Gabor Analysis and Wavelet-Like Transforms on Some Non-Euclidean 2-Dimensional Manifolds

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Abstract

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Many problems in physics require the crafting of suitable tools for the analysis of data emanating from various non-Euclidean manifolds. The main tools, currently employed for this purpose, are Gabor type frames or general frames, and wavelets. Given this backdrop, the primary objective of this thesis is the development of wavelet-like and time frequency type transforms on certain non-Euclidean manifolds. An immediate example of such a manifold (in the sense that it is homeomorphic to several other two-dimensional manifolds of revolution) is the two-dimensional infinite cylinder, for which we construct here Gabor type frames and wavelets. The two-dimensional cylinder, as a surface of revolution, is naturally homeomorphic to several other two-dimensional manifolds (themselves also surfaces of revolution). Examples are the one-sheeted hyperboloid, the paraboloid with its apex removed, the sphere with two points removed, the ellipsoid with two points removed, the plane with the origin removed, the upper sheet, of the two sheeted hyperboloid, with one point removed, and so on. Using this fact, in this thesis we build Gabor type frames and wavelets on these manifolds. We also present a method for constructing wavelet-like transforms on a large class of such surfaces of revolution using a group theoretic approach. Finally, as a beginning to a related but different sort of study, we construct some localization operators associated to group representations, using symbols (in the sense of pseudo-differential operators) which are operator valued functions on the group.

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Contents

Chapter 1

Introduction

1.1 Time-frequency and wavelet analysis

The Fourier series, the Fourier transform and Fourier analysis in general, have played an immense role in many areas of present day mathematical and scientific research – far beyond what could have been anticipated when the theory, now popularly known as Fourier analysis, was first created by Joseph Fourier in 1822 (see [3]). As is well known, the Fourier series was devised to enable one to study signals in detail by decomposing them into their frequency components. The idea here is that any 2π-periodic function $f(x)$ can be written as the sum

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.1.1}$$

of its Fourier components a_n and b_n,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx \tag{1.1.2}$$

when the function is developed in the standard trigonometric basis. For many years the Fourier transform remained the tool of choice for analyzing data arising from problems in applied mathematics and signal processing. Apart from the analysis of signals, the method also enabled one (e.g., in the case of stationary signals) to work backwards and reconstruct the signal from a knowledge of its frequency distribution. However, as is well known, the theory suffers from the drawback that for a non-stationary signal, such as that representing, for example, a musical score, an analysis into the component frequencies suppresses all knowledge of its profile in time. Using the frequency distribution, one is unable to say, for example, when the musical piece began. In other words, one does not, strictly speaking, have a time-frequency analysis in this case – one either has a signal which is a fuction of time alone or of its frequency components alone.

The first successful attempt at constructing a genuine time-frequency transform of a signal appeared in a work by D. Gabor, published in [27]. Gabor suggested that a more appropriate way to analyze a signal would be to use a "sliding window" in time and to analyze the frequency content of the signal within this window at each point of time. Specifically, Gabor proposed expanding a signal f as a series $f(x) = \sum c_{m,n} e^{2\pi imbx} g(x - na)$, where g is the Gaussian function (the window). His basic idea was to use the expansion for data transmission: instead of transmitting the function f, one could send the coefficients $c_{m,n}$. Subsequent development of the theory led, on the one hand to a continuous version of Gabor analysis and, on the other hand, to wavelet analysis (both discrete and continuous). With hindsight, the two approaches can now be subsumed within the theory of time-frequency (Gabor) and time-scale (wavelet) analysis.

The idea behind the continuous Gabor or time-frequency transform is to take the function f, representing the signal in time, and then to analyze the frequency content of $\chi_{[t-r,t+r]} \cdot f$, (with $\chi_{[t-r,t+r]}$ denoting the characteristic function of the set $[t - r, t + r]$). This, to a good approximation, yields information on the frequency content of the signal in the corresponding neighbourhood of t. The resulting transform then gives a proper time-frequency transform of the signal f. The technique can, furthermore, be generalized by replacing the characteristic function $\chi_{[t-r,t+r]}$ of the set $[t-r, t+r]$ by a more general function, satisfying some additional technical conditions and with support more or less concentrated in $[t - r, t + r]$. The supplementary technical conditions are needed to allow for a reconstruction of the signal from its time-frequency transform. The Gabor transform, as this time-frequency transform is often called, basically appears as the window function, modulated and translated, and then convolved with the signal.

The notion of a wavelet first appeared in an appendix of the thesis of A. Haar (1909) (see [3]). In a related development, between 1960 and 1980, the mathematicians Guido Weiss and Ronald R. Coifman (see [3]) studied the problem of analyzing elements of certain function spaces in terms of atoms, or elementary constituents of these spaces, the goal being to find an appropriate set of atoms and a set of rules using which any function in the space could be reconstructed in terms of the atoms. Such studies fall within the scope of discrete wavelet and discrete time-frequency analyses. The continuous wavelet transform, which may be viewed as a modification of the continuous Gabor transform, also works with a basic window, satisfying certain technical conditions, but with the modulation being replaced by a scaling or dilation. In a remarkable series of papers, Grossmann and Morlet in 1980 and Grossmann, Morlet and Paul in 1985 ([29], [30],[32]) clarified and elaborated on the essential structure of the wavelet transform, giving it, in the process, an elegant group theoretical interpretation. The idea here was to consider a family of vectors $\psi_{b,a}$, on the Hilbert space $L^2(\mathbb{R})$ of signals, generated by a single vector ψ, called the mother wavelet, in the Hilbert space. The parameter b runs through $\mathbb{R}$ while a is a non-zero real number. Specifically,

$$\psi_{b,a}(x) = |a|^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right), \tag{1.1.3}$$

and the overlap integral of $\psi_{b,a}$ with the signal f then gives the wavelet transform, as a function of the time b and inverse frequency a. It can then be shown that if the mother wavelet satisfies the admissibility condition,

$$\int_{\mathbb{R}} \frac{|\hat{\psi}(w)|^2}{|w|} dw = 1, \tag{1.1.4}$$

the signal can be reconstructed from its wavelet transform $\langle f, \psi_{b,a}\rangle$ in the manner,

$$f = \int_{\mathbb{R}}\int_{\mathbb{R}} \langle f, \psi_{b,a}\rangle \psi_{b,a} \frac{dadb}{a^2}, \tag{1.1.5}$$

with this latter equation being now called the reconstruction formula.

The original theory of the wavelet transform underwent rapid development in the late 1980's and 1990's through the work of Daubechies, Mallat and Meyer, among others. For details one may refer to [15], [16], [18], [19]. Today, the theory of time-frequency and time-scale analysis, apart from its mathematical interest, has an impressive range of practical applications, encompassing astronomy, acoustics, nuclear engineering, sub-band coding, signal and image processing, neurophysiology, music, magnetic resonance imaging, speech discrimination, optics, the study of fractals, turbulence, earthquake prediction, radar imaging and human vision.

Gabor analysis took an entirely new direction from 1986 with the fundamental paper [20], by Daubechies, Grossmann and Meyer where they developed an idea of combining Gabor analysis with frame theory. The authors constructed tight frames for $L^2(\mathbb{R})$ having the form $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, and this contribution was the beginning of an intense activity on wavelets which is still ongoing. For a collection of research articles about Gabor systems we refer to [25]. The book by Gröchenig [31] is also a good source of information.

Nowadays, one of the current and important problems in signal analysis is the representation and analysis of signals arising from non-Euclidean geometries (for example in geophysics). This problem was considered by the research group of Jean-Pierre Antoine and Pierre Vandergheynst ([5], [6], [8],[9],[10], [4], [7]) by constructing wavelets on a sphere and on the upper sheet of a one-sheeted hyperboloid. For a collection of research articles in the same direction we refer also to [11], [17], [28], [42],[43],[45].

With the aim of constructing wavelets on various non-Euclidean manifolds, Iva Bogdanova in her thesis, *Wavelets on non-Euclidean manifolds*, constructed wavelets on the upper sheet of the two-sheeted hyperboloid.

The present thesis is an attempt to continue this work on two-dimensional manifolds in a systematic way. The content of this thesis can be summarized as follow:

Going back to a group theoretical treatment of a Gabor system, leads to a discretezation of the wavelet transform associated to the Schrödinger representation of the Weyl-Heisenberg group. Recently, a generalization of the Weyl-Heisenberg group has been presented in [36, 37]. Such a generalized Weyl-Heisenberg group is the central extension of the direct product of a locally compact abelian group G with its dual group $\hat{G}$. By analogy with the standard Weyl-Heisenberg group, it is then possible to construct Schrödinger-type representations in these general situations, which are again continuous, unitary and irreducible. Since a Gabor system can be considered as the orbit of a discrete subset of the Weyl-Heisenberg group, under the Schrödinger action, this thesis presents a generalization of such a system using a discrete subset of the generalized Weyl-Heisenberg group. The Walnut type representation of the corresponding frame operator is also presented.

Next, the fact that a 2D-cylinder $\mathfrak{C} = S^1 \times \mathbb{R}$ is a locally compact abelian group, is used to construct a time-frequency transform on it. On the other hand, using a particular group, we also construct a wavelet transform on a cylinder. We then transfer these transforms (time-frequency and wavelet) to some non-Euclidean manifolds which are topologically homeomorphic to the cylinder.

The basic idea is as follows:

Let $\boldsymbol{\ell}$ be a smooth curve in $\mathbb{R}^2$, which is parametrized as

$$z \longmapsto \begin{pmatrix} u(z) \\ v(z) \end{pmatrix}, \quad z \in \mathbb{R},$$

where u and v are two smooth functions. We assume that this map is a homeomorphism between $\mathbb{R}$ and $\boldsymbol{\ell}$. The map

$$V : \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \longmapsto \begin{pmatrix} u(z)\cos\theta \\ u(z)\sin\theta \\ v(z) \end{pmatrix} \tag{1.1.6}$$

transforms the cylinder, homeomorphically to a surface of revolution $\mathfrak{S}$ about the z-axis.

The surface element $d\sigma_3$ on the cylinder transforms to

$$d\sigma_{\mathfrak{S}}(\theta, z) = w(z)\ d\theta\ dz, \tag{1.1.7}$$

on this surface, where $w(z) = |u(z)|\ [u'(z)^2 + v'(z)^2]^{\frac{1}{2}}$. The mapping V then induces a unitary map $\tilde{V} : L^2(\mathfrak{C}, dzd\theta) \rightarrow L^2(\mathfrak{S}, d\sigma_{\mathfrak{S}})$, defined by

$$\left(\tilde{V} f\right)\left(\begin{pmatrix} u(z)\cos\theta \\ u(z)\sin\theta \\ v(z) \end{pmatrix}\right) = (w(z))^{-\frac{1}{2}} f\left(\begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix}\right). \tag{1.1.8}$$

By using this approach, we present a family of wavelet transforms on some surfaces of revolution such as the one-sheeted hyperboloid, two-sheeted hyperboloid, paraboloid, ellipsoid, sphere, 2-D plane, etc.

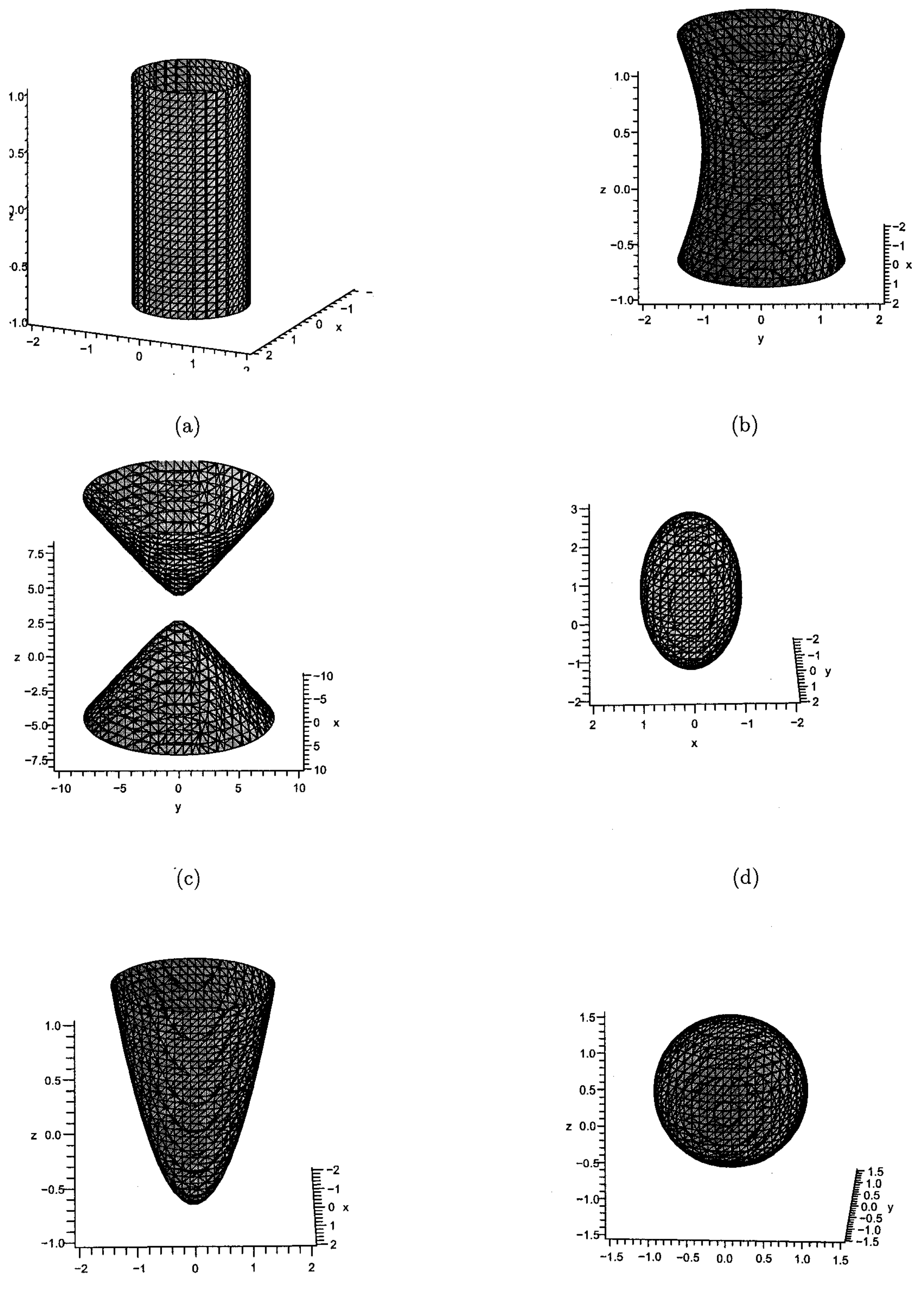


Figure 1.1: Examples of non-Euclidean manifolds: (a) Cylinder, (b) One-sheeted 2-hyperboloid, (c)Two-sheeted 2-hyperboloid, (d) Ellipsoid, (e)Paraboloid, (f) Sphere

Using Mackey's theory of the induced representations, we present a wavelet transform on some general two-dimensional surfaces given by the equation $z = (x^2 + y^2)^\alpha$, where α is a real number, by using a specific group. The wavelet transform on the paraboloid is obtained by taking $\alpha = 1$.

Finally, a short study of a localization operator associated to a group representation where the symbol is assumed to be an operator valued function on the group is presented. This generalizes the standard one when the value of the symbol is a multiple of the identity on the Hilbert space of the representation. An application is given by using a spin representation of the Galilei group.

1.2 Main contributions

The contents of this thesis may be summarized as follows:

1. In chapters 2 and 3, after introducing the Gabor system associated to any abstract abelian group, we generalize some necessary and sufficient conditions for the system to be a frame. As an example, the Gabor type frame on the torus is constructed. The Walnut representation is also presented. We define a generalized shift-invariant system on $L^2(G)$ and present some results associated to it.

2. In chapter 4, the idea of the first chapter is used to construct Gabor type frames on various surfaces, namely the sphere, the ellipsoid, the one-sheeted hyperboloid, the upper sheet of a two-sheeted hyperboloid, paraboloid and plane. The idea is that the infinite 2D-cylinder can be mapped homeomorphically to all those surfaces, possibly with one or two points removed.

3. In chapter 5, as in the fourth chapter, we construct affine type wavelets on the infinite cylinder and map it to various surfaces.

4. In chapter 6, we present a wavelet transform on a paraboloid through a representation of a particular group.

5. Localization operators associated to group representations and their boundedness is presented in [46], in which the symbol is taken as a complex valued function on a group. In chapter 7, we present localization operators with the symbol as an operator valued function on a group. An application is given by using the spin representation of the extended Galilei group.

Chapter 2

Gabor-type Frames Associated to a Generalized Weyl-Heisenberg Group

2.1 Preliminaries

Gabor systems nowadays play a central role in e.g. signal processing, image and data compression. A deep analysis of necessary and sufficient conditions for the convergence of the frame operator is a fundamental problem in Gabor frame analysis. In [33], some necessary and sufficient conditions are given on the Gabor system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a frame. In [12], weaker sufficient conditions are given and this result is based on the Ron and Shen characterization of Gabor frames [40].

There exists an abundance of frame-related results, in the literature, for Gabor frames. A good sampling of these may be found, for example, in [12, 20, 25, 31, 33, 40]. In the present chapter and the following, we work out extensions of several of these results, in particular those dealing with the boundedness and invertibility of the frame operator, to generalized Gabor systems. Specifically, we refer to Theorems 2.2.1, 2.3.1, 2.3.2 and 2.4.1 below. We ought to also mention that generalized Weyl-Heisenberg groups have also been looked at for other and related studies. A good reference is the work edited by H. Feichtinger and T. Strohmer [25] and in particular, Chapter 7 of that book. We begin by giving the central definitions and some necessary and sufficient conditions for a standard Gabor system to be a frame.

2.1.1 Some definitions

A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in a Hilbert space $\mathcal{H}$ is called a *Bessel sequence* if there exists a constant $B > 0$ such that

$$\sum_{k=1}^{\infty} | \langle f, f_k \rangle |^2 \leq B \| f \|^2, \quad \forall f \in \mathcal{H}. \tag{2.1.1}$$

A *Riesz basis* for $\mathcal{H}$ is a family of the form $\{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded bijective operator.

(i) A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in a Hilbert space $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A \| f \|^2 \leq \sum_{k=1}^{\infty} | \langle f, f_k \rangle |^2 \leq B \| f \|^2, \quad \forall f \in \mathcal{H}. \tag{2.1.2}$$

The numbers A and B are called frame bounds.

(ii) A frame is *tight* if we can choose $A = B$ as frame bounds.

The condition (2.1.2) is equivalent to say that, the following operator, the so call the frame operator $S = \sum_{k=1}^{\infty} | f_k \rangle \langle f_k |$ associated to $\{f_k\}_{k=1}^{\infty}$ is bounded above and below:

$$A\mathbb{I} \leq S \leq B\mathbb{I}. \tag{2.1.3}$$

Lemma 2.1.1 *Assume that* $\{f_k\}_{k=1}^{\infty}$, *is an overcomplete frame with frame operator* S. *Then there exist frames* $\{g_k\}_{k=1}^{\infty} \neq \{S^{-1}f_k\}_{k=1}^{\infty}$ *for which*

$$f = \sum_{k \in \mathbb{Z}} \langle f \mid g_k \rangle f_k, \quad f \in \mathcal{H}. \tag{2.1.4}$$

A frame $\{g_k\}_{k=1}^{\infty}$ satisfying (2.1.1) is call a dual frame of $\{f_k\}_{k=1}^{\infty}$.

The following Lemma will be useful in the sequel.

Lemma 2.1.2 *Let* $\{f_k\}_{k=1}^{\infty}$, *be a frame. Then the following are equivalent:*

(i) $\{f_k\}_{k=1}^{\infty}$ *is tight.*

(ii) $\{f_k\}_{k=1}^{\infty}$ *has a dual frame of the form* $g_k = Cf_k$, *for some constant* $C > 0$.

2.1.2 Weyl-Heisenberg frame

Let x, w be real numbers. The unitary operators defined on $L^2(\mathbb{R})$ by $T_x f(y) = f(y-x)$, and $E_w f(y) = e^{2i\pi w.y} f(y)$, are called translation and modulation operators, respectively.

A Weyl-Heisenberg frame, or synonymously, a Gabor frame is a frame for $L^2(\mathbb{R})$ of the form $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, where $a, b > 0$ and $g \in L^2(\mathbb{R})$ is a fixed function. Explicitly,

$$E_{mb}T_{na}g(x) = e^{2i\pi mbx} g(x - na).$$

The function g is called the window function or the frame generator. For an exhaustive list of papers dealing with such frames we refer to the monograph [25].

Our main results in this chapter will consist of generalizations of the following four theorems for standard Gabor or Weyl-Heisenberg frames [14].

Theorem 2.1.1 *Let* $g \in L^2(\mathbb{R})$ *and* $a, b > 0$ *be given. Then the following holds:*

(i) If $ab > 1$, *then* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is not a frame for* $L^2(\mathbb{R})$.

(ii) If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a frame, then*

$ab = 1 \Leftrightarrow \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a Riesz basis.*

In [39], there is the following stronger result than (i) : when $ab > 1$, the family $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ cannot even be complete in $L^2(\mathbb{R})$. The assumption $ab \leq 1$ is not enough for $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a frame, even if $g \neq 0$. For example, if $a \in [\frac{1}{2}, 1[$, the set of functions $\{E_m T_{na}\chi_{[0,\frac{1}{2}]}\}_{m,n\in\mathbb{Z}}$ is not complete in $L^2(\mathbb{R})$ and cannot form a frame.

Proposition 8.3.2 in [14] gives a necessary condition for a Gabor system to be a frame. Sufficient conditions for $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a frame for $L^2(\mathbb{R})$ have been known since 1988, the basic insight being provided by Daubechies [18]. A slight improvement was proved in [33]. Later, Ron and Shen [40] were able to give a complete characterization of Gabor frames, spelled out in the next theorem. Given $g \in L^2(\mathbb{R})$, consider the matrix-valued function

$$M(x) = \{g\,(x - na - m/b)\}_{m,n\in\mathbb{Z}}\,. \tag{2.1.5}$$

The matrix $M(x)M^*(x)$ is positive.

Theorem 2.1.2 *$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B if and only if*

$$bA\mathbb{I} \leq M(x)M^*(x) \leq bB\mathbb{I},\ a.e., \tag{2.1.6}$$

where $\mathbb{I}$ is the identity operator on $\ell^2(\mathbb{Z})$.

This theorem is a special case of the following characterization [14] of a shift-invariant system to be a frame.

Recall that if $\{g_m\}_{m\in I}$ is a collection of functions in $L^2(\mathbb{R})$, the shift-invariant system generated by
$\{g_m\}_{m\in I}$ and some $a \in \mathbb{R}$ is the collection of functions $\{g_m(. - na)\}_{m\in I n\in\mathbb{Z}}$. Usually we will set $I = \mathbb{Z}$.

Given a shift-invariant system $\{g_m(. - na)\}_{n,m\in\mathbb{Z}}$ for $L^2(\mathbb{R})$, define the matrix-valued function $H(\nu)$, $\nu \in \mathbb{R}$, by

$$H(\nu) \;=\; (\hat{g}_m(\nu - k/a))_{k,m\in\mathbb{Z}}\,, \quad a.e.\,, \tag{2.1.7}$$

$\hat{g}$ denoting the Fourier transform of g. The following theorem then contains a generalization of Theorem 2.1.2 to any shift-invariant system $\{g_m(. - na)\}_{n,m\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ [14]:

Theorem 2.1.3 *With the above setting, the following hold:*

(i) $\{g_{n,m}\}$ *is a Bessel sequence with upper bound* B *if and only if* $H(\nu)$*, for almost all* ν*, defines a bounded operator on* $l^2(\mathbb{Z})$ *of norm at most* $\sqrt{aB}$*.*

(ii) $\{g_{n,m}\}$ *is a frame for* $L^2(\mathbb{R})$ *with frame bounds* A*,* B *if and only if*

$$aA\mathbb{I} \leq H(\nu)H^*(\nu) \leq aB\mathbb{I}, \quad a.e.\ \nu. \tag{2.1.8}$$

(iii) $\{g_{n,m}\}$ *is a tight frame for* $L^2(\mathbb{R})$ *if and only if there is a constant* $c > 0$ *such that*

$$\sum_{m\in\mathbb{Z}} \overline{\hat{g}_m(\nu)}\hat{g}_m(\nu + k/a) = c\delta_{k,0}, \quad k \in \mathbb{Z},\ a.e.\ \nu. \tag{2.1.9}$$

In case (2.1.9) *is satisfied, the frame bound is* $A = c/a$*.*

(iv) *Two shift-invariant systems* $\{g_{n,m}\}$ *and* $\{h_{n,m}\}$*, which form Bessel sequences, are dual frames if and only if*

$$\sum_{m\in\mathbb{Z}} \overline{\hat{g}_m(\nu)}\hat{h}_m(\nu + k/a) = a\delta_{k,0}, \quad k \in \mathbb{Z},\ a.e.\ \nu. \tag{2.1.10}$$

Theorem 2.1.2 is difficult to apply. This leads to a sufficient condition [12] for $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a frame for $L^2(\mathbb{R})$.

Theorem 2.1.4 *Let* $g \in L^2(\mathbb{R})$ *and* $a, b > 0$ *and suppose that*

$$B := \frac{1}{b} \sup_{x\in[0,a]} \sum_{k\in\mathbb{Z}} \Big| \sum_{n\in\mathbb{Z}} g(x - na)\overline{g(x - na - k/b)} \Big| < \infty. \tag{2.1.11}$$

Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a Bessel sequence with upper frame bound* B*. If also*

$$A := \frac{1}{b} \inf_{x\in[0,a]} \left[\sum_{n\in\mathbb{Z}} | g(x - na) |^2 - \sum_{0\neq k\in\mathbb{Z}} \Big| \sum_{n\in\mathbb{Z}} g(x - na)\overline{g(x - na - k/b)} \Big| \right] > 0. \tag{2.1.12}$$

Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a frame for* $L^2(\mathbb{R})$ *with bounds* A, B*.*

2.2 Generalized Weyl-Heisenberg group

Definition 2.2.1 *Let G be a locally compact abelian (LCA) group, $\widehat{G}$ its dual group, dx and $d\xi$ their Haar measures, respectively. Let $\mathbb{T}$ be the unit circle and put $H_G = G \times \widehat{G} \times \mathbb{T}$. For (g_1, w_1, z_1) and (g_2, w_2, z_2) in H_G, define the following composition:*

$$(g_1, w_1, z_1).(g_2, w_2, z_2) = \big(g_1 g_2, w_1 w_2, z_1 z_2 w_2(g_1)\big). \tag{2.2.13}$$

H_G is closed under this action, which is also associative. Equipped with this product, H_G is a group, called the generalized Weyl-Heisenberg group, associated with G. This group is nonabelian, locally compact, and unimodular [41], with invariant measure $dxd\xi d\theta$ (where $z = e^{i\theta}$).

A uniform lattice in G is a discrete subgroup K of G such that G/K is compact. For a uniform lattice K in G, $Ann(K)$ denotes the annihilator of K, *i.e.*, $Ann(K) := \{\gamma \in \widehat{G} \ : \ \gamma(k) = 1, \forall\, k \in K\}$.

By Lemma 24.5 of [34], we know that $Ann(K) \simeq \widehat{G/K}$, so that $Ann(K)$ is a discrete subgroup of $\widehat{G}$.

Let $\pi : H_G \longrightarrow U\Big(L^2(G)\Big)$ be the Schrödinger representation of H_G, which is a unitary, irreducible representation, given explicitly by

$$\Big(\pi(x, \gamma, z)g\Big)(t) = z\gamma(t)g(tx^{-1}), \tag{2.2.14}$$

for all $(x, \gamma, z) \in H_G$ and almost all $g \in L^2(G)$. In [36] frames of $L^2(G)$ of the type

$$\Big\{\Theta^g_{(k,\gamma)} = \Big(\pi(k, \gamma, 1)g\Big)\Big\}_{(k,\gamma)\in K\times Ann(K)}, \tag{2.2.15}$$

where K is a uniform lattice in G, have been studied. In this case, taking $G = \mathbb{R}$ and $K = a\mathbb{Z}$ and defining the dual pairing in the usual way:

$$\xi(x) = e^{2\pi i x\xi}, \tag{2.2.16}$$

we obtain $Ann(K) = \dfrac{1}{a}\mathbb{Z}$. Thus, the Gabor system defined by (2.2.15) is

$$\Big\{e^{\frac{2\pi i m x}{a}} g(x - na)\Big\}_{m,n\in\mathbb{Z}}, \tag{2.2.17}$$

which is a particular case ($ab = 1$) of the standard Gabor system:

$$\left\{e^{2\pi imbx}g(x-na)\right\}_{m,n\in\mathbb{Z}}. \tag{2.2.18}$$

In this chapter, we study frames for $L^2(G)$ of the form

$$\left\{\Theta^{g}_{(k_1,\gamma_2)} = \left(\pi(k_1,\gamma_2,1)g\right)\right\}_{(k_1,\gamma_2)\in K_1\times Ann(K_2)}, \tag{2.2.19}$$

where K_1 and K_2 are two lattices in G. Such a frame is clearly a generalization of a Gabor frame, because if we take $K_1 = a\mathbb{Z}$ and $K_2 = \frac{1}{b}\mathbb{Z}$ and use the same dual pairing as above, we get exactly the standard Gabor system (2.2.18).

Definition 2.2.2 *The set defined by*

$$\left\{\Theta^{g}_{(k_1,\gamma_2)}\right\}_{(k_1,\gamma_2)\in K_1\times Ann(K_2)} \tag{2.2.20}$$

will be called the generalized Gabor system for $L^2(G)$ associated to the uniform lattices K_1 and K_2, and the window function g.

Let K be a uniform lattice of a LCA group G. The lattice size is defined as the measure of the a fundamental domain U of K, this latter being a measurable subset of G such that every x in G can be uniquely written as $x = uk^{-1}$; $\quad k \in K$ and $u \in U$. Equivalently, G is a disjoint union of the sets $k^{-1}U = T_k(U)$.

Once the Haar measure dx on G, is fixed, there exists a unique invariant measure ν on G/K which satisfies the Weil formula (see, e.g, Chapter 6 of [25]), so that if $f \in L^1(G, dx)$, then $\sum_{k\in K} f(xk) \in L^1(G/K, d\nu)$ and

$$\int_G f(x)dx = \int_{G/K} \sum_{k\in K} f(yk)d\nu(y), \tag{2.2.21}$$

Also, $\nu(G/K)$ is the lattice size of K in G.

Now, let K_1 and K_2 be two uniform lattices in G and let ν_1 and ν_2 be the unique invariant measures on G/K_1 and G/K_2 respectively, which satisfy Weil's formula (2.2.21). Also, let $\hat{\nu}_1$ and $\hat{\nu}_2$ be the unique invariant measures on $\widehat{G}/Ann(K_1)$ and $\widehat{G}/Ann(K_2)$ respectively, which satisfy the Weil formula,

$$\int_{\hat{G}} f(\xi)d\xi = \int_{\hat{G}/Ann(K_i)} \sum_{\gamma_i\in Ann(K_i)} f(y\gamma_i)d\hat{\nu}_i(y); \quad i = 1,2;\ \forall\, f \in L^1(\widehat{G}). \tag{2.2.22}$$

The following Lemmas are essential for this work:

Lemma 2.2.1 *Let $f, g \in L^2(G, dx)$ and let K_1 and K_2 be two uniform lattices in G. Then, for any $k_2 \in K_2$, the series*

$$\sum_{k_1 \in K_1} f(xk_1^{-1})\overline{g(xk_1^{-1}k_2^{-1})} \tag{2.2.23}$$

converges absolutely for almost all $x \in G$ and the function

$$x \longmapsto \sum_{k_1 \in K_1} \mid f(xk_1^{-1})\overline{g(xk_1^{-1}k_2^{-1})} \mid \in L^1(G/K_1, d\nu_1), \tag{2.2.24}$$

Proof. Since $f \in L^2(G)$ and $T_{k_2}g \in L^2(G)$, then $f.T_{k_2}g \in L^1(G)$. But,

$$\int_{G/K_1} \sum_{k_1 \in K_1} \mid f(\xi k_1^{-1})\overline{g(\xi k_1^{-1}k_2^{-1})} \mid d\nu_1(\xi) = \int_G \mid f(x)\overline{g(xk_2^{-1})} \mid dx < \infty, \tag{2.2.25}$$

by Hölder's inequality. This implies that $\sum_{k_1 \in K_1} \mid f(xk_1^{-1})\overline{g(xk_1^{-1}k_2^{-1})} \mid < \infty$, *a.e.* ∎

Lemma 2.2.2 *Let $f, g \in L^2(G, dx)$ and let K_1, K_2 be two uniform lattices in G. For a fixed $k_1 \in K_1$, consider the function $F_{k_1} \in L^1(G/K_2, d\nu_2)$, defined by*

$$F_{k_1}(x) = \sum_{k_2 \in K_2} f(xk_2^{-1})\overline{g(xk_1^{-1}k_2^{-1})}. \tag{2.2.26}$$

Then, for any $(k_1, \gamma_2) \in K_1 \times Ann(K_2)$, we have

$$\langle f | \Theta^g_{(k_1,\gamma_2)} \rangle = \int_{G/K_2} \overline{\gamma_2(\xi)} F_{k_1}(\xi) d\nu_2(\xi). \tag{2.2.27}$$

Proof.

$$\begin{aligned}
\langle f | \Theta^g_{(k_1,\gamma_2)} \rangle &= \int_G \overline{\gamma_2(x)} f(x) \overline{g(xk_1^{-1})} dx \\
&= \int_{G/K_2} \sum_{k_2 \in K_2} \overline{\gamma_2(\xi k_2^{-1})} f(\xi k_2^{-1}) \overline{g(\xi k_1^{-1} k_2^{-1})} d\nu_2(\xi) \\
&= \int_{G/K_2} \overline{\gamma_2(\xi)} \sum_{k_2 \in K_2} f(\xi k_2^{-1}) \overline{g(\xi k_1^{-1} k_2^{-1})} d\nu_2(\xi) \\
&= \int_{G/K_2} \overline{\gamma_2(\xi)} F_{k_1}(\xi) d\nu_2(\xi).
\end{aligned} \tag{2.2.28}$$

∎

Let us now present the generalization of the WH-frame identity (see [14], Lemma 8.4.3) for an arbitrary LCA group G. Our generalization appears in Lemma 2.2.3 below. Consider the function H_{1_G} defined by

$$H_{1_G}(x) = \sum_{k_1 \in K_1} \mid g(xk_1^{-1}) \mid^2 \ . \tag{2.2.29}$$

Lemma 2.2.3 *Let $f, g \in L^2(G, dx)$ and K_1, K_2 be two uniform lattices in G. Suppose that f is a bounded measurable function with compact support and that the function H_{1_G} defined by (2.2.29) is bounded. Then*

$$\begin{aligned}
&\sum_{k_1 \in K_1} \sum_{\gamma_2 \in Ann(K_2)} \mid \langle f | \Theta^g_{(k_1,\gamma_2)} \rangle \mid^2 \qquad (2.2.30)\\
= \ & \nu_2\left(G/K_2\right) \int_G \mid f(x) \mid^2 \sum_{k_1 \in K_1} \mid g(xk_1^{-1}) \mid^2 dx \\
+ \ & \nu_2\left(G/K_2\right) \sum_{1_G \neq k_2 \in K_2} \int_G \overline{f(x)} f(xk_2^{-1}) \sum_{k_1 \in K_1} g(xk_1^{-1}) \overline{g(xk_1^{-1}k_2^{-1})} dx \ ,
\end{aligned}$$

where $\nu_2\left(G/K_2\right)$ denotes the measure of G/K_2.

Proof. From Theorem 4.26 in [26], we conclude that the set of functions

$$\{\nu_2\left(G/K_2\right)^{-\frac{1}{2}} \gamma_2\}_{\gamma_2 \in Ann(K_2)}$$

is an orthonormal basis for $L^2(G/K_2, d\nu_2)$. By Parseval's theorem, we have

$$\begin{aligned}
\sum_{\gamma_2 \in Ann(K_2)} \mid \int_{G/K_2} \overline{\gamma_2(x)} F_{k_1}(x) d\nu_2(x) \mid^2 \ &= \ \nu_2\left(G/K_2\right) \\
&\times \int_{G/K_2} \mid F_{k_1}(x) \mid^2 d\nu_2(x). \qquad (2.2.31)
\end{aligned}$$

which implies that

$$
\begin{aligned}
& \sum_{k_1 \in K_1} \sum_{\gamma_2 \in Ann(K_2)} \mid \langle f | \Theta^g_{(k_1,\gamma_2)} \rangle \mid^2 \qquad (2.2.32) \\
= & \sum_{k_1 \in K_1} \sum_{\gamma_2 \in Ann(K_2)} \mid \int_{G/K_2} \overline{\gamma_2(\xi)} F_{k_1}(\xi) d\nu_2(\xi) \mid^2 \\
= & \; \nu_2(G/K_2) \sum_{k_1 \in K_1} \int_{G/K_2} \mid F_{k_1}(\xi) \mid^2 d\nu_2(\xi) \\
= & \; \nu_2(G/K_2) \sum_{k_1 \in K_1} \int_{G/K_2} F_{k_1}(\xi) \sum_{k_2 \in K_2} \overline{f(\xi k_2^{-1})} g(\xi k_2^{-1} k_1^{-1}) d\nu_2(\xi) \\
= & \; \nu_2(G/K_2) \sum_{k_1 \in K_1} \int_G F_{k_1}(x) \overline{f(x)} g(x k_1^{-1}) dx \\
= & \; \nu_2(G/K_2) \sum_{k_1 \in K_1} \int_G \overline{f(x)} g(x k_1^{-1}) \sum_{k_2 \in K_2} f(x k_2^{-1}) \overline{g(x k_2^{-1} k_1^{-1})} dx \\
= & \; \nu_2(G/K_2) \int_G \mid f(x) \mid^2 \sum_{k_1 \in K_1} \mid g(x k_1^{-1}) \mid^2 dx \\
+ & \; \nu_2(G/K_2) \sum_{1_G \neq k_2 \in K_2} \int_G \overline{f(x)} f(x k_2^{-1}) \sum_{k_1 \in K_1} g(x k_1^{-1}) \overline{g(x k_1^{-1} k_2^{-1})} dx.
\end{aligned}
$$

■

The following is a generalization of Theorem 2.1.4 to any LCA group G.

