Linking Procedural and Conceptual Understandings of Fractions During Learning and Instruction with Fifth- and Sixth-Grade Students: An Evaluation of Hiebert's *Sites* Approach

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Abstract

Linking Procedural and Conceptual Understandings of Fractions During Learning and Instruction with Fifth- and Sixth-Grade Students: An Evaluation of Hiebert's *Sites* Approach

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The study's objective was to empirically investigate an instructional approach (Hiebert, 1984) theorized to foster students' abilities to link conceptual and procedural understandings during the learning of mathematics. Seventy-two fifth- and sixth-grade students were randomly assigned to the treatment (n = 36) and a control condition (n = 36) and were administered a validated paper and pencil test (Saxe, Gearhart, & Nasir, 2001) of fractions knowledge and a researcher-developed fractions knowledge interview before and after instruction. Doubly multivariate analyses of covariance revealed that, consistent with the research hypotheses, students in the treatment condition demonstrated greater improvements in conceptual understandings of fractions ($p < .01$) and in ability to link their knowledge of fractions procedures to their knowledge of fractions concepts ($p < .05$). Contrary to the research hypotheses, control group students demonstrated comparable gains to students in the treatment group in procedural knowledge of fractions. The findings point to important instructional considerations for the effective support of students' development of conceptual knowledge and procedural skill related to specific points in the problem-solving process, and, more broadly, to a specific time in the course of their mathematics education.

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And lastly, but of no lesser importance, to my husband Peter, thank you for teaching me to keep my eye on the bigger picture and how to get there. And thank you, Peter, for your everlasting patience.

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Statement of the Problem

Research conducted over the past several decades by the mathematics education community has revealed a widespread and alarming trend: Canadian and U.S. students are not learning mathematics well. This commonly reported finding, revealing unacceptable levels of achievement and performance, has been shown to be true at both the elementary and secondary levels (Kilpatrick, Swafford, & Findell, 2001; National Assessment of Educational Progress [NAEP], 2005; Organization for Economic Co-operation and Development [OECD], 2004; Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999) and across a range of mathematical topics such as place-value (Fuson & Briars, 1990), addition and subtraction (Hiebert & Wearne, 1996), fractions (Ball, 1993; Bright, Behr, Post, & Wachsmuth, 1988; Mack, 1990, 1995, 2001; Saxe, Taylor, McIntosh, & Gearhart, 2005), and decimals (Hiebert, Wearne, & Taber, 1992). Empirical evidence and reports on student assessments all converge at the same place: most students develop a strong command of low-level procedural skills but few are proficient at higher levels of performance in mathematics (Kilpatrick et al., 2001; NAEP, 2005; OECD, 2004).

Findings from the 2005 NAEP mathematics report illustrate this well. The results revealed that 80% of fourth grade students and 69% of eighth grade students function at, or just above, the most basic proficiency level in mathematics. It should come as no surprise, then, that only 56% of fourth grade students could correctly identify a point on a number line (NAEP, 2005). Similarly, the 2003 Program for International Student Assessment (PISA) results indicated that only 6% of Canada's 15- and 16-year-olds achieved the highest level of performance (Level 6) on assessment items measuring quantitative reasoning, a strand of mathematics that involves number and operational sense (OECD, 2004). In the realm of problem solving, only one quarter of Canadian young adults demonstrated Level 6 performance (OECD, 2004). The OECD characterizes Level 6 performance in mathematics as "advanced mathematical thinking and reasoning" (p.47), as demonstrated by such activities as the ability to conceptualize, generalize, utilize, link, and apply mathematical knowledge with understanding (OECD, 2004).

In other words, most students can "do" mathematics – that is, they can perform computations and use algorithms or formulas to solve routine problems, but they cannot really "do" mathematics in the true sense – they cannot apply their mathematical knowledge to more complex problems or novel situations, nor can they use it in importantly versatile ways. The picture that emerges is, in effect, one of contrasting doing with thinking in mathematics. Children may know what to do when they encounter mathematical symbols or a mathematical task, but they are less successful at thinking mathematically – at reasoning, interpreting, and evaluating (Bruer, 1993; Fuson, 1991; NAEP, 2005; OECD, 2004).

