On the Stability of the Absolutely Continuous Invariant Measures of a Certain Class of Maps with Deterministic Perturbation

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ABSTRACT

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Keller [8] showed the instability of the absolutely continuous invariant measure (acim) for a family of W-shaped maps. This instability is the result of the invariant neighborhood of the fixed turning point at $1/2$. The construction of these W-maps, for which the Lasota-Yorke inequality fails to prove stability, has recently been generalized. In the Eslami-Misiurewicz paper [4], a map was defined, whose third iterate has a fixed turning point at $1/2$, raising the question of the stability of the map.

The goal of this thesis is to show the stability of this map. We define a family of deterministic perturbations of the map and express their invariant densities as an infinite sum with the purpose of showing that the normalized invariant densities are uniformly bounded. This result is used to show the stability of absolutely continuous invariant measure of this transformation.

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Chapter 1

Introduction

The question of stability of an absolutely continuous invariant measure (acim) arises from the introduction of perturbed maps. Given a transformation τ with its invariant measure f, and a family of perturbed maps $\{\tau_\epsilon\}_{\epsilon\geq 0}$ with the corresponding invariant measures $\{f_\epsilon\}_{\epsilon\geq 0}$, the question is: assuming the distance of τ and τ_ϵ approaches 0, does $f_\epsilon \to f$ as $\epsilon \to 0$? Maps satisfying this property are called acim-stable maps. The existence of the acim has been established by the famous Lasota-Yorke inequality [11]. For piecewise expanding maps on intervals, with slopes greater than 2 in magnitude, this inequality guarantees that they have an absolutely continuous invariant measure.

Keller [8] introduced a family of maps $W_{a,b,r}$ depending on three parameters, such that in the limit these maps are approach a W_0 map with slopes equal to 2. Keller concluded that this map is not acim-stable. This kind of behavior is caused by the existence of a small neighborhood around the fixed turning point at 1/2 which stays invariant under perturbation. It was conjectured that this kind of construction of the perturbed maps is the only way to show that acim stability fails.

This conjecture was proven wrong in [4] and [12]. In [4] the authors posed the question whether the map $\tau : [0,1] \to [0,1], \tau = (x+1/2)\chi[0,1/2] + (2-2x)\chi[1/2,1]$, with invariant density f is acim-stable. This question was motivated by the third iteration of τ, which has a fixed turning point at 1/2.

The main goal of this thesis is to prove the stability of the absolutely continuous invariant measure for this map. We define a family of deterministically perturbed maps $\{\tau_\epsilon\}_{\epsilon\geq 0}$ with the corresponding family of their invariant densities $\{f_\epsilon\}_{\epsilon\geq 0}$ and show that $\tau_\epsilon \to \tau$ almost uniformly.

The family of the normalized densities of τ_ϵ is denoted as $\{\widetilde{f}_\epsilon\}_{\epsilon\geq 0}$. We show that $\widetilde{f}_\epsilon$ is uniformly bounded, which implies that it forms a weakly precompact set in L^1. Lemma 2.6, [1], says that given $\tau_\epsilon \to \tau$ almost uniformly and $\widetilde{f}_\epsilon \to f$ weakly in L^1, then $P_\tau f = f$. Hence, as $\epsilon \to 0$, any convergent subsequence of $\widetilde{f}_\epsilon$ converges to f, the unique invariant measure of τ. This proves that $\lim_{\epsilon\to 0} \widetilde{f}_\epsilon = f$.

In Chapter 2 we review basic measure-theoretic properties. Then the invariant density of τ is calculated. We define deterministic perturbations of τ, τ_ϵ. We state the Lasota-Yorke inequality and elaborate on the nature of our problem. We also state Lemma 2.6, [1], upon which the proof of the stability of the absolutely continuous invariant measure of τ is based.

Chapter 3 deals with the general formula of $\{f_\epsilon\}_\epsilon$, the family of invariant densities of τ_ϵ. This section is based on the work in [6] where a formula is developed for the invariant densities of piecewise linear maps of the unit interval.

In Chapter 4, we show that f_ϵ and $\widetilde{f}_\epsilon$, the normalized invariant measures of τ_ϵ, are uniformly bounded. The acim-stability of τ is proven.

Recently, two different papers, [5] and [7] offered an answer about the stability of the acim for the map defined above. In [5], the Lasota-Yorke inequality [11] was improved for piecewise expanding $\mathcal{C}^{1,1}$ maps. The constraints in the Lasota-Yorke inequality were a motivation for the work done in [7], where it is shown that the harmonic average of slopes is sufficient for Rychlik's theorem [1] to hold. The conclusions of these two papers are outlined in Chapter 5.

Chapter 2

Measure-Theoretic properties of Maps of Intervals

In this chapter we present some basic notions of measure theory upon which the introduction of acim will be established. In addition to this, we introduce the Frobenius-Peron operator which plays an important role in dealing with acims and we introduce a special class of maps on intervals, called Markov maps, for which we have a very convenient representation of the Frobenius-Perron operator.

In the last section of this chapter we introduce the idea of acim and we talk discuss their existence and stability. We also elaborate on the main problem of this thesis, and how this problem originates from the related question about the acim of one-dimensional transformations.

2.1 Measure Theory

Definition 2.1. *A collection of subsets $\mathcal{B} \in X$ is called a σ-algebra if the following conditions are satisfied:*

(a) $X \in \mathcal{B}$,

(b) whenever $B \in \mathcal{B}$, *then* $X \backslash B \in \mathcal{B}$,

(c) if $B_n \in \mathcal{B}$, *for* $n = 1, 2, ...,$ *then* $\cup_{n=1}^{\infty} B_n \in \mathcal{B}$.

Definition 2.2. *A real valued function* $\mu : \mathcal{B} \to \mathbb{R}$ *is called a measure on* $\mathcal{B}$ *if:*
(a) $\mu(B) \geq 0$ *for any* $B \in \mathcal{B}$*;*
(b) for any sequence of disjoint sets $\{B_n\}$*,* $B_n \in \mathcal{B}, n = 1, 2, ...,$ $\mu(\cup_{i=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n)$*.*

The triplet $(X, \mathcal{B}, \mu)$ is called a measure space. In the case where $\mu(X) = 1$, we are talking about a probability space or a normalized space.

Definition 2.3. *Let* $(X, \mathcal{B}, \mu)$ *be a measure space. We call a function* $f : X \to \mathbb{R}$ *a measurable function if for all* $x \in \mathbb{R}$*,* $f^{-1}(x, \infty) \in \mathcal{B}$*, or, equivalently, if* $f^{-1}(B) \in \mathcal{B}$ *for any Borel set* $B \subset \mathbb{R}$*.*

Definition 2.4. *Let* μ *and* ν *be two measures on the same measure space* $(X, \mathcal{B})$*. Then* ν *is absolutely continuous with respect to* μ *if for any* $B \in \mathcal{B}$*,* $\mu(B) = 0$ *implies* $\nu(B) = 0$*. In this case we write* $\nu << \mu$*.*

For two measures ν and μ, such that $\nu << \mu$, the following theorem states a possibility to represent ν in terms of μ.

Theorem 2.1 (Radon-Nikodym Theorem)**.** *[1] Let* $(X, \mathcal{B}, \mu)$ *be a measure space and let* ν *be a finite measure on the same space such that* $\nu << \mu$*. Then, there exists a unique* $f \in \mathcal{L}^1(X, \mathcal{B}, \mu)$ *such that for all* $B \in \mathcal{B}$*,*

$$\nu(B) = \int_B f d\mu.$$

The function f *is called the Radon-Nikodym derivative.*

Definition 2.5. *[1] Consider the normalized measure space* $(X, \mathcal{B}, \mu)$*. For a map* $\tau : X \to X$ *we say that* τ *is nonsingular if and only if for any* $B \in \mathcal{B}$ *such that* $\mu(B) = 0$ *we have* $\mu(\tau^{-1}(B)) = 0$*.*

Definition 2.6. *[1] Let $(X, \mathcal{B}, \mu)$ be a measure space. We call a transformation $\tau : X \to X$ a measure μ-preserving transformation if*

$$\mu(\tau^{-1}(B)) = \mu(B),$$

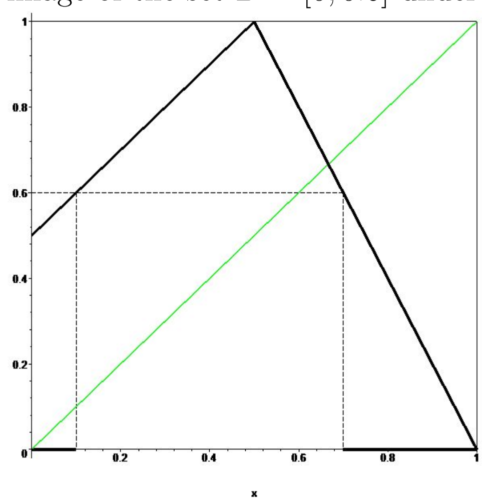
for all $B \in \mathcal{B}$. Equivalently, we say that τ preserves measure μ or μ is τ invariant.

Example 2.1. *Consider the transformation $\tau : [0,1] \to [0,1]$ defined by,*

$$\tau(x) = \begin{cases} x + \frac{1}{2} & \text{if } 0 \le x \le \frac{1}{2} \\ -2x + 2 & \text{if } \frac{1}{2} < x \le 1. \end{cases} \tag{2.1}$$

We will show that τ does not preserve the Lebesgue measure μ.

Figure 2.1: The inverse image of the set $B = [0, 0.6]$ under the transformation $\tau(x)$



Let $B = [0, x]$, where $x \le 1/2$. Then $\tau^{-1}(B) = \tau^{-1}([0,x]) = [1 - \frac{x}{2}, 1]$. We can see that the Lebesgue measure of the preimage of B is not equal to the Lebesgue measure of B:

$$\mu(\tau^{-1}(B)) = 1 - (1 - \frac{x}{2}) = \frac{x}{2} \neq x = \mu(B). \tag{2.2}$$

Now let $1/2 < x \le 1$. Then, $\tau^{-1}(B) = \tau^{-1}([0,x]) = [0, x - \frac{1}{2}] \cup [1 - \frac{x}{2}, 1]$. Again,

$$\mu(\tau^{-1}(B)) = x - \frac{1}{2} + 1 - (1 - \frac{x}{2}) = \frac{3x}{2} - \frac{1}{2} \neq x = \mu(B). \tag{2.3}$$

We conclude that τ does not preserve the Lebesgue measure.

Here we use Devaney's definition of chaos and we show that the map τ from Example 2.1. is a chaotic map.

Definition 2.7. *[2] Let A be a set. We say that $\tau : A \to A$ is a chaotic map if: 1) τ has sensitive dependence on initial conditions; 2) τ is topologically transitive and 3) periodic points are dense in A.*

In order to clarify the definition above, we state the following three definitions:

Definition 2.8. *[2] A map $\tau : A \to A$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in A$ and every neighborhood N of x, there exists $y \in N$ and $n \geq 0$ such that $|\tau^n(x) - |\tau^y(x)| > \delta$.*

Definition 2.9. *[2] A map $\tau : A \to A$ is called topologically transitive if for any pair of open sets $U, V \subset A$ there exists $k > 0$ such that $\tau^k(U) \cap V \neq \emptyset$.*

Definition 2.10. *[2] A point $x \in A$ is called a periodic point of period n if $\tau^n(x) = x$.*

In [13] it was shown that if a certain map τ has a point of period 3, then we can say that τ is a chaotic map. Looking back at Example 2.1, we can easily see that $\tau(0) \to 1/2$, $\tau(1/2) \to 1$ and $\tau(1) \to 0$. Since τ has a point of period 3, we can conclude that τ is a chaotic map.

Consider $\tau : X \to X$ to be a chaotic map. Then, because of the sensitive dependence on the initial conditions, the map is unpredictable, i.e., for almost every $x \in I$, $I \subset X$, it is impossible to predict the set where the n-th iteration of the map τ will belong. However, we can ask whether the trajectory visits certain sets more often than others. In other words, we are interested in the average amount of time the trajectory of a map spends in a certain subset.

Definition 2.11. *The characteristic function of a set $B \subset X$ is defined as,*

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B. \end{cases} \tag{2.4}$$

Let $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measure preserving transformation and let $B \in \mathcal{B}$. Then, the total amount of the first n iterations of x under τ that visit the subset B is equal to $\sum_{k=0}^{n-1} \chi_B(\tau^k(x))$. Consequently, the relative frequency of the points $\tau^i(x)$, $i = 0, 1, ..., n-1$ that visit B, is defined as,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_B(\tau^k(x)), \tag{2.5}$$

when the limit exists.

Theorem 2.2 (Birkhoff Ergodic Theorem)**.** *[1] Suppose $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a measure preserving transformation, where $(X, \mathcal{B}, \mu)$ is σ-finite, and $f \in \mathcal{L}^1(\mu)$. Then, there exists a function $f^* \in \mathcal{L}^1$ such that*

$$\frac{1}{n} \sum_{k=0}^{n-1} f(\tau^k(x)) \to f^*, \ \ \mu - a.e.,$$

Furthermore, $f^* \circ \tau = f^*$ μ-a.e. and if $\mu(X) < \infty$, then $\int_X f^* d\mu = \int_X f d\mu$.

As we shall see later in this chapter, our study will rely heavily on the idea of densities. For certain one dimensional maps, especially those that exhibit chaotic behavior, the study of certain properties becomes much easier if we study their invariant densities. Here, we define the notion of density.

Definition 2.12. *Let $(X, \mathcal{B}, \mu)$ be a measure space and define the set $D(X, \mathcal{B}, \mu)$ as*

$$D(X, \mathcal{B}, \mu) = \{f \in \mathcal{L}^1(X, \mathcal{B}, \mu) : f \geq 0 \text{ and } \|f\|_1 = 1\}. \tag{2.6}$$

A function $f \in D(X, \mathcal{B}, \mu)$ is called a density.

2.1.1 Precompactness of densities

Let $(X, \mathcal{B}, \mu)$ be a measure space and let $\mathcal{F}$ be a set of functions in $\mathcal{L}^p$. The notions of strong convergence and weak convergence are defined in [1]. We define strong convergence as: $f_n \to f$ in $\mathcal{L}^p$-norm $\Leftrightarrow ||f_n - f||_p \to 0, n \to +\infty$; and we say that $f_n \to f$ weakly in $\mathcal{L}^p, 1 \leq p < +\infty \Leftrightarrow \forall g \in \mathcal{L}^p$, $\int f_n g d\mu \to \int f g d\mu$ where $\frac{1}{p} + \frac{1}{q} = 1$. We state the notion of precompactness of sets in L^p [10].

Definition 2.13. *The set $\mathcal{F}$ is called strongly precompact if every sequence of functions $\{f_n\}$, $f_n \in \mathcal{F}$, contains a strongly convergent subsequence $\{f_{a_n}\}$ that converges to some $\bar{f} \in L^p$.*

Definition 2.14. *The set $\mathcal{F}$ is called weakly precompact if every sequence of functions $\{f_n\}$, $f_n \in \mathcal{F}$, contains a weakly convergent subsequence $\{f_{a_n}\}$ that converges to some $\bar{f} \in L^p$.*

In definitions (2.13) and (2.14) we write $\bar{f}$ instead of f, because $\bar{f} \in L^p$ rather than $\bar{f} \in \mathcal{F}$.

The following lemma [3], states a condition for weak precompactness of sets in L^1.

Lemma 2.1. *If $g \in L^1$ is a nonnegative function, then the set of all functions $f \in L^1$ for which*

$$|f(x)| < g(x), \text{ for } x \in X \, a.e. \tag{2.7}$$

is weakly precompact in L^1.