Theorem 2.2.1 *Let K_1 and K_2 be two uniform lattices of the LCA group G. Let $g \in L^2(G, dx)$ such that:*

$$
B := \nu_2(G/K_2) \sup_{x \in G/K_1} \sum_{k_2 \in K_2} \mid \sum_{k_1 \in K_1} g(x k_1^{-1}) \overline{g(x k_1^{-1} k_2^{-1})} \mid < \infty. \qquad (2.2.33)
$$

Then $\{\Theta^g_{k_1,\gamma_2}\}_{(k_1,\gamma_2) \in K_1 \times Ann(K_2)}$ is a Bessel sequence with upper frame bound B. If also

$$
\begin{aligned}
& A := \nu_2(G/K_2) \times \\
& \inf_{x \in G/K_1} \left[\sum_{k_1 \in K_1} \mid g(x k_1^{-1}) \mid^2 - \sum_{1_G \neq k_2 \in K_2} \mid \sum_{k_1 \in K_1} g(x k_1^{-1}) \overline{g(x k_1^{-1} k_2^{-1})} \mid \right] > 0\,,
\end{aligned}
\qquad (2.2.34)
$$

then $\{\Theta^g_{k_1,\gamma_2}\}_{(k_1,\gamma_2) \in K_1 \times Ann(K_2)}$ is a frame for $L^2(G, dx)$ with bounds A, B.

Proof.

For $k_2 \in K_2$, fixed, define the function H_{k_2} by

$H_{k_2}(x) = \sum_{k_1 \in K_1} T_{k_1} g(x) \overline{T_{k_1 k_2} g(x)}$. We have:

$$\begin{aligned}\sum_{1_G \neq k_2 \in K_2} \mid T_{k_2^{-1}} H_{k_2}(x) \mid &= \sum_{1_G \neq k_2 \in K_2} \mid T_{k_2^{-1}} \sum_{k_1 \in K_1} T_{k_1} g(x) \overline{T_{k_1 k_2} g(x)} \mid \\ &= \sum_{1_G \neq k_2 \in K_2} \mid \sum_{k_1 \in K_1} T_{k_1 k_2^{-1}} g(x) \overline{T_{k_1} g(x)} \mid . \qquad (2.2.35)\end{aligned}$$

Replacing k_2 by k_2^{-1}, we have

$$\begin{aligned}\sum_{1_G \neq k_2 \in K_2} \mid T_{k_2^{-1}} H_{k_2}(x) \mid &= \sum_{1_G \neq k_2 \in K_2} \mid T_{k_2^{-1}} \sum_{k_1 \in K_1} T_{k_1} g(x) \overline{T_{k_1 k_2} g(x)} \mid \\ &= \sum_{1_G \neq k_2 \in K_2} \mid \sum_{k_1 \in K_1} T_{k_1 k_2^{-1}} g(x) \overline{T_{k_1} g(x)} \mid . \\ &= \sum_{1_G \neq k_2 \in K_2} \mid \sum_{k_1 \in K_1} T_{k_1} g(x) \overline{T_{k_1 k_2} g(x)} \mid . \\ &= \sum_{1_G \neq k_2 \in K_2} \mid H_{k_2}(x) \mid . \qquad (2.2.36)\end{aligned}$$

Thus,

$$\begin{aligned}& \mid \sum_{1_G \neq k_2 \in K_2} \int_G \overline{f(x)} f(x k_2^{-1}) \sum_{k_1 \in K_1} g(x k_1^{-1}) \overline{g(x k_1^{-1} k_2^{-1})} dx \mid \qquad (2.2.37) \\ = & \mid \sum_{1_G \neq k_2 \in K_2} \int_G \overline{f(x)} T_{k_2} f(x) H_{k_2}(x) dx \mid, \\ \leq & \sum_{1_G \neq k_2 \in K_2} \int_G \mid f(x) \mid . \mid T_{k_2} f(x) \mid . \mid H_{k_2}(x) \mid dx\end{aligned}$$

(using the Cauchy-Schwarz inquality twice: with respect to the integral and the sum)

$$\leq \int_G \mid f(x) \mid^2 . \sum_{1_G \neq k_2 \in K_2} \mid H_{k_2}(x) \mid dx.$$

By Lemma 2.2.3, we have

$$\begin{aligned}
&\sum_{k_1\in K_1}\sum_{\gamma_2\in Ann(K_2)} \mid \langle f|\Theta^g_{(k_1,\gamma_2)}\rangle \mid^2 \\
\leq\ & \nu_2(G/K_2)\int_G \left(\mid f(x)\mid^2 \left[H_{1_G}(x) + \sum_{1_G\neq k_2\in K_2} \mid \sum_{k_1\in K_1} T_{k_1}g(x)\overline{T_{k_1k_2}g(x)} \mid \right]\right) dx \\
=\ & \nu_2(G/K_2)\int_G dx \mid f(x)\mid^2 \sum_{k_2\in K_2} \mid \sum_{k_1\in K_1} T_{k_1}g(x)\overline{T_{k_1k_2}g(x)} \mid \\
\leq\ & B \parallel f \parallel^2 .
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\sum_{k_1\in K_1}\sum_{\gamma_2\in Ann(K_2)} \mid \langle f|\Theta^g_{(k_1,\gamma_2)}\rangle \mid^2 \\
\geq\ & \nu_2(G/K_2)\int_G \left(\mid f(x)\mid^2 \left[H_{1_G}(x) - \sum_{1_G\neq k_2\in K_2} \mid \sum_{k_1\in K_1} T_{k_1}g(x)\overline{T_{k_1k_2}g(x)} \mid \right]\right) dx \\
\geq\ & A \parallel f \parallel^2 .
\end{aligned}$$

Since the frame conditions hold for all f in a dense subspace of $L^2(G)$, it is true for any element of $L^2(G)$. ■

Remark 2.2.1 *The above result is more general than, and is in fact an extension to other classes of groups, of the results in [12, 33]. By taking $G = \mathbb{R}$, $K_1 = a\mathbb{Z}$, and $K_2 = \frac{1}{b}\mathbb{Z}$, we recover Theorem 2.1.4.*

2.3 Frames on the torus $\mathbb{T}^d$

Let $G = \mathbb{T}^d$ be the torus in d dimensions. Let $N_i, M_i \in \mathbb{N}^*$, $i = 1, 2, ..., d$, be $2d$ positive integers. For simplicity, we adopt the following notation in this section:

Let $\underline{\mathfrak{n}} = (n_1, ..., n_d)$, $\underline{\mathfrak{N}} = (N_1, ..., N_d)$ and $\underline{\mathfrak{M}} = (M_1, ..., M_d)$. Set $(\underline{\mathfrak{n}}, \underline{\mathfrak{N}}) = \left(\frac{n_1}{N_1}, \frac{n_2}{N_2}, ..., \frac{n_d}{N_d}\right)$ and $(\underline{\mathfrak{m}}, \underline{\mathfrak{M}}) = \left(\frac{m_1}{M_1}, \frac{m_2}{M_2}, ..., \frac{m_d}{M_d}\right)$ and consider the following two uniform lattices in $\mathbb{T}^d$:

$$\mathcal{K}_1^{\mathfrak{N}} = \{(\underline{\mathfrak{n}}, \mathfrak{N}) : n_i = 0, 1, ..., N_i - 1, \text{ for } i = 1, ..., d\} \tag{2.3.38}$$

and

$$\mathcal{K}_2^{\mathfrak{M}} = \{(\underline{\mathfrak{m}}, \mathfrak{M}) : m_i = 0, 1, ..., M_i - 1, \text{ for } i = 1, ..., d\} . \tag{2.3.39}$$

Using these, we form the sets

$$\mathbb{T}^d/\mathcal{K}_1^{\mathfrak{N}} = \left[0, \frac{1}{N_1}\right] \times ... \times \left[0, \frac{1}{N_d}\right] \equiv \Delta_1,$$

$$\mathbb{T}^d/\mathcal{K}_2^{\mathfrak{M}} = \left[0, \frac{1}{M_1}\right] \times ... \times \left[0, \frac{1}{M_d}\right] \equiv \Delta_2$$

and

$$Ann(\mathcal{K}_2^{\mathfrak{M}}) = \left\{\gamma_{\underline{k}}(\underline{x}) = e^{2\pi i \sum_{j=1}^d M_j k_j x_j}; \quad \underline{k} = (k_1, ..., k_d) \in \mathbb{Z}^d\right\} .$$

Note that $\left\{ \left(\prod_{j=1}^d M_i\right)^{\frac{1}{2}} \gamma \right\}_{\gamma \in Ann(\mathcal{K}_2^{\mathfrak{M}})}$ is an orthonormal basis of $L^2(\Delta_2)$, and we have:

Corollary 2.3.1 *Let $g \in L^2(\mathbb{T}^d)$, and $N_i, M_i,\ i = 1, 2, \ldots, d$, be $2d$ positive integers such that:*

$$\begin{aligned} B : &= \frac{1}{\prod_{j+1}^d M_j} \sup_{\underline{x} \in \Delta_1} \sum_{(\underline{\mathfrak{m}}, \mathfrak{M}) \in \mathcal{K}_2^{\mathfrak{M}}} \mid \sum_{(\underline{\mathfrak{n}}, \mathfrak{N}) \in \mathcal{K}_1^{\mathfrak{M}}} g\left([\underline{x} - (\underline{\mathfrak{n}}, \mathfrak{N})]\right) \\ &\times \overline{g\left([\underline{x} - (\underline{\mathfrak{n}}, \mathfrak{N}) - (\underline{\mathfrak{m}}, \mathfrak{M})]\right)} \mid < \infty . \end{aligned} \tag{2.3.40}$$

Then $\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}}, \mathfrak{N})} g\right\}_{((\underline{\mathfrak{n}}, \mathfrak{N}), \underline{k}) \in \mathcal{K}_1^{\mathfrak{N}} \times \mathbb{Z}^d}$ is a Bessel sequence for $L^2(\mathbb{T}^d)$ with upper bound B. If also

$$\begin{aligned} A := \frac{1}{\prod_{j+1}^d M_j} \inf_{\underline{x} \in \Delta_1} \Bigg[&\sum_{(\underline{\mathfrak{n}}, \mathfrak{N}) \in \mathcal{K}_1^{\mathfrak{N}}} \mid g\left([\underline{x} - (\underline{\mathfrak{n}}, \mathfrak{N})]\right) \mid^2 \\ - &\sum_{\underline{o} \neq (\underline{\mathfrak{m}}, \mathfrak{M}) \in \mathcal{K}_2^{\mathfrak{M}}} \mid \sum_{(\underline{\mathfrak{n}}, \mathfrak{N}) \in \mathcal{K}_1^{\mathfrak{M}}} g\left([\underline{x} - (\underline{\mathfrak{n}}, \mathfrak{N})]\right) \overline{g\left([\underline{x} - (\underline{\mathfrak{n}}, \mathfrak{N}) - (\underline{\mathfrak{m}}, \mathfrak{M})]\right)} \mid \Bigg] > 0 , \end{aligned} \tag{2.3.41}$$

then $\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}}, \mathfrak{N})} g\right\}_{((\underline{\mathfrak{n}}, \mathfrak{N}), \underline{k}) \in \mathcal{K}_1^{\mathfrak{N}} \times \mathbb{Z}^d}$ is a frame for $L^2(\mathbb{T}^d)$ with bounds A, B.

We observe immediately that the canonical commutation relations,

$$T_{(\underline{\mathfrak{n}},\mathfrak{N})}\gamma_{\underline{k}} = e^{2\pi i \sum_{j=1}^{d} M_j (\underline{\mathfrak{n}},\mathfrak{N})_j k_j}\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} \ ,$$

hold in this case. Indeed, for $g \in L^2(\mathbb{T}^d)$ we see that,

$$\begin{aligned}
\left(T_{(\underline{\mathfrak{n}},\mathfrak{N})}\gamma_{\underline{k}} g\right)(\underline{x}) &= T_{(\underline{\mathfrak{n}},\mathfrak{N})}\left(e^{2\pi i \sum_{j=1}^{d} M_j x_j k_j} g\left(\underline{x}\right)\right) = e^{2\pi i \sum_{j=1}^{d} M_j \left(x_j - \frac{n_j}{N_j}\right) k_j} g\left([\underline{x} - (\underline{\mathfrak{n}},\mathfrak{N})]\right) \\
&= e^{-2\pi i \sum_{j=1}^{d} \frac{M_j n_j k_j}{N_j}} \gamma_{\underline{k}}(\underline{x}) g\left([\underline{x} - (\underline{\mathfrak{n}},\mathfrak{N})]\right) \\
&= e^{-2\pi i \sum_{j=1}^{d} M_j (\underline{\mathfrak{n}},\mathfrak{N})_j k_j}\gamma_{\underline{k}}(\underline{x}) T_{(\underline{\mathfrak{n}},\mathfrak{N})} g\left(\underline{x}\right) .
\end{aligned}$$

It will be useful, for the purposes of the next section, to note that the frame operator, $S = \sum_{((\underline{\mathfrak{n}},\mathfrak{N}),\underline{k}) \in \mathcal{K}_1^{\mathfrak{N}} \times \mathbb{Z}^d} \mid \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} g\rangle\langle \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} g \mid$, commutes with the corresponding modulation and translation operators.

Lemma 2.3.1 *Let $g \in L^2(\mathbb{T}^d)$ and let $N_i, M_i,\ i = 1, 2, \ldots, d$, be $2d$ positive integers such that $\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} g\right\}_{((\underline{\mathfrak{n}},\mathfrak{N}),\underline{k}) \in \mathcal{K}_1^{\mathfrak{N}} \times \mathbb{Z}^d}$ is a frame for $L^2(\mathbb{T}^d)$. If S is the corresponding frame operator then,*

$$S T_{\left(\underline{\mathfrak{n}_0},\mathfrak{N}\right)} \gamma_{\underline{k}_0} = T_{\left(\underline{\mathfrak{n}_0},\mathfrak{N}\right)} \gamma_{\underline{k}_0} S, \quad \text{for all} \quad \underline{k_0} \in \mathbb{Z}^d, \text{ and all } \left(\underline{\mathfrak{n}_0}, \mathfrak{N}\right) \in \mathcal{K}_1^{\mathfrak{N}}.$$

Proof. Let $f \in L^2(\mathbb{T}^d)$. We know that

$$S(f) \ = \sum_{\underline{k} \in \mathbb{Z}^d} \sum_{(\underline{\mathfrak{n}},\mathfrak{N}) \in \mathcal{K}_1^{\mathfrak{N}}} \langle f \mid \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} g\rangle \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} g \ , \qquad (2.3.42)$$

so that,

$$
\begin{aligned}
\left(S\gamma_{\underline{k}_0}T_{\left(\underline{\mathfrak{n}_0},\mathfrak{N}\right)}\right)f &= \sum_{\underline{k}\in\mathbb{Z}^d}\sum_{(\underline{\mathfrak{n}},\mathfrak{N})\in\mathcal{K}_1^{\mathfrak{N}}} \langle \gamma_{\underline{k}_0}T_{\left(\underline{\mathfrak{n}_0},\mathfrak{N}\right)}f \mid \gamma_{\underline{k}}T_{(\underline{\mathfrak{n}},\mathfrak{N})}g\rangle \gamma_{\underline{k}}T_{(\underline{\mathfrak{n}},\mathfrak{N})}g \\
&= \sum_{\underline{k}\in\mathbb{Z}^d}\sum_{(\underline{\mathfrak{n}},\mathfrak{N})\in\mathcal{K}_1^{\mathfrak{N}}} \langle f \mid T_{\left(-\underline{\mathfrak{n}_0},\mathfrak{N}\right)}\gamma_{-\underline{k_0}}\gamma_{\underline{k}}T_{(\underline{\mathfrak{n}},\mathfrak{N})}g\rangle \gamma_{\underline{k}}T_{(\underline{\mathfrak{n}},\mathfrak{N})}g \\
&= \sum_{\underline{k}\in\mathbb{Z}^d}\sum_{(\underline{\mathfrak{n}},\mathfrak{N})\in\mathcal{K}_1^{\mathfrak{N}}} \langle f \mid e^{2\pi i\sum_{j=1}^d M_j\left(\underline{\mathfrak{n}_0},\mathfrak{N}\right)_j (k_j-k_{0j})}\gamma_{\underline{k}-\underline{k_0}}T_{\left(\underline{\mathfrak{n}-\mathfrak{n}_0},\mathfrak{N}\right)}g\rangle \\
&\times\ \gamma_{\underline{k}}T_{(\underline{\mathfrak{n}},\mathfrak{N})}g \\
&= \sum_{\underline{k}\in\mathbb{Z}^d}\sum_{(\underline{\mathfrak{n}},\mathfrak{N})\in\mathcal{K}_1^{\mathfrak{N}}} \langle f \mid e^{2\pi i\sum_{j=1}^d M_j\left(\underline{\mathfrak{n}_0},\mathfrak{N}\right)_j k_j}\gamma_{\underline{k}}T_{(\underline{\mathfrak{n}},\mathfrak{N})}g\rangle \\
&\times\ \gamma_{\underline{k+k_0}}T_{\left(\underline{\mathfrak{n}+n_0},\mathfrak{N}\right)}g \\
&= \sum_{\underline{k}\in\mathbb{Z}^d}\sum_{(\underline{\mathfrak{n}},\mathfrak{N})\in\mathcal{K}_1^{\mathfrak{N}}} \langle f \mid e^{2\pi i\sum_{j=1}^d M_j\left(\underline{\mathfrak{n}_0},\mathfrak{N}\right)_j k_j}\gamma_{\underline{k}}T_{(\underline{\mathfrak{n}},\mathfrak{N})}g\rangle \\
&\times\ e^{2\pi i\sum_{j=1}^d M_j\left(\underline{\mathfrak{n}_0},\mathfrak{N}\right)_j k_j}\gamma_{\underline{k_0}}T_{\left(\underline{\mathfrak{n}_0},\mathfrak{N}\right)}\gamma_{\underline{k}}T_{(\underline{\mathfrak{n}},\mathfrak{N})}g \\
&= \sum_{\underline{k}\in\mathbb{Z}^d}\sum_{(\underline{\mathfrak{n}},\mathfrak{N})\in\mathcal{K}_1^{\mathfrak{N}}} \langle f \mid \gamma_{\underline{k}}T_{(\underline{\mathfrak{n}},\mathfrak{N})}g\rangle \gamma_{\underline{k_0}}T_{\left(\underline{\mathfrak{n}_0},\mathfrak{N}\right)}\gamma_{\underline{k}}T_{(\underline{\mathfrak{n}},\mathfrak{N})}g \\
&= \left(\gamma_{\underline{k_0}}T_{\left(\underline{\mathfrak{n}_0},\mathfrak{N}\right)}S\right)f.
\end{aligned}
$$

■

2.3.1 Necessary condition for having frames on the torus $\mathbb{T}^d$

We derive, in the next theorem, conditions for the existence of frames on $L^2(\mathbb{T}^d)$, which will be analogues of the conditions imposed by the product ab, in Theorem 2.1.1, for $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a Gabor frame for $L^2(\mathbb{R})$.

We start by fixing a certain partition of $\mathbb{T}^d$. For $k=1,...,d$, let $i_k\in\{0,1,...,M_k-1\}$. For a $d-$tuplet $(i_1,i_2,...,i_d)$, of such i_k, let us define the subset $\Gamma_{(i_1,i_2,...,i_d)}$ by

$$
\Gamma_{(i_1,i_2,...,i_d)} = \left[\frac{i_1}{M_1},\frac{i_1+1}{M_1}\right]\times....\times\left[\frac{i_d}{M_d},\frac{i_d+1}{M_d}\right]. \tag{2.3.43}
$$

It is easy to see that these subsets have the following properties:

$$
\cup_{i_k=0,...,M_k-1;k=1,...,d}\Gamma_{(i_1,i_2,...,i_d)} = \mathbb{T}^d\ , \tag{2.3.44}
$$

$$\Gamma_{(i_1,i_2,\dots,i_d)} \cap \Gamma_{(i'_1,i'_2,\dots,i'_d)} = \emptyset \; a.e. \text{ if } (i_1,i_2,\dots,i_d) \neq (i'_1,i'_2,\dots,i'_d)\,, \tag{2.3.45}$$

$$T_{(\underline{\mathfrak{m}},\mathfrak{M})}\left(\Gamma_{(i_1,i_2,\dots,i_d)}\right) \cap \Gamma_{(i_1,i_2,\dots,i_d)} = \emptyset,\ a.e. \tag{2.3.46}$$

for all $(\underline{\mathfrak{o}},\mathfrak{M}) \neq (\underline{\mathfrak{m}},\mathfrak{M}) \in \mathcal{K}_2^{\mathfrak{M}}$ and all $(i_1,i_2,\dots,i_d)$.

Let

$$\mathcal{F}(\mathbb{T}^d) = \left\{f \in L^2(\mathbb{T}^d) \ : \ \exists (i_1,i_2,\dots,i_d) \text{ and } supp(f) \subset \Gamma_{(i_1,i_2,\dots,i_d)}\right\}$$

Then, by virtue of (2.3.44),

$$\mathcal{F}(\mathbb{T}^d) \quad \text{is dense in} \quad L^2(\mathbb{T}^d)\,, \tag{2.3.47}$$

and by (2.3.46),

$$\forall f \in \mathcal{F}(\mathbb{T}^d),\, f(\underline{x}).\overline{T_{(\underline{\mathfrak{m}},\mathfrak{M})} f(\underline{x})} = 0, \tag{2.3.48}$$

for almost all $\underline{x} \in \mathbb{T}^d$ and all $(\underline{\mathfrak{m}},\mathfrak{M}) \in \mathcal{K}_2^{\mathfrak{M}}$ and $(\underline{\mathfrak{o}},\mathfrak{M}) \neq (\underline{\mathfrak{m}},\mathfrak{M})$.

The following gives a necessary condition for having frame on $L^2(\mathbb{T}^d)$

Theorem 2.3.1 *Let $g \in L^2(\mathbb{T}^d)$, and let $N_i, M_i,\ i = 1,2,\dots,d$, be $2d$ positive integers. Then the following hold:*

(i) If $\left(\prod_{i=1}^d M_i\right) > \left(\prod_{j=1}^d N_j\right)$, *then*
$\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} g\right\}_{((\underline{\mathfrak{n}},\mathfrak{N}),\underline{k}) \in \mathcal{K}_1^{\mathfrak{N}} \times \mathbb{Z}^d}$ *is a not a frame for* $L^2(\mathbb{T}^d)$.

(ii) If $\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} g\right\}_{((\underline{\mathfrak{n}},\mathfrak{N}),\underline{k}) \in \mathcal{K}_1^{\mathfrak{N}} \times \mathbb{Z}^d}$ *is a frame for* $L^2(\mathbb{T}^d)$, *then*
$\prod_{i=1}^d M_i = \prod_{j=1}^d N_j \Leftrightarrow \left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} g\right\}_{((\underline{\mathfrak{n}},\mathfrak{N}),\underline{k}) \in \mathcal{K}_1^{\mathfrak{N}} \times \mathbb{Z}^d}$ *is a Riesz basis.*

Proof. Let us assume that $\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} g\right\}_{((\underline{\mathfrak{n}},\mathfrak{N}),\underline{k}) \in \mathcal{K}_1^{\mathfrak{N}} \times \mathbb{Z}^d}$ is a frame for $L^2(\mathbb{T}^d)$ with frame operator S. Then $\left\{S^{-\frac{1}{2}}\left(\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} g\right)\right\}_{((\underline{\mathfrak{n}},\mathfrak{N}),\underline{k}) \in \mathcal{K}_1^{\mathfrak{N}} \times \mathbb{Z}^d}$ is a tight frame for $L^2(\mathbb{T}^d)$

with frame bounds 1. Let $f \in \mathcal{F}(\mathbb{T}^d)$. Using *Lemma* 2.2.3, *Lemma* 2.3.1, and (2.3.48), we have:

$$\begin{aligned}
\int_{\mathbb{T}^d} | f(\underline{x}) |^2 \, d\underline{x} &= \sum_{\underline{k}\in\mathbb{Z}^d} \sum_{(\underline{\mathfrak{n}},\mathfrak{N})\in\mathcal{K}_1^{\mathfrak{N}}} | \langle f \mid S^{-\frac{1}{2}} \left(\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} g\right) \rangle |^2 \\
&= \sum_{\underline{k}\in\mathbb{Z}^d} \sum_{(\underline{\mathfrak{n}},\mathfrak{N})\in\mathcal{K}_1^{\mathfrak{N}}} | \langle f \mid \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} S^{-\frac{1}{2}} g \rangle |^2 \\
&= \frac{1}{\prod_{j=1}^{d} M_j} \times \\
&\quad \int_{\mathbb{T}^d} | f(\underline{x}) |^2 \sum_{(\underline{\mathfrak{n}},\mathfrak{N})\in\mathcal{K}_1^{\mathfrak{M}}} | S^{-\frac{1}{2}} g\left([\underline{x} - (\underline{\mathfrak{n}},\mathfrak{N})]\right) |^2 \, dx, \quad (2.3.49)
\end{aligned}$$

which implies that

$$\sum_{(\underline{\mathfrak{n}},\mathfrak{N})\in\mathcal{K}_1^{\mathfrak{M}}} | S^{-\frac{1}{2}} g\left([\underline{x} - (\underline{\mathfrak{n}},\mathfrak{N})]\right) |^2 = \prod_{j=1}^{d} M_j, \quad a.e., \ in \ \mathbb{T}^d. \tag{2.3.50}$$

Since

$$\left\{ S^{-\frac{1}{2}} \left(\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} g\right) \right\}_{((\underline{\mathfrak{n}},\mathfrak{N}),\underline{k})\in\mathcal{K}_1^{\mathfrak{N}}\times\mathbb{Z}^d}$$

is a tight frame, we have

$$\begin{aligned}
1 &\geq \| S^{-\frac{1}{2}} \left(\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} g\right) \|^2 \\
&= \int_{\mathbb{T}^d} | S^{-\frac{1}{2}} g(\underline{x}) |^2 \, d\underline{x} \\
&= \int_0^{\frac{1}{N_1}} \int_0^{\frac{1}{N_2}} \ldots \int_0^{\frac{1}{N_d}} \sum_{(\underline{\mathfrak{n}},\mathfrak{N})\in\mathcal{K}_1^{\mathfrak{M}}} | S^{-\frac{1}{2}} g\left([\underline{x} - (\underline{\mathfrak{n}},\mathfrak{N})]\right) |^2 \, d\underline{x} \\
&= \int_0^{\frac{1}{N_1}} \int_0^{\frac{1}{N_2}} \ldots \int_0^{\frac{1}{N_d}} \prod_{j=1}^{d} M_j d\underline{x} \\
&= \left(\prod_{j=1}^{d} M_j\right) \left(\prod_{j=1}^{d} N_j\right)^{-1},
\end{aligned}$$

which proves (i). In order to prove part (ii), let us assume that $\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} g\right\}_{((\underline{\mathfrak{n}},\mathfrak{N}),\underline{k})\in\mathcal{K}_1^{\mathfrak{N}}\times\mathbb{Z}^d}$ is a Riesz Basis. Then $\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} S^{-\frac{1}{2}} g\right\}_{((\underline{\mathfrak{n}},\mathfrak{N}),\underline{k})\in\mathcal{K}_1^{\mathfrak{N}}\times\mathbb{Z}^d}$ is a Riesz Basis having bounds $A = B = 1$, which implies that $\| S^{-\frac{1}{2}} g \|^2 = 1$. So we have $\prod_{j=1}^{d} M_j = \prod_{j=1} N_j$. For the second implication, let us assume that $\prod_{j=1}^{d} M_j = \prod_{j=1} N_j$. Then $\| \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\mathfrak{N})} S^{-\frac{1}{2}} g \|^2 =$

1, for all $\underline{k} \in Z^d$ and all $(\underline{\mathfrak{n}}, \underline{\mathfrak{N}}) \in \mathcal{K}_1^{\mathfrak{N}}$. Thus, $\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} S^{-\frac{1}{2}} g\right\}_{((\underline{\mathfrak{n}},\underline{\mathfrak{N}}),\underline{k}) \in \mathcal{K}_1^{\mathfrak{N}} \times \mathbb{Z}^d}$ is a an orthonormal Basis for $L^2(\mathbb{T}^d)$, and therefore,

$$\left\{\gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} g\right\}_{((\underline{\mathfrak{n}},\underline{\mathfrak{N}}),\underline{k}) \in \mathcal{K}_1^{\mathfrak{N}} \times \mathbb{Z}^d} = \left\{S^{\frac{1}{2}} \gamma_{\underline{k}} T_{(\underline{\mathfrak{n}},\underline{\mathfrak{N}})} S^{-\frac{1}{2}} g\right\}_{((\underline{\mathfrak{n}},\underline{\mathfrak{N}}),\underline{k}) \in \mathcal{K}_1^{\mathfrak{N}} \times \mathbb{Z}^d} \tag{2.3.51}$$

is a Riesz basis. ■

2.3.2 Generalization to arbitrary LCA groups

We show next that the above theorem can be extended to any LCA group G, thus constituting a generalization of Theorem 2.1.1 to any such group. Before stating the result, we note two commutation relations.

For $(k_1, \gamma_2) \in K_1 \times Ann(K_2)$, we have

$$T_{k_1} \gamma_2 = \gamma_2 \left(k_1^{-1}\right) \gamma_2 T_{k_1}. \tag{2.3.52}$$

Also, for $(k_1^0, \gamma_2^0) \in K_1 \times Ann(K_2)$, we have

$$S \gamma_2^0 T_{k_1^0} = \gamma_2^0 T_{k_1^0} S\ . \tag{2.3.53}$$

Indeed, for $f, g \in L^2(G)$,

$$\begin{aligned}
\left(S \gamma_2^0 T_{k_1^0}\right) f &= \sum_{k_1 \in K_1} \sum_{\gamma_2 \in Ann(K_2)} \langle \gamma_2^0 T_{k_1^0} f | \gamma_2 T_{k_1} g \rangle \gamma_2 T_{k_1} g \\
&= \sum_{k_1 \in K_1} \sum_{\gamma_2 \in Ann(K_2)} \langle f | T_{(k_1^0)^{-1}} \left(\gamma_2^0\right)^{-1} \gamma_2 T_{k_1} g \rangle \gamma_2 T_{k_1} g \\
&= \sum_{k_1 \in K_1} \sum_{\gamma_2 \in Ann(K_2)} \langle f | \left(\left(\gamma_2^0\right)^{-1} \gamma_2\right) (k_1^0) \left(\gamma_2^0\right)^{-1} \gamma_2 T_{k_1 (k_1^0)^{-1}} g \rangle \gamma_2 T_{k_1} g \\
&= \sum_{\tilde{k}_1 \in K_1} \sum_{\tilde{\gamma}_2 \in Ann(K_2)} \langle f | \tilde{\gamma}_2 T_{\tilde{k}_1} g \rangle \gamma_2^0 T_{k_1^0} \tilde{\gamma}_2 T_{\tilde{k}_1} g\ ,
\end{aligned}$$

whence the result.

Theorem 2.3.2 *Let $g \in L^2(G, dx)$ and let K_1 and K_2 be two uniform lattices in G. Then, the following hold:*

(i) If $\frac{\nu_1(G/K_1)}{\nu_2(G/K_2)} > 1$, *then*

$\{\Theta^g_{(k_1,\gamma_2)}\}_{(k_1,\gamma_2)\in K_1\times Ann(K_2)}$ *is a not a frame for* $L^2(G,dx)$.

(ii) If $\{\Theta^g_{(k_1,\gamma_2)}\}_{(k_1,\gamma_2)\in K_1\times Ann(K_2)}$ *is a frame for* $L^2(G,dx)$, *then*

$\nu_1(G/K_1) = \nu_2(G/K_2) \Leftrightarrow \{\Theta^g_{(k_1,\gamma_2)}\}_{(k_1,\gamma_2)\in K_1\times Ann(K_2)}$ *is a Riesz basis.*

Proof. Let U be a fundamental domain of K_2 in G. For $k_2 \in K_2$, let $U_{k_2} = T_{k_2}U = k_2U$.

We have:

(i) $\cup_{k_2\in K_2} U_{k_2} = G$

(ii) $U_{k_2} \cap U_{k_2'} = \emptyset$, for $k_2 \neq k_2'$

Let

$$\mathcal{F}(G) = \left\{f \in L^2(G,dx) \ : \ \exists k_2 \in K_2 \quad \text{and} \quad supp(f) \subset U_{k_2}\right\}$$

and assume that $\{\Theta^g_{k_1,\gamma_2}\}_{(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(\mathcal{K}_2)}$ is a frame for $L^2(G)$. Then, the set of vectors $\{\Theta^{S^{-\frac{1}{2}}g}_{k_1,\gamma_2}\}_{(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(\mathcal{K}_2)}$ is a tight frame with bounds 1. Let $f \in \mathcal{F}(G)$. Using Lemma 2.2.3, and the statements (i) and (ii), we have:

$$\begin{aligned}\int_G |\, f(\underline{x})\,|^2\, d\underline{x} &= \sum_{k_1\in\mathcal{K}_1}\ \sum_{\gamma_2 Ann(\mathcal{K}_2)} |\, \langle f \mid \gamma_2 T_{k_1} S^{-\frac{1}{2}} g\rangle\,|^2 \\ &= \nu_2\left(G/K_2\right) \int_G |\, f(\underline{x})\,|^2 \sum_{k_1\in\mathcal{K}_1} |\, S^{-\frac{1}{2}} g\left(xk_1^{-1}\right)|^2\, dx, \quad (2.3.54)\end{aligned}$$

which implies that

$$\sum_{k_1\in\mathcal{K}_1} |\, S^{-\frac{1}{2}} g\left(xk_1^{-1}\right)|^2 = \nu_2\left(G/K_2\right)^{-1}, \quad \text{a.e. in} \quad G. \qquad (2.3.55)$$

Since $\left\{S^{-\frac{1}{2}}\left(\gamma_2 T_{k_1} g\right)\right\}_{(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(\mathcal{K}_2)}$ is a tight frame, we have

$$\begin{aligned}
1 &\geq \| S^{-\frac{1}{2}}\left(\gamma_2 T_{k_1} g\right) \|^2 \\
&= \int_G | S^{-\frac{1}{2}} g(\underline{x}) |^2 \, d\underline{x} \\
&= \int_{G/\mathcal{K}_1} \sum_{k_1\in\mathcal{K}_1} | S^{-\frac{1}{2}} g\left(\xi k_1^{-1}\right) |^2 \, d\nu_1(\xi) \\
&= \int_{G/\mathcal{K}_1} \nu_2\left(G/K_2\right)^{-1} d\nu_1(\xi) \\
&= \nu_1\left(G/K_1\right).\nu_2\left(G/K_2\right)^{-1} = \frac{\nu_1\left(G/K_1\right)}{\nu_2\left(G/K_2\right)},
\end{aligned}$$

which proves (i). For part (ii) let assume that $\{\gamma_2 T_{k_1} g\}_{(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(K_2)}$ is a Riesz basis. Then the set of vectors $\left\{S^{-\frac{1}{2}}\left(\gamma_2 T_{k_1} g\right)\right\}_{(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(\mathcal{K}_2)}$ is a Riesz Basis having bounds $A = B = 1$, which implies that $\| S^{-\frac{1}{2}} g \|^2 = 1$. So, we have $\nu_1\left(G/K_1\right) = \nu_2\left(G/K_2\right)$. For the second implication, let assume that $\nu_1\left(G/K_1\right) = \nu_2\left(G/K_2\right)$, then $\| \gamma_2 T_{k_1} S^{-\frac{1}{2}} g \|^2 = 1$, for all $(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(\mathcal{K}_2)$. Then $\left\{\gamma_2 T_{k_1} S^{-\frac{1}{2}} g\right\}_{(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(\mathcal{K}_2)}$ is a an orthonormal Basis for $L^2(G,dx)$, and therefore,

$$\{\gamma_2 T_{k_1} g\}_{(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(\mathcal{K}_2)} = \left\{S^{\frac{1}{2}}\gamma_2 T_{k_1} S^{-\frac{1}{2}} g\right\}_{(k_1,\gamma_2)\in\mathcal{K}_1\times Ann(\mathcal{K}_2)} \tag{2.3.56}$$

is a Riesz basis. ■

2.4 General shift-invariant systems

In this section, using an obvious generalization of the notion of a shift invariant system, defined in Section 6.1, we present a complete characterization of a generalized Gabor frame on $L^2(G)$, where G is any locally compact Abelian group. The result will be an extension of the result of Ron and Shen [40] on $L^2(\mathbb{R})$. Recall that for an LCA group the Fourier transform is a map $\mathcal{F}$ defined from $L^1(G) \longrightarrow C(\widehat{G})$ by

$$(\mathcal{F}f)(\xi) := \hat{f}(\xi) = \int_G \overline{\langle x,\xi\rangle} f(x)dx. \tag{2.4.57}$$

Let define the following operations:

$$\langle yx,\xi\rangle = \langle y,\xi\rangle\langle x,\xi\rangle,\ \forall x,y\in G \ \text{ and}\,\forall\,\xi\in\widehat{G}. \tag{2.4.58}$$

$$\langle x,\xi\rangle = \overline{\langle y,\xi^{-1}\rangle},\ \forall x\in G \ \text{ and}\,\forall\,\xi\in\widehat{G}. \tag{2.4.59}$$

$$(\eta f)(x) = \langle x,\eta\rangle f(x),\ \forall x\in G\,\forall\,\eta\in\widehat{G},\ \text{where f is complex valued function on } G. \tag{2.4.60}$$

This can be extended to a map $L^2(G)\longrightarrow L^2(\widehat{G})$, satisfying the well-known Plancherel identity. The following properties of the Fourier transform will be required in the sequel:

$$\begin{aligned}\left(\widehat{T_y f}\right)(\xi) &= \int_G \overline{\langle x,\xi\rangle} f(y^{-1}x)dx = \int_G \overline{\langle yx,\xi\rangle} f(x)dx \\ &= \overline{\langle y,\xi\rangle}\hat{f}(\xi),\end{aligned} \tag{2.4.61}$$

$$\text{For } \eta\in\widehat{G},\ \text{ we have : } \left(\widehat{\eta f}\right)(\xi) = \int_G \overline{\langle x,\xi\rangle}\langle x,\eta\rangle f(x)dx = \hat{f}\left(\eta^{-1}\xi\right). \tag{2.4.62}$$

Let $\{g_m\}_{m\in\mathbb{Z}}$ be a collection of functions in $L^2(G)$ and K_1 a uniform lattice in G. For $m\in\mathbb{Z}$ and $k_1\in K_1$, consider the function $g_{k_1,m}$ defined on G by $g_{k_1,m}(x) = g_m(xk_1^{-1})$.