Hatano (2003) has conceptualized the inconsistency in students' mathematical performance as routine versus adaptive expertise. Routine expertise is the ability to complete mathematics exercises, usually quickly and accurately, but with little or no understanding of the underlying concepts. Routine expertise, although it has its particular merits, is insufficient in and of itself because of its narrow and limited applicability; it is useful only in familiar mathematical tasks and situations. Adaptive expertise, on the other hand, is the ability to apply meaningfully learned mathematical knowledge flexibly and creatively. This type of mathematical knowledge is considerably more valuable because of its wide applicability; it can be applied in the context of new and unfamiliar mathematical tasks and situations. Because it is unreasonable to expect every student to be familiar with all mathematical problems or situations, or even to assume that all mathematical problems can be solved by using a procedure, algorithm, or rule, the true value in adaptive expertise becomes evident. To be successful in mathematics, children need to learn how to think mathematically for themselves, something that routine expertise does not afford.

In an effort to explain the phenomenon of students' inefficient grasp of mathematics –indeed, some have even called it deficient (Kilpatrick et al., 2001) – many individuals, including teachers, parents, and students themselves, have come to varying conclusions regarding mathematics. These conclusions tend to focus on the apparent enigmatic nature of the subject: that mathematics is complex and difficult to understand; that it is the domain of only a privileged few; and that it has little applicability to everyday life (Ashcraft, 2002; Bruer, 1993). Although such stereotypical perspectives of mathematics are common among the general public (Ashcraft, 2002), researchers have looked at the problem of mathematics learning from a different standpoint. These scholars have theorized about and examined the role of instruction in students' learning and performance in mathematics. Today, it is commonly recognized that instructional practice is a major factor in any attempt to understand student outcomes and performances in mathematics (Kilpatrick et al., 2001). This conclusion has stemmed largely from research revealing discrepancies between children's intuitive and informal mathematical knowledge and their performance in the classroom. For example, although children can use their informal knowledge to successfully solve mathematics problems upon entering school, making few, if any errors, they encounter many difficulties and become unable to perform at the same level of proficiency once formal instruction begins (Gutstein & Romberg, 1995; Hiebert, 1984; Mack, 1990, 1995, 2001).

In response to the problem of students' inadequate learning and performance in mathematics, and to the association made with inadequate teaching, many reform efforts have been proposed in an effort to improve the quality of instruction in mathematics classrooms. Earlier views of what mathematics instruction should look like – a focus on computational skill with successful performance defined as the ability to be quick and accurate at executing procedures (Hiebert & Lefevre, 1986) – have more recently been overridden by reforms that emphasize teaching for understanding, largely through the process of problem-solving (National Council of Teachers of Mathematics, [NCTM], 2000). Teaching for understanding, however, has not been articulated particularly well in many reform documents. Teaching for understanding has been expressed in generalities and few descriptions of what effective instruction actually looks like in practice have been offered (Brown, 1993; Fuson, 2004; Hiebert & Carpenter, 1992; Stigler et al., 1999). As a result, teachers are left to their own devices in interpreting what such ideals mean, with few concrete examples available to them. Clearly, the problem of students' underachievement in mathematics must also be viewed as the problem of teaching in mathematics, and more elaborated and tangible ideas of what it means to teach for understanding are needed.

Given the call for improved instruction aimed at building students' understanding, and the need to make clear what this looks like in practice, my purpose for conducting this research was to empirically examine a teaching method that has been theorized to be an effective approach to teaching mathematics for understanding. Although the proposed theory assumes the instructional approach to be applicable globally to the teaching of mathematics – that is, across all content areas in mathematics – in this study, I examined it within the specific topic area of fractions. I specifically chose the topic of fractions for this research because it is one that is particularly important to students' mathematical education (Behr, Lesh, Post, & Silver, 1983; Saxe, Gearhart, & Nasir, 2001) and one that is especially difficult to teach and to learn with understanding (Ball, 1990, 1993; D'Ambrosio & Mewborn, 1994; Saxe et al., 2001; Tirosh, 2000).

Review of the Literature

What Does it Mean to Understand Mathematics?

