and $U''' \geq 0$). Further, if that F dominates G by TSD, we write it as FD_3G where the subscript 3 indicates a third order stochastic dominance.

[7] The quadratic utility function obviously violates the DARA property.

V. Linear Programming Tests

The Linear Programming (LP) Tests developed by Post (2003) are computationally tractable methods for determining whether SSD efficiency holds for a given portfolio. Therefore, we apply the Post's (2003) LP tests to our empirical study. In this section, we present the two Post's LP tests for SSD efficiency: i) a primal test, and ii) an equivalent dual test.

We consider a single-period, portfolio-based model of investment that satisfies the following assumptions: [8]

Assumption 1: Investors are non-satiable and risk-averse, and they choose investment portfolios to maximize the expected utility associated with the return of their portfolios. Throughout the text, we will denote utility functions by $u:\Re \rightarrow P, u \in U_2$, with U_2 for the set of increasing and concave, continuously differentiable, von Neumann-Morgenstern utility functions, and P for a nonempty, closed, and convex subset of $\Re$.[9]

Assumption 2: The investment universe consists of N assets, associated with returns $x \in \Re^N$. Throughout the text, we will use the index set $I \equiv \{1,...,N\}$ to denote the different assets. The return vector is a random vector with a continuous joint cumulative

[8] The assumptions here are not entirely accordance with the assumptions in Post (2003). Rather, some parts have been modified in accordance with the working paper of Post and Versijp (2006) for the purpose of our empirical study.

[9] Throughout the text, we will use R^N for an N-dimension Euclidean space, and R_+^N denotes the positive orthant. We also use x^T for the transpose of x.

distribution function (CDF) $G:\Re^N \to [0,1]$. Further, the returns have a vector of means $E[x]=\mu$ and a variance-covariance matrix $E[(x-\mu)(x-\mu)^T]=\Omega$.

Assumption 3: Investors may diversify between the assets, and we will use $\lambda \in \Re^N$ for a vector of portfolio weights. We consider the case where short sales are not allowed, because short selling typically is difficult to implement in practice due to margin requirements and explicit or implicit restrictions on short selling for institutional investors. In addition, the portfolio weights belong to the portfolio possibilities set $\Lambda \equiv \{\lambda \in \Re_+^N : e^T\lambda = 1\}$, where e indicates a unity of dimension N. A given portfolio $\tau \in \Lambda$ is optimal for an investor with utility $u \in U$ if and only if

$$\int u(x^T\tau)dG(x) = \max_{\lambda\in\Lambda} \int u(x^T\lambda)dG(x) \tag{10}$$

Assumption 4: The observations are serially independently and identically distributed (IID) random draws from the CDF. Throughout the text, we will represent the observations by the matrix $X \equiv (x_1 \cdots x_T)$, with $x_t \equiv (x_{1t} \cdots x_{Nt})^T$. Since the timing of the draws is inconsequential, we are free to label the observation by their ranking with respect to the evaluated portfolio, that is, $x_1^T\tau < x_2^T\tau < \cdots < x_T^T\tau$.

Using the observations, we can construct the empirical distribution function (EDF):

$$F(x) \equiv card\{t \in \Theta : x_t \le x\}/T \tag{11}$$

with $\Theta \equiv \{1,...,T\}$ for the index set of ranked observations, and card $\{\bullet\}$ for the number of elements of a set. Our empirical tests will use the EDF in place of the CDF. Based on these assumptions, we define SSD for pairwise comparisons as follows:

Definition 1: Portfolio $\lambda \in \Lambda$ dominates portfolio $\tau \in \Lambda$ by SSD if and only if, for all utility functions $u \in U_2$, λ has a higher expected utility than τ, that is,

$$\int u(x^T\lambda)dF(x) - \int u(x^T\tau)dF(x) > 0 \quad \forall u \in U_2 \quad \Leftrightarrow \tag{12}$$

$$\sum_{t\in\Theta}[u(x_t^T\lambda) - u(x_t^T\tau)]/T > 0 \quad \forall u \in U_2 \tag{13}$$

In Post (2003) this definition of SSD uses strict inequalities for all $u \in U_2$. By contrast, the traditional definition[10] uses weak inequalities with a strict inequality for at least one $u \in U_2$. The theoretical differences between those two definitions can be illustrated by the example in Post (2003). For example, using Definition 1, $\lambda \in \Lambda$ does not dominate mean-preserving spreads of λ, because risk-neutral investors are indifferent between alternatives that have identical means. On the other hand, dominance does exist using the traditional definition, because all strictly risk-averse investors do prefer λ to its mean-preserving spreads. However, as Post has stated, from an empirical perspective, the definitions are indistinguishable, because arbitrary small data perturbations to the evaluated portfolio can make the classification consistent. Related to this, data sets where this theoretical issue has a decisive impact are extremely unlikely for return distributions that are continuous by approximation.

[10] See Levy (1998).

The following is a straightforward generalization of Definition 1 to the case where full diversification is allowed.

Definition 2: Portfolio $\tau \in \Lambda$ is SSD-inefficient if and only if some portfolio $\lambda \in \Lambda$ SSD dominates it. Alternatively, portfolio $\tau \in \Lambda$ is SSD-efficient if and only if no portfolio $\lambda \in \Lambda$ SSD dominates it.

Post (2003) also rephrases this definition in terms of a minimax formulation.

Theorem 4: Portfolio $\tau \in \Lambda$ is SSD-inefficient if and only if, for all utility functions $u \in U_2$, the maximum expected utility of λ is greater than the expected utility of τ, that is,

$$\min_{u \in U_2} \left\{ \max_{\lambda \in \Lambda} \left\{ \int u(x^T \lambda) dF(x) - \int u(x^T \tau) dF(x) \right\} \right\} > 0 \Leftrightarrow \tag{14}$$

$$\min_{u \in U_2} \left\{ \max_{\lambda \in \Lambda} \left\{ \sum_{t \in \Theta} [u(x_t^T \lambda) - u(x_t^T \tau)] / T \right\} \right\} > 0 \tag{15}$$

Alternatively, portfolio $\tau \in \Lambda$ is SSD-efficient if and only if it is optimal relative to some utility functions $u \in U_2$, that is,

$$\min_{u \in U_2} \left\{ \max_{\lambda \in \Lambda} \left\{ \int u(x^T \lambda) dF(x) - \int u(x^T \tau) dF(x) \right\} \right\} = 0 \Leftrightarrow \tag{16}$$

$$\min_{u \in U_2} \left\{ \max_{\lambda \in \Lambda} \left\{ \sum_{t \in \Theta} [u(x_t^T \lambda) - u(x_t^T \tau)] / T \right\} \right\} = 0 \tag{17}$$

A. Primal Test

The primal test verifies whether we can construct piecewise-linear utility functions $p \in U_2$ that rationalize the evaluated portfolio $\tau \in \Lambda$. All relevant piecewise-linear utility functions $p \in U_2$ can be constructed from a series of T linear support lines characterized by intercept coefficients $\alpha \equiv (\alpha_1 \cdots \alpha_T)^T \in \Re^T$ and normalized slope coefficients $\beta \in B \equiv \{\beta \in \Re_+^T : \beta_1 \geq \beta_2 \geq \cdots \geq \beta_T = 1\}$ as

$$p(x|\alpha, \beta) \equiv \min_{t \in \Theta}(\alpha_t + \beta_t x) \tag{18}$$

Based on this central idea, Post (2003) develops the primal test statistics as follows:

$$\xi(\tau) \equiv \min_{\beta \in B, \theta}\left\{\theta : \sum_{t \in \Theta} \beta_t (x_t^T \tau - x_{it})/T + \theta \geq 0 \forall i \in I\right\} \tag{19}$$

$$\text{subject to} \qquad \beta_1 \geq \beta_2 \geq \ldots \geq \beta_T = 1$$

Post (2003) interprets the primal test statistics $\xi(\tau)$ as the "least amount of disutility" that any non-satiated and risk-averse investor would suffer by holding on to the evaluated portfolio τ. The primal problem involves T variables and $N + T - 1$ constraints. According to Post (2003), a possible primal solution is $\beta = e$, associated with upper bound $\xi(\tau) = \max_{i \in I}\left\{\sum_{t \in \Theta}(x_{it} - x_t^T \tau)/T\right\}$. This solution effectively represents the utility function $u(x) = x$ or the risk-neutral investor who cares about expected return only.

B. Dual Test

The dual test involves the Ordered Mean Difference (OMD) introduced by Bowden (2000). The formula is:

$$\rho_t(\lambda,\tau) \equiv \sum_{s=1}^{t}(x_s^T\lambda - x_s^T\tau)/t \qquad t \in \Theta \tag{20}$$

The OMD represents a running mean for the difference between the return of the evaluated portfolio τ and a benchmark portfolio $\lambda \in \Lambda$. The dual test statistic is presented as follows:

$$\psi(\tau) \equiv \max_{\lambda\in\Lambda}\{\rho_T(\lambda,\tau) : \rho_t(\lambda,\tau) \geq 0 \forall t \in \{1,\ldots,T-1\}\} \tag{21}$$

$$\text{subject to} \quad \sum_{i=1}^{N}\lambda_i = 1,\ \lambda_i \geq 0,\ \text{i=1, … , N}$$

The dual problem involves $N+T-1$ variables and T constraints. A possible dual solution is $\lambda = \tau$, associated with the lower bound $\psi(\tau) = 0$. The dual test statistic $\psi(\tau)$ gives the maximum possible increase in average return, that is, $\rho_T(\lambda,\tau)$, achieved by a benchmark portfolio $\lambda \in \Lambda$ that outperforms the evaluated portfolio in terms of the OMDs. These test statistics give necessary and sufficient conditions that can separate efficient portfolios from inefficient ones. The central theorem developed by Post (2003) states the following:

Theorem 5: Portfolio $\tau \in \Lambda$ is SSD-inefficient if and only if $\xi(\tau) > 0$ or, equivalently, $\psi(\tau) > 0$. Alternatively, portfolio $\tau \in \Lambda$ is SSD-efficient if and only if $\xi(\tau) = 0$ or, equivalently, $\psi(\tau) = 0$.

VI. Polynomial Utilities

Perrakis has developed a series of inequality conditions on the portfolios' central moments arising out of SSD and TSD. The following discussion of the relations arising out of polynomial utilities follows the notation from Perrakis' unpublished paper.

Let y denote the one-period random terminal wealth of an investor whose utility function is u(y). In our discussion, we consider only investment prospects whose terminal wealth distributions have finite ranges. Given such assets, it is always possible to normalize[11] the distributions of their random returns to make them lie between 0 and 1. Without loss of generality it will be assumed that such normalization has already been carried out.

The behavioral postulates in the stochastic dominance imply some restrictions on the shape of the utility function, which is assumed continuous and differentiable and whose first, second, and third derivatives are assumed to exist almost everywhere. These are $u' \geq 0$, $u'' \leq 0$, and $u''' \geq 0$ for all y in the [0,1] interval. The utilities that we consider will satisfy either the first two or all three of the above restrictions.[12] We denote by $\{U^i\}$, i=2, 3 the corresponding sets of utility functions, i.e. $u \in \{U^2\}$ if $u' \geq 0$, $u'' \leq 0$, and $u \in \{U^3\}$ if $u \in \{U^2\}$ and $u''' \geq 0$.

Suppose that the normalization has been carried out, and that the random return of any asset in the choice set is equal to K+y, where $K \geq 0$ is the normalized smallest possible return for all assets in the choice set and $y \in [0,1]$. Then, it may be possible to

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[11] Such normalization involves dividing each return by the largest possible return for all assets in the choice set; if K now denotes the smallest possible return for the assets in the choice set, then all random returns are equal to K+y, where y has a [0,1] domain.

[12] FSD is ignored here. In our opinion, it is too weak a condition to impose on admissible investors' utilities.

expand an investor's utility function u(y) in a Taylor series around u(K): $u(y) = u(K) + u'y + \frac{u''y^2}{2} + \frac{u'''y^3}{3} + \ldots$, where all derivatives have been evaluated at K. Such an expansion, if it converges, can be truncated at any desired degree of approximation.

A polynomial utility of degree n must belong to the set $\{U^2\}$ or $\{U^3\}$. This means that its coefficients obey some restrictions. These can be expressed by the following results.

Definition 3: A polynomial utility of degree n belongs to $\{U^i\}$ if it can be written in the form P_n^i, i=2, 3, where

$$P_n^2 = \sum_{j=0}^{n-2} \frac{a_j}{(j+1)(j+2)}[(j+2)y - y^{j+2}] + by + u_0, \quad n=2, 3\ldots \tag{22a}$$

$$P_n^3 = \sum_{j=0}^{n-3} \frac{a_j}{(j+1)(j+2)(j+3)}[(j+1)(j+3)y - \frac{1}{2}(j+3)(j+2)y^2 + y^{j+3}] + b_1 y - b_2 y^2 + u_0, \quad n=3, 4\ldots \tag{22b}$$

where the coefficients are such that

$$u_0 \geq 0,\ b \geq 0,\ b_1 \geq 2b_2 \geq 0,\ \sum_{j=0}^{n-i} a_j y^j \geq 0, \text{ for all } y \in [0,1]. \tag{23}$$

Definition 3 allows us to represent by polynomials admissible utility functions within $\{U^2\}$ or $\{U^3\}$ up to any desired degree of accuracy. The truncation of the polynomials after the nth power means that the moments of order higher than n are irrelevant to the investor's decision-making. An investor with a utility within the set

$\{P_n^2\}$ or $\{P_n^3\}$ is represented by a set of coefficients satisfying (23), in which case the behavioral postulates are satisfied.

Since $\{P_n^i\} \in \{U_n^i\}$, i=2, 3, it follows that if asset A is preferred to asset B by all investors with $u(y) \in \{U^i\}$, then the preference is also going to hold for all $u(y) \in \{P_n^i\}$. Consequently, if a given condition on the assets' moments is necessary for the SSD (TSD) of A over B for $u(y) \in \{P_n^2\}$ ($\{P_n^3\}$), it will also be necessary for the SSD (TSD) over $\{U^2\}$ ($\{U^3\}$). Hence, the use of the polynomial representation allows us to derive general results that are valid for dominance over the entire set of utilities. For this to be true, however, an inequality relationship on the assets' moments must hold for all possible coefficients satisfying (23).

Hereafter we use the general notation u_i to denote the i^{th} moment from the origin $E[y^i]$, i=2, 3, 4,.... The unsubscripted μ denotes E[y], the portfolio mean, σ^2 is the portfolio variance, $S \equiv E[(y-\mu)^3]$ is the third central moment or skewness, and $K \equiv E[(y-\mu)^4]$ is the fourth central moment or kurtosis. Subscripts A and B on the central moments denote two portfolios, one of which may dominate the other over the appropriate utility set.

If A stochastically dominates B in the second or third degree, it is well-known that this implies that $\mu_A \geq \mu_B$, and that the following necessary condition must be satisfied [Jean (1984) and Whitmore (1970)]:

$$\sigma_A^2 - \sigma_B^2 \leq (\mu_A - \mu_B)[2-(\mu_A + \mu_B)] \tag{24}$$

Note that this inequality allows dominance to exist even if σ_A^2 is larger than σ_B^2, provided $\mu_A > \mu_B$ and the difference of variance does not exceed the right-hand-side (RHS). The moment inequalities to be derived in the next subsections are similar in spirit to (24), involving the second, third, and fourth central moments.