Lemma 2.4.1 *Let $\{g_{k_1,m}\}_{m\in\mathbb{Z};k_1\in K_1}$ and $\{h_{k_1,m}\}_{m\in\mathbb{Z};k_1\in K_1}$ be two shift invariant systems and assume that they are Bessel sequences. Then, for $e,f\in L^2(G)$, the function,*

$$\begin{aligned} P(e,f):\ G &\longrightarrow \mathbb{C} \\ x &\longmapsto \sum_{m\in\mathbb{Z}}\sum_{k_1\in K_1}\langle T_x e\mid g_{k_1,m}\rangle\langle h_{k_1,m}\mid T_x f\rangle \end{aligned} \tag{2.4.63}$$

is continuous and well defined on G/K_1. Its Fourier series in $L^2(G/K_1)$ is

$$P(e,f)(x) = \sum_{\gamma_1\in Ann(K_1)} c_{\gamma_1}\gamma_1(x)\ , \tag{2.4.64}$$

where,

$$c_{\gamma_1} = \nu_1(G/K_1)^{-1}\int_{\widehat{G}}\hat{e}(\xi)\overline{\hat{f}(\xi\gamma_1)}\sum_{m\in\mathbb{Z}}\overline{\hat{g_m}(\xi)}\hat{h}_i(\xi\gamma_1)d\xi,\quad \gamma_1\in Ann(K_1).$$

Proof. Using the Cauchy-Schwarz inequality and the fact that the sets $\{g_{k_1,m}\}_{m\in\mathbb{Z};k_1\in K_1}$ and $\{h_{k_1,m}\}_{m\in\mathbb{Z};k_1\in K_1}$ are Bessel sequences, we conclude that the series defined by $P(e,f)(x)$ converges absolutely. Also, for any $k_1 \in K_1$, we have $P(e,f)(xk_1) = P(e,f)(x)$, for almost all $x \in G$. Hence $P(e,f)$ is well defined as a function on G/K_1. For the determination of the Fourier coefficients, let assume that e, f are continuous and have compact supports. Then the coefficients c_{γ_1}, with respect to $\{\gamma_1(x)\}_{\gamma_1\in Ann(K_1)}$, are given by

$$\begin{aligned} c_{\gamma_1} &= \nu_1(G/K_1)^{-1}\int_{G/K_1}\rho(e,f)(\xi)\overline{\gamma_1(\xi)}d\nu_1(\xi) \\ &= \nu_1(G/K_1)^{-1}\sum_{m\in\mathbb{Z}}\sum_{k_1\in K_1}\int_{G/K_1}\langle T_\xi e \mid g_m\left(.k_1^{-1}\right)\rangle\langle h_m\left(.k_1^{-1}\right) \mid T_\xi f\rangle\overline{\gamma_1(\xi)}d\nu_1(\xi) \\ &= \nu_1(G/K_1)^{-1}\sum_{m\in\mathbb{Z}}\int_G\langle T_x e \mid g_m\rangle\langle h_m \mid T_x f\rangle\overline{\gamma_1(x)}dx \\ &= \nu_1(G/K_1)^{-1}\sum_{m\in\mathbb{Z}}\int_G\langle T_x e \mid g_m\rangle\overline{\langle T_x f \mid h_m\rangle\gamma_1(x)}dx\,. \end{aligned} \tag{2.4.65}$$

For an arbitrary $\phi \in L^2(G)$,

$$\langle T_x e \mid \phi\rangle = \langle \mathcal{F}T_x e \mid \mathcal{F}\phi\rangle = \int_{\widehat{G}}\overline{\xi(x)}\hat{e}(\xi)\overline{\hat{\phi}(\xi)}d\xi = \mathcal{F}\left(\hat{e}.\overline{\hat{\phi}}\right)(x). \tag{2.4.66}$$

The last equality is justified by identifying G and $\widehat{\widehat{G}}$ and adopting the natural dual pairing.

Also, using (2.4.66) and (2.4.61), we have:

$$\begin{aligned} \langle T_x f \mid h_m\rangle\gamma_1(x) &= \gamma_1(x)\mathcal{F}\left(\hat{f}.\overline{\hat{h}_m}\right)(x) \\ &= \langle x,\gamma_1\rangle\mathcal{F}\left(\hat{f}.\overline{\hat{h}_m}\right)(x) = \overline{\langle x,\gamma_1^{-1}\rangle}\mathcal{F}\left(\hat{f}.\overline{\hat{h}_m}\right)(x) \\ &= \mathcal{F}\left(T_{\gamma_1^{-1}}\hat{f}.\overline{\hat{h}_m}\right)(x). \end{aligned} \tag{2.4.67}$$

Using (2.4.66) and (2.4.67) in (2.4.65), we have

$$\begin{aligned} c_{\gamma_1} &= \nu_1(G/K_1)^{-1}\sum_{m\in\mathbb{Z}}\int_G\mathcal{F}\left(\hat{e}.\overline{\hat{g}_m}\right)(x)\overline{\mathcal{F}\left(T_{\gamma_1^{-1}}\hat{f}.\overline{\hat{h}_m}\right)(x)}dx \\ &= \nu_1(G/K_1)^{-1}\sum_{m\in\mathbb{Z}}\int_{\widehat{G}}\left(\hat{e}.\overline{\hat{g}_m}\right)(\xi)\left[\overline{T_{\gamma_1^{-1}}\left(\hat{f}.\overline{\hat{h}_m}\right)}\right](\xi)d\xi \\ &= \nu_1(G/K_1)^{-1}\int_{\widehat{G}}\hat{e}(\xi).\overline{\hat{f}(\xi\gamma_1)}\sum_{m\in\mathbb{Z}}\overline{\hat{g}_m}(\xi)\hat{h}_m(\xi\gamma_1)d\xi. \end{aligned} \tag{2.4.68}$$

■

We proceed to characterize the frame properties of shift-invariant systems for $L^2(G)$, where G is an arbitrary LCA group. Let $\{g_m\}_{m\in\mathbb{Z}}$, be a collection of functions in $L^2(G)$ and K_1 a uniform lattice in G. For $\xi \in \widehat{G}$, consider the matrix valued function $H(\xi) = \{\hat{g}_{\gamma_1,m}(\xi) = \hat{g}_m\left(\xi\gamma_1^{-1}\right)\}_{m\in\mathbb{Z};\gamma_1\in Ann(K_1)}$.

Proposition 2.4.1 *Assume that the system $\{g_{\gamma_1,m}\}_{(\gamma_1,m)\in Ann(K_1)\times\mathbb{Z}}$, has finite upper frame bound B. Then, for almost all $\xi \in \widehat{G}$, $H(\xi)$ defines a bounded linear operator from $\ell^2(\mathbb{Z})$ into $\ell^2(Ann(K_1))$ with operator norm $\leq (\nu_1(G/K_1).B)^{1/2}$. Explicitly,*

$$\sum_{\gamma_1\in Ann(K_1)} \mid \sum_{m\in\mathbb{Z}} \hat{g}_m(\xi\gamma_1)\beta_m \mid^2 \leq \nu_1(G/K_1).B \parallel \underline{\beta} \parallel^2, \tag{2.4.69}$$

for almost all $\xi \in \widehat{G}$ and $\underline{\beta} \in \ell^2(\mathbb{Z})$.

Proof. Using the fact that $\widehat{\left(\frac{\widehat{G}}{Ann(K_1)}\right)} = Ann\,(Ann(K_1)) = K_1$, we see that K_1 is an orthonormal basis of $L^2\left(\widehat{G}/Ann\,(K_1), \nu_1\,(G/K_1)^{-1}\,d\hat{\nu}_1\right)$, where the action is defined in a natural way by $k_1(\xi) := \xi(k_1)$.

Let $\alpha_{k_1,m} \neq 0$ for only finitely many $(k_1,m) \in K_1 \times \mathbb{Z}$, and let

$$\alpha_m(\xi) \;=\; \sum_{k_1\in K_1} \alpha_{k_1,m}\overline{\xi(k_1)} = \sum_{k_1\in K_1} \alpha_{k_1,m}\overline{k_1(\xi)}. \tag{2.4.70}$$

For any $\gamma_1 \in Ann\,(K_1)$, we have $\alpha_m(\xi\gamma_1) = \alpha_m(\xi)$. Thus, α_m is well defined as a function on $\widehat{G}/Ann\,(K_1)$, and we have:

$$\int_{\widehat{G}/Ann(K_1)} \sum_{\gamma_1\in Ann(K_1)} \mid \sum_{m\in\mathbb{Z}} \alpha_m(\xi)\hat{g}_m(\xi\gamma_1) \mid^2 d\hat{\nu}_1(\xi) = \tag{2.4.71}$$

$$\int_{\widehat{G}} \mid \sum_{m\in\mathbb{Z}} \alpha_m(\xi)\hat{g}_m(\xi) \mid^2 d\xi.$$

Using Parseval's theorem (or the fact that the Fourier transform is a unitary operator) and (2.4.61), we have:

$$\int_{\widehat{G}} \mid \sum_{k_1\in K_1;m\in\mathbb{Z}} \alpha_{k_1,m}\xi(k_1)\hat{g}_m(\xi) \mid^2 d\xi = \parallel \sum_{k_1\in K_1;m\in\mathbb{Z}} \alpha_{k_1,m}g_{k_1,m} \parallel^2. \tag{2.4.72}$$

Also, we have

$$\| \sum_{k_1 \in K_1; m \in \mathbb{Z}} \alpha_{k_1,m} g_{k_1,m} \|^2 \leq B \sum_{k_1 \in K_1; m \in \mathbb{Z}} | \alpha_{k_1,m} |^2, \tag{2.4.73}$$

and

$$\begin{aligned} \| \underline{\alpha} \|^2 &= \sum_{k_1 \in K_1; m \in \mathbb{Z}} | \alpha_{k_1,m} |^2 \\ &= \nu_1 (G/K_1) \int_{\widehat{G}/Ann(K_1)} \sum_{m \in \mathbb{Z}} | \alpha_m(\xi) |^2 \, d\hat{\nu}_1(\xi). \end{aligned} \tag{2.4.74}$$

Using (2.4.72), (2.4.73) and(2.4.74), we obtain

$$\begin{aligned} &\int_{\widehat{G}/Ann(K_1)} \sum_{\gamma_1 \in Ann(K_1)} | \sum_{m \in \mathbb{Z}} \alpha_m(\xi) \hat{g}_m(\xi \gamma_1^{-1}) |^2 \, d\hat{\nu}_1(\xi) \leq \\ &\nu_1 (G/K_1) \int_{\widehat{G}/Ann(K_1)} \sum_{m \in \mathbb{Z}} | \alpha_m(\xi) |^2 \, d\hat{\nu}_1(\xi). \end{aligned} \tag{2.4.75}$$

For $\underline{\beta} \in \ell^2(\mathbb{Z})$, with $\beta_m \neq 0$ for only finitely many $m \in \mathbb{Z}$, choose $\alpha_m(\xi) = \beta_m \rho(\xi)$, where $\rho(\xi) = \sum_{k_1 \in K_1} \rho_{k_1} k_1(\xi)$, with $\rho_{k_1} \neq 0$ for only finitely many $k_1 \in K_1$. Thus, we get

$$\begin{aligned} &\int_{\widehat{G}/Ann(K_1)} \sum_{\gamma_1 \in Ann(K_1)} | \rho(\xi) |^2 . | \sum_{m \in \mathbb{Z}} \hat{g}_m(\xi \gamma_1^{-1}) \beta_m |^2 \, d\hat{\nu}_1(\xi) \leq \\ &\nu_1 (G/K_1) \, B \, \| \underline{\beta} \|^2 \int_{\widehat{G}/Ann(K_1)} | \rho(\xi) |^2 \, d\hat{\nu}_1(\xi). \end{aligned} \tag{2.4.76}$$

Since the set of such ρ is dense in $L^2 \left(\widehat{G}/Ann\,(K_1), \nu_1\,(G/K_1)^{-1} \, d\hat{\nu}_1 \right)$ (because of the fact that K_1 is an orthonormal basis $L^2 \left(\widehat{G}/Ann\,(K_1), \nu_1\,(G/K_1)^{-1} \, d\hat{\nu}_1 \right)$), we have

$$\sum_{\gamma_1 \in Ann(K_1)} | \sum_{m \in \mathbb{Z}} \hat{g}_m(\xi \gamma_1^{-1}) \beta_m |^2 \leq \nu_1 (G/K_1) \, B \, \| \underline{\beta} \|^2 ,$$

for almost all $\xi \in \widehat{G}/Ann\,(K_1)$. Let V be a countable, dense subset of $\ell^2(\mathbb{Z})$ of $\underline{\beta}$'s with $\beta_m \neq 0$ for only finitely many $m \in \mathbb{Z}$, and let $N_1 \subset \widehat{G}/Ann\,(K_1)$ be the null set outside of which

$$\sum_{\gamma_1 \in Ann(K_1)} | \sum_{m \in \mathbb{Z}} \hat{g}_m(\xi \gamma_1^{-1}) \beta_m |^2 \leq \nu_1 (G/K_1) \, B \, \| \underline{\beta} \|^2, \ \forall \underline{\beta} \in V. \tag{2.4.77}$$

Also, let $N_2 \subset \widehat{G}/Ann\,(K_1)$ be a null set outside of which

$$\sum_{m\in\mathbb{Z}} \mid \hat{g}_m(\xi\gamma_1^{-1}) \mid^2 \;\leq\; \nu_1\,(G/K_1)\,.B,\; \forall\, \gamma_1 \in Ann\,(K_1)\,. \tag{2.4.78}$$

Letting $\underline{\beta} \in \ell^2(\mathbb{Z})$ and $\underline{\beta}^{(M)} \in V$ such that $\underline{\beta}^{(M)} \longrightarrow \underline{\beta}$, and applying Fatou's Lemma, we arrive at

$$\begin{aligned}\sum_{\gamma_1\in Ann(K_1)} \mid \sum_{m\in\mathbb{Z}} \hat{g}_m(\xi\gamma_1^{-1})\beta_m \mid^2 \;&\leq\; \liminf_{M\longrightarrow\infty} \sum_{\gamma_1\in Ann(K_1)} \mid \sum_{m\in\mathbb{Z}} \hat{g}_m(\xi\gamma_1^{-1})\beta_m^{(M)} \mid^2 \\ &\leq\; \nu_1\,(G/K_1)\,.B \parallel \underline{\beta} \parallel^2 .\end{aligned} \tag{2.4.79}$$

Finally, we have

$$\sum_{\gamma_1\in Ann(K_1)} \mid \sum_{m\in\mathbb{Z}} \hat{g}_m(\xi\gamma_1^{-1})\beta_m \mid^2 \leq \nu_1\,(G/K_1)\,.B \parallel \underline{\beta} \parallel^2 \;\; \forall\, \underline{\beta} \in \ell^2(\mathbb{Z})\;,$$

for almost all $\xi \in \left\{\widehat{G}/Ann\,(K_1)\right\} \setminus N$, where $N = N_1 \cup N_2$. Since any element $\nu \in \widehat{G}$ can be written as $\nu = \xi\gamma_1$, where $\xi \in \widehat{G}/Ann\,(K_1)$ and $\gamma_1 \in Ann\,(K_1)$ and since the first sum in (2.4.80) is taken over all elements of $Ann\,(K_1)$, we have

$$\sum_{\gamma_1\in Ann(K_1)} \mid \sum_{m\in\mathbb{Z}} \hat{g}_m(\xi\gamma_1^{-1})\beta_m \mid^2 \leq \nu_1\,(G/K_1)\,.B \parallel \underline{\beta} \parallel^2;\; \underline{\beta} \in \ell^2(\mathbb{Z});\;\; a.e.\;\, \xi \in \widehat{G}.$$

■

The following is a generalization of the Theorem 2.1.3 to $L^2(G)$, for any LCA group G.

Theorem 2.4.1 *With the same setting as above, the following hold:*

(i) $\{g_{\gamma_1,m}\}$ *is a Bessel sequence with upper bound* B *if and only if for almost all* ξ, $H(\xi)$ *defines a bounded operator from* $\ell^2(\mathbb{Z})$ *into* $\ell^2\,(Ann(K_1))$ *of norm at most* $\sqrt{\nu_1\,(G/K_1)\,.B}$.

(ii) $\{g_{\gamma_1,m}\}$ *is a frame for* $L^2(G,dx)$ *with frame bounds* A, B *if and only if*

$$\nu_1\,(G/K_1)\,.A\mathbb{I} \leq H(\xi)H^*(\xi) \leq \nu_1\,(G/K_1)\,.B\mathbb{I}, \tag{2.4.80}$$

for almost all ξ, *where* $\mathbb{I}$ *is identity operator on* $\ell^2\,(Ann(K_1))$.

(iii) $\{g_{\gamma_1,m}\}$ *is a tight frame for* $L^2(G, dx)$ *if and only if there is a constant* $c > 0$ *such that*

$$\sum_{m\in\mathbb{Z}} \overline{\hat{g}_m(\xi)}\hat{g}_m(\xi\gamma_1) = c\delta_{\gamma_1,1_G}\ , \quad \gamma_1 \in Ann(K_1)\ , \tag{2.4.81}$$

for almost all ξ. *In this case, the frame bound is* $A = \dfrac{c}{\nu_1\left(G/K_1\right)}$.

(iv) *Two shift-invariant systems* $\{g_{\gamma_1,m}\}$ *and* $\{h_{\gamma_1,m}\}$, *which form Bessel sequences, are dual frames if and only if*

$$\sum_{m\in\mathbb{Z}} \overline{\hat{g}_m(\xi)}\hat{h}_m(\xi\gamma_1) = \nu_1\left(G/K_1\right)\delta_{\gamma_1,1_{\hat{G}}}\ , \quad \gamma_1 \in Ann(K_1)\ . \tag{2.4.82}$$

for almost all ξ.

Proof. For part (iv), it's known that $\{g_{\gamma_1,m}\}$ and $\{h_{\gamma_1,m}\}$ are dual frames if and only if

$$\langle e \mid f\rangle = \sum_{m\in\mathbb{Z}} \sum_{k_1\in Ann(K_1)} \langle e \mid g_{\gamma_1,m}\rangle\langle h_{\gamma_1,m} \mid f\rangle, \quad \forall e,\ f \in\ L^2(G), \tag{2.4.83}$$

and we have $\rho\left(e,f\right)(x) = \langle T_x e \mid T_x f\rangle = \langle e \mid f\rangle, \quad \forall x \in G/K_1$. Hence the functions $\rho\left(e,f\right)(x)$ and $\langle e \mid f\rangle$ have the same Fourier coefficients in $L^2\left(G/K_1, d\nu_1\right)$, whence

$$\begin{aligned} c_{\gamma_1} &= \frac{1}{\nu_1\left(G/K_1\right)}\int_{\hat{G}} \hat{e}(\xi).\overline{\hat{f}(\xi\gamma_1)}\sum_{m\in\mathbb{Z}} \overline{\hat{g}_m}(\xi)\hat{h}_m(\xi\gamma_1)d\xi \\ &= \langle e \mid f\rangle\delta_{\gamma_1,1_{\hat{G}}} = \delta_{\gamma_1,1_{\hat{G}}}\int_{\hat{G}} \hat{e}(\xi)\overline{\hat{f}(\xi)}d\xi. \end{aligned} \tag{2.4.84}$$

Since (2.4.84) holds for all $e \in L^2\left(G\right)$, we have

$$\overline{\hat{f}(\xi\gamma_1)}\sum_{m\in\mathbb{Z}} \overline{\hat{g}_m}(\xi)\hat{h}_m(\xi\gamma_1) \ = \ \nu_1\left(G/K_1\right)\delta_{\gamma_1,1_{\hat{G}}}\overline{\hat{f}(\xi)},\ a.e.\ \xi \in \widehat{G}. \tag{2.4.85}$$

If $\gamma_1 = 1_{\hat{G}}$, we get

$$\sum_{m\in\mathbb{Z}} \overline{\hat{g}_m}(\xi)\hat{h}_m(\xi) \ = \ \nu_1\left(G/K_1\right), \quad a.e.\ \xi \in \widehat{G}. \tag{2.4.86}$$

If $\gamma_1 \neq 1_{\hat{G}}$, then $\gamma_1\xi \neq \xi, \quad \forall\xi \in \widehat{G}$. Since $\widehat{G}$ is Hausdorff, there exists an open neighbourhood, $\mathcal{O}_{\gamma_1\xi}$ of $\gamma_1\xi$, such that $\xi \notin \mathcal{O}_{\gamma_1\xi}$. By taking $\hat{f}(s) = \chi_{\mathcal{O}_{\gamma_1\xi}}(s)$, the characteristic function of $\mathcal{O}_{\gamma_1\xi}$, and using this function in (2.4.85), we obtain

$$\sum_{m\in\mathbb{Z}} \overline{\hat{g}_m}(\xi)\hat{h}_m(\xi\gamma_1) \ = \ 0, \tag{2.4.87}$$

for almost all $\xi \in \widehat{G}$ and for all $\gamma_1 \neq 1_{\widehat{G}}$. Thus,

$$\sum_{m\in\mathbb{Z}} \overline{\hat{g}_m}(\xi)\hat{h}_m(\xi\gamma_1) = \nu_1\left(G/K_1\right)\delta_{\gamma_1, 1_{\widehat{G}}}, \quad \gamma_1 \in Ann(K_1), \tag{2.4.88}$$

for almost all $\xi \in \widehat{G}$. For the opposite implication let us assume that (2.1.10) holds. It follows that the function $\rho\left(e, f\right)$ and $\langle e \mid f\rangle$ have the same Fourier coefficients and then $\rho\left(e, f\right)(x) = \langle e \mid f\rangle$, $\forall x \in G$. By taking $x = 1_G$, we obtain (2.4.83).

For the first part of (iii), we use (iv) and Lemma 2.1.2. For the second part, we use (ii) and the fact that the γ_1-st row and γ_2-nd column of $H(\xi)H^*(\xi)$ are $\sum_{m\in\mathbb{Z}} \hat{g}_m(\xi\gamma_1^{-1})\overline{\hat{g}_m(\xi\gamma_2^{-1})} = c\delta_{\gamma_1,\gamma_2}$, to get $A = \frac{c}{\nu_1(G/K_1)}$.

For part (ii), if $\sum_{\gamma_1\in Ann(K_1), m\in\mathbb{Z}} \mid \langle f, g_{\gamma_1,m}\rangle \mid^2 \leq B \parallel f \parallel^2$, then by virtue of Proposition 2.4.1, $H(\xi)H^*(\xi) \leq \nu_1\left(G/K_1\right)B\mathbb{I}$. Let f be an element in $L^2(G, dx)$ with $\hat{f}$ compactly supported and let $m \in \mathbb{Z}$. The series

$$\sum_{\gamma_1\in Ann(K_1)} \hat{g}_m(\xi\gamma_1)\hat{f}^*(\xi\gamma_1) \tag{2.4.89}$$

defines a function in $L^2\left(\widehat{G}/Ann(K_1)\right)$.

Since K_1 is an orthogonal basis of $L^2\left(\widehat{G}/Ann(K_1), \nu_1\left(\widehat{G}/Ann(K_1)\right)^{-1} d\hat{\nu}_1\right)$, it follows that

$$\sum_{\gamma_1\in Ann(K_1)} \hat{g}_m(\xi\gamma_1)\hat{f}^*(\xi\gamma_1) = \sum_{k_1\in K_1} c_{k_1,m}\xi(k_1), \tag{2.4.90}$$

where

$$\begin{aligned}
c_{k_1,m} &= \nu_1\left(G/K_1\right)\int_{\widehat{G}/Ann(K_1)} \sum_{\gamma_1\in Ann(K_1)} \overline{\xi(k_1)}\hat{g}_m(\xi\gamma_1)\hat{f}^*(\xi\gamma_1)d\hat{\nu}_1(\xi) \\
&= \nu_1\left(G/K_1\right)\int_{\widehat{G}} \overline{\xi(k_1)}\hat{g}_m(\xi)\hat{f}^*(\xi)d\xi \\
&= \nu_1\left(G/K_1\right)\int_{\widehat{G}} \mathcal{F}\left(T_{k_1}g_m\right)(\xi)\mathcal{F}\left(f^*\right)(\xi)d\xi \\
&= \nu_1\left(G/K_1\right)\int_{\widehat{G}} g_m(\xi {k_1}^{-1})f^*(\xi)d\xi \\
&= \nu_1\left(G/K_1\right)\langle g_{k_1,m}, f\rangle .
\end{aligned} \tag{2.4.91}$$

Thus,

$$\int_{\widehat{G}/Ann(K_1)} \mid \sum_{\gamma_1\in Ann(K_1)} \hat{g}_m(\xi\gamma_1)\hat{f}^*(\xi\gamma_1) \mid^2 d\hat{\nu}_1(\xi) = \nu_1\left(G/K_1\right)\sum_{k_1\in K_1} \mid \langle g_{k_1,m}, f\rangle \mid^2 . \tag{2.4.92}$$

Let $\xi \in \widehat{G}$ and set $\underline{\hat{f}}(\xi) = \{\underline{f}(\xi\gamma_1^{-1})\}_{\gamma_1 \in K_1}$. We then have

$$\begin{aligned}\int_{\widehat{G}/Ann(K_1)} \| \underline{\hat{f}}(\xi) \|^2 \, d\hat{\nu}_1(\xi) &= \int_{\widehat{G}/Ann(K_1)} \sum_{\gamma_1 \in Ann(K_1)} | \hat{f}(\xi\gamma_1^{-1}) |^2 \, d\hat{\nu}_1(\xi) \\ &= \int_{\widehat{G}} | \hat{f}(\xi) |^2 \, d\xi \\ &= \| f \|^2 , \end{aligned} \tag{2.4.93}$$

while use of (2.4.92) yields

$$\begin{aligned}&\int_{\widehat{G}/Ann(K_1)} \| H^*(\xi)\underline{\hat{f}}(\xi) \|^2 \, d\hat{\nu}_1(\xi) \\ &= \sum_{m\in\mathbb{Z}} \int_{\widehat{G}/Ann(K_1)} | \sum_{\gamma_1 \in Ann(K_1)} \hat{g}_m(\xi\gamma_1)\hat{f}^*(\xi\gamma_1) |^2 \, d\hat{\nu}_1(\xi) \\ &= \nu_1(G/K_1) \sum_{m\in\mathbb{Z}} \sum_{k_1 \in K_1} | \langle g_{k_1,m}, f \rangle |^2 . \end{aligned} \tag{2.4.94}$$

Let $\hat{\rho} \in L^2\left(\widehat{G}/Ann(K_1)\right)$ and let $\underline{\beta} \in \ell(\mathbb{Z})$ with $\beta_k \neq 0$ for only finitely many $k \in \mathbb{Z}$. Let U_{K_1} be a fundamental domain in $\widehat{G}$ associated to $Ann(K_1)$. It is easy to see that $\hat{\rho}$ can also be considered as a function on U_{K_1}.

Assuming that $| Ann(K_1) | = \infty$, let us write $Ann(K_1)$ in the form

$$Ann(K_1) = \{\gamma_{1,k} : k \in \mathbb{Z}\} . \tag{2.4.95}$$

Define a function $\hat{f}$ on $\widehat{G}$ by $\hat{f}(\xi) = \beta_k \hat{\rho}(\xi\gamma_{1,k})$, where k is such that $\xi\gamma_{1,k} \in U_{K_1}$.

We have

$$\underline{\hat{f}}(\xi) = \underline{\beta}\hat{\rho}(\xi) . \tag{2.4.96}$$

Furthermore, from (2.4.93) and (2.4.94) and the first (2.4.80), we have

$$\begin{aligned}
&\int_{\widehat{G}/Ann(K_1)} \| H^*(\xi)\underline{\hat{f}}(\xi) \|^2 \, d\hat{\nu}_1(\xi) \qquad (2.4.97)\\
= &\int_{\widehat{G}/Ann(K_1)} | \hat{\rho}(\xi) |^2 \| H^*(\xi)\underline{\beta} \|^2 \, d\hat{\nu}_1(\xi)\\
= &\ \nu_1 (G/K_1) . \sum_{m\in\mathbb{Z}} \sum_{k_1\in K_1} | \langle g_{k_1,m}, f\rangle |^2\\
\geq &\ \nu_1 (G/K_1) . A \| f \|^2\\
= &\ \nu_1 (G/K_1) . A \int_{\widehat{G}/Ann(K_1)} \| \underline{f}(\xi) \|^2 \, d\hat{\nu}_1(\xi)\\
= &\ \nu_1 (G/K_1) . A \int_{\widehat{G}/Ann(K_1)} \sum_{k\in\mathbb{Z}} | \beta_k . \hat{\rho}(\xi) |^2 \, d\hat{\nu}_1(\xi)\\
= &\ \nu_1 (G/K_1) . A \| \underline{\beta} \|^2 \int_{\widehat{G}/Ann(K_1)} | \hat{\rho}(\xi) |^2 \, d\hat{\nu}_1(\xi).
\end{aligned}$$

Letting $\hat{\rho}$ run over all of $L^2\left(\widehat{G}/Ann(K_1)\right)$, we obtain

$$\| H^*(\xi)\underline{\beta} \|^2 \geq \nu_1 (G/K_1) . A \| \underline{\beta} \|^2 , \qquad (2.4.98)$$

for almost all $\xi \in \widehat{G}/Ann(K_1)$, where the null set involved in (2.4.98) may depend on $\underline{\beta}$. Let V be a countable dense set of $\underline{\beta}$'s in $\ell^2(\mathbb{Z})$ such that $\beta_k \neq 0$ for only finitely many $k \in \mathbb{Z}$ and let $N_1 \subset \widehat{G}/Ann(K_1)$ be a null set such that

$$\| H^*(\xi)\underline{\beta} \|^2 \geq \nu_1 (G/K_1) . A \| \underline{\beta} \|^2; \ \beta \in V, \ \xi \in \hat{G}/Ann(K_1)/N_1. \qquad (2.4.99)$$

Also, let $N_2 \subset \widehat{G}/Ann(K_1)$ be a null set such that

$$\| H^*(\xi)\underline{\beta} \|^2 \leq \nu_1 (G/K_1) . B \| \underline{\beta} \|^2; \ \underline{\beta} \in \ell^2(\mathbb{Z}), \ \xi \in \widehat{G}/Ann(K_1)/N_2. \qquad (2.4.100)$$

Take $\xi \in \left(\widehat{G}/Ann(K_1)\right) - (N_2 \cup N_2)$, $\underline{\beta} \in \ell^2(\mathbb{Z})$ and $\underline{\beta}^{(M)} \in V$ such that $\beta^{(M)} \longrightarrow \underline{\beta}$. Then, from (2.4.99) and (2.4.100), we conclude that

$$\begin{aligned}
\| H^*(\xi)\underline{\beta} \|^2 &= \lim_{M\to\infty} \| H^*(\xi)\underline{\beta}^{(M)} \|^2 \geq \nu_1 (G/K_1) . A \lim_{M\to\infty} \| \underline{\beta}^{(M)} \|^2\\
&= \nu_1 (G/K_1) . A \| \underline{\beta} \|^2 . \qquad (2.4.101)
\end{aligned}$$

This completes the proof of the implication "$\Rightarrow$" of part (ii). To prove the opposite implication, let $f \in L^2(G)$ such that $\hat{f}$ is compactly supported in $\widehat{G}$. Then (2.4.93) and (2.4.94), imply

$$A \| f \|^2 \leq \sum_{\gamma_1\in Ann(K_1), m\in\mathbb{Z}} | \langle f, g_{\gamma_1,m}\rangle |^2 \leq B \| f \|^2 .$$

■

Chapter 3

Walnut Representation of Generalized Gabor Frame Operators

3.1 Introduction

The Walnut representation of the frame operator (see [33] [44]), is often much easier to compute and work with. This representation was done under the assumption that the generating function for the frame has rapid decay. Recently [12], Christensen and Casazza gave the same representation of this operator under a weaker condition: (CC)-condition. The authors presented some results about the convergence properties of this series representation of the frame operator [13]: they show that the weak and norm symmetric convergence of the Walnut series are equivalent; also weak and norm convergence of the Walnut series are equivalent. The analogue result on the unconditionally case is also presented. Using this representation of the frame operator, the authors show that under some condition, this operator can be extented as a bounded linear operator on $L^p(\mathbb{R})$, for $1 \leq p \leq \infty$.

More recently [35], Honnouvo and Ali gave a generalization of the Gabor frame construction for the generalized Weyl-Heisenberg group which is a central extension of the direct product of a locally compact abelian group G with its dual group $\hat{G}$. Under a condition which we can call generalized (CC)-condition, (GCC)-condition for short, we present a generalization of the Walnut representation using this generalized Gabor frame. The representation of the frame operator in the frequency domain is also presented.

3.1.1 Walnut representation of Weyl-Heisenberg frame operators

It's known that the frame operator associated to the Gabor system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, is given by

$$S = \sum_{m,n\in\mathbb{Z}} \mid E_{mb}T_{na}g\rangle\langle E_{mb}T_{na}g \mid . \tag{3.1.1}$$

For a, b fixed, the class of functions g for which the operator (3.1.1) defines a bounded linear operator on $L^2(\mathbb{R})$ is called the class of *preframe functions* and is denoted by PF.

In 1992, it was proved by Walnut [44] that if the generating function g has rapid decay, the operator (3.1.1) can be represented as follow:

$$Sf = b^{-1} \sum_{k\in\mathbb{Z}} \left(T_{k/b}f\right) G_k, \tag{3.1.2}$$

where

$$G_k(x) = \sum_{n\in\mathbb{Z}} g\,(x - na)\,\overline{g\,(x - na - k/b)}. \tag{3.1.3}$$

In this case, the Walnut series (3.1.2) converges rapidly in norm. The Walnut representation of the frame operator can often be much easier to compute and work with. But the rapid decay assumptions limit its use. Recently, Casazza and Christensen [12] gave much weaker assumptions on the window function g which still ensures that the corresponding Weyl-Heisenberg system has a finite upper frame bound.

Let us recall the following important convergence results about the operator defined by the series (3.1.2) [13]:

Proposition 3.1.1 *Let $a, b \in \mathbb{R}$ with $ab \leq 1$ and $g \in L^2(\mathbb{R})$ and assume that*

$$\sum_{k\in\mathbb{Z}} | G_k(x) |^2 \leq B, \quad a.e. x \in \mathbb{R}, \tag{3.1.4}$$

for some $B > 0$.

Then for all bounded, compactly supported function $f \in L^2(\mathbb{R})$, the series

$$Lf = b^{-1} \sum_{k\in\mathbb{Z}} \left(T_{k/b} f\right) G_k, \tag{3.1.5}$$

converges unconditionally in norm in $L^2(\mathbb{R})$. Moreover,

$$\langle Lf, f\rangle = \sum_{m,n\in\mathbb{Z}} | \langle f, E_{mb}T_{na}g\rangle |^2 . \tag{3.1.6}$$

Finally if $g \in PF$, so that the series

$$Sf = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle E_{mb}T_{na}g, \tag{3.1.7}$$

converges unconditionally in $L^2(\mathbb{R})$, we have $Lf = Sf$.

Let us recall some convergence results which will be used in the study of Weyl-Heisenberg frames.

If x_n are elements of some Banach space X, a series $\sum_n x_n$ is said to be unconditionally convergent if for every increasing sequence of natural numbers (k_n) we have that

$$\lim_n \sum_{j=1}^{n} x_{k_j}, \quad \text{(strong limit)}$$

exists.

Let us recall a result which can be found in [38], Proposition 1.c.1

Proposition 3.1.2 *For x_n in a Banach space X, the following are equivalent:*

(1) $\sum_n x_n$ *is unconditionally convergent.*

(2) $\sum_n x_{\sigma(n)}$ *converges for every permutation* σ.

(3) $\sum_n \theta_n x_n$ *converges for every choice of complex* $| \theta_n | \leq 1$. *Moreover, in this case there is a constant K such that for every choice of numbers (a_n) we have*

$$\| \sum_n a_n x_n \| \leq K \sup_n | a_n | . \| \sum_n x_n \| .$$

The following is the celebrated Orlicz-Pettis theorem [22]:

ORLICZ-PETTIS Theorem 3.1.1 *If x_n are elements of a Banach space such that for every increasing sequence of natural numbers (k_n)*

$$weak \lim_n \sum_{j=1}^{n} x_{k_j}$$

exists, then the series $\sum_n x_n$ is unconditionally convergent.