One common element among most definitions of mathematical understanding is the idea of connections. For example, one is said to have mathematical understanding if she or he can make connections between informal and formal mathematical ideas (Ball, 1993), between pieces of mathematical information (Fuson, 2004), between mathematical symbols and notations and the ideas those represent (Hiebert, 1984, 1988), and between mathematical procedures and their underlying rationales (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). In other words, one is viewed to have mathematical understanding if one's mathematical knowledge is well related, if one has built a network of relationships between different pieces of mathematical information (Fuson, 2004; Hiebert & Lefevre, 1986). When mathematical knowledge or information is rich in relationships, that knowledge becomes meaningful: it is understood (or has meaning) from a variety of perspectives and its integrated nature allows it to be used flexibly so that it becomes applicable in many different mathematical contexts or situations (Fuson, 2004). Understanding, therefore, can be characterized as a meaningful integration of mathematical information. By contrast, when mathematical knowledge is not integrated or connected, such as when one's mathematical knowledge consists of discrete pieces of information that are not related, then it is said that meaningful understanding is not present (Fuson, 2004; Hiebert & Lefevre, 1986). Discrete and unconnected pieces of knowledge are relevant only in the contexts in which they were learned and therefore do not lend themselves well to application in novel situations.

Meaningful understanding therefore refers to the ability to use the mathematical knowledge one possesses in many different ways and not just in the context in which it was learned, an ability made possible by the connected or related nature of the knowledge. Referring back to an earlier made point, meaningful understanding illustrates the difference between doing and thinking in mathematics. Discrete or isolated knowledge may provide one with the basic tools to do mathematics – to solve computations and utilize procedures – but meaningful knowledge allows one to think about the mathematics with which he or she is engaging, and to see the relevance of what one knows in different mathematical situations.

Conceptual and Procedural Knowledge Defined

Conceptual and procedural knowledge are two terms frequently used in the mathematics literature to contrast meaningful understanding with knowledge of a more discrete or isolated nature. According to recent literature (e.g., Star, 2005), the current meanings associated with the terms conceptual and procedural knowledge can be traced back to the work of Hiebert and Lefevre (1986). Hiebert and Lefevre defined conceptual knowledge as "knowledge that is rich in relationships… a connected web of knowledge" (p.3), and explained that conceptual knowledge is developed through the construction of relationships, either between extant pieces of information or between already existing and new pieces of information. They further explained that conceptual knowledge, or knowledge that is rich in relationships, denotes understanding because relationship building involves the tying together of pieces of information.

Procedural knowledge, according to Hiebert and Lefevre (1986) is defined in two parts as (a) knowledge of mathematical symbols, and (b) knowledge of the algorithms or rules used in solving mathematical tasks. Knowledge of mathematical symbols involves knowledge of both the particular symbol system of mathematics and the conventions of its use, and does not necessarily imply meaning: it generally refers to surface feature knowledge, such as the style and form of written mathematical equations, and not to mathematical content. Knowledge of algorithms or rules involves knowledge of the step-by-step procedures used to solve mathematical problems and similarly does not necessarily imply meaning: knowledge of procedures, as Hiebert and Lefevre (1986) noted, is likely little more than an awareness of instructions used to manipulate mathematical symbols.

These definitions imply that the difference between conceptual and procedural knowledge in mathematics is the difference between knowledge that is meaningful and knowledge that is not meaningful. Conceptual knowledge, knowledge that is well related, is meaningful because the many links between the pieces of knowledge make it both accessible and applicable in many ways. Procedural knowledge, on the other hand, because it is not related to other pieces of information and consists mainly of surface level knowledge, is not as widely accessible or applicable and therefore not as meaningful.

The above definitions of conceptual and procedural knowledge appear to treat the two as distinct and separate entities. For example, conceptual knowledge appears to imply knowing why, and procedural knowledge appears to imply knowing how, with little mention of any relationship between the two. A review of the literature tracing the historical development of these two notions provides some explanation (see for example Hiebert & Lefevre, 1986). Traditionally, concepts and procedures, often distinguished by the words understanding and skill, respectively, have been viewed as separate entities in an effort to explain student learning in mathematics and determine which should take instructional prominence (Hiebert & Lefevre, 1986). In other words, researchers and educators have historically debated between the importance of teaching skills in mathematics over understanding, and vice versa. Although at different times different conclusions were reached, the debate always ended in the same place: viewing conceptual and procedural knowledge as divided and the related implications this had for instructional decision-making (Hiebert & Lefevre, 1986). A major change in that "either-or" conclusion has been considering the ways in which the two types of knowledge interact and relate. Whereas traditionally conceptual and procedural knowledge were viewed as distinct, with the same distinction carried over to the teaching of mathematics, more recently the focus has been on acknowledging the relationship between the two and consequently, learning more about that relationship to inform instruction and learning (Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler, & Alibali, 2001).