2.2 Functions of Bounded Variations in One Dimension

The famous Lasota-Yorke inequality, stated in the following section, relies on the notion of functions of bounded variation. In this section we define what it means for a one dimensional function f to be of bounded variation.

Let $I = [a, b]$ be a bounded interval on the real line. We define a partition $\mathcal{P}$ in the following way: let $a = x_0 < x_1 < ... < x_n = b$ be a sequence of points and denote $I_i = [x_{i-1}, x_i)$; then $\mathcal{P} = \{I_i\}$ is called a partition of $[a, b]$ and the points $\{x_0, x_1, ...x_n\}$ are called the endpoints of $\mathcal{P}$. We also write $\mathcal{P} = \mathcal{P}\{x_0, x_1, ...x_n\}$.

Definition 2.15. *Let $f : [a, b] \to \mathcal{R}$ and let $\mathcal{P} = \mathcal{P}\{x_0, x_1, ...x_n\}$ be a partition of the domain of f. We say that f is of bounded variation on $[a, b]$ if there is an $M > 0$ such that for all partitions $\mathcal{P}$,*

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq M. \tag{2.8}$$

Definition 2.16. *For a function of bounded variation $f : [a, b] \to \mathcal{R}$, the number*

$$V_{[a,b]}f = sup_{\mathcal{P}}\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|\} \tag{2.9}$$

is defined as the total variation or simply the variation of f.

2.3 Ergodicity, Mixing and Exactness

We extend our study of measure-preserving maps by introducing the notion of ergodicity. Ergodic maps are a special type of maps that need to be studied on the whole space. This restriction becomes clearer once we define what an ergodic map is.

Definition 2.17. *[1] Let $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measure preserving transformation. If for any invariant set under τ, $B \in \mathcal{B}$, either $\mu(B) = 0$ or $\mu(X \setminus B) = 0$, we call τ an ergodic transformation.*

Ergodicity along with the notions of mixing and exactness are the three most basic features of maps with irregular behavior. Here we define the concepts of mixing and exactness and we state the connection between them.

Definition 2.18. *[1] Let $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measure preserving transformation on a normalized space. If*

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1}\left|\mu(A\cap\tau^{-i}(B)) - \mu(A)\mu(B)\right|\right) = 0, \text{ for all } A,\ B \in \mathcal{B},$$

we call τ weakly mixing, and if

$$\lim_{n\to\infty}\mu(A\cap\tau^{-n}(B)) = \mu(A)\mu(B),$$

then we say that τ is strongly mixing.

Definition 2.19. *[1] We say that the map $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is exact if*

$$\lim_{n\to\infty}\mu(\tau^{n}(B)) = 1$$

for any $B \in \mathcal{B}$ with τ-invariant measure $\mu(B) > 0$.

It is known that if τ is strongly mixing, then τ is ergodic, and that the exactness of τ implies strong mixing. The following proposition uses this idea to show that the map defined in Example 2.1 is an ergodic map.

Proposition 2.1. *The map $\tau : [0, 1] \to [0, 1]$, defined as,*

$$\tau(x) = \begin{cases} x + \frac{1}{2} & \text{if } 0 \le x < \frac{1}{2}, \\ -2x + 2 & \text{if } \frac{1}{2} \le x \le 1, \end{cases}$$

is an ergodic map with respect to the Lebesgue measure μ.

Proof. Let us denote $I = [0, 1]$, $I_1 = [0, 1/2)$ and $I_2 = [1/2, 1]$. Let $a, b \in I_1$ and $a', b' \in I_2$ such that $a < b < a' < b'$. Since,

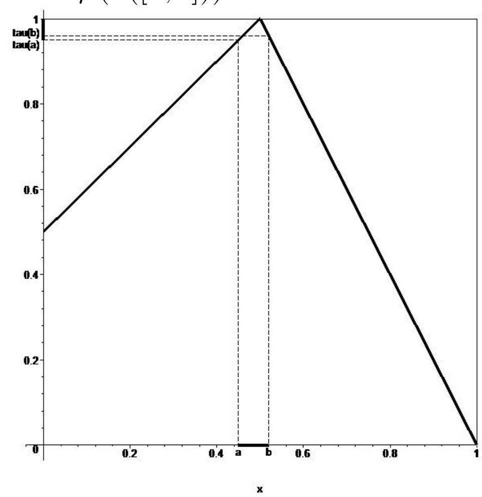
$$\mu(\tau([a, b])) = b + 1/2 - a - 1/2 = b - a = \mu([a, b]), \text{ and}$$

$$\mu(\tau([a', b'])) = |-2b' + 2 + 2a' - 2| = 2(b' - a') = 2\mu([a', b']),$$

we can say that the measure is being preserved on $[0, 1/2)$ and gets doubled on $[1/2, 1]$.

In addition to this, we note that every interval $B \subset I_1$ gets mapped to I_2 and, moreover, once an interval is mapped on the left branch of τ, the next iteration maps the interval to the right branch, the one that doubles the measure. Hence, there exists an n such that $\tau^n(B) \supset [1/2, 1]$, which implies that $\tau^{n+1}(B) = [0, 1]$.

Figure 2.2: The image of the interval $[a, b] = [0.45, 0.52]$ and $\tau([a, b]) = [0.95, 1]$, such that $\mu([a, b]) = 0.07 > 0.05 = \mu(\tau([a, b]))$



The only difficult case happens when $\mu(\tau(B)) < \mu(B)$. This occurs only when $B = [a, b]$, $a < 1/2 < b$ such that $1/2 - a > 2(b - 1/2)$. Let $\mu(B) = p + q$, where $p = 1/2 - a$ and $q = [b - 1/2]$. In order to obtain the measure of the image being smaller than the measure of the interval, we let $p > 2q$.

Then, $\tau(B) = [1 - p, 1]$ and hence $\mu(\tau(B)) = \max(p, 2q) = p < \mu(B)$. Then, $\tau^2(B) = [0, 2p], \tau^3(B) = [1/2, 1/2 + 2p], \tau^4(B) = [1 - 4p, 1], \tau^5(B) = [0, 8p]$ and so on. We notice that all of these intervals stay away from the critical interval $B^* = [a, b], a < 1/2 < b$, and at most after one step the measure gets doubled. That means that there is an n such that $\mu(\tau^n(B)) = 1$.

We have shown that $\tau(x)$ is an exact map which implies that this map is strongly mixing, which proves the ergodicity of τ. $\square$

2.4 Frobenius-Perron Operator

2.4.1 Motivation

One of the essential tools used in the study of absolutely continuous invariant measures is the Frobenius-Peron (F-P) operator. Here we give a short introduction of this operator and in the following section we state its precise definition.

For a chaotic one dimensional map $\tau : I \to I$, it is impossible to follow the trajectories of the consecutive iterations τ^n when $n \to \infty$. Instead of dealing with every single iteration $\{\tau^n\}_{n=1}^{\infty}$, we can try to find the probability of τ^n falling into a certain subinterval I_i of I. In other words, let us divide the interval $I = [a, b]$ into m subintervals where $I_i = [x_{i-1}, x_i]$, $a = x_0 < x_1 < ... < x_m = b$. For the map τ, the probability with which $\tau(x) \in I_i$ is equal to the probability of $x \in \tau^{-1}(I_i)$. If f is the probability density function of the variable $\tau(x)$, we can write the last statement as

$$\int_{I_i} f d\lambda = \int_{\tau^{-1}(I_i)} f d\lambda,$$

where λ is the normalized Lebesgue measure on I.

For τ being a non-singular map and $f \in \mathcal{L}^1$, let us define,

$$\mu(I_i) = \int_{\tau^{-1}(I_i)} f d\lambda. \tag{2.10}$$

Since τ is non-singular with respect to λ, $\lambda(I_i) = 0$ implies $\lambda(\tau^{-1}(I_i)) = 0$. Thus, $\mu(I_i) = 0$ which means that μ is absolutely continuous with respect to λ, i.e., $\mu << \lambda$.

Then, by the Radon-Nikodym theorem, there exists a function $\phi \in \mathcal{L}^1$, such that for any measurable set $I_i \subset I$,

$$\mu(I_i) = \int_{I_i} \phi \, d\lambda. \tag{2.11}$$

We let $\phi = P_\tau f$. Then from (2.10) and (2.11) we obtain,

$$\int_{\tau^{-1}(I_i)} f d\lambda = \int_{I_i} P_\tau f d\lambda.$$

2.4.2 Definition and Properties

Following the motivation in the previous section, we can give a precise definition of the Frobenius-Perron operator:

Definition 2.20. *[1] Let $(X, \mathcal{B}, \mu)$ be a measure space and let $\tau : X \to X$ be a nonsingular transformation. For any $f \in \mathcal{L}^1$, we define the value of the Frobenius-Perron operator $\mathrm{P}_\tau f$ as the unique function in $\mathcal{L}^1$ such that for any $A \in \mathcal{B}$ the following equation holds:*

$$\int_A \mathrm{P}_\tau f d\mu = \int_{\tau^{-1}(A)} f d\mu. \tag{2.12}$$

The most basic properties of P_τ follow directly from the definition of the operator itself.

Lemma 2.2. *[1] For any nonnegative f_1, $f_2 \in \mathcal{L}^1$, and any nonsingular transformations $\tau : I \to I$ and $\sigma : I \to I$, the Frobenius-Perron operator P has the following properties:*

1) $\mathrm{P}_\tau f : \mathcal{L}^1 \to \mathcal{L}^1$ *is a linear operator, i.e.,*

$$\mathrm{P}_\tau(c_1 f_1 + c_2 f_2) = c_1 \mathrm{P}_\tau f_1 + c_2 \mathrm{P}_\tau f_2$$

for $c_1, c_2 \in \mathbb{R}$*;*

2) $\mathrm{P}_\tau f \geq 0$*, for* $f \geq 0$*;*

3) $\mathrm{P}_\tau f$ *preserves integrals, i.e.,*

$$\int_I \mathrm{P}_\tau f d\mu = \int_I f d\mu;$$

4) $\mathrm{P}_{\tau\circ\sigma} f = \mathrm{P}_\tau \circ \mathrm{P}_\tau f$*. In particular,* $\mathrm{P}_{\tau^n} f = \mathrm{P}_\tau^n f$*.*

The following Lemma states a very important relation between P_τ and the density of the measure μ that is τ-invariant and absolutely continuous with respect to the measure λ.

Lemma 2.3. *[1] Let* $\tau : I \to I$ *be a nonsingular map and consider a nonnegative* $f^* \in \mathcal{L}^1$*. Then* $\mathrm{P}_\tau f^* = f^*$*, i.e.,* f^* *is the fixed point of* P_τ *if and only if* μ *is* τ*-invariant where* μ *is defined as:*

$$\mu(A) = \int_A f^* d\lambda. \tag{2.13}$$

Proof. Assume μ is τ-invariant, i.e. for any $A \in \mathcal{B}$

$$\mu(\tau^{-1}(A)) = \mu(A).$$

Then, by (2.13),

$$\int_{\tau^{-1}(A)} f^* d\lambda = \int_A f^* d\lambda. \tag{2.14}$$

However, by the definition of the Frobenius-Perron operator we have

$$\int_{\tau^{-1}(A)} f^* d\lambda = \int_A P_\tau f^* d\lambda = \int_A f^* d\lambda.$$

Since this holds for any $A \in B$, we have $P_\tau f^* = f^*$.

Now, we assume that f^* is a fixed point of P_τ, i.e.,

$$\int_A f^* d\lambda = \int_A P_\tau f^* d\lambda, \text{ for any } A \in \mathcal{B}.$$

By definition (2.20) and equation (2.13) we have

$$\mu(A) = \int_A f^* d\lambda = \int_A P_\tau f^* d\lambda = \int_{\tau^{-1}(A)} f^* d\lambda = \mu(\tau^{-1}(A)).$$

Hence, we conclude that μ is invariant under the transformation τ. □

2.4.3 Representations of the Frobenius-Perron Operator

For one dimensional nonsingular transformations that are piecewise monotonic, the Frobenius-Perron operator has a very convenient representation. Moreover, based on this form of the F-P, the study of this operator and its fixed points becomes easier when we deal with so called Markov maps. In this section, we define the class of piecewise expanding maps as well as the Markov maps and we derive the representation of the Frobenuis-Perron operator for these classes of maps.

Representations of the Frobenius-Perron Operator for Piecewise Monotonic Maps

Definition 2.21. *We call the transformation $\tau : I \to I$, $I = [a, b]$, a piecewise monotonic transformation if there is a partition of $I, a = a_0 < a_1 < a_2 < ... < a_n = b$, and a number $r \geq 1$ such that:*

1) τ is a C^r function on each subinterval $(a_{i-1}, a_i), i = 1, ..., n$, that can be extended to a C^r function on $[a_{i-1}, a_i], i = 1, ..., n$, and

2) $|\tau'(x)| > 0$ on $(a_{i-1}, a_i), i = 1, ..., n$.

This class of functions allows a very convenient and useful representation of the Frobenius-Perron operator. Let $I_i = [a_{i-1}, a_i]$ and $\tau_i(x) = \{\tau(x) : x \in I_i\}$, $i = 1, ..., n$. Since τ is piecewise monotonic, we can find $\tau_i^{-1}(x)$ for each $1 \leq i \leq n$. Call $\tau_i(I_i) = B_i$ and hence $\tau_i^{-1}(B_i) = I_i$. Then for any Borel set A in I,

$$\tau_i^{-1}(A) = \bigcup_{i=1}^{n} \tau_i^{-1}(B_i \cap A), \tag{2.15}$$

where the sets $\{\tau_i^{-1}(B_i \cap A)\}_{i=1}^{n}$ are mutually disjoint.

Then,by the definition of P_τ we have

$$\int_A P_\tau f d\lambda = \int_{\tau^{-1}(A)} f d\lambda = \int_{\cup_{i=1}^{n}\tau_i^{-1}(B_i\cap A))} f d\lambda = \sum_{i=1}^{n} \int_{\tau_i^{-1}(B_i\cap A))} f d\lambda.$$

Using a change of variables, the last expression can be written as

$$\begin{aligned}
\int_A P_\tau f(x) d\lambda &= \sum_{i=1}^{n} \int_{B_i\cap A} f(\tau_i^{-1}(x))|(\tau_i^{-1}(x))'| d\lambda \\
&= \sum_{i=1}^{n} \int_A f(\tau_i^{-1}(x))|(\tau_i^{-1}(x))'|\chi_{B_i}(x) d\lambda \\
&= \int_A \sum_{i=1}^{n} f(\tau_i^{-1}(x))|(\tau_i^{-1}(x))'|\chi_{B_i}(x) d\lambda \\
&= \int_A \sum_{i=1}^{n} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|}\chi_{\tau(I_i)}(x) d\lambda.
\end{aligned}$$

Hence, since A is arbitrary, we are allowed to write a more concise version of the Frobenius-Perron operator for piecewise monotonic maps:

$$P_\tau f(x) = \sum_{i=1}^{n} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|}\chi_{\tau(I_i)}(x). \tag{2.16}$$

Example 2.2. *Recall the map τ from Example 2.1. Here we will derive the Frobenius-Perron operator for this map. Let $I_1 = [0, 1/2]$ and $I_2 = [1/2, 1]$. We define*

$$\begin{aligned}
\tau_1(x) &= x + \frac{1}{2} \text{ if } 0 \le x \le \frac{1}{2}, \\
\tau_2(x) &= -2x + 2 \text{ if } \frac{1}{2} < x \le 1.
\end{aligned}$$

Consequently, $\tau_1^{-1}(x) = x - \frac{1}{2}$ *and* $\tau_2^{-1}(x) = 1 - \frac{x}{2}$*. Since* $|\tau_1'(x)| = 1$ *and* $|\tau_2'(x)| = 2$*, using (2.16) we write the Frobenius-Perron operator for the transformation* τ *as*

$$\mathrm{P}_\tau f(x) = f(x - \frac{1}{2})\chi_{[1/2,1]} + \frac{1}{2}f(1 - \frac{x}{2}). \tag{2.17}$$

2.4.4 Markov Maps and the Matrix Representation of the Frobenius-Perron Operator

Conveniently, the Frobenius-Perron operator can be represented as a finite dimensional matrix for a class of transformations known as Markov maps. In this section we define the Markov maps and the corresponding representation of the Frobenius -Perron operator.

