ON THE FIRST RANGE TIME OF DIFFUSION PROCESSES

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ABSTRACT

On the First Range Time of Diffusion Processes

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First range time is the first time when the range of a stochastic process reaches a certain level. The first range time for Brownian motion has already been studied in several papers. In this thesis we will use a different approach to derive a joint Laplace transform on the first range time for a general diffusion process. This derivation is more intuitive than that presented in previous papers. From this main result we will see that the problems on the first range time could be transferred to the problem of solving an ordinary differential equation.

We will also apply the result to some well-known diffusions, such as Brownian motion, geometric Brownian motion, Ornstein-Uhlenbeck processes and squared Bessel processes.

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Contents

List of Figures

1. Introduction

1.1. Definitions. The *range* process of a stochastic process X is defined by

$$R_t = S_t - I_t, \qquad t \geq 0,$$

where $S_t = \sup_{0 \leq s \leq t} X_s$ and $I_t = \inf_{0 \leq s \leq t} X_s$. It is an increasing process which vanishes at 0. Its left-continuous inverse, called the *first range time* for range r, is defined by

$$\theta_r = \inf\{t \geq 0 : R_t \geq r\}, \qquad r \geq 0.$$

Sometimes it is also called the *cover time*. Intuitively, θ_r is the first time when the range of the process X reaches r. Suppose $X_0 = x$. We define another time related to the first range time as

$$\eta_r = \inf\{t \geq 0 : X_t = (X_{\theta_r} - r)\mathbf{1}_{\{X_{\theta_r} > x\}} + (X_{\theta_r} + r)\mathbf{1}_{\{X_{\theta_r} < x\}}\}.$$

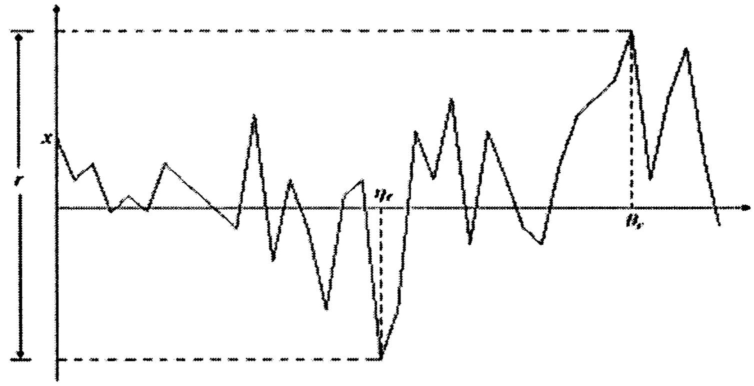

Figure 1. First range time.

We can see that $\eta_r < \theta_r$ holds for any $r > 0$. Since θ_r is the first time when the range of the process reaches r, the process should be necessarily at an extremum at time θ_r. Correspondingly, η_r is the first time when the process is at the other extremum up to time θ_r, i.e. if X_{θ_r} is at the minimum, then X_{η_r} is at the maximum, and conversely, if X_{θ_r} is at the maximum then X_{η_r} is at the minimum. Thus the two endpoints of the coverage interval with length r are visited at times η_r and θ_r. Note that θ_r is a stopping time, but η_r is not a stopping time.

Define the *first hitting time* at level b as

$$T_b = \inf\{t : X_t = b\}$$

with the convention that $\inf\{\emptyset\} = \infty$. Throughout the paper the notation θ_r and T_b will be reserved for the first range time for range r and the first hitting time at level b respectively.

1.2. **Previous Results on Range Time.** Previous results on the range process and the first range time could be found in Feller (1951), Imhof (1985), Vallois (1993), Vallois (1995), Vallois (1996), Borodin (1999), Chong, Cowan and Holst (2000), Salminen and Vallois (2005), Tanré and Vallois (2006) and so on.

In Feller (1951), the density function $f(t; r)$ of the range R_t for standard Brownian motion B_t was given by

$$f(t; r) = 8 \sum_{k=1}^{\infty} (-1)^{k-1} k^2 \phi\left(kr / \sqrt{t}\right)$$

where $\phi(x)$ is the density function of the standard normal distribution. Note that in this form it is not even obvious that the function is positive. To derive this density, the author first found $\mathbb{P}\{B_t \in dx, S_t \leq v, I_t \geq -u\}$, and then the density function $g(t; u, v)$ of (S_t, I_t). The result was finally obtained by $f(t; r) = \int_0^r g(t; u, r-u)du$.

In Imhof (1985), the process of the first range time θ_r for standard Brownian motion B_t starting at 0 (or a three-dimensional Bessel process) was considered. It is a pure jump process with independent and non-stationary increments. The density function of θ_r was given by

$$\mathbb{P}_0\{\theta_r \in dt\} = 2(\partial/\partial r)Q_t(r/2, r/2, r)dt, \quad t > 0$$

where the explicit expression of $Q_t(x, y, z)dy = \mathbb{P}_x\{B_t \in dy, T_0 \wedge T_z > t\}$, $0 < x, y < z$, could be found. It was derived by using

$$G_t(x, r)dt := \mathbb{P}_0\{T_x \in dt, R_{T_x} < r\} = \mathbb{P}_0\{T_x \in dt, T_{x-r} > t\}, \quad 0 < x < r$$

of which the right side was already known (for example, see Chung 1976) and then using

$$\mathbb{P}_0\{\theta_r \in dt, X_{\theta_r} \in dx\} = (\partial/\partial r)G_t(x, r)dtdx.$$

Furthermore, the Laplace transform of θ_r could be obtained by direct computation on its density function, and we have

$$\mathbb{E}[e^{-\lambda\theta_r}] = \cosh^{-2}[r\sqrt{\lambda/2}].$$

Since the process θ_r for Brownian motion has independent increments, the Laplace transform of $\theta_{r_2} - \theta_{r_1}$ for $0 < r_1 < r_2$ is

$$\mathbb{E}[e^{-\lambda(\theta_{r_2}-\theta_{r_1})}] = \cosh^2[r_1 \sqrt{\lambda/2}] \cosh^{-2}[r_2 \sqrt{\lambda/2}].$$

The density of $\theta_{r_2} - \theta_{r_1}$ was given by

$$\mathbb{P}\{\theta_{r_2} - \theta_{r_1} \in dt\}$$
$$= \partial/\partial r_2\{Q_t((r_2 - r_1)/2, (r_2 - r_1)/2, r_2) + Q_t((r_2 - r_1)/2, (r_2 + r_1)/2, r_2)\}dt.$$

It was also proved that $\{|B_{\eta_r+t} - B_{\eta_r}|; 0 \le t \le \theta_r - \eta_r\}$ is a three-dimensional Bessel process stopped at its first hitting time of level r.

In Vallois (1993), the first range time with randomized range level was considered. It was derived that, when U is uniformly distributed on $(0, a)$, the Brownian motion stopped when R_t exceeds U for the first time is identical in law with the Brownian motion stopped when it first exits the interval $(U, a - U)$.

In Vallois (1995), the Brownian motion were further considered. Note that in this paper the first range time was defined as the right-continuous inverse of the range R_t, which is slightly different from the previous definition. Given $C > 0$ fixed, a family $\{\hat{\eta}_{-n}; n \ge 0\}$ of positive random times was defined by induction as

$$\hat{\eta}_0 = \eta_C, \qquad \hat{\eta}_{-n-1} = \eta_{R_{\hat{\eta}_{-n}}}, \quad n \ge 0.$$

By this way the Brownian motion path up to time η_C was split into countable parts. Then an intrinsic decomposition of the Brownian motion followed. It was also pointed out that for $0 < r_1 < r_2$,

$$\theta_{r_2} - \theta_{r_1} \stackrel{D}{=} \inf\{t \geq 0; -I_t + max(r_1, S_t) > r_2\}.$$

Another result was deduced from the scaling property of Brownian motion that $\{\theta_{\lambda t}; t \geq 0\}$ has the same law as $\{\lambda^2\theta_t; t \geq 0\}$ for any $\lambda > 0$. Moreover, a square-integrable martingale was constructed by

$$M_t = \sqrt{2}B_{\theta_{\sqrt{t}}}, \quad t \geq 0.$$

M was connected to the parabolic martingale (see Emery 1989) and after that it was proved that M had the chaotic property representation (see Emery 1989).