Although Hiebert and Lefevre's (1986) influential definitions seem to keep conceptual and procedural knowledge divided, they clearly stated that they did not view the two as separate entities, but rather as distinct yet linked in critical ways. They further stated that it was their view that both types of knowledge, as well as a strong relationship between the two, are necessary to make the claim that one has mathematical understanding. To develop clear definitions that can allow for the systematic investigation of such abstract constructs as knowledge types, however, Hiebert and Lefevre (1986) explained that their use of the above definitions was meant simply to distinguish and not to separate the two ideas. Indeed, others, too, have recognized the notion that a significant relationship between conceptual and procedural knowledge exists, and that this should be an important consideration related to teaching and learning in mathematics (Siegler, 2003). Current perspectives, as described in the research that follows, appear to have reached at least some consensus on a departure from the view of the two as distinct toward a view of the two as intertwined.

The Conceptual-Procedural Debate

Much of the research leading to the conclusion that procedural and conceptual knowledge is linked in important ways has dealt with theorizing about the development of each. More precisely, the research has focused on investigating the order in which procedural and conceptual knowledge develop, and whether or not one type of knowledge forms the basis of the other. This debate has taken the form of what some scholars have called the procedures-first theory versus the concepts-first theory (Rittle-Johnson et al., 2001), and Silver (1986) has offered some explanation of the basics of this debate. For example, Silver described the idea behind the argument that procedures are based on concepts and explained that many tend to take as evidence the finding that students make errors in procedural computations, or cannot apply their procedural knowledge in new situations, as confirmation that conceptual knowledge must form the base of procedural knowledge. In other words, the view is that conceptual knowledge must be lacking if students cannot solve computations accurately or understand what it is they are doing.

Although intuitively the argument seems reasonable – if one does not understand what he or she is doing then one must be lacking the relevant conceptual knowledge – there is much evidence to support the contrary. One convincing finding is that many students can use procedures and perform computations fluently but still show little understanding of what they are doing or why (Bright et al., 1988; Franke & Grouws, 1997; Hiebert & Wearne, 1996; Saxe et al., 2005). As such, Silver (1986) explained that it may be more sensible to consider the problem as one of a lack of strong connections between the two types of knowledge rather than to view it as one of a lack of conceptual knowledge in particular. He argued that this latter argument is simplistic because it portrays procedural and conceptual knowledge as static and unrelated elements of one's knowledge base. Instead, Silver contended that knowledge is used dynamically when an individual engages cognitive activity to solve problems or complete a mathematical task, and dynamic knowledge necessarily entails evoking the relationships between the different elements of one's knowledge.

Silver (1986) also described the flip side of this argument, that conceptual knowledge rests on a procedural knowledge base, and explained that this perspective stems largely from research that has examined the role of procedural knowledge in students' concept attainment. Stated differently, the argument is that if procedural knowledge is used to further one's understandings of a concept, then the case that conceptual knowledge is based on knowledge of procedures can be made (Silver, 1986). In earlier work, Silver (1979) reported findings from his own examination of this hypothesis in which he investigated students' understanding of the concept of problem structure. More specifically, his research looked at whether or not students would determine the mathematical relatedness of problem types by the procedures they used to solve the various classes of problems. Silver (1979) found that the students in his study did not use knowledge of procedures to classify problems conceptually even though they employed similar procedures to solve the different types of problems. Rather, the students used conceptual knowledge of the underlying structure of the problems to categorize them. In addition to finding no evidence in support of the claim that procedural knowledge underpins conceptual knowledge, Silver (1979) did find evidence to support his notion of dynamic knowledge – that relationships are enacted between the different elements of one's knowledge base. That is, although the task was such that it demanded the students to evoke mathematical concepts, they also drew on their procedural knowledge as they worked through it. Silver (1979) took this as evidence that it is indeed the relationships between an individual's procedural and conceptual knowledge that are paramount to understanding learning and performance.