A. Quadratic Utility and SSD Efficiency on the EV Frontier

Perrakis and Zerbinis (1978) have provided an exact theoretical justification showing that a portion of the EV frontier is a subset of the SSD-efficient set. This theoretical proof is based on the empirical findings that the upper portion of the EV frontier involves the portfolios which are also SSD-efficient; however, the portfolios on the lower tier of the EV frontier where they have lower-mean and lower-variance may be SSD-inefficient.

As we mentioned before, U_2 denote the class of increasing and concave utility functions, on which SSD is normally defined. For the case where portfolio returns are bounded above by b, the following class of quadratic utility functions is contained in U_2.

$$U_q = \left\{ u(x) \middle| u(x) = x - Kx^2, K \in [0, \frac{1}{2b}) \right\}. \tag{25}$$

The portfolios that maximize uniquely E[u(x)] for all $u(x) \in U_q$ are both EV- and SSD- efficient.

In the case of an EV frontier that is continuous, differentiable, and strictly convex everywhere within its domain of definition.

$$\sigma^2 = f(\mu),\ f' \text{ piece-wise continuous and increasing,} \tag{26}$$

where μ and σ^2 are the portfolio's mean and variance respectively. The necessary and sufficient conditions on points of the EV frontier so that the corresponding portfolios would be quadratic utility-efficient are developed as follows:

Theorem 6: Given a point $(\mu, f(\mu))$ on the EV frontier, a necessary and sufficient condition for the corresponding portfolio to be quadratic utility-efficient is:

$$f' > 2(b - \mu). \tag{27}$$

Note that this result (27) can be easily extracted from inequality (24) in Perrakis' unpublished paper: first we normalize the returns by dividing all of them by b, so that they now lie in the interval [0,1]; then we denote the variance along the EV frontier as $f(\mu)$ and set $\mu_A - \mu_B$ equal to $\Delta\mu$; last, we divide both sides by $\Delta\mu$ and take the limit for $\Delta\mu$ tending to zero. Then (27) becomes $f' \leq 2(1-\mu)$, which is a condition for quadratic utility dominance of two adjacent portfolios on the EV frontier. Hence, its reversed form:

$$f' > 2(1 - \mu) \tag{28}$$

is the condition for the absence of dominance, thus establishing the above result (28) for the normalized returns.

B. The Preference for Skewness

From (22a) and (22b), eliminating the constants and simplifying, we get:

$$E[u(y)|u(y)\in\{P_3^2\}]=(a_0+\frac{a_1}{2}+b)\mu-\frac{a_0}{2}\mu_2-\frac{a_1}{6}\mu_3 \tag{29}$$

$$E[u(y)|u(y)\in\{P_3^3\}]=(\frac{a_0}{2}+b_1)\mu-(\frac{a_0}{2}+b_2)\mu_2+\frac{a_0}{6}\mu_3 \tag{30}$$

where the coefficients in (29) and (30) obey the restrictions in (23). Setting $\mu_2=\mu^2+\sigma^2$, and $\mu_3=S+\mu^3+3\mu\sigma^2$, we get the expected utilities in terms of the third central moments:

$$E[u(y)|u(y)\in\{P_3^2\}]=(a_0+\frac{a_1}{2}+b)\mu-\frac{a_0}{2}\mu^2-\frac{a_1}{6}\mu^3-\frac{\sigma^2}{2}(a_0+a_1\mu)-\frac{a_1}{6}S \tag{31}$$

$$E[u(y)|u(y)\in\{P_3^3\}]=(\frac{a_0}{2}+b_1)\mu-(\frac{a_0}{2}+b_2)\mu^2+\frac{a_0}{6}\mu^3-\frac{\sigma^2}{2}[\frac{a_0}{2}(1-\mu)+b_2)]+\frac{a_0}{6}S \tag{32}$$

The restrictions in (23) imply that, with the exception of a_1, all coefficients must be non-negative in (31) and (32), and that $b_1\geq 2b_2$. On the other hand, even though a_1 can be negative, we must have $a_0+a_1y\geq 0$ for all $y\in[0,1]$.

Jean (1984) has proved the following necessary condition[13] for SSD and TSD of A over B:

$$3\mu_B(1-\mu_B)+\mu_B^3-3\sigma_B^2(1-\mu_B)+S_B\leq 3\mu_A(1-\mu_A)+\mu_A^3-3\sigma_A^2(1-\mu_A)+S_A \tag{33}$$

An additional moment condition for SSD of A over B has been proven by Perrakis in the following theorem.

[13] The form of the condition proven by Jean (1984) is slightly different from (33) because the random returns of A and B were not normalized.

Theorem 7: If A and B are portfolios with $\mu_A \geq \mu_B$, then necessary conditions for second-degree stochastic dominance of A over B are that (33) must hold, together with

$$3\mu_B(1-\sigma_B^2) - \mu_B^3 - S_B \leq 3\mu_A(1-\sigma_A^2) - \mu_A^3 - S_A \tag{34}$$

Together with (24), (33), and (34) are also sufficient conditions for A to be preferred over B by all investors with utilities $u(y) \in \{P_3^2\}$, as defined in Definition 3.

C. The Fourth Central Moment

From (22a) and (22b) we get, after eliminating the constants and simplifying:

$$E[u(y)|u(y)\in\{P_4^2\}]=(a_0+\frac{a_1}{2}+\frac{a_2}{2}+b)\mu-\frac{a_0}{2}\mu_2-\frac{a_1}{6}\mu_3-\frac{a_2}{12}\mu_4 \quad (35)$$

$$E[u(y)|u(y)\in\{P_4^3\}]=(\frac{a_0}{2}+\frac{a_1}{3}+b_1)\mu-(\frac{a_0}{2}+\frac{a_1}{4}+b_2)\mu_2+\frac{a_0}{6}\mu_3+\frac{a_1}{24}\mu_4 \quad (36)$$

These results can be expressed in terms of the central moments, which are linked to the μ_i's, the moments from the origin, by the following relations: $\mu_2=\mu^2+\sigma^2$, $\mu_3=S+\mu^3+3\mu\sigma^2$, and $\mu_4=K+4\mu S+6\mu^2\sigma^2+\mu^4$. We get:

$$\begin{aligned}E[u(y)|u(y)\in\{P_4^2\}]&=(a_0+\frac{a_1}{2}+\frac{a_2}{3}+b)\mu-\frac{a_0}{2}\mu^2-\frac{a_1}{6}\mu^3-\frac{a_2}{12}\mu^4\\&-\frac{\sigma^2}{2}(a_0+a_1\mu+a_2\mu^2)-\frac{S}{6}(a_1+2a_2\mu)-\frac{a_2}{12}K\end{aligned} \quad (37)$$

$$\begin{aligned}E[u(y)|u(y)\in\{P_4^3\}]&=(\frac{a_0}{2}+\frac{a_1}{3}+b_1)\mu-(\frac{a_0}{2}+\frac{a_1}{4}+b_2)\mu^2+\frac{a_0}{6}\mu^3+\frac{a_1}{24}\mu^4\\&-\frac{\sigma^2}{2}[(a_0(1-\mu)+\frac{a_1}{2}(1-\mu^2)+b_2)]+\frac{S}{6}(a_0+a_1\mu)+\frac{a_1}{24}K\end{aligned} \quad (38)$$

where the restrictions (23) on the coefficients imply, among others, that $a_0+a_1y+a_2y^2\geq 0$ for all $y\in[0,1]$ in (37) and $a_0+a_1y\geq 0$ for all $y\in[0,1]$ in (38).

From Perrakis' unpublished paper, the necessary conditions for SSD are expressed as follows:

Theorem 8: If A and B are assets or portfolios with $\mu_A \geq \mu_B$, then if A dominates B in the second degree and relation (24) holds as a strict inequality we must have:

$$K_B - K_A \geq \frac{3[\frac{S_A - S_B}{3} + (\sigma_A^2\mu_A - \sigma_B^2\mu_B) - (\mu_A - \mu_B) + \frac{\mu_A^3 - \mu_B^3}{3}]^2}{(\mu_A - \mu_B)[1 - \frac{(\mu_A + \mu_B)}{2}] + \frac{\sigma_B^2 - \sigma_A^2}{2}}$$
$$+ 4[S_A\mu_A - S_B\mu_B - (\mu_A - \mu_B)] + \mu_A^4 - \mu_B^4 + 6(\sigma_A^2\mu_A^2 - \sigma_B^2\mu_B^2) \tag{39a}$$

$$2\mu_A - 2\mu_A^3 + \mu_A^4 - 6\sigma_A^2\mu_A(1-\mu_A) - 2S_A(1-\mu_A) + K_A$$
$$\geq 2\mu_B - 2\mu_B^3 + \mu_B^4 - 6\sigma_B^2\mu_B(1-\mu_B) - 2S_B(1-\mu_B) + K_B \tag{39b}$$

Together with (24), (33), and (34), the above conditions are also sufficient for A to be preferred over B by all investors with utilities $u(y) \in \{P_4^2\}$ defined in Definition 3.

Theorem 9: If A and B are assets or portfolios with $\mu_A \geq \mu_B$, then necessary conditions for third-degree stochastic dominance of A over B are:

$$8\mu_B - 6\mu_B^2 + \mu_B^4 - 6\sigma_B^2(1 - {\mu_B}^2) + 4S_B\mu_B + K_B$$
$$\leq 8\mu_A - 6\mu_A^2 + \mu_A^4 - 6\sigma_A^2(1-\mu_A^2) + 4S_A\mu_A + K_A \tag{40a}$$

$$4\mu_B - 6\mu_B^2 + 4\mu_B^3 - \mu_B^4 - 6\sigma_B^2(1-\mu_B)^2 + 4S_B(1-\mu_B) - K_B$$
$$\leq 4\mu_A - 6\mu_A^2 + 4\mu_A^3 - \mu_A^4 - 6\sigma_A^2(1-\mu_A)^2 + 4S_A(1-\mu_A) - K_A \tag{40b}$$

Together with (24) and (33), these necessary conditions are also sufficient for A to be preferred over B by all investors with utilities $u(y) \in \{P_4^3\}$ as in Definition 3.

VII. SSD Efficiency of the EV Frontier – A New Test

In this section, we are going to apply the results of the third and fourth degree moment inequalities derived from polynomial utility functions and try to find some new optimal portfolios which may or may not be on the EV frontier.

In Perrakis' unpublished paper, a third degree polynomial utility satisfying the appropriate restrictions (23) has the form given by equation (31). Then, the moment inequalities (33) and (34) are extracted, which are the necessary conditions for the second-degree stochastic dominance of A over B on the third central moment. For notational convenience, we first define the following functions from inequalities (33) and (34):

$$3\mu(1-\mu)+\mu^3-3\sigma^2(1-\mu)+S \equiv A_1(\mu) \tag{41a}$$

$$3\mu(1-\sigma^2)-\mu^3-S \equiv A_2(\mu) \tag{41b}$$

These inequalities imply that portfolio A dominates by SSD portfolio B if the above moment functions are higher for A than for B.

Suppose now that we consider portfolios along the EV frontier, in which case $\sigma^2 = f(\mu)$. Let $S(\mu)$ denote the third central moment or skewness of these portfolios. By the same reasoning as in the derivation of the quadratic utility-efficient portion, we must have either one of $dA_i / d\mu$, i=1, 2, negative for absence of dominance of two adjacent portfolios on the EV frontier. Setting again $\sigma^2 = f(\mu)$ and differentiating, we get:

$$\frac{dA_1}{d\mu} < 0 \Leftrightarrow \frac{dS}{d\mu} < 3[f'(1-\mu) - f - \mu^2 + 2\mu - 1] \equiv B_1 \tag{42a}$$

$$\frac{dA_2}{d\mu} < 0 \Leftrightarrow \frac{dS}{d\mu} > 3(1 - f - \mu f' - \mu^2) \equiv B_2 \tag{42b}$$

If either one of B_1 or B_2 is verified along the entire EV frontier up to a given value of μ then that portfolio of the frontier is undominated by higher mean portfolios on the frontier. Unfortunately, this solution still leaves open the possibility that the lower-mean frontier portfolios may be dominated by higher mean portfolios not on the frontier.

However, since there are no a priori restrictions on the sign of $dS/d\mu$, it is possible that inequalities (42a) and (42b) change direction along various parts of the frontier. Therefore, we use the following programs to guarantee that its solutions would generate SSD-undominated portfolios that may or may not be on the EV frontier. We denote by M the dimension of the portfolio vector and by N the dimension of the sample.[14] Let $\lambda \equiv (\lambda_1, ..., \lambda_M)$ denote the (unknown) portfolio vector that is potentially SSD-efficient. Let also the moments be:

$$Mean \equiv m(\lambda) = \frac{\sum_{j=1}^{N}\sum_{i=1}^{M}\lambda_i x_{ij}}{N} \tag{43}$$

$$Variance \equiv \sigma^2(\lambda) = [(R-\mu)^2] = \frac{\sum_{j=1}^{N}[\sum_{i=1}^{M}\lambda_i x_{ij} - m(\lambda)]^2}{N}$$

$$Skewness \equiv S(\lambda) = [(R-\mu)^3] = \frac{\sum_{j=1}^{N}[\sum_{i=1}^{M}\lambda_i x_{ij} - m(\lambda)]^3}{N}$$

[14] Later in our empirical tests, M=45 and N=104 for our weekly data; M=24 and N=60 for our monthly data. In general, M refers to the number of funds and N refers to the number of time periods.

$$Kurtosis \equiv K(\lambda) = [(R-\mu)^4] = \frac{\sum_{j=1}^{N}[\sum_{i=1}^{M}\lambda_i x_{ij} - m(\lambda)]^4}{N}$$

where x denotes the normalized returns in the interval [0, 1] in our sample.

Then, we solve the programs for descending values of μ and $f(\mu)$ along the EV frontier, for $A_1(\mu)$ and $A_2(\mu)$ given by (41ab), and with $\sigma^2(\lambda)$ and $S(\lambda)$ given by (43).

$$Max_\lambda \{A_k(m(\lambda))\} \qquad \text{k=1, 2} \tag{44}$$

$$\text{subject to} \qquad m(\lambda) \geq \mu,\ \sigma^2(\lambda) \geq f(\mu),\ \lambda_i \geq 0,\ \sum_{i=1}^{M}\lambda_i = 1$$

The optimal portfolios λ^* are SSD-undominated. We want to determine whether they also lie on the EV frontier beyond the quadratic utility-efficient segment identified earlier. Let also $A_i{}^*(\mu)$ denotes the optimal values of the objective functions in these maximizations.

For the kurtosis, the fourth central moment, the fourth degree polynomial utility is reproduced from Perrakis' unpublished paper equation (37), where the coefficients must meet the restrictions in equation (23) of that same paper. The necessary conditions are inequalities (39a) and (39b) in the paper, which must be satisfied by the moments of two portfolios A and B if A is to dominate B by SSD. While (39a) is too complex to generate undominated portfolios by a program, we can extract the following function from (39b):

$$2\mu - 2\mu^3 + \mu^4 - 6\sigma^2\mu(1-\mu) - 2S(1-\mu) + K \equiv D(\mu) \tag{45}$$

Portfolio A dominates by SSD portfolio B if $D(\mu)$ is higher for A than for B. Hence, we proceed again as before, solving the following program for descending values of μ and

$f(\mu)$ along the EV frontier, for $D(\mu)$ given by (45) and with $\sigma^2(\lambda), S(\lambda)$, and $K(\lambda)$ given by (43).

$$Max_{\lambda}\{D(m(\lambda))\} \tag{46}$$

$$\text{subject to} \qquad m(\lambda) \geq \mu\,,\ \sigma^2(\lambda) \geq f(\mu)\,,\ \lambda_i \geq 0\,,\ \sum_{i=1}^{M} \lambda_i = 1$$

The optimal portfolios given by (46) are also SSD-undominated. However, we notice that the programs (44) and (46) may or may not generate new SSD-efficient portfolios. This is what we are going to verify in our empirical applications. Also, we denote by $D_i*(\mu)$ the optimal values of the objective functions in these maximizations.