Define the map $\tau : I \to I$, where $I = [a, b]$. Let $\mathcal{P}$ be a partition of I given by $a = a_0 < a_1 < ... < a_n = b$ and for $i = 1, ..., n$ denote the subintervals I_i as $I_i = (a_i, a_{i-1})$. The map τ on each I_i is called τ_i. A map τ is called a Markov map if τ_i is a homeomorphism from I_i onto a connected union of intervals of $\mathcal{P}$. Such a partition is called a Markov partition with respect to τ.

Definition 2.22. *Let* $\tau : I \to I$ *be a piecewise monotonic transformation and let* $\mathcal{P} = \{I\}_{i=1}^n$ *be a partition of* I*. We define the incidence matrix* $A_\tau = (a_{ij})_{1\le i,j\le n}$*, with entries*

$$a_{ij} = \begin{cases} 1 & \text{if } I_j \subset \tau(I_i), \\ 0 & \text{otherwise .} \end{cases}$$

Note that if τ is Markov, then $a_{ij} = 0$ means that $I_j \cap \tau(I_i)$ contains at most only one point, more precisely an endpoint of I_i.

When we are dealing with piecewise linear Markov transformation, the Frobenius-Perron operator has a very simple matrix representation. In the rest of this section we will show this representation of the Frobenius-Perron operator and we will use it to find the fixed point for $P_\tau f$ that was calculated in Example 2.2.

Let us fix a partition $\mathcal{P}$ on I and let S denote the class of all the functions that are piecewise constant on the partition $\mathcal{P}$. Then,

$$f \in S \qquad \text{if and only if} \qquad f = \sum_{i=1}^{n} \pi_i \chi_{I_i},$$

for some constants $\pi_1, ..., \pi_n$. This f can be also represented as a column vector $\pi^f = (\pi_1, ..., \pi_n)^T$.

Theorem 2.3 ([1]). *Let $\tau : I \to I$ be a piecewise linear Markov map on the partition $\mathcal{P} = \{I_i\}_{i=1}^n$. Then there exists an $n \times n$ matrix M_τ such that $\mathrm{P}_\tau f = M_\tau^T \pi^f$ for every $f \in S$ and π^f is the column vector obtained from f.*
The matrix M_τ is of the form $M_\tau = (m_{ij})_{1 \le i,j \le n}$, where

$$m_{ij} = \frac{a_{ij}}{|\tau_i'|} = \frac{\lambda(I_i \cap \tau^{-1}(I_j))}{\lambda(I_i)}, \qquad 1 \le i, j \le n,$$

where $A_\tau = (a_{ij})_{1 \le i,j \le n}$ is the incidence matrix induced by τ and $\mathcal{P}$.

Proof. Recall the equation (2.16), and let us define the function f by

$$f(x) = \chi_{I_k}(x) = \begin{cases} 1 & \text{if } x \in I_k, \\ 0 & \text{otherwise} . \end{cases} \tag{2.18}$$

Then, the Frobenius-Perron operator can be expressed as,

$$P_\tau f(x) = \sum_{i=1}^{n} \frac{\chi_{I_k}(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|} \chi_{\tau(I_i)}(x). \tag{2.19}$$

Since τ is Markov, the range of τ_i^{-1} is I_i and

$$\chi_{I_k}(\tau_i^{-1}(x)) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise} . \end{cases} \tag{2.20}$$

Thus, we rewrite (2.19) as,

$$P_\tau f(x) = |\tau'(\tau_k^{-1}(x))|^{-1} \chi_{\tau(I_k)}(x). \tag{2.21}$$

Since τ is a piecewise linear function, τ_i' is a constant on I_i. Moreover, since $\tau_k^{-1}(x) \in I_k$, we state Perron-Frobenius operator as

$$P_\tau f(x) = |\tau_k'|^{-1}\chi_{\tau_k(I_k)}(x). \tag{2.22}$$

We represent f as

$$f = \sum_{k=1}^{n} \pi_k \chi_{I_k} = (\pi_1, ..., \pi_n)^T.$$

Then, we use the property of linearity of the Frobenius-Perron operator from Lemma 2.2, and equation (2.22) to write

$$P_\tau f(x) = P_\tau\left(\sum_{k=1}^{n} \pi_k \chi_{I_k}(x)\right) = \sum_{k=1}^{n} \pi_k P_\tau(\chi_{I_k}(x)) = \sum_{k=1}^{n} \pi_k |\tau_k'|^{-1}\chi_{\tau_k(I_k)}(x). \tag{2.23}$$

Thus, we have expressed the P-F operator as a step function on the partition $\mathcal{P}$, i.e., $P_\tau f \in S$. That means that $P_\tau f$ can be expressed in terms of a column vector. Let us say $P_\tau f = (d_1, ..., d_n)^T$.

Let $x \in I_j$ and let $P_\tau f = d_j$. The k-th term in (2.23) is equal to $\pi_k|\tau_k'|^{-1}$ whenever $x \in \tau_k(I_k)$, i.e., whenever $I_j \subset \tau_k(I_k)$. Now we define

$$\triangle_{jk} = \begin{cases} 1 & \text{if } I_j \subset \tau_k(I_k), \\ 0 & \text{otherwise} \end{cases}$$

as well as the $n \times n$ matrix

$$M_\tau^T = (m_{jk}) = \triangle_{jk}|\tau_k'|^{-1}.$$

Then, for $x \in I_j$, we rewrite (2.23)

$$P_\tau f = d_j = \sum_{k=1}^{n} \pi_k(m_{jk})$$

and since $P_\tau f = (d_1, ..., d_n)^T$,

$$P_\tau f = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = M_\tau^T \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_n \end{pmatrix}.$$

□

Example 2.3. *We continue working on the map τ from Example 2.1. We define the partition $\mathrm{P} = \{I_1, I_2\}$, where $I_1 = [0, 1/2]$ and $I_2 = (1/2, 1]$, and respectively we define $\tau_1(x) = x + \frac{1}{2}$ on I_1 and $\tau_2(x) = -2x + 2$ on I_2. Since τ_1 is a homeomorphism of I_1 onto I_2 and τ_2 is a homeomorphism of I_2 onto $I_1 \cup I_2$, we see that τ is a Markov map.*

Moreover, since $\tau(I_1) = I_2$ and $\tau(I_2) = I_1 \cup I_2$, we define the incidence matrix for the map τ,

$$A_\tau = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

The map has slopes 1 and -2 on I_1 and I_2 respectively and so the matrix M_τ is of the form,

$$M_\tau = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \text{and} \qquad M_\tau^T = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}.$$

As we know, the invariant density of τ is the fixed point of $\mathrm{P}_\tau f$. From Theorem 2.3, the solution of $\mathrm{P}_\tau f = f$ can be obtained by solving

$$M_\tau^T \pi = \pi,$$

where $\pi = [\pi_1, \pi_2]^T$. Hence,

$$\begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}, \text{ or } \pi_1 = 1 \text{ and } \pi_2 = 2.$$

Hence, the unique normalized invariant density of τ is given by

$$\frac{(1,2)}{\sum_{i=1}^{2} \pi_i \lambda(I_i)} = \frac{(1,2)}{1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2}} = (2/3, 4/3).$$

If we look back at the Example 2.2, we we can see that the density defined as

$$f = \begin{cases} 2/3 & \text{if } x \in [0, 1/2), \\ 4/3 & \text{if } x \in [1/2, 1]. \end{cases} \tag{2.24}$$

is indeed the fixed point of $\mathrm{P}_\tau f$, where $\mathrm{P}_\tau f$ is defined in (2.20). In order to check this result, we can see that if $x \in I_1$, then $(1 - \frac{x}{2}) \in [1/2, 1]$ and $f(1 - \frac{x}{2}) = 4/3$, so

$P_\tau f = 1/2 \cdot 4/3 = 2/3 = f$. *On the other hand, if* $x \in I_2$ *then* $(x - 1/2) \in [0, 1/2]$ *and so* $P_\tau f = 2/3 + 1/2 \cdot 4/3 = 4/3 = f$.

Now let us recall Lemma (2.3) and use it in this example. We have found a function f *such that* $P_\tau f = f$. *Thus, the measure*

$$\mu(A) = \int_A f d\lambda \tag{2.25}$$

is invariant under the transformation τ *and it is absolutely continuous with respect to* λ.

2.5 Existence and Stability of Absolutely Continuous Invariant Measures

In [11], the authors showed the existence of invariant measures for piecewise expanding C^2 transformations. Here we state the original result as well as the inequality contained in the proof of the theorem that became known as the Lasota-Yorke inequality.

Theorem 2.4 ([11]). *Let* $\tau : [0,1] \to [0,1]$ *be a piecewise* C^2 *function such that* $\inf |\tau'| > 1$. *Then for any* $f \in \mathcal{L}^1$, *the sequence*

$$\frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f$$

converges in norm to a function $f^* \in \mathcal{L}^1$. *This limit function has the following properties:*

a) $f \geq 0 \Rightarrow f^* \geq 0$;

b) $\int_0^1 f^* dm = \int_0^1 f dm$;

c) $P_\tau f^* = f^*$ *and the measure* $d\mu^* = f^* dm$ *is invariant under* τ;

d) The function f^* is of bounded variation and, moreover, there exists a constant c, independent of the choice of the initial f, such that the variation of the limiting f^* satisfies the inequality

$$\bigvee_0^1 f^* \le c||f||_{\mathcal{L}_1},$$

where $\bigvee_a^b f$ denotes the variation of f over $[a,b]$.

One of the conditions required by this theorem is that there is a number N such that $s^N > 2$, where $s = \inf|\tau'|$. We denote $\tau^N = \phi$, which is also a C^2 piecewise map. The measure of each subinterval $I_i = [a_{i-1}, a_i], i = 1, ..., q$ of the map ϕ is denoted by $m_i = m(I_i)$. In order for the Lasota-Yorke inequality to hold, we need $|\phi_i'(x)| \ge s^N$ for each i. We also denote $\psi_i = \phi_i^{-1}$ and $\sigma_i(x) = |\psi_i'(x)|$. Then the Frobenius-Perron operator for the map ϕ is

$$P_\phi f(x) = \sum_{i=1}^{q} f(\psi_i(x))\sigma_i(x)\chi_i(x),$$

where $\chi_i(x)$ is the characteristic function of the interval I_i.

Theorem 2.5 (Lasota-Yorke inequality)**.** [11] *For a piecewise expanding C^2 map τ on the interval $I = [0,1]$ the following holds*

$$\bigvee_0^1 \mathrm{P}_\phi f \le 2s^{-N} \bigvee_0^1 f + \alpha||f||_{\mathcal{L}_1}, \tag{2.26}$$

where $\alpha = (K + \frac{2}{\beta})$, $K := \frac{\max|\sigma_i'|}{\min(\sigma_i)}$ and $\beta := \min_{1\le i\le q}(m_i)$.

2.5.1 Stability of Absolutely Continuous Invariant Measures for Piecewise Monotonic Transformation

The problem of the stability of acims poses the following question: instead of, being given a piecewise expanding map τ and its invariant density f, what happens with the invariant densities of the family of perturbed maps $\{\tau_\epsilon\}$, when $\epsilon > 0$ is very small? This can be stated in the following definition:

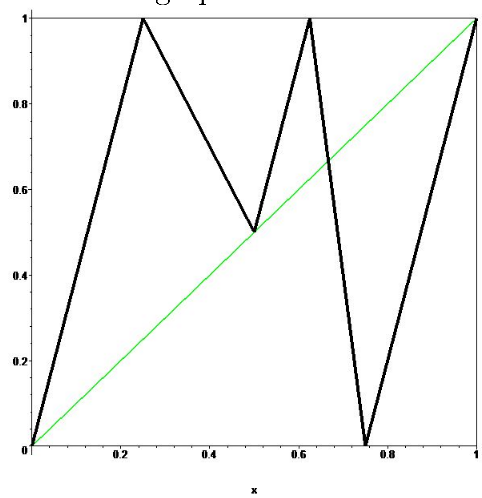
Definition 2.23. *Let $\tau : X \to X$ be a piecewise expanding map with its corresponding invariant density f. We say that τ is acim-stable if $\lim_{\epsilon\to 0} \tau_\epsilon = \tau$ implies that $\lim_{\epsilon\to 0} f_\epsilon = f$, where $\{\tau_\epsilon\}$ represents a family of ϵ-perturbed τ maps, and $\{f_\epsilon\}$ their respective invariant densities.*

The Lasota-Yorke inequality (Theorem 2.4) establishes the existence of an acim for maps with slopes strictly greater than 2 in magnitude. The problem studied in this thesis originates from a problem posed by Keller [4]. There, the author studies a family of maps that have slope equal to 2 in the limit and hence the standard Lasota-Yorke inequality cannot be used. This family of maps has another interesting feature. Namely, the point 1/2 is a fixed turning point which causes interesting behavior of these maps.

The main characteristic of these W-maps is the fact that as the parameters get very small, this family of maps converges to the unperturbed W-map (W_0), but their acims do not converge to the one of W_0. Keller concluded that this behavior occurs because of the invariant neighborhood around the fixed point 1/2, and conjectured that this construction is the only way that the acims will not converge to the one of the unperturbed map.

In [4] and [12] this conjecture was proven wrong: it was shown that for a family of W_a-maps, all with slopes greater than 2 in magnitude, as $a \to 0$, $W_a \to W_0$, f_a does not converge to f, the invariant density of W_0. The same behavior of this map was concluded in [2], where the authors presented a different family of perturbed three parameter Markov W-transformations. In this paper, the question raised is whether the map $\tau = (x + \frac{1}{2})\chi_{[0,1/2]} + (2 - 2x)\chi_{[1/2,1]}$ is acim-stable. The question is inspired by the third iterate of the τ-map, since looks similar to the W-map with a turning, fixed point at 1/2.

Figure 2.3: The graph of the third iterate of τ



In Chapter 4, we answer this question by introducing a family of perturbation maps $\{\tau_\epsilon\}_{\epsilon \geq 0}$. Below, we define the notion of deterministic perturbation and the necessary Skorokhod metric on $\mathcal{T}$, but first we define $\mathcal{T}$, the class of all piecewise expanding transformations.

Definition 2.24. *[1] Consider the interval $I = [a, b]$ with the normalized Lebesgue measure μ on I. We say that the map $\tau : I \to I$ belongs to the class of maps $\mathcal{T}$ if:*

a) τ is piecewise expanding map, which means that there exists a partition $\mathcal{P} = \{I_i = [a_{i-1}, a_i], i = 1, ..., q\}$ of I such that $\tau_i := \tau|_{I_i}$ is C^1 and $|\tau_i'(x)| \geq s_i > 1$ for any i and for all $x \in (a_{i-1}, a_i)$;

b) $g(x) = \frac{1}{|\tau'(x)|}$ is of bounded variation, where $\tau'(x)$ is the appropriate one-sided derivative at the endpoints of $\mathcal{P}$.