Other findings also add to the conceptual-procedural debate. In a series of experiments, Byrnes and Wasik (1991) investigated fourth-, fifth-, and sixth-grade students' development of conceptual and procedural knowledge in the domain of fractions. Findings reported from their first experiment with fourth- and sixth-graders demonstrated somewhat ambiguous results. For example, Byrnes and Wasik found that conceptual knowledge was at times related to procedural performance and at other times not, and that this depended on the particular task. More specifically, for tasks involving the multiplication of fractions, students who showed both high and low levels of conceptual understanding demonstrated skilled performance. The researchers thus concluded that conceptual understanding was not necessary for procedural competence in this context. In contrast, for tasks involving the addition of fractions with unlike denominators, the majority of participants (93%) who showed high conceptual knowledge did not demonstrate skilled performance. Only a very small percentage (approximately 4%) of students who showed high levels of conceptual understandings were successful at this task and none of the students who demonstrated low conceptual knowledge were able to solve the addition tasks correctly. In this case then, Byrnes and Wasik concluded that conceptual understanding appeared to be necessary, but not sufficient, for procedural understanding in this context.

In their second experiment with fifth-graders, Byrnes and Wasik examined conceptual and procedural knowledge within the context of instruction designed to repair students' difficulties with addition of fractions problems. Students were randomly assigned to one of three instructional conditions and Byrnes and Wasik reported that there were no significant differences in the students' levels of procedural and conceptual knowledge prior to instruction. The instruction consisted of three different approaches: (a) teaching the students a procedure for solving addition of fractions items, (b) demonstrating the underlying meaning of the procedure with concrete models, and (c) telling them the rule in addition to drawing on the analogy that the procedure for adding fractions is similar to the idea of adding subclasses of things. Several findings are noteworthy. First, Byrnes and Wasik reported significant increases in all students' procedural performance, regardless of instructional condition. Second, they also reported that the majority of students whose performance increased on procedural tasks showed stable or increased growth on conceptual knowledge tasks. These findings seem to suggest two things. First, both procedurally and conceptually based instruction appeared to be effective at increasing students' procedural knowledge. Second, increases in procedural understanding appeared to relate to increases in conceptual understanding. Thus, in the context of this investigation, it may be generally concluded that conceptual understanding appears to underpin procedural understanding, but that on the other hand, procedural understanding is also important for conceptual knowledge growth.

In a longitudinal investigation, Hiebert and Wearne (1996) examined first- and second-graders' procedural and conceptual knowledge in the domain of multidigit addition and subtraction. At the beginning of the research, participants were classified as either understanders or nonunderstanders, depending on whether they exhibited high or low levels of conceptual understanding, respectively. Hiebert and Wearne (1996) tracked the development of the participants' procedural and conceptual knowledge over a three-year period and across different instructional contexts (traditional instruction versus instruction that focused on conceptual understanding). Overall, their findings indicated that conceptual knowledge was an important precursor to the development of both understanding and skill.

For example, at the beginning of the study, only those students who were classified as understanders could demonstrate procedural skill, as evidenced by their ability to use invented procedures. After instruction, both understanders and nonunderstanders were able to demonstrate procedural skill with standard algorithms, but upon closer examination, Hiebert and Wearne found that nonunderstanders could not do so with understanding. That is, although both the understanders and nonunderstanders could use the standard algorithm to complete a multidigit addition, only the understanders could demonstrate its underlying rationale. Moreover, increases in procedural skill developed gradually over the course of instruction for the understanders who were able to use invented procedures, whereas for nonunderstanders, this only happened after instruction, after the standard algorithms were taught. This would suggest that it was the understanders' higher levels of conceptual understanding that allowed them to build their procedural skill. In addition to being able to invent procedures, Hiebert and Wearne also found that the understanders in their study were more capable of effectively adopting demonstrated procedures, suggesting again that it was their increased conceptual understandings that allowed them to do so.