An additional set of undominated portfolios may be determined by examining necessary conditions for the third-degree stochastic dominance involving the fourth central moment. According to Perrakis' unpublished paper, the fourth degree polynomial utility for $u(y) \in \{P_4^3\}$ is expressed by equation (38) in that paper, and the coefficients must satisfy the restrictions in (23). Further, the necessary conditions for portfolio A stochastically dominates B by TSD on the fourth central moment are presented in inequalities (40a) and (40b) in the paper. We extract from inequalities (40a) and (40b) the following expressions:

$$8\mu - 6\mu^2 + \mu^4 - 6\sigma^2(1-\mu^2) + 4S\mu + K \equiv E_1(\mu) \tag{47a}$$

$$4\mu - 6\mu^2 + 4\mu^3 - \mu^4 - 6\sigma^2(1-\mu)^2 + 4S(1-\mu) - K \equiv E_2(\mu) \tag{47b}$$

These two functions also indicate that portfolio A dominates by TSD portfolio B if the above two fourth central moment functions are higher for A than for B.

As before, we solve the following programs for descending values of μ and $f(\mu)$ along the EV frontier, for $E_1(\mu)$ and $E_2(\mu)$ given by (47ab) and with $\sigma^2(\lambda), S(\lambda)$, and $K(\lambda)$ given by (43).

$$Max_{\lambda}\{E_k(m(\lambda))\} \qquad \text{k=1, 2} \tag{48}$$

$$\text{subject to} \qquad m(\lambda) \geq \mu,\ \sigma^2(\lambda) \geq f(\mu),\ \lambda_i \geq 0,\ \sum_{i=1}^{M} \lambda_i = 1$$

The optimal portfolios given by (48) should be TSD-undominated. Since the lack of dominance by TSD implies also lack of dominance by SSD, whatever portfolios we may find by (48) belong also to the SSD-efficient set. We denote $E_i*(\mu)$ the optimal values of the objective functions in these maximizations.

VIII. Empirical Application

A. Data

In our empirical test, we have used two sets of data – weekly returns and monthly returns of a sample of U.S. equity mutual funds.[15] In particular, the weekly data is composed of returns adjusted for dividends of 45 U.S. equity funds from January, 2003 to December, 2004. The monthly data consists of returns adjusted for dividends of 24 U.S. equity funds from January, 2001 to December, 2005. We select these 24 funds from those among the 45 funds for which 5-year data exists. Hence, the sample size for the weekly data is: $45 \times 104 = 4680$ observations, while the sample size for the monthly data is: $24 \times 60 = 1440$ observations.

In addition, these funds are well-diversified and randomly chosen from different fund categories (i.e. large blend, large growth, large value, mid-cap blend, mid-cap growth, mid-cap value, small blend, small growth, and small value). They are also considered to be popular funds due to their high ranking by Morning Star, usually among the four or five star rankings.[16] Therefore, our empirical results can provide investors with typical and beneficial information to select their optimal portfolios.

The fund profiles of the weekly and monthly data are summarized in Table 1 and Table 2, respectively. Also, their simple statistic descriptions are listed in Table 3 for weekly data and Table 4 for monthly data.

(Table 1, Table 2, Table 3, and Table 4)

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[15] Our source of data is from Yahoo, Finance.

[16] The exceptions are: 2 funds from our weekly data are ranked 3 stars.

B. Results

1. The Mean-Variance (EV) Frontier

Using the Markowitz two-moment EV theory, we compute the EV frontier for both sets of data during the sample period. The plots of the EV frontier with scattered funds are shown in the Figure 1 for weekly data, and Figure 2 for monthly data.

(Figure 1 and Figure 2)

2. LP Tests for SSD Efficiency of the EV Efficient Portfolios

In this test, we select 17 portfolios on different parts of the EV frontier for each set of data, in order to test whether these EV efficient portfolios are SSD-efficient by using the LP tests, both the primal and the dual tests, developed by Post (2003). The portfolios chosen for the SSD efficiency test on the EV frontier are plotted in Figure 3 and Figure 4 for our weekly and monthly data, respectively. The results from the LP tests for each data set are reported in Table 5 and Table 6.

(Figure 3, Table 5, Figure 4, and Table 6)

According to Theorem 5 the tested portfolio is SSD-efficient if and only if $\xi(\tau)=0$ or $\psi(\tau)=0$; otherwise, the portfolio is SSD-inefficient. The results show that among the 17 EV efficient portfolios chosen for each set of data, there are 5 EV-efficient portfolios in the weekly data and 6 EV-efficient portfolios in the monthly data that are SSD-inefficient. . In other words, those 11 portfolios on the two EV frontiers are SSD-inefficient or dominated by other portfolios. We notice that all those SSD-inefficient portfolios are along the lower portion of the EV frontier where the portfolios have lower-mean and lower-variance. This finding is consistent with Porter (1973), Porter and Bey

(1974), Porter and Carey (1974), Porter and Gaumnitz (1972), and Perrakis and Zerbinis (1978).

3. Quadratic Utility Efficiency

The Post (2003) tests evaluate SSD efficiency of a target portfolio by using the historical sample distribution as the true distribution of the data. They also assume that no short sales are allowed. For these reasons it is possible to convert our data into returns lying within the interval [0,1]. In our next tests, we need first to convert all our returns to be normalized within the entire [0,1] interval. Such normalization process involves dividing each return by the largest possible return for all assets in the choice. In our case, if b denotes the highest observed return in each set of data, then we have $b(weekly) = 1.089674$ and $b(monthly) = 1.248736$. Hence, we get all the random returns $\mu \in [0,1]$, and corresponding variances σ^2 or $f(\mu)$. The results for the normalized data sets compared with the non-normalized data are shown in Table 7 and Table 8. Please note that all the results in this and the following parts of our tests are reported in the new normalized returns for both weekly and monthly data.

(Table 7 and Table 8)

Then, we plot the normalized means and variances by the function of $\sigma^2 = f(\mu)$ which is continuous, differentiable, and strictly convex everywhere within its domain of definition in Figure 5 for weekly data and Figure 6 for monthly data.

(Figure 5 and Figure 6)

Next, we conduct the quadratic utility efficiency tests for the chosen EV-efficient portfolios. By the Perrakis-Zerbinis (1978) results, if $f' \leq 2(1-\mu)$ for a portfolio on the

EV frontier then that portfolio is SSD-dominated. On the other hand, if $f' > 2(1-\mu)$[17], the portfolio on the EV frontier is SSD-undominated. In each case, f' is the slope of the line tangent to the EV frontier passing at the tested portfolio. The results of our test are shown in Table 9 and Table 10.

(Table 9 and Table 10)

From the two tables we can see that only the top portion of the EV frontier is quadratic utility-efficient. In particular, 6 of 17 EV efficient portfolios are quadratic utility-efficient for weekly data, while only 3 of 17 EV efficient portfolios are quadratic utility-efficient for monthly data. We know that those quadratic utility-efficient portfolios are also SSD-undominated and that such portfolios lie on the higher-mean and higher-variance portion of the EV frontier. This is the result of Perrakis and Zerbinis (1978). The LP tests and the quadratic utility efficiency of the EV frontier are shown in Tables 11 and 12.

(Table 11 and Table 12)

Since the LP tests produce more SSD-efficient portfolios than the quadratic utility test, we seek SSD-undominated portfolios by examining the third central moment.

4. SSD Efficiency Using the Third Central Moment

We provide the first four central moments which are the mean (normalized), variance, skewness, and kurtosis[18] of our sample EV efficient portfolios in each set of data. The moments are evaluated according to (43). The results are shown in Table 13 for weekly data and Table 14 for monthly data.

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[17] See inequality (28).

[18] In this test we only examine the first three central moments. We present the kurtosis here for use in our following tests of the fourth degree polynomial utility.

(Table 13 and Table 14)

According to the expressions in (42a) and (42b), if either B_1 or B_2 is satisfied along the entire portion of the EV frontier up to a given value of μ, then we can conclude that portion of the EV frontiers is SSD-undominated. The results are shown in Table 15 and Table 16.

(Table 15 and Table 16)

From these two tables we can see that the third moment test produces more SSD-undominated portfolios along the EV frontier following the quadratic utility-efficient portfolios, which means that the SSD-efficient portion along the EV frontier becomes larger. This result indicates that although these portfolios are not SSD-efficient on the basis of the quadratic utility function, they are efficient within the class of third degree polynomial utilities which are also increasing and concave as required by the SSD assumptions; consequently, these portfolios are also SSD-efficient and on the EV frontier.

In the tables below we provide a comparison of the results from our three tests: the LP tests, the quadratic utility test, and the third moment test.

(Table 17 and Table 18)

5. New Maximization Programs on the Third and Fourth Central Moments

Here we derive SSD-undominated portfolios within the third degree polynomial utility class by applying the formulas provided in (41a), (41b), (43), and (44). As defined, the optimal portfolios created in this program are SSD-undominated, and may or may not be on the EV frontier. Since we have already proved that some portfolios on the upper part of the EV frontier are quadratic utility-efficient, here we exclude these portfolios which have already satisfied the quadratic utility efficiency criterion , and focus our attention on the portfolios will be potentially satisfied with the higher degree polynomial utility functions. The results of the maximization for SSD on the third central moment are provided in Table 19 and 20 for weekly data, and Table 21 and 22 for monthly data.

(Table 19, Table 20, Table 21, and Table 22)

We find that for the weekly data, $A_1(m(\lambda))$ and $A_2(m(\lambda))$ retain the same values given different values of μ in our program. As a result, there is only one new optimal portfolios found in the weekly data, and it is portfolio (0.92459899, 0.00024802). In terms of the monthly data, the program provides better results and produces more new optimal portfolios.

Next, we apply a similar maximization program for SSD on the fourth central moment. The formulas we use were introduced in (43), (45), and (46). The results are shown in Table 23 for the weekly data and Table 24 for the monthly data.

(Table 23 and Table 24)

Similarly, we find that for the weekly data, $D(m(\lambda))$ retains the same value for different μ. The only optimal portfolio found is the same as the one found on the third central moment. It is portfolio (0.92459899, 0.00024802). As for the monthly data, roughly 4 new optimal portfolios are found with this program.

To provide a better picture regarding the relative positions of the newly found optimal portfolios compared with the portfolios on the EV frontier, we plot those new SSD-efficient portfolios on the third and fourth central moments with the EV frontier. The plots are provided in Figure 7 and Figure 8 for our weekly and monthly data, respectively.

(Figure 7 and Figure 8)

With the help of plots, we can see that although the polynomial utilities of the new optimal portfolios are maximized, these SSD-undominated portfolios still seem to be on the EV frontier. Such findings are consistent with the results of Kroll, Levy, and Markowitz (1984).

For a further step, we also program the maximization for TSD on the fourth central moment provided by the formulas in (43), (47a), (47b), and (48). Similarly, the new optimal portfolios are TSD-undominated and may or may not be on the EV frontier. The results for the weekly data are shown in Table 25 and 26, and for the monthly in Table 27 and 28.

(Table 25, Table 26, Table 27, and Table 28)

From the tables we can see that for the weekly data, the maximization also generates only one new TSD-undominated portfolio which is the same portfolio generated for SSD on the third and fourth moments. However, the program generates more new optimal portfolios for our monthly data.

We also plot the new TSD-undominated portfolios with EV frontier in Figure 9 and Figure 10 for the weekly and monthly data, respectively.

(Figure 9 and Figure 10)

All these newly TSD-efficient portfolios appear to be on the EV frontier, which is not surprising given that they are also SSD-efficient portfolios. It is likely that even though we have maximized the utilities of those portfolios without constraining them to lie on the EV-efficient set, the actually maximized utilities are very close to the ones generated with the EV-efficient set, which is only restricted to the first two moments, the mean and the variance.

We have also noticed that for the monthly data, the TSD-undominated portfolios do move downwards on the EV frontier. Therefore, the SSD-efficient portion of the EV frontier has been further extended.

IX. Conclusions and Future Research

In this paper, we have examined the SSD efficiency of the EV frontier by both theoretical and empirical arguments. The theory asserts that if the EV efficient portfolios are quadratic utility-efficient, they are also SSD-efficient. Those SSD- (and quadratic utility-) efficient portfolios have higher-means and higher-variances on the upper portion of the EV frontier, which may indicate that investors with low risk aversion make similar decisions in selecting their portfolios with the EV or the SD approach.

Then, we introduced the concept of polynomial utility functions on the central moments. We used some of Perrakis' unpublished results to develop the theoretical inequalities on the central moments of return distributions of a pair of portfolios. These are the necessary conditions for one portfolio to dominate the other in the second- or third-degree. Based on these inequalities, we developed the conditions for SSD efficiency on the third central moment, which can extend the SSD-efficient portion of the EV frontier following the portion which is quadratic utility-efficient. We do a similar extension with the fourth central moment without constraining the portfolios to lie on the EV efficient set. Therefore, those new generated portfolios may or may not be on the EV frontier.

Empirically, we used Post's LP tests to examine the SSD efficiency for the portfolios on the EV frontier. Our tests include the weekly returns of U.S. equity funds from 2003 to 2004, and the monthly returns from 2001 to 2005. The LP tests show that the higher portion of the EV frontier is SSD-efficient while the lower portion is not. Then, our quadratic utility test confirms a very top portion of the EV frontier is SSD- (and quadratic utility-) efficient. The test for SSD on the third central moment has found more undominated-portfolios on the EV frontier following the quadratic utility-efficient portfolios, but there is still a portion of the EV frontier which has not been proven to be SSD-efficient by the moment tests, even though they are SSD-efficient in the LP tests. Finally, we conducted the maximization tests for SSD on the third and the fourth moments and for TSD on the fourth moment. We found that the new optimal SSD- and TSD-undominated portfolios appear to be very close to or on the EV frontier, but the SSD-efficient portion of the EV frontier has been further extended. Presumably, higher moment functions may be able to account for the remaining undominated portion of the EV frontier.