Definition 2.25. *Let $\tau : I \to I$ be a transformation and $\{\tau_n\}$ a family of maps on the same space. We say that $\{\tau_n\}$ is a small deterministic perturbation of τ, if $d_S(\tau, \tau_n) \to 0$ as $n \to \infty$, where $d_S(\tau_1, \tau_2)$ is the Skorokhod metric on $\mathcal{T}(I)$, defined as*

$$d_S(\tau_1, \tau_2) = \inf\{\delta > 0 : \exists A \subset I \text{ and } \exists \sigma : I \to I \text{ such that } \lambda(A) > 1 - \delta,$$
$$\sigma \text{ is a diffeomorphism}, \tau_1|_A = \tau_2 \circ \sigma|_A \text{ and } \forall x \in A, |\sigma(x) - x| < \delta,$$
$$\left|\frac{1}{\sigma'(x)} - 1\right| < \delta\}.$$

Lemma 2.4. *For each $\epsilon \geq 0$, the map $\tau_\epsilon : [0,1] \to [0,1]$, defined as*

$$\tau_\epsilon(x) = \begin{cases} (1+2\epsilon)x + \frac{1}{2} - \epsilon & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2(1-x) & \text{if } \frac{1}{2} < x \leq 1, \end{cases} \tag{2.27}$$

represents a small deterministic perturbation of the map $\tau = (x+1/2)\chi[0,1/2] + (2-2x)\chi[1/2,1]$. The graph of this map is shown in Fig. 4.1.

Proof. Here we show that that $d_S(\tau, \tau_\epsilon) \to 0$ as $\epsilon \to 0$, where $d_S(\tau, \tau_\epsilon)$ denotes the Skorokhod metric. For that purpose, we have to find a diffeomorphism σ, that will satisfy the conditions of the metric mentioned above.

The function σ is defined on $I = [0,1]$, such that $\tau_\epsilon|_A = \tau \circ \sigma|_A$ for $A \subset I$. Since on the interval $[1/2, 1]$, $\tau_\epsilon = \tau$ we define $\sigma(x) = x$ on $I_2 = [1/2, 1]$. On $[0, 1/2]$, σ has to be a linear function and hence on I_1, we write $\sigma(x) = ax + b$, for $a, b \in \mathcal{R}$. Then,

$$\tau \circ \sigma = ax + b + \frac{1}{2} = (1+2\epsilon)x + \frac{1}{2} - \epsilon = \tau_\epsilon,$$

or,

$$a = (1+2\epsilon) \text{ and } b = -\epsilon.$$

Hence, $\tau_\epsilon|_A = \tau \circ \sigma|_A$ where,

$$\sigma(x) = \begin{cases} (1+2\epsilon)x - \epsilon & \text{if } 0 \le x \le \frac{1}{2}, \\ x & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

Taking $A = I$, the Lebesgue measure $\lambda(A) = 1 > 1-\delta$ for $\delta > 0$. Let $\epsilon < \frac{\delta}{2}$. Then, for $x \in I_2$, $|\sigma(x) - x| = 0 < \delta$, and for $x \in I_1$, $|\sigma(x) - x| = |2\epsilon x - \epsilon| = |\epsilon(2x-1)| < \delta$. Moreover, on I_2, $\left|\frac{1}{\sigma'(x)} - 1\right| = 0 < \delta$, and on I_1, $\left|\frac{1}{\sigma'(x)} - 1\right| = \left|\frac{1}{1+2\epsilon} - 1\right| = \left|\frac{2\epsilon}{1+2\epsilon}\right| < 2\epsilon < \delta$. Hence, $d_S(\tau_1, \tau_2) < 2\epsilon$, and therefore $d_S(\tau_1, \tau_2) \to 0$ as $\epsilon \to 0$. □

Our goal is to express f_ϵ, as an infinite sum. In this way, we can show that the normalized densities $\widetilde{f}_\epsilon$ are uniformly bounded, which, by Lemma 2.1, implies weak precompactness in $\mathcal{L}^1$.

Definition 2.26. *For $\tau_\epsilon, \tau \in \mathcal{T}$ we say that $\tau_\epsilon \to \tau$ almost uniformly if for any $\epsilon > 0$, there exists a measurable set $A_\epsilon \subset I, \lambda(A_\epsilon) > 1 - \epsilon$, such that $\tau_\epsilon \to \tau$ uniformly on A_ϵ.*

Lemma 2.5. *The family of perturbed maps $\{\tau_\epsilon\}_{\epsilon \ge 0}$ defined in Lemma 2.4, converges almost uniformly to the limiting map τ.*

Proof. Let $A_\epsilon = A \subset I$ such that, on the set A, $\tau \to \tau\circ$ as $\epsilon \to 0$. Then, for $x \in A_\epsilon$,

$$|\tau_\epsilon(x) - \tau(x)| \le |\tau_\epsilon(x) - \tau \circ \sigma(x)| + |\tau \circ \sigma(x) - \tau(x)| \tag{2.28}$$

$$\le |\tau_\epsilon(x) - \tau \circ \sigma(x)| + M|\sigma(x) - x|, \tag{2.29}$$

where $M = \tau'(x^*), x^* \in (\sigma(x), x) \cup (x, \sigma(x))$. By Lemma 2.4, since $d_S(\tau, \tau_\epsilon) \to 0$, the first summand of (2.29) approaches 0 and $|\sigma(x) - x| < \delta \to 0$ as $\epsilon \to 0$. Hence, we conclude that $\tau_\epsilon \to \tau$ almost uniformly. □

The following lemma says that if we can establish that τ_ϵ converges to τ almost uniformly and $f_\epsilon \to f$ weakly in $\mathcal{L}^1$, then the invariant density of τ_ϵ converges to the invariant density of τ.

Lemma 2.6. *[1] Assume that τ_ϵ converges to τ almost uniformly. Let f_ϵ be a fixed point of $\mathrm{P}_\epsilon = \mathrm{P}_{\tau_\epsilon}$, i.e., $\mathrm{P}_\epsilon f_\epsilon = f_\epsilon$. If $f_\epsilon \to f$ weakly in $\mathcal{L}^1$, then $\mathrm{P}_\tau f = f$.*

In [9], the authors state a stability theorem of isolated eigenvalues of the linear operators satisfying the Lasota-Yorke inequality. A family of linear operators $(P_\epsilon)_{\epsilon \geq 0}$ is considered on the Banach space $(\mathcal{B}, || \cdot ||)$. These operators satisfy the following conditions: there are constants $C_1, M > 0$ such that for all $\epsilon \geq 0$,

$$||P_\epsilon^n||_{\mathcal{L}^1} \leq C_1 M \ \forall n \in \mathbb{N}; \tag{2.30}$$

there are $C_2, C_3 > 0$ and $\alpha \in (0,1), \alpha < M$, such that for all $\epsilon \geq 0$

$$||P_\epsilon^n f||_{BV} \leq C_2 \alpha^n ||f||_{BV} + C_3 M^n ||f||_{\mathcal{L}^1} \ \forall n \in \mathbb{N} \ \forall f \in \mathcal{B}; \tag{2.31}$$

as well as

$$\text{if } z \ \in \sigma(\mathcal{P}_\epsilon), |z| > \alpha, \text{then } z \text{ is not in the residual spectrum of } \mathcal{P}_\epsilon; \tag{2.32}$$

and there is a monotone upper-semicontinuous function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi_\epsilon > 0$ if $\epsilon > 0$, where $\varphi_\epsilon = 12 d_S(\tau, \tau_\epsilon)$, and

$$|||P_0 - P_\epsilon||| \leq \varphi_\epsilon \to 0, \tag{2.33}$$

where for any bounded linear operator $P : \mathcal{B} \to \mathcal{B}$, $|||P||| := \sup\{|Pf| : f \in \mathcal{B}, ||f|| \leq 1\}$.

Let us assume that the Frobenius-Perron operator P of a piecewise-expanding map τ and the family $(P_\epsilon)_{\epsilon > 0}$ corresponding to the family of perturbed maps τ_ϵ all satisfy the conditions (2.30)-(2.33). The "closeness" of τ_ϵ to τ is defined in the sense of the Skorokhod metric.

Theorem 2.6 ([9])**.** *Suppose that* $(\mathrm{P}_\epsilon)_{\epsilon>0}$ *is a family of linear operators on* $\mathcal{B}$ *satisfying (2.30)-(2.33). Fix* $\delta > 0$ *and* $r \in (\alpha, M)$ *and let* $\eta := \frac{\log r/\alpha}{\log M/\alpha}$. *Then* $\eta > 0$ *and there are constants* $\epsilon_0 = \epsilon_0(\delta, r) > 0, a = a(r) > 0, b = b(\delta, r) > 0$ *and* $d = d(\delta, r) > 0$ *such that for* $0 \leq \epsilon \leq \epsilon_0$ *and* $z \in \mathbb{C}\ V_{\delta,r}$

$$||(z - \mathrm{P}_\epsilon)^{-1} f||_{BV} \leq \alpha ||f||_{BV} + b||f||_{\mathcal{L}^1} \text{ for all} f \in \mathcal{B} \tag{2.34}$$

and

$$|||(z - \mathrm{P}_\epsilon)^{-1} - (z - \mathrm{P}_0)^{-1}||| \leq \varphi_\epsilon^\eta \big(c||(z - \mathrm{P}_0)^{-1}||_{BV} + d||(z - \mathrm{P}_0)^{-1}||_{BV}^2\big). \tag{2.35}$$

Let λ be an isolated eigenvalue of P_0 such that $|\lambda| > \alpha$. Then δ can be chosen very small so that $B_\delta \cap \delta_\alpha(P_0) = \{\lambda\}$ and we define

$$\Pi_\epsilon^{(\lambda,\delta)} := \frac{1}{2\pi\iota} \int_{\partial B_\delta(\lambda)} (z - P_\epsilon)^{-1} dz. \tag{2.36}$$

Corollary 2.1. *With the same assumptions as Theorem (2.6), if* λ *is an isolated eigenvalue of* P_0 *with* $|\lambda| > 0$ *and if* $\delta > 0$ *is such that* $\mathrm{B}_\delta \cap \delta_\alpha(\mathrm{P}_0) = \{\lambda\}$*, we have:*

1) There is a constant $K_1 = K_1(\delta, r) > 0$ *such that* $|||\Pi_\epsilon^{(\lambda,\delta)} - \Pi_0^{(\lambda,\delta)}||| \leq K_1 \cdot \varphi_\epsilon^\eta$ *for all* $\epsilon \in [0, \epsilon_0]$.

2)There are constants $K_2 = K_2(\delta, r) > 0$ *and* $\delta_0 = \delta_0(r) > 0$ *such that* $||\Pi_\epsilon^{(\lambda,\delta)} f||_{BV} \leq K_2 \cdot ||\Pi_\epsilon^{(\lambda,\delta)} f||_{\mathcal{L}^1}$ *for all* $f \in \mathrm{B}, \delta \in (0, \delta_0]$ *and* $\epsilon \in [0, \epsilon_1]$.

3) If $\delta \in (0, \delta_0]$, *then* $\mathrm{rank}(\Pi_\epsilon^{(\lambda,\delta)}) = \mathrm{rank}(\Pi_0^{(\lambda,\delta)})$ *for* ϵ *very small.*

Chapter 3

General Formula for Invariant Densities of Piecewise Linear Maps

In [6] the author developed a formula for the invariant density of a linear map $\tau : [0,1] \to [0,1]$, where the only restrictions on τ are that the map is onto and eventually piecewise expanding.For this class of maps the invariant density can be represented as an infinite series.

3.1 Definitions and notation

As it was already mentioned, we are dealing with a piecewise linear map $\tau : [0,1] \to [0,1]$. There is no restriction on the slopes of the branches as long as $|(\tau^n)'| > 1$, for some $n \geq 1$. By N we denote the total number of branches of τ. K represents the number of not onto branches and L represents the number of branches that do not touch zero and one. Obviously $L \leq K \leq N$.

We also define three sequences of numbers: the length of the branches α_i with $0 < \alpha_i \leq 1$ and $i = 1, ...N$; the heights of the lower endpoints of the branches γ_i with $0 \leq \gamma_i \leq 1 - \alpha_i$ and $i = 1, ...N$; and the slopes of the branches β_i with $i = 1, ...N$ and $\beta_i \neq 0$.

Depending on the lengths of the branches and the heights of the endpoints, we group the branches that are not onto in three different groups: "lazy" branches are the ones such that $\gamma_i + \alpha_i = 1$; "greedy" branches are the one such that $\gamma_i = 0$; and the ones such that $0 < \gamma_i$ and $\gamma_i + \alpha_i < 1$ are called "hanging" branches [6].

For example, let us look at the graph of the piecewise map ϕ, whose graph is given in Figure (3.1). The first two branches are examples of "hanging" branches, the third one is "greedy" and the last branch is an example of a "lazy" branch.

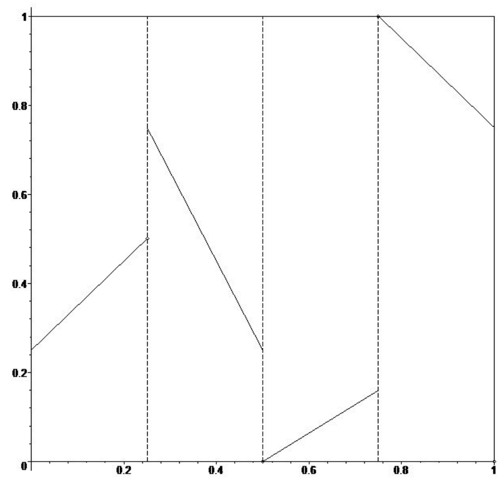


Figure 3.1: The graph of the map of ϕ

Here we define the points c_i that play a crucial role in deriving the invariant density of τ. The points c_i are the endpoints of the domains of the shorter branches that do not touch zero or one (or simply not onto branches). Since one c_i can represent two endpoints for two different branches, we define a pair (c, i), $c \in [0, 1]$, $1 \leq i \leq N$ and c is one of the endpoints of the interval I_i. We also define the index function on points: $c_i : j(c_i, k = k)$.

We define c_i to be the endpoints of the not onto branches such that either: they are the right-hand side endpoints of increasing or the left-hand side endpoints of the decreasing 'greedy' branches; are the left-hand side endpoints of increasing or the right-hand side endpoints of the decreasing 'lazy' branches; or the both endpoints of the 'hanging' branches. We enumerate them in a way that $c_1 < c_2 < ... < c_{K+L}$, such that $(c, j) < (d, k)$ if either $c < d$ or $c = d$ and $j < k$.

We also group c_i into two disjoint sets. W_u contains the c_i that are the upper endpoints of the 'greedy' branches, the right-hand side endpoints of the domains of the 'hanging' branches and the left-hand side endpoints of the domains of decreasing 'hanging' branches. On the other hand, W_l contains the c_i that are the lower endpoints of the 'lazy' branches, the left-hand side endpoints of the domains of the 'hanging' branches and the right-hand side endpoints of the domains of decreasing 'hanging' branches. Also we group c_i into U_l and U_r depending on whether c_i is the left-hand side or the right-hand side endpoint of the domain I_i of a certain branch.

The endpoints of of the domains of the branches are denoted by b_i, where $b_1 = 0$ and $b_j = \frac{\alpha_1}{|\beta_1|} + ... + \frac{\alpha_{j-1}}{|\beta_{j-1}|}$, for $J = 1, 2, ..., N + 1$. Depending on the slopes β_j, we define the set of numbers $A = \{a_1, a_2, ...a_N\}$ in such a way that:
if $\beta_j > 0$, then $a_j = \beta_j b_j - \gamma_j = \beta_j b_{j+1} - (\gamma_j + \alpha_j)$, and
if $\beta_j < 0$, then $a_j = \beta_j b_j - (\gamma_j + \alpha_j) = \beta_j b_{j+1} - \gamma_j$.

The set A gives a convenient way of representing the map τ in the following way:

$$\tau(x) = \beta_j \cdot x - a_j.$$

Finally, we define the cumulative slopes for iterates of points as follows:

$$\beta(x, 1) = \beta_{j(x)},$$

$$\beta(x, n) = \beta(x, n-1) \cdot \beta_{j(\tau^{n-1}(x))}, \; n \geq 2.$$

3.2 Formula for the invariant density of τ

We define the δ(condition) that equals one whenever the condition is satisfied and it is equal to zero otherwise.