The conceptual-procedural debate in mathematics is not a new one. Historically, it has received much attention in the research literature (e.g., Baroody & Ginsburg, 1986; Byrnes & Wasik, 1991; Davis, 1986; Hiebert, 1984; Hiebert & Lefevre, 1986; Hiebert & Wearne, 1996; Silver, 1986). The findings from such research have led to uncertain conclusions regarding which precedes and underpins the other, as evidenced from the research cited above. In a review of the literature that has examined the development of procedural and conceptual knowledge in mathematics, however, Rittle-Johnson and Siegler (1998) made clear that methodological issues in the investigation of conceptual and procedural knowledge make it difficult to resolve the debate. Of particular significance, they examined the type of evidence often used by researchers to draw conclusions and provided important insights as to how such evidence can be countered. Along this dimension, the first noteworthy observation is that few of the existing studies have offered much credible evidence. In fact, in their extensive review of the literature, Rittle-Johnson and Siegler concluded that most examinations provided weak evidence for the claims they made.

The two investigations by Byrnes and Wasik (1991) and Hiebert and Wearne (1996) cited above, however, provide better evidence given the methodologies used (Rittle-Johnson and Siegler, 1998). Rittle-Johnson and Siegler (1998) have nevertheless pointed to important considerations within these particular investigations that should be taken into account when interpreting the results, such as the lack of random assignment of students to instructional conditions, important differences in instruction across conditions, and the sensitivity of the outcome measures used. In their concluding arguments, Rittle-Johnson and Siegler (1998) provided some resolve to the conceptual-procedural debate. They stated that given the existing evidence, and taking into account methodology, conceptual and procedural knowledge develop in tandem. They did however note that the iterative development of conceptual and procedural knowledge was an especially plausible one and that further research needed to be conducted to examine that idea empirically.

In a recent study, Rittle-Johnson and her colleagues (2001) investigated the iterative hypothesis of the development of conceptual and procedural knowledge and found support for this argument. Rittle-Johnson et al. (2001) examined fifth- and sixth-graders' procedural and conceptual knowledge in the domain of decimal fractions. More specifically, their research looked at whether or not conceptual knowledge would predict gains in procedural knowledge, and whether in turn, gains in procedural knowledge would improve conceptual understandings. In two different experiments, Rittle-Johnson and her colleagues (2001) concluded that initial conceptual understanding was in fact related to improved procedural performance and that gains in procedural skill were also related to improved conceptual understandings. Given these findings, the researchers noted that the procedures-first versus the concepts-first debate is an invalid one and that the iterative model provides a more accurate description. The authors also argued that either type of knowledge may develop first and thus serve to support the development of the other, but that the sequence largely depends on students' experiences. For example, if students have many experiences with concepts before procedures are taught, conceptual knowledge will develop first. On the other hand, if they have more experiences with procedures before concepts are taught, then procedural knowledge will develop first. Rittle-Johnson et al. (2001) further noted that the iterative explanation applies regardless of whether it is conceptual or procedural knowledge that develops first.

The findings from earlier research examining the developmental sequence of conceptual and procedural knowledge offer no simple answers as to which type of knowledge develops first or which acts as a foundation for the other. Empirical evidence provides support that students can employ known procedures without using knowledge of concepts, and vice versa (Byrnes & Wasik, 1991; Hiebert & Wearne, 1996; Silver, 1979), making the perspective that one type of knowledge must necessarily rest on the other a narrow one. More recent evidence indicates that in some instances and for some learners, depending on experience and prior knowledge, either form of knowledge may develop first, but that regardless of the order, either can serve to strengthen the development of the other, which in turn can serve to further strengthen the development of the first (Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2001).

Taken together, these findings indicate what Silver (1986) and Rittle-Johnson and her colleagues (2001) summed up well: The conceptual-procedural debate in mathematics education, though intended to advance our knowledge and understanding of learning, in effect may serve to distance us from it. The debate, by pitting concepts against procedures, has resulted in a line of research in which each type of knowledge is both regarded and examined as separate entities, thus blurring the more appropriate view that conceptual and procedural knowledge develop in an ongoing and interactive nature. In other words, scholars believe that pure forms of either type of knowledge are seldom used when students are involved in mathematical tasks, nor that one is necessarily preferable to the other (Fuson, 2004; Silver, 1986; Wu, 1999). Though more emphasis in mathematics learning appears to be placed on having a firm grasp of conceptual knowledge, as evidenced by reform movements for teaching for understanding, many also acknowledge the necessity of procedural knowledge for competent mathematical functioning (Hiebert & Lefevre, 1986; Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2001; Star, 2005; Wu, 1999).