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Table 1: Fund profiles for the 45 U.S. equity funds in our weekly data

Fund Name	Symbol	Fund Family	Category	Morning Star Rating
Analytic Defensive Equity Instl	ANDEX	Analytical Funds	Large Blend	5 Stars
Cambiar Opportunity Inst	CAMOX	Cambiar Funds	Large Blend	5 Stars
Everest America	EVAMX	Everest	Large Blend	5 Stars
Exeter Equity	EXEYX	Exeter Funds	Large Blend	4 Stars
Exeter Pro-Blend Maximum Term A	EXHAX	Exeter Funds	Large Blend	5 Stars
Amana Mutual Funds Trust Growth	AMAGX	Amana	Large Growth	5 Stars
Brandywine Blue	BLUEX	Brandywine	Large Growth	5 Stars
FMI Provident Trust Strategy	FMIRX	FMI Funds	Large Growth	5 Stars
Fidelity Contrafund	FCNTX	Fidelity Group	Large Growth	5 Stars
Gartmore U.S. Growth Leaders C	GXXCX	Gartmore	Large Growth	5 Stars
American Beacon Lg Cap Value AMR	AAGAX	American Beacon	Large Value	5 Stars
Diamond Hill Large Cap A	DHLAX	Diamond Hill Funds	Large Value	5 Stars
DFA U.S. Large Cap Value II	DFCVX	Dimensional Investment Group	Large Value	5 Stars
Dodge & Cox Stock	DODGX	Dodge & Cox	Large Value	5 Stars
AXA Enterprise Deep Value Y	EDVYX	Enterprise	Large Value	3 Stars
CGM Focus	CGMFX	CGM	Mid-Cap Blend	5 Stars
Fidelity Advisor Leveraged Co Stk A	FLSAX	Fidelity Advisor Funds	Mid-Cap Blend	5 Stars
Kinetics Paradigm	WWNPX	Kinetics	Mid-Cap Blend	5 Stars
Kinetics Paradigm Adv A	KNPAX	Kinetics	Mid-Cap Blend	5 Stars
Fidelity Advisor Leveraged Co Stk T	FLSTX	Fidelity Advisor Funds	Mid-Cap Blend	5 Stars
Bridgeway Aggressive Investors 1	BRAGX	Bridgeway	Mid-Cap Growth	4 Stars
First American Mid Cap Growth Opp B	FMQBX	First American	Mid-Cap Growth	5 Stars
Leuthold Select Industries	LSLTX	Leuthold	Mid-Cap Growth	4 Stars
Rainier Small/Mid Cap Equity	RIMSX	Rainer	Mid-Cap Growth	5 Stars
AFBA Five Star Mid Cap A	AFMAX	AFBA Five Star Fund	Mid-Cap Growth	3 Stars
Artisan Mid Cap Value	ARTQX	Artisan	Mid-Cap Value	5 Stars
Fidelity Select Construction&Housing	FSHOX	Fidelity Group	Mid-Cap Value	4 Stars
Goldman Sachs Mid Cap Value Instl	GSMCX	Goldenman Sachs	Mid-Cap Value	5 Stars
Goldman Sachs Mid Cap Value Service	GSMSX	Goldenman Sachs	Mid-Cap Value	5 Stars
AXP Mid Cap Value A	AMVAX	RiverSource	Mid-Cap Value	5 Stars
Gartmore Small Cap C	GSXCX	Gartmore	Small Blend	5 Stars
Harbor Small Cap Value Instl	HASCX	Harbor	Small Blend	5 Stars
Keeley Small Cap Value	KSCVX	Keeley	Small Blend	5 Stars
Oppenheimer Small & Mid Cap Value N	QSCNX	Oppenheimer Funds	Small Blend	5 Stars
RS Partners	RSPFX	RS Funds	Small Blend	5 Stars
Bridgeway Micro-Cap Limited	BRMCX	Bridgeway	Small Growth	5 Stars
Managers AMG Essex Sm/Mic Cp Gr	MBRSX	Managers Funds	Small Growth	5 Stars
Turner Micro Cap Growth	TMCGX	Turner Investment Partners	Small Growth	5 Stars
Wasatch Micro Cap	WMICX	Wasatch	Small Growth	5 Stars
Bridgeway Ultra-Small Company	BRUSX	Bridgeway	Small Growth	5 Stars
Constellation TIP Small Cap Value Opp	TSVOX	Constellation	Small Value	5 Stars
Hotchkis and Wiley Small Cap Value A	HWSAX	Hotchkis and Wiley	Small Value	5 Stars
Hotchkis and Wiley Small Cap Value C	HWSCX	Hotchkis and Wiley	Small Value	5 Stars
Allianz NFJ Small Cap Value A	PCVAX	Allianz Funds	Small Value	4 Stars
Allianz NFJ Small Cap Value Admin	PVADX	Allianz Funds	Small Value	4 Stars

Table 2: Fund profiles for the 24 U.S. equity funds in our monthly data

Fund Name	Symbol	Fund Family	Category	Morning Star Rating
Analytic Defensive Equity Instl	ANDEX	Analytical Funds	Large Blend	5 Stars
Exeter Pro-Blend Maximum Term A	EXHAX	Exeter Funds	Large Blend	5 Stars
Amana Mutual Funds Trust Growth	AMAGX	Amana	Large Growth	5 Stars
Brandywine Blue	BLUEX	Brandywine	Large Growth	5 Stars
FMI Provident Trust Strategy	FMIRX	FMI Funds	Large Growth	5 Stars
Fidelity Contrafund	FCNTX	Fidelity Group	Large Growth	5 Stars
American Beacon Lg Cap Value AMR	AAGAX	American Beacon	Large Value	5 Stars
DFA U.S. Large Cap Value II	DFCVX	Dimensional Investment Group	Large Value	5 Stars
Dodge & Cox Stock	DODGX	Dodge & Cox	Large Value	5 Stars
CGM Focus	CGMFX	CGM	Mid-Cap Blend	5 Stars
Kinetics Paradigm	WWNPX	Kinetics	Mid-Cap Blend	5 Stars
Bridgeway Aggressive Investors 1	BRAGX	Bridgeway	Mid-Cap Growth	4 Stars
Rainier Small/Mid Cap Equity	RIMSX	Rainer	Mid-Cap Growth	5 Stars
Fidelity Select Construction&Housing	FSHOX	Fidelity Group	Mid-Cap Value	4 Stars
Goldman Sachs Mid Cap Value Instl	GSMCX	Goldenman Sachs	Mid-Cap Value	5 Stars
Goldman Sachs Mid Cap Value Service	GSMSX	Goldenman Sachs	Mid-Cap Value	5 Stars
Keeley Small Cap Value	KSCVX	Keeley	Small Blend	5 Stars
RS Partners	RSPFX	RS Funds	Small Blend	5 Stars
Bridgeway Micro-Cap Limited	BRMCX	Bridgeway	Small Growth	5 Stars
Turner Micro Cap Growth	TMCGX	Turner Investment Partners	Small Growth	5 Stars
Wasatch Micro Cap	WMICX	Wasatch	Small Growth	5 Stars
Bridgeway Ultra-Small Company	BRUSX	Bridgeway	Small Growth	5 Stars
Allianz NFJ Small Cap Value A	PCVAX	Allianz Funds	Small Value	4 Stars
Allianz NFJ Small Cap Value Admin	PVADX	Allianz Funds	Small Value	4 Stars

Table 3: Simple statistic description for the weekly data

Fund	Mean	Std Dev	Minimum	Maximum	Skewness	Kurtosis
ANDEX	1.002680	0.011910	0.963160	1.042010	-0.157316	1.259687
CAMOX	1.004220	0.018840	0.955720	1.079670	0.201190	2.366611
EVAMX	1.003340	0.015440	0.963470	1.055200	-0.003877	0.838520
EXEYX	1.003210	0.018150	0.954110	1.069320	-0.032392	1.343178
EXHAX	1.003610	0.016120	0.958680	1.060690	-0.000519	1.015005
AMAGX	1.004810	0.018680	0.952460	1.048510	-0.203052	0.140037
BLUEX	1.004070	0.018580	0.948740	1.048700	-0.506343	0.820118
FMIRX	1.003780	0.016790	0.973930	1.072120	0.654917	1.488968
FCNTX	1.003660	0.016020	0.959010	1.053050	-0.027231	0.629467
GXXCX	1.005060	0.024600	0.940510	1.081630	0.080304	0.431946
AAGAX	1.004270	0.017920	0.950930	1.071120	0.154478	1.684610
DHLAX	1.004470	0.016680	0.958900	1.068590	0.026529	1.879961
DFCVX	1.004140	0.019580	0.949760	1.079210	0.106386	1.689832
DODGX	1.004250	0.017620	0.955570	1.069480	0.222784	1.449907
EDVYX	1.003280	0.017280	0.954970	1.072860	0.333034	2.170313
CGMFX	1.006160	0.031130	0.923800	1.084840	-0.127966	-0.270200
FLSAX	1.008280	0.026720	0.927660	1.078390	-0.218596	0.488926
WWNPX	1.005610	0.016810	0.966220	1.048670	-0.111659	-0.248492
KNPAX	1.005560	0.016850	0.965090	1.047970	-0.141481	-0.252886
FLSTX	1.008230	0.026730	0.926990	1.079030	-0.217782	0.521336
BRAGX	1.005400	0.031020	0.912140	1.074810	-0.241686	0.351272
FMQBX	1.004500	0.021500	0.939000	1.060170	-0.216117	0.788386
LSLTX	1.005380	0.024970	0.930970	1.060770	-0.275612	0.465621
RIMSX	1.005300	0.023640	0.934830	1.061610	-0.211054	0.353958
AFMAX	1.004940	0.025090	0.941130	1.084110	0.079510	0.641087
ARTQX	1.005080	0.015430	0.965920	1.054450	0.115920	0.406986
FSHOX	1.006140	0.023420	0.952230	1.086670	0.233610	0.964184
GSMCX	1.003890	0.017820	0.930390	1.059440	-0.466498	2.654938
GSMSX	1.003800	0.017830	0.929480	1.059560	-0.488136	2.782229
AMVAX	1.005810	0.022730	0.943790	1.089670	0.176138	1.239766
GSXCX	1.004780	0.026720	0.899880	1.059360	-0.594206	1.659173
HASCX	1.005730	0.020070	0.954960	1.053080	-0.017820	0.246085
KSCVX	1.006010	0.019450	0.944620	1.046670	-0.208171	0.347768
QSCNX	1.006010	0.018940	0.961270	1.057610	0.133173	-0.026064
RSPFX	1.007510	0.017240	0.963590	1.041430	-0.281116	-0.163914
BRMCX	1.006160	0.030890	0.912610	1.073730	-0.265355	0.255688
MBRSX	1.006460	0.029570	0.917860	1.065010	-0.103482	0.141389
TMCGX	1.006050	0.024970	0.936610	1.058870	-0.198363	0.353270
WMICX	1.005260	0.024310	0.940000	1.057550	-0.175573	-0.087828
BRUSX	1.008380	0.027080	0.920000	1.076830	-0.502744	0.775387
TSVOX	1.006360	0.020820	0.944960	1.052560	-0.125707	0.157873
HWSAX	1.007090	0.021150	0.955700	1.069470	-0.039428	0.347052
HWSCX	1.006950	0.021140	0.955670	1.069300	-0.030277	0.329713
PCVAX	1.004540	0.017480	0.962990	1.045750	0.086396	-0.116052
PVADX	1.004480	0.017440	0.963400	1.045430	0.105154	-0.104891

Table 4: Simple statistic description for the monthly data

Fund	Mean	Std Dev	Minimum	Maximum	Skewness	Kurtosis
ANDEX	1.004360	0.027110	0.930300	1.058760	-0.572632	0.273508
EXHAX	1.002540	0.042700	0.892830	1.091740	-0.352407	0.404024
AMAGX	1.005230	0.051290	0.886380	1.110600	-0.549696	0.149544
BLUEX	1.002630	0.038640	0.914530	1.073440	-0.130299	-0.765256
FMIRX	1.003160	0.039680	0.859810	1.134770	-0.137022	3.429875
FCNTX	1.006410	0.030200	0.931760	1.064870	-0.447162	-0.336950
AAGAX	1.005840	0.040910	0.883060	1.086920	-0.562412	0.900881
DFCVX	1.006260	0.044270	0.888390	1.093750	-0.636883	0.911205
DODGX	1.009090	0.038460	0.902990	1.081620	-0.565732	0.920444
CGMFX	1.019760	0.088340	0.760180	1.248740	-0.494675	1.185555
WWNPX	1.012280	0.032630	0.940730	1.103870	0.192841	0.656755
BRAGX	1.006190	0.060320	0.862400	1.134000	-0.211542	-0.305014
RIMSX	1.008320	0.052410	0.891280	1.109650	-0.288130	-0.607590
FSHOX	1.014970	0.056360	0.867950	1.130540	-0.543479	0.450478
GSMCX	1.009430	0.035390	0.919040	1.074280	-0.603853	0.106137
GSMSX	1.009050	0.035320	0.918580	1.073840	-0.600438	0.096605
KSCVX	1.014540	0.045250	0.878090	1.116420	-0.639579	0.983352
RSPFX	1.016040	0.049330	0.856290	1.110750	-0.806119	1.413181
BRMCX	1.014670	0.063530	0.832950	1.116190	-0.625626	0.373735
TMCGX	1.012500	0.053330	0.833940	1.096640	-0.704132	1.008122
WMICX	1.013560	0.063310	0.830290	1.130100	-0.542006	0.134257
BRUSX	1.020040	0.061930	0.843900	1.177720	-0.289101	0.385919
PCVAX	1.012030	0.039350	0.900870	1.084640	-0.856156	1.282240
PVADX	1.011950	0.039300	0.899530	1.084980	-0.838780	1.295119