In Vallois (1996), a simple random walk on $\mathbb{Z}$ was considered. The generating function of the first range time θ_r for the non-symmetric cases with $\mathbb{P}\{X_{n+1}-X_n = 1\} = p$ and $\mathbb{P}\{X_{n+1} - X_n = -1\} = q$ $(p + q = 1)$ was obtained, and it was also proved that it is a rational function. This allowed us to invert the generating function to find the explicit distributions of θ_r and R_n. The asymptotic behavior of θ_r for non-symmetric cases was also investigated. Two results were given as follows. The first result looks like the law of large number and the second one is similar to the central limit theorem. More precisely,

$$\theta_n/n \xrightarrow{P} 1/|p-q|,$$

$$(\theta_n - n/|p-q|) \,/\, \sqrt{n} \xrightarrow{D} \mathcal{N}(0, 4pq/|p-q|^3).$$

In Borodin (1999), methods for computation on distributions of two classes of functionals of Brownian motion stopped at the first range time were developed. Both classes involved the Brownian local time. The basic idea was to transfer the problems on the first range time to the problems on the first exit time. This idea is very important when we prove some results on the distributions of functionals stopped at the first range time. We will also use this idea in the following sections.

In Chong, Cowan and Holst (2000), the joint Laplace transform of $(\theta_r, \eta_r, X_{\theta_r})$ was derived for Brownian motion cases. In their paper they first derived such a joint Laplace transform for simple random walks, and then applied the property that Brownian motion is a scaling limit of random walks to find the desired result. It was given as the following Theorem.

Theorem 1.1. *[Chong, Cowan and Holst (2000)] For a Brownian motion X_t with drift μ, variance σ^2 and $X_0 = 0$ we have*

$$\begin{aligned} &\mathbb{E}\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)-vX_{\theta_r}}\right] \\ &= \frac{2\kappa(\beta)}{\sinh[\kappa(\alpha)]\sinh[\kappa(\beta)]}\left[\frac{\sinh^2[\frac{1}{2}(\kappa(\alpha)+\rho)]}{\kappa(\alpha)+\rho} + \frac{\sinh^2[\frac{1}{2}(\kappa(\alpha)-\rho)]}{\kappa(\alpha)-\rho}\right] \end{aligned} \tag{1.1}$$

for any $\alpha, \beta \geq 0$ and v with $\rho = r\mu/\sigma^2 - rv$ and $\kappa(x) = r/\sigma\sqrt{\mu^2/\sigma^2 + 2x}$.

The method in Chong, Cowan and Holst (2000) works only when the process is spacially homogenous. In this thesis we will consider a general diffusion which is not necessarily spacially homogenous.

There are also some recent papers on the first range time. In Salminen and Vallois (2005), the first range time (with randomized range level) of a linear diffusion on $\mathbb{R}$ was considered. Inspired by the observation that the exponentially randomized first range time has the same law as a similarly randomized first exit time from an interval (see Vallois 1993), they studied a large family of non-negative two-dimensional random variables (X, X') instead of $(U, a - U)$ (see Vallois 1993) with this property. The feature of this family is $F^c(x, y) = F^c(x + y, 0), \forall x, y \geq 0$, where $F^c(x, y) := \mathbb{P}\{X > x, X' > y\}$. In particular, X and X' could be taken to be i.i.d. exponential distributed random variables.

In Tanré and Vallois (2006), the Brownian motion with drift μ staring at 0 was considered. The law of R_t and θ_r was obtained. The asymptotic behavior of θ_r was also investigated as $r \to \infty$ and the results as follows were similar to those for the simple random walks in Vallois (1996):

$$\theta_r / r \xrightarrow{P} 1/\mu,$$

$$\sqrt{r}(\theta_r / r - 1/\mu) \xrightarrow{D} \mathcal{N}(0, 1/\mu^3).$$

So far we have seen that most results on the first range time were derived for some specific process(es) only, such as Brownian motion or random walks. In this paper we will use a different and simple approach to derive a general result of a joint Laplace transform on the first range time for diffusion processes. The above Theorem 1.1 will be given as an example in Corollary 4.1.

In section 2, some basic settings and an important lemma will be first stated. The lemma gives the Laplace transform of the exit time from a finite interval for diffusion processes. Then two propositions will follow.

In section 3, the main result will be presented in Theorem 3.1. It gives the joint Laplace transform of $(\theta_r, X_{\theta_r}, \eta_r)$ for general diffusion processes. The derivation is more intuitive than that presented in previous papers. From this main result we will see that the problems on the first range time can be transferred to the problem of solving an ordinary differential equation in which the infinitesimal generator is involved. After that some corollaries will be discussed.

In section 4, we will apply the main result to some well-known diffusion processes, such as Brownian motion, geometric Brownian motion, Ornstein-Uhlenbeck processes and squared Bessel processes.

2. Basic Settings and Preliminary Results

2.1. Basic Settings. Let $\{X_t; t \geq 0\}$ be a one-dimensional stochastic process satisfying the time homogeneous Itô stochastic differential equation

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dW_t, \qquad t \geq 0 \tag{2.1}$$

with $X_0 = x$ a.s., $\{W_t; t \geq 0\}$ a standard Wiener process, and $\sigma(x) > 0$. It is further assumed that $\alpha(x)$ and $\sigma(x)$ are measurable and defined in $(-\infty, +\infty)$ and satisfy the conditions of the existence and uniqueness theorem for stochastic differential equations, i.e. there exists a constant K such that for all x, y in $(-\infty, +\infty)$,

$$|\alpha(x) - \alpha(y)| + |\sigma(x) - \sigma(y)| \leq K|x - y|,$$

$$\alpha^2(x) + \sigma^2(x) \leq K^2(1 + x^2) \tag{2.2}$$

(see Gihman and Skorohod, page 40). Then solutions to (2.1) are strong solutions.

The special case that $\alpha(x) = \mu$ and $\sigma(x) = \sigma$ gives the Brownian motion.

2.2. A Lemma. We first state a lemma which could be found in Breiman (1968). It gave a Laplace transform of the first exit time for a diffusion process from a finite interval. This lemma will be used in the derivation of the succeeding results. We present a proof in Lehoczky (1977) for completeness.

Lemma 2.1. *Under the above settings, for $a \le x \le b$, we have*

$$\mathbb{E}_x\left[e^{-\beta T_a}; T_a < T_b\right] = \frac{p_\beta(x,b)}{p_\beta(a,b)} \tag{2.3}$$

and

$$\mathbb{E}_x\left[e^{-\beta T_b}; T_b < T_a\right] = \frac{p_\beta(a,x)}{p_\beta(a,b)} \tag{2.4}$$

where

$$p_\beta(x,y) := g_\beta(x)h_\beta(y) - g_\beta(y)h_\beta(x)$$

and g_β and h_β are any two independent solutions of the ordinary differential equation

$$\frac{1}{2}\sigma^2(x)f''(x) + \alpha(x)f'(x) = \beta f(x). \tag{2.5}$$

Proof. Since X_t satisfies (2.1) and (2.2), we integrate (2.1) to find

$$X_t = x + \int_0^t \alpha(X_s)ds + \int_0^t \sigma(X_s)dW_s.$$

Let $f(x)$ be any solution of (2.5) and consider the transformation $Y_t = e^{-\beta t}f(X_t)$. Using Itô's formula (see Karatzas and Shreve, page153), Y_t satisfies

$$Y_t = f(x) + \int_0^t -\beta e^{-\beta s}f(X_s)ds + \int_0^t e^{-\beta s}f'(X_s)\alpha(X_s)ds + \int_0^t e^{-\beta s}f'(X_s)\sigma(X_s)dW_s + \frac{1}{2}\int_0^t e^{-\beta s}f''(X_s)\sigma^2(X_s)ds.$$

It follows that

$$dY_t = e^{-\beta t}\left(\frac{1}{2}\sigma^2(X_t)f''(X_t) + \alpha(X_t)f'(X_t) - \beta f(X_t)\right)dt + e^{-\beta t}\sigma(X_t)f'(X_t)dW_t.$$

However, $f(x)$ satisfies (2.5), which means that the expression inside the parentheses is equal to zero almost surely. So we have

$$dY_t = e^{-\beta t}\sigma(X_t)f'(X_t)dW_t$$

with $Y_0 = f(x)$ a.s. or equivalently,

$$Y_t - f(x) = \int_0^t e^{-\beta s}\sigma(X_s)f'(X_s)dW_s.$$

Truncate $T_a \wedge T_b$, forming $\tau_u = T_a \wedge T_b \wedge u$, replace t by τ_u and take the expectation of both sides,

$$\mathbb{E}_x[Y_{\tau_u}] - f(x) = \mathbb{E}_x\left[\int_0^{\tau_u} e^{-\beta s}\sigma(X_s)f'(X_s)dW_s\right]$$

where the right side is equal to zero by applying the optional sampling theorem (see Karatzas and Shreve, page19) and the fact that $\int_0^{\tau_u} e^{-\beta s}\sigma(X_s)f'(X_s)dW_s$ is a martingale, and furthermore, the integrand is bounded for $s \le T_a \wedge T_b$.