Definition 3.1.1 *A series $\sum_n x_n$ is said to be weakly unconditionally Cauchy (wuC) if given any permutation σ of the natural numbers we have that $\left(\sum_{j=1}^{n} x_{\sigma(j)}\right)$ is a weakly Cauchy sequence.*

Recall the Banach space c_0 :

$$c_0 = \left\{ x = (a_n) : \| x \| =: \sup_n | a_n | < \infty \text{ and } \lim_{n\to\infty} a_n = 0 \right\}. \tag{3.1.8}$$

Let recall the following theorem [22].

Theorem 3.1.2 *The following are equivalent for a series $\sum_n x_n$ in a Banach space:*

(1) $\sum_n x_n$ *is wuC.*

(2) *For every $x^* \in X^*$ we have $\sum_n | x^*(x_n) | < \infty$.*

(3) For every $(a_n) \in c_0$,

$$\sum_n a_n x_n$$

converges.

(4) There is a constant C so that for every finite subset M of the natural numbers we have

$$\| \sum_{n \in M} x_n \| \leq C.$$

3.2 Walnut representation of the generalized Gabor frame operator

Let K_1 and K_2 be uniform lattices in G. For any $k_1 \in K_1$ and $k_2 \in K_2$, from the proof of the lemma 2.2.1, the function $G_{k_2}(x) = \sum_{k_1 \in K_1} g(xk_1^{-1})\overline{g(xk_1^{-1}k_2^{-1})}$ converges absolutely for almost all $x \in G$ and we have

$$\sum_{k_2 \in K_2} | T_{k_2^{-1}} G_{k_2}(x) |^2 = \sum_{k_2 \in K_2} | G_{k_2}(x) |^2, \quad a.e. x \in G. \tag{3.2.9}$$

Proposition 3.2.1 *Let K_1 and K_2 be two uniform lattices of the LCA group G. Let $g \in L^2(G, dx)$ such that*

$$\sum_{k_2 \in K_2} | G_{k_2}(x) |^2 \leq B, \quad a.e. x \in G \tag{3.2.10}$$

for some $B > 0$.

Then for all bounded, compactly supported functions $f \in L^2(G, dx)$, the series

$$Lf = \nu_2 (G/K_2) \sum_{k_2 \in K_2} (T_{k_2} f) G_{k_2}, \tag{3.2.11}$$

converges unconditionally in norm in $L^2(G, dx)$ *and*

$$\langle Lf, f\rangle = \sum_{k_1\in K_1;\gamma_2\in Ann(K_2)} |\langle f, \gamma_2 T_{k_1} g\rangle|^2 . \tag{3.2.12}$$

Finally if $g \in PF$, *so that the series*

$$Sf = \sum_{k_1\in K_1,\gamma_2\in Ann(K_2)} \langle f, \gamma_2 T_{k_1} g\rangle \gamma_2 T_{k_1} g, \tag{3.2.13}$$

converges unconditionally in $L^2(G, dx)$, *we have* $Lf = Sf$.

Proof. Let prove that the series (3.2.11) converges unconditionally for all bounded compactly supported functions $f \in L^2(G, dx)$.

Let U be a fundamental domain of K_2 and for $k_2 \in K_2$, let $U_{k_2} = T_{k_2}(U)$.

We have

(i) $\cup_{k_2\in K_2} U_{k_2} = G$

(ii) $U_{k_2} \cap U_{k_2'} = \emptyset$, for $k_2 \neq k_2'$

Since any bounded compactly supported function f can be written as a finite sum of bounded functions supported on some U_{k_2}, let f be supported in $U_{k_2^0}$ such that $| f(x) | \leq D$ on $U_{k_2^0}$ and let $M \subset K_2$ such that $| M | < \infty$.

The functions $\{(T_{k_2} f)\, G_{k_2}\}_{k_2\in K_2}$ are disjointly supported and we have

$$\begin{aligned} \| \sum_{k_2\in M} (T_{k_2} f)\, G_{k_2} \|^2_{L^2(G)} &= \sum_{k_2\in M} \int_G |(T_{k_2} f)(x)|^2 . |G_{k_2}(x)|^2\, dx \\ &= \sum_{k_2\in M} \int_{U_{k_2^{-1}k_2^0}} |(T_{k_2} f)(x)|^2 . |G_{k_2}(x)|^2\, dx \\ &= \sum_{k_2\in M} \int_{U_{k_2^0}} |f(x)|^2 . |T_{k_2^{-1}} G_{k_2}(x)|^2\, dx \\ &= \int_{U_{k_2^0}} |f(x)|^2 . \sum_{k_2\in M} |T_{k_2^{-1}} G_{k_2}(x)|^2\, dx \\ &\leq D^2 \int_{U_{k_2^0}} \sum_{k_2\in M} |T_{k_2^{-1}} G_{k_2}(x)|^2\, dx. \end{aligned} \tag{3.2.14}$$

So, Lf converges unconditionally by the monotone convergence theorem.

$$\begin{aligned}
\langle Lf, f\rangle &= \left\langle \nu_2\left(G/K_2\right) \sum_{k_2\in K_2} (T_{k_2} f) G_{k_2}, f \right\rangle \\
&= \nu_2\left(G/K_2\right) \sum_{k_2\in K_2} \langle (T_{k_2} f) G_{k_2}, f\rangle \\
&= \nu_2\left(G/K_2\right) \sum_{k_2\in K_2} \int_G \overline{f(x)} G_{k_2}(x) f(x k_2^{-1}) dx \\
&\quad (\text{ using the generalized W H-frame Identity}), \\
&= \sum_{k_1\in K_1} \sum_{\gamma_2\in Ann(K_2)} \mid \langle f, \gamma_2 T_{k_1} g \mid^2 . \qquad (3.2.15)
\end{aligned}$$

To see that $Sf = Lf$, we just take f to be bounded and compactly supported and $\forall\, h \in L^2(G)$, we have

$$\left\langle \nu_2\left(G/K_2\right) \sum_{k_2\in K_2} (T_{k_2} f) G_{k_2}, h \right\rangle = \langle Sf, h\rangle. \qquad (3.2.16)$$

■

3.2.1 Remarks:

By looking carefully at the proof of the first part of this theorem, we do not need the fact that $\frac{\nu_1(G/K_1)}{\nu_2(G/K_2)} \le 1$. So, in the proof of the original theorem the authors use the assumption $ab \le 1$ which they do not really need.

Theorem 3.2.1 *Let K_1 and K_2 be two uniform lattices on LCA group G and Let $g \in L^2(G)$ such that $\sum_{m\in\mathbb{Z}} \mid G_{k_2^{(m)}}(x) \mid \le B$, a.e.*

Then the Walnut series for every $f \in L^2(G)$ converges unconditionally in L^2 norm, where $K_2 = \left\{k_2^{(m)}\right\}_{m\in\mathbb{Z}}$.

Proof. We know that

$$\sum_{m\in\mathbb{Z}} \mid T_{k_2^{(m)-1}} G_{k_2^{(m)}}(x) \mid = \sum_{m\in\mathbb{Z}} \mid G_{k_2^{(m)}}(x) \mid, \quad a.e. \qquad (3.2.17)$$

Let $h \in L^2(G)$, we have

$$\begin{aligned}
\sum_{|m|=l}^{p} \mid \langle h, \left(T_{k_2^{(m)}} f\right) G_{k_2^{(m)}} \rangle \mid &= \sum_{|m|=l}^{p} \mid \int_G \overline{h(x)} \left(T_{k_2^{(m)}} f\right)(x) G_{k_2^{(m)}}(x) dx \mid \qquad (3.2.18) \\
&\leq \sum_{|m|=l}^{p} \int_G \mid \overline{h(x)} \mid . \mid \left(T_{k_2^{(m)}} f\right)(x) \mid . \mid G_{k_2^{(m)}}(x) \mid dx \\
&= \sum_{|m|=l}^{p} \int_G \mid \overline{h(x)} \mid . \sqrt{\mid G_{k_2^{(m)}}(x) \mid} . \mid \left(T_{k_2^{(m)}} f\right)(x) \mid \\
&\times \sqrt{\mid G_{k_2^{(m)}}(x) \mid} dx \\
&\leq \left(\int_G \mid h(x) \mid^2 . \sum_{|m|=l}^{p} \mid G_{k_2^{(m)}}(x) \mid dx \right)^{\frac{1}{2}} \\
&\times \left(\int_G \mid f(x) \mid^2 . \sum_{|m|=l}^{p} \mid \left(T_{k_2^{(m)^{-1}}} G_{k_2^{(m)}} \right)(x) \mid dx \right)^{\frac{1}{2}},
\end{aligned}$$

which goes to zero when $l \to \infty$. So, $\sum_{m\in\mathbb{Z}} \left(T_{k_2^{(m)}} f\right) G_{k_2^{(m)}}$ is weakly unconditionally convergent in $L^2(G)$. ■

Theorem 3.2.2 *Let K_1 and K_2 be two uniform lattices of a LCA group G. If there is a constant $B > 0$ so that*

$$\sum_{k_2\in K_2} \mid G_{k_2}(x) \mid \leq B, \qquad a.e. \qquad (3.2.19)$$

then $g \in PF$.

The condition (3.2.19), will be call the generalized CC-condition.

Here, we give a complete characterization of the preframe functions for some class of group: compact abelian groups.

Proposition 3.2.2 *Let K_1 and K_2 be two uniform lattices of a LCA group G. The preframe functions PF of the generalized Gabor system $\{\gamma_2 T_{k_1} g\}_{\gamma_2\in Ann(K_2), k_1\in K_1}$, is a subset of $L^\infty(G, dx)$:*

$$PF \subseteq L^\infty(G). \qquad (3.2.20)$$

Proof. Let $g \in PF$ and assume that $\{\gamma_2 T_{k_1} g\}_{\gamma_2 \in Ann(K_2), k_1 \in K_1}$ has a finite upper bound B. Let us prove that

$$\sum_{k_2 \in K_2} \mid g(xk_1^{-1}) \mid^2 \leq \frac{B}{\nu_2\left(G/K_2\right)}, \; a.e. \tag{3.2.21}$$

Let U be a fundamental domain of K_2 and for $k_2 \in K_2$, let $U_{k_2} = T_{k_2}\left(U\right)$.

We have

(i) $\cup_{k_2 \in K_2} U_{k_2} = G$

(ii) $U_{k_2} \cap U_{k_2'} = \emptyset$, for $k_2 \neq k_2'$

Let assume that (3.2.21) is violated. Then there exists a measurable set $\Delta \subseteq G$ such that

$$H(x) := \sum_{k_2 \in K_2} \mid g(xk_1^{-1}) \mid^2 > \frac{B}{\nu_2\left(G/K_2\right)}$$

on Δ. We can assume that there exists some $k_2^0 \in K_2$ such that $\Delta \subseteq U_{k_2^0}$.

Let

$$\Delta_0 = \left\{ x \in \Delta : \sum_{k_2 \in K_2} \mid g(xk_1^{-1}) \mid^2 \geq 1 + \frac{B}{\nu_2\left(G/K_2\right)} \right\} \tag{3.2.22}$$

and for $n \in \mathbb{N}^*$,

$$\Delta_n = \left\{ x \in \Delta : \frac{1}{n+1} + \frac{B}{\nu_2\left(G/K_2\right)} \leq H(x) \leq \frac{1}{n} + \frac{B}{\nu_2\left(G/K_2\right)} \right\} \tag{3.2.23}$$

It follows that $\cup_{n \in \mathbb{N}} \Delta_n = \Delta$ and we can conclude that there exists $n_0 \in \mathbb{N}$ such that $\mid \Delta_0 \mid \neq 0$.

Let $f = \chi_{\Delta_{n_0}}$. Let $\gamma_2 \in Ann(K_2)$ and let $\gamma_2^{k_2^0} = \gamma_{2|_{U_{k_2^0}}}$. From the proof of the theorem 2.3 of [41] it is easy to see that $\left\{ \nu_2\left(G/K_2\right)^{-\frac{1}{2}} \gamma_2^{k_2^0} \right\}_{\gamma_2 \in Ann(K_2)}$ is an orthonormal basis of $L^2\left(U_{k_2^0}, dx\right)$ and we have:

$$\sum_{\gamma_2\in Ann(K_2)} |\langle f,\gamma_2 T_{k_1}g\rangle|^2 = \sum_{\gamma_2\in Ann(K_2)} \left|\int_G f(x)\overline{g(xk_1^{-1})}\overline{\gamma_2(x)}dx\right|^2 \quad (3.2.24)$$
$$= \sum_{\gamma_2\in Ann(K_2)} \left|\int_{U_{k_2^0}} f(x)\overline{g(xk_1^{-1})}\overline{\gamma_2^{k_2^0}(x)}dx\right|^2$$
$$= \nu_2(G/K_2)\int_{\Delta_{n_0}} |f(x)|^2 . |g(xk_1^{-1})|^2\, dx.$$

So, we have:

$$\sum_{k_1\in K_1}\sum_{\gamma_2\in Ann(K_2)} |\langle f,\gamma_2 T_{k_1}g\rangle|^2 = \nu_2(G/K_2)\int_{\Delta_{n_0}} |f(x)|^2 . \sum_{k_1\in K_1} |g(xk_1^{-1})|^2\, dx$$
$$\geq \nu_2(G/K_2)\left(\frac{1}{n_0+1}+\frac{B}{\nu_2(G/K_2)}\right) \| f\|^2$$
$$= \left(B+\frac{\nu_2(G/K_2)}{n_0+1}\right)\| f\|^2, \quad (3.2.25)$$

contradiction.

So, if $\{\gamma_2 T_{k_1}g\}_{\gamma_2\in Ann(K_2),k_1\in K_1}$ has a finite upper bound B then,

$$\sum_{k_2\in K_2} |g(xk_1^{-1})|^2 \leq \frac{B}{\nu_2(G/K_2)},$$

which proves that $g\in L^\infty(G)$. So, $PF\subseteq L^\infty(G)$. ■

Corollary 3.2.1 *Let K_1 and K_2 be two uniform lattices of a compact abelian group G. The preframe functions PF of the generalized Gabor system $\{\gamma_2 T_{k_1}g\}_{\gamma_2\in Ann(K_2),k_1\in K_1}$, coincide with the space $L^\infty(G,dx)$:*

$$PF = L^\infty(G,dx). \quad (3.2.26)$$

Proof. Since the group G is compact, K_1 and K_2 are finite. So, let $|K_1|$and $|K_2|$ be there cardinality respectively. Let $g\in L^\infty(G)$, we have:

$$\sum_{k_2\in K_2} \left|\sum_{k_1\in K_1} g(xk_1^{-1})\overline{g(xk_1^{-1}k_2^{-1})}\right| \leq |K_1| . |K_2| . \| g\|_\infty^2 . \quad (3.2.27)$$

So, g satisfiyes the generalized CC-condition and by [35], $g\in PF$, and $L^\infty(G)\subseteq PF$. Using the previous proposition, we have the equality. ■

3.2.2 Representation of the frame operator in the frequency domain

Lemma 3.2.1 *Let G be a LCA group and K be a uniform lattice in G. For all $f \in L^2(G, dx)$, we have*

$$\sum_{\gamma \in Ann(K)} \mid \hat{f}(\xi\gamma) \mid^2 < \infty \quad a.e. \quad \xi \in \widehat{G}. \tag{3.2.28}$$

Proof. Let show that the function $\widehat{F}$ defined on $\widehat{G}$ by $\widehat{F}(\xi) = \sum_{\gamma \in Ann(K)} \mid \hat{f}(\xi\gamma) \mid^2$ is well defined as a function on $\widehat{G}/Ann(K)$.

$$\begin{aligned}
\int_{\widehat{G}/Ann(K)} \widehat{F}(\xi) d\hat{\nu}(\xi) &= \int_{\widehat{G}/Ann(K)} \sum_{\gamma \in Ann(K)} \mid \hat{f}(\xi\gamma) \mid^2 d\hat{\nu}(\xi), \\
&= \int_{\widehat{G}} \mid \hat{f}(\xi) \mid^2 d\xi \\
&= \parallel f \parallel^2_{L^2(G)} < \infty.
\end{aligned} \tag{3.2.29}$$

So, $\sum_{\gamma \in Ann(K)} \mid \hat{f}(\xi\gamma) \mid^2 < \infty \quad a.e. \quad \xi \in \widehat{G}/Ann(K)$. Since $\widehat{F}(\xi\gamma) = \widehat{F}(\xi)$ *a.e.* $\xi \in \widehat{G}$ and $\gamma \in Ann(K)$, we have $\widehat{F}(\xi) < \infty$ *a.e.* $\xi \in \widehat{G}$. ■

Let $\{g_m\}_{m \in \mathbb{Z}}$ be a collection of functions in $L^2(G)$ and let $\{g_{m,k_1}\}_{k_1 \in K_1; m \in \mathbb{Z}} = \{g_m(-.k_1^{-1})\}_{k_1 \in K_1; m \in \mathbb{Z}}$ the corresponding shift-invariant system. The generalized (CC) condition (GCC), is

$$\nu_1 (G/K_1)^{-1} \sum_{\gamma_1 \in Ann(K_1)} \mid \sum_{m \in \mathbb{Z}} \hat{g}_m(\xi) \overline{\hat{g}_m(\xi\gamma_1^{-1})} \mid \leq B \tag{3.2.30}$$

Let now give a representation of the frame operator

$$Sf = \sum_{k_1 \in K_1, m \in \mathbb{Z}} \langle f, g_{k_1,m} \rangle g_{k_1,m}, \quad f \in L^2(G), \tag{3.2.31}$$

in frequency domain.

Theorem 3.2.3 *Let K be a uniform lattice of a LCA group G. Assume that the shift invariant system $g_{k_1,m}, (k_1, m) \in K_1 \times \mathbb{Z}$ has a finite frame upper bound B and let $f \in L^2(G)$. Then we have*

$$\widehat{Sf}(\xi) = \frac{1}{\nu_1 (G/K_1)} \sum_{\gamma_1 \in Ann(K_1)} d_{\gamma_1}(\xi) \hat{f}(\xi\gamma_1^{-1}) \tag{3.2.32}$$

with absolute convergence for a.e. $\xi \in \widehat{G}$. *Here*

$$d_{\gamma_1}(\xi) = \sum_{m\mathbb{Z}} \hat{g}_m(\xi)\hat{g}_m^*(\xi\gamma_1^{-1}), \quad \gamma_1 \in Ann(K_1). \tag{3.2.33}$$

Proof. From [35] proposition 5.2, we have

$$\begin{aligned}
\sum_{\gamma_1 \in Ann(K_1)} \mid d_{\gamma_1}(\xi) \mid^2 &= \sum_{\gamma_1 \in Ann(K_1)} \mid \sum_{m\mathbb{Z}} \hat{g}_m(\xi)\hat{g}_m^*(\xi\gamma_1^{-1}) \mid^2 \\
&\leq \nu_1 (G/K_1) .B \sum_{m\in\mathbb{Z}} \mid \hat{g}_m(\xi) \mid^2 \\
&\leq (\nu_1 (G/K_1) .B)^2, \quad a.e. \, \xi \in \widehat{G}.
\end{aligned} \tag{3.2.34}$$

Since $\sum_{\gamma_1 \in Ann(K_1)} \mid \hat{f}(\xi\gamma_1) \mid^2 < \infty \quad a.e. \xi \in \widehat{G}$, the series (3.2.32) converges absolutely and the right hand side of (3.2.32) is in $L^2_{loc}(G)$. Now, let $h \in L^2(G) \cup L^1(G)$ such that $\hat{h}$ is compactly supported in $\hat{G}$.

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\int_{\widehat{G}} \mid \sum_{\gamma_1 \in Ann(K_1)} d_{\gamma_1}(\xi)\hat{f}(\xi\gamma_1^{-1})\hat{h}(\xi) \mid &= \int_{Supp(\hat{h})} \mid \sum_{\gamma_1 \in Ann(K_1)} d_{\gamma_1}(\xi)\hat{f}(\xi\gamma_1^{-1}) \mid . \mid \hat{h}(\xi) \mid d\xi \\
&\leq \parallel \hat{h} \parallel_\infty \int_{Supp(\hat{h})} \mid \sum_{\gamma_1 \in Ann(K_1)} d_{\gamma_1}(\xi)\hat{f}(\xi\gamma_1^{-1}) \mid d\xi,
\end{aligned} \tag{3.2.35}$$

which is finite because the right hand side of (3.2.32) is in $L^2_{loc}(G)$.

Using $g_m = h_m$ in Lemma 5.1 of [35], we have

$$\begin{aligned}
\langle S_G^{K_1} f, h\rangle &= p(f,h)(1_G) = \sum_{\gamma_1 \in Ann(K_1)} C_{\gamma_1} \\
&= \nu_1 (G/K_1)^{-1} \sum_{\gamma_1 \in Ann(K_1)} \int_{\widehat{G}} \hat{f}(\xi)\hat{h}^*(\xi\gamma_1) \sum_{m\in\mathbb{Z}} \hat{g}_m(\xi)\hat{g}_m^*(\xi\gamma_1)d\xi \\
&= \nu_1 (G/K_1)^{-1} \sum_{\gamma_1 \in Ann(K_1)} \int_{\widehat{G}} d_{\gamma_1}(\xi)\hat{f}(\xi\gamma_1^{-1})\hat{h}(\xi)d\xi.
\end{aligned} \tag{3.2.36}$$

Since the set of such h is dense in $L^2(G)$, we have,

$$\widehat{Sf}(\xi) = \frac{1}{\nu_1 (G/K_1)} \sum_{\gamma_1 \in Ann(K_1)} d_{\gamma_1}(\xi) \hat{f}(\xi\gamma_1^{-1}). \tag{3.2.37}$$

■

Theorem 3.2.4 *Let $\{g_{m,k_1}\}_{k_1 \in K_1; m \in \mathbb{Z}}$ be a shift-invariant system with finite upper frame bound and frame operator S. If the system satisfies the (GCC) condition, then $\{Sg_m(-.k_1^{-1})\}_{k_1 \in K_1; m \in \mathbb{Z}}$ satisfies also the (GCC)condition.*

Proof. Let B be the upper frame bound of our system. We have for all $f \in L^2(G)$ and $a.e.\ \xi \in \hat{G}$,

$$\widehat{Sf}(\xi) = \sum_{\gamma_1 \in Ann(K_1)} \widehat{S}_{1_{\hat{G}},\gamma_1}(\xi) \hat{f}(\xi\gamma_1^{-1}), \tag{3.2.38}$$

where

$$\widehat{S}_{1_{\hat{G}},\gamma_1}(\xi) = \nu_2 (G/K_2)^{-1} \sum_{m \in \mathbb{Z}} \hat{g}_m(\xi) \overline{\hat{g}_m(\xi\gamma_1^{-1})}. \tag{3.2.39}$$

So, we have

$$\left(\widehat{Sf}(\xi\gamma_1^{'-1})\right)_{\gamma_1^{'-1} \in Ann(K_1)} = \widehat{S}(\xi) \left(\hat{f}(\xi\gamma_1^{-1})\right) \tag{3.2.40}$$

where

$$\widehat{S}(\xi) = \left(\nu_2 (G/K_2)^{-1} \sum_{m \in \mathbb{Z}} \hat{g}_m(\xi\gamma_1^{-1}) \overline{\hat{g}_m(\xi\gamma_1^{'-1})}\right)_{\gamma_1^{'},\gamma_1 \in Ann(K_1)}. \tag{3.2.41}$$

So,

$$\widehat{S}(\xi) = \left(\widehat{S}_{\gamma_1^{'},\gamma_1}(\xi)\right)_{\gamma_1^{'},\gamma_1 \in Ann(K_1)}. \tag{3.2.42}$$

Since T_{k_1} commute with S, the frame operator of the system $\{Sg_m(-.k_1^{-1})\}_{k_1 \in K_1; m \in \mathbb{Z}}$ is S^3.

So, we need to show that there exists $C > 0$ such that

$$\sum_{\gamma_1 \in Ann(K_1)} \mid \widehat{S}_{1_{\hat{G}},\gamma_1}(\xi) \mid^3 \leq C. \tag{3.2.43}$$

From (3.2.39), by replacing ξ by $\xi\gamma_1^{'-1}$ we have,

$$\sum_{\gamma_1 \in Ann(K_1)} | \widehat{S}_{\gamma_1', \gamma_1}(\xi) |^3 \leq B, \tag{3.2.44}$$

which implies that

$$\begin{aligned} \sum_{\gamma_1} | \widehat{S}^2_{\gamma', \gamma_1}(\xi) | &= \sum_{\gamma_1} | \sum_{\alpha} \widehat{S}_{\gamma', \alpha}(\xi) \widehat{S}_{\alpha, \gamma_1}(\xi) | \\ &\leq \sum_{\alpha} | \widehat{S}_{\gamma_1', \alpha}(\xi) | \cdot \sum_{\gamma_1} | \widehat{S}_{\alpha, \gamma_1}(\xi) | \leq B^2. \end{aligned} \tag{3.2.45}$$

From that, we arrived at

$$\begin{aligned} \sum_{\gamma_1} | \widehat{S}^3_{\gamma', \gamma_1}(\xi) | &= \sum_{\gamma_1} | \sum_{\alpha} \widehat{S}^2_{\gamma', \alpha}(\xi) \widehat{S}_{\alpha, \gamma_1}(\xi) | \\ &\leq \sum_{\alpha} | \widehat{S}^2_{\gamma_1', \alpha}(\xi) | \cdot \sum_{\gamma_1} | \widehat{S}_{\alpha, \gamma_1}(\xi) | \leq B^2 B = B^3. \end{aligned} \tag{3.2.46}$$

■

3.3 Extending the frame operator

In this section, we present the extension of a generalized frame operator as a bounded linear operator on $L^p(G)$.

Theorem 3.3.1 *Let G be a LCA group and K_1 and K_2 be two uniform lattices in G. Let $g \in L^2(G)$ such that*

$$\sum_{m \in \mathbb{Z}} | G_{k_2^{(m)}}(x) | \leq B, \quad a.e. \tag{3.3.47}$$

Then, the operator

$$Sf = \sum_{k_1 \in K_1, \gamma_2 \in Ann(K_2)} \langle f, \gamma_2 T_{k_1} g \rangle \gamma_2 T_{k_1} g \tag{3.3.48}$$

extends to a bounded linear operator from $L^p(G)$ to $L^p(G)$ for every $1 \leq p \leq \infty$.

Proof.

(i) Let f be a bounded and compactly supported. The using the Walnut representation of the operator S, we have

$$\begin{aligned}
\| Sf \|_{L^1(G)} &= \| Lf \|_{L^1(G)} = \nu_2 (G/K_2) \int_G | \sum_{k_2 \in K_2} (T_{k_2 f}) (x) G_{k_2}(x) | \, dx \\
&\leq \nu_2 (G/K_2) \sum_{k_2 \in K_2} \int_G | (T_{k_2 f}) (x) | . | G_{k_2}(x) | \, dx \\
&= \nu_2 (G/K_2) \sum_{k_2 \in K_2} \int_G | f(x) | . | T_{k_2^{-1}} G_{k_2}(x) | \, dx \\
&\leq \nu_2 (G/K_2) . B \int_G | f(x) | \, dx \\
&\leq \nu_2 (G/K_2) . B . \| f \|_{L^1(G)} . \qquad (3.3.49)
\end{aligned}$$

So, S is a bounded linear operator from a dense subset of $L^1(G)$ into $L^1(G)$, and it can be extended uniquely to a bounded linear operator on $L^1(G)$.

(ii) Let $f \in L^\infty$ with compactly supported, we have

$$\begin{aligned}
\| Sf \|_\infty &= \| Lf \|_\infty = \nu_2 (G/K_2) . \| \sum_{k_2 \in K_2} (T_{k_2 f}) G_{k_2} \|_\infty \\
&= \nu_2 (G/K_2) \, ess \sup | \sum_{k_2 \in K_2} (T_{k_2 f}) G_{k_2} | \\
&\leq \nu_2 (G/K_2) \, ess \sup \sum_{k_2 \in K_2} | (T_{k_2 f}) | . | G_{k_2} | \\
&\leq \nu_2 (G/K_2) . \| f \|_\infty \, ess \sup \sum_{k_2 \in K_2} | G_{k_2} | \\
&\leq \nu_2 (G/K_2) . \| f \|_\infty . B \qquad (3.3.50)
\end{aligned}$$

So, S can be extended uniquely to a bounded linear operator on L^∞.

Using (i), (ii) and the Riesz-Thorin interpolation theorem (Theorem 7.1.1), S is bounded linear operator from L^p to L^p for all $1 \leq p \leq \infty$.

■

Chapter 4

Time-frequency-like Transforms of Square-integrable Functions on Some Non-Euclidean Manifolds

4.1 Preliminaries

This chapter is devoted to a time-frequency analysis on some non-Euclidean manifolds such as : sphere, one and two sheeted hyperboloid, ellipsoid, paraboloid, etc. Since it is not easy to find an underlying group structure for the translation and modulation on these manifolds, we construct a time-frequency transform on an infinite cylinder and map it homeomorphically to these manifolds.

4.2 Time-frequency-like transforms on some non-Euclidean manifolds

In reality this is done by mapping homeomorphically these non-Euclidean manifolds to a simple one (infinite cylinder).

We take $\boldsymbol{\ell}$ to be a smooth curve in $\mathbb{R}^2$, which is parametrized as

$$z \longmapsto \begin{pmatrix} u(z) \\ v(z) \end{pmatrix}, \quad z \in \mathbb{R},$$

where u and v are two smooth functions. We assume that this map is a homeomorphism between $\mathbb{R}$ and $\boldsymbol{\ell}$. The map

$$V : \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \longmapsto \begin{pmatrix} u(z)\cos\theta \\ u(z)\sin\theta \\ v(z) \end{pmatrix} \tag{4.2.1}$$

transforms the cylinder homeomorphically to a surface of revolution $\mathfrak{S}$ about the z-axis.

The surface element $d\sigma_3$ on the cylinder transforms in to

$$d\sigma_{\mathfrak{S}}(\theta, z) = w(z)\ d\theta\ dz, \tag{4.2.2}$$

on this surface, where $w(z) = |u(z)|\ [u'(z)^2 + v'(z)^2]^{\frac{1}{2}}$. The mapping V then induces a unitary map $\tilde{V} : L^2(\mathfrak{C}, dzd\theta) \to L^2(\mathfrak{S}, d\sigma_{\mathfrak{S}})$, defined by

$$\left(\tilde{V}f\right)\left(\begin{pmatrix} u(z)\cos\theta \\ u(z)\sin\theta \\ v(z) \end{pmatrix}\right) = (w(z))^{-\frac{1}{2}} f\left(\begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix}\right). \tag{4.2.3}$$

For a special case of the first chapter, let $G = \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ and $H_G = (\mathbb{R} \times \mathbb{R}/\mathbb{Z}) \times \left(\hat{\mathbb{R}} \times \mathbb{Z}\right) \times \mathbb{T}$, the corresponding Weyl Heisenberg group. Let U be the unitary irreducible representation of H_G on the Hilbert space $L^2\left(\mathbb{R} \times \mathbb{R}/\mathbb{Z}, dzd\theta\right)$, defined by

$$\left(U\left((x,\varphi);(w,k);\eta\right)g\right)(z,\theta) = \eta e^{2\pi i(w.z+k.\theta)} g\left(z-x, \theta-\varphi\right). \tag{4.2.4}$$

it's known that this representation is square integrable.

Using the following unitary map $\tilde{V}$ between the Hilbert spaces $L^2(C, dzd\theta)$ and $L^2(\mathfrak{S}, d\sigma_{\mathfrak{S}})$, we obtain a representation $\tilde{U}$ on $L^2(\mathfrak{S}, d\sigma_{\mathfrak{S}})$, defined by $\tilde{U} = \tilde{V}U\tilde{V}^*$; which underline a time frequency transform on $\mathfrak{S}$.

The representation $\tilde{U}$ is defined by

$$\left(\tilde{U}\left((x,\varphi);(w,k);\eta\right)f\right)\begin{pmatrix} u(z)\cos\theta \\ u(z)\sin\theta \\ v(z) \end{pmatrix} =$$

$$= \eta\left[\frac{w\left(z-x\right)}{w(z)}\right]^{\frac{1}{2}} e^{2\pi i(w.z+k.\theta)} f\begin{pmatrix} u\left(z-x\right)\cos(\theta-\varphi) \\ u\left(z-x\right)\sin(\theta-\varphi) \\ v\left(z-x\right) \end{pmatrix}. \tag{4.2.5}$$

In the next section, we will give an explicit expression of this representation depending on the manifold in question.

4.3 Examples of wavelet-like transform on some manifolds

we construct a time frequency transform on some known non-Euclidean manifolds by giving some explicit form of $u(z)$ and $v(z)$ in the parametrization of $\mathfrak{S}$. Thereby we build a time frequency transform on those manifolds, namely: Sphere, Ellipsoid, one and two sheeted hyperboloid, paraboloid, Plane, and so on.

4.3.1 Time-frequency-like transform on the sphere, S^2

This section is devoted to a construction of transform on $L^2(\mathcal{S}^2, d\mu)$, where S^2 is two-dimensional sphere in $\mathbb{R}^3$.

Let $\mathcal{S}^2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x^2 + y^2 + z^2 = 1 \right\}$, be the two dimensional unit sphere.

Let consider the map $V : \mathfrak{C} \longrightarrow \mathcal{S}^2$, defined by

$$V : \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \cosh^{-1}(z)\cos\theta \\ \cosh^{-1}(z)\sin\theta \\ \tanh(z) \end{pmatrix}, \tag{4.3.6}$$

This map induces a unitary map,

$$\tilde{V} : L^2(\mathfrak{C}, dzd\theta) \longrightarrow L^2\left(\mathcal{S}^2, \cosh^{-2}(z)dzd\theta\right) :$$

defined by

$$\left(\tilde{V}f\right) \begin{pmatrix} \cosh^{-1}(z)\cos\theta \\ \cosh^{-1}(z)\sin\theta \\ \tanh(z) \end{pmatrix} = \cosh(z).f \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \tag{4.3.7}$$

The representation $\tilde{U}$, of H_G on $L^2\left(\mathcal{S}^2, \cosh^{-2}(z)dzd\theta\right)$, is defined by

$$\left(\tilde{U}\left((x,\varphi);(w,k);\eta\right)f\right)\begin{pmatrix}\cosh^{-1}(z)\cos\theta\\ \cosh^{-1}(z)\sin\theta\\ \tanh(z)\end{pmatrix}=$$

$$=\eta\left[\frac{\cosh(z)}{\cosh(z-x)}\right]e^{2\pi i(w.z+k.\theta)}f\begin{pmatrix}\cosh^{-1}(z-x)\cos(\theta-\varphi)\\ \cosh^{-1}(z-x)\sin(\theta-\varphi)\\ \tanh(z-x)\end{pmatrix} \quad (4.3.8)$$

Gabor systems

Let now introduce the Gabor system on $L^2\left(\mathcal{S}^2,\cosh^{-2}(z)dzd\theta\right)$. So, let $K_1 = a\mathbb{Z}\times\{0,\frac{1}{L},\frac{2}{L},\frac{3}{L},...,\frac{L-1}{L}\}$ and $K_2=\frac{1}{b}\mathbb{Z}\times\{0,\frac{1}{M},\frac{2}{M},\frac{3}{M},...,\frac{M-1}{M}\}$ be two uniform lattices in $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$. Let $g\in L^2\left(\mathcal{S}^2,\cosh^{-2}(z)dzd\theta\right)$. The corresponding Gabor system is

$$\mathcal{S}^2_{g;L,M;a,b}= \quad (4.3.9)$$

$$\left\{\left[\frac{\cosh(z)}{\cosh(z-na)}\right]e^{2\pi i(mb.z+M.k.\theta)}g\begin{pmatrix}\cosh^{-1}(z-na)\cos(\theta-\frac{2\pi l}{L})\\ \cosh^{-1}(z-na)\sin(\theta-\frac{2\pi l}{L})\\ \tanh(z-na)\end{pmatrix}\right\},$$

$$(m,n,k)\in\mathbb{Z}^3,\ l=0,1,...,L-1. \quad (4.3.10)$$

Using theorem 2.3.2, we have the following:

Theorem 4.3.1 *Let $a,b>0$ and $(M,L)\in\mathbb{N}^*\times\mathbb{N}^*$. Let $K_1=a\mathbb{Z}\times\{0,\frac{1}{L},\frac{2}{L},\frac{3}{L},...,\frac{L-1}{L}\}$ and $K_2=\frac{1}{b}\mathbb{Z}\times\{0,\frac{1}{M},\frac{2}{M},\frac{3}{M},...,\frac{M-1}{M}\}$ be two uniform lattices in $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$. Let $g\in L^2\left(\mathcal{S}^2,\cosh^{-2}(z)dzd\theta\right)$. The following hold:*

(i) If $\frac{ab}{ML}>1$, *then*

$\mathcal{S}^2_{g;L,M;a,b}$ *is a not a frame for* $L^2\left(\mathcal{S}^2,\cosh^{-2}(z)dzd\theta\right)$.

(ii) If $\mathcal{S}^2_{g;L,M;a,b}$ *is a frame for* $L^2\left(\mathcal{S}^2, \cosh^{-2}(z)dzd\theta\right)$*, then* $ab = ML \Leftrightarrow \mathcal{S}^2_{g;L,M;a,b}$ *is a Riesz basis.*

4.3.2 Time-frequency-like transforms on the ellipsoid

Let $\alpha, \gamma > 0$, and let $\mathcal{E}_{\alpha,\gamma} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \frac{x^2+y^2}{\alpha^2} + \frac{z^2}{\gamma^2} = 1 \right\}$, be a two-dimensional ellipsoid of revolution.