Figure 1: EV frontier for the weekly data with 45 scattered funds

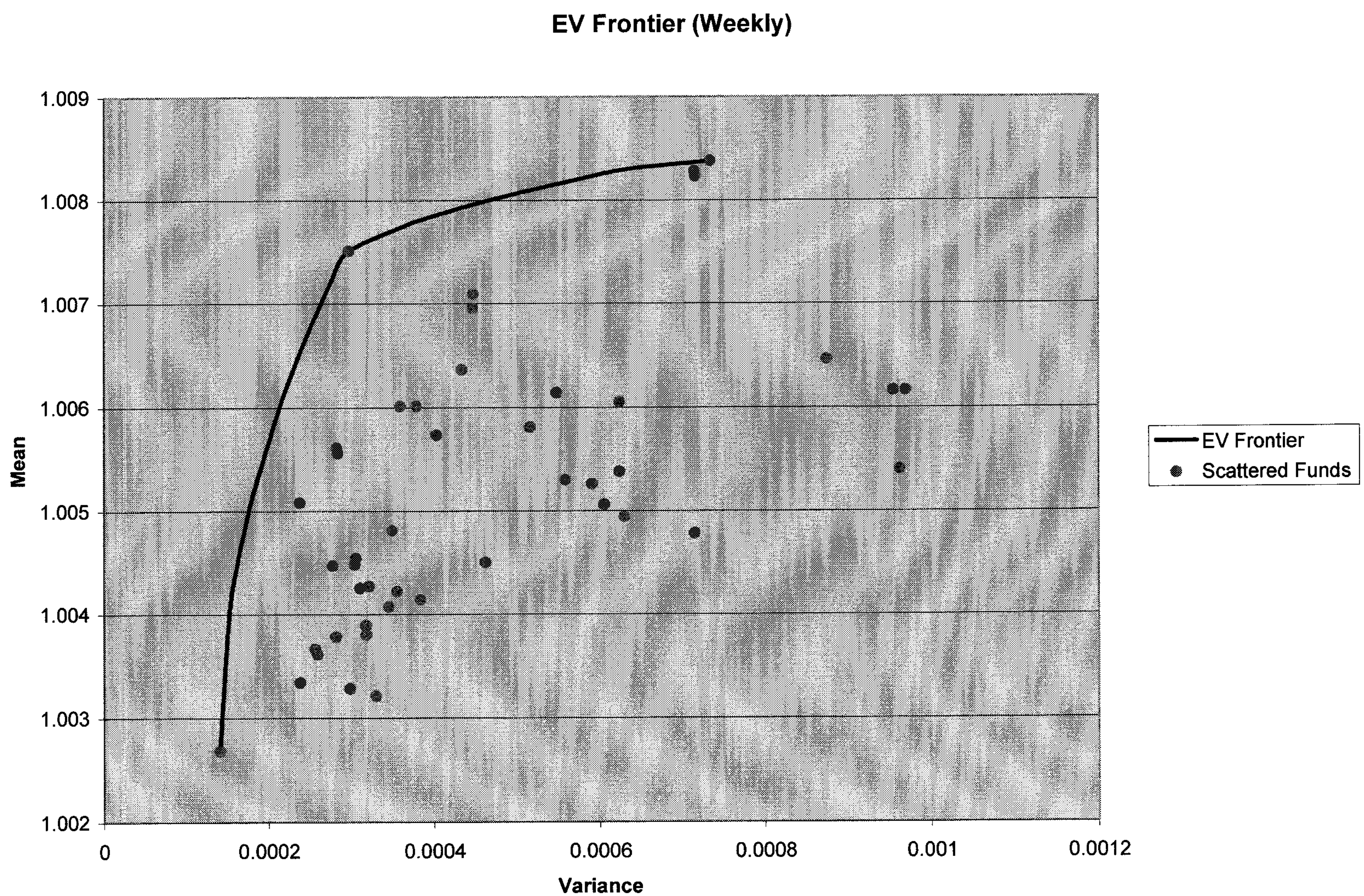

Figure 2: EV frontier for the monthly data with 24 scattered funds

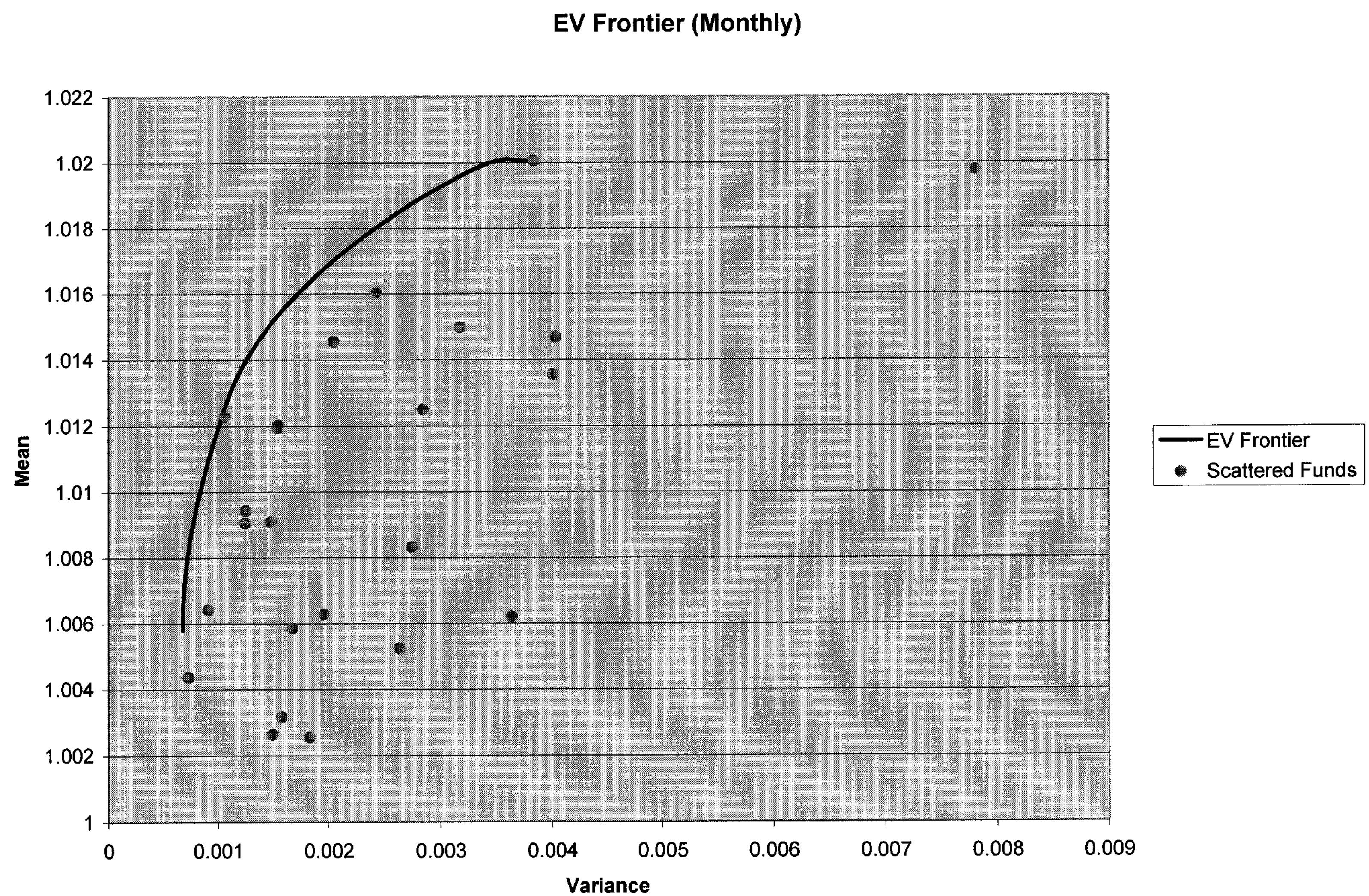

Figure 3: EV frontier for the weekly data with of the selected EV efficient portfolios for SSD efficiency test

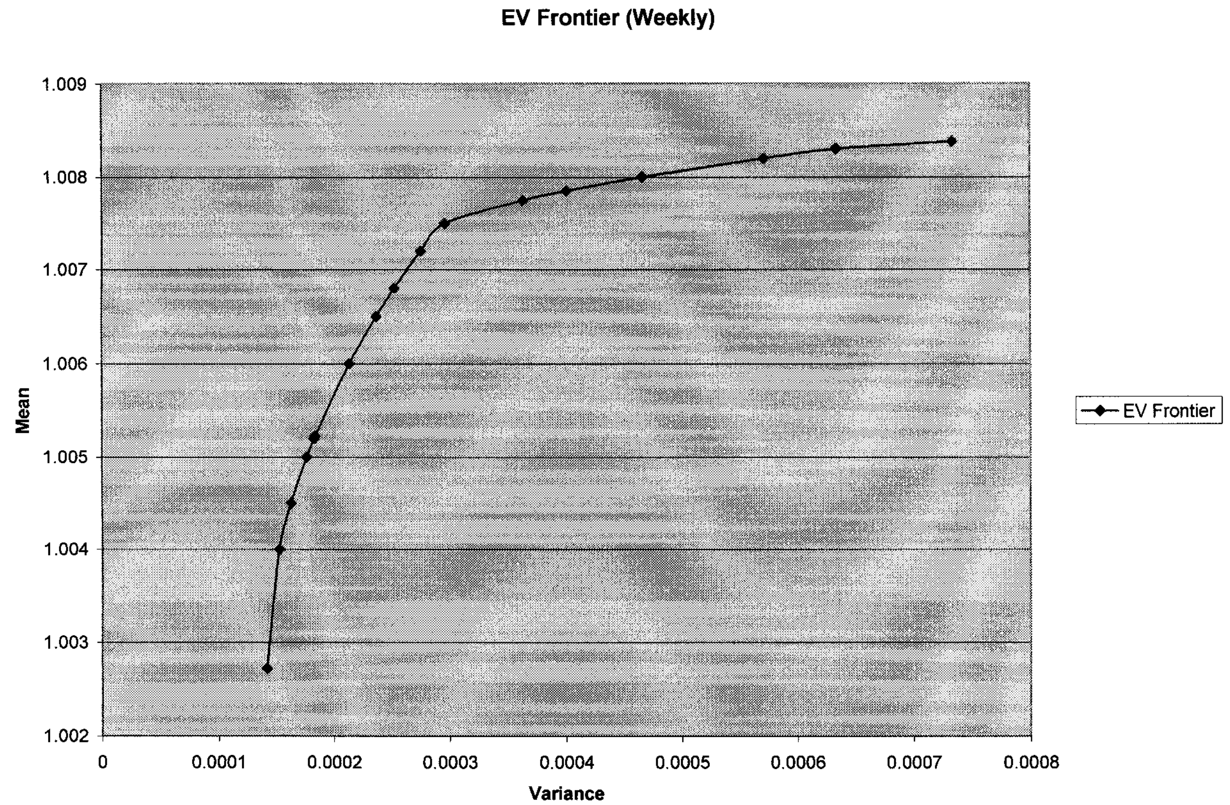

Table 5: Results from the LP tests for the weekly data

μ	σ^2	LP Tests: $\xi(\tau)$, $\psi(\tau)$	SSD Efficiency
1.008380	0.000734	0	Yes
1.008301	0.000634	0	Yes
1.008199	0.000572	0	Yes
1.008001	0.000466	0	Yes
1.007851	0.000401	0	Yes
1.007750	0.000364	0	Yes
1.007500	0.000297	0	Yes
1.007200	0.000276	0	Yes
1.006800	0.000253	0	Yes
1.006500	0.000237	0	Yes
1.006000	0.000214	0	Yes
1.005229	0.000184	0	Yes
1.005199	0.000183	0.000682	No
1.005000	0.000176	0.009641	No
1.004500	0.000163	0.032138	No
1.004000	0.000153	0.054642	No
1.002723*	0.000142*	0.111859	No

* The last portfolio is the minimum-variance portfolio (MVP) on the EV frontier

Figure 4: EV frontier for the monthly data with the selected EV efficient portfolios for SSD efficiency test

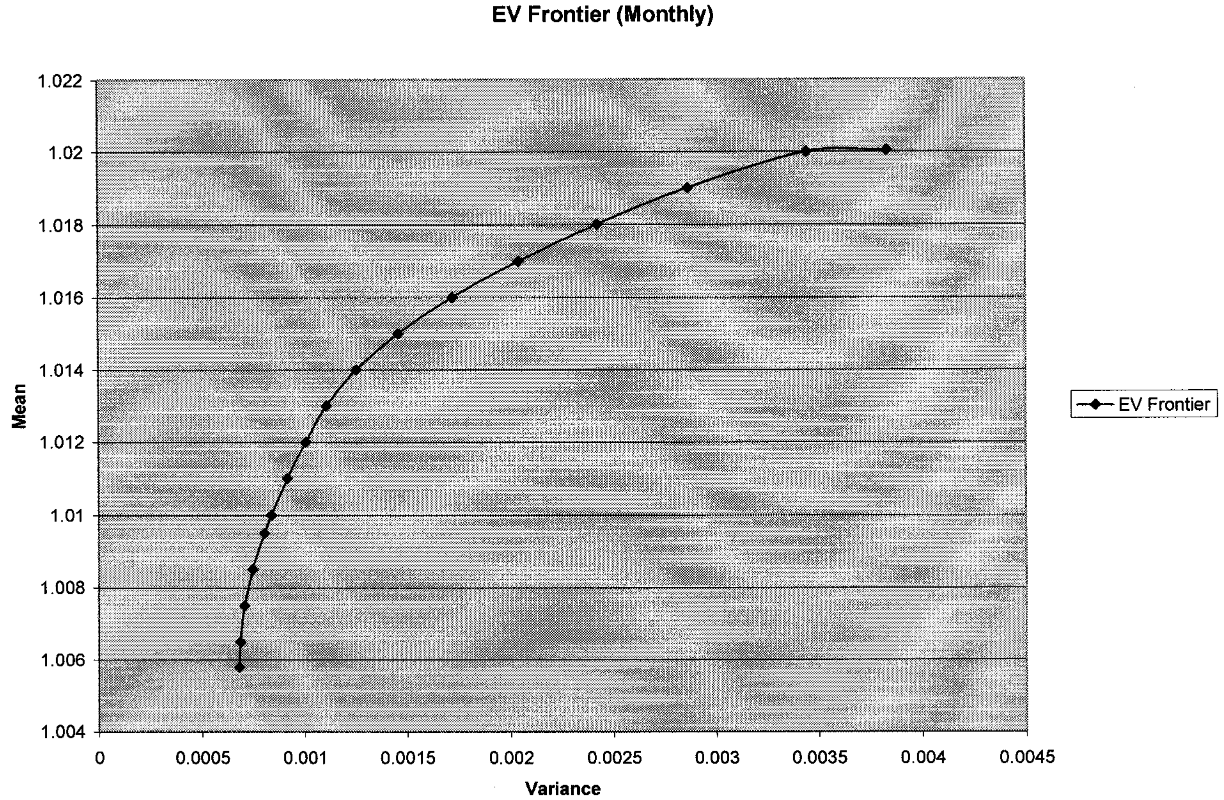

Table 6: Results from the LP tests for the monthly data

μ	σ^2	LP Tests: $\xi(\tau)$, $\psi(\tau)$	SSD Efficiency
1.020040	0.003835	0	Yes
1.020001	0.003447	0	Yes
1.019001	0.002868	0	Yes
1.018001	0.002426	0	Yes
1.017001	0.002044	0	Yes
1.016001	0.001722	0	Yes
1.015001	0.001461	0	Yes
1.014001	0.001257	0	Yes
1.013001	0.001111	0	Yes
1.012000	0.001009	0	Yes
1.011000	0.000919	0	Yes
1.010000	0.000840	0.000885	No
1.009500	0.000806	0.012881	No
1.008500	0.000748	0.036870	No
1.007500	0.000708	0.060855	No
1.006500	0.000686	0.084856	No
1.005796*	0.000681*	0.101719	No

* The last portfolio is the minimum-variance portfolio (MVP) on the EV frontier

Table 7: Comparison of the non-normalized weekly data with the normalized weekly data: b=1.089674

Before Normalization		After Normalization	
μ	σ^2	μ	σ^2
1.008380	0.000734	0.925396	0.000612
1.008301	0.000634	0.925324	0.000529
1.008199	0.000572	0.925230	0.000478
1.008001	0.000466	0.925048	0.000388
1.007851	0.000401	0.924911	0.000334
1.007750	0.000364	0.924818	0.000303
1.007500	0.000297	0.924589	0.000247
1.007200	0.000276	0.924313	0.000230
1.006800	0.000253	0.923946	0.000211
1.006500	0.000237	0.923671	0.000198
1.006000	0.000214	0.923212	0.000178
1.005229	0.000184	0.922504	0.000153
1.005199	0.000183	0.922477	0.000152
1.005000	0.000176	0.922294	0.000147
1.004500	0.000163	0.921835	0.000136
1.004000	0.000153	0.921377	0.000127
1.002723	0.000142	0.920205	0.000118

Table 8: Comparison of the non-normalized monthly data with the normalized monthly data: b=1.248736

Before Normalization		After Normalization	
μ	σ^2	μ	σ^2
1.020040	0.003835	0.816858	0.002418
1.020001	0.003447	0.816827	0.002174
1.019001	0.002868	0.816026	0.001809
1.018001	0.002426	0.815225	0.001530
1.017001	0.002044	0.814424	0.001289
1.016001	0.001722	0.813623	0.001086
1.015001	0.001461	0.812823	0.000921
1.014001	0.001257	0.812022	0.000793
1.013001	0.001111	0.811221	0.000701
1.012000	0.001009	0.810419	0.000636
1.011000	0.000919	0.809619	0.000579
1.010000	0.000840	0.808818	0.000530
1.009500	0.000806	0.808417	0.000508
1.008500	0.000748	0.807617	0.000472
1.007500	0.000708	0.806816	0.000446
1.006500	0.000686	0.806015	0.000432
1.005796	0.000681	0.805451	0.000430