Let $u \to \infty$ to find

$$\begin{aligned}
f(x) &= \mathbb{E}_x[Y_{T_a \wedge T_b}] \\
&= \mathbb{E}_x[e^{-\beta(T_a \wedge T_b)} f(X_{T_a \wedge T_b})]dt \\
&= \mathbb{E}_x[e^{-\beta T_a} f(a); T_a < T_b] + \mathbb{E}_x[e^{-\beta T_b} f(b); T_b < T_a] \\
&= f(a)\mathbb{E}_x[e^{-\beta T_a}; T_a < T_b] + f(b)\mathbb{E}_x[e^{-\beta T_b}; T_b < T_a].
\end{aligned}$$

Let g_β and h_β be two independent solutions of (2.5). Then we have two linear equations

$$g_\beta(x) = g_\beta(a)\mathbb{E}_x[e^{-\beta T_a}; T_a < T_b] + g_\beta(b)\mathbb{E}_x[e^{-\beta T_b}; T_b < T_a],$$

and

$$h_\beta(x) = h_\beta(a)\mathbb{E}_x[e^{-\beta T_a}; T_a < T_b] + h_\beta(b)\mathbb{E}_x[e^{-\beta T_b}; T_b < T_a],$$

from which we can easily find

$$\mathbb{E}_x\left[e^{-\beta T_a}; T_a < T_b\right] = \frac{g_\beta(x)h_\beta(b) - g_\beta(b)h_\beta(x)}{g_\beta(a)h_\beta(b) - g_\beta(b)h_\beta(a)},$$

and

$$\mathbb{E}_x\left[e^{-\beta T_b}; T_b < T_a\right] = \frac{g_\beta(a)h_\beta(x) - g_\beta(x)h_\beta(a)}{g_\beta(a)h_\beta(b) - g_\beta(b)h_\beta(a)}.$$

□

Remark 2.2. It is not difficult to see that $p_\beta(x, y) \neq 0$ for any $x \neq y$ because g_β and h_β are independent. Given g_β and h_β, then the sign of $p_\beta(x, y)$ is fixed if the magnitude of x and y is fixed. Moreover, if $p_\beta(x, y) > 0$, then $\partial p_\beta(x, y)/\partial x < 0$ and $\partial p_\beta(x, y)/\partial y > 0$.

Remark 2.3. From the proof we can see that even if the conditions (2.2) are not satisfied, we may still find the Laplace transform of the first exit times, because we may find two independent solutions g and h to the ordinary differential equation (2.5). So the conditions (2.2) are sufficient but not necessary.

2.3. **Other Preliminary Results.** In the following we will use Lemma 2.1 to derive two propositions related to the exit time.

In the first proposition, we consider a process that will never go across the starting level before it reaches another given level.

Proposition 2.4. *Under the above settings, for $a \leq x \leq b$ and $dx > 0$,*

$$\mathbb{E}_x\left[e^{-\beta T_a}; T_a < T_{x+dx}\right] = \frac{q_\beta(x)}{p_\beta(a,x)}dx \tag{2.6}$$

and

$$\mathbb{E}_x\left[e^{-\beta T_b}; T_b < T_{x-dx}\right] = \frac{q_\beta(x)}{p_\beta(x,b)}dx \tag{2.7}$$

where

$$q_\beta(x) := g_\beta(x)h'_\beta(x) - g'_\beta(x)h_\beta(x).$$

Proof. The left side of (2.6) indicates that starting at x the process should never go above level x before it reaches level a, while the left side of (2.7) indicates that starting at x the process should never go below level x before it reaches level b.

To prove (2.6), we directly apply Lemma 2.1 to get

$$\begin{aligned}\mathbb{E}_x\left[e^{-\beta T_a}; T_a < T_{x+dx}\right]/dx &= \lim_{\epsilon\to 0+}\mathbb{E}_x\left[e^{-\beta T_a}; T_a < T_{x+\epsilon}\right]/\epsilon \\ &= \lim_{\epsilon\to 0+}\frac{p_\beta(x,x+\epsilon)/\epsilon}{p_\beta(a,x+\epsilon)}.\end{aligned}$$

The numerator has a limit of

$$
\begin{aligned}
&\lim_{\epsilon \to 0+} p_\beta(x, x+\epsilon)/\epsilon \\
&= \lim_{\epsilon \to 0+} \left[g_\beta(x) h_\beta(x+\epsilon) - g_\beta(x+\epsilon) h_\beta(x) \right] /\epsilon \\
&= \lim_{\epsilon \to 0+} \left[g_\beta(x)(h_\beta(x+\epsilon) - h_\beta(x)) - h_\beta(x)(g_\beta(x+\epsilon) - g_\beta(x)) \right] /\epsilon \\
&= \lim_{\epsilon \to 0+} \left[g_\beta(x) h'_\beta(x) \cdot \epsilon - h_\beta(x) g'_\beta(x) \cdot \epsilon \right] /\epsilon \\
&= g_\beta(x) h'_\beta(x) - g'_\beta(x) h_\beta(x) \\
&= q_\beta(x).
\end{aligned}
$$

Therefore,

$$
\mathbb{E}_x \left[e^{-\beta T_a}; T_a < T_{x+dx} \right] = \frac{q_\beta(x)}{p_\beta(a, x)} dx.
$$

Then we follow the same procedure to prove (2.7). By Lemma 2.1,

$$
\begin{aligned}
\mathbb{E}_x \left[e^{-\beta T_b}; T_b < T_{x-dx} \right] /dx &= \lim_{\epsilon \to 0+} \mathbb{E}_x \left[e^{-\beta T_b}; T_b < T_{x-\epsilon} \right] /\epsilon \\
&= \lim_{\epsilon \to 0+} \frac{p_\beta(x-\epsilon, x)/\epsilon}{p_\beta(x-\epsilon, b)}.
\end{aligned}
$$

The numerator has a limit of

$$\begin{aligned}
&\lim_{\epsilon\to 0+} p_\beta(x-\epsilon, x)/\epsilon \\
&= \lim_{\epsilon\to 0+} \left[g_\beta(x-\epsilon)h_\beta(x) - g_\beta(x)h_\beta(x-\epsilon)\right]/\epsilon \\
&= \lim_{\epsilon\to 0+} \left[g_\beta(x-\epsilon)(h_\beta(x) - h_\beta(x-\epsilon)) - h_\beta(x-\epsilon)(g_\beta(x) - g_\beta(x-\epsilon))\right]/\epsilon \\
&= \lim_{\epsilon\to 0+} \left[g_\beta(x-\epsilon)h'_\beta(x-\epsilon)\cdot\epsilon - h_\beta(x-\epsilon)g'_\beta(x-\epsilon)\cdot\epsilon\right]/\epsilon \\
&= g_\beta(x)h'_\beta(x) - g'_\beta(x)h_\beta(x) \\
&= q_\beta(x).
\end{aligned}$$

Therefore,

$$\mathbb{E}_x\left[e^{-\beta T_b}; T_b < T_{x-dx}\right] = \frac{q_\beta(x)}{p_\beta(x,b)}dx.$$

□

In the following proposition we will consider a process that will not exit an interval, one endpoint of which is the starting point, before time T. We choose T exponentially distributed because of its nice property.