Definition 3.1. *Let* $\mathbf{S}$ *be a matrix* $(S_{i,j})_{1\le j,k\le K+L}$ *with entries*

$$S_{i,j} = \sum_{n=1}^{\infty} \frac{1}{|\beta(c_i,n)|}\left[\delta(\beta(c_i,n)>0)\delta(\tau^n(c_i)>c_j)+\delta(\beta(c_i,n)<0)\delta(\tau^n(c_i)<c_j),\right.$$

for $c_i \in U_r$ *and all* c_j*, and,*

$$S_{i,j} = \sum_{n=1}^{\infty} \frac{1}{|\beta(c_i,n)|}\left[\delta(\beta(c_i,n)<0)\delta(\tau^n(c_i)>c_j)+\delta(\beta(c_i,n)>0)\delta(\tau^n(c_i)<c_j),\right.$$

for $c_i \in U_l$ *and all* c_j*.*

Definition 3.2. *Let* $\mathbf{Id}$ *be a* $(K+L)\times(K+L)$ *identity matrix and let* $\mathbf{v} = [1,1,...1,1]$ *be a* $K+L$*-dimensional vector. Let* $D = [D_1, ..., D_{K+L}]$ *be the solution of the system*

$$(-\mathbf{S}^T + \mathbf{Id})D^T = D_0\mathbf{v}^T \tag{3.1}$$

where D_0 *is either zero or one.*

We also define,

$$\chi^s(\beta,x) = \begin{cases} \chi[0,x] & \text{for } \beta>0 \\ \chi[x,1] & \text{for } \beta<0. \end{cases} \tag{3.2}$$

Now we state the main theorem of [6] that gives us the invariant density function for eventually expanding, piecewise linear maps.

Theorem 3.1. *([6]) Let* τ *be an eventually expanding, piecewise linear map. Then system (3.1) always has a non-vanishing solution. If* 1 *is not an eigenvalue of* $\mathbf{S}$*, then with* $D_0 = 1$*. If* 1 *is an eigenvalue of* $\mathbf{S}$*, then at least with* $D_0 = 0$*. Let*

$$h = D_0 + \sum_{c_i\in U_r} D_i \sum_{n=1}^{\infty} \frac{\chi^s(\beta(c_i,n),\tau^n(c_i))}{|\beta(c_i,n)|} + \sum_{c_i\in U_l} D_i \sum_{n=1}^{\infty} \frac{\chi^s(-\beta(c_i,n),\tau^n(c_i))}{|\beta(c_i,n)|}, \tag{3.3}$$

where the constants $D_i, i = 1, ..., K$ satisfy the system (3.1). Then h is τ-invariant. If all values $\tau(c_i), i = 1, ..., K+L$, are different, then the inverse statement also holds. In particular, the system (3.1) is uniquely solvable (i.e., 1 in not an eigenvalue of $\mathbf{S}$) if $\min_{1\leq j\leq N} |\beta_j| > K + L + 1$.

Example 3.1. *Consider the family of maps $\{\tau_\epsilon\}_{\epsilon\geq 0}$, introduced in Lemma 2.4. For each ϵ, τ_ϵ is defined as*

$$\tau_\epsilon(x) = \begin{cases} (1+2\epsilon)x + \frac{1}{2} - \epsilon & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2(1-x) & \text{if } \frac{1}{2} < x \leq 1. \end{cases} \tag{3.4}$$

From the graph of τ_ϵ, [Figure 4.1], it is obvious that $N = 2, K = 1$ and $L = 0$. The length of the branches are $\alpha_1 = \frac{1}{2} + \epsilon$ and $\alpha_2 = 1$. The slopes of the the corresponding branches are given by $\beta = (1 + 2\epsilon, -2)$, and the heights of the lower end-points are given by $\gamma = (\frac{1}{2} - \epsilon, 0)$.

From the values of β_1 and β_2 we derive the corresponding digits $A = (a_1, a_2)$ such that

$$a_1 = \beta_1 b_1 - \gamma_1 = (1+2\epsilon)0 - (\frac{1}{2} - \epsilon) = -\frac{1}{2} + \epsilon,$$
$$\text{and } a_2 = \beta_2 b_2 - (\gamma_2 + \alpha_2) = -2 \cdot 1/2 - (0+1) = -2.$$

Since there is only one branch of τ_ϵ that is not onto, there is only one $c_1 = (0, 1)$. That means that $W_l = \{c_1\}$, $W_u = \emptyset$, $U_l = \{c_1\}$ and $U_r = \emptyset$. The cumulative slope is given by $\beta(0,1) = \frac{1}{2} + \epsilon$, $\beta(0,2) = (\frac{1}{2} + \epsilon)^2$, $\beta(0,3) = -2(\frac{1}{2} + \epsilon)^2$ and so on. Since $U_l = \{c_1\}$ and $U_r = \emptyset$, and since $\tau^n(0) > 0$ for $n = 1, 2, ...,$ i.e., $\delta(\tau^n(0) < 0) = 0$ for any $n \geq 1$, $\mathbf{S}$ is represented by the constant

$$S_\epsilon = \sum_{n=1}^{\infty} \frac{1}{|\beta(0,n)|} \big[\delta(\beta(0,n) < 0)\delta(\tau^n(0) > 0)\big]. \tag{3.5}$$

Using this fact, we rewrite the system (3.1) for the map τ_ϵ as

$$(-S_\epsilon + 1)D = 1. \tag{3.6}$$

Then the invariant density of the map τ_ϵ is given by,

$$h = 1 + D\sum_{n=1}^{\infty} \frac{\chi^s(-\beta(0,n), \tau^n(0))}{|\beta(0,n)|}. \tag{3.7}$$

Chapter 4

Deriving the Invariant Density for τ_ϵ

The main goal of this thesis is to show that τ is acim-stable (Definition 2.23). We recall the family of deterministically perturbed maps τ_ϵ,

$$\tau_\epsilon(x) = \begin{cases} (1+2\epsilon)x + \frac{1}{2} - \epsilon & \text{if } 0 \le x \le \frac{1}{2}, \\ 2(1-x) & \text{if } \frac{1}{2} < x \le 1, \end{cases}$$

and try to express the family of measures f_ϵ.

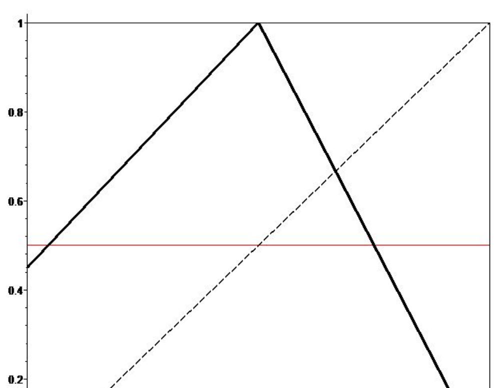


Figure 4.1: The graph of τ_ϵ for $\epsilon = 0.05$

We recall from the previous chapter that the invariant density is given by

$$f_\epsilon = 1 + D\sum_{n=1}^{\infty}\frac{\chi^s(-\beta(0,n),\tau^n(0))}{|\beta(0,n)|},$$

where $D = \frac{1}{-S_\epsilon+1}$ and,

$$S_\epsilon = \sum_{n=0}^{\infty}\frac{\delta(-\beta(0,n)<0)\delta(\tau^n(0)>0)}{|\beta(0,n)|}.$$

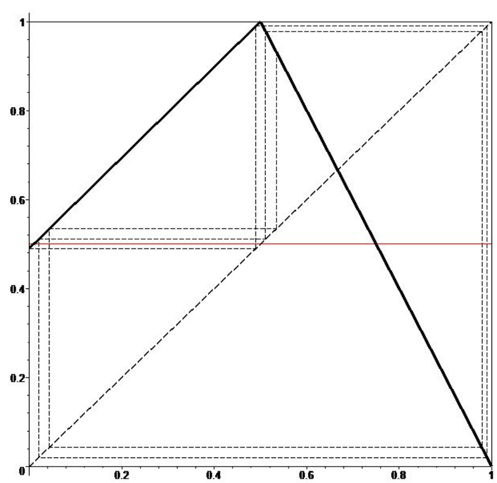
Recall the function $\chi^s(\beta,x)$,

$$\chi^s(\beta,x) = \begin{cases} \chi[0,x] & \text{for } \beta>0, \\ \chi[x,1] & \text{for } \beta<0. \end{cases} \tag{4.1}$$

4.1 Estimates on the Constant D

We follow the trajectory of $\tau_\epsilon(0)$, stating the intervals in which $\tau_\epsilon^i(0)$ belongs for $i = 1,2,3...l$, where l is defined below. Let $I_1 = [0,\frac{1}{2}]$ and $I_2 = [\frac{1}{2},1]$. Then $\tau(0)\in I_1$, $\tau^2(0)\in I_2$, $\tau^3(0)\in I_1$, $\tau^4(0)\in I_2$, $\tau^5(0)\in I_2$, $\tau^6(0)\in I_1$, and the iterations of $\tau_\epsilon(0)$ will follow this pattern up to a certain iteration.

Figure 4.2: The first 8 iterations of $\tau_\epsilon(0)$



We define the l-th iteration as the first time $\tau_\epsilon^i(0)$ will abandon this pattern, i.e., $\tau_\epsilon^i(0) \in I_1, I_2, I_1, I_2, I_2, I_1, I_2, I_2, I_1, ...$ for $1 \le i < l$. Then,

$$\begin{aligned}
S_\epsilon = \sum_{n=0}^{\infty} \frac{\delta(\beta(0,n)<0)\delta(\tau^n(0)>0)}{|\beta(0,n)|} &= 0 + 0 + \frac{1}{2(1+2\epsilon)^2} + \frac{1}{2(1+2\epsilon)^3} + 0 + \frac{1}{2^3(1+2\epsilon)^3} \\
&+ \frac{1}{2^3(1+2\epsilon)^4} + 0 + \frac{1}{2^5(1+2\epsilon)^4} + \frac{1}{2^5(1+2\epsilon)^5} + 0 + \frac{1}{2^7(1+2\epsilon)^5} + ... \\
&= \frac{1}{2(1+2\epsilon)^2} + \frac{1}{2(1+2\epsilon)^3}(1+\frac{1}{2^2}) + \frac{1}{2^3(1+2\epsilon)^4}(1+\frac{1}{2^2}) + \frac{1}{2^5(1+2\epsilon)^5}(1+\frac{1}{2^2}) + ... \\
&= \frac{1}{2(1+2\epsilon)^2} + (1+\frac{1}{4}) \underbrace{\left[\frac{1}{2(1+2\epsilon)^3} + \frac{1}{2^3(1+2\epsilon)^3} + \frac{1}{2^5(1+2\epsilon)^5} + ...\right]}_{\frac{2}{3}(l-3)\text{-times}} + ...
\end{aligned}$$

Since, $\frac{1}{2^i(1+2\epsilon)^j} < \frac{1}{2^i}$ for any $i, j = 1, 2, ...$ we can bound S_ϵ by

$$\begin{aligned}
S_\epsilon &< \frac{1}{2} + \frac{5}{4}(\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + ...) + ... \\
&= \frac{1}{2} + \frac{5}{4}\sum_{i=1}^{k_1}(\frac{1}{2})^{2i-1} + ... \sum_{i=k_1+1}^{\infty} \frac{\delta(\beta(0,n)<0)\delta(\tau^n(0)>0)}{|\beta(0,n)|},
\end{aligned}$$

where $k_1 = [\frac{2}{3}l - 2]$.

Next we need to calculate how big is the tail of S_ϵ, i.e., what happens with this sum after the l-th iteration. The cumulative slope up to l is $\frac{1}{2(1+2\epsilon)^2}\left[\frac{1}{4(1+2\epsilon)}\right]^{l-3}$.

Hence, S_ϵ is bounded by

$$S_\epsilon < \frac{1}{2} + \frac{5}{4}\sum_{i=1}^{k_1}(\frac{1}{2})^{2i-1} + \frac{1}{2(1+2\epsilon)^2}\left[\frac{1}{4(1+2\epsilon)}\right]^{l-3} \sum_{j=l+1}^{\infty} \frac{1}{|\beta(\tau^j(0), i)|}. \tag{4.2}$$

The following lemmas will be important for bounding S_ϵ:

Lemma 4.1. *For $l = \min\{j \ge 1 : \tau_\epsilon^j$ abandons the pattern$\}$, the following holds:*

a) $\lim_{\epsilon\to 0} l = +\infty$;

b) $\lim_{\epsilon\to 0}(l\epsilon) = 0$;

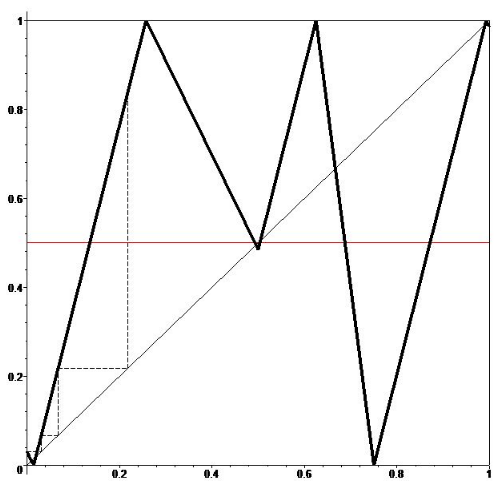
c) $\lim_{\epsilon\to 0} \frac{1}{(1+2\epsilon)^l} = 1$.

Proof. a) The third iterate of τ_ϵ is given by the formula:

$$\tau_\epsilon^3(x) = \begin{cases} -2(1+2\epsilon)^2(x - \frac{\epsilon}{1+2\epsilon}) & \text{if } x \in [0, \frac{\epsilon}{1+2\epsilon}), \\ 4(1+2\epsilon)(x - \frac{\epsilon}{1+2\epsilon}) & \text{if } x \in [\frac{\epsilon}{1+2\epsilon}, \frac{1+4\epsilon}{4(1+2\epsilon)}), \\ -2^2(x - \frac{\epsilon}{1+2\epsilon}) + 1/2 - \epsilon & \text{if } x \in [\frac{1+4\epsilon}{4(1+2\epsilon)}), 1/2), \\ 4(1+2\epsilon)(x - 5/8) - 1 & \text{if } x \in [1/2, 5/8), \\ -8(x - 3/4) & \text{if } x \in [5/8, 3/4), \\ 4(1+2\epsilon)(x - 3/4) & \text{if } x \in [3/4, \frac{2+7\epsilon+6\epsilon^2}{2(1+2\epsilon)^2}), \\ -2(1+2\epsilon)^2 x + 2 + 7\epsilon + 6\epsilon^2 & \text{if } x \in [\frac{2+7\epsilon+6\epsilon^2}{2(1+2\epsilon)^2}, 1]. \end{cases} \tag{4.3}$$

Observing the graph of $\tau_\epsilon^{(3)}(x)$, we see that $\tau_\epsilon^3(0) = 2\epsilon(1+2\epsilon)$ and that the next iterations are below 1/2 and they land on the line $y = 4(1+2\epsilon)x - 4\epsilon$. This behavior coincides with the defined pattern where every third iteration is less than 1/2. The orbit will abandon the defined pattern the moment when the third iterate will be greater than $\frac{1+4\epsilon}{4(1+2\epsilon)}$. We define $m = \min\{i : (\tau_\epsilon^3)^i \geq \frac{1+4\epsilon}{4(1+2\epsilon)}\}$. This means that we can prove $\lim_{\epsilon\to 0} l = +\infty$ by proving that $\lim_{\epsilon\to 0} m = +\infty$, where $m = \frac{l-3}{3}$.