Figure 5: Convex EV frontier for the weekly data with the function: $\sigma^2 = f(\mu)$

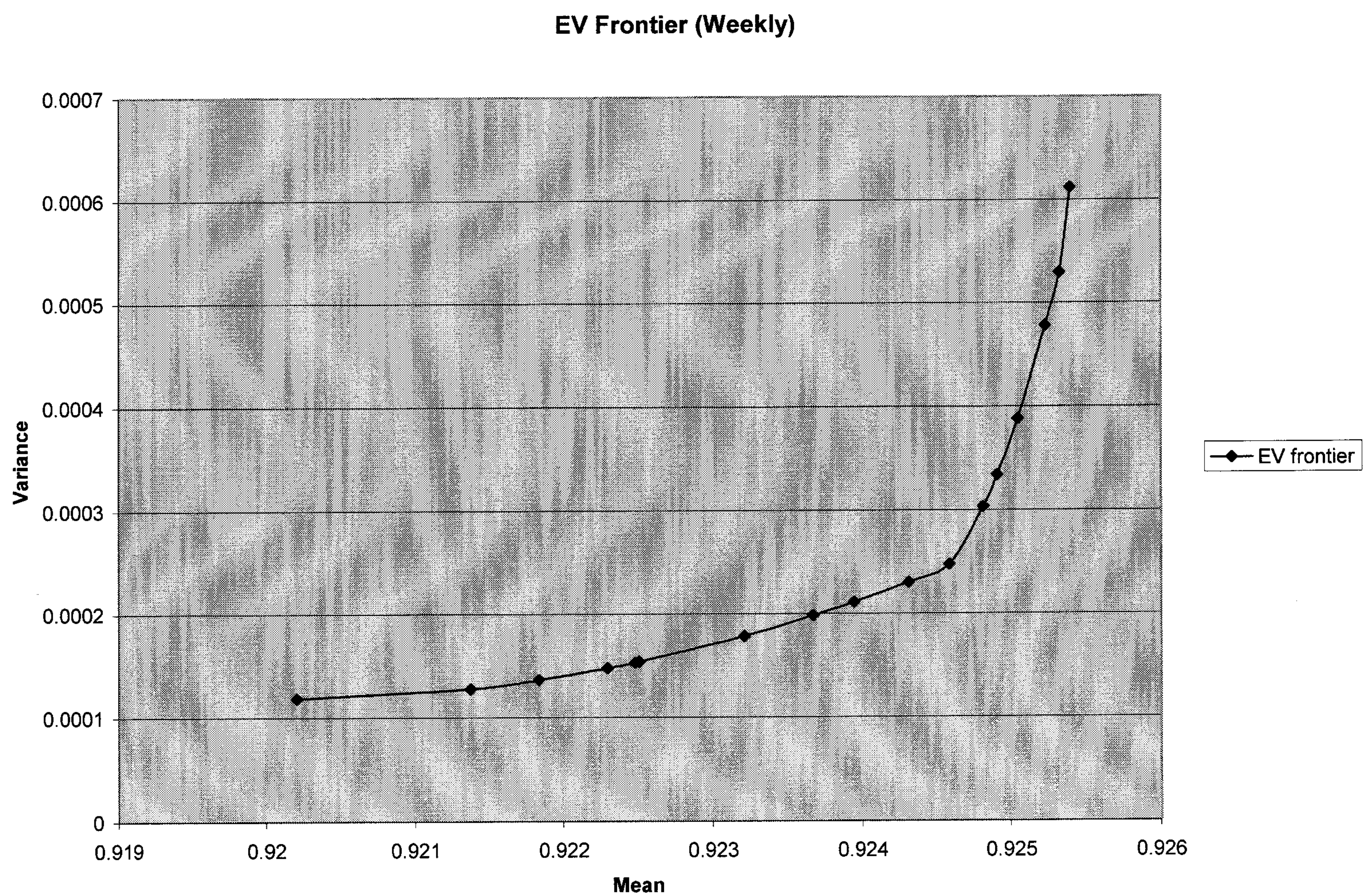

Figure 6: Convex EV frontier for the monthly data with the function: $\sigma^2 = f(\mu)$

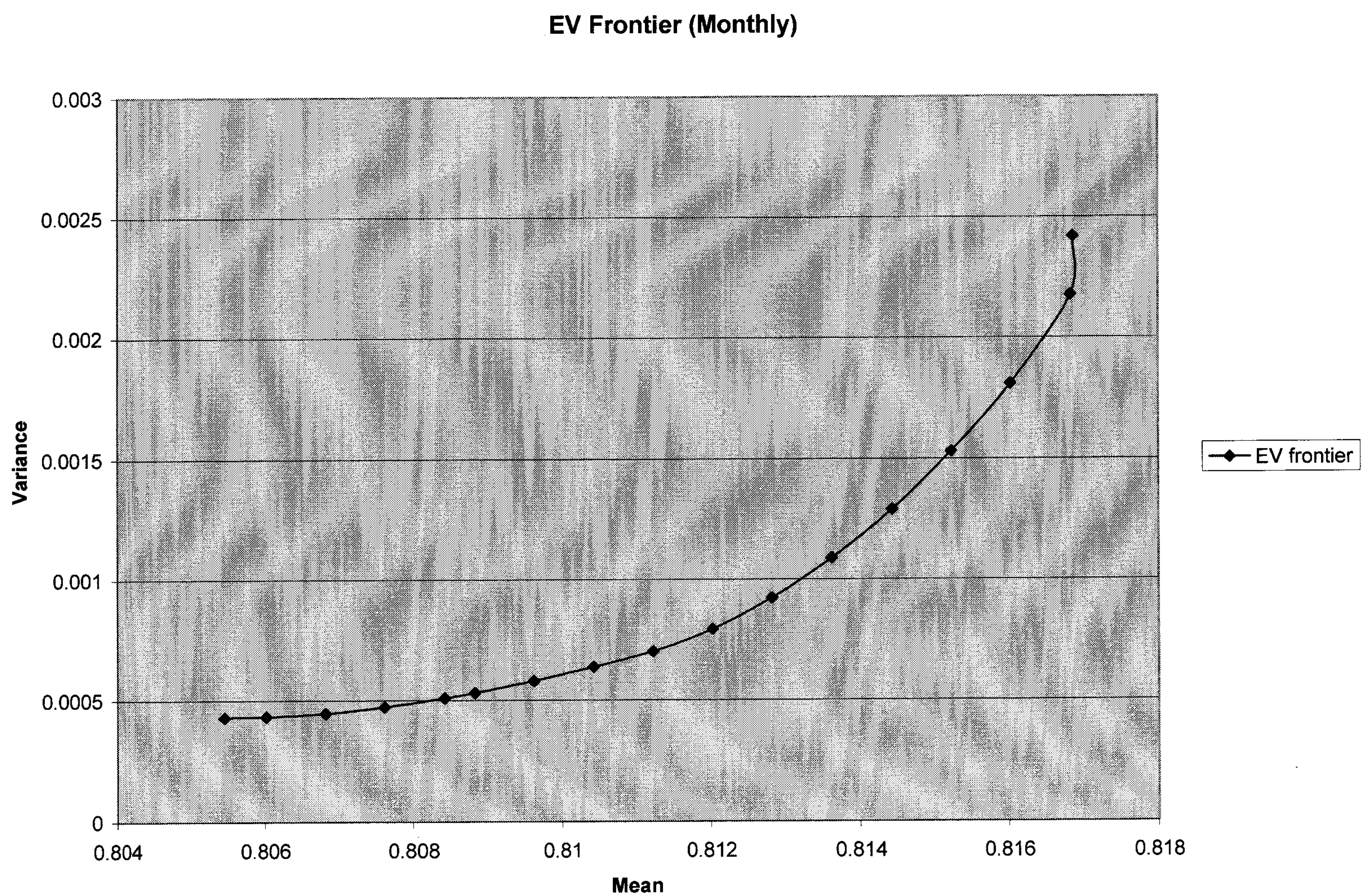

Table 9: Results of the quadratic utility efficiency for the EV efficient portfolios (weekly data)

μ	f'	$2(1-\mu)$	$f'-2(1-\mu)$	**Quadratic Utility Efficiency**
0.925396	1.771172	0.149208	1.621964	Yes
0.925324	0.651571	0.149353	0.502218	Yes
0.925230	0.559800	0.149540	0.410261	Yes
0.925048	0.477207	0.149903	0.327304	Yes
0.924911	0.385436	0.150179	0.235258	Yes
0.924818	0.321197	0.150364	0.170833	Yes
0.924589	0.110125	0.150823	-0.040698	No
0.924313	0.056898	0.151374	-0.094476	No
0.923946	0.050474	0.152108	-0.101634	No
0.923671	0.045885	0.152658	-0.106773	No
0.923212	0.041297	0.153576	-0.112279	No
0.922504	0.032120	0.154991	-0.122871	No
0.922477	0.028449	0.155046	-0.126597	No
0.922294	0.027531	0.155411	-0.127880	No
0.921835	0.022025	0.156329	-0.134304	No
0.921377	0.016519	0.157247	-0.140728	No
0.920205	0.000143	0.159590	-0.159448	No

Table 10: Results of the quadratic utility efficiency for the EV efficient portfolios (monthly data)

μ	f'	$2(1-\mu)$	$f'-2(1-\mu)$	**Quadratic Utility Efficiency**
0.816858	76.773628	0.366284	76.407343	Yes
0.816827	1.585603	0.366347	1.219257	Yes
0.816026	0.384389	0.367948	0.016440	Yes
0.815225	0.336340	0.369550	-0.033210	No
0.814424	0.288291	0.371151	-0.082860	No
0.813623	0.248251	0.372753	-0.124502	No
0.812823	0.192194	0.374355	-0.182160	No
0.812022	0.144146	0.375956	-0.231811	No
0.811221	0.096097	0.377558	-0.281461	No
0.810419	0.080081	0.379161	-0.299080	No
0.809619	0.072073	0.380763	-0.308690	No
0.808818	0.056857	0.382364	-0.325507	No
0.808417	0.053654	0.383165	-0.329511	No
0.807617	0.042443	0.384767	-0.342324	No
0.806816	0.025626	0.386368	-0.360743	No
0.806015	0.008008	0.387970	-0.379962	No
0.805451	0.000596	0.389097	-0.388502	No

Table 11: Comparison of the SSD efficiency from LP tests and quadratic utility test (weekly data)

μ	σ^2	SSD Efficiency (LP tests)	SSD Efficiency (Quadratic utility)
0.925396	0.000612	Yes	Yes
0.925324	0.000529	Yes	Yes
0.925230	0.000478	Yes	Yes
0.925048	0.000388	Yes	Yes
0.924911	0.000334	Yes	Yes
0.924818	0.000303	Yes	Yes
0.924589	0.000247	Yes	No
0.924313	0.000230	Yes	No
0.923946	0.000211	Yes	No
0.923671	0.000198	Yes	No
0.923212	0.000178	Yes	No
0.922504	0.000153	Yes	No
0.922477	0.000152	No	No
0.922294	0.000147	No	No
0.921835	0.000136	No	No
0.921377	0.000127	No	No
0.920205	0.000118	No	No

Table 12: Comparison of the SSD efficiency from LP tests and quadratic utility test (monthly data)

μ	σ^2	SSD Efficiency (LP tests)	SSD Efficiency (Quadratic utility)
0.816858	0.002418	Yes	Yes
0.816827	0.002174	Yes	Yes
0.816026	0.001809	Yes	Yes
0.815225	0.001530	Yes	No
0.814424	0.001289	Yes	No
0.813623	0.001086	Yes	No
0.812823	0.000921	Yes	No
0.812022	0.000793	Yes	No
0.811221	0.000701	Yes	No
0.810419	0.000636	Yes	No
0.809619	0.000579	Yes	No
0.808818	0.000530	No	No
0.808417	0.000508	No	No
0.807617	0.000472	No	No
0.806816	0.000446	No	No
0.806015	0.000432	No	No
0.805451	0.000430	No	No

Table 13: The first four central moments of the tested EV efficient portfolios for the weekly data

μ	σ^2	$S(\mu)$	$K(\mu)$
0.925396	0.000612	-7.50E-06	1.38E-06
0.925324	0.000529	-6.47E-06	9.25E-07
0.925230	0.000478	-5.68E-06	7.49E-07
0.925048	0.000388	-4.16E-06	4.81E-07
0.924911	0.000334	-3.12E-06	3.45E-07
0.924818	0.000303	-2.46E-06	2.77E-07
0.924589	0.000247	-1.08E-06	1.70E-07
0.924313	0.000230	-1.05E-06	1.43E-07
0.923946	0.000211	-9.74E-07	1.20E-07
0.923671	0.000198	-9.20E-07	1.06E-07
0.923212	0.000178	-8.29E-07	8.63E-08
0.922504	0.000153	-6.90E-07	6.58E-08
0.922477	0.000152	-6.84E-07	6.53E-08
0.922294	0.000147	-6.48E-07	6.17E-08
0.921835	0.000136	-5.55E-07	5.53E-08
0.921377	0.000127	-4.61E-07	5.21E-08
0.920205	0.000118	-2.12E-07	5.75E-08

Table 14: The first four central moments of the tested EV efficient portfolios for the monthly data

μ	σ^2	$S(\mu)$	$K(\mu)$
0.816858	0.002418	-3.35E-05	1.90E-05
0.816827	0.002174	-5.66E-05	1.45E-05
0.816026	0.001809	-4.58E-05	9.97E-06
0.815225	0.001530	-3.04E-05	6.95E-06
0.814424	0.001289	-1.87E-05	4.88E-06
0.813623	0.001086	-1.00E-05	3.50E-06
0.812823	0.000921	-4.33E-06	2.61E-06
0.812022	0.000793	-8.96E-07	2.04E-06
0.811221	0.000701	1.16E-06	1.69E-06
0.810419	0.000636	5.00E-07	1.40E-06
0.809619	0.000579	-1.26E-07	1.14E-06
0.808818	0.000530	-7.01E-07	9.23E-07
0.808417	0.000508	-9.90E-07	8.33E-07
0.807617	0.000472	-1.76E-06	6.89E-07
0.806816	0.000446	-2.64E-06	5.94E-07
0.806015	0.000432	-3.31E-06	5.42E-07
0.805451	0.000430	-3.65E-06	5.30E-07

Table 15: Test results for the SSD efficiency on the third moment for the weekly data. Note: Concerning the format, we here denote $B_1 = 3[f'(1-\mu) - f - \mu^2 + 2\mu - 1]$ and $B_2 = 3[1 - f - \mu f' - \mu^2]$. Hence, either $dS/d\mu - B_1 < 0$ or $dS/d\mu - B_2 > 0$, the SSD efficiency can be established on the third central moment.