Proposition 2.5. *Given a random variable T which is independent of X and follows an exponential distribution with parameter λ, we have for $y < z$, $dy > 0$ and $dz > 0$,*

$$\mathbb{P}_z\{S_T \le z + dz, I_T \ge y\} = \frac{\partial p_\lambda(y,z)/\partial z - q_\lambda(z)}{p_\lambda(y,z)}dz \tag{2.8}$$

and

$$\mathbb{P}_y\{S_T \leq z, I_T \geq y - dy\} = \frac{-\partial p_\lambda(y,z)/\partial y - q_\lambda(y)}{p_\lambda(y,z)} dy. \tag{2.9}$$

Proof. Notice that for an exponential random variable T with parameter λ and an independent random variable Y, we have

$$\mathbb{P}\{Y < T\} = \mathbb{E}\left[e^{-\lambda Y}\right].$$

For the event on the left side of (2.8) to occure, the process starting at z cannot exit the interval between y and $z + dz$ before time T. Then

$$\begin{aligned}
&\mathbb{P}_z\{S_T \leq z + dz, I_T \geq y\}/dz \\
&= \lim_{\epsilon \to 0+} \mathbb{P}_z\{T_y \wedge T_{z+\epsilon} > T\}/\epsilon \\
&= \lim_{\epsilon \to 0+} \left(1 - \mathbb{E}_z\left[e^{-\lambda(T_y \wedge T_{z+\epsilon})}\right]\right)/\epsilon \\
&= \lim_{\epsilon \to 0+} \left(1 - \mathbb{E}_z\left[e^{-\lambda T_y}; T_y < T_{z+\epsilon}\right] - \mathbb{E}_z\left[e^{-\lambda T_{z+\epsilon}}; T_{z+\epsilon} < T_y\right]\right)/\epsilon.
\end{aligned}$$

By Proposition 2.4, we have

$$\lim_{\epsilon \to 0+} \mathbb{E}_z[e^{-\lambda T_y}; T_y < T_{z+\epsilon}]/\epsilon = \frac{q_\lambda(z)}{p_\lambda(y,z)},$$

and by Lemma 2.1,

$$\begin{aligned}
&\lim_{\epsilon\to 0+}\left(1-\mathbb{E}_z[e^{-\lambda T_{z+\epsilon}}; T_{z+\epsilon} < T_y]\right)/\epsilon \\
&= \lim_{\epsilon\to 0+}\left(1-\frac{p_\lambda(y,z)}{p_\lambda(y,z+\epsilon)}\right)/\epsilon \\
&= \lim_{\epsilon\to 0+}\frac{p_\lambda(y,z+\epsilon)-p_\lambda(y,z)}{p_\lambda(y,z+\epsilon)\cdot\epsilon} \\
&= \frac{\partial p_\lambda(y,z)/\partial z}{p_\lambda(y,z)}.
\end{aligned}$$

Therefore,

$$\mathbb{P}_z\{S_T \le z+dz, I_T \ge y\} = \frac{\partial p_\lambda(y,z)/\partial z - q_\lambda(z)}{p_\lambda(y,z)}dz.$$

For the event on the left side of (2.9) to occure, the process starting at y cannot exit the interval between $y-dy$ and z before time T.

$$\begin{aligned}
&\mathbb{P}_y\{S_T \le z, I_T \ge y-dy\}/dy \\
&= \lim_{\epsilon\to 0+}\mathbb{P}_y\{T_{y-\epsilon}\wedge T_z > T\}/\epsilon \\
&= \lim_{\epsilon\to 0+}\left(1-\mathbb{E}_y\left[e^{-\lambda(T_{y-\epsilon}\wedge T_z)}\right]\right)/\epsilon \\
&= \lim_{\epsilon\to 0+}\left(1-\mathbb{E}_y[e^{-\lambda T_{y-\epsilon}}; T_{y-\epsilon} < T_z] - \mathbb{E}_y[e^{-\lambda T_z}; T_z < T_{y-\epsilon}]\right)/\epsilon.
\end{aligned}$$

By Proposition 2.4, we have

$$\lim_{\epsilon\to 0+}\mathbb{E}_y[e^{-\lambda T_z}; T_z < T_{y-\epsilon}]/\epsilon = \frac{q_\lambda(y)}{p_\lambda(y,z)},$$

and by Lemma 2.1,

$$\begin{aligned}
&\lim_{\epsilon\to 0+}\left(1-\mathbb{E}_y[e^{-\lambda T_{y-\epsilon}}; T_{y-\epsilon} < T_z]\right)/\epsilon \\
&= \lim_{\epsilon\to 0+}\left(1-\frac{p_\lambda(y,z)}{p_\lambda(y-\epsilon,z)}\right)/\epsilon \\
&= \lim_{\epsilon\to 0+}\frac{p_\lambda(y-\epsilon,z)-p_\lambda(y,z)}{p_\lambda(y-\epsilon,z)\epsilon} \\
&= \frac{-\partial p_\lambda(y,z)/\partial y}{p_\lambda(y,z)}.
\end{aligned}$$

Note that the last expression above is positive due to Remark 2.2. Finally we have

$$\mathbb{P}_y\{S_T \le z, I_T \ge y-dy\} = \frac{-\partial p_\lambda(y,z)/\partial y - q_\lambda(y)}{p_\lambda(y,z)}dy.$$

□

3. Main Results

The joint Laplace transform of $(\eta_r, \theta_r, X_{\theta_r})$ for general diffusion processes is given in the following Theorem.

Theorem 3.1. *Under the settings in Section 2, for any y such that $|y - x| < r$ and $\alpha, \beta \geq 0$, we have*

$$\begin{aligned} \mathbb{E}_x\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)}; X_{\theta_r} \in dy\right] = & \frac{p_\alpha(x,y)q_\beta(y-r)}{p_\alpha(y-r,y)p_\beta(y-r,y)}1_{\{y\geq x\}}dy \\ & + \frac{p_\alpha(y,x)q_\beta(y+r)}{p_\alpha(y,y+r)p_\beta(y,y+r)}1_{\{y<x\}}dy. \end{aligned} \tag{3.1}$$

Proof. We have to consider the following two cases separately as we did in the previous section. In the first case, the range of the diffusion process reaches r from below, i.e. X_t first reaches the minimum and then at time θ_r it is at the maximum, so $X_{\theta_r} \geq x$. In the second case, the range reaches r from above, i.e. X_t first reaches the maximum and then at time θ_r it is at the minimum, so $X_{\theta_r} < x$.

We first prove (3.1) for $y \geq x$. Note that if $X_{\theta_r} \in dy$, then $X_{\eta_r} \in dy - r$. Therefore, the process must first reach level $y - r$ before it reaches level y, and since then it should never go below level $y - r$ before it reaches level y. Apply the strong Markov property, we have

$$\begin{aligned} & \mathbb{E}_x\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)}; X_{\theta_r} \in dy\right]/dy \\ & = \mathbb{E}_x\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)}; X_{\eta_r} \in dy - r, X_{\theta_r} \in dy\right]/dy \\ & = \mathbb{E}_x\left[e^{-\alpha T_{y-r}}; T_{y-r} < T_y\right] \cdot \mathbb{E}_{y-r}\left[e^{-\beta T_y}; T_y < T_{y-r-dy}\right]/dy. \end{aligned}$$

By Lemma 2.1 and Proposition 2.4, it follows that

$$\mathbb{E}_x\left[e^{-\alpha T_{y-r}}; T_{y-r} < T_y\right] = \frac{p_\alpha(x,y)}{p_\alpha(y-r,y)},$$

and

$$\mathbb{E}_{y-r}\left[e^{-\beta T_y}; T_y < T_{y-r-dy}\right]/dy = \frac{q_\beta(y-r)}{p_\beta(y-r,y)}.$$