Let consider the map $V : \mathfrak{C} \longrightarrow \mathcal{E}_{\alpha,\gamma}$, defined by

$$V : \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \alpha\cosh^{-1}(z)\cos\theta \\ \alpha\cosh^{-1}(z)\sin\theta \\ \gamma\tanh(z) \end{pmatrix}, \tag{4.3.11}$$

This map induces a unitary map

$$\tilde{V} : L^2\left(\mathfrak{C}, dzd\theta\right) \longrightarrow L^2\left(\mathcal{E}_{\alpha,\gamma},\ \rho_{\mathcal{E}_{\alpha,\gamma}}(z)dzd\theta\right),$$

where

$$\rho_{\mathcal{E}_{\alpha,\gamma}}(z) = \frac{\alpha}{\cosh^3(z)}\sqrt{\alpha^2\sinh^2(z) + \gamma^2}, \tag{4.3.12}$$

defined by

$$\left(\tilde{V}f\right)\begin{pmatrix} \alpha\cosh^{-1}(z)\cos\theta \\ \alpha\cosh^{-1}(z)\sin\theta \\ \gamma\tanh(z) \end{pmatrix} = \left[\rho_{\mathcal{E}_{\alpha,\gamma}}(z)\right]^{-\frac{1}{2}}.f\begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix}. \tag{4.3.13}$$

The representation $\tilde{U}$, of H_G on $L^2\left(\mathcal{E}_{\alpha,\gamma},\ \rho_{\mathcal{E}_{\alpha,\gamma}}(z)dzd\theta\right)$, is defined by

$$\left(\tilde{U}\left((x,\varphi);(w,k);\eta\right)f\right)\begin{pmatrix}\alpha\cosh^{-1}(z)\cos\theta\\ \alpha\cosh^{-1}(z)\sin\theta\\ \gamma\tanh(z)\end{pmatrix}=\eta\left[\frac{\cosh(z)}{\cosh(z-x)}\right]^{\frac{3}{2}}\times$$

$$\left[\frac{\alpha^2\sinh^2(z-x)+\gamma^2}{\alpha^2\sinh^2(z)+\gamma^2}\right]^{\frac{1}{4}}e^{2\pi i(w.z+k.\theta)}f\begin{pmatrix}\alpha\cosh^{-1}(z-x)\cos(\theta-\varphi)\\ \alpha\cosh^{-1}(z-x)\sin(\theta-\varphi)\\ \gamma\tanh(z-x)\end{pmatrix}\quad(4.3.14)$$

Gabor systems

Let now introduce the Gabor system on $L^2\left(\mathcal{E}_{\alpha,\gamma},\ \rho_{\mathcal{E}_{\alpha,\gamma}}(z)dzd\theta\right)$. So, let $K_1=a\mathbb{Z}\times\{0,\frac{1}{L},\frac{2}{L},\frac{3}{L},...,\frac{L-1}{L}\}$ and $K_2=\frac{1}{b}\mathbb{Z}\times\{0,\frac{1}{M},\frac{2}{M},\frac{3}{M},...,\frac{M-1}{M}\}$ be two uniform lattices in $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$. Let $g\in L^2\left(\mathcal{E}_{\alpha,\gamma},\ \rho_{\mathcal{E}_{\alpha,\gamma}}(z)dzd\theta\right)$. The corresponding Gabor system is

$$\mathcal{E}_{\alpha,\gamma}^{g;N,M;a,b}=\quad(4.3.15)$$

$$\left\{\Upsilon_{(\alpha,\gamma)}^{(g;N,M;a,b)}(z).e^{2\pi i(mb.z+M.k.\theta)}g\begin{pmatrix}\alpha\cosh^{-1}(z-na)\cos(\theta-\frac{2\pi l}{L})\\ \alpha\cosh^{-1}(z-na)\sin(\theta-\frac{2\pi l}{L})\\ \gamma\tanh(z-na)\end{pmatrix}\right\},$$

$$(m,n,k)\in\mathbb{Z}^3,\ l=0,1,...,L-1.\quad(4.3.16)$$

where

$$\Upsilon_{(\alpha,\gamma)}^{(g;N,M;a,b)}(z)=\left[\frac{\cosh(z)}{\cosh(z-na)}\right]^{\frac{3}{2}}\left[\frac{\alpha^2\sinh^2(z-na)+\gamma^2}{\alpha^2\sinh^2(z)+\gamma^2}\right]^{\frac{1}{4}}.\quad(4.3.17)$$

Using theorem 2.3.2, we have the following:

Theorem 4.3.2 *Let $a,b>0$ and $(M,L)\in\mathbb{N}^*\times\mathbb{N}^*$. Let $K_1=a\mathbb{Z}\times\{0,\frac{1}{L},\frac{2}{L},\frac{3}{L},...,\frac{L-1}{L}\}$ and $K_2=\frac{1}{b}\mathbb{Z}\times\{0,\frac{1}{M},\frac{2}{M},\frac{3}{M},...,\frac{M-1}{M}\}$ be two uniform lattices in $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$. Let $g\in L^2\left(\mathcal{E}_{\alpha,\gamma},\ \rho_{\mathcal{E}_{\alpha,\gamma}}(z)dzd\theta\right)$. The following hold:*

(i) If $\frac{ab}{ML} > 1$, *then*

$\mathcal{E}_{\alpha,\gamma}^{g;L,M;a,b}$ *is a not a frame for* $L^2\left(\mathcal{E}_{\alpha,\gamma},\ \rho_{\mathcal{E}_{\alpha,\gamma}}(z)dzd\theta\right)$.

(ii) If $\mathcal{E}_{\alpha,\gamma}^{g;L,M;a,b}$ *is a frame for* $L^2\left(\mathcal{E}_{\alpha,\gamma},\ \rho_{\mathcal{E}_{\alpha,\gamma}}(z)dzd\theta\right)$, *then*

$ab = ML \Leftrightarrow \mathcal{E}_{\alpha,\gamma}^{g;L,M;a,b}$ *is a Riesz basis.*

4.3.3 Time-frequency-like transforms on one-sheeted hyperboloid

Let $\mathcal{H} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x^2 + y^2 - z^2 = 1 \right\}$, be the two dimensional one sheeted hyperboloid.

Let consider the map $V : \mathfrak{C} \longrightarrow \mathcal{H}$, defined by

$$V : \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \cosh(z)\cos\theta \\ \cosh(z)\sin\theta \\ \sinh(z) \end{pmatrix}. \tag{4.3.18}$$

This map induces a unitary map

$$\tilde{V} : L^2\left(\mathfrak{C}, dzd\theta\right) \longrightarrow L^2\left(\mathcal{H}, \cosh(z)\cosh^{\frac{1}{2}}(2z)dzd\theta\right) :$$

defined by

$$\left(\tilde{V}f\right)\begin{pmatrix} \cosh(z)\cos\theta \\ \cosh(z)\sin\theta \\ \sinh(z) \end{pmatrix} = \cosh(z)^{-\frac{1}{2}}\cosh^{-\frac{1}{4}}(2z).f\begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix}. \tag{4.3.19}$$

The representation $\tilde{U}$, of H_G on $L^2\left(\mathcal{H}, \cosh(z)\cosh^{\frac{1}{2}}(2z)dzd\theta\right)$, is defined by defined by

$$\left(\tilde{U}\left((x,\varphi);(w,k);\eta\right)f\right)\begin{pmatrix}\cosh(z)\cos\theta\\ \cosh(z)\sin\theta\\ \sinh(z)\end{pmatrix} = \tag{4.3.20}$$

$$= \eta\left[\frac{\cosh(z-x)}{\cosh(z)}\right]^{\frac{1}{2}}\left[\frac{\cosh(2(z-x))}{\cosh(2z)}\right]^{\frac{1}{4}} e^{2\pi i(w.z+k.\theta)} f\begin{pmatrix}\cosh(z-x)\cos(\theta-\varphi)\\ \cosh(z-x)\sin(\theta-\varphi)\\ \sinh(z-x)\end{pmatrix}.$$

Gabor systems

Let now introduce the Gabor system on $L^2\left(\mathcal{H}, \cosh(z)\cosh^{\frac{1}{2}}(2z)dzd\theta\right)$. So, let $K_1 = a\mathbb{Z}\times\{0, \frac{1}{L}, \frac{2}{L}, \frac{3}{L}, ..., \frac{L-1}{L}\}$ and $K_2 = \frac{1}{b}\mathbb{Z}\times\{0, \frac{1}{M}, \frac{2}{M}, \frac{3}{M}, ..., \frac{M-1}{M}\}$ be two uniform lattices in $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$. Let $g \in L^2\left(\mathcal{H}, \cosh(z)\cosh^{\frac{1}{2}}(2z)dzd\theta\right)$. The corresponding Gabor system is

$$\mathcal{H}_{g;N,M;a,b} = \left\{ A_{(g;N,M;a,b)}(z).e^{2\pi i(mb.z+M.k.\theta)} g\begin{pmatrix}\cosh(z-na)\cos(\theta-\frac{2\pi l}{L})\\ \cosh(z-na)\sin(\theta-\frac{2\pi l}{L})\\ \sinh(z-na)\end{pmatrix}\right\},$$
$$(m,n,k)\in\mathbb{Z}^3,\ l=0,1,...,L-1. \tag{4.3.21}$$

where

$$A_{(g;N,M;a,b)}(z) = \left[\frac{\cosh(z-na)}{\cosh(z)}\right]^{\frac{1}{2}}\left[\frac{\cosh(2(z-na))}{\cosh(2z)}\right]^{\frac{1}{4}}. \tag{4.3.22}$$

Using theorem 2.3.2, we have the following:

Theorem 4.3.3 *Let $a, b > 0$ and $(M, L) \in \mathbb{N}^* \times \mathbb{N}^*$. Let $K_1 = a\mathbb{Z} \times \{0, \frac{1}{L}, \frac{2}{L}, \frac{3}{L}, ..., \frac{L-1}{L}\}$ and $K_2 = \frac{1}{b}\mathbb{Z} \times \{0, \frac{1}{N}, \frac{2}{M}, \frac{3}{M}, ..., \frac{M-1}{M}\}$ be two uniform lattices in $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$. Let $g \in L^2\left(\mathcal{H}, \cosh(z)\cosh^{\frac{1}{2}}(2z)dzd\theta\right)$. The following hold:*

(i) If $\frac{ab}{ML} > 1$, *then*
$\mathcal{H}_{g;L,M;a,b}$ *is a not a frame for* $L^2\left(\mathcal{H}, \cosh(z)\cosh^{\frac{1}{2}}(2z)dzd\theta\right)$.

(ii) If $\mathcal{H}_{g;L,M;a,b}$ *is a frame for* $L^2\left(\mathcal{H}, \cosh(z)\cosh^{\frac{1}{2}}(2z)dzd\theta\right)$, *then*
$ab = ML \Leftrightarrow \mathcal{H}_{g;L,M;a,b}$ *is a Riesz basis.*

4.3.4 Time-frequency-like transforms on the plane without the origin

Let $\mathbb{R}^2_*$ be the two-dimensional flat space without the origin.

Let consider the map $V : \mathfrak{C} \longrightarrow \mathbb{R}^2_*$, defined by

$$V : \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \longmapsto \begin{pmatrix} e^z\cos\theta \\ e^z\sin\theta \end{pmatrix}. \tag{4.3.23}$$

This map induces a unitary map

$$\tilde{V} : L^2(\mathfrak{C}, dzd\theta) \longrightarrow L^2\left(\mathbb{R}^2_*, e^{2z}dzd\theta\right),$$

defined by

$$\left(\tilde{V}f\right)\begin{pmatrix} e^z\cos\theta \\ e^z\sin\theta \end{pmatrix} = e^{-z}.f\begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix}. \tag{4.3.24}$$

The representation $\tilde{U}$, of H_G on $L^2\left(\mathbb{R}^2_*, e^{2z}dzd\theta\right)$, is defined by

$$\left(\tilde{U}\left((x,\varphi);(w,k);\eta\right)f\right)\begin{pmatrix} e^{z}\cos\theta \\ e^{z}\sin\theta \end{pmatrix} =$$

$$= \eta \left[\frac{e^{(z-x)}}{e^{z}}\right]^{\frac{1}{2}} e^{2\pi i(w.z+k.\theta)} f \begin{pmatrix} e^{z-x}\cos(\theta-\varphi) \\ e^{z-x}\sin(\theta-\varphi \end{pmatrix}. \quad (4.3.25)$$

Gabor systems

Let now introduce the Gabor system on $L^2\left(\mathbb{R}^2_*, e^{2z}dzd\theta\right)$. So, let $K_1 = a\mathbb{Z}\times\{0, \frac{1}{L}, \frac{2}{L}, \frac{3}{L}, ..., \frac{L-1}{L}\}$ and $K_2 = \frac{1}{b}\mathbb{Z}\times\{0, \frac{1}{M}, \frac{2}{M}, \frac{3}{M}, ..., \frac{M-1}{M}\}$ be two uniform lattices in $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$. Let $g \in L^2\left(\mathbb{R}^2_*, e^{2z}dzd\theta\right)$. The corresponding Gabor system is

$$\mathcal{P}^*_{g;L,M;a,b} = \left\{ \left[\frac{e^{(z-na)}}{e^{z}}\right]^{\frac{1}{2}} e^{2\pi i(mb.z+M.k.\theta)} g \begin{pmatrix} e^{z-na}\cos(\theta-\frac{2\pi l}{L}) \\ e^{z-na}\sin(\theta-\frac{2\pi l}{L} \end{pmatrix} \right\},$$
$$(m,n,k)\in\mathbb{Z}^3,\ l = 0,1,...,L-1. \quad (4.3.26)$$

Using theorem 2.3.2, we have the following:

Theorem 4.3.4 *Let $a,b>0$ and $(M,L)\in\mathbb{N}^*\times\mathbb{N}^*$. Let $K_1 = a\mathbb{Z}\times\{0, \frac{1}{L}, \frac{2}{L}, \frac{3}{L}, ..., \frac{L-1}{L}\}$ and $K_2 = \frac{1}{b}\mathbb{Z}\times\{0, \frac{1}{M}, \frac{2}{M}, \frac{3}{M}, ..., \frac{M-1}{M}\}$ be two uniform lattices in $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$. Let $g \in L^2\left(\mathbb{R}^2_*, e^{2z}dzd\theta\right)$. The following hold:*

(i) If $\frac{ab}{ML} > 1$, *then*
$\mathcal{P}^*_{g;L,M;a,b}$ *is a not a frame for* $L^2\left(\mathbb{R}^2_*, e^{2z}dzd\theta\right)$.

(ii) If $\mathcal{P}^*_{g;L,M;a,b}$ *is a frame for* $L^2\left(\mathbb{R}^2_*, e^{2z}dzd\theta\right)$, *then*
$ab = ML \Leftrightarrow \mathcal{P}^*_{g;L,M;a,b}$ *is a Riesz basis.*

4.3.5 Time-frequency-like transforms on the paraboloid without the origin

Let $\mathcal{P}_* = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x^2 + y^2 - z = 0 \right\}$, be the paraboloid without the origin

Let consider the map $V : \mathfrak{C} \longrightarrow \mathcal{P}_*$, defined by

$$V : \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \longmapsto \begin{pmatrix} e^z \cos\theta \\ e^z \sin\theta \\ e^{2z} \end{pmatrix}. \tag{4.3.27}$$

This map induces a unitary map

$$\tilde{V} : L^2(\mathfrak{C}, dzd\theta) \longrightarrow L^2\left(\mathcal{P}_*, e^{2z}\left(1 + 4e^{2z}\right)^{\frac{1}{2}} dzd\theta\right),$$

defined by

$$\left(\tilde{V}f\right) \begin{pmatrix} \cosh(z)\cos\theta \\ \cosh(z)\sin\theta \\ \sinh(z) \end{pmatrix} = e^{-z}\left(1 + 4e^{2z}\right)^{-\frac{1}{4}} .f \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix}. \tag{4.3.28}$$

The representation $\tilde{U}$, of H_G on $L^2\left(\mathcal{P}_*, e^{2z}\left(1 + 4e^{2z}\right)^{\frac{1}{2}} dzd\theta\right)$, is defined by

$$\left(\tilde{U}\left((x,\varphi);(w,k);\eta\right)f\right)\begin{pmatrix} e^{z}\cos\theta \\ e^{z}\sin\theta \\ e^{2z}\end{pmatrix} =$$

$$= \eta\left[\frac{e^{z-x}}{e^{z}}\right]\left[\frac{1+4e^{2(z-x)}}{1+4e^{2z}}\right]^{\frac{1}{4}} e^{2\pi i(w.z+k.\theta)} f\begin{pmatrix} e^{(z-x)}\cos(\theta-\varphi) \\ e^{(z-x)}\sin(\theta-\varphi) \\ e^{2(z-x)}\end{pmatrix}. \quad (4.3.29)$$

Gabor systems

Let now introduce the Gabor system on $L^2\left(\mathcal{P}_*, e^{2z}\left(1+4e^{2z}\right)^{\frac{1}{2}} dzd\theta\right)$. So, let $K_1 = a\mathbb{Z}\times\{0,\frac{1}{L},\frac{2}{L},\frac{3}{L},...,\frac{L-1}{L}\}$ and $K_2 = \frac{1}{b}\mathbb{Z}\times\{0,\frac{1}{M},\frac{2}{M},\frac{3}{M},...,\frac{M-1}{M}\}$ be two uniform lattices in $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$. Let $g\in L^2\left(\mathcal{P}_*, e^{2z}\left(1+4e^{2z}\right)^{\frac{1}{2}} dzd\theta\right)$. The corresponding Gabor system is

$$\mathcal{P}_a^{g;L,M;a,b} = \left\{\left[\frac{e^{z-na}}{e^{z}}\right]\left[\frac{1+4e^{2(z-na)}}{1+4e^{2z}}\right]^{\frac{1}{4}} e^{2\pi i(mb.z+M.k.\theta)} g\begin{pmatrix} e^{(z-na)}\cos(\theta-\frac{2\pi l}{L}) \\ e^{(z-na)}\sin(\theta-\frac{2\pi l}{L}) \\ e^{2(z-na)}\end{pmatrix}\right\},$$

$$(m,n,k)\in\mathbb{Z}^3,\ l=0,1,...,L-1. \quad (4.3.30)$$

Using theorem 2.3.2, we have the following:

Theorem 4.3.5 *Let $a,b>0$ and $(M,L)\in\mathbb{N}^*\times\mathbb{N}^*$. Let $K_1 = a\mathbb{Z}\times\{0,\frac{1}{L},\frac{2}{L},\frac{3}{L},...,\frac{L-1}{L}\}$ and $K_2 = \frac{1}{b}\mathbb{Z}\times\{0,\frac{1}{M},\frac{2}{M},\frac{3}{M},...,\frac{M-1}{M}\}$ be two uniform lattices in $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$. Let $g\in L^2\left(\mathcal{P}_*, e^{2z}\left(1+4e^{2z}\right)^{\frac{1}{2}} dzd\theta\right)$. The following hold:*

(i) If $\frac{ab}{ML}>1$, *then*
$\mathcal{P}_a^{g;L,M;a,b}$ *is a not a frame for* $L^2\left(\mathcal{P}_*, e^{2z}\left(1+4e^{2z}\right)^{\frac{1}{2}} dzd\theta\right)$.

(ii) If $\mathcal{P}_a^{g;L,M;a,b}$ *is a frame for* $L^2\left(\mathcal{P}_*, e^{2z}\left(1+4e^{2z}\right)^{\frac{1}{2}}dzd\theta\right)$*, then* $ab = ML \Leftrightarrow \mathcal{P}_a^{g;L,M;a,b}$ *is a Riesz basis.*

4.3.6 Time-frequency-like transforms on the two-sheeted hyperboloid

Let $\mathcal{H}_+ = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x^2 - y^2 - z^2 = 1; z \gneqq 1 \right\}$, be the upper sheet of the two-sheeted hyperboloid with one point removed.

Let consider the map $V : \mathfrak{C} \longrightarrow \mathcal{H}_+$, defined by

$$V : \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \sinh(e^z)\cos\theta \\ \sinh(e^z)\sin\theta \\ \cosh(e^z) \end{pmatrix}. \tag{4.3.31}$$

This map induces a unitary map

$$\tilde{V} : L^2(\mathfrak{C}, dzd\theta) \longrightarrow L^2\left(\mathcal{H}_+, e^z\sinh(e^z)\cosh^{\frac{1}{2}}(2e^z)dzd\theta\right),$$

defined by

$$\left(\tilde{V}f\right)\begin{pmatrix} \sinh(e^z)\cos\theta \\ \sinh(e^z)\sin\theta \\ \cosh(e^z) \end{pmatrix} = e^{-z}\sinh^{-\frac{1}{2}}(e^z)\cosh^{-\frac{1}{4}}(2e^z).f\begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix}. \tag{4.3.32}$$

The representation $\tilde{U}$, of H_G on $L^2\left(\mathcal{H}_+, e^z\sinh(e^z)\cosh^{\frac{1}{2}}(2e^z)dzd\theta\right)$, is defined by

$$\left(\tilde{U}\left((x,\varphi);(w,k);\eta\right)f\right)\begin{pmatrix}\cosh(z)\cos\theta\\ \cosh(z)\sin\theta\\ \sinh(z)\end{pmatrix} = \tag{4.3.33}$$

$$= \eta\left[\frac{\cosh(z-x)}{\cosh(z)}\right]^{\frac{1}{2}}\left[\frac{\cosh(2(z-x))}{\cosh(2z)}\right]^{\frac{1}{4}} e^{2\pi i(w.z+k.\theta)} f\begin{pmatrix}\cosh(z-x)\cos(\theta-\varphi)\\ \cosh(z-x)\sin(\theta-\varphi)\\ \sinh(z-x)\end{pmatrix}.$$

Gabor systems

Let now introduce the Gabor system on $L^2\left(\mathcal{H}_+, e^z\sinh(e^z)\cosh^{\frac{1}{2}}(2e^z)dzd\theta\right)$. So, let $K_1 = a\mathbb{Z}\times\{0,\frac{1}{L},\frac{2}{L},\frac{3}{L},...,\frac{L-1}{L}\}$ and $K_2 = \frac{1}{b}\mathbb{Z}\times\{0,\frac{1}{M},\frac{2}{M},\frac{3}{M},...,\frac{M-1}{M}\}$ be two uniform lattices in $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$. Let $g\in L^2\left(\mathcal{H}_+, e^z\sinh(e^z)\cosh^{\frac{1}{2}}(2e^z)dzd\theta\right)$. The corresponding Gabor system is

$$\mathcal{H}_+^{g;L,M;a,b} = \left\{ \mathcal{B}_+^{g;L,M;a,b}(z)e^{2\pi i(mb.z+M.k.\theta)}g\begin{pmatrix}\cosh(z-na)\cos(\theta-\frac{2\pi l}{L})\\ \cosh(z-na)\sin(\theta-\frac{2\pi l}{L})\\ \sinh(z-na)\end{pmatrix}\right\},$$
$$(m,n,k)\in\mathbb{Z}^3,\ l=0,1,...,L-1. \tag{4.3.34}$$

where

$$\mathcal{B}_+^{g;L,M;a,b}(z) = \left[\frac{\cosh(z-na)}{\cosh(z)}\right]^{\frac{1}{2}}\left[\frac{\cosh(2(z-na))}{\cosh(2z)}\right]^{\frac{1}{4}}. \tag{4.3.35}$$

Using theorem 2.3.2, we have the following:

Theorem 4.3.6 *Let $a,b>0$ and $(M,L)\in\mathbb{N}^*\times\mathbb{N}^*$. Let $K_1 = a\mathbb{Z}\times\{0,\frac{1}{L},\frac{2}{L},\frac{3}{L},...,\frac{L-1}{L}\}$ and $K_2 = \frac{1}{b}\mathbb{Z}\times\{0,\frac{1}{L},\frac{2}{M},\frac{3}{M},...,\frac{M-1}{M}\}$ be two uniform lattices in $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$. Let $g\in L^2\left(\mathcal{H}_+, e^z\sinh(e^z)\cosh^{\frac{1}{2}}(2e^z)dzd\theta\right)$. The following hold:*

(i) If $\frac{ab}{ML} > 1$, *then*
$\mathcal{H}_{+}^{g;L,M;a,b}$ *is a not a frame for* $L^2\left(\mathcal{H}_+, e^z\sinh(e^z)\cosh^{\frac{1}{2}}(2e^z)dzd\theta\right)$.

(ii) If $\mathcal{H}_{+}^{g;L,M;a,b}$ *is a frame for* $L^2\left(\mathcal{H}_+, e^z\sinh(e^z)\cosh^{\frac{1}{2}}(2e^z)dzd\theta\right)$, *then*
$ab = ML \Leftrightarrow \mathcal{H}_{+}^{g;L,M;a,b}$ *is a Riesz basis.*

Chapter 5

Wavelet-like Transforms on 2-dimensional Surfaces

5.1 Wavelet-like transforms

For constructing wavelet-like transforms for a similar class of 2-dimensional surfaces, we are suggesting that one starts with the infinite cylinder, construct a wavelet-like transform on it in a natural way and then homeomorphically map the cylinder to the various surfaces.

5.1.1 The cylinder

We use the following parametrization for the cylinder:

$$\mathfrak{C} = \left\{ \mathbf{X}(\theta, z) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \theta \in [0, 2\pi), \quad z \in \mathbb{R} \right\}. \tag{5.1.1}$$

In these coordinates, the surface element on $\mathfrak{C}$ is

$$d\sigma_{\mathfrak{C}}(\theta, z) = d\theta\, dz.$$

The group $G_{\mathfrak{C}} = SO(2) \times G_{\text{Aff}}$, where G_{Aff} is the full affine group of of the line, acts on $\mathfrak{C}$ in the manner,

$$\mathbf{X}(\theta, z) \longmapsto \mathbf{X}(g(\theta, z)) = \mathbf{X}(\theta', z') = \mathbf{X}(\theta + \varphi \mod 2\pi,\ az + b), \tag{5.1.2}$$

where, $g = (\varphi, b, a) \in G_{\mathfrak{C}}$. The left Haar measure on this group is

$$d\mu_\ell(g) = d\mu_\ell(\varphi, b, a) = \frac{1}{a^2}\, d\varphi\, db\, da. \tag{5.1.3}$$

It is non-unimodular and has square-integrable representations. We shall work on the Hilbert space $\mathfrak{H}_{\mathfrak{C}} = L^2(\mathfrak{C}, d\sigma_{\mathfrak{C}})$, consisting of functions $F : \mathfrak{C} \longmapsto \mathbb{C}$, which are 2π-periodic with respect to the variable θ, and hence have the Fourier decomposition

$$\begin{aligned} F(\mathbf{X}(\theta, z)) &:= F(\theta, z) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{in\theta} f_n(z), \\ \text{with} \qquad \|F\|^2_{\mathfrak{H}_{\mathfrak{C}}} &= \sum_{n=-\infty}^{\infty} \|f_n\|^2_{L^2(\mathbb{R}, dz)} < \infty. \end{aligned} \tag{5.1.4}$$

For fixed n, the functions $F_n(\theta, z) = \frac{1}{\sqrt{2\pi}} e^{in\theta} f_n(z)$ form a closed subspace $\mathfrak{H}_n$ of $\mathfrak{H}_{\mathfrak{C}}$ and in fact one has the orthogonal decomposition,

$$\mathfrak{H}_{\mathfrak{C}} = \bigoplus_{n=-\infty}^{\infty} \mathfrak{H}_n.$$

The subspaces, $\mathfrak{H}_n$, each carry a unitary irreducible representation of $G_{\mathfrak{C}}$ given by the operators U_n:

$$(U_n(g)F_n(\theta, z) = a^{-\frac{1}{2}} F_n(g^{-1}(\theta, z)) = a^{-\frac{1}{2}} F_n(\theta - \varphi, \frac{z - b}{a}). \tag{5.1.5}$$

This representation is clearly square-integrable and there exist admissible vectors $\Psi_n \in \mathfrak{H}_n,\ \Psi_n(\theta, z) = \frac{1}{\sqrt{2\pi}} e^{in\theta}\psi_n(z)$, satisfying

$$c(\Psi_n) = 2\pi \int_{-\infty}^{\infty} \frac{|\widehat{\psi}_n(z)|^2}{|z|}\, dz < \infty \tag{5.1.6}$$

On $\mathfrak{H}_{\mathfrak{C}}$ we have the reducible representation

$$U(\varphi, b, a) = \bigoplus_{n=-\infty}^{\infty} U_n(\varphi, b, a).$$

A general admissibility condition for vectors in $\mathfrak{H}_{\mathfrak{C}}$ can be obtained as follows: let $F(\theta, z) = \sum_{n=-\infty}^{\infty} e^{in\theta} f_n(z)$ be an arbitrary element in $\mathfrak{H}_{\mathfrak{C}}$ and fix a vector $\Psi(\theta, z) = \sum_{n=-\infty}^{\infty} e^{in\theta}\psi_n(z)$ in $\mathfrak{H}_{\mathfrak{C}}$ such that each ψ_n satisfies the admissibility condition (5.1.6). Then, it is easy to see that

$$\int_{G_{\mathfrak{C}}} |\langle U(g)\Psi \mid F\rangle_{\mathfrak{H}_{\mathfrak{C}}}|^2\, d\mu_\ell(g) = 2\pi \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} |f_n(z)|^2\, dz \int_{\mathbb{R}} \frac{|\widehat{\psi}_n(u)|^2}{|u|}\, du, \tag{5.1.7}$$

provided the sum on the RHS converges. Since $\Psi \in \mathfrak{H}_{\mathfrak{C}}$,

$$\int_{\mathbb{R}} \frac{|\widehat{\psi}_n(u)|^2}{|u|}\, du \longrightarrow 0, \quad \text{as } n \to \pm\infty.$$

Thus, $c(\Psi_n) \to 0$ as $n \to \pm\infty$, so that,

$$T := \int_{G_{\mathfrak{C}}} |U(g)\Psi\rangle\langle U(g)\Psi|\, d\mu_\ell(g) = \sum_{n=-\infty}^{\infty} c(\Psi_n)\mathbb{P}_n, \tag{5.1.8}$$

where $\mathbb{P}_n$ is the projector on $\mathfrak{H}_{\mathfrak{C}}$ which projects onto the subspace $\mathfrak{H}_n$. Clearly, the operator T is bounded. However, its inverse, if it exists, is not bounded. Thus, it is not possible to have a resolution of the identity on the entire Hilbert space. On the other hand, let J be a finite discrete index set, $\mathfrak{H}_J = \oplus_{n\in J}\mathfrak{H}_n$ and U_J the restriction of U to $\mathfrak{H}_J$. Then, choosing $\Psi_J(\theta, z) = \sum_{n\in J} \Psi_n(\theta, z) = \sum_{n\in J} e^{in\theta}\psi_n(z)$ in $\mathfrak{H}_J$, such that $c(\Psi_n) = 1$ for all $n \in J$, we immediately get

$$\int_{G_{\mathfrak{C}}} |U_J(g)\Psi_J\rangle\langle U_J(g)\Psi_J|\, d\mu_\ell(g) = I_J. \tag{5.1.9}$$

Thus, using Ψ_J we can define a wavelet-like transform, S_J, of a vector in $F_J \in \mathfrak{H}_J$:

$$\begin{aligned} S_J(\varphi, b, a) &= \langle U_J(\varphi, b, a)\Psi_J \mid F_J\rangle_{\mathfrak{H}_J} = a^{-\frac{1}{2}} \int_{\mathfrak{C}} \overline{\Psi(\theta - \varphi, \frac{z-b}{a})} F(\theta, z)\, d\theta\, dz \\ &= a^{-\frac{1}{2}} \sum_{n\in J} e^{in\varphi} \int_{\mathbb{R}} \overline{\psi_n(\frac{z-b}{a})} f_n(z)\, dz. \end{aligned} \tag{5.1.10}$$

For practical purposes, it is just a question of choosing J large enough to include all relevant angular momenta.

5.1.2 Other surfaces

The construction of the transform (5.1.10) on the cylinder can now be easily transferred to surfaces which are topologically homeomorphic to the cylinder. Let $\boldsymbol{\ell}$ be a smooth open curve in $\mathbb{R}^2$, parametrized as

$$z \longmapsto \begin{pmatrix} u(z) \\ v(z) \end{pmatrix}, \quad z \in \mathbb{R},$$

u and v being two smooth functions. (We are assuming that this map is a homeomorphism between $\mathbb{R}^2$ and $\boldsymbol{\ell}$). The map

$$V : \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \longmapsto \begin{pmatrix} u(z)\cos\theta \\ u(z)\sin\theta \\ v(z) \end{pmatrix}, \tag{5.1.11}$$

then transforms the cylinder homeomorphically to a surface of revolution $\mathfrak{S}$ about the z-axis. The surface element $d\sigma_{\mathfrak{C}}$ on the cylinder transforms to

$$d\sigma_{\mathfrak{S}}(\theta, z) = w(z)\ d\theta\ dz, \qquad w(z) = |u(z)|\ [u'(z)^2 + v'(z)^2]^{\frac{1}{2}} \tag{5.1.12}$$

on this surface. The mapping V then induces an isometric map W of $\mathfrak{H}_{\mathfrak{C}}$ onto $\mathfrak{H}_{\mathfrak{S}} = L^2(\mathfrak{S}, d\sigma_{\mathfrak{S}})$, according to which $F(\mathbf{X}(\theta, z)) \longmapsto F \circ V(\mathbf{X}(\theta, z))$ and

$$\int_{\mathfrak{S}} |F \circ V(\mathbf{X}(\theta, z))|^2\ d\sigma_{\mathfrak{S}} = \int_{\mathfrak{C}} |F(\mathbf{X}(\theta, z))|^2\ d\sigma_{\mathfrak{C}}\,.$$

More explicitly, let $\mathbf{X}_{\mathfrak{C}}(\theta, z)$ be a point on the cylinder and $\mathbf{X}_{\mathfrak{S}}(\theta, z) = (V\mathbf{X}_{\mathfrak{C}})(\theta, z)$ the corresponding point on the surface $\mathfrak{S}$. Then for any $F \in \mathfrak{H}_{\mathfrak{C}}$, we have

$$(WF)(\mathbf{X}_{\mathfrak{S}}(\theta, z)) = [w(z)]^{-\frac{1}{2}} F((V^{-1}\mathbf{X}_{\mathfrak{S}})(\theta, z)).$$

The action of the group $G_{\mathfrak{C}}$ on the cylinder is similarly transferred to an action on the surface $\mathfrak{S}$, under which,

$$\begin{pmatrix} u(z)\cos\theta \\ u(z)\sin\theta \\ v(z) \end{pmatrix} \xrightarrow{(\varphi,b,a)} \begin{pmatrix} u(az+b)\cos(\theta+\varphi) \\ u(az+b)\sin(\theta+\varphi) \\ v(az+b) \end{pmatrix}. \tag{5.1.13}$$

Functions in $\mathfrak{H}_{\mathfrak{S}}$ are again periodic in θ. Since V induces an isometry between $\mathfrak{H}_{\mathfrak{C}}$ and $\mathfrak{H}_{\mathfrak{S}}$, the representations U_n in (5.1.5) carry over to irreducible and square-integrable representations $U_n^{\mathfrak{S}} = WU_nW^{-1}$on $\mathfrak{H}_{\mathfrak{S}}$ in the manner

$$(U_n^{\mathfrak{S}}(\varphi, b, a)F)(\theta, z) = \left[\frac{w(\frac{z-b}{a})}{a\,w(z)}\right]^{\frac{1}{2}} F(\theta - \varphi, \frac{z-b}{a}) , \qquad F \in \mathfrak{H}_{\mathfrak{S}} . \tag{5.1.14}$$

Let us look at a few special cases.

5.1.3 One-sheeted hyperboloid

Define the two matrices of rigid and hyperbolic rotations, $R(\varphi)$ and $\Lambda(b)$, respectively,

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Lambda(b) = \begin{pmatrix} \cosh b & 0 & \sinh b \\ 0 & 1 & 0 \\ \sinh b & 0 & \cosh b \end{pmatrix},$$

where $\varphi \in [0, 2\pi), \quad b \in \mathbb{R}$. The first matrix represents a rigid rotation about the z-axis and the second a Lorentz transformation in the xz-plane. Also, let $D(a)$ denote the linear transformation,

$$D(a)\mathbf{X}(\theta, z) = \mathbf{X}(\theta, az), \quad a \neq 0. \tag{5.1.15}$$

From (5.1.13), the action of a group element (φ, b, a) on $\mathbf{X}(\theta, z)$ is seen to be,

$$\begin{aligned} \mathbf{X}(\theta, z) \quad \stackrel{(\varphi,b,a)}{\longrightarrow} & \begin{pmatrix} \cosh(az+b)\cos(\theta+\varphi) \\ \cosh(az+b)\sin(\theta+\varphi) \\ \sinh(az+b) \end{pmatrix} \\ = \quad & R(\varphi)\,[R(\theta)\;\Lambda(b)\;R(-\theta)]\;D(a)\;\mathbf{X}(\theta, z). \end{aligned} \tag{5.1.16}$$

The matrix $R(\theta)\;\Lambda(b)\;R(-\theta)$ performs a Lorentz transformation in the plane of the vectors $(\cos\theta, \sin\theta, 0)^T$ and $(0, 0, 1)^T$, while the transformation $D(a)$ induces a hyperbolic dilation. Physically, since

$$\mathbf{X}(\theta, z) = R(\theta)\Lambda(z)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

the quantity $\mathfrak{v} = \tanh z$ represents the velocity of a relativistic particle at the point $\mathbf{X}(\theta, z)$. Hence, $z = \tanh^{-1}\mathfrak{v}$ represents its rapidity (assuming the velocity of light $c = 1$). Thus the dilation $D(a)$ simply scales the rapidity. In other words, on the one-sheeted hyperboloid, the three transformations induced by the group $G_{\mathfrak{C}}$, amount to a rotation, a Lorentz transformation and a rapidity scaling.