Figure 4.3: The first 4 iterations of $\tau_\epsilon^3(0)$ for $\epsilon = 0.015$



In order to see when the third iterate of τ_ϵ will be greater than $\frac{1+4\epsilon}{4(1+2\epsilon)}$, we use the general formula for iterating a linear function. If $T(x) = ax + b$, then

$$T^m(x) = a^m x + b \sum_{i=1}^{m} a^{i-1}.$$

In our case $x = 2\epsilon(1+2\epsilon)$ and the third iterate up to l has the form

$$\tau^m(2\epsilon(1+2\epsilon)) = [4(1+2\epsilon)]^m [2\epsilon(1+2\epsilon)] + (-4\epsilon) \sum_{i=1}^{m} [4(1+2\epsilon)]^{i-1}.$$

Then we can express the following inequality:

$$\tau^m(2\epsilon(1+2\epsilon)) = [4(1+2\epsilon)]^m [2\epsilon(1+2\epsilon)] + (-4\epsilon) \sum_{i=1}^{m} [4(1+2\epsilon)]^{i-1} \geq \frac{1+4\epsilon}{4(1+2\epsilon)}, \text{or,}$$

$$[4(1+2\epsilon)]^m (2\epsilon)(1+2\epsilon) - 4\epsilon \left[\frac{1 - [4(1+2\epsilon)]^m}{1 - 4(1+2\epsilon)} \right] \geq \frac{1+4\epsilon}{4(1+2\epsilon)}, \text{or,}$$

$$[4(1+2\epsilon)]^m (2\epsilon)(1+2\epsilon) - \frac{4\epsilon}{1 - 4(1+2\epsilon)} + \frac{4\epsilon[4(1+2\epsilon)]^m}{1 - 4(1+2\epsilon)} \geq \frac{1+4\epsilon}{4(1+2\epsilon)}, \text{or,}$$

$$[4(1+2\epsilon)]^m \left[\frac{-2\epsilon(16\epsilon^2 + 14\epsilon + 1)}{-(3+8\epsilon)} \right] \geq \frac{1+4\epsilon}{4(1+2\epsilon)} - \frac{4\epsilon}{3+8\epsilon}, \text{or,}$$

$$[4(1+2\epsilon)]^m \left[\frac{2\epsilon(16\epsilon^2 + 14\epsilon + 1)}{3+8\epsilon} \right] \geq \frac{3+4\epsilon}{4(1+2\epsilon)(3+8\epsilon)}, \text{or,}$$

$$[4(1+2\epsilon)]^m \geq \frac{3+4\epsilon}{8\epsilon(1+2\epsilon)(16\epsilon^2 + 14\epsilon + 1)}, \text{or,}$$

$$m \cdot \ln[4(1+2\epsilon)] \geq \ln \frac{3+4\epsilon}{8\epsilon(1+2\epsilon)(16\epsilon^2 + 14\epsilon + 1)}, \text{or,}$$

$$m \geq \frac{\ln[\frac{3+4\epsilon}{8\epsilon(1+2\epsilon)(16\epsilon^2+14\epsilon+1)}]}{\ln[4(1+2\epsilon)]}.$$

Taking the limit as $\epsilon \to 0$, we get

$$\lim_{\epsilon \to 0} m \geq \lim_{\epsilon \to 0} \left\{ \frac{\ln(3+4\epsilon) - \ln(8\epsilon) - \ln(1+2\epsilon) - \ln(16\epsilon^2 + 14\epsilon + 1)}{\ln[4(1+2\epsilon)]} \right\},$$

$$\text{or, } \lim_{\epsilon \to 0} m \geq \lim_{\epsilon \to 0} \left\{ \frac{\ln(3+4\epsilon)}{\ln[4(1+2\epsilon)]} - \frac{\ln(8\epsilon)}{\ln[4(1+2\epsilon)]} \right\} = +\infty.$$

Since $\lim_{\epsilon\to 0} m = +\infty$ then, $\lim_{\epsilon\to 0} l = +\infty$.

b) As in the proof in a), since $\tau : [0,1] \to [0,1]$, we will find the relation between l and ϵ by solving the inequality:

$$\tau^m(2\epsilon(1+2\epsilon)) = [4(1+2\epsilon)]^m[2\epsilon(1+2\epsilon)] + (-4\epsilon)\sum_{i=1}^{m}[4(1+2\epsilon)]^{i-1} \le 1, \text{or},$$

$$[4(1+2\epsilon)]^m(2\epsilon)(1+2\epsilon) - 4\epsilon\left[\frac{1-[4(1+2\epsilon)]^m}{1-4(1+2\epsilon)}\right] \le 1, \text{or},$$

$$[4(1+2\epsilon)]^m(2\epsilon)(1+2\epsilon) - \frac{4\epsilon}{1-4(1+2\epsilon)} + \frac{4\epsilon[4(1+2\epsilon)]^m}{1-4(1+2\epsilon)} \le 1, \text{or},$$

$$[4(1+2\epsilon)]^m\left[\frac{-2\epsilon(16\epsilon^2+14\epsilon+1)}{-(3+8\epsilon)}\right] \le 1 - \frac{4\epsilon}{3+8\epsilon}, \text{or},$$

$$[4(1+2\epsilon)]^m \le \frac{3+4\epsilon}{2\epsilon(16\epsilon^2+14\epsilon+1)}, \text{or},$$

$$[4(1+2\epsilon)]^{\frac{l-3}{3}} \le \frac{3+4\epsilon}{2\epsilon(16\epsilon^2+14\epsilon+1)}, \text{ or},$$

$$[4(1+2\epsilon)]^l \le \frac{[4(3+4\epsilon)(1+2\epsilon)]^3}{[2\epsilon(16\epsilon^2+14\epsilon+1)]^3}, \text{or},$$

$$l\ln 4(1+2\epsilon) \le 3\ln\frac{4(3+4\epsilon)(1+2\epsilon)}{2\epsilon(16\epsilon^2+14\epsilon+1)}, \text{or},$$

$$\epsilon l\ln 4(1+2\epsilon) \le 3\epsilon\ln\frac{4(3+4\epsilon)(1+2\epsilon)}{2\epsilon(16\epsilon^2+14\epsilon+1)}, \text{or},$$

$$\epsilon l \le 3\epsilon\frac{\ln\frac{4(3+4\epsilon)(1+2\epsilon)}{2\epsilon(16\epsilon^2+14\epsilon+1)}}{\ln 4(1+2\epsilon)}.$$

Taking the limit as $\epsilon \to 0$, we get

$$\lim_{\epsilon\to 0}\epsilon l \le \lim_{\epsilon\to 0} 3\epsilon\frac{\ln\frac{2(3+4\epsilon)(1+2\epsilon)}{\epsilon(16\epsilon^2+14\epsilon+1)}}{\ln 4(1+2\epsilon)} \tag{4.4}$$

or,

$$\lim_{\epsilon\to 0}\epsilon l \le \lim_{\epsilon\to 0} 3\epsilon\left[\frac{\ln 2(3+4\epsilon)}{\ln 4(1+2\epsilon)} + \cancelto{0}{\frac{\ln(1+2\epsilon)}{\ln 4(1+2\epsilon)}} - \frac{\ln\epsilon}{\ln 4(1+2\epsilon)} - \cancelto{0}{\frac{\ln(16\epsilon^2+14\epsilon+1)}{\ln 4(1+2\epsilon)}}\right]$$

$$\text{or } \lim_{\epsilon\to 0}\epsilon l \le \cancelto{0}{\lim_{\epsilon\to 0}\frac{3\epsilon\ln 2(3+4\epsilon)}{\ln 4(1+2\epsilon)}} - \lim_{\epsilon\to 0}\frac{3\epsilon\ln\epsilon}{\ln 4(1+2\epsilon)}$$

$$\text{or } \lim_{\epsilon\to 0}\epsilon l \le -\lim_{\epsilon\to 0}\frac{3\epsilon\ln\epsilon}{\ln 4(1+2\epsilon)} = 0.$$

c)

$$\lim_{\epsilon\to 0}\frac{1}{(1+2\epsilon)^l} = \lim_{\epsilon\to 0}(1+2\epsilon)^{-l} = \lim_{\epsilon\to 0} e^{\ln(1+2\epsilon)^{-l}} = \lim_{\epsilon\to 0} e^{-l\ln(1+2\epsilon)}.$$

Now,

$$\lim_{\epsilon\to 0}\big[-l\ln(1+2\epsilon)\big] = \lim_{\epsilon\to 0}\Big[-l(2\epsilon)\frac{\ln(1+2\epsilon)}{2\epsilon}\Big] =$$

$$\lim_{\epsilon\to 0}\big[(-2l\epsilon)\ln(1+2\epsilon)^{\frac{1}{2\epsilon}}\big] = \lim_{\epsilon\to 0}(-2l\epsilon)\cdot\lim_{\epsilon\to 0}\big[\ln(1+2\epsilon)^{\frac{1}{2\epsilon}}\big].$$

Then, we can write

$$\lim_{\epsilon\to 0}\big[-l\ln(1+2\epsilon)\big] = -2\lim_{\epsilon\to 0}(l\epsilon)\cdot\ln\lim_{\epsilon\to 0}(1+2\epsilon)^{\frac{1}{2\epsilon}} = -2\lim_{\epsilon\to 0}(l\epsilon)\cdot\ln(e^{-1}) = 2\lim_{\epsilon\to 0}(l\epsilon). \tag{4.5}$$

Using the first part of Lemma (4.1), we can conclude that,

$$\lim_{\epsilon\to 0}\big[-l\ln(1+2\epsilon)\big] = 0. \tag{4.6}$$

Hence,

$$\lim_{\epsilon\to 0}\frac{1}{(1+2\epsilon)^l}=\lim_{\epsilon\to 0}e^{-l\ln(1+2\epsilon)}=e^0=1.$$

$\square$

Lemma 4.2. *Consider the map $\tau_\epsilon(x)$ on $[0,1]$. Then, the following two results hold:*

a)The trajectory of 0 does not visit the left branch for more than two consecutive iterations, i.e., for any $i=1,2,...,$ if $\tau_\epsilon^i(0)\in I_1$, then either $\tau_\epsilon^{i+1}(0)\in I_2$ or $\tau_\epsilon^{i+2}(0)\in I_2$ and;

b) For any $i\geq l+1$, for the cumulative slope $\beta(\tau^i(0),i)$, the following holds: $\frac{1}{|\beta(\tau^i(0),i)|}\leq(\frac{1}{2})^{\frac{i}{3}}$

Proof. a) We take into consideration that,

$$\tau_\epsilon([0,\frac{\epsilon}{1+2\epsilon}])=[1/2-\epsilon,1/2]\in[\frac{\epsilon}{1+2\epsilon},1/2],$$

and

$$\tau_\epsilon([\frac{\epsilon}{1+2\epsilon},1/2])=[1/2,1].$$

Then, for any $\tau_\epsilon^i(0)\in I_1$ either $\tau_\epsilon^i(0)\in[0,\frac{\epsilon}{1+2\epsilon})$ or $\tau_\epsilon^i(0)\in[\frac{\epsilon}{1+2\epsilon},1/2]$. If $\tau_\epsilon^i(0)\in[\frac{\epsilon}{1+2\epsilon},1/2]$, then $\tau_\epsilon^{i+1}(0)\in[1/2,1]$. Hence if $\tau_\epsilon^i(0)\in[\frac{\epsilon}{1+2\epsilon},1/2]$, then the next image lands on the right branch.

If $\tau_\epsilon^i(0)\in[0,\frac{\epsilon}{1+2\epsilon}]$, then $\tau_\epsilon^{i+1}(0)\in[\frac{\epsilon}{1+2\epsilon},1/2]$ and $\tau_\epsilon^{i+2}(0)\in[1/2.1]$. Hence if $\tau_\epsilon^i(0)\in[0,\frac{\epsilon}{1+2\epsilon}]$ then after two iterations the image lands on the right branch.

b) The cumulative slope $\beta(\tau^i(0),i)$ for every $i\geq l+1$ is defined as

$$\beta(\tau^i(0),i)=\beta^*(0)\cdot\tau_\epsilon'(\tau_\epsilon^{l+1}(0))\cdot\tau_\epsilon'(\tau_\epsilon^{l+2}(0))\cdot\tau_\epsilon'(\tau_\epsilon^{l+3}(0))\cdot...$$

where $\beta^*(0)$ is the cumulative slope up to the l-th iteration. Then,

$$\frac{1}{|\beta(\tau^i(0),i)|} = \frac{1}{|\beta^*(0)|} \cdot \frac{1}{|\tau'_\epsilon(\tau_\epsilon^{l+1}(0))|} \cdot \frac{1}{|\tau'_\epsilon(\tau_\epsilon^{l+2}(0))|} \cdot \frac{1}{|\tau'_\epsilon(\tau_\epsilon^{l+3}(0))|} \cdot ... \tag{4.7}$$

Grouping every three consecutive multiplicands after l and following (4.2,a), we get

$$\frac{1}{|\tau'_\epsilon(\tau_\epsilon^{l+j}(0))|} \cdot \frac{1}{|\tau'_\epsilon(\tau_\epsilon^{l+j+1}(0))|} \cdot \frac{1}{|\tau'_\epsilon(\tau_\epsilon^{l+j+2}(0))|} \leq \frac{1}{2(1+2\epsilon)^2}, \tag{4.8}$$

for $j = 1, 2....$ Then from (4.6), we obtain the desired inequality

$$\frac{1}{|\beta(\tau^i(0),i)|} \leq \frac{1}{|\beta^*(0)|} \cdot \frac{1}{2(1+2\epsilon)^2} \cdot \frac{1}{2(1+2\epsilon)^2} \cdot \frac{1}{2(1+2\epsilon)^2} \cdot ... \leq$$

$$\frac{1}{|\beta^*(0)|} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot ... = \frac{1}{|\beta^*(0)|} \cdot \frac{1}{2^{\frac{i}{3}}},$$

for $i \geq l+1$. □

Proposition 4.1. *For any* $\epsilon > 0$, $-4 < D < -3$.

Proof. We start by bounding the constant S_ϵ from above. Using Lemmas (4.1) and (4.2) we can rewrite the inequality (4.11) as

$$S_\epsilon < \frac{1}{2} + \frac{5}{4}\sum_{i=1}^{k_1}(\frac{1}{2})^{2i-1} + \frac{1}{2(1+2\epsilon)^2}\Big[\frac{1}{4(1+2\epsilon)}\Big]^{\frac{l-3}{3}} \sum_{j=l+1}^{\infty}(\frac{1}{2})^{\frac{i}{3}} \tag{4.9}$$

$$= \frac{1}{2} + \frac{5}{4}(2 \cdot \frac{1}{4} \cdot \frac{1-(\frac{1}{4})^{k_1}}{1-\frac{1}{4}}) + \frac{1}{2(1+2\epsilon)^2}\Big[\frac{1}{4(1+2\epsilon)}\Big]^{\frac{l-3}{3}}. \tag{4.10}$$

By Lemma 4.1 c), $\lim_{\epsilon\to 0}\big[\frac{1}{4(1+2\epsilon)}\big]^{\frac{l-3}{3}}=0$, and hence for small ϵ,

$$S_\epsilon < \frac{1}{2} + \frac{5}{4}(\frac{1}{2} \cdot \frac{4}{3}) = \frac{4}{3}. \tag{4.11}$$

On the other hand, for small ϵ,

$$S_\epsilon \geq \frac{1}{2(1+2\epsilon)^2} + \frac{1}{2(1+2\epsilon)^3} + \frac{1}{2^3(1+2\epsilon)^3} + \frac{1}{2^3(1+2\epsilon)^4} \geq \frac{10}{8}. \tag{4.12}$$

Substituting (4.11) and (4.12) in (3.6), we obtain a bound on D:

$$-4 < D < -3. \tag{4.13}$$

□

4.2 Estimates on the density f_ϵ

Proposition 4.2. *For the deterministically perturbed map τ_ϵ, the following holds:*

a) its invariant density f_ϵ is uniformly bounded, and uniformly bounded away from 0;

b) the normalized invariant density $\widetilde{f_\epsilon}$ is uniformly bounded.