Therefore, for $y \geq x$,

$$\mathbb{E}_x\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)}; X_{\theta_r} \in dy\right] = \frac{p_\alpha(x,y)q_\beta(y-r)}{p_\alpha(y-r,y)p_\beta(y-r,y)}dy.$$

Then we prove (3.1) for $y < x$. Note that if $X_{\theta_r} \in dy$, then $X_{\eta_r} \in dy + r$. This means that the process must first reach level $y + r$ before it reaches level y, and then it should never go above level $y + r$ before it reaches level y. Similarly, we apply the strong Markov property to get

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)}; X_{\theta_r} \in dy\right]/dy \\
&= \mathbb{E}_x\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)}; X_{\eta_r} \in dy + r, X_{\theta_r} \in dy\right]/dy \\
&= \mathbb{E}_x\left[e^{-\alpha T_{y+r}}; T_{y+r} < T_y\right]\cdot\mathbb{E}_{y+r}\left[e^{-\beta T_y}; T_y < T_{y+r+dy}\right]/dy.
\end{aligned}$$

By Lemma 2.1 and Proposition 2.4, we have

$$\mathbb{E}_x\left[e^{-\alpha T_{y+r}}; T_{y+r} < T_y\right] = \frac{p_\alpha(y,x)}{p_\alpha(y,y+r)},$$

and

$$\mathbb{E}_{y+r}\left[e^{-\beta T_y}; T_y < T_{y+r+dy}\right]/dy = \frac{q_\beta(y+r)}{p_\beta(y, y+r)}.$$

Therefore, for $y < x$,

$$\mathbb{E}_x\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)}; X_{\theta_r} \in dy\right] = \frac{p_\alpha(y, x)q_\beta(y+r)}{p_\alpha(y, y+r)p_\beta(y, y+r)}dy.$$

Putting both parts together, we obtain the result (3.1).

□

Remark 3.2. From Theorem 3.1 we can see that the problem on the first range time is transferred to the problem of solving an ordinary differential equation (2.5).

Remark 3.3. If the diffusion process X_t is a circular diffusion on a circle with perimeter r, then the first range time θ_r is actually the first time when the process finishes visiting every point on the circle.

Remark 3.4. Usually θ_r does not have independent increments. However, in the Brownian motion cases, it does have because a Brownian motion is spacially homogenous.

Given x and $X_{\theta_r} \in dy$, the factorization of the joint Laplace transform (3.1) into factors depending solely on α and on β shows that η_r and $\theta_r - \eta_r$ are conditionally independent. Then for fixed x, the Laplace transform of η_r given $X_{\theta_r} \in dy$ and the Laplace transform of $\theta_r - \eta_r$ given $X_{\theta_r} \in dy$ could be obtained directly from (3.1).

Corollary 3.5. *For $\alpha, \beta, \gamma \geq 0$,*

$$\mathbb{E}_x\left[e^{-\alpha\eta_r}; X_{\theta_r} \in dy\right] = \frac{p_\alpha(x,y)}{p_\alpha(y-r,y)} 1_{\{y \geq x\}} dy + \frac{p_\alpha(y,x)}{p_\alpha(y,y+r)} 1_{\{y<x\}} dy, \tag{3.2}$$

$$\mathbb{E}_x\left[e^{-\beta(\theta_r-\eta_r)}; X_{\theta_r} \in dy\right] = \frac{q_\beta(y-r)}{p_\beta(y-r,y)} 1_{\{y \geq x\}} dy + \frac{q_\beta(y+r)}{p_\beta(y,y+r)} 1_{\{y<x\}} dy, \tag{3.3}$$

$$\mathbb{E}_x\left[e^{-\gamma\theta_r}; X_{\theta_r} \in dy\right] = \frac{p_\gamma(x,y) q_\gamma(y-r)}{(p_\gamma(y-r,y))^2} 1_{\{y \geq x\}} dy + \frac{p_\gamma(y,x) q_\gamma(y+r)}{(p_\gamma(y,y+r))^2} 1_{\{y<x\}} dy. \tag{3.4}$$

Remark 3.6. Comparing (3.2) with the results in Lemma 2.1, we can see that given $X_{\theta_r} \in dy$, η_r is just the exit time from the interval between X_{η_r} and X_{θ_r} when exiting occurs at X_{η_r}.

Remark 3.7. If we integrate (3.4) with respect to y, we will obtain the Laplace transform of the first range time θ_r. It also implies a result on the distribution of the range R_t since we know that $\mathbb{P}\{R_t < r\} = \mathbb{P}\{\theta_r > t\}$.

If we further investigate (3.3), we would see that $\theta_r - \eta_r$ is actually the duration from the time at which the process is at the maximum (or minimum) to the time at which the process is at the minimum (or maximum). In the following corollary, we discuss a related result. A time interval [0, T] is given in advance with T exponentially distributed. The process starting at x cannot exit a given interval before time T.

Proposition 3.8. *Given a random variable T which is independent of X and follows an exponential distribution with parameter λ, define τ_1 and τ_2 as*

$$\tau_1 = \inf\{t \geq 0 : X_t = I_T\} \qquad \text{and} \qquad \tau_2 = \inf\{t \geq 0 : X_t = S_T\}.$$

Then we have for $y < x < z$,

$$\begin{aligned} &\mathbb{E}_x\left[e^{-\alpha\tau_1-\beta(\tau_2-\tau_1)}; X_{\tau_1} \in dy, X_{\tau_2} \in dz, \tau_1 < \tau_2 < T\right] \\ &= \frac{p_{\alpha+\lambda}(x,z)}{p_{\alpha+\lambda}(y,z)} \cdot \frac{q_{\beta+\lambda}(y)}{p_{\beta+\lambda}(y,z)} \cdot \frac{\partial p_\lambda(y,z)/\partial z - q_\lambda(z)}{p_\lambda(y,z)} dydz \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &\mathbb{E}_x\left[e^{-\alpha\tau_2-\beta(\tau_1-\tau_2)}; X_{\tau_1} \in dy, X_{\tau_2} \in dz, \tau_2 < \tau_1 < T\right] \\ &= \frac{p_{\alpha+\lambda}(y,x)}{p_{\alpha+\lambda}(y,z)} \cdot \frac{q_{\beta+\lambda}(z)}{p_{\beta+\lambda}(y,z)} \cdot \frac{-\partial p_\lambda(y,z)/\partial y - q_\lambda(y)}{p_\lambda(y,z)} dydz. \end{aligned} \tag{3.6}$$

Proof. Notice that for an exponential random variable T with parameter λ and an independent random variable Y, we have $\mathbb{P}\{Y < T\} = \mathbb{E}\left[e^{-\lambda Y}\right]$.

Apply the strong Markov property and the memoryless property of the exponential distribution to get

$$\mathbb{E}_x\left[e^{-\alpha\tau_1-\beta(\tau_2-\tau_1)}; X_{\tau_1}\in dy, X_{\tau_2}\in dz, \tau_1<\tau_2<T\right]$$

$$= \mathbb{E}_x\left[e^{-\alpha\tau_1}1_{\{\tau_1<T\}}1_{\{X_{\tau_1}\in dy\}}\cdot e^{-\beta(\tau_2-\tau_1)}1_{\{\tau_1+(\tau_2-\tau_1)<T\}}1_{\{\tau_2-\tau_1>0\}}1_{\{X_{\tau_1+(\tau_2-\tau_1)}\in dz\}}\right]$$

$$= \mathbb{E}_x\left[e^{-\alpha T_y}1_{\{T_y<T\}}; T_y<T_z\right]\cdot\mathbb{E}_y\left[e^{-\beta T_z}1_{\{T_z<T\}}; T_z<T_{y-dy}\right]\cdot\mathbb{P}_z\{S_T\le z+dz, I_T\ge y\}$$

$$= \mathbb{E}_x\left[e^{-(\alpha+\lambda)T_y}; T_y<T_z\right]\cdot\mathbb{E}_y\left[e^{-(\beta+\lambda)T_z}; T_z<T_{y-dy}\right]\cdot\mathbb{P}_z\{S_T\le z+dz, I_T\ge y\}.$$

