ON THE SYMMETRIZING TRANSFORMATIONS OF RANDOM VARIABLES

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ABSTRACT

The variance stabilizing transformations formally introduced by Bartlett (1947) are often seen to also approximately normalize the random variable. This is so not due to the variance stabilizing property, but because these transformations often induce approximate symmetry. In this note, we obtain a condition under which the variance stabilizing transformation is also an approximately symmetrizing transformation and examine some familiar transformations in this light. We also construct a differential equation, analogous to Bartlett's for obtaining an approximately symmetrizing transformation and illustrate it in terms of common cases.

Key words: variance stabilizing transformation, symmetrizing transformation.

1. INTRODUCTION

The earliest consideration of a transformation which stabilizes the variance is due to R.A. Fisher when he proposed $Z = \tanh^{-1} r$ in 1915 and $\sqrt{2\chi_\nu^2 - 1}$ in 1922 as approximately normalizing transformations of the correlation coefficient r and the χ_ν^2 variable respectively. Bartlett (1947) introduced variance stabilizing transformations formally for the purpose of utilizing the usual analysis of variance in the absence of homoscedasticity. He showed how to derive these using a differential equation, and, as illustrations, confirmed the variance stabilizing character of z and $\sqrt{2\chi_\nu^2}$ and gave many additional examples including the square root of a Poisson random variable and the function arcsin $\sqrt{p}$ of the binomial sample proportion p. Since then, these transformations have been variously studied and refined essentially with a view to improving normality. Thus, Anscomb (1948) improved $\sqrt{X}$ of the Poisson variable X to $\sqrt{X + (3/8)}$, arcsin $\sqrt{p}$ to arcsin $\sqrt{(p + (3/8))/\ (1 + (3/4))}$, and Hotelling (1953) in his definitive study of the distribution of the correlation coefficient, proposed numerous improvements of Z.

Now, we note that even though many variance stabilizing transformations of random variables have near normal distributions and they simplify the inference problems such as confidence interval estimation of the parameter, the stability of variance is not necessary for normality. However, approximate symmetry is clearly a prerequisite of any approximately normalizing transformation. Hence, an approximately symmetrizing transformation of a random variable may be a more effective method of normalizing it than stabilizing its variance. Historically, this was first illustrated by Wilson and Hilferty (1931), who showed that the cube root of a chi square variable obtained by them as an approximately symmetrizing power-transformation provides a normal approximation superior to that based on Fisher's variance stabilizing transformation. Their approach of constructing a skewness reducing power transformation has now been extended to many other distributions, e.g. to noncentral chi square by Sankaran (1959), to quadratic forms by Jensen and Solomon (1972), to sample variance from non-normal populations and multivariate likelihood ratio statistics by Mudholkar and Trivedi (1980, 1981, 1982). In the next section, we present a condition under which the variance stabilizing transformation is also an approximately symmetrizing transformation, and examine some familiar transformations in this light. The final section, contains a differential equation constructed analogous to Bartlett's, which gives an approximately symmetrizing transformation.

2. A CONDITION UNDER WHICH A VARIANCE STABILIZING TRANSFORMATION IS ALSO AN APPROXIMATE SYMMETRIZING TRANSFORMATION.

Let $\{X_n\}_1^\infty$ be a sequence of random variables such that $\sqrt{n}\,(X_n - \theta) \sim N(0, \sigma^2(\theta))$. Let also $\xi_3(\theta)$ and $\xi_4(\theta)$ be moments of X_n around θ, then using Taylor's formula we can write for any smooth function $g(X_n)$,

$$g(X_n) - g(\theta) = \sum_{j=1}^{4} (X_n - \theta)^j \frac{1}{j!} \frac{d^j g(\theta)}{d\theta^j} + R_n, \qquad (2.1)$$

where $R_n \to 0$ in probability as $n \to \infty$. Using (2.1) we can write, approximately (upto $0(\frac{1}{n^2})$)

$$\mu_3\,(g(X_n)) = g'^2\{g' f_1(\theta) + \frac{3}{2} g'' f_2(\theta)\}, \qquad (2.2)$$

where $f_1(\theta) = \xi_3(\theta) - 3\xi_1(\theta)\xi_2(\theta)$, $f_2(\theta) = \xi_4(\theta) - \xi_2^2(\theta)$, $\xi_j(\theta) = E\,(X_n - \theta)^j$, $j = 1, 2, 3, 4$. Now if $g(X_n)$ is a variance stabilizing transformation, we have,

$$g'(\theta) = c/\sigma(\theta). \qquad (2.3)$$

If this g is also approximately symmetrizing, then

$$\mu_3\,(g(X_n)) = 0$$

i.e. $\frac{1}{\sigma(\theta)}\left\{f_1(\theta) - \frac{3}{2} f_2(\theta) \frac{d}{d\theta} \ln \sigma(\theta)\right\} = 0$

i.e. $$\frac{d}{d\theta} \ln \sigma(\theta) = \frac{2}{3} f_1(\theta)/f_2(\theta). \qquad (2.4)$$