It will be more convenient to work with the variable $v = \sinh z$ itself, rather than z. We write

$$\mathbf{X}(\theta, v) = \begin{pmatrix} v_0 \cos\theta \\ v_0 \sin\theta \\ v \end{pmatrix}, \quad v_0 = \sqrt{1 + v^2},$$

so that in these coordinates,

$$d\sigma_{\mathcal{H}_1} = \sqrt{1 + 2v^2}\, d\theta\, dv.$$

Under the action of $(\varphi, b, a) \in G_{\mathfrak{C}}$

$$\mathbf{X}(\theta, v) \longrightarrow \mathbf{X}(\theta + \varphi, v') = \begin{pmatrix} v_0' \cos(\theta + \varphi) \\ v_0' \sin(\theta + \varphi) \\ v' \end{pmatrix}, \quad v_0' = \sqrt{1 + v'^2},$$

where,

$$\begin{aligned} v' &= \sinh b\, \cosh(a \sinh^{-1} v) + \cosh b\, \sinh(a \sinh^{-1} v) \\ v_0' &= \cosh b\, \cosh(a \sinh^{-1} v) + \sinh b\, \sinh(a \sinh^{-1} v). \end{aligned} \tag{5.1.17}$$

5.1.4 Wavelets on the sphere

Let $\mathcal{S}^2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x^2+y^2+z^2=1 \right\}$, be the two dimensional unit sphere.

Let consider the map $V : \mathfrak{C} \longrightarrow \mathcal{S}^2$, defined by

$$V : \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \cosh^{-1}(z)\cos\theta \\ \cosh^{-1}(z)\sin\theta \\ \tanh(z) \end{pmatrix}. \tag{5.1.18}$$

This map induces a unitary map

$$\tilde{V} : L^2\left(\mathfrak{C}, dzd\theta\right) \longrightarrow L^2\left(\mathcal{S}^2, \cosh^{-2}(z)dzd\theta\right),$$

defined by

$$\left(\tilde{V}f\right)\begin{pmatrix} \cosh^{-1}(z)\cos\theta \\ \cosh^{-1}(z)\sin\theta \\ \tanh(z) \end{pmatrix} = \cosh(z).f\begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix}. \tag{5.1.19}$$

Using the representation U_J, the restriction of U to $\mathfrak{H}_J$, we get a representation $\tilde{U}_J$ of G on a subspace of $L^2\left(\mathcal{S}^2, \cosh^{-2}(z)dzd\theta\right)$, characterized by J, given by

$$\tilde{U}_J(g) = \tilde{V}U_J(g)\tilde{V}^{-1}, \tag{5.1.20}$$

defined by

$$\left(\tilde{U}_J(g)f\right)\begin{pmatrix}\cosh(z)\cos\theta\\ \cosh(z)\sin\theta\\ \sinh(z)\end{pmatrix} =$$

$$|a|^{-\frac{1}{2}}\left[\frac{\cosh(z)}{\cosh(\frac{z-b}{a})}\right] f \begin{pmatrix}\cosh^{-1}\left(\frac{z-b}{a}\right)\cos(\theta-\psi)\\ \cosh^{-1}\left(\frac{z-b}{a}\right)\sin(\theta-\psi)\\ \tanh\left(\frac{z-b}{a}\right)\end{pmatrix} \qquad (5.1.21)$$

Admissibility conditions

Let $F_n(z,\theta) = \frac{1}{\sqrt{2}}e^{in\theta}\phi_n(z)$, be an element of $L^2\left(\mathcal{S}^2, \cosh^{-2}(z)dzd\theta\right)$, for $n \in J$.

Its easy to see that the admissibility condition on F_n is given by

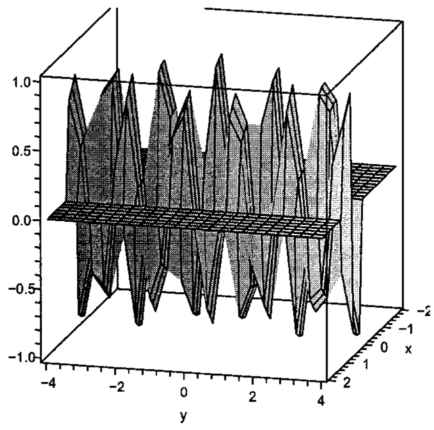
$$\int_{\mathbb{R}} \frac{|\hat{\Phi}_n(\gamma)|^2}{|\gamma|} d\gamma < \infty, \qquad (5.1.22)$$

where $\Phi_n(z) = \phi_n(z).\cosh^{-1}(z)$.

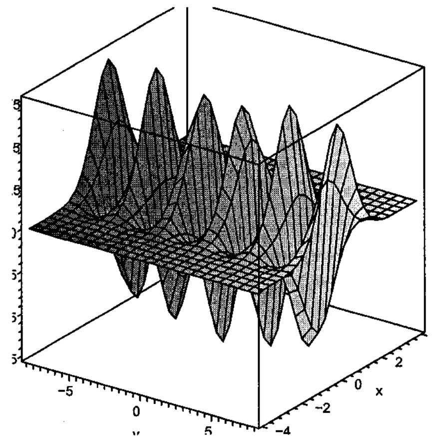
Let us plot the graphs of some families of admissible wavelets: $\Phi_{\alpha,\beta,\gamma}$, defined by

$$\Phi_{\alpha,\beta,\gamma}(x,y) = \left[\alpha\cos(y)H(x) + \beta\cos(2y)Me(x) + \gamma\cos(4y)Mo(x)\right].\cosh(x), \quad (5.1.23)$$

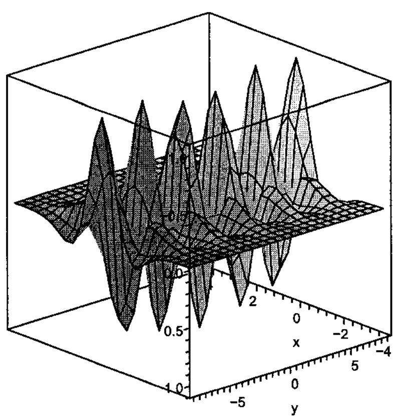
by using the following changing of variable $(z,\theta) \mapsto (x,y)$, where α, β, γ are real parameters, H stands for the standard Haar wavelet, Me stands for the standard Mexican hat, and $Mo(z)$ stands for the standard Morlet wavelet.



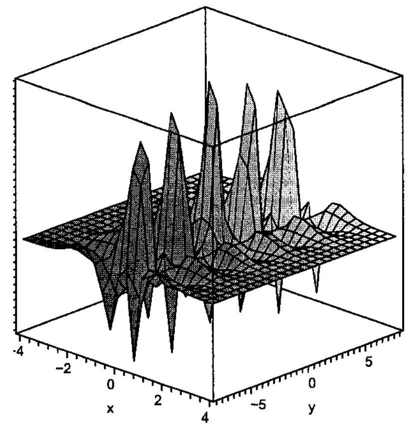
(a) $(\alpha, \beta, \gamma) = (1, 0, 0)$



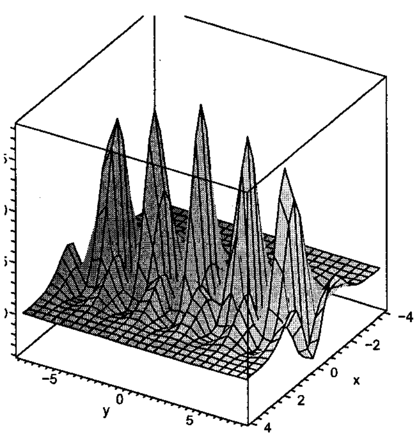
(b) $(\alpha, \beta, \gamma) = (0, 0, 1)$



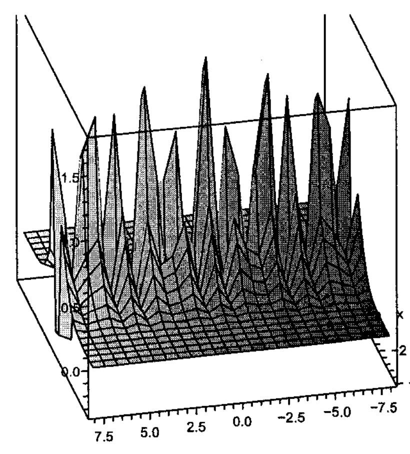
(c) $(\alpha, \beta, \gamma) = (0, 1, 0)$



(d) $(\alpha, \beta, \gamma) = (1, 1, 0)$



(e) $(\alpha, \beta, \gamma) = (0, 1, 1)$



(f) $(\alpha, \beta, \gamma) = (1, 0, 1)$

Figure 5.1: The wavelet $\Phi_{\alpha,\beta,\gamma}$ (5.1.23), at different values of the parameter (α, β, γ).

5.1.5 Wavelets on an ellipsoid of revolution

The previous construction of wavelets on $2D$ can be generalized to any ellipsoid of revolution.

Let $\alpha, \gamma > 0$, and let $\mathcal{E}_{\alpha,\gamma} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \frac{x^2+y^2}{\alpha^2} + \frac{z^2}{\gamma^2} = 1 \right\}$, be the two dimensional ellipsoid of revolution.

Let consider the map $V : \mathfrak{C} \longrightarrow \mathcal{E}_{\alpha,\gamma}$, defined by

$$V : \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \alpha\cosh^{-1}(z)\cos\theta \\ \alpha\cosh^{-1}(z)\sin\theta \\ \gamma\tanh(z) \end{pmatrix}. \tag{5.1.24}$$

This map induces a unitary map

,

$$\tilde{V} : L^2\left(\mathfrak{C}, dzd\theta\right) \longrightarrow L^2\left(\mathcal{E}_{\alpha,\gamma},\ \rho_{\mathcal{E}_{\alpha,\gamma}}(z)dzd\theta\right) :$$

where

$$\rho_{\mathcal{E}_{\alpha,\gamma}}(z) = \frac{\alpha}{\cosh^3(z)}\sqrt{\alpha^2\sinh^2(z) + \gamma^2}, \tag{5.1.25}$$

defined by

$$\left(\tilde{V}f\right)\begin{pmatrix} \alpha\cosh^{-1}(z)\cos\theta \\ \alpha\cosh^{-1}(z)\sin\theta \\ \gamma\tanh(z) \end{pmatrix} = \left[\rho_{\mathcal{E}_{\alpha,\gamma}}(z)\right]^{\frac{1}{2}}.f\begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix}. \tag{5.1.26}$$

Using the representation U_J, the restriction of U to $\mathfrak{H}_J$, we get a representation $\tilde{U}_J$ of G on a subspace of $L^2\left(\mathcal{E}_{\alpha,\gamma},\ \rho_{\mathcal{E}_{\alpha,\gamma}}(z)dzd\theta\right)$, characterized by J, given by

$$\tilde{U}_J(g) = \tilde{V}U_J(g)\tilde{V}^{-1}, \tag{5.1.27}$$

defined by

$$\left(\tilde{U}_J(g)f\right)\begin{pmatrix}\alpha\cosh^{-1}(z)\cos\theta\\ \alpha\cosh^{-1}(z)\sin\theta\\ \gamma\tanh(z)\end{pmatrix} = |a|^{-\frac{1}{2}}\left[\frac{\cosh(z)}{\cosh(\frac{z-b}{a})}\right]^{\frac{3}{2}}\left[\frac{\alpha^2\sinh^2(\frac{z-b}{a})+\gamma^2}{\alpha^2\sinh^2(z)+\gamma^2}\right]^{\frac{1}{4}}\times$$

$$f\begin{pmatrix}\alpha\cosh^{-1}\left(\frac{z-b}{a}\right)\cos(\theta-\psi)\\ \alpha\cosh^{-1}\left(\frac{z-b}{a}\right)\sin(\theta-\psi)\\ \gamma\tanh\left(\frac{z-b}{a}\right)\end{pmatrix} \tag{5.1.28}$$

Admissibility conditions

Let $F_n(z,\theta)=\frac{1}{\sqrt{2}}e^{in\theta}\phi_n(z)$, be an element of $L^2\left(\mathcal{E}_{\alpha,\gamma},\ \rho_{\mathcal{E}_{\alpha,\gamma}}(z)dzd\theta\right)$, for $n\in J$.

Its easy to see that the admissibility condition on F_n is given by

$$\int_{\mathbb{R}}\frac{|\hat{\Phi}_n(\gamma)|^2}{|\gamma|}d\gamma<\infty, \tag{5.1.29}$$

where $\Phi_n(z)=\phi_n(z).\rho_{\mathcal{E}_{\alpha,\gamma}}(z)^{\frac{1}{2}}$.

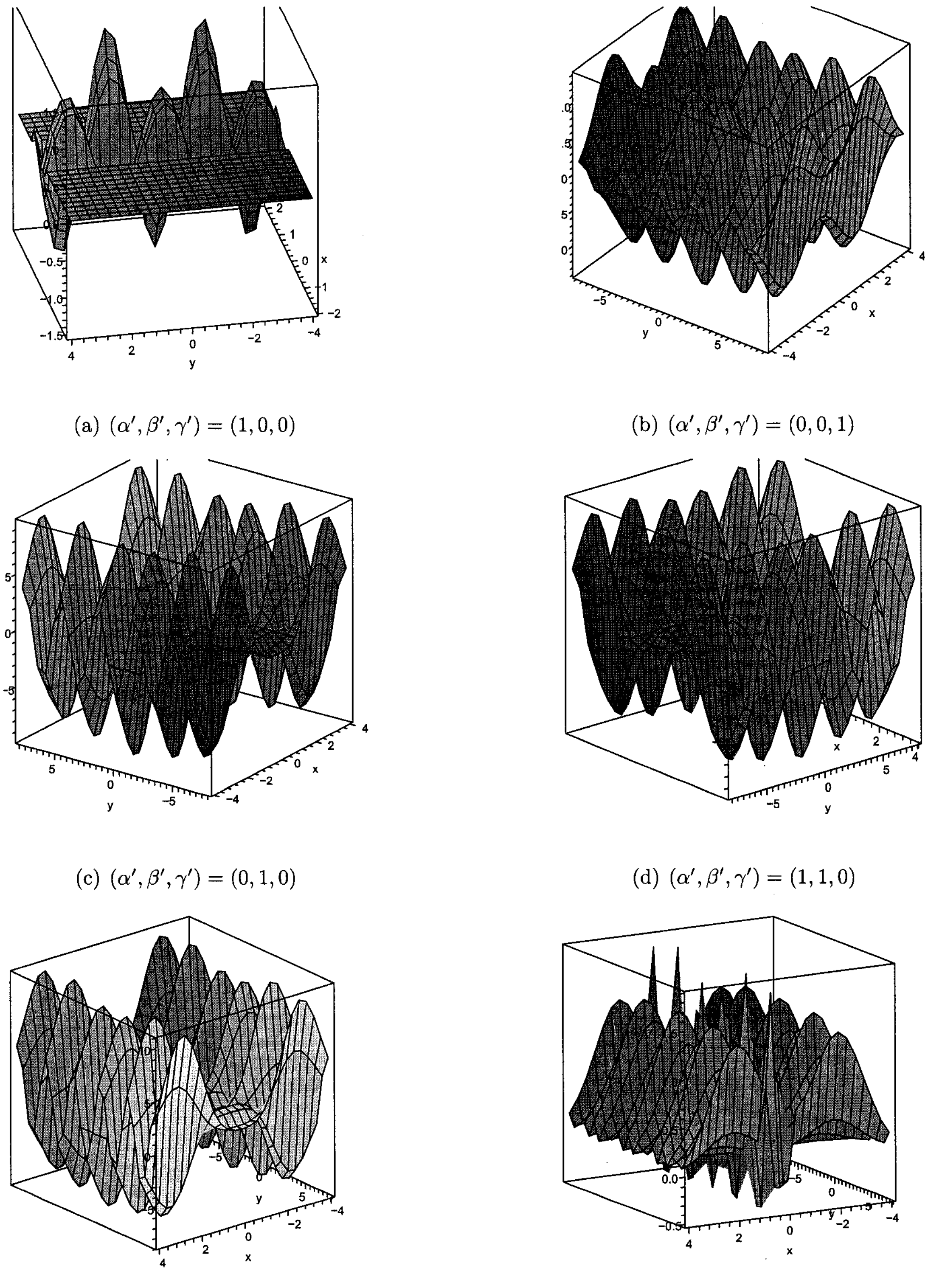
Remark When $\alpha=\gamma=1$, we get the previous result on the sphere.

Let now examine the case $\alpha=1,\ \gamma=2$.

Let us plot the graphs of some families of admissible wavelets: $\Phi_{\alpha',\beta',\gamma'}$, defined by

$$\Phi_{\alpha',\beta',\gamma'}(x,y)=\frac{\left[\alpha'\cos(y)H(x)+\beta'\cos(2y)Me(x)+\gamma'\cos(4y)Mo(x)\right]}{\rho_{\mathcal{E}_{1,2}}(x)^{\frac{1}{2}}} \tag{5.1.30}$$

where α',β',γ' are real parameters.



(a) $(\alpha', \beta', \gamma') = (1, 0, 0)$

(b) $(\alpha', \beta', \gamma') = (0, 0, 1)$

(c) $(\alpha', \beta', \gamma') = (0, 1, 0)$

(d) $(\alpha', \beta', \gamma') = (1, 1, 0)$

(e) $(\alpha', \beta', \gamma') = (0, 1, 1)$

(f) $(\alpha', \beta', \gamma') = (1, 0, 1)$

Figure 5.2: The wavelet $\Phi_{\alpha',\beta',\gamma'}$ (5.1.30), at different values of the parameter $(\alpha', \beta', \gamma')$.

5.1.6 Wavelets on the one-sheeted hyperboloid

Let $\mathcal{H} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x^2 + y^2 - z^2 = 1 \right\}$, be the two dimensional one sheeted hyperboloid.

Let consider the map $V : \mathfrak{C} \longrightarrow \mathcal{H}$, defined by

$$V : \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \cosh(z)\cos\theta \\ \cosh(z)\sin\theta \\ \sinh(z) \end{pmatrix}. \qquad (5.1.31)$$

This map induces a unitary map

$$\tilde{V} : L^2(\mathfrak{C}, dzd\theta) \longrightarrow L^2\left(\mathcal{H}, \cosh(z)\cosh^{\frac{1}{2}}(2z)dzd\theta\right) :$$

defined by

$$\left(\tilde{V}f\right)\begin{pmatrix} \cosh(z)\cos\theta \\ \cosh(z)\sin\theta \\ \sinh(z) \end{pmatrix} = \cosh(z)^{-\frac{1}{2}}\cosh^{-\frac{1}{4}}(2z).f\begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix}. \qquad (5.1.32)$$

Using the representation U_J, the restriction of U to $\mathfrak{H}_J$, we get a representation $\tilde{U}_J$ of G on a subspace of $L^2\left(\mathcal{H}, \cosh(z)\cosh^{\frac{1}{2}}(2z)dzd\theta\right)$, characterized by J, given by

$$\tilde{U}_J(g) = \tilde{V}U_J(g)\tilde{V}^{-1}, \qquad (5.1.33)$$

defined by

$$\left(\tilde{U}_J(g)f\right)\begin{pmatrix}\cosh(z)\cos\theta\\ \cosh(z)\sin\theta\\ \sinh(z)\end{pmatrix} =$$

$$|a|^{-\frac{1}{2}}\left[\frac{\cosh\left(\frac{z-b}{a}\right)}{\cosh(z)}\right]^{\frac{1}{2}}\left[\frac{\cosh\left(2\frac{z-b}{a}\right)}{\cosh(2z)}\right]^{\frac{1}{4}} f\begin{pmatrix}\cosh\left(\frac{z-b}{a}\right)\cos(\theta-\psi)\\ \cosh\left(\frac{z-b}{a}\right)\sin(\theta-\psi)\\ \sinh\left(\frac{z-b}{a}\right)\end{pmatrix}. \quad (5.1.34)$$

Admissibility conditions

Let $F_n(z,\theta) = \frac{1}{\sqrt{2}}e^{in\theta}\phi_n(z)$, be an element of $L^2\left(\mathcal{H}, \cosh(z)\cosh^{\frac{1}{2}}(2z)dzd\theta\right)$, for $n \in J$.

Its easy to see that the admissibility condition on F_n is given by

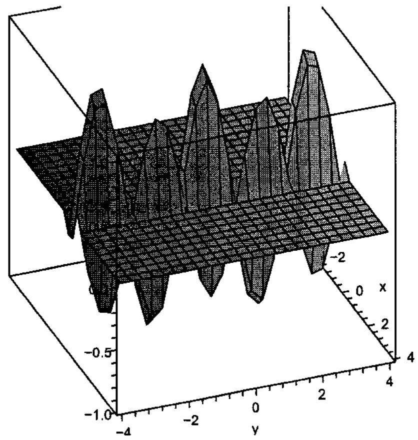
$$\int_{\mathbb{R}} \frac{|\hat{\Phi}_n(\gamma)|^2}{|\gamma|}d\gamma < \infty, \quad (5.1.35)$$

where $\Phi_n(z) = \phi_n(z).\cosh^{\frac{1}{2}}(z)\cosh^{\frac{1}{4}}(2z)$.

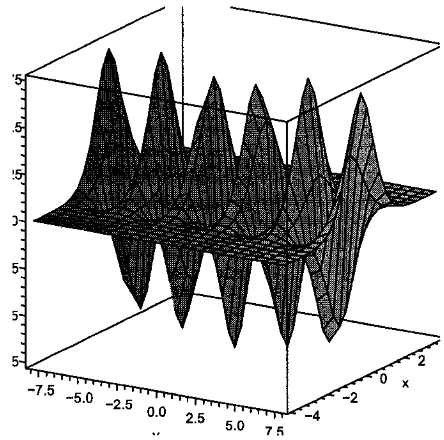
Let us plot the graphs of some families of admissible wavelets: $\Phi_{\alpha,\beta,\gamma}$, defined by

$$\Phi_{\alpha,\beta,\gamma}(x,y) = \frac{\left[\alpha\cos(y)H(x) + \beta\cos(2y)Me(x) + \gamma\cos(4y)Mo(x)\right]}{\cosh^{\frac{1}{2}}(x)\cosh^{\frac{1}{4}}(2x)}, \quad (5.1.36)$$

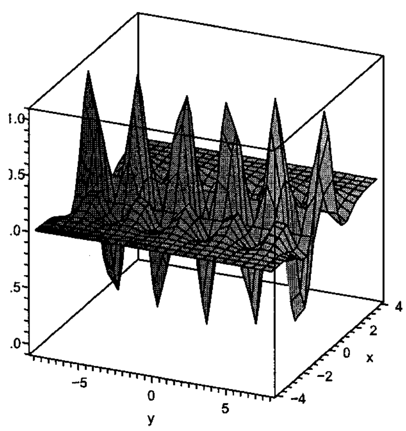
where α, β, γ are real parameters.



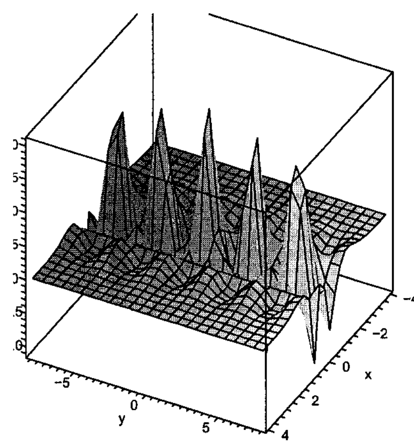
(a) $(\alpha, \beta, \gamma) = (1, 0, 0)$



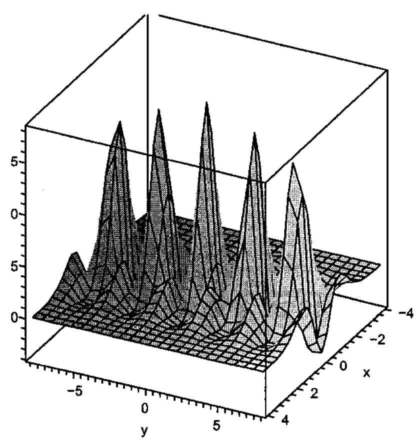
(b) $(\alpha, \beta, \gamma) = (0, 0, 1)$



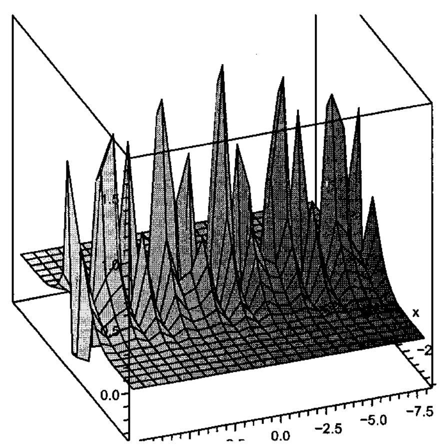
(c) $(\alpha, \beta, \gamma) = (0, 1, 0)$



(d) $(\alpha, \beta, \gamma) = (1, 1, 0)$



(e) $(\alpha, \beta, \gamma) = (0, 1, 1)$



(f) $(\alpha, \beta, \gamma) = (0, 1, 1)$

Figure 5.3: The wavelet $\Phi_{\alpha,\beta,\gamma}$ (5.1.36), at different values of the parameter (α, β, γ).

5.1.7 Wavelets on the plane without the origin

Let $\mathbb{R}^2_*$ be the plane without the origin.

Let consider the map $V : \mathfrak{C} \longrightarrow \mathbb{R}^2_*$, defined by

$$V : \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \longmapsto \begin{pmatrix} e^z \cos\theta \\ e^z \sin\theta \end{pmatrix}. \tag{5.1.37}$$

This map induces a unitary map

$$\tilde{V} : L^2(\mathfrak{C}, dzd\theta) \longrightarrow L^2\left(\mathbb{R}^2_*, e^{2z} dzd\theta\right) :$$

defined by

$$\left(\tilde{V}f\right)\begin{pmatrix} e^z \cos\theta \\ e^z \sin\theta \end{pmatrix} = e^{-z}.f\begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix}. \tag{5.1.38}$$

Using the representation U_J, the restriction of U to $\mathfrak{H}_J$, we get a representation $\tilde{U}_J$ of G on a subspace of $L^2(\mathbb{R}^2_*, e^{2z} dzd\theta)$, characterized by J, given by

$$\tilde{U}_J(g) = \tilde{V} U_J(g) \tilde{V}^{-1}, \tag{5.1.39}$$

defined by

$$\left(\tilde{U}_J(g)f\right)\begin{pmatrix} e^z \cos\theta \\ ez \sin\theta \end{pmatrix} = |a|^{-\frac{1}{2}} \left[\frac{e^{\left(\frac{z-b}{a}\right)}}{e^z}\right]^{\frac{1}{2}} f\begin{pmatrix} e^{\left(\frac{z-b}{a}\right)} \cos(\theta-\psi) \\ e^{\left(\frac{z-b}{a}\right)} \sin(\theta-\psi) \end{pmatrix} \tag{5.1.40}$$

Admissibility conditions

Let $F_n(z,\theta) = \frac{1}{\sqrt{2}} e^{in\theta}\phi_n(z)$, be an element of $L^2\left(\mathbb{R}^2_*, e^{2z}dzd\theta\right)$, for $n \in J$.

Its easy to see that the admissibility condition on F_n is given by

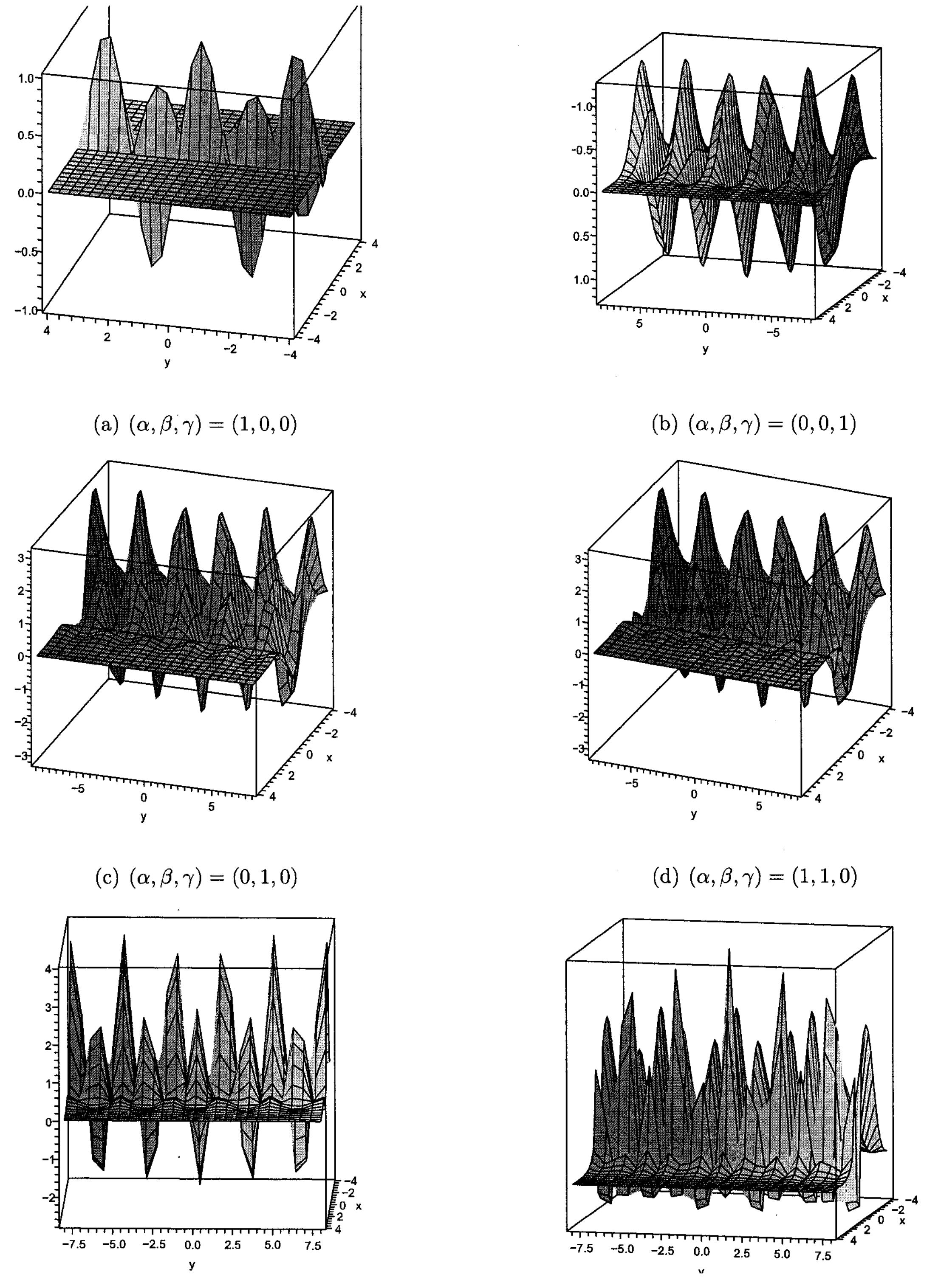
$$\int_{\mathbb{R}} \frac{\mid \hat{\Phi}_n(\gamma) \mid^2}{|\gamma|} d\gamma < \infty, \tag{5.1.41}$$

where $\Phi_n(z) = \phi_n(z).e^z$.

Let us plot the graphs of some families of admissible wavelets: $\Phi_{\alpha,\beta,\gamma}$, defined by

$$\Phi_{\alpha,\beta,\gamma}(x,y) = \left[\alpha\cos(y)H(x) + \beta\cos(2y)Me(x) + \gamma\cos(4y)Mo(x)\right].e^x, \tag{5.1.42}$$

where α, β, γ are real parameters.



(a) $(\alpha, \beta, \gamma) = (1, 0, 0)$

(b) $(\alpha, \beta, \gamma) = (0, 0, 1)$

(c) $(\alpha, \beta, \gamma) = (0, 1, 0)$

(d) $(\alpha, \beta, \gamma) = (1, 1, 0)$

(e) $(\alpha, \beta, \gamma) = (0, 1, 1)$

(f) $(\alpha, \beta, \gamma) = (1, 0, 1)$

Figure 5.4: The wavelet $\Phi_{\alpha,\beta,\gamma}$ (5.1.42), at different values of the parameter (α, β, γ).

5.1.8 Wavelets on the paraboloid without the origin

Let $\mathcal{P}_*$ be the paraboloid without the origin

Let consider the map $V : \mathfrak{C} \longrightarrow \mathcal{P}_*$, defined by

$$V : \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \longmapsto \begin{pmatrix} e^z \cos\theta \\ e^z \sin\theta \\ e^{2z} \end{pmatrix}. \tag{5.1.43}$$

This map induces a unitary map

$$\tilde{V} : L^2(\mathfrak{C}, dzd\theta) \longrightarrow L^2\left(\mathcal{P}_*, e^{2z}\left(1+4e^{2z}\right)^{\frac{1}{2}} dzd\theta\right) :$$

defined by

$$\left(\tilde{V}f\right)\begin{pmatrix} \cosh(z)\cos\theta \\ \cosh(z)\sin\theta \\ \sinh(z) \end{pmatrix} = e^{-z}\left(1+4e^{2z}\right)^{-\frac{1}{4}}.f\begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix}. \tag{5.1.44}$$

Using the representation U_J, the restriction of U to $\mathfrak{H}_J$, we get a representation $\tilde{U}_J$ of G on a subspace of $L^2\left(\mathcal{P}_*, e^{2z}\left(1+4e^{2z}\right)^{\frac{1}{2}} dzd\theta\right)$, characterized by J, given by

$$\tilde{U}_J(g) = \tilde{V}U_J(g)\tilde{V}^{-1}, \tag{5.1.45}$$

defined by

$$\left(\tilde{U}_J(g)f\right)\begin{pmatrix} e^z\cos\theta \\ e^z\sin\theta \\ e^{2z} \end{pmatrix} =$$

$$|a|^{-\frac{1}{2}}\left[\frac{e^{\frac{z-b}{a}}}{e^z}\right]\left[\frac{1+4e^{2\frac{z-b}{a}}}{1+4e^{2z}}\right]^{\frac{1}{4}} f\begin{pmatrix} e^{\left(\frac{z-b}{a}\right)}\cos(\theta-\psi) \\ e^{\left(\frac{z-b}{a}\right)}\sin(\theta-\psi) \\ e^{2\left(\frac{z-b}{a}\right)} \end{pmatrix} \tag{5.1.46}$$

Admissibility conditions

Let $F_n(z,\theta) = \frac{1}{\sqrt{2}} e^{in\theta}\phi_n(z)$, be an element of $L^2\left(\mathcal{P}_*, e^{2z}\left(1+4e^{2z}\right)^{\frac{1}{2}} dz d\theta\right)$, for $n \in J$.

Its easy to see that the admissibility condition on F_n is given by

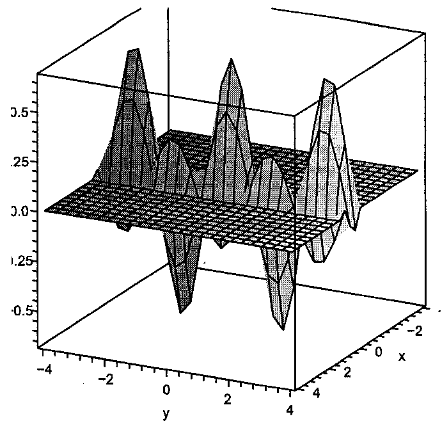
$$\int_{\mathbb{R}} \frac{\mid \hat{\Phi}_n(\gamma) \mid^2}{|\gamma|} d\gamma < \infty, \tag{5.1.47}$$

where $\Phi_n(z) = \phi_n(z).e^z\left(1+4e^{2z}\right)^{\frac{1}{4}}$

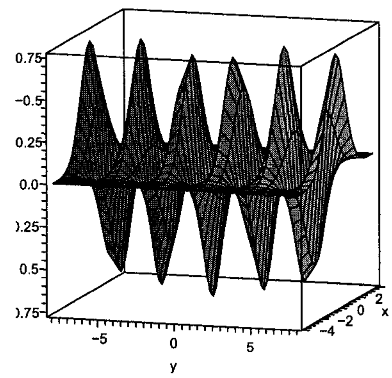
Let us plot the graphs of some families of admissible wavelets: $\Phi_{\alpha,\beta,\gamma}$, defined by

$$\Phi_{\alpha,\beta,\gamma}(x,y) = \frac{\left[\alpha\cos(y)H(x) + \beta\cos(2y)Me(x) + \gamma\cos(4y)Mo(x)\right]}{e^x\left(1+4e^{2x}\right)^{\frac{1}{4}}}, \tag{5.1.48}$$

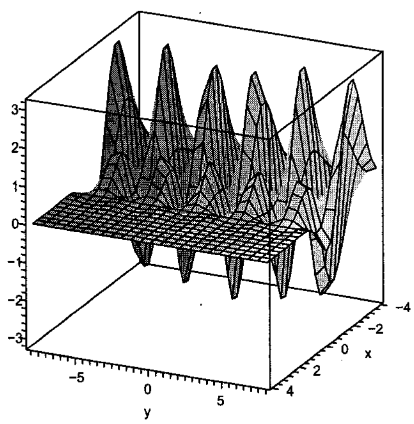
where α, β, γ are real parameters.