By Lemma 2.1, we have

$$\mathbb{E}_x\left[e^{-(\alpha+\lambda)T_y}; T_y<T_z\right] = \frac{p_{\alpha+\lambda}(x,z)}{p_{\alpha+\lambda}(y,z)}.$$

By Proposition 2.4 and Proposition 2.5, we have for $dy, dz > 0$,

$$\mathbb{E}_y\left[e^{-(\beta+\lambda)T_z}; T_z<T_{y-dy}\right] = \frac{q_{\beta+\lambda}(y)}{p_{\beta+\lambda}(y,z)}dy,$$

and

$$\mathbb{P}_z\{S_T\le z+dz, I_T\ge y\} = \frac{\partial p_\lambda(y,z)/\partial z - q_\lambda(z)}{p_\lambda(y,z)}dz.$$

Therefore,

$$\mathbb{E}_x\left[e^{-\alpha\tau_1-\beta(\tau_2-\tau_1)}; X_{\tau_1}\in dy, X_{\tau_2}\in dz, \tau_1<\tau_2<T\right]$$
$$= \frac{p_{\alpha+\lambda}(x,z)}{p_{\alpha+\lambda}(y,z)}\cdot\frac{q_{\beta+\lambda}(y)}{p_{\beta+\lambda}(y,z)}\cdot\frac{\partial p_\lambda(y,z)/\partial z - q_\lambda(z)}{p_\lambda(y,z)}dydz.$$

Then we prove the second part, i.e. $\tau_2<\tau_1$, by a similar approach.

$$\mathbb{E}_x\left[e^{-\alpha\tau_2-\beta(\tau_1-\tau_2)}; X_{\tau_2}\in dz, X_{\tau_1}\in dy, \tau_2<\tau_1<T\right]$$

$$= \mathbb{E}_x\left[e^{-\alpha\tau_2}1_{\{\tau_2<T\}}1_{\{X_{\tau_2}\in dz\}}\cdot e^{-\beta(\tau_1-\tau_2)}1_{\{\tau_2+(\tau_1-\tau_2)<T\}}1_{\{\tau_1-\tau_2>0\}}1_{\{X_{\tau_2+(\tau_1-\tau_2)}\in dy\}}\right]$$

$$= \mathbb{E}_x\left[e^{-\alpha T_z}1_{\{T_z<T\}}; T_z<T_y\right]\cdot\mathbb{E}_z\left[e^{-\beta T_y}1_{\{T_y<T\}}; T_y<T_{z+dz}\right]\cdot\mathbb{P}_y\{S_T\le z, I_T\ge y-dy\}$$

$$= \mathbb{E}_x\left[e^{-(\alpha+\lambda)T_z}; T_z<T_y\right]\cdot\mathbb{E}_z\left[e^{-(\beta+\lambda)T_y}; T_y<T_{z+dz}\right]\cdot\mathbb{P}_y\{S_T\le z, I_T\ge y-dy\}.$$

By Lemma 2.1, we have

$$\mathbb{E}_x\left[e^{-(\alpha+\lambda)T_z}; T_z<T_y\right] = \frac{p_{\alpha+\lambda}(y,x)}{p_{\alpha+\lambda}(y,z)}.$$

By Proposition 2.4 and Proposition 2.5, we have for $dy, dz>0$,

$$\mathbb{E}_z\left[e^{-(\beta+\lambda)T_y}; T_y<T_{z+dz}\right] = \frac{q_{\beta+\lambda}(z)}{p_{\beta+\lambda}(y,z)}dz,$$

and

$$\mathbb{P}_y\{S_T\le z, I_T\ge y-dy\} = \frac{-\partial p_\lambda(y,z)/\partial y - q_\lambda(y)}{p_\lambda(y,z)}dy.$$

Therefore,

$$\mathbb{E}_x\left[e^{-\alpha\tau_2-\beta(\tau_1-\tau_2)}; X_{\tau_2}\in dz, X_{\tau_1}\in dy, \tau_2<\tau_1<T\right]$$
$$= \frac{p_{\alpha+\lambda}(y,x)}{p_{\alpha+\lambda}(y,z)}\cdot\frac{q_{\beta+\lambda}(z)}{p_{\beta+\lambda}(y,z)}\cdot\frac{-\partial p_\lambda(y,z)/\partial y - q_\lambda(y)}{p_\lambda(y,z)}dydz.$$

□

4. Some examples

In this section we will show some examples by applying Theorem 3.1 to some well-known diffusion processes. We first consider Brownian motion for which the result has already been obtained by Chong, Cowan and Holst (2000), and then we consider geometric Brownian motion, Ornstein-Uhlenbeck processes and squared Bessel processes.

4.1. **Brownian motion with drift.** For Brownian motion X_t with drift such that

$$dX_t = \mu dt + \sigma dW_t, \tag{4.1}$$

we have $\alpha(x) = \mu$ and $\sigma(x) = \sigma$. The ordinary differential equation (2.5) becomes

$$\frac{1}{2}\sigma^2 f''(x) + \mu f'(x) = \beta f(x).$$

It has two independent solutions given by

$$g_\beta(x) = e^{-(\gamma-\delta_\beta)x} \qquad \text{and} \qquad h_\beta(x) = e^{-(\gamma+\delta_\beta)x},$$

where $\gamma = \mu/\sigma^2$ and $\delta_\beta = \sqrt{\gamma^2 + 2\beta/\sigma^2}$.

In the following we will first apply the result in Theorem 3.1, and then take an integral to obtain the result in Theorem 1.1 [Chong, Cowan and Holst (2000)].

Corollary 4.1. *For Brownian motion X_t with drift μ and variance $\sigma^2 > 0$, we have*

$$\mathbb{E}_x\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)}; X_{\theta_r} \in dy\right] = \frac{\delta_\beta e^{-\gamma(x-y)}\sinh[|x-y|\delta_\alpha]}{\sinh[r\delta_\alpha]\sinh[r\delta_\beta]}dy \tag{4.2}$$

with $\gamma = \mu/\sigma^2$ *and* $\delta_\beta = \sqrt{\gamma^2 + 2\beta/\sigma^2}$, *and*

$$\mathbb{E}_x\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)-\nu X_{\theta_r}}\right] = \frac{2\kappa(\beta)e^{-\nu x}}{\sinh[\kappa(\alpha)]\sinh[\kappa(\beta)]}\left[\frac{\sinh^2[\frac{1}{2}(\kappa(\alpha)+\rho)]}{\kappa(\alpha)+\rho} + \frac{\sinh^2[\frac{1}{2}(\kappa(\alpha)-\rho)]}{\kappa(\alpha)-\rho}\right] \tag{4.3}$$

for any $\alpha, \beta \geq 0$ *and* ν *with* $\rho = r\mu/\sigma^2 - r\nu$ *and* $\kappa(x) = r/\sigma\sqrt{\mu^2/\sigma^2 + 2x}$.

Proof. We start with the first part of (3.1) in Theorem 3.1. For $y \geq x$,

$$\begin{aligned}
&\frac{p_\alpha(x,y)q_\beta(y-r)}{p_\alpha(y-r,y)p_\beta(y-r,y)} \\
&= \frac{g_\alpha(x)h_\alpha(y) - g_\alpha(y)h_\alpha(x)}{g_\alpha(y-r)h_\alpha(y) - g_\alpha(y)h_\alpha(y-r)} \cdot \frac{g_\beta(y-r)h'_\beta(y-r) - g'_\beta(y-r)h_\beta(y-r)}{g_\beta(y-r)h_\beta(y) - g_\beta(y)h_\beta(y-r)} \\
&= \frac{e^{-(\gamma-\delta_\alpha)x}e^{-(\gamma+\delta_\alpha)y} - e^{-(\gamma-\delta_\alpha)y}e^{-(\gamma+\delta_\alpha)x}}{e^{-(\gamma-\delta_\alpha)(y-r)}e^{-(\gamma+\delta_\alpha)y} - e^{-(\gamma-\delta_\alpha)y}e^{-(\gamma+\delta_\alpha)(y-r)}} \\
&\quad \cdot \frac{e^{-(\gamma-\delta_\beta)(y-r)}e^{-(\gamma+\delta_\beta)(y-r)}[-(\gamma+\delta_\beta)] - e^{-(\gamma-\delta_\beta)(y-r)}e^{-(\gamma+\delta_\beta)(y-r)}[-(\gamma-\delta_\beta)]}{e^{-(\gamma-\delta_\beta)(y-r)}e^{-(\gamma+\delta_\beta)y} - e^{-(\gamma-\delta_\beta)y}e^{-(\gamma+\delta_\beta)(y-r)}} \\
&= \frac{e^{-\gamma(x+y)} \cdot 2\sinh[(x-y)\delta_\alpha]}{e^{-\gamma(2y-r)} \cdot 2\sinh[(-r)\delta_\alpha]} \cdot \frac{-2\delta_\beta e^{-2\gamma(y-r)}}{e^{-\gamma(2y-r)} \cdot 2\sinh[(-r)\delta_\beta]} \\
&= \frac{e^{\gamma(y-x)} \cdot 2\sinh[(y-x)\delta_\alpha]}{2\sinh[r\delta_\alpha]} \cdot \frac{\delta_\beta}{\sinh[r\delta_\beta]} \\
&= \frac{\delta_\beta e^{-\gamma(x-y)}\sinh[(y-x)\delta_\alpha]}{\sinh[r\delta_\alpha]\sinh[r\delta_\beta]}.
\end{aligned}$$