μ	$\sigma^2(\mu)$	$S(\mu)$	f'	$\frac{dS}{d\mu}$	B_1	$\frac{dS}{d\mu} - B_1$	B_2	$\frac{dS}{d\mu} - B_2$	SSD Efficiency (on the third moment)
0.925396	0.000612	-7.50E-06	1.771172		0.377877	-0.377877	-4.488016	4.488016	Yes
0.925324	0.000529	-6.47E-06	0.651571	-0.014158	0.127654	-0.141812	-1.379001	1.364842	Yes
0.925230	0.000478	-5.68E-06	0.559800	-0.008447	0.107364	-0.115810	-1.123418	1.114971	Yes
0.925048	0.000388	-4.16E-06	0.477207	-0.008369	0.089284	-0.097654	-0.892626	0.884257	Yes
0.924911	0.000334	-3.12E-06	0.385436	-0.007556	0.068909	-0.076465	-0.636864	0.629308	Yes
0.924818	0.000303	-2.46E-06	0.321197	-0.007101	0.054579	-0.061680	-0.457920	0.450819	Yes
0.924589	0.000247	-1.08E-06	0.110125	-0.006020	0.007111	-0.013131	0.129206	-0.135226	Yes
0.924313	0.000230	-1.05E-06	0.056898	-0.000127	-0.004956	0.004829	0.278471	-0.278599	No
0.923946	0.000211	-9.74E-07	0.050474	-0.000198	-0.006469	0.006272	0.298432	-0.298630	No
0.923671	0.000198	-9.20E-07	0.045885	-0.000197	-0.007565	0.007368	0.312754	-0.312951	No
0.923212	0.000178	-8.29E-07	0.041297	-0.000197	-0.008710	0.008514	0.328128	-0.328324	No
0.922504	0.000153	-6.90E-07	0.032120	-0.000198	-0.011009	0.010811	0.357605	-0.357803	No
0.922477	0.000152	-6.84E-07	0.028449	-0.000200	-0.011870	0.011671	0.367922	-0.368121	No
0.922294	0.000147	-6.48E-07	0.027531	-0.000200	-0.012138	0.011938	0.371503	-0.371702	No
0.921835	0.000136	-5.55E-07	0.022025	-0.000202	-0.013572	0.013370	0.389341	-0.389542	No
0.921377	0.000127	-4.61E-07	0.016519	-0.000205	-0.015031	0.014825	0.407154	-0.407359	No
0.920205	0.000118	-2.12E-07	0.000143	-0.000212	-0.019423	0.019210	0.458921	-0.459134	No

Table 16: Test results for the SSD efficiency on the third moment for the monthly data. Note: Concerning the format, we here denote $B_1 = 3[f'(1-\mu) - f - \mu^2 + 2\mu - 1]$ and $B_2 = 3[1 - f - \mu f' - \mu^2]$. Hence, either $dS/d\mu - B_1 < 0$ or $dS/d\mu - B_2 > 0$, the SSD efficiency can be established on the third central moment.

μ	$\sigma^2(\mu)$	$S(\mu)$	f'	$\frac{dS}{d\mu}$	B_1	$\frac{dS}{d\mu} - B_1$	B_2	$\frac{dS}{d\mu} - B_2$	SSD Efficiency (on the third moment)
0.816858	0.002418	-3.35E-05	76.773628		42.073562	-42.073562	-187.148468	187.148468	Yes
0.816827	0.002174	-5.66E-05	1.585603	0.737814	0.764142	-0.026328	-2.893628	3.631442	Yes
0.816026	0.001809	-4.58E-05	0.384389	-0.013385	0.105187	-0.118572	0.055866	-0.069251	Yes
0.815225	0.001530	-3.04E-05	0.336340	-0.019232	0.079427	-0.098659	0.179056	-0.198288	Yes
0.814424	0.001289	-1.87E-05	0.288291	-0.014701	0.053318	-0.068019	0.301898	-0.316599	Yes
0.813623	0.001086	-1.00E-05	0.248251	-0.010820	0.031337	-0.042157	0.404843	-0.415663	Yes
0.812823	0.000921	-4.33E-06	0.192194	-0.007076	0.000053	-0.007129	0.546534	-0.553610	Yes
0.812022	0.000793	-8.96E-07	0.144146	-0.004292	-0.027097	0.022805	0.668334	-0.672627	No
0.811221	0.000701	1.16E-06	0.096097	-0.002565	-0.054591	0.052026	0.789792	-0.792357	No
0.810419	0.000636	5.00E-07	0.080081	0.000822	-0.064185	0.065007	0.833055	-0.832234	No
0.809619	0.000579	-1.26E-07	0.072073	0.000782	-0.069309	0.070091	0.856761	-0.855979	No
0.808818	0.000530	-7.01E-07	0.056857	0.000718	-0.078631	0.079349	0.897890	-0.897172	No
0.808417	0.000508	-9.90E-07	0.053654	0.000721	-0.080798	0.081519	0.907735	-0.907014	No
0.807617	0.000472	-1.76E-06	0.042443	0.000956	-0.087953	0.088909	0.939019	-0.938062	No
0.806816	0.000446	-2.64E-06	0.025626	0.001107	-0.098448	0.099555	0.983780	-0.982673	No
0.806015	0.000432	-3.31E-06	0.008008	0.000839	-0.109527	0.110366	1.030358	-1.029519	No
0.805451	0.000430	-3.65E-06	0.000596	0.000595	-0.114489	0.115084	1.051016	-1.050421	No

Table 17: Comparison of the SSD efficiency from LP tests, quadratic utility test, and on the third moment (weekly data)

μ	$\sigma^2(\mu)$	SSD Efficiency (LP tests)	SSD Efficiency (Quadratic utility)	SSD Efficiency (Third moment)
0.925396	0.000612	Yes	Yes	Yes
0.925324	0.000529	Yes	Yes	Yes
0.925230	0.000478	Yes	Yes	Yes
0.925048	0.000388	Yes	Yes	Yes
0.924911	0.000334	Yes	Yes	Yes
0.924818	0.000303	Yes	Yes	Yes
0.924589	0.000247	Yes	No	Yes
0.924313	0.000230	Yes	No	No
0.923946	0.000211	Yes	No	No
0.923671	0.000198	Yes	No	No
0.923212	0.000178	Yes	No	No
0.922504	0.000153	Yes	No	No
0.922477	0.000152	No	No	No
0.922294	0.000147	No	No	No
0.921835	0.000136	No	No	No
0.921377	0.000127	No	No	No
0.920205	0.000118	No	No	No

Table 18: Comparison of the SSD efficiency from LP tests, quadratic utility test, and on the third moment (monthly data)

μ	$\sigma^2(\mu)$	SSD Efficiency (LP tests)	SSD Efficiency (Quadratic utility)	SSD Efficiency (Third moment)
0.816858	0.002418	Yes	Yes	Yes
0.816827	0.002174	Yes	Yes	Yes
0.816026	0.001809	Yes	Yes	Yes
0.815225	0.001530	Yes	No	Yes
0.814424	0.001289	Yes	No	Yes
0.813623	0.001086	Yes	No	Yes
0.812823	0.000921	Yes	No	Yes
0.812022	0.000793	Yes	No	No
0.811221	0.000701	Yes	No	No
0.810419	0.000636	Yes	No	No
0.809619	0.000579	Yes	No	No
0.808818	0.000530	No	No	No
0.808417	0.000508	No	No	No
0.807617	0.000472	No	No	No
0.806816	0.000446	No	No	No
0.806015	0.000432	No	No	No
0.805451	0.000430	No	No	No

Table 19: The maximization for SSD on the third central moment with weekly data: Results from maximizing $A_1(m(\lambda))$.

μ	$f(\mu)$	$A_1(m(\lambda))$	$m(\lambda)$	$\sigma^2(\lambda)$
0.92458854	0.00024734	0.99951414	0.92459899	0.00024802
0.92431322	0.00022999	0.99951414	0.92459899	0.00024802
0.92394617	0.00021091	0.99951414	0.92459899	0.00024802
0.92367088	0.00019777	0.99951414	0.92459899	0.00024802
0.92321198	0.00017810	0.99951414	0.92459899	0.00024802
0.92250442	0.00015324	0.99951414	0.92459899	0.00024802
0.92247689	0.00015241	0.99951414	0.92459899	0.00024802
0.92229429	0.00014713	0.99951414	0.92459899	0.00024802
0.92183541	0.00013584	0.99951414	0.92459899	0.00024802
0.92137658	0.00012733	0.99951414	0.92459899	0.00024802
0.92020479	0.00011827	0.99951414	0.92459899	0.00024802

Table 20: The maximization for SSD on the third central moment with weekly data: Results from maximizing $A_2(m(\lambda))$.

μ	$f(\mu)$	$A_2(m(\lambda))$	$m(\lambda)$	$\sigma^2(\lambda)$
0.92458854	0.00024734	1.98268587	0.92459899	0.00024802
0.92431322	0.00022999	1.98268587	0.92459899	0.00024802
0.92394617	0.00021091	1.98268587	0.92459899	0.00024802
0.92367088	0.00019777	1.98268587	0.92459899	0.00024802
0.92321198	0.00017810	1.98268587	0.92459899	0.00024802
0.92250442	0.00015324	1.98268587	0.92459899	0.00024802
0.92247689	0.00015241	1.98268587	0.92459899	0.00024802
0.92229429	0.00014713	1.98268587	0.92459899	0.00024802
0.92183541	0.00013584	1.98268587	0.92459899	0.00024802
0.92137658	0.00012733	1.98268587	0.92459899	0.00024802
0.92020479	0.00011827	1.98268587	0.92459899	0.00024802

Table 21: The maximization for SSD on the third central moment with monthly data: Results from maximizing $A_1(m(\lambda))$.

μ	$f(\mu)$	$A_1(m(\lambda))$	$m(\lambda)$	$\sigma^2(\lambda)$
0.81522509	0.00152972	0.99281350	0.81522542	0.00153066
0.81442428	0.00128885	0.99287305	0.81442453	0.00128955
0.81362347	0.00108616	0.99290854	0.81362300	0.00108664
0.81282266	0.00092143	0.99292069	0.81282300	0.00092220
0.81202185	0.00079288	0.99292069	0.81281709	0.00092113
0.81122098	0.00070052	0.99292068	0.81281706	0.00092112
0.81041945	0.00063604	0.99292069	0.81281706	0.00092112
0.80961863	0.00057924	0.99292069	0.81281704	0.00092112
0.80881781	0.00052972	0.99292069	0.81281704	0.00092112
0.80841741	0.00050801	0.99292069	0.81281707	0.00092112
0.80761660	0.00047159	0.99292069	0.81281704	0.00092112
0.80681579	0.00044626	0.99292069	0.81281708	0.00092113
0.80601498	0.00043242	0.99292069	0.81281707	0.00092112
0.80545140	0.00042957	0.99292069	0.81281707	0.00092112

Table 22: The maximization for SSD on the third central moment with monthly data: Results from maximizing $A_2(m(\lambda))$.

μ	$f(\mu)$	$A_2(m(\lambda))$	$m(\lambda)$	$\sigma^2(\lambda)$
0.81522509	0.00152972	1.90034520	0.81678891	0.00210993
0.81442428	0.00128885	1.90034525	0.81676028	0.00209801
0.81362347	0.00108616	1.90034525	0.81677088	0.00210242
0.81282266	0.00092143	1.90034526	0.81676346	0.00209934
0.81202185	0.00079288	1.90034519	0.81679238	0.00211138
0.81122098	0.00070052	1.90034518	0.81679212	0.00211127
0.81041945	0.00063604	1.90034519	0.81679213	0.00211127
0.80961863	0.00057924	1.90034519	0.81679186	0.00211116
0.80881781	0.00052972	1.90034519	0.81679199	0.00211122
0.80841741	0.00050801	1.90034519	0.81679213	0.00211127
0.80761660	0.00047159	1.90034519	0.81679186	0.00211116
0.80681579	0.00044626	1.90034519	0.81679225	0.00211133
0.80601498	0.00043242	1.90034519	0.81679225	0.00211133
0.80545140	0.00042957	1.90034519	0.81679213	0.00211127

Table 23: The maximization for SSD on the fourth central moment with weekly data

μ	$f(\mu)$	$D(m(\lambda))$	$m(\lambda)$	$\sigma^2(\lambda)$
0.92458854	0.00024734	0.999071555	0.92459899	0.00024802
0.92431322	0.00022999	0.999071555	0.92459899	0.00024802
0.92394617	0.00021091	0.999071555	0.92459899	0.00024802
0.92367088	0.00019777	0.999071555	0.92459899	0.00024802
0.92321198	0.00017810	0.999071555	0.92459899	0.00024802
0.92250442	0.00015324	0.999071555	0.92459899	0.00024802
0.92247689	0.00015241	0.999071555	0.92459899	0.00024802
0.92229429	0.00014713	0.999071555	0.92459899	0.00024802
0.92183541	0.00013584	0.999071555	0.92459899	0.00024802
0.92137658	0.00012733	0.999071555	0.92459899	0.00024802
0.92020479	0.00011827	0.999071555	0.92459899	0.00024802

Table 24: The maximization for SSD on the fourth central moment with monthly data

μ	$f(\mu)$	$D(m(\lambda))$	$m(\lambda)$	$\sigma^2(\lambda)$
0.81522509	0.00152972	0.98718420	0.81522513	0.00152988
0.81442428	0.00128885	0.98724711	0.81442475	0.00128920
0.81362347	0.00108616	0.98727739	0.81362300	0.00108631
0.81282266	0.00092143	0.98728022	0.81329478	0.00101429
0.81202185	0.00079288	0.98728022	0.81329477	0.00101429
0.81122098	0.00070052	0.98728022	0.81329476	0.00101428
0.81041945	0.00063604	0.98728022	0.81329476	0.00101428
0.80961863	0.00057924	0.98728022	0.81329476	0.00101428
0.80881781	0.00052972	0.98728022	0.81329476	0.00101428
0.80841741	0.00050801	0.98728022	0.81329476	0.00101428
0.80761660	0.00047159	0.98728022	0.81329476	0.00101428
0.80681579	0.00044626	0.98728022	0.81329476	0.00101428
0.80601498	0.00043242	0.98728022	0.81329476	0.00101428
0.80545140	0.00042957	0.98728022	0.81329476	0.00101428

Figure 7: Plots of the maximized SSD-undominated portfolios on the third and fourth central moments with the EV frontier (weekly data)