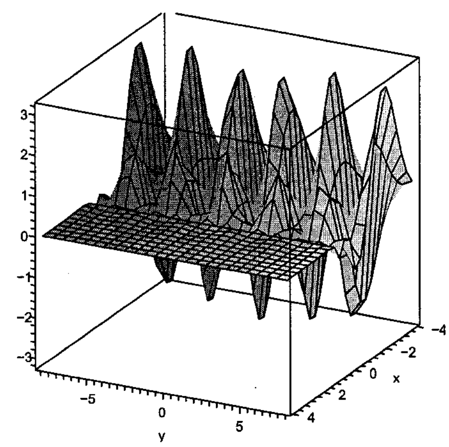
(a) $(\alpha, \beta, \gamma) = (1, 0, 0)$



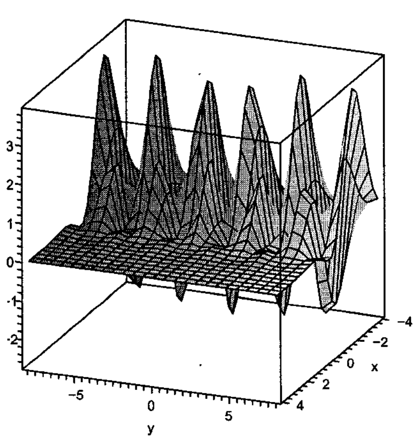
(b) $(\alpha, \beta, \gamma) = (0, 0, 1)$



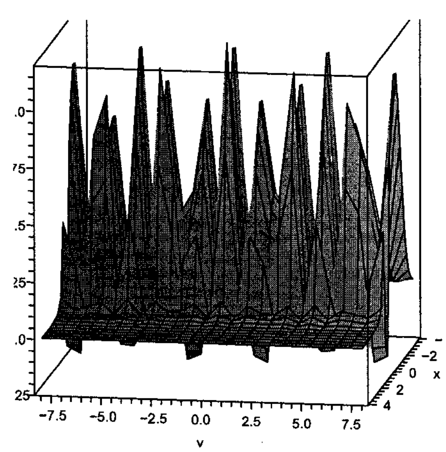
(c) $(\alpha, \beta, \gamma) = (0, 1, 0)$



(d) $(\alpha, \beta, \gamma) = (1, 1, 0)$



(e) $(\alpha, \beta, \gamma) = (0, 1, 1)$



(f) $(\alpha, \beta, \gamma) = (1, 0, 1)$

Figure 5.5: The wavelet $\Phi_{\alpha,\beta,\gamma}$ (5.1.48), at different values of the parameter (α, β, γ).

5.1.9 Wavelets on the two-sheeted hyperboloid

Let consider the map $V : \mathfrak{C} \longrightarrow \mathcal{H}_+$, defined by

$$V : \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \sinh(e^z)\cos\theta \\ \sinh(e^z)\sin\theta \\ \cosh(e^z) \end{pmatrix}. \tag{5.1.49}$$

This map induces a unitary map

$$\tilde{V} : L^2\left(\mathfrak{C}, dzd\theta\right) \longrightarrow L^2\left(\mathcal{H}_+, e^z\sinh(e^z)\cosh^{\frac{1}{2}}(2e^z)dzd\theta\right):$$

defined by

$$\left(\tilde{V}f\right)\begin{pmatrix} \sinh(e^z)\cos\theta \\ \sinh(e^z)\sin\theta \\ \cosh(e^z) \end{pmatrix} = e^{-z}\sinh^{-\frac{1}{2}}(e^z)\cosh^{-\frac{1}{4}}(2e^z).f\begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix}. \tag{5.1.50}$$

Using the representation U_J, the restriction of U to $\mathfrak{H}_J$, we get a representation $\tilde{U}_J$ of G on a subspace of $L^2\left(\mathcal{H}_+, e^z\sinh(e^z)\cosh^{\frac{1}{2}}(2e^z)dzd\theta\right)$, characterized by J, given by

$$\tilde{U}_J(g) = \tilde{V}U_J(g)\tilde{V}^{-1}, \tag{5.1.51}$$

defined by

$$\left(\tilde{U}_J(g)f\right)\begin{pmatrix} \cosh(z)\cos\theta \\ \cosh(z)\sin\theta \\ \sinh(z) \end{pmatrix} =$$

$$|a|^{-\frac{1}{2}}\left[\frac{\cosh\left(\frac{z-b}{a}\right)}{\cosh(z)}\right]^{\frac{1}{2}}\left[\frac{\cosh\left(2\frac{z-b}{a}\right)}{\cosh(2z)}\right]^{\frac{1}{4}} f\begin{pmatrix} \cosh\left(\frac{z-b}{a}\right)\cos(\theta-\psi) \\ \cosh\left(\frac{z-b}{a}\right)\sin(\theta-\psi) \\ \sinh\left(\frac{z-b}{a}\right) \end{pmatrix}. \tag{5.1.52}$$

Admissibility conditions

Let $F_n(z,\theta) = \frac{1}{\sqrt{2}} e^{in\theta}\phi_n(z)$, be an element of $L^2\left(\mathcal{H}_+, e^z \sinh(e^z)\cosh^{\frac{1}{2}}(2e^z)dzd\theta\right)$, for $n \in J$.

Its easy to see that the admissibility condition on F_n is given by

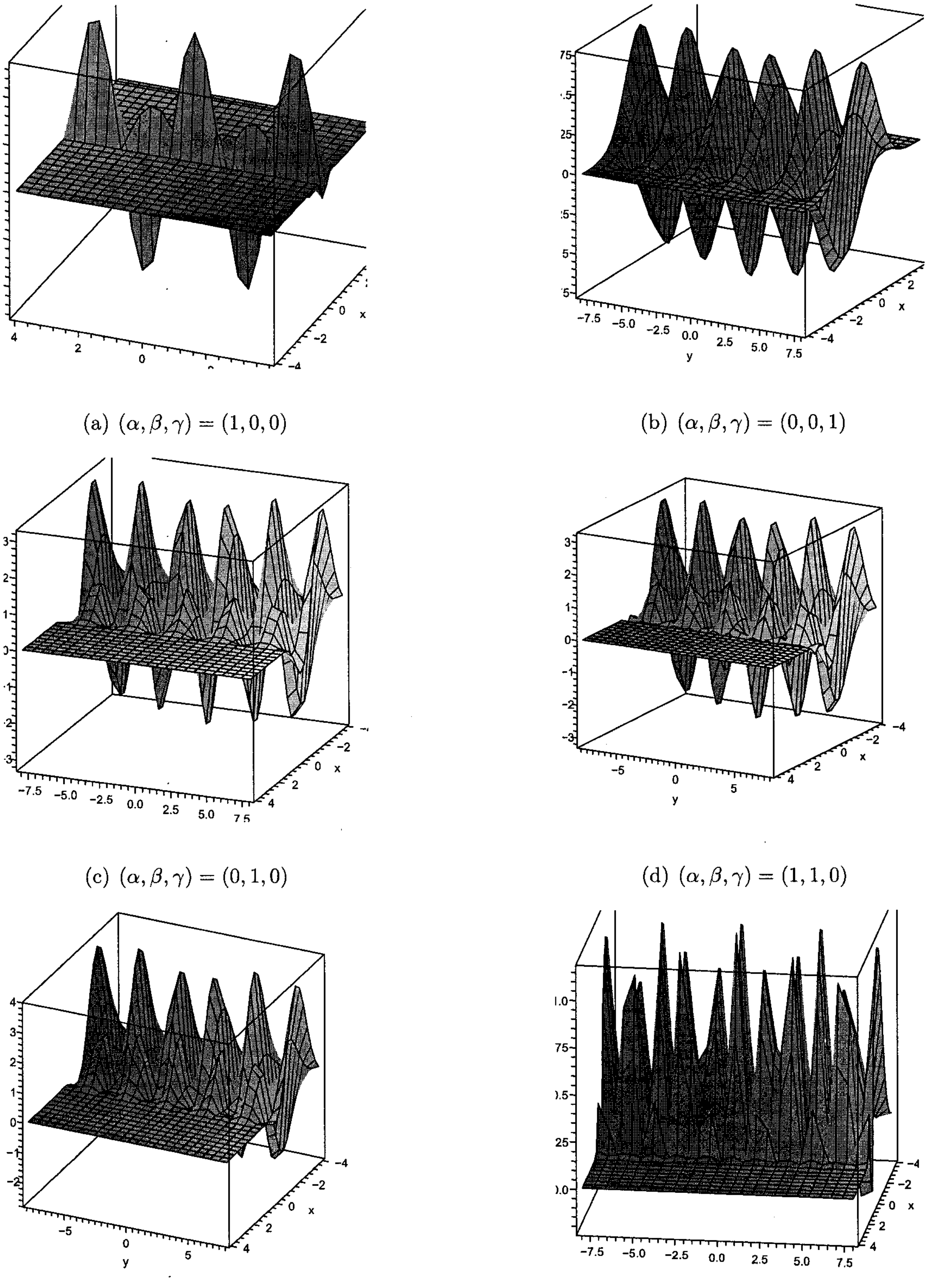
$$\int_{\mathbb{R}} \frac{\mid \hat{\Phi}_n(\gamma) \mid^2}{|\gamma|} d\gamma < \infty, \tag{5.1.53}$$

where $\Phi_n(z) = \phi_n(z).e^{\frac{z}{2}} \sinh^{\frac{1}{2}}(e^z)\cosh^{\frac{1}{4}}(2e^z)$.

Let us plot the graphs of some families of admissible wavelets: $\Phi_{\alpha,\beta,\gamma}$, defined by

$$\Phi_{\alpha,\beta,\gamma}(x,y) = \frac{\left[\alpha\cos(y)H(x) + \beta\cos(2y)Me(x) + \gamma\cos(4y)Mo(x)\right]}{e^{\frac{x}{2}} \sinh^{\frac{1}{2}}(e^x)\cosh^{\frac{1}{4}}(2e^x)}, \tag{5.1.54}$$

where α, β, γ are real parameters.



(a) $(\alpha, \beta, \gamma) = (1, 0, 0)$

(b) $(\alpha, \beta, \gamma) = (0, 0, 1)$

(c) $(\alpha, \beta, \gamma) = (0, 1, 0)$

(d) $(\alpha, \beta, \gamma) = (1, 1, 0)$

(e) $(\alpha, \beta, \gamma) = (0, 1, 1)$

(f) $(\alpha, \beta, \gamma) = (1, 0, 1)$

Figure 5.6: The wavelet $\Phi_{\alpha,\beta,\gamma}$ (5.1.54), at different values of the parameter (α, β, γ).

5.2 Existence of affine-type frames on the previous 2d-surfaces

5.2.1 Affine frame on a cylinder

As it was proved before, we consider our Hilbert space to be of the form

$$\mathfrak{H}_J = \bigoplus_{j\in J}\left[\frac{e^{ij\theta}}{\sqrt{2\pi}} \otimes L^2(\mathbb{R}, dz)\right], \tag{5.2.55}$$

and let $\Psi_J(z,\theta) = \sum_{j\in J} \frac{e^{ij\theta}}{\sqrt{2\pi}}\phi_j(z) \in \mathfrak{H}_J$ be an admissible vector of U_J.

Let consider the two subspaces of $\mathfrak{H}_J^{\pm}$ of $\mathfrak{H}_J$, defined by

$$\mathfrak{H}_J^{\pm} = \bigoplus_{j\in J}\left[\frac{e^{ij\theta}}{\sqrt{2\pi}} \otimes H^2_{\pm}(\mathbb{R})\right], \tag{5.2.56}$$

where

$$H^2_{\pm}(\mathbb{R}) = \left\{f \in L^2(\mathbb{R}) : \hat{f}(k) = 0 \text{ for } k \lessgtr 0\right\}, \tag{5.2.57}$$

are the known Hardy spaces. Let $N \in \mathbb{N}^*$. For $k = 1, ..., N; m, n \in \mathbb{Z}$, let us consider the set of functions $\Psi_J^{k,n,m}$ defined by

$$\Psi_J^{k,n,m}(z,\theta) = \sum_{j\in J} |a|^{-\frac{n}{2}} \frac{e^{ij\left(\theta - \frac{2\pi k}{N}\right)}}{\sqrt{2\pi}}\phi_j(a^{-n}z - mb) \tag{5.2.58}$$

Theorem 5.2.1 *Let $\Psi_J \in \mathfrak{H}_J^{+}$ be such that $\forall j \in J$, supp$\left(\hat{\phi}_j\right) \subset [l_j, L_j]$, where $0 \le l_j < L_j < \infty$ and let $a > 1$ and $b > 0$ be such that*

(1) There exist A, B such that

$$0 < A \le \sum_{n\in\mathbb{Z}} |\,\hat{\phi}_j(a^n\gamma)\,|^2 \le B < \infty, \quad \forall\gamma \ge 0, \forall j \in J, \tag{5.2.59}$$

(2) $(L_j - l_j) \le 1/b, \quad \forall j \in J.$

Then for all $f_J(z,\theta) = \sum_{j\in J} \frac{e^{ij\theta}}{\sqrt{2\pi}} f_j(z) \in \mathfrak{H}_J$,

$$b^{-1}AN\sum_{j\in J}\int_0^\infty |\hat{f}_j(\gamma)|^2\,d\gamma \le \sum_{k,nm} |\langle f, \Psi_J^{k,n,m}\rangle|^2 \le b^{-1}BN\sum_{j\in J}\int_0^\infty |\hat{f}_j(\gamma)|^2\,d\gamma. \tag{5.2.60}$$

Thus, $\left\{\Psi_J^{k,n,m}\right\}_{k=1,\ldots,N;n,m\in\mathbb{Z}}$ *is a frame for* $\mathfrak{H}_J^+$ *with bounds* $b^{-1}AN, \quad b^{-1}BN$.

Proof. The important and easy thing to see here is that

$$|\langle f_J, \Psi_J^{k,n,m}\rangle|^2 = \sum_{j\in J} |\langle f_j, D_{a^n}T_{mb}\phi_j\rangle|^2, \tag{5.2.61}$$

where $D_{a^n}T_{mb}\phi_j(z) = |a|^{-\frac{n}{2}}\phi_j\left(a^{-n}z - mb\right)$,

and it's known that

$$\sum_{m\in\mathbb{Z}} |\langle f_j, D_{a^n}T_{mb}\phi_j\rangle|^2 = b^{-1}\int_0^\infty |\hat{f}_j(\gamma)|^2 |\hat{\phi}_j(a^n\gamma)|^2\,d\gamma. \tag{5.2.62}$$

So, we have

$\sum_{m,n\in\mathbb{Z}} |\langle f_j, D_{a^n}T_{mb}\phi_j\rangle|^2 = b^{-1}\int_0^\infty |\hat{f}_j(\gamma)|^2 \sum_{n\in\mathbb{Z}} |\hat{\phi}_j(a^n\gamma)|^2\,d\gamma.$

Now, using the fact that $A \le \sum_{n\in\mathbb{Z}} |\hat{\phi}_j(a^n\gamma)|^2 \le B$, we have

$$b^{-1}AN\sum_{j\in J}\int_0^\infty |\hat{f}_j(\gamma)|^2\,d\gamma \le \sum_{k,nm} |\langle f, \Psi_J^{k,n,m}\rangle|^2 \le b^{-1}BN\sum_{j\in J}\int_0^\infty |\hat{f}_j(\gamma)|^2\,d\gamma. \tag{5.2.63}$$

As before, the same theorem can be proved for $\mathfrak{H}_J^-$

The following gives a frame for $\mathfrak{H}_J$,

Theorem 5.2.2 *Let* $\Psi_J^1, \Psi_J^2 \in \mathfrak{H}_J$, *with* $\Psi_J^1(z,\theta) = \sum_{j\in J} \frac{e^{ij\theta}}{\sqrt{2\pi}}\phi_j^1(z)$ *and* $\Psi_J^2(z,\theta) = \sum_{j\in J} \frac{e^{ij\theta}}{\sqrt{2\pi}}\phi_j^2(z) \quad \forall j \in J$ *and supp* $\left(\hat{\phi}_j^1\right) \subset [-L_j, -l_j]$ *and supp* $\left(\hat{\phi}_j^2\right) \subset [l_j, L_j]$, *where* $0 \le l_j < L_j < \infty\, \forall j \in J$. *Let* $a > 1$, $b > 0$ *be such that*

(1) There exist A, B *such that*

$$A \le \sum_{n\in\mathbb{Z}} |\hat{\phi}_j^1(a^n\gamma)|^2 \le B, \quad \gamma > 0\; a.e.\, \forall j \in J \tag{5.2.64}$$

$$A \le \sum_{n\in\mathbb{Z}} |\hat{\phi}_j^2(a^n\gamma)|^2 \le B, \quad \gamma < 0\; a.e.\, \forall j \in J. \tag{5.2.65}$$

(2) $(L_j - l_j) \leq 1/b, \quad j \in J.$

Then, $\left\{\Psi_J^{1;k,n,m}, \Psi_J^{2;k,n,m}\right\}_{k=1,\ldots,N;n,m\in\mathbb{Z}}$ *is a frame for* $\mathfrak{H}_J$ *for all* $1 < a < \min\{L_j/l_j\}$ *and* $o < b < \min\{L_j - l_j\}$.

This theorem gives a condition on Ψ_J whose Fourier transforms of ϕ_j are not necessary compactly supported so that $\left\{\Psi_J^{k,n,m}, a, b\right\}$ generates a frame for $\mathfrak{H}_J$.

Theorem 5.2.3 *Let* $\Psi_J(z,\theta) = \sum_{j\in J} \frac{e^{ij\theta}}{\sqrt{2\pi}}\phi_j(z) \in \mathfrak{H}_j$ *and* $a > 1$ *such that*

(1) There exist A, B *such that*

$$A \leq \sum_{n\in\mathbb{Z}} \mid \hat{\phi}_j(a^n\gamma) \mid^2 \leq B, \quad \gamma \in \hat{\mathbb{R}}\, a.e., \forall j \in J. \tag{5.2.66}$$

(2)

$$\lim_{b\longrightarrow 0} \sum_{k\neq 0} \beta_j\,(k/b)^{\frac{1}{2}}\,\beta_j\,(-k/b)^{\frac{1}{2}} = 0, \quad \forall j \in J, \tag{5.2.67}$$

where $\beta_j(s) = ess\sup_{|\gamma|\in[1,a]} \sum_{n\in\mathbb{Z}} \mid \hat{\phi}_j(a^n\gamma)\hat{\phi}_j(a^n\gamma - s) \mid$.

Then, there exists $b_0 > 0$ *such that* $\left\{\Psi_J^{k,n,m}, a, b\right\}$ *generates a frame for* $\mathfrak{H}_J$, *for each* $0 < b < b_0$.

Proof. Let $f_J(z,\theta) = \sum_{j\in J} \frac{e^{ij\theta}}{\sqrt{2\pi}} f_j(z) \in \mathfrak{H}_J$.

By following step by step the proof of Theorem 5.1.6 in [33], it follows that there exits $b_j > 0$ such that

$$b^{-1}A \parallel f_j \parallel^2_{L^2(\mathbb{R})} \leq \sum_{n,m\in\mathbb{Z}} \mid \langle f_j, D_{a^n}T_{mb}\phi_j\rangle \mid^2 \leq b^{-1}B \parallel f_j \parallel^2_{L^2(\mathbb{R})}, \tag{5.2.68}$$

for $0 < b < b_j$.

By taking $b_0 = \min_{j\in J}\{b_j\}$, the proof is complete.

5.3 Multiresolution analysis

Definition 5.3.1 *A frame multiresolution analysis for $L^2(\mathbb{R})$ consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ and a function $\phi \in V_0$ such that [14]*

1. $...V_{-1} \subset V_0 \subset V_1...$
2. $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$
3. $V_j = D^j V_0$
4. $f \in V_0 \Rightarrow T_k f \in V_0 \quad k \in \mathbb{Z}$
5. $\{T_k\phi\}_{k\in\mathbb{Z}}$ *is a frame for* V_0

5.3.1 A multiresolution analysis on $L^2(\mathfrak{C}, dzd\theta)$

Let $J \subset \mathbb{Z}$ such that $| J |< \infty$ and let consider the Hilbert space $\mathcal{H}_J$

$$\mathcal{H}_J = \bigoplus_{j\in J} \left[\frac{e^{ij\theta}}{\sqrt{2\pi}} \otimes L^2(\mathbb{R})\right]. \tag{5.3.69}$$

Let $\tilde{V}_J$ be a closed subspace of $\mathcal{H}_J$. So, there exist $\{V_j\}_{j\in J}$, a sequence of closed subspace of $L^2(\mathbb{R})$ such that

$$\tilde{V}_J = \bigoplus_{j\in J} \left[\frac{e^{ij\theta}}{\sqrt{2\pi}} \otimes V_j\right]. \tag{5.3.70}$$

We will say that $\tilde{V}_J$ is nowhere trivial if $\{0\} \neq V_j \neq L^2(\mathbb{R})$ for all $j \in J$.

Let $N \in \mathbb{N}^*$, $\quad k \in \mathbb{Z}$ and let $\phi \in \mathcal{H}_J$ such that $\phi(z,\theta) = \sum_{j\in J} \frac{e^{ij\theta}}{\sqrt{2\pi}}\phi_j(z)$. For $l = 1, ..., N$, let

$$\left(T_{(k,l)}\phi\right)(z,\theta) = \frac{e^{ij(\theta-\frac{2\pi l}{N})}}{\sqrt{2\pi}}\phi_j(z-k). \tag{5.3.71}$$

Definition 5.3.2 *Let $J \subset \mathbb{Z}$ such that $\mid J \mid < \infty$. A frame multiresolution analysis for $\mathcal{H}_J = \bigoplus_{j \in J} \left[\frac{e^{ij\theta}}{\sqrt{2\pi}} \otimes L^2(\mathbb{R}) \right]$ consists of a sequence of nowhere trivial closed subspace $\left\{ \tilde{V}_J^n = \bigoplus_{j \in J} \left[\frac{e^{ij\theta}}{\sqrt{2\pi}} \otimes V_j^n \right] \right\}_{n \in \mathbb{Z}}$ of $\mathcal{H}_J$ and a function $\phi \in \tilde{V}_J^0$ such that*

1. $...\tilde{V}_J^{-1} \subset \tilde{V}_J^0 \subset \tilde{V}_J^1 ...$
2. $\overline{\cup_{n \in \mathbb{Z}} \tilde{V}_J^n} = \mathcal{H}_J$
3. $\tilde{V}_J^n = D^n \tilde{V}_J^0$
4. $f \in \tilde{V}_J^0 \Rightarrow T_{(k,l)} f \in \tilde{V}_J^0 \quad k \in \mathbb{Z}, l = 1, ..., N$
5. $\left\{ T_{(k,l)} \phi \right\}_{k \in \mathbb{Z};\, l=1,\dots,N}$ *is a frame for* $\tilde{V}_J^0$

Theorem 5.3.1 *A sequence of nowhere trivial closed subspace $\left\{ \tilde{V}_J^n \right\}_{n \in \mathbb{Z}}$ of $\mathcal{H}_J$ and a function $\phi(z, \theta) = \sum_{j \in J} \frac{e^{ij\theta}}{\sqrt{2\pi}} \phi_j z$ is frame multiresolution analysis for $\mathcal{H}_J$ if and only if for all $j \in J$, the sequence $\left\{ V_j^n \right\}_{n \in \mathbb{Z}}$ and the function ϕ_j is a frame multiresolution analysis of $L^2(\mathbb{R})$*

From this theorem, we can conclude that the problem of having a frame multiresolution analysis of $\mathcal{H}_J$ is reduced to $|J|$ copies of a frame multiresolution analysis of $L^2(\mathbb{R})$.

So, we can have a frame multiresolution analysis on the Hilbert spaces $L^2(\mathfrak{S}, d\sigma)$ by transporting the one on the cylinder using the unitary maps $\tilde{V}$.

Chapter 6

Wavelets on the Paraboloid Using a Group-theoretical Approach

6.1 Preliminaries

Using Mackey's theory of induced representations, we present a wavelet transform on some general two-dimensional surfaces given by the equation $z = (x^2 + y^2)^\alpha$, where α is a real number, by using a specific group. The wavelet transform on the paraboloid is obtained by taking $\alpha = 1$.

6.2 Wavelet transforms on the parabola

For time-frequency like transforms, the generic group, consisting of 3×3 matrices, that we are suggesting is:

$$G = \left\{ \begin{pmatrix} ar_\theta & \mathbf{b} \\ \mathbf{0}^T & f(a) \end{pmatrix} \middle| \; a > 0, \;\; \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2, \;\; \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \;\; \theta \in [0, 2\pi) \right\}, \quad (6.2.1)$$

where $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ satisfies $f(a_1 a_2) = f(a_1)f(a_2)$, $f(a) \neq 0$ for any a, and

$$r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Note that if f is a continuous function, then the only possibility is $f(a) = a^\alpha$ for some real number α.

The subgroup of G with $\mathbf{b} = \mathbf{0}$, then leaves the surface $\mathfrak{S}$, defined by the equation

$$z = f(\|\mathbf{v}\|), \quad \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \qquad (6.2.2)$$

invariant.

6.3 Paraboloid with apex removed

Consider the special case where $f(a) = a^2$, with typical group element:

$$g = (\mathbf{b}, a, \theta) = \begin{pmatrix} ar_\theta & \mathbf{b} \\ \mathbf{0}^T & a^2 \end{pmatrix} \qquad (6.3.3)$$

and multiplication rule:

$$(\mathbf{b}, a, \theta)(\mathbf{b}', a', \theta') = (ar_\theta \mathbf{b}' + a'^2 \mathbf{b}, aa', \theta + \theta').$$

We denote this group by $G_{\mathfrak{P}}$. Its subgroup, with $\mathbf{b} = \mathbf{0}$, leaves the paraboloid $z = x^2 + y^2$, with the point $(0,0,0)$ removed, invariant. We denote the resulting two-dimensional surface by $\mathfrak{P}$. The two invariant measures on $G_{\mathfrak{P}}$ are:

$$d\mu_\ell(g) = \frac{d\mathbf{b}\ da\ d\theta}{a^3} \qquad \text{and} \qquad d\mu_r(g) = \frac{d\mathbf{b}\ da\ d\theta}{a^5}, \qquad d\mathbf{b} = db_1\ db_2. \tag{6.3.4}$$

Define the two abelian subgroups of $G_{\mathfrak{P}}$:

$$H = \left\{ h = \begin{pmatrix} \mathbb{I}_2 & \mathbf{b} \\ \mathbf{0}^T & 1 \end{pmatrix} \mid \mathbf{b} \in \mathbb{R}^2 \right\}, \quad \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{6.3.5}$$

and

$$P = \left\{ p = \begin{pmatrix} ar_\theta & \mathbf{0} \\ \mathbf{0}^T & a^2 \end{pmatrix} \mid a > 0, \quad \theta \in [0, 2\pi) \right\}. \tag{6.3.6}$$

Then for any $g \in G_{\mathfrak{P}}$, one has a decomposition $g = ph$, with

$$p = \begin{pmatrix} ar_\theta & \mathbf{0} \\ \mathbf{0}^T & a^2 \end{pmatrix} \qquad h = \begin{pmatrix} \mathbb{I}_2 & a^{-1}r_{-\theta}\mathbf{b} \\ \mathbf{0}^T & 1 \end{pmatrix}. \tag{6.3.7}$$

The subgroup H acts as a *shear* group on $\mathbb{R}^3$, while as noted earlier, P leaves $\mathfrak{P}$ invariant. In fact $P \simeq G_{\mathfrak{P}}/H$ and is homeomorphic to $\mathfrak{P}$, which can also be looked upon as the orbit:

$$\mathcal{O} = \left\{ p \begin{pmatrix} \mathbf{e}_1 \\ 1 \end{pmatrix} \middle| \ p \in P, \ \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \tag{6.3.8}$$

i.e., $P \simeq G_{\mathfrak{P}}/H \simeq \mathfrak{P} \simeq \mathcal{O}$. Thus P has a natural action on $\mathfrak{P}$,

$$p \begin{pmatrix} \mathbf{v} \\ z \end{pmatrix} = \begin{pmatrix} ar_\theta \mathbf{v} \\ a^2 z \end{pmatrix}, \qquad p \in P, \qquad z = \|\mathbf{v}\|^2,$$

under which the measure

$$d\mu_{\mathfrak{P}} = \frac{d\mathbf{v}}{\|\mathbf{v}\|^2} = \frac{dx\ dy}{z} \tag{6.3.9}$$

is invariant.

6.3.1 Lie algebra and coadjoint action

The Lie algebra $\mathfrak{g}_{\mathfrak{P}}$ of $G_{\mathfrak{P}}$ is generated by the four elements:

$$D = \begin{pmatrix} \mathbb{I}_2 & \mathbf{0} \\ \mathbf{0}^T & 2 \end{pmatrix}, \quad J = \begin{pmatrix} -\omega & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix}, \quad P_i = \begin{pmatrix} \mathbb{O}_2 & \mathbf{e}_i \\ \mathbf{0}^T & 0 \end{pmatrix}, \quad i = 1, 2, \tag{6.3.10}$$

where,

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbb{O}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad (\mathbf{e}_i)_j = \delta_{ij}.$$

The generators satisfy the commutation relations,

$$\begin{aligned} [D, J] &= 0 = [P_1, P_2], \qquad [D, P_i] = -P_i, \qquad i = 1, 2, \\ [J, P_1] &= P_2, \qquad [J, P_2] = -P_1. \end{aligned} \tag{6.3.11}$$

It ought to be noted here that $\mathfrak{g}_{\mathfrak{P}}$ looks similar to the Lie algebra $sim(2)$ of the $SIM(2)$ group, which consists of 2×2 matrices of the type

$$\begin{pmatrix} ar_\theta & \mathbf{b} \\ \mathbf{0}^T & 1 \end{pmatrix},$$

which can indeed be obtained, as a special case of the generic group G in (6.2.1), by choosing $f(a) = 1$ for all a. However, in the case of $sim(2)$, the commutator between D and P_i is $[D, P_i] = P_i$ whereas in the present case it is $[D, P_i] = -P_i$

A general Lie algebra element has the form

$$X = \alpha_1 D + \alpha_2 J + \beta_1 P_1 + \beta_2 P_2 = \begin{pmatrix} \mathfrak{s}(\boldsymbol{\alpha}) & \boldsymbol{\beta} \\ \mathbf{0}^T & 2\alpha_1 \end{pmatrix}, \quad \alpha_i, \beta_i \in \mathbb{R}, \tag{6.3.12}$$

with

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \quad \text{and} \quad \mathfrak{s}(\boldsymbol{\alpha}) = \begin{pmatrix} \alpha_1 & -\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix}.$$

The adjoint action of $g \in G_{\mathfrak{P}}$ on $X \in \mathfrak{g}_{\mathfrak{P}}$ is then:

$$X \longrightarrow \mathrm{Ad}_g(X) := gXg^{-1} = X' = \begin{pmatrix} \mathfrak{s}(\boldsymbol{\alpha}) & a^{-1}r_\theta\boldsymbol{\beta} + a^{-2}[\alpha_1\mathbb{I}_2 + \alpha_2\omega]\mathbf{b} \\ \mathbf{0}^T & 2\alpha_1 \end{pmatrix}. \quad (6.3.13)$$

Thus, the action on the coordinate vectors $\mathbf{x} = \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} \in \mathbb{R}^4$ is then given by the 4×4 matrix $\mathrm{Ad}(g)$:

$$\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} \longrightarrow \begin{pmatrix} \boldsymbol{\alpha}' \\ \boldsymbol{\beta}' \end{pmatrix} = \mathrm{Ad}(g) \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}$$

with

$$\mathrm{Ad}(g) = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbb{O}_2 \\ a^{-2}\mathbf{b} & a^{-2}\omega\mathbf{b} & a^{-1}r_\theta \end{pmatrix}. \quad (6.3.14)$$

From this, the coadjoint action on the dual vectors $\mathbf{x}^T = (\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T) = (\alpha_1, \alpha_2, \beta_1, \beta_2)$ is calculated to be

$$\mathbf{x}^T \longrightarrow \mathbf{x}^{T\,\prime} = \mathbf{x}^T\mathrm{Ad}(g)^{-1} = (\alpha_1 - a^{-1}\boldsymbol{\beta}^T r_{-\theta}\mathbf{b},\ \alpha_2 - a^{-1}\boldsymbol{\beta}^T r_{-\theta}\omega\mathbf{b},\ a\boldsymbol{\beta}^T r_{-\theta}). \quad (6.3.15)$$

In particular, the action on the vector $(0, 0, \mathbf{e}_1^T)$ is seen to be

$$(0, 0, \mathbf{e}_1^T) \longrightarrow (-a^{-1}\mathbf{e}_1^T r_{-\theta}\mathbf{b},\ -a^{-1}\mathbf{e}_1^T r_{-\theta}\omega\mathbf{b},\ a\mathbf{e}_1^T r_{-\theta}), \quad (6.3.16)$$

and consequently the corresponding coadjoint orbit can be identified with $\mathbb{R}^2 \times (\mathbb{R}^2\backslash\{\mathbf{0}\})$ which is homeomorphic to the cotangent bundle $T^*\mathfrak{P}$. From (6.3.15) it can easily be deduced that this is the only non-trivial coadjoint orbit for this group and hence, up to unitary equivalence, there is only one unitary irreducible representation of $G_{\mathfrak{P}}$. Note also that the group $SIM(2)$ also has a single non-trivial coadjoint orbit, $\mathbb{R}^2 \times (\mathbb{R}^2\backslash\{\mathbf{0}\})$. Moreover, in both cases, the group space itself can be identified with this orbit.

6.3.2 UIR and TF-transform

The single UIR of $G_{\mathfrak{P}}$ can be obtained by inducing from a character of the abelian subgroup H. The representation is carried by the Hilbert space $L^2(\mathfrak{P}, d\mu_{\mathfrak{P}})$ (see (6.3.9)) and it has the form

$$(U(g)\psi)(\mathbf{x}) = (U(\mathbf{b}, a, \theta)\psi)(\mathbf{x}) = \exp(i\frac{\mathbf{v}\cdot\mathbf{b}}{a^2})\psi(g^{-1}\mathbf{x}), \quad \mathbf{x} = \begin{pmatrix} \mathbf{v} \\ z \end{pmatrix} \in \mathfrak{P}, \tag{6.3.17}$$

for all $\psi \in L^2(\mathfrak{P}, d\mu_{\mathfrak{P}})$ and with

$$g^{-1}\mathbf{x} = \begin{pmatrix} a^{-1}r_{-\theta}\mathbf{v} \\ a^{-2}z \end{pmatrix}.$$

6.3.3 Construction of the representation

Since $G_{\mathfrak{P}}/H \simeq P$ we identify points in $G_{\mathfrak{P}}/H$ with $\mathbf{x} \in \mathfrak{P}$ in the manner

$$p \longmapsto \mathbf{x} = \begin{pmatrix} ar_\theta\mathbf{e}_1 \\ a^2 \end{pmatrix} \in \mathfrak{P},$$

with $p \in P$ as in (6.3.6). Next we define the section $\sigma : G_{\mathfrak{P}}/H \longrightarrow G_{\mathfrak{P}}$:

$$\sigma(\mathbf{x}) = \begin{pmatrix} ar_\theta & \mathbf{0} \\ \mathbf{0}^T & a^2 \end{pmatrix}.$$

Thus, for $g_0 \in G_{\mathfrak{P}}$,

$$\sigma(g_0\mathbf{x}) = \begin{pmatrix} a_0 a r_{\theta_0+\theta} & \mathbf{0} \\ \mathbf{0}^T & (a_0 a)^2 \end{pmatrix},$$

from which we compute the cocycle $h : G_{\mathfrak{P}} \times \mathfrak{P} \longrightarrow H$,

$$h(g_0, \mathbf{x}) = \sigma(g_0\mathbf{x})^{-1} g_0 \sigma(\mathbf{x}) = \begin{pmatrix} \mathbb{I}_2 & {a_0}^{-1} a r_{-\theta_0-\theta}\mathbf{b}_0 \\ \mathbf{0}^T & 1 \end{pmatrix}. \tag{6.3.18}$$

We now take the one-dimensional unitary representation of H:

$$V(h) = e^{i\mathbf{e}_1 \cdot \mathbf{b}}, \tag{6.3.19}$$

where h is as in (6.3.5). Then, writing $\mathbf{x}$ as in (6.3.17), so that

$$V(h(g_0^{-1}, \mathbf{x}) = \exp(-i\frac{\mathbf{v}\cdot\mathbf{b}_0}{a_0^2}),$$

we construct the UIR of $G_{\mathfrak{P}}$ on $L^2(\mathfrak{P}, d\mu_{\mathfrak{P}})$ in the standard manner:

$$(U(g)\psi)(\mathbf{x}) = V(h(g^{-1}, \mathbf{x}))^{-1}\psi(g^{-1}\mathbf{x}), \qquad \psi \in L^2(\mathfrak{P}, d\mu_{\mathfrak{P}}),$$

yielding (6.3.17).

6.3.4 Square-integrability and coherent states

This representation is square-integrable, the admissibility condition being

$$c_\psi = (2\pi)^2 \int_{\mathfrak{P}} \frac{|\psi(\mathbf{x})|^2}{\|\mathbf{v}\|^2}\, d\mu_{\mathfrak{P}}(\mathbf{x}) < \infty. \tag{6.3.20}$$

To see this, consider the integral

$$I(\psi, \phi) = \int_{G_{\mathfrak{P}}} |\langle\phi|U(\mathbf{b}, a, \theta)\psi\rangle|^2\, d\mu_\ell(\mathbf{b}, a, \theta), \quad \phi, \psi \in L^2(\mathfrak{P}, d\mu_{\mathfrak{P}}),$$

where $d\mu_\ell$ is the left Haar measure in (6.3.4). After some straightforward manipulations, this integral can be brought into the form:

$$I(\psi, \phi) = (2\pi)^2 \int_0^\infty \int_0^{2\pi} \int_{\mathbb{R}^2} \frac{|\phi(\mathbf{x})|^2}{(x^2+y^2)^2}\, |\psi(a^{-1}r_{-\theta}\mathbf{v},\ a^{-2}z)|^2\ a\ da\ d\theta\ dx\ dy,$$

with

$$\mathbf{x} = \begin{pmatrix} \mathbf{v} \\ z \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad z = \|\mathbf{v}\|^2.$$

Writing $\boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = a^{-1}r_{-\theta}\mathbf{v}$, changing variables, $(a, \theta) \longrightarrow (\xi_1, \xi_2)$, and noting that $\|\boldsymbol{\xi}\|^2 = a^{-2}\|\mathbf{v}\|^2$, the above integral transforms to

$$I(\psi, \phi) = (2\pi)^2\|\phi\|^2 \int_{\mathfrak{P}} \frac{|\psi(\mathbf{y})|^2}{\|\boldsymbol{\xi}\|^2}\, d\mu_{\mathfrak{P}}(\mathbf{y}), \quad \text{where} \quad \mathbf{y} = \begin{pmatrix} \boldsymbol{\xi} \\ \|\boldsymbol{\xi}\|^2 \end{pmatrix}.$$

From this follows the admissibility condition (6.3.20).