Proof. a) From the definition of $\chi^s(\beta, x)$, we express

$$\chi^s(-\beta(0,n), \tau^n(0)) = \begin{cases} \chi_{[0,\tau^n(0)]} & \text{for } -\beta > 0, \\ \chi_{[\tau^n(0),1]} & \text{for } -\beta < 0. \end{cases} = \begin{cases} \chi_{[0,\tau^n(0)]} & \text{for } \beta < 0, \\ \chi_{[\tau^n(0),1]} & \text{for } \beta > 0. \end{cases} \tag{4.14}$$

Then,

$$\sum_{n=1}^{\infty} \frac{\chi^s(-\beta(0,n), \tau^n(0))}{|\beta(0,n)|} = \frac{\chi_{[\tau(0),1]}}{|\beta(0,1)|} + \frac{\chi_{[\tau^2(0),1]}}{|\beta(0,2)|} + \frac{\chi_{[0,\tau^3(0)]}}{|\beta(0,3)|} + \frac{\chi_{[0,\tau^4(0)]}}{|\beta(0,4)|} + \frac{\chi_{[\tau^5(0),1]}}{|\beta(0,5)|} + ...$$

$$= \frac{\chi_{[\frac{1}{2}-\epsilon,1]}}{1+2\epsilon} + \frac{\chi_{[1-\epsilon-2\epsilon^2,1]}}{(1+2\epsilon)^2} + \frac{\chi_{[0,2(\epsilon+2\epsilon^2)]}}{2(1+2\epsilon)^2} + \frac{\chi_{[0,\frac{1}{2}+\epsilon+8\epsilon^3+O(3)]}}{2(1+2\epsilon)^3} + ...$$

$$\geq \frac{\chi_{[\frac{1}{2}-\epsilon,1]}}{1+2\epsilon} + \frac{\chi_{[0,\frac{1}{2}+\epsilon+8\epsilon^3+O(3)]}}{2(1+2\epsilon)^3} > \frac{1}{2(1+2\epsilon)^3}.$$

The last inequality along with the bound of D in (4.12) enables us to give a proper upper bound on the invariant density f_ϵ,

$$f_\epsilon < 1 - 3 \cdot \frac{1}{2} = -\frac{1}{2}. \tag{4.15}$$

On the other hand,

$$\sum_{n=1}^{\infty} \frac{\chi^s(-\beta(0,n), \tau^n(0))}{|\beta(0,n)|} = \frac{\chi_{[\tau(0),1]}}{|\beta(0,n)|} + \frac{\chi_{[\tau^2(0),1]}}{|\beta(0,n)|} + \frac{\chi_{[0,\tau^3(0)]}}{|\beta(0,n)|} + \frac{\chi_{[0,\tau^4(0)]}}{|\beta(0,n)|} + \frac{\chi_{[\tau^5(0),1]}}{|\beta(0,n)|} + ...$$

$$= \frac{\chi_{[\frac{1}{2}-\epsilon,1]}}{1+2\epsilon} + \frac{\chi_{[1-\epsilon-2\epsilon^2,1]}}{(1+2\epsilon)^2} + \frac{\chi_{[0,2(\epsilon+2\epsilon^2)]}}{2(1+2\epsilon)^2} + \frac{\chi_{[0,\frac{1}{2}+\epsilon+8\epsilon^3+O(3)]}}{2(1+2\epsilon)^3} + ...$$

$$< \frac{1}{1+2\epsilon} + \frac{1}{(1+2\epsilon)^2} + \frac{1}{2(1+2\epsilon)^2} + \frac{1}{2(1+2\epsilon)^3} + ...$$

$$< \frac{1}{1+2\epsilon} + \frac{1}{(1+2\epsilon)^2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^4} + ...$$

$$= \frac{2(1+2\epsilon)}{(1+2\epsilon)^2} + 1 + \sum_{i=1}^{l} 2 \cdot (\frac{1}{4})^k + ... + \sum_{i=l+1}^{\infty} \frac{1}{2(1+2\epsilon)^3} \cdot (\frac{1}{4(1+2\epsilon)})^{\frac{l-4}{3}} \cdot \frac{1}{|\beta(\tau^i(0),i)|}$$

$$< \frac{2}{1+2\epsilon} + 1 + \frac{8(1-(1/4)^l)}{3} + \frac{1}{2(1+2\epsilon)^3} \cdot (\frac{1}{4(1+2\epsilon)})^{\frac{l-4}{3}} \cdot \sum_{i=l+1}^{\infty} (\frac{1}{2})^{\frac{i}{3}}.$$

For small ϵ the last inequality approaches $\frac{17}{3}$, i.e.,

$$\sum_{n=1}^{\infty} \frac{\chi^s(-\beta(0,n), \tau^n(0))}{|\beta(0,n)|} < 2 + 1 + \frac{8}{3} = \frac{17}{3}. \tag{4.16}$$

Using (4.15) we give a lower bound on h_ϵ,

$$-16 = 1 - 3 \cdot \frac{17}{3} < f_\epsilon. \tag{4.17}$$

Finally we can conclude that

$$-16 < f_\epsilon < -\frac{1}{2}. \tag{4.18}$$

b) Once we have shown that f_ϵ is bounded and uniformly bounded away from 0, what remains is to show that that the normalized density $\widetilde{f}_\epsilon$ is uniformly bounded.

Let $\widetilde{f}_\epsilon$ denote the normalized density f_ϵ. From (4.17) we have

$$-\int 16 d\lambda < \int f_\epsilon d\lambda < -\int 1/2 d\lambda.$$

Taking the absolute value, we obtain

$$1/2 < \left| \int f_\epsilon d\lambda \right| < 16. \tag{4.19}$$

Inequality (4.18) implies that the normalized density $\widetilde{f}_\epsilon$ is bounded by

$$\frac{\inf|f_\epsilon|}{\left|\int f_\epsilon d\lambda\right|} < \widetilde{f}_\epsilon < \frac{\sup|f_\epsilon|}{\left|\int f_\epsilon d\lambda\right|},$$

$$\frac{\inf|f_\epsilon|}{16} < \widetilde{f}_\epsilon < \frac{\sup|f_\epsilon|}{1/2},$$

$$\frac{1}{32} < \widetilde{f}_\epsilon < 32.$$

□

4.3 The continuity of the invariant density of τ

In the previous section we proved that the normalized invariant densities $\widetilde{f}_\epsilon$ of τ_ϵ are uniformly bounded. In this section we use Lemma (2.5) to show that the normalized density of τ_ϵ converges to f as $\epsilon \to 0$.

Proposition 4.3. *The map* $\tau : [0,1] \to [0,1]$, $\tau(x) = (x+1/2)\chi_{[0,1/2]} + (2-2x)\chi_{[1/2,1]}$ *is acim-stable.*

Proof. Since $\widetilde{f}_\epsilon$ is uniformly bounded, by Lemma 2.1 we say that $\{\widetilde{f}_\epsilon\}$ is weakly precompact in $\mathcal{L}^1$. Hence, $\{\widetilde{f}_\epsilon\}$ contains a weakly convergent subsequence $\{\widetilde{f}_{\epsilon_n}\}$ such that $\widetilde{f}_{\epsilon_n} \to f$, $f \in \mathcal{L}^1$.

Since $\tau_\epsilon \to \tau$ almost uniformly (Lemma 2.5), by Lemma 2.6 we conclude that $P_\epsilon f = f$, the unique, invariant normalized density of τ. Hence, the whole family $\widetilde{f}_\epsilon \to f$ weakly in $\mathcal{L}^1$.

□

Chapter 5

Theoretical Proofs

Recent papers have extended the study on the stability of acim for piecewise expanding maps. In this chapter we elaborate on the work done in [5] and [7], where an answer to the problem in this thesis is also offered.

As we saw in Chapter 2, the Lasota-Yoke inequality is satisfied for piecewise expanding transformations with slopes strictly greater than 2. In [5], the authors developed a more general Lasota-Yorke inequality where the condition $s = \inf |\tau'| > 2$ is improved. In other words, the constant $2s^{-1}$ in (2.26) in replaced with a smaller one. In this paper, a class of piecewise expanding $C^{1,1}$ maps is defined as follows:

Definition 5.1. *We say that the map $\tau : I \to I$, $I = [0,1]$ belongs to the class of piecewise expanding $C^{1,1}$ maps $\mathcal{T}(I)$ if it satisfies the following conditions:*

a) There exists a partition $\mathcal{P} = \{I_i = (a_{i-1}, a_i), i = 1, ..., q\}$ such that $\tau_i := \tau|_{I_i}$ is monotonic, C^1, and it can be extended to the closed interval $[a_{i-1}, a_i]$ as a C^1 function;

b) τ_i' is Lipschitz, i.e., there exists a constant M_i such that $|\tau_i'(x) - \tau_i'(y)| \leq M_i|x - y|$ for al $x, y \in I_i$;

c) $|\tau_i'(x)| \geq s_i > 1$ for any i and for all $x \in (a_{i-1}, a_i)$.

Similarly, a family of maps $\{\tau_\epsilon\} \subset \mathcal{T}(I)$ if the above conditions are satisfied with uniform constants s_i and M_i. We define $s^* := \min_{1\leq i\leq q} s_i$ and $M := \max_{1\leq i\leq q} M_i$ and we define the δ-condition, which helps us to differentiate the hanging branches of τ from the others:

$$\delta_i^{\pm} = \delta_{\{\tau(a_i^{\pm})\notin\{0,1\}\}} = \begin{cases} 0 & \text{if } \tau(a_i^{\pm}) \in \{0,1\}, \\ 1 & \text{if } \tau(a_i^{\pm}) \notin \{0,1\}, \end{cases}$$

where $\tau(a_i^{\pm})$ denotes $\lim_{x\to a_i^{\pm}} \tau(x)$. The new version of the Lasota-Yorke inequality is given by:

Proposition 5.1. *[5] Suppose τ belongs to the class of piecewise expanding $C^{1,1}$ maps on I. Then, for every $f \in BV(I)$,*

$$\bigvee_I \mathrm{P}_\tau f \le \max_{1\le i\le q}\left\{\frac{1}{s_i} + \eta_i\right\} \bigvee_I f + \left[\frac{M}{s^2} + \frac{2\max_{1\le i\le q}\eta_i}{\min_{1\le i\le q} m(I_i)}\right]\int_I |f|dm, \tag{5.1}$$

where

$$\eta_i := \begin{cases} \max\{\frac{\delta_0^+}{s_1}, \frac{\delta_1^+}{s_2}\} & \text{if } i = 1, \\ \max\{\frac{\delta_{q-1}^-}{s_{q-1}}, \frac{\delta_q^-}{s_q}\} & \text{if } i = q, \\ \max\{\frac{\delta_{i-1}^-}{s_{i-1}}, \frac{\delta_i^+}{s_{i+1}}\} & \text{if } i = 2, ..., q-1. \end{cases}$$

Inequality (5.1) contains the improvement of the original Lasota-Yorke inequality (2.26) in terms of the coefficient

$$\max_{1\le i\le q}\left\{\frac{1}{s_i} + \eta_i\right\} \le \alpha < 1, \alpha > 0. \tag{5.2}$$

The condition (5.2) is satisfied in the following case:

Theorem 5.1. *[5] We say that the inequality (5.2) holds for a transformation $\tau \in \mathcal{T}(I)$, or for an extension (τ^*, I^*) that has (τ, I) as an attractor, if the following is satisfied:*

$$\frac{1}{s_i} + \frac{1}{s_{i+1}} \le \alpha < 1, \textit{ for } i = 1, ..., q-1. \tag{5.3}$$

The following two theorems, given in [5], state the existence and the stability of absolutely continuous invariant measures of maps $\tau \in \mathcal{T}(I)$, for which the condition $\max_{1\leq i\leq q}\left\{\frac{1}{s_i}+\eta_i\right\} \leq \alpha < 1$ is satisfied.

Theorem 5.2. *[5] If a map $\tau \in \mathcal{T}(I)$ satisfies the inequality (5.1) with the coefficient $\max_{1\leq i\leq q}\left\{\frac{1}{s_i}+\eta_i\right\} \leq \alpha < 1$, for some $\alpha > 0$, then for any $f \in BV(I)$ and $n \in \mathbb{N}$,*

$$||\mathrm{P}_\tau^n f||_{BV} \leq \alpha^n ||f||_{BV} + \left(1 + \frac{K + 2\beta^{-1}}{1-\alpha}\right)||f||_{\mathcal{L}^1},$$

where $K := M/s^2$ and $\beta := \min_{1\leq i\leq q} m(I_i)$. Furthermore, τ admits an acim with a density of bounded variation and $\mathrm{P}_\tau : BV(I) \to BV(I)$ is quasicompact.

Theorem 5.3. *[5] Consider the one parameter family of maps $\{\tau_\epsilon\}_{\epsilon\geq 0}$, where $\{\tau_\epsilon\}_{\epsilon\geq 0} \subset \mathcal{T}(I)$ uniformly. Suppose that there is an α, for which the condition, $\max_{1\leq i\leq q}\left\{\frac{1}{s_i}+\eta_i\right\} \leq \alpha < 1$, $0 < \alpha < 1$ is satisfied. Let f_ϵ be a τ_ϵ-invariant density whose existence is satisfied by Theorem (5.2). If $d_S(\tau_\epsilon, \tau_0) \to 0$ as $\epsilon \to 0$, then the following statements hold.*

1)The family $\{f_\epsilon\}_{\epsilon>0}$ is relatively compact in $\mathcal{L}^1$ and any of its limit functions is a τ_0-invariant density.

2) If τ_0 is ergodic, then so is τ_ϵ for small ϵ and $f_\epsilon \to f_0$ in $\mathcal{L}^1$ as $\epsilon \to 0$, (i.e. τ_0 is acim-stable).