For the second part, we can use the same procedure as we did in the first part. However, we can also choose a shortcut by the interplay of the dual processes. Let $X_t^* = -X_t$, then X_t^* starts at $-x$ with drift $\mu^* = -\mu$. But the corresponding θ_r and η_r will stay the same because a dual process has the same range process R_t.

Then we have

$$\mathbb{E}_x\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)}; X_{\theta_r}\in dy\right]1_{\{y<x\}} = \mathbb{E}^*_{-x}\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)}; X_{\theta_r}\in d(-y)\right]1_{\{-y>-x\}}.$$

Taking use of the result in the first part, replacing μ with $-\mu$, x with $-x$ and y with $-y$, we obtain

$$\mathbb{E}_x\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)}; X_{\theta_r}\in dy\right]1_{\{y<x\}} = \frac{\delta_\beta e^{-\gamma(x-y)}\sinh[(x-y)\delta_\alpha]}{\sinh[r\delta_\alpha]\sinh[r\delta_\beta]}.$$

Hence (4.2) follows.

To prove (4.3), we integrate (4.2) with respect to y . First add the term X_{θ_r} to the exponent. For the first part,we integrate y over the interval $[x, x+r]$.

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)-\nu X_{\theta_r}}; X_{\theta_r}\geq x\right]\\
&= \int_x^{x+r}\frac{p_\alpha(x,y)q_\beta(y-r)}{p_\alpha(y-r,y)p_\beta(y-r,y)}e^{-\nu y}dy\\
&= \frac{\delta_\beta e^{-\gamma x}}{2\sinh[r\delta_\alpha]\sinh[r\delta_\beta]}\cdot\int_x^{x+r}e^{(\gamma-\nu)y}\left(2\sinh[(y-x)\delta_\alpha]\right)dy\\
&= \frac{\delta_\beta e^{-\gamma x}}{2\sinh[r\delta_\alpha]\sinh[r\delta_\beta]}\cdot\int_x^{x+r}e^{(\gamma-\nu)y}\left(e^{(y-x)\delta_\alpha}-e^{-(y-x)\delta_\alpha}\right)dy\\
&= \frac{\delta_\beta e^{-\gamma x}}{2\sinh[r\delta_\alpha]\sinh[r\delta_\beta]}\cdot\int_x^{x+r}\left(e^{(\delta_\alpha+\gamma-\nu)y}\cdot e^{-x\delta_\alpha}-e^{(-\delta_\alpha+\gamma-\nu)y}\cdot e^{x\delta_\alpha}\right)dy\\
&= \frac{\delta_\beta e^{-\nu x}}{2\sinh[r\delta_\alpha]\sinh[r\delta_\beta]}\cdot\left[\frac{e^{r(\delta_\alpha+\gamma-\nu)}-1}{\delta_\alpha+\gamma-\nu}+\frac{e^{r(-\delta_\alpha+\gamma-\nu)}-1}{\delta_\alpha-\gamma+\nu}\right].
\end{aligned}$$

For the second part, we can also use the dual process. We only need to replace μ by $-\mu$, x by $-x$ and ν by $-\nu$. The last replacement is because that the sign of

X_{θ_r} changes. Thus

$$\mathbb{E}_x\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)-\nu X_{\theta_r}}; X_{\theta_r} < x\right]$$
$$= \frac{\delta_\beta e^{-\nu x}}{2\sinh[r\delta_\alpha]\sinh[r\delta_\beta]} \cdot \left[\frac{e^{-r(-\delta_\alpha+\gamma-\nu)}-1}{(\delta_\alpha-\gamma+\nu)} + \frac{e^{-r(\delta_\alpha+\gamma-\nu)}-1}{(\delta_\alpha+\gamma-\nu)}\right].$$

Then we add the two parts to obtain

$$\mathbb{E}_x\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)-\nu X_{\theta_r}}\right]$$
$$= \frac{\delta_\beta e^{-\nu x}}{2\sinh[r\delta_\alpha]\sinh[r\delta_\beta]}\left[\frac{e^{r(\delta_\alpha+\gamma-\nu)}+e^{-r(\delta_\alpha+\gamma-\nu)}-2}{(\delta_\alpha+\gamma-\nu)} + \frac{e^{r(-\delta_\alpha+\gamma-\nu)}+e^{-r(-\delta_\alpha+\gamma-\nu)}-2}{(\delta_\alpha-\gamma+\nu)}\right]$$
$$= \frac{r\delta_\beta e^{-\nu x}}{2\sinh[r\delta_\alpha]\sinh[r\delta_\beta]}\left[\frac{2\cosh[r(\delta_\alpha+\gamma-\nu)]-2}{r(\delta_\alpha+\gamma-\nu)} + \frac{2\cosh[r(\delta_\alpha-\gamma+\nu)]-2}{r(\delta_\alpha-\gamma+\nu)}\right]$$
$$= \frac{2r\delta_\beta e^{-\nu x}}{\sinh[r\delta_\alpha]\sinh[r\delta_\beta]}\left[\frac{\sinh^2[\frac{1}{2}r(\delta_\alpha+\gamma-\nu)]}{r(\delta_\alpha+\gamma-\nu)} + \frac{\sinh^2[\frac{1}{2}r(\delta_\alpha-\gamma+\nu)]}{r(\delta_\alpha-\gamma+\nu)}\right].$$

Finally the result (4.3) follows after denoting

$$\rho := r(\gamma-\nu) = r\mu/\sigma^2 - r\nu,$$

$$\kappa(x) := r\delta_x = r/\sigma\sqrt{\mu^2/\sigma^2+2x}.$$

□

Remark 4.2. There are two results (4.2) and (4.3) in the above corollary. (4.3) is the same as Theorem 1.1 except that $X_0 = x$ instead of $X_0 = 0$. Then the term $e^{-\nu x}$ appears, which indicates that it is actually only a shift from the case $X_0 = 0$ to the case $X_0 = x$. The reason is that a Brownian motion is a spacially homogenous Lévy process. We can see this from the SDE (4.1) that $\alpha(x) = \mu$

and $\sigma(x) = \sigma$, both of which do not depend on x. It follows that η_r and $\theta_r - \eta_r$ do not depend on x either. Therefore, it can be treated as a shift if we change the starting point.

4.2. **Geometric Brownian motion.** Consider geometric Brownian motion X_t with

$$dX_t = \mu X_t dt + \sigma X_t dW_t\,, \tag{4.4}$$

or equivalently,

$$X_t = X_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right). \tag{4.5}$$

Then we know that $\alpha(x) = \mu x$ and $\sigma(x) = \sigma x$. The ordinary differential equation (2.5) becomes

$$\frac{1}{2}\sigma^2 x^2 f''(x) + \mu x f'(x) = \beta f(x).$$

To solve this differential equation, let $x = e^t$ and we will obtain the following two independent solutions

$$g_\beta(x) = e^{-(\gamma-\delta_\beta)\ln x} = x^{-(\gamma-\delta_\beta)}, \qquad h_\beta(x) = e^{-(\gamma+\delta_\beta)\ln x} = x^{-(\gamma+\delta_\beta)},$$

where $\gamma = \mu/\sigma^2 - 1/2$ and $\delta_\beta = \sqrt{\gamma^2 + 2\beta/\sigma^2}$.