If ψ is an admissible vector, we define coherent states $\eta_{\mathbf{b},a,\theta}$ in the usual way

$$\eta_{\mathbf{b},a,\theta} = \frac{1}{\sqrt{C_\psi}}\, U(\mathbf{b},a,\theta)\psi, \qquad (\mathbf{b},a,\theta) \in G_{\mathfrak{P}}, \tag{6.3.21}$$

which will then satisfy the resolution of the identity,

$$\int_{G_{\mathfrak{P}}} |\eta_{\mathbf{b},a,\theta}\rangle\langle\eta_{\mathbf{b},a,\theta}|\; d\mu_\ell(\mathbf{b},a,\theta) = I. \tag{6.3.22}$$

6.3.5 TF-like transform

Using the coherent states $\eta_{\mathbf{b},a,\theta}$ we now define a generalized "time-frequency transform" on $\mathfrak{P}$, the paraboloid with the vertex removed. For $\phi \in L^2(\mathfrak{P}, d\mu_{\mathfrak{P}})$, we define its TF-transform S_ϕ by

$$S_\phi(\mathbf{b},a,\theta) = \langle\eta_{\mathbf{b},a,\theta}|\phi\rangle = \int_{\mathfrak{P}} \exp(-i\frac{\mathbf{v}\cdot\mathbf{b}}{a^2})\; \overline{\psi(a^{-1}r_{-\theta}\mathbf{v},\; a^{-2}z)}\; \phi(\mathbf{x})\; d\mu_{\mathfrak{P}}(\mathbf{x}). \tag{6.3.23}$$

It is also clear now that the same method could be applied, to build analogous transforms, for any of the surfaces (6.2.2), in particular, for surfaces of the type, $z = \|\mathbf{v}\|^\alpha,\;\; \alpha \in \mathbb{R}$. It would be interesting to study in detail the case where $\alpha = -\frac{1}{2}$, i.e., of the surface $z = \dfrac{1}{\sqrt{x^2+y^2}}$, formed by rotating the hyperbola $xy = 1$ about the z axis.

6.4 Some general considerations

It is worthwhile re-deriving the above results for the general group G in (6.2.1). The group multiplication rule now reads:

$$(\mathbf{b},a,\theta)(\mathbf{b}',a',\theta') = ar_\theta\mathbf{b}' + f(a')\mathbf{b}, aa', \theta+\theta') \tag{6.4.24}$$

The invariant measures for this group are:

$$d\mu_\ell(g) = \frac{d\mathbf{b}\; da\; d\theta}{a^3} \qquad \text{and} \qquad d\mu_r(g) = \frac{d\mathbf{b}\; da\; d\theta}{[f(a)]^2\; a}, \qquad d\mathbf{b} = db_1\, db_2, \tag{6.4.25}$$

so that the left invariant measure is the same for all such groups and the modular function $\Delta(g)$ (such that $d\mu_\ell(g) = \Delta(g)\, d\mu_r(g)$) is given by

$$\Delta(g) = \Delta(a) = \left[\frac{f(a)}{a}\right]^2 . \tag{6.4.26}$$

Thus, the the group is unimodular only in the case where $f(a) = a$, i.e., when the invariant surface (6.2.2) is $\mathfrak{C}$, the cone with the apex removed. Of the four generators, $D, J, P_i,\ i = 1, 2$, in the Lie algebra $\mathfrak{g}$ of the group, the last three are the same as in (6.3.10), while the first one has the form

$$D = \begin{pmatrix} \mathbb{I}_2 & \mathbf{0} \\ \mathbf{0}^T & f'(1) \end{pmatrix}, \qquad f'(1) = \frac{d}{da} f(a)\,|_{a=1} . \tag{6.4.27}$$

This leads to the commutation relation

$$[D, P_i] = cP_i, \quad i = 1, 2, \quad c = 1 - f'(1) \tag{6.4.28}$$

all the other commutation relations in (6.3.11) remaining the same. Note that this commutator also vanishes when $f(a) = a$ (i.e., in the situation where the invariant surface is $\mathfrak{C}$). Furthermore, in this case, D is an element in the centre of the Lie algebra $\mathfrak{g}$, which now is just the (trivial) central extension of the Lie algebra of the two-dimensional Euclidean group. We don't expect this group to have square integrable representations and this indeed is the case, as we shall see later.

A general Lie algebra element (see (6.3.12) has the form:

$$X = \begin{pmatrix} \mathfrak{s}(\boldsymbol{\alpha}) & \boldsymbol{\beta} \\ \mathbf{0}^T & f'(1)\alpha_1 \end{pmatrix}, \tag{6.4.29}$$

which under the adjoint action (see (6.3.13)) changes to:

$$X' = \mathrm{Ad}_g(X) = \begin{pmatrix} \mathfrak{s}(\boldsymbol{\alpha}) & \frac{1}{f(a)}[ar_\theta\boldsymbol{\beta} + (\alpha_2\omega - c\alpha_1\mathbb{I}_2)\mathbf{b}] \\ \mathbf{0}^T & f'(1)\alpha_1 \end{pmatrix} . \tag{6.4.30}$$

Similarly, the matrix (6.3.14) changes to

$$\mathrm{Ad}(g) = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbb{O}_2 \\ \dfrac{-c\mathbf{b}}{f(a)} & \dfrac{\omega\mathbf{b}}{f(a)} & \dfrac{ar_\theta}{f(a)} \end{pmatrix}, \tag{6.4.31}$$

yielding the coadjoint action on the dual vectors:

$$\mathbf{x}^T \longrightarrow \mathbf{x}^{T\,\prime} = \mathbf{x}^T \mathrm{Ad}(g)^{-1} = (\alpha_1 + \frac{c}{a}\boldsymbol{\beta}^T r_{-\theta}\mathbf{b},\ \alpha_2 - \frac{1}{a}\boldsymbol{\beta}^T r_{-\theta}\omega\mathbf{b},\ \frac{f(a)}{a}\boldsymbol{\beta}^T r_{-\theta}). \tag{6.4.32}$$

This leads to the action on the vector $(0, 0, \mathbf{e}_1^T)$:

$$(0, 0, \mathbf{e}_1^T) \longrightarrow (ca\mathbf{e}_1^T r_{-\theta}\mathbf{b},\ -a^{-1}\mathbf{e}_1^T r_{-\theta}\omega\mathbf{b},\ \frac{f(a)}{a}\mathbf{e}_1^T r_{-\theta}). \tag{6.4.33}$$

Thus, unless $f(a) = a$, implying $c = 0$, the corresponding coadjoint orbit is identifiable with $\mathbb{R}^2 \times (\mathbb{R}^2 \backslash \{\mathbf{0}\})$ which, in turn, is homeomorphic to the cotangent bundle $T^*\mathfrak{S}$ of the invariant surface (6.2.2). Moreover, this is the only non-trivial coadjoint orbit for this group and hence G has only one unitary irreducible representation.

The invariant measure on $\mathfrak{S}$ is, once again (see (6.3.9)),

$$d\mu_{\mathfrak{S}}(\mathbf{x}) = \frac{d\mathbf{v}}{\|\mathbf{v}\|^2}, \qquad \mathbf{x} = \begin{pmatrix} \mathbf{v} \\ f(\|\mathbf{v}\|) \end{pmatrix} \in \mathfrak{S}, \quad \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2. \tag{6.4.34}$$

The cocycle h (see (6.3.18)) now reads

$$h(g_0, \mathbf{x}) = \sigma(g_0\mathbf{x})^{-1} g_0 \sigma(\mathbf{x}) = \begin{pmatrix} \mathbb{I}_2 & \dfrac{f(a)}{a_0 a} r_{-\theta-\theta_0}\mathbf{b}_0 \\ \mathbf{0}^T & 1 \end{pmatrix}, \tag{6.4.35}$$

with

$$\sigma(x) = \begin{pmatrix} ar_\theta & \mathbf{0} \\ \mathbf{0}^T & f(a) \end{pmatrix}.$$

Once again, we induce a representation of G from the representation (6.3.19) of H. This resulting representation is carried by the Hilbert space $L^2(\mathfrak{S}, d\mu_{\mathfrak{S}})$ and has the form (see (6.3.17)),

$$(U(\mathbf{b}, a, \theta)\psi)(\mathbf{x}) = \exp\left[i\,\frac{f(a^{-1}\|\mathbf{v}\|)}{\|\mathbf{v}\|^2}\ \mathbf{v}\cdot\mathbf{b}\right] \psi(a^{-1}r_{-\theta}\mathbf{v},\ f(a^{-1}\|\mathbf{v}\|)). \tag{6.4.36}$$

If we choose $f(a) = a^\alpha$, $\alpha \in \mathbb{R}$, this representation is square-integrable, provided $\alpha \neq 1$, the admissibility condition being,

$$\frac{(2\pi)^2}{|\alpha - 1|} \int_{\mathfrak{S}} \frac{|\psi(\mathbf{x})|^2}{\|\mathbf{v}\|^{2\alpha-2}} \, d\mu_{\mathfrak{S}} < \infty. \tag{6.4.37}$$

Also, in this case, the representation (6.4.36) takes the form

$$(U(\mathbf{b}, a, \theta)\psi)(\mathbf{x}) = \exp\left[i\frac{v^{\alpha-2}}{a^\alpha}\, \mathbf{v}\cdot\mathbf{b}\right] \psi(a^{-1}r_{-\theta}\mathbf{v},\, (a^{-1}v)^\alpha), \qquad v = \|\mathbf{v}\|. \tag{6.4.38}$$

Furthermore, for $\alpha \neq 1$, the Fourier-like map, $\mathcal{F} : L^2(\mathfrak{S}, d\mu_{\mathfrak{S}}) \longrightarrow L^2(\mathbb{R}^2, d\mathbf{u})$,

$$(\mathcal{F}\psi)(\mathbf{u}) = \frac{|\alpha - 1|^{\frac{1}{2}}}{2\pi} \int_{\mathfrak{S}} e^{-iv^{\alpha-2}\mathbf{v}\cdot\mathbf{u}}\, \psi(\mathbf{v}, v^\alpha)\, v^{\alpha-1}\, d\mu_{\mathfrak{S}}, \tag{6.4.39}$$

is unitary and writing $\widehat{U}(\mathbf{b}, a, \theta) = \mathcal{F}(U(\mathbf{b}, a, \theta)\mathcal{F}^{-1}$, we get for $\widehat{\psi} \in L^2(\mathbb{R}^2, d\mathbf{u})$,

$$(\widehat{U}(\mathbf{b}, a, \theta)\widehat{\psi})(\mathbf{u}) = a^{\alpha-1}\,\widehat{\psi}(g^{-1}\mathbf{u}), \quad \text{where} \quad g\mathbf{u} = \frac{ar_\theta\mathbf{u} + \mathbf{b}}{a^\alpha}. \tag{6.4.40}$$

6.5 Special cases

It is worthwhile to look at a few special cases.

6.5.1 Case of the cone without the apex

When $f(a) = a$, i.e., $\alpha = 1$ and $\mathfrak{S} = \mathfrak{C}$, the cone without the apex, the admissibility condition above cannot be satisfied and hence, in this case, the representation is not square-integrable. Also, now $c = 0$ and from (6.4.32) we see that there is a continuum of coadjoint orbits (one for each value of α_1). The transformation (6.4.33) now becomes

$$(0, 0, \mathbf{e}_1^T) \longrightarrow (0,\ -a^{-1}\mathbf{e}_1^T r_{-\theta}\omega\mathbf{b},\ \mathbf{e}_1^T r_{-\theta}), \tag{6.5.41}$$

implying that each one of these orbits, is identifiable with the cotangent bundle of the circle or the cone without the apex itself.

6.5.2 Case of the 2-dimensional plane with the origin removed

In this case, $\alpha = 0$ and we recover the affine group in two dimensions. The transformation $\|\mathbf{v}\| \longrightarrow \dfrac{1}{\|\mathbf{v}\|}$ and the unitary map $\dfrac{\psi(\mathbf{x})}{\|\mathbf{v}\|} \longrightarrow \psi(\mathbf{v})$ between $L^2(\mathfrak{S}, d\mu_{\mathfrak{S}})$ and $L^2(\mathbb{R}^2, d\mathbf{v})$ brings (6.4.36) and (6.4.37) into their well-known standard forms

6.6 A more general situation

A much larger class of surfaces can be obtained if we look at a group of the type:

$$G = \left\{ \begin{pmatrix} A & \mathbf{b} \\ \mathbf{0}^T & \Delta(A)^{\frac{\alpha}{2}} \end{pmatrix} \middle| \ \alpha \in \mathbb{R}, \ \ \Delta(A) = \det[A], \ \ \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2, \ \ \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}, \tag{6.6.42}$$

where $A \in K$, which is a subgroup of $GL(2, \mathbb{R})$ such that it has open free orbits in $\mathbb{R}^2$ and $\det A > 0$.

The action of G on the homogeneous space G/H, with H as in (6.3.5), is given by:

$$\begin{pmatrix} \mathbb{I}_2 & \mathbf{x} \\ \mathbf{0}^T & 1 \end{pmatrix} \xrightarrow{g} \begin{pmatrix} \mathbb{I}_2 & \dfrac{A\mathbf{x} + \mathbf{b}}{\Delta(A)^{\frac{\alpha}{2}}} \\ \mathbf{0}^T & 1 \end{pmatrix}. \tag{6.6.43}$$

Hence G can also be looked upon as a group of transformations of $\mathbb{R}^2$ of the type: $\mathbf{x} \longrightarrow \dfrac{A\mathbf{x} + \mathbf{b}}{\Delta(A)^{\frac{\alpha}{2}}}$, which should be compared to the action given in (6.4.40).

Clearly, the groups considered in the previous sections are all of this type. Another obvious example of a group of this general type is obtained by taking the Lorentz group in two-dimensions with dilations, so that

$$A = a \begin{pmatrix} \cosh\vartheta & \sinh\vartheta \\ \sinh\vartheta & \cosh\vartheta \end{pmatrix}, \qquad a > 0, \quad \vartheta \in \mathbb{R}. \tag{6.6.44}$$

Then $\Delta(A) = a^2$ and the surface $\mathfrak{S} : z = (x^2 - y^2)^{\frac{\alpha}{2}}$ is invariant under the action of the subgroup

$$P = \left\{ p = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0}^T & \Delta(A)^{\frac{\alpha}{2}} \end{pmatrix} \right\}. \tag{6.6.45}$$

It is also the orbit of $\mathbf{e}_1$ under P.

Chapter 7

Localization Operators Associated to Group Representations: Application to the Galilei Group

7.1 Preliminaries and motivation

A Weyl-Heisenberg group $(WH)^n$, is the group $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}/2\pi\mathbb{Z}$, with the binary operation

$$(q_1, p_1, t_1).(q_2, p_2, t_2) = (q_1 + q_2, p_1 + p_2, t_1 + t_2 + q_1p_2), \tag{7.1.1}$$

for all points (q_1, p_1, t_1) and $(q_2, p_2, t_2) \in (WH)^n$, and $t_1 + t_2 + q_1p_2$ is cocycle in quotient group $\mathbb{R}/2\pi\mathbb{Z}$. This group is unimodular with the Haar measure *dqdpdt*. The representation (Schrödinger representation) U of this group on the Hilbert space $L^2(\mathbb{R}^n)$ is defined by:

$$\left(U(q,p,t)f\right)(x) = e^{i(p.x-qp+t)}f(x-q), \quad x \in \mathbb{R}^n. \tag{7.1.2}$$

It is a square integrable, unitary irreducible representation of $(WH)^n$. Daubechies in [18], defined a class of bounded linear operators

$D_{F,\varphi} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ associated to F in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$ and φ in $L^2(\mathbb{R}^n)$ with $\| \varphi \|_{L^2(\mathbb{R}^n)} = 1$ is studied in the context of signal analysis , and

$$\langle D_{F,\varphi}u, v\rangle_{L^2(\mathbb{R}^n)} = (2\pi)^{-n} \int_{L^2(\mathbb{R}^n)} \int_{L^2(\mathbb{R}^n)} F(q,p)\,(u, \varphi_{q,p})_{L^2(\mathbb{R}^n)}\,(\varphi_{q,p}, v)_{L^2(\mathbb{R}^n)}\,dqdp, \tag{7.1.3}$$

for all u, v in $L^2(\mathbb{R}^n)$, where $\varphi_{q,p} = e^{ip.x}\varphi(x-q)$.

Later, it was proved that ([46])

$$\langle D_{F,\varphi}u, v\rangle_{L^2(\mathbb{R}^n)} = \tag{7.1.4}$$
$$\frac{1}{\sqrt{c_\varphi}} \int_{L^2(\mathbb{R}^n} \int_{L^2(\mathbb{R}^n} F(q,p)\,(u, (U(q,p,t)\varphi)_{L^2(\mathbb{R}^n}\,((U(q,p,t)\varphi, v)_{L^2(\mathbb{R}^n}\,dqdp,$$

for all u, v in $L^2(\mathbb{R}^n)$. So, the linear operator $D_{F,\varphi}$ which is called Daubechies operator in [[23], [24]], is the same as the localization operator $L_{F,\varphi} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ associated to the symbol F and admissible wavelet φ for the Schrödinger representation for the Weyl-Heisenberg group $(WH)^n$ on $L^2(\mathbb{R}^n)$.

This idea was generalized to any abstract locally compact group which has a square integrable unitary irreducible representation [46].

Let φ be an admissible vector for an irreducible and square-integrable representation $\pi : G \longrightarrow U(\mathcal{H})$ of a locally compact and Hausdorff group G on a Hilbert space $\mathcal{H}$. Then, we have the resolution of identity:

$$I_{\mathcal{H}} = \frac{1}{C_\varphi} \int_G \mid \pi(g)(\varphi)\rangle\langle\pi(g)(\varphi) \mid d\mu(g), \tag{7.1.5}$$

where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$ and μ is the Haar measure on G.

Let $F \in L^1(G) \cap L^\infty(G)$. Then for all ϕ in $\mathcal{H}$, we define the operator $L_{\varphi,F}\phi$ by:

$$\langle L_{\varphi,F}\phi \mid \psi\rangle = \frac{1}{C_\varphi} \int_G F(g)\,\langle\phi \mid \pi(g)\varphi\rangle\,\langle\pi(g)\varphi \mid \psi\rangle\,d\mu(g), \tag{7.1.6}$$

for all $\psi \in \mathcal{H}$.

The operator $L_{\varphi,F}$ is called a localization operator associated to the symbol F. The following results on $L_{\varphi,F}$ are known [46]

Proposition 7.1.1 *Let $F \in L^p(G)$, $1 \leq p < \infty$. Then there exists a unique bounded linear operator $L_{\varphi,F} : \mathcal{H} \longrightarrow \mathcal{H}$ such that*

$$\| L_{\varphi,F} \|_{\mathcal{H}} \leq \left(\frac{1}{C_\varphi}\right)^{\frac{1}{p}} \| F \|_{L^p(G)},$$

and $L_{\varphi,F}\phi$ is given by (7.1.6) for all ϕ in $\mathcal{H}$ and all simple functions F on G for which $\mu\{g \in G : F(g) \neq 0\} < \infty$.

The proof of this proposition is actually based on the following theorem, the so-call interpolation theorem ,e.g, Chapter 10 of the book [47]:

Riesz-Thorin Theorem 7.1.1 *Let (X, μ) be a measure space and (Y, ν) a σ-finite measure space. Let T be a linear transformation with domain $\mathcal{D}$ consisting of all simple functions f on X such that*

$$\mu\{s \in X : f(s) \neq 0\} < \infty \tag{7.1.7}$$

and such that the range of T is contained in the set of all measurable functions on Y. Suppose that $\alpha_1, \alpha_2, \beta_1$ and β_2 are numbers in $[0, 1]$ and there exist positive constants M_1 and M_2 such that

$$\| Tf \|_{L^{\frac{1}{\beta_j}}(Y)} \leq M_j \| f \|_{L^{\frac{1}{\alpha_j}}(X)}, \; f \in \mathcal{D}, j = 1, 2. \tag{7.1.8}$$

Then for $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_1 + \theta\alpha_2$ and $\beta = (1-\theta)\beta_1 + \theta\beta_2$, we have

$$\| Tf \|_{L^{\frac{1}{\beta}}(Y)} \leq M_1^{(1-\theta)} M_2^{\theta} \| f \|_{L^{\frac{1}{\alpha}}(X)}, \; f \in \mathcal{D}. \tag{7.1.9}$$

In this chapter, we try to consider the symbol σ associated to a localization operator L_σ to be more general. So, let us consider it in the following form: $\sigma : G \longrightarrow \mathcal{B}(\mathcal{H})$, we mean by that $\sigma(g)$ is a.e. a bounded linear operator on $\mathcal{H}$,i.e. $\|\sigma(g)\|_{\mathcal{H}}$ is a measurable function on G.

Now let us define the following objects which are necessary for the next sections

$$\begin{aligned} \Xi^p_{\mathcal{H}}(G) &= \{\sigma : G \longrightarrow \mathcal{B}(\mathcal{H}) : \sigma(g) \in \mathcal{B}(\mathcal{H}), \| \sigma(g) \|_{\mathcal{H}} \in L^p(G, d\mu)\} \\ &= \mathcal{B}(\mathcal{H}) \otimes L^p(G, d\mu) \end{aligned} \tag{7.1.10}$$

For $\sigma \in \Xi^p_{\mathcal{H}}(G)$, let us define the following operator on $\mathcal{H}$ by

$$L_{\phi,\sigma} f = \frac{1}{C_\phi} \int_G \langle f, \sigma(g) [\pi(g)\phi] \rangle \pi(g)\phi d\mu(g). \tag{7.1.11}$$

For any complex valued measurable function λ on G, by taking σ to be $\sigma(g) = \lambda(g)\mathbb{I}_{\mathcal{H}}$, the operator (7.1.11) defines the standard localization operator associated to the symbol λ. So, this definition of the Localization operator is more general than the one known so far.

7.2 Main results

Under the assumptions stated above, we have the following propositions

Proposition 7.2.1 *Let* $\sigma \in \Xi^1_{\mathcal{H}}(G)$. *Then*

$$L_{\phi,\sigma} f = \frac{1}{C_\phi} \int_G \langle f, \sigma(g) [\pi(g)\phi] \rangle \pi(g)\phi d\mu(g)$$

is a bounded linear operator and

$$\|L_{\phi,\sigma}\|_{\mathcal{H}} \leq \frac{1}{C_\phi} \| \|\sigma\|_{\mathcal{H}} \|_{L^1(G)}. \tag{7.2.12}$$

Proof. Let suppose that $\sigma \in \Xi^1_{\mathcal{H}}(G)$. Then for $f, h \in \mathcal{H}$, we have

$$\begin{aligned} | \langle L_{\phi,\sigma} f, h \rangle | &\leq \frac{1}{C_\phi} \int_G | \langle f, \sigma(g) [\pi(g)\phi] \rangle \langle \pi(g)\phi, h \rangle | \, d\mu(g) \\ &\leq \frac{1}{C_\phi} \| f \| . \| h \| \int_G \| \sigma(g) \|_{\mathcal{H}} \, d\mu(g) \\ &= \frac{1}{C_\phi} \| f \| . \| h \| . \| \| \sigma \|_{\mathcal{H}} \|_{L^1(G)}, \end{aligned} \tag{7.2.13}$$

where $\| \phi \| = 1$.

■

Proposition 7.2.2 *Let* $\sigma \in \Xi^2_{\mathcal{H}}(G)$. *Then*

$$L_{\phi,\sigma} f = \frac{1}{C_\phi} \int_G \langle f, \sigma(g) [\pi(g)\phi] \rangle \pi(g)\phi d\mu(g)$$

is a bounded linear operator and

$$\| L_{\phi,\sigma} \|_{\mathcal{H}} \leq \left(\frac{1}{C_\phi} \right)^{\frac{1}{2}} \| \| \sigma \|_{\mathcal{H}} \|_{L^2(G)} . \tag{7.2.14}$$

Proof.

Let $\sigma \in \Xi^2_{\mathcal{H}}(G)$. Using Holder inequality, we have

$$\begin{aligned} | \langle L_{\phi,\sigma} f, h \rangle |^2 &\leq \left(\frac{1}{C_\phi} \right)^2 \left(\int_G | \langle f, \sigma(g) [\pi(g)\phi] \rangle |^2 \, d\mu(g) \right) \times \\ &\quad \left(\int_G | \langle \pi(g)\phi, h \rangle |^2 \, d\mu(g) \right) \\ &\leq \left(\frac{1}{C_\phi} \right)^2 \| f \|^2 \int_G \| \sigma(g) \|^2_{\mathcal{H}} \, d\mu(g) \left(\int_G | \langle \pi(g)\phi, h \rangle |^2 \, d\mu(g) \right) \\ &= \frac{1}{C_\phi} \| f \|^2 . \| h \|^2 . \| \| \sigma \|_{\mathcal{H}} \|^2_{L^2(G)} . \end{aligned} \tag{7.2.15}$$

So, $\| L_{\phi,\sigma} \|_{\mathcal{H}} \leq \left(\frac{1}{C_\phi}\right)^{\frac{1}{2}} \| \| \sigma \|_{\mathcal{H}} \|_{L^2(G)}$.

■

The following is our main result

Theorem 7.2.1 *If $\sigma \in \Xi^p_{\mathcal{H}}(G)$, for $p \in [1,2]$, the operator defined by (7.1.11) is a bounded linear operator on $\mathcal{H}$ and*

$$\|L_{\phi,\sigma}\|_{\mathcal{H}} \leq \left(\frac{1}{C_\phi}\right)^{\frac{1}{p}} \| \|\sigma\|_{\mathcal{H}}\|_{L^p(G)}. \tag{7.2.16}$$

Proof. The proof comes from the proposition 7.2.1 , proposition 7.2.2 and the interpolation theorem 7.1.1. ■

7.3 Application to extended Galilei Group

Now, let G be a locally compact group and H be a closed subgroup of G and $X = G/H$, let ν be a quasi-invariant measure on X, and $\lambda(g,.)$ be the Radon-Nikodym derivative of the transformed measure ν_g, $g \in G$ with respect to ν. Fix a Borel section $\sigma : X \longrightarrow G$. For $g \in G$ and $x \in X$, we define $h(g,x) = \sigma(gx)^{-1} g\sigma(x)$.

Suppose that H has a strongly continuous unitary representation $h \longmapsto V(h), \quad h \in H$, on the Hilbert space $\mathfrak{K}$. Let $B(g,x) = [\lambda(g,x)]^{\frac{1}{2}} V\left(h(g^{-1},x)^{-1}\right)$ and consider the Hilbert space $\tilde{\mathfrak{H}} = \mathfrak{K} \otimes L^2(X, d\nu)$, of functions $\Phi : X \longrightarrow \mathfrak{K}$, which are square integrable in the norm

$$\| \Phi \|^2_{\tilde{\mathfrak{H}}} = \int_X \| \Phi \|^2_{\mathfrak{K}} \, d\nu(x). \tag{7.3.17}$$

The operators $\tilde{U}(g)$, $g \in G$, defined on $\tilde{\mathfrak{H}}$ by $\left(\tilde{U}(g)\Phi\right)(x) = B(g,x)\Phi(g^{-1}x)$, are unitary on $\tilde{\mathfrak{H}}$. Moreover, they are a strongly continuous representation of G. The representation $g \longrightarrow \tilde{U}(g)$ so constructed is called the representation of G induced from the representation V of the subgroup H.

The Galilei group is a ten-parameter group $\mathfrak{G}$ of transformations of Newtonian space-time. An element $g \in \mathfrak{G}$ is of the form

$$g = (b, \mathbf{a}, \mathbf{v}, R)\,, \quad \mathbf{a}, \mathbf{v} \in \mathbb{R}^3,\ R \in \mathcal{SO}(\mathbf{3}), \tag{7.3.18}$$

with the group operation defined by

$$gg' = (b + b', \mathbf{a}, \mathbf{v} + \mathbf{R}\mathbf{v}', RR') \tag{7.3.19}$$

where b is a time parameter and $\mathbf{a}$ a spatial translation, $\mathbf{v}$ is the velocity boost, and R is the spatial rotation.

In quantum mechanics, one needs to work with a central extension of $\mathfrak{G}$ denoted by $\tilde{\mathfrak{G}}$ defined by

$$\tilde{g} = (\theta, b, \mathbf{a}, \mathbf{v}, R)\,, \quad \theta \in \mathbb{R},\ \mathbf{a}, \mathbf{v} \in \mathbb{R}^3,\ R \in \mathcal{SO}(\mathbf{3}), \tag{7.3.20}$$

with the group law defined by

$$\tilde{g}\tilde{g}' = (\theta + \theta' + \xi_{g,g'}, gg')\,, \tag{7.3.21}$$

where $\xi_{g,g'} = m\left[\frac{1}{2}\mathbf{v_1}^2 b_2 + \mathbf{v_1}.R_1.\mathbf{a_2}\right], \quad m = const. > 0.$

It is easy to see the Group $\tilde{\mathfrak{G}}$ can be looked upon as the semidirect product, see [1]:

$$\tilde{\mathfrak{G}} = (\Theta \times \mathcal{T} \times \mathcal{S}) \ltimes \mathcal{K}, \quad \mathcal{K} = \mathcal{V} \ltimes SU(2), \tag{7.3.22}$$

where $\Theta \times \mathcal{T} \times \mathcal{S}$, is an abelian subgroup of $\tilde{\mathfrak{G}}$.

Using Mackey's theory of the induced representation, we get the spin representation $\mathcal{U}^j$ of the group $\tilde{\mathfrak{G}}$ on the Hilbert space $\mathbb{C}^{2j+1} \times L^2(\mathbb{R}^3, dx)$ given by [1]:

$$\left(\mathcal{U}^j(\theta, b, \mathbf{a}, \mathbf{v}, \rho)\,\phi\right)(x) = exp\left[i\left\{\theta + \frac{\mathbf{P^2}}{2m}b + m\mathbf{v}.(\mathbf{x} - \mathbf{a})\right\}\right] \times \mathfrak{D}^j(\rho)\phi\left(R(\rho^{-1})(\mathbf{x} - \mathbf{a})\right). \tag{7.3.23}$$

The resolution of the identity give rise the the reconstruction formula

$$f = \frac{1}{C_\phi} \int_{\tilde{\mathfrak{G}}} \langle f, U^j(\theta, b, \mathbf{a}, \mathbf{v}, \rho)\,\phi\rangle U^j(\theta, b, \mathbf{a}, \mathbf{v}, \rho)\,\phi d\tilde{g}. \qquad (7.3.24)$$

According to what we have proved so far, we can take a symbol σ to have a matrix form as follow:

$$\sigma(\tilde{g}) = \{\sigma_{i,k}(\tilde{g})\}_{1\le i,k\le 2j+1}, \qquad (7.3.25)$$

which is a linear operator on $\mathbb{C}^{2j+1}$, with norm defined by

$$\|\sigma(\tilde{g})\|_{\mathbb{C}^{2j+1}} = \sum_{1\le i,k\le 2j+1} |\sigma_{i,k}(\tilde{g})|. \qquad (7.3.26)$$

In this form, $\sigma(\tilde{g})$ can be defined as a bounded linear operator on $\mathbb{C}^{2j+1}\times L^2(\mathbb{R}^3, dx)$ as follow:

$$(\sigma(\tilde{g})\phi)(x) = \sigma(\tilde{g})(\phi(x)), \text{ which is well defined.} \qquad (7.3.27)$$

So, the corresponding localization operatorL_σ is defined by

$$L_\sigma f = \frac{1}{C_\phi} \int_{\tilde{\mathfrak{G}}} \langle f, U^j_\sigma(\theta, b, \mathbf{a}, \mathbf{v}, \rho)\,\phi\rangle U^j(\theta, b, \mathbf{a}, \mathbf{v}, \rho)\,\phi d\tilde{g}, \qquad (7.3.28)$$

where

$$\left(\mathcal{U}^j_\sigma(\theta, b, \mathbf{a}, \mathbf{v}, \rho)\,\phi\right)(x) = \qquad (7.3.29)$$
$$exp\left[i\left\{\theta + \frac{\mathbf{P^2}}{2m}b + m\mathbf{v}.(\mathbf{x}-\mathbf{a})\right\}\right] \times \sigma(\tilde{g})\left[\mathfrak{D}^j(\rho)\phi\left(R(\rho^{-1})(\mathbf{x}-\mathbf{a})\right)\right].$$

Corollary 7.3.1 *If $\|\sigma(.)\|_{\mathbb{C}^{2j+1}} \in L^p(\tilde{\mathfrak{G}})$ for $p \in [1,2]$,then the Localization operator defined by* (7.3.28) *is a bounded linear operator on $\mathbb{C}^{2j+1}\times L^2(\mathbb{R}^3, dx)$ and*

$$\|L_\sigma\|_{\mathbb{C}^{2j+1}\times L^2(\mathbb{R}^3,dx)} \le \left(\frac{1}{C_\phi}\right)^{1/p} . \|\|\sigma(.)\|_{\mathbb{C}^{2j+1}}\|_{L^p(\tilde{\mathfrak{G}})}, \qquad (7.3.30)$$

and $L_\sigma f$ is given by (7.3.28) *for all f in $\mathbb{C}^{2j+1}\times L^2(\mathbb{R}^3, dx)$ and all simple functions σ on $\tilde{\mathfrak{G}}$ for which $\mu\left\{\tilde{g}\in\tilde{\mathfrak{G}} : \sigma(\tilde{g}) \neq \mathbb{O}\right\} < \infty$.*

Remark : If σ is such that $\sigma_{i,k}(\tilde{g}) = \lambda(\tilde{g})\delta_{i,k}$, for some measurable function λ on $\tilde{\mathfrak{G}}$, we get the standard localization operator.

Conclusion

The theme of this dissertation has been to develop a time-frequency-like and wavelet-like analysis on certain non-Euclidean manifolds.

We first generalized the Gabor-type frame to an arbitrary locally compact abelian group, and presented necessary and sufficient conditions for the convergence of the corresponding frame operators.

Since the 2D-cylinder can be considered as a locally compact abelian group, we associated to it the corresponding Weyl-Heisenberg group, and then constructed a Gabor type frame on it. Next we obtained time-frequency type transforms on several non-Euclidean manifolds which are homeomorphic to the 2D-cylinder.

Thereafter we presented a group theoretical approach for the construction of a wavelet-like transform on the cylinder, and then transferred it to non-Euclidean manifolds as before.

Using the above two techniques, wavelet and time frequency analyses were presented on a number of 2D-manifolds namely, the sphere, the ellipsoid, the one-sheeted hyperboloid, the two sheeted hyperboloid, the paraboloid the and plane. We also constructed a wavelet-like transform on the 2D-paraboloid using a representation of a particular group.

Finally, we studied localization operators associated to certain group representations.

Scientific Publications

1. G. Honnouvo and Ali, S. T." Gabor-type Frames from Generalized Weyl-Heisenberg Groups. cond-mat/0510378 ¡http://front.math.ucdavis.edu/cond-mat/0510378.(submitted)

2. 1 K. Thirulogasanthar, G. Honnouvo and A. Krzyzak" Multi-Matrix Vector Coherent States,"*Annals of Physics,* **314**, 2004, 119-144.

3. 2 K. Thirulogasanthar and G. Honnouvo," Coherent States with Complex Functions,"*International Journal of Theoretical Physics,* vol.**43** No.4, April 2004.

4. 3 G. Honnouvo, M. N. Hounkonnou, "Asymptotics of eigenvalues of the Aharonov-Bohm operator with a strong δ-interaction on a loop," *J. Phys. A: Math. Gen.***37***, 2004,* 693-700.

5. 4 G. Honnouvo, M. N. Hounkonnou and G. Y. H. Avossevou, "von Neumann Quantization of Bohm-Aharonov Operator with with δ-interaction: Scattering Theory, Spectral and Resonance Properties," *J. of Nonlinear Math. Phys.* vol.**11**, 2004, 66-71.

6. K. Thirulogasanthar and G. Honnouvo, " Coherent States for Tensoring Matrices," *Proceedings of the third International Worksop On Contemporary Problems in Mathematical Physics*, 2003.

7. G. Honnouvo, M. N. Hounkonnou, "Spectral Properties of Landau Operator with $\delta-$cylinder interaction ," *Proceedings: Physical and Mathematical Aspects of Symmetries*, edited by J-P. Gazeau, R. Kerner, J-P. Antoine, S. Metens, J-Y. Thibon; IOP Publishing, Bristol 2003.

8. G. Honnouvo, M. N. Hounkonnou, "On a Generalization of the Method by Barbaroux et al. for the Improvement on the Rate of Decay of an Operator Resolvent," *Proceedings of the Seconde International Worksop On Contemporary Problems in Mathematical Physics*, edited by Jan Govaerts, M. Norbert Hounkonnou, Alfred Z. Msezane; Published, 2001.

9. G. Honnouvo, *Groupe des Difféomorphismes préservant la forme volume, Report of D.E.A., 1999.*

Preprints 2006-2007

1. G. Honnouvo and Ali, S. T." Gabor-type Frames from Generalized Weyl-Heisenberg Groups."

2. G. Honnouvo and Ali, S. T. "Walnut representation of the generalized Gabor-type frame operator."

3. G. Honnouvo and Ali, S. T. "Time-frequency-like transforms of square integrable functions on some non-Euclidean manifolds."

4. G. Honnouvo and Ali, S. T. "Wavelet-like transforms on 2-dimensional Surfaces."

5. G. Honnouvo and Ali, S. T. "Wavelets on the parabola using a group theoretic approach."

6. G. Honnouvo and Ali, S. T. "Localization Operators Associated to Group Representation: Application to the Galilei Group."

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