3)If τ_0 is weakly mixing, then the eigenvalue gaps of $\{\mathrm{P}_{\tau_\epsilon}\}_\epsilon$, for small enough ϵ, are uniformly bounded, i.e., $0 < \gamma < 1 - |\lambda_2^\epsilon|$. As a consequence, there exists a constant $C > 0$ such that for all small enough ϵ and all densities $f \in BV$

$$||\mathrm{P}_{\tau_\epsilon}^n f - f_\epsilon||_{\mathcal{L}^1} \leq C(1-\gamma)^n ||f||_{BV}. \tag{5.4}$$

Example 5.1. *The last theorem offers another way of proving the stability of the map $\tau = (x + 1/2)\chi_{[0,1/2]} + (2 - 2x)\chi_{[1/2,1]}$. We observe the third iterate of $\tau_\epsilon =$*

$[(1+2\epsilon)x+1/2-\epsilon]\chi_{[0,1/2]}+(2-2x)\chi_{[1/2,1]}$, *whose graph of* $\tau_\epsilon^3(x)$ *is shown in figure (5.1). This map is represented by the piecewise equation*

$$\tau_\epsilon^3(x)=\begin{cases} -2(1+2\epsilon)^2(x-\frac{\epsilon}{1+2\epsilon}) & \text{if } x\in[0,\frac{\epsilon}{1+2\epsilon}), \\ 4(1+2\epsilon)(x-\frac{\epsilon}{1+2\epsilon}) & \text{if } x\in[\frac{\epsilon}{1+2\epsilon},\frac{1+4\epsilon}{4(1+2\epsilon)}), \\ -2^2(x-\frac{\epsilon}{1+2\epsilon})+1/2-\epsilon & \text{if } x\in[\frac{1+4\epsilon}{4(1+2\epsilon)}),1/2), \\ 4(1+2\epsilon)(x-5/8)-1 & \text{if } x\in[1/2,5/8), \\ -8(x-3/4) & \text{if } x\in[5/8,3/4), \\ 4(1+2\epsilon)(x-3/4) & \text{if } x\in[3/4,\frac{2+7\epsilon+6\epsilon^2}{2(1+2\epsilon)^2}), \\ -2(1+2\epsilon)^2x+2+7\epsilon+6\epsilon^2 & \text{if } x\in[\frac{2+7\epsilon+6\epsilon^2}{2(1+2\epsilon)^2},1]. \end{cases} \quad (5.5)$$

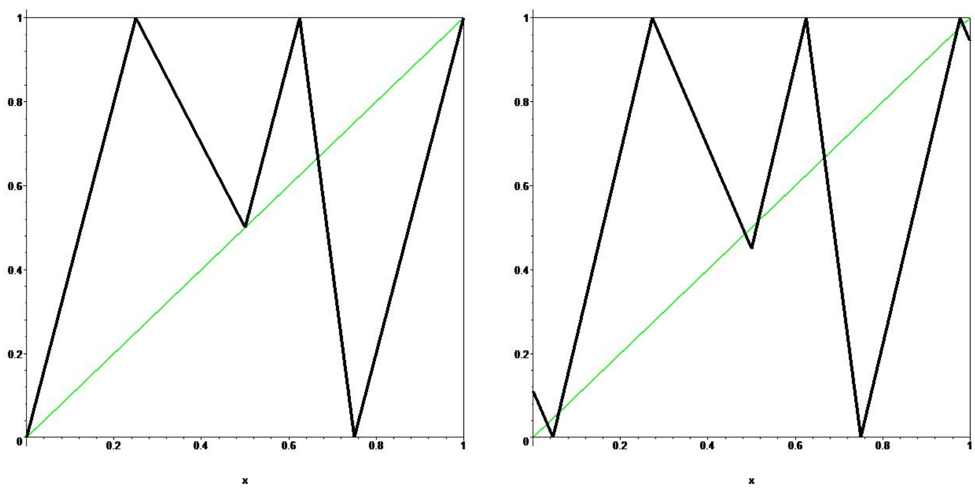


Figure 5.1: The graphs of $\tau^3(x)$ and $\tau_\epsilon^3(x)$

The partition on which τ_ϵ^3 *is defined does not converge to the partition of* τ^3. *In other words, if* $\mathcal{P}=\{I_1,...,I_5\}$, $I_i=(a_{i-1},a_i)$, $i=0,...5$, *and* $\mathcal{P}_\epsilon=\{I_1^\epsilon,...,I_7^\epsilon\}$, $I_i=(a_{i-1}^\epsilon,a_i^\epsilon)$, $i=0,...7$, *are the partitions of* τ^3 *and* τ_ϵ^3 *respectively, then* $a_i^\epsilon\nrightarrow a_i$ *as* $\epsilon\to 0$. *Hence, we cannot directly apply Theorem (5.3).*

In order to solve this problem, we define a map $\tau_\epsilon^*(x)$ *that has* $\tau_\epsilon^3(x)$ *as an attractor in the following way: We extend the interval on which* $\tau_\epsilon^*(x)$ *is defined, such that*

$\tau_\epsilon^*(x) : [a_0^\epsilon, a_7^\epsilon] \to [a_0^\epsilon, a_7^\epsilon]$, where $g_1^\epsilon(a_0^\epsilon) = a_7^\epsilon$ and $g_7^\epsilon(a_7^\epsilon) = a_0^\epsilon$. The functions g_1^ϵ and g_7^ϵ are nothing else but the first and the last branch of $\tau_\epsilon^3(x)$, i.e, $g_1^\epsilon(x) = -2(1+2\epsilon)^2(x - \frac{\epsilon}{1+2\epsilon})$ and $g_2^\epsilon = -2(1+2\epsilon)^2 x + 2 + 7\epsilon + 6\epsilon^2$. The fact that $s_1 = s_7 > 2$ guarantees that $\tau_\epsilon^3(x)$ is the attractor for $\tau_\epsilon^*(x)$.

Let $s_i^* = |(\tau_\epsilon^*)_i'|$ for $i = 1, ..., 7$. Since $s_i^* > 2$, for $i = 1, ..., 7$, and for every $\epsilon > 0$, we can say that condition (5.3) is satisfied, which implies that condition (5.2) is satisfied.

Similarly, using $\tau^3(x)$ we define $\tau^*(x) : [a_0, a_7] \to [a_0, a_7]$, with $|g_1'| > 4/3$ and $|g_7'| > 4/3$ so that condition (5.3) will be satisfied, and $\tau^3(x) : [0,1] \to [0,1]$ will be its attractor. The map $\tau^*(x)$ is piecewise expanding on the partition $\mathcal{P} = \{a_0, 0, 1/4, 1/2, 5/8, 3/4, 1, a_7\}$.

The map $\tau_\epsilon^*(x)$ is also piecewise expanding on the partition $\mathcal{P}_\epsilon = \{a_0^\epsilon, \frac{\epsilon}{1+2\epsilon}, \frac{1+4\epsilon}{4(1+2\epsilon)}, 1/2, 5/8, 3/4, \frac{2+7\epsilon+6\epsilon^2}{2(1+2\epsilon)^2}, a_7^\epsilon\}$, and $\tau_\epsilon^*(x) \subset \mathcal{T}([a_0^\epsilon, a_7^\epsilon])$ for every $\epsilon > 0$. As $\epsilon \to 0$, $\mathcal{P}_\epsilon \to \mathcal{P}$ and $d_S(\tau^*, \tau_\epsilon^*) \to 0$.

Hence, using Theorem (5.3), we have proven the acim-stability for the map τ^*. As τ^3 is the attractor of τ^*, we obtain the acim-stability for τ^3. Since τ is an exact map, it has a unique acim which means that the stability of τ^3 implies the acim-stability of τ.

While in [5], a stronger Lasota-Yoke inequality was proven, in [7] the authors showed that the harmonic average condition is sufficient for the use of the Rychlik's Theorem. That way, we can prove the acim-stability of the map τ without evoking the Lasota-Yoke inequality.

The class of piecewise expanding maps $\mathcal{T}$ is defined in Definition (2.21). Moreover, suppose that τ satisfies the condition (5.3), here denoted as

$$s_H = \max_{i=1,\dots,q-1} \{\frac{1}{s_i} + \frac{1}{s_{i=1}}\}. \tag{5.6}$$

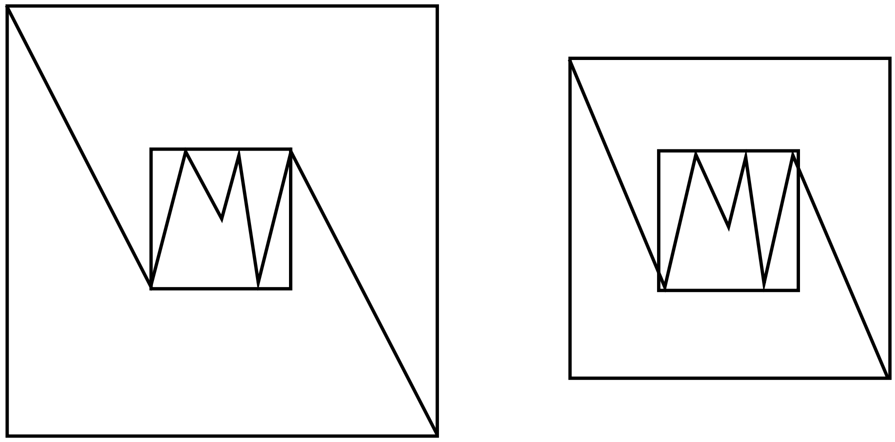


Figure 5.2: The graphs of τ^* and τ^*_ϵ shown in [5]

Let $\delta := \min_{2\leq i\leq q-1} m(I_i)$. It is important to note that in order to calculate δ, we do not take into consideration the first and the last subinterval of the partition. We also define $g_n = \frac{1}{|(\tau^n)'|}$, for each n such that $(\tau^n)'$ is defined.

Let $\mathcal{P}^{(n)} = \bigvee_{i=0}^{n-1} \tau^{-i}(\mathcal{P})$, and $\mathcal{P} = \mathcal{P}^{(1)}$. For a measurable subset A of $[a,b]$, let $\mathcal{P}(A) = \{J \in \mathcal{P} : \lambda(J \cap A) > 0\}$. Let $\gamma_n = \sum_{J\in\mathcal{P}^{(n)}} \sup_J g_n$.

For $J \in \mathcal{P}^{(n)}$, we define $osc_J \frac{1}{|\tau'|} = \max_J \frac{1}{|\tau'|} - \min_J \frac{1}{|\tau'|}$, and we define

$$d_n = \max_{J\in\mathcal{P}^{(n)}} osc_J \frac{1}{|\tau'|}. \tag{5.7}$$

Definition 5.2. *We say that a map $\tau \in \mathcal{T}(I)$ satisfies the summable oscillation condition, or $\tau \in \mathcal{T}_{\sum}$, if*

$$\sum_{n\geq 1} d_n \leq D < +\infty. \tag{5.8}$$

Here we state the original Rychlik's Theorem and then we explain how a piecewise expanding map τ that satisfies the harmonic average of slopes condition, satisfies the assumptions of the Rychlik's Theorem.

Theorem 5.4. *[7] A piecewise expanding map τ on an interval $[a,b]$ admits an absolutely continuous invariant measure if it satisfies the following conditions:*

1) There exists $d > 0$ such that for any $n \geq 1$ and any $J \in \mathcal{P}^{(n)}$,

$$\sup_J g_n \leq d \cdot \inf_J g_n; \tag{5.9}$$

2) There exists $\epsilon > 0$ and $r \in (0, 1)$ such that for any $n \geq 1$ and any $J \in \mathcal{P}^{(n)}$,

$$m(\tau^n(J)) < \epsilon \Rightarrow \sum_{J' \in \mathcal{P}(\tau^n(J))} \sup_{J'} g \leq r; \tag{5.10}$$

3) $\gamma_1 = \sum_{J \in \mathcal{P}} \sup_J g < +\infty$. Moreover, if f is an τ-invariant density then

$$||f||_\infty \leq \gamma_1 \frac{d}{\epsilon(1-r)}. \tag{5.11}$$

Theorem 5.5. *[7] If $\tau \in \mathcal{T}_{\sum}$ and satisfies the harmonic average of slopes condition $s_H < 1$, then it satisfies the assumption of the Rychlik's Theorem.*

Theorem 5.6. *[7] Let the family $\{\tau_\gamma\}_{\gamma>0} \subset \mathcal{T}_{\sum}$ satisfy the assumption of Rychlik's Theorem in a uniform way, i.e, with the same constants and $\tau_\gamma \to \tau_0$ almost uniformly as $\gamma \to 0$. If τ_0 has exactly one acim, then $f_\gamma \to f_0$ in $\mathcal{L}^1$ as $\gamma \to 0$. In the general case every limit point of the family $\{f_\gamma\}$, as $\gamma \to 0$, is an invariant density of τ_0.*

Theorem 5.7. *[7] Let $\tau_\gamma \in \mathcal{T}, \gamma \geq 0$. Let the invariant densities of $\{f_\gamma\}_{\gamma \geq 0}$ be uniformly bounded in L^∞. If $\tau_\gamma \to \tau_0$ almost uniformly as $\gamma \to 0$, then any limit point of $\{f_\gamma\}_{\gamma>0}$, as $\gamma \to 0$, is a τ_0-invariant density. If $\{\tau_0, f \cdot m\}$ is ergodic, then $f_\gamma \to f_0$ in $\mathcal{L}^1$.*

Example 5.2. *Recall that in Example (2.3) we have found that the probability density for the map* $\tau = \tau_0 = (x+1/2)\chi_{[0,1/2]} + (2-2x)\chi_{[1/2,1]}$ *is* $f_0 = \frac{2}{3}\chi_{[0,1/2]} + \frac{4}{3}\chi_{[1/2,1]}$. *The question being posed in [2] is whether this map is acim-stabe i.e, whether the family of densities* $\{f_\epsilon\}_{\epsilon>0}$ *of the family of maps* $\{\tau_\epsilon\}_{\epsilon>0}$, *where* $\tau_\epsilon = \big((1+2\epsilon)x + \frac{1}{2} - \epsilon\big)\chi_{[0,1/2]} + 2(1-x)\chi_{[1/2,1]}$ *converges to* f_0. *As we said, Theorem (5.6) gives us a tool to answer this question without referring to the Lasota-Yorke inequality.*

Here again, it is easier to work with the third iterates of τ *and* τ_ϵ. *Since* τ *is exact it means that* τ^3 *is also exact and has the same stability of the absolutely continuous invariant measure. Hence, if we prove that* τ^3 *is acim-stable, then we conclude the same for* τ.

The third iterate of τ *given by,*

$$\tau^3(x) = \begin{cases} 4x & \text{if } x \in [0,1/4), \\ -2x+3/2 & \text{if } x \in [1/4,1/2), \\ 4x-3/2 & \text{if } x \in [1/2,5/8), \\ -8x+6 & \text{if } x \in [5/8,3/4), \\ 4(x-3/4) & \text{if } x \in [3/4,1]. \end{cases}$$

On the partition $\mathcal{P} = \{0, 1/4, 1/2, 5/8, 3/4, 1\}$, $d_n = \max_{J\in\mathcal{P}^{(n)}} osc_J \frac{1}{|\tau'|} = 0$ *which implies that* $\sum_{n\geq 1} d_n = 0 < \infty$, *i.e,* $\tau_0 \in \mathcal{T}_{\sum}(I)$. *Moreover, since the slopes of* τ^3 *are* $s_1 = s_3 = s_5 = 4$, $s_2 = 2$ *and* $s_4 = 8$, $s_H < 1$ *for* $i = 1, ..5$, *it satisfies the conditions of the Rychlik's Theorem.*

Now we observe the third iterate of τ_ϵ, *given by equation (5.5) and whose graph is shown in Figure [5.1].*

We define the partition $\mathcal{P}_\epsilon = \{0, \frac{\epsilon}{1+2\epsilon}, \frac{1+4\epsilon}{4(1+2\epsilon)}, 1/2, 5/8, 3/4, \frac{2+3\epsilon}{2(1+2\epsilon)}, 1\}$, *on which* τ_ϵ^3 *is piecewise expanding. Note that as* $\epsilon \to 0$, $\frac{\epsilon}{1+2\epsilon} \to 0$, $\frac{1+4\epsilon}{4(1+2\epsilon)} \to 1/4$ *and* $\frac{2+3\epsilon}{2(1+2\epsilon)} \to 1$. *The absolute values of the slopes of* τ_ϵ^3 *are* $s_1 = s_3 = s_7 = 2(1+2\epsilon)^2$, $s_2 = s_4 = s_6 = 4(1+2\epsilon)$ *and* $s_5 = 8$ *and hence, as* $\epsilon \to 0$,

$$s_H \to \frac{3}{4} < 1.$$

Here again $\sum_{n\geq 1} d_n = 0 < \infty$ *and we conclude that* τ_ϵ^3 *uniformly satisfies the summable oscillation condition and the harmonic average condition. Since* $\tau_\epsilon^3 \to \tau^3$ *as* $\epsilon \to 0$*, the family* $\{\tau_\epsilon^3\}_{\epsilon\geq 0}$ *satisfies the conditions of Theorem 5.3. Since* τ^3 *is acim-stable, we conclude that the same holds for* τ.

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