Corollary 4.3. *For geometric Brownian motion defined via (4.4) and $x, y > 0$, we have*

$$\begin{aligned}\mathbb{E}_x\left[e^{-\alpha\eta_r-\beta(\theta_r-\eta_r)}; X_{\theta_r}\in dy\right] &= \frac{2\delta_\beta(\frac{x}{y})^{-\gamma}\psi_\alpha(x,y)}{(y-r)\psi_\alpha(y-r,y)\psi_\beta(y-r,y)}1_{\{y\geq x\}}dy\\ &+\frac{2\delta_\beta(\frac{x}{y})^{-\gamma}\psi_\alpha(y,x)}{(y+r)\psi_\alpha(y,y+r)\psi_\beta(y,y+r)}1_{\{y<x\}}dy\end{aligned} \tag{4.6}$$

for $t, u \geq 0$ with $\psi_\beta(x,y) = (y/x)^{\delta_\beta} - (y/x)^{-\delta_\beta}$, $\gamma = \mu/\sigma^2 - \frac{1}{2}$ and $\delta_\beta = \sqrt{\gamma^2 + 2\beta/\sigma^2}$.

Proof. We can use Theorem 3.1 to derive this result. However, we choose another way by using (4.5) which indicates that for fixed t geometric Brownian motion is a one-to-one mapping of Brownian motion. Let $\tilde{\mu} = \mu - \sigma^2/2$, then $\gamma = \tilde{\mu}/\sigma^2$ which is of the same form as that for the Brownian motion. δ_β also has the same form.

After comparing the Brownian motion with the geometric Brownian motion, we will see that we only need to replace x with 0, y with $\ln(y/x)$, and r with

$$(\ln(y/x) - \ln((y-r)/x))1_{\{y\geq x\}} + (\ln((y+r)/x) - \ln(y/x))1_{\{y<x\}}$$

in (4.2) to find the result for the geometric Brownian motion. Notice that r is random w.r.t. y instead of being a constant, and

$$dr = (1/y - 1/(y-r))1_{\{y\geq x\}}dy + (1/(y+r) - 1/y)1_{\{y<x\}}dy.$$

When we apply Theorem 3.1 we need to pay attention to the right side of (3.1) which is actually

$$\frac{p_\alpha(x,y)q_\beta(y-r)}{p_\alpha(y-r,y)p_\beta(y-r,y)}1_{\{y\geq x\}}d(y-r)+\frac{p_\alpha(y,x)q_\beta(y+r)}{p_\alpha(y,y+r)p_\beta(y,y+r)}1_{\{y<x\}}d(y+r).$$

Hence in (4.2) we replace dy with

$$1/(y-r)1_{\{y\geq x\}}dy+1/(y+r)1_{\{y<x\}}dy.$$

After careful substitution we will obtain the result (4.6). It can also be checked by direct computation from Theorem 3.1.

□

Remark 4.4. We will not consider the integral for the geometric Brownian motion case as we did in the Brownian motion case, because a geometric Brownian motion is always positive, but $x-r$ might be negative or zero. However, this problem does not exist in the Brownian motion case.

4.3. **Ornstein-Uhlenbeck processes.** Consider an Ornstein-Uhlenbeck process X_t with

$$dX_t=-\lambda X_t dt+\sigma dW_t. \tag{4.7}$$

To solve this stochastic differential equation, we apply the product rule to get

$$d(e^{\lambda t}X_t)=e^{\lambda t}dX_t+e^{\lambda t}\lambda X_t dt=e^{\lambda t}\sigma dW_t.$$

Then take an integral,

$$e^{\lambda t}X_t - X_0 = \int_0^t \sigma e^{\lambda s} dW_s.$$

Thus an equivalent form to (4.7) is

$$X_t = e^{-\lambda t}X_0 + e^{-\lambda t}\int_0^t \sigma e^{\lambda s} dW_s. \tag{4.8}$$

From (4.7) we have $\alpha(x) = -\lambda x$ and $\sigma(x) = \sigma$. Thus the ordinary differential equation (2.5) becomes

$$\frac{1}{2}\sigma^2 f''(x) - \lambda x f'(x) = \beta f(x).$$

Then we solve this equation to obtain the following two independent solutions (see Zauderer 1989)

$$g_\beta(x) = e^{\frac{\gamma x^2}{2}} D_{-\delta_\beta}(-\sqrt{2\gamma}x), \qquad h_\beta(x) = e^{\frac{\gamma x^2}{2}} D_{-\delta_\beta}(\sqrt{2\gamma}x)$$

where $\gamma = \lambda/\sigma^2$, $\delta_\beta = \beta/\lambda$ and $D_{-\nu}(x)$ is the parabolic cylinder function (see Zauderer 1989) defined as

$$D_{-\nu}(x) := e^{-x^2/4}2^{-\nu/2}\sqrt{\pi}\left\{\frac{1}{\Gamma((\nu+1)/2)}\left(1 + \sum_{k=1}^{\infty}\frac{\nu(\nu+2)\cdots(\nu+2k-2)}{(2k)!}x^{2k}\right)\right.$$
$$\left. - \frac{x\sqrt{2}}{\Gamma(\nu/2)}\left(1 + \sum_{k=1}^{\infty}\frac{(\nu+1)(\nu+3)\cdots(\nu+2k-1)}{(2k+1)!}x^{2k}\right)\right\}.$$

The joint Laplace transform for the Ornstein-Uhlenbeck processes could be obtained, but the expression cannot be simplified further.

4.4. **Squared Bessel processes.** Consider a squared Bessel process X_t with

$$dX_t = (2\nu + 2)dt + 2\sqrt{X_t}dW_t. \tag{4.9}$$

Now we have $\alpha(x) = 2\nu + 2$ and $\sigma(x) = 2\sqrt{x}$. The ordinary differential equation (2.5) becomes

$$2xf''(x) + (2\nu + 2)f'(x) = \beta f(x). \tag{4.10}$$

To solve this equation, by Zauderer (1989) we would obtain the following two independent solutions

$$g_\beta(x) = x^{-\frac{\nu}{2}} I_\nu(\sqrt{2\beta x}), \qquad h_\beta(x) = x^{-\frac{\nu}{2}} K_\nu(\sqrt{2\beta x})$$

where $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions (see Zauderer 1989) defined as

$$I_\nu(x) := \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{k!\Gamma(\nu + k + 1)},$$

and

$$K_\nu(x) := \frac{\pi}{2\sin(\nu\pi)}(I_{-\nu}(x) - I_\nu(x)).$$

The joint Laplace transform for the squared Bessel processes could be obtained, but the expression cannot be simplified further. However, we can still obtain some simpler results.

Corollary 4.5. *For a squared Bessel process with (4.9), we have for $v \geq 0$,*

$$\begin{aligned} \mathbb{P}_x\{X_{\theta_r} \in dy\} = & \frac{-v(y^{-v} - x^{-v})(y - r)^{-v-1}}{(y^{-v} - (y - r)^{-v})^2} 1_{\{y \geq x\}} dy \\ & + \frac{-v(x^{-v} - y^{-v})(y + r)^{-v-1}}{((y + r)^{-v} - y^{-v})^2} 1_{\{y < x\}} dy \end{aligned} \tag{4.11}$$

Proof. Set $\alpha = \beta = 0$ in (3.1), then we obtain the result after some computation.

□

Notice that for a squared Bessel process (4.9), $\alpha(x)$ and $\sigma(x)$ do not satisfy the conditions (2.2) of the existence and uniqueness theorem, but we can still find two independent solutions to (4.10), and then find the joint Laplace transform. Therefore, the conditions (2.2) are sufficient but not necessary. From Remark 2.3 we can see that the joint Laplace transform can be obtained as long as we can find two independent solutions to the ordinary differential equation (2.5).

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