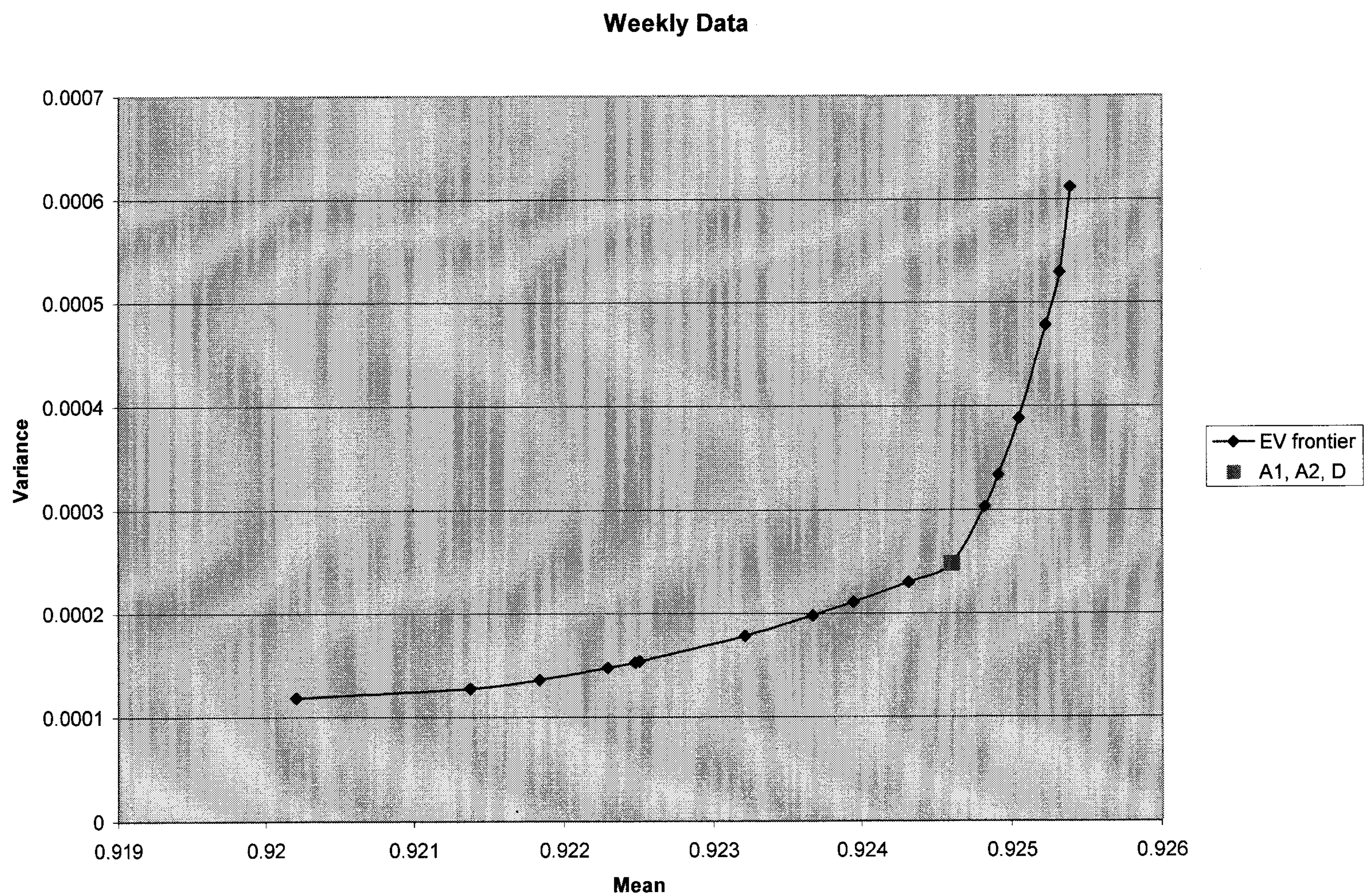

Figure 8: Plots of the maximized SSD-undominated portfolios on the third and fourth central moments with the EV frontier (monthly data)

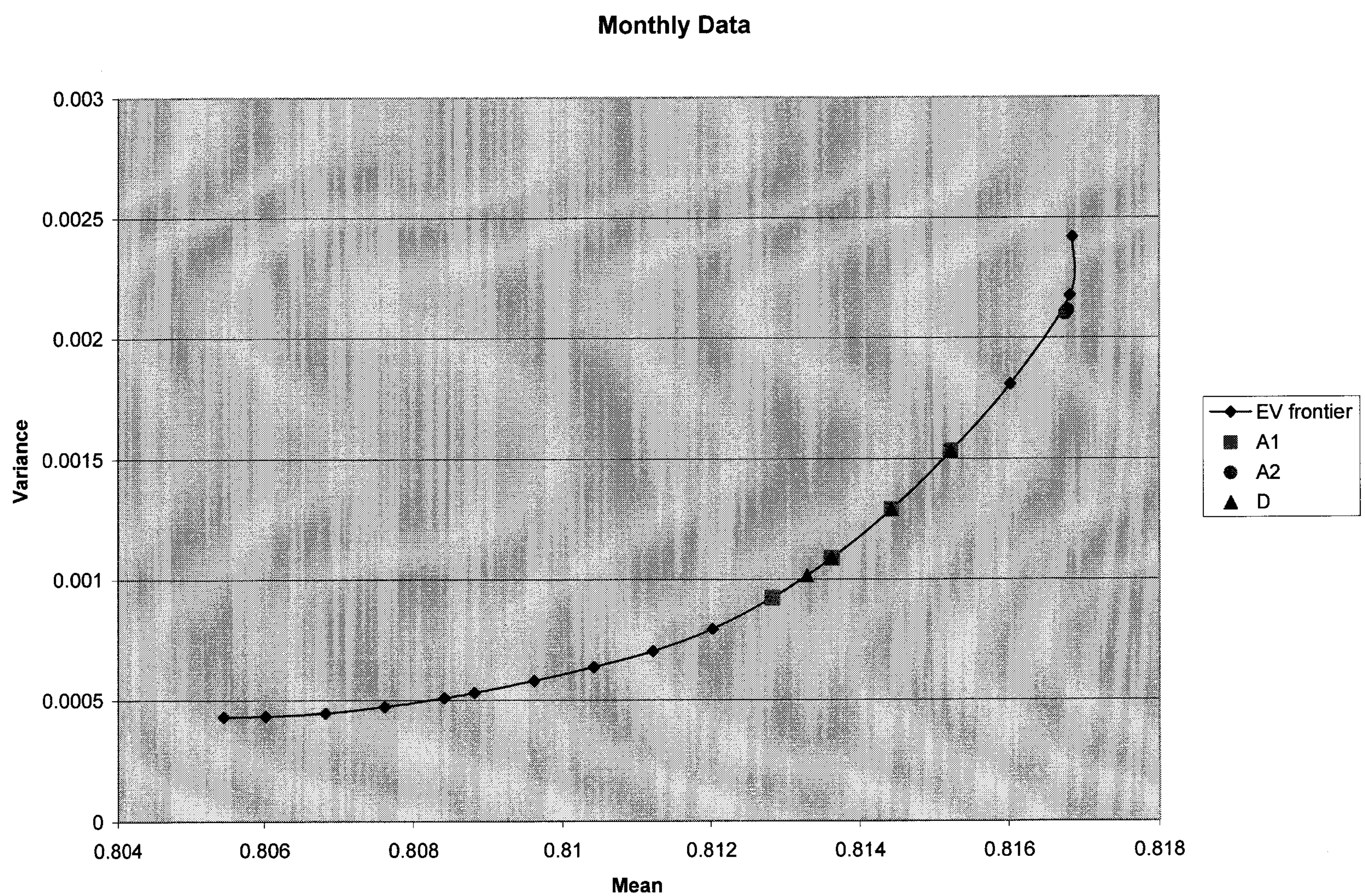

Table 25: The maximization for TSD on the fourth central moment with weekly data: Results from maximizing $E_1(m(\lambda))$.

μ	$f(\mu)$	$E_1(m(\lambda))$	$m(\lambda)$	$\sigma^2(\lambda)$
0.92458854	0.00024734	2.99809783	0.92459899	0.00024802
0.92431322	0.00022999	2.99809783	0.92459899	0.00024802
0.92394617	0.00021091	2.99809783	0.92459899	0.00024802
0.92367088	0.00019777	2.99809783	0.92459899	0.00024802
0.92321198	0.00017810	2.99809783	0.92459899	0.00024802
0.92250442	0.00015324	2.99809783	0.92459899	0.00024802
0.92247689	0.00015241	2.99809783	0.92459899	0.00024802
0.92229429	0.00014713	2.99809783	0.92459899	0.00024802
0.92183541	0.00013584	2.99809783	0.92459899	0.00024802
0.92137658	0.00012733	2.99809783	0.92459899	0.00024802
0.92020479	0.00011827	2.99809783	0.92459899	0.00024802

Table 26: The maximization for TSD on the fourth central moment with weekly data: Results from maximizing $E_2(m(\lambda))$.

μ	$f(\mu)$	$E_2(m(\lambda))$	$m(\lambda)$	$\sigma^2(\lambda)$
0.92458854	0.00024734	0.99995872	0.92459899	0.00024802
0.92431322	0.00022999	0.99995872	0.92459899	0.00024802
0.92394617	0.00021091	0.99995872	0.92459899	0.00024802
0.92367088	0.00019777	0.99995872	0.92459899	0.00024802
0.92321198	0.00017810	0.99995872	0.92459899	0.00024802
0.92250442	0.00015324	0.99995872	0.92459899	0.00024802
0.92247689	0.00015241	0.99995872	0.92459899	0.00024802
0.92229429	0.00014713	0.99995872	0.92459899	0.00024802
0.92183541	0.00013584	0.99995872	0.92459899	0.00024802
0.92137658	0.00012733	0.99995872	0.92459899	0.00024802
0.92020479	0.00011827	0.99995872	0.92459899	0.00024802

Table 27: The maximization for TSD on the fourth central moment with monthly data: Results from maximizing $E_1(m(\lambda))$.

μ	$f(\mu)$	$E_1(m(\lambda))$	$m(\lambda)$	$\sigma^2(\lambda)$
0.81522509	0.00152972	2.97276118	0.81522598	0.00153053
0.81442428	0.00128885	2.97296223	0.81442429	0.00128948
0.81362347	0.00108616	2.97307868	0.81362347	0.00108666
0.81282266	0.00092143	2.97311021	0.81293270	0.00094244
0.81202185	0.00079288	2.97311020	0.81294217	0.00094418
0.81122098	0.00070052	2.97311021	0.81293142	0.00094219
0.81041945	0.00063604	2.97311021	0.81293185	0.00094227
0.80961863	0.00057924	2.97311021	0.81293242	0.00094238
0.80881781	0.00052972	2.97311021	0.81293183	0.00094227
0.80841741	0.00050801	2.97311021	0.81293243	0.00094238
0.80761660	0.00047159	2.97311021	0.81293184	0.00094227
0.80681579	0.00044626	2.97311021	0.81293181	0.00094227
0.80601498	0.00043242	2.97311021	0.81293181	0.00094227
0.80545140	0.00042957	2.97311021	0.81293185	0.00094227

Table 28: The maximization for TSD on the fourth central moment with monthly data: Results from maximizing $E_2(m(\lambda))$.

μ	$f(\mu)$	$E_2(m(\lambda))$	$m(\lambda)$	$\sigma^2(\lambda)$
0.81522509	0.00152972	0.99849211	0.81522500	0.00153290
0.81442428	0.00128885	0.99852935	0.81442428	0.00129151
0.81362347	0.00108616	0.99855634	0.81362347	0.00108726
0.81282266	0.00092143	0.99857342	0.81282266	0.00092312
0.81202185	0.00079288	0.99858091	0.81202185	0.00079489
0.81122098	0.00070052	0.99858127	0.81181549	0.00076986
0.81041945	0.00063604	0.99858127	0.81181519	0.00076983
0.80961863	0.00057924	0.99858127	0.81181569	0.00076989
0.80881781	0.00052972	0.99858127	0.81181506	0.00076982
0.80841741	0.00050801	0.99858127	0.81181485	0.00076979
0.80761660	0.00047159	0.99858127	0.81181518	0.00076983
0.80681579	0.00044626	0.99858127	0.81181562	0.00076989
0.80601498	0.00043242	0.99858127	0.81181526	0.00076984
0.80545140	0.00042957	0.99858127	0.81181527	0.00076984

Figure 9: Plots of the maximized TSD-undominated portfolios on the fourth moment with the EV frontier (weekly data)

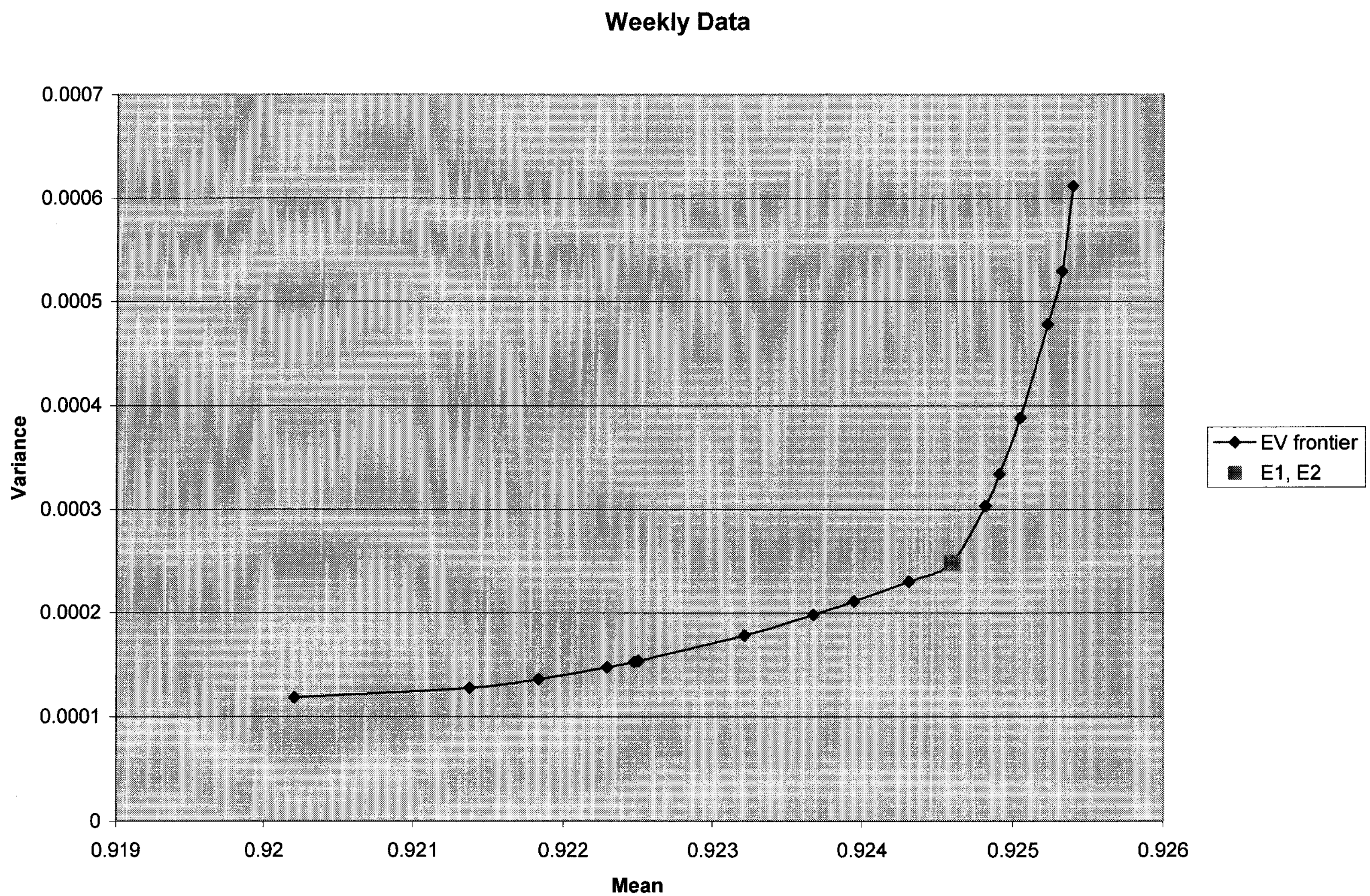

Figure 10: Plots of the maximized TSD-undominated portfolios on the fourth moment with the EV frontier (monthly data)