Thus, to be able to get a variance stabilizing transformation which will also be symmetric, 1st four moments of X_n must satisfy (2.4). Now, we examine some transformations in this light.

2.2.1 Fisher transformation of Correlation Coefficient

From Hotelling (1953) we get up to order $0(\frac{1}{n^2})$,

$$f_1(\rho) = -6\rho(1-\rho^2)^3/n^2,$$

$$f_2(\rho) = 2(1-\rho^2)^4/n^2,$$

and $\sigma(\rho) = (1-\rho^2)$. We find that (2.4) is satisfied because both sides of (2.4) equal $-2\rho/(1-\rho^2)$.

2.2.2 Binomial Proportion

Let p be the sample proportion from a Binomial population with n trials and probability of success θ, then from Kendall (1979, p.p. 121) we obtain

$$f_1(\theta) = \theta(1-\theta)(1-2\theta)/n^2,$$

$$f_2(\theta) = 2\theta^2(1-\theta)^2/n^2,$$

and $\sigma(\theta) = \sqrt{\theta(1-\theta)}$. In this case the condition (2.4) is not satisfied because $\frac{2}{3}\, f_1(\theta)/f_2(\theta) = \frac{2}{3}\, \frac{d\ln\sigma(\theta)}{d\theta}$ Hence arcsin $\sqrt{p}$ is not a "good" symmetrizing transformation. This comment should be understood in view of the fact that there may be other transformations with considerably smaller skewness than arcsin $\sqrt{p}$.

2.2.3 Poisson Variable

Let X have a Poisson variable with mean θ, then $f_1(\theta) = \theta$, $f_2(\theta) = \theta + 2\theta^2$ and $\sigma(\theta) = \sqrt{\theta}$. Since,

$\frac{d \ \ln \sqrt{\theta}}{d\theta} = \frac{1}{2\theta}$ and $\frac{2}{3} f_1(\theta)/f_2(\theta) = \frac{2}{3}/(1+2\theta)$, the condition in (2.4) is not satisfied. Hence, there is no transformation which is variance stabilizing as well as approximately symmetrizing.

2.2.4 Chi Square Random Variable

Let X has a chi square distribution with parameter $n\theta$. Then letting $X_n = X/n$, we have $f_1(\theta) = 8\theta^3/n^2$,

$f_2(\theta) = 8\,\theta^4/n^2 + O(\frac{1}{n^3})$. Hence, $\frac{2}{3} f_1(\theta)/f_2(\theta) = 2/3\theta$

and $\frac{d}{d\theta} \ln \sigma(\theta) = (1/\theta)$. Again, we see that condition (2.4) is not satisfied.

The examples in Secs. 2.2.2 through 2.2.4 demonstrate that there may be a possibility to get a better normalizing transformation than given by the variance stabilizing transformation. To accomplish this, we construct a differential equation to obtain such a transformation and examine its solutions for the cases discussed in section 2.

3. Symmetrizing Transformation

In order to get a functional form of the transformation $g(X_n)$ such that $\mu_3\,(g(X_n)) \approx 0$, we may use equation (2.2) for $\mu_3\,(g(X_n))$ and thereby obtain (if $g' \neq 0$),

$$\frac{g''}{g'} = -\frac{2}{3}\,\frac{f_1(\theta)}{f_2(\theta)}\,. \qquad (3.1)$$

Hence, solution for g(.) can be obtained by solving

$$g(\theta) = \int e^{-\frac{2}{3}\int\left\{\frac{f_1(\theta)}{f_2(\theta)}\right\}d\theta}\, d\theta, \qquad (3.2)$$

which can be regarded as an approximately normalizing transformation. Now we examine (3.2) for the cases considered in section 2.

3.1 Correlation Coefficient

In this case we have

$$g(\rho) = \int e^{\int \frac{2\rho}{1-\rho^2}\, d\rho}\, d\rho$$

$$= \int e^{-\log\,[(1+\rho)/(1-\rho)]}\, d\rho$$

$$= \int \frac{1}{(1-\rho^2)}\, d\rho$$

$$= \frac{1}{2}\ln\,[(1+\rho)/(1-\rho)]$$, which is Fisher's Z

and it confirms our conclusion in (2.1).

3.2 Binomial Proportion

In this case the symmetrizing transformation $g(\theta)$ is given by

$$g(\theta) = \int \theta^{-1/3}(1-\theta)^{-1/3}\, d\theta, \qquad (3.3)$$

This transformation may be contrasted with the variance stabilizing transformation $g_v(\theta)$

$$g_v(\theta) = \int \theta^{-1/2}(1-\theta)^{-1/2}\, d\theta\,. \qquad (3.4)$$

We note that (3.3) does not have an explicit solution but can be numerically evaluated for any argument.

3.3 Poisson Variable

Let X be a Poisson variable with mean $n\theta$ then consider, $X_n = \frac{X}{n}$, we have $f_1(\theta) = \theta/n^2$, $f_2(\theta) = \frac{2\theta^2}{n^2} + O(\frac{1}{n^3})$ Hence,

$$g(\theta) = \int e^{-2/3 \int \frac{1}{2\theta} d\theta}\, d\theta$$

$$= \frac{3}{2}\,\theta^{2/3}. \qquad (3.5)$$

Thus, Poisson variable is better normalized by a power transformation with power = 2/3 as compared to the square root transformation, which is variance stabilizing.

3.4 Chi Square Random Variable

Consider the set up in (2.4), we have

$$g(\theta) = \int e^{-2/3 \ln \theta}\, d\theta$$

$$= 3\,\theta^{1/3}. \qquad (3.6)$$

The above transformation is the well known Wilson-Hilferty cube-root transformation.

4. DISCUSSION

We have given an easily verifiable condition which tells whether variance stabilizing transformation is a good normalizing transformation. In case this condition is not met, it may be possible to get an explicit normalizing transformation through a differential equation. We may also use the condition derived here to propose a measure η of the normalizing strength of the variance stabilizing transformation, which is given by

$$\eta = \frac{2}{3} \frac{f_1(\theta)}{f_2(\theta)\frac{d \ln \sigma(\theta)}{d\theta}} . \qquad (4.1)$$

A value of η close to 1 gives good normalizing strength, η less than 1 indicates negative skewness and a value greater than 1 indicates a positive skewness. This measure for various examples given here is presented below.

Random Variable	η
Correlation Coefficient	1
Binomial Proportion	2/3
Poisson	1/3
Chi-Square	2/3

REFERENCES

Anscombe, F.J. (1948). The transformation of Poisson, Binomial, Negative Binomial data. Biometrika, 35, 246-254.

Bartlett, M.S. (1947). The use of transformations. Biometrics, 3, 39-52.

Fisher, R.A. (1915). Frequency distribution of the values of correlation coefficient from an indefinitely large population. Biometrika, 10, 507-521.

Fisher, R.A. (1922). On the interpretation of χ^2 from contingency tables and calculation of p. J. Roy. Statist. Soc. Ser. A, 85, 87-94.

Hotelling, H. (1953). New light on the correlation coefficient and its transforms, J. Roy. Statist. Soc., Ser. B, 15, 193-224.

Jensen, D.R. and Solomon, H. (1972). A Gaussian approximation to the distribution of a Quadratic form in normal variables. J. Amer. Statist. Assoc., 67, 898-902.

Kendall, M.G. (1979). The advanced theory of Statistics, McMillan Publishing Company, New York.

Mudholkar, G.S. and Trivedi, M.C. (1980). A normal approximation for the distribution of the likelihood ratio statistic in multivariate analysis of variance. Biometrika, 67, 485-488.

Mudholkar, G.S. and Trivedi, M.C. (1981). A Gaussian approximation to the distribution of sample variance for non normal populations. J. Amer. Statist. Assoc., 76, 479-485.

Mudholkar, G.S. and Trivedi, M.C. (1981). A normal approximation for the multivariate likelihood ratio statistics. Commun. Statist.

Sankaran, M.S. (1959). On the noncentral χ^2 distribution. Biometrika 46, 235-237.

Wilson, E.B. and Hilfesty, M.M. (1931). The distribution of Chi-square. Proc. Nat. Acad. Sc. 17, 684-688.