In essence, the current research that informs us appears to be moving away from the traditional view of the two knowledge types as distinct – with the related ideas about which to teach first – toward the notion of the two as linked in fundamental and critical ways, and the related idea that the focus should no longer be about teaching one over the other but about teaching both together (Carpenter, 1986; Fuson, 2004; Hiebert, 1984; Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2001).

The Nature and Role of Instruction

The move toward the perspective that procedural and conceptual knowledge interact in important ways and that both should be emphasized during instruction is reflected in current reform ideas. For example, new goals for mathematics instruction that have been established in curriculum documents such as the Principles and Standards for School Mathematics (NCTM, 2000) have called for an increased focus on understanding, problem-solving, reasoning, and making connections between mathematical ideas. These new goals have been proposed in response to the troubling state of student performance and achievement in mathematics. For instance, the results of both national (Kilpatrick et al., 2001; NAEP, 2005) and international assessments (OECD, 2004; Stigler et al., 1999), in addition to those reported by numerous researchers who have examined student performance in mathematics, consistently reveal that students show many weaknesses in understanding basic mathematical concepts, applying their mathematical skills to solve both simple and complex problems, and using their mathematical knowledge in flexible and spontaneous ways.

Though popular belief may attribute the problem to the complex nature of the subject of mathematics (Aschraft, 2002), educational researchers believe that the problem is one of teaching in mathematics. Several scholars claim that formal mathematics instruction is inadequate because although many students have strong and mathematically-sound intuitive understandings, they experience much difficulty with the mathematics they are exposed to in school (Gutstein & Romberg, 1995; Hiebert, 1984; Mack, 1990, 1995, 2001). Others also argue that students are shortchanged in the mathematics classroom by curricula that probe mathematical content at superficial levels and are thus undemanding and underestimating of students' abilities (Fuson, 2004; Kilpatrick et al., 2001). Still others have reported that mathematics instruction does not afford students the opportunity to think about the mathematics they are learning (Behr et al., 1983; Bruer, 1993; Hiebert, 1984; Kilpatrick et al., 2001); on the contrary, it tends to emphasize lower-level basic skills such as performing computations and memorizing rules and facts.

New instructional goals found in reform documents have therefore been set forth in opposition to the way that mathematics has traditionally been taught and the resultant learning opportunities afforded by such instruction. Though one might like to assume that instructional methods have changed given the increased attention being paid to this matter over recent years, current research shows that traditional methods, at least within the discussion of procedural and conceptual understanding, still dominate most of classroom teaching (Fuson, 2004; Kilpatrick et al., 2001; Stigler et al., 1999). Examining instructional practice, as I will do below, may help illustrate and perhaps even explain the perpetuation of such traditional methods. In addition, it may also serve to support the argument that the need for meaningful mathematics instruction does indeed exist.

Standard instruction. It is clear that instructional practices differ in many ways along many different dimensions; teachers have their own personal styles, student characteristics vary across schools and classrooms, and curricula are not the same everywhere. As such, it is difficult to classify and define instruction in very neat and simple terms. With respect to the way procedural and conceptual knowledge are (and are not) approached during instruction, however, the literature, both past and current, demonstrates that instruction can be typified by its procedural focus (Davis, 1986; Kilpatrick et al., 2001; Stigler et al., 1999). Therefore, in this thesis, I use the term standard instruction specifically in reference to this typical focus on procedural instruction and not to presume that all teachers in all classrooms teach in the same way.

Perhaps the most commonly reported problem about student ability (or lack thereof) in mathematics is that although most are quite proficient at executing procedures and performing computations, few are able to think at more complex levels. In other words, students appear to be functioning at very basic, skill-oriented levels. How might instruction result in such student behavior? One plausible answer is the goal of formal mathematics education and the ensuing instructional approaches that are adopted. For example, in his analysis of the problem with the existing mathematics programs of his time, Davis (1986) noted that instructional goals tended to overemphasize the manipulation of written symbols, with little importance placed on understanding both the process and outcome of such manipulations. Mathematics, he argued, was thus regarded as an exercise in following the rules for writing symbols and performing symbolic computations.

Indeed, recent evidence shows that mathematics instruction has not kept pace with reform ideas; it can still be characterized in much the same way as it was by Davis (1986) over 20 years ago. For instance, the majority (61%) of U.S. teachers who participated in the 1999 Third International Mathematics and Science Study, when asked to comment on the goals of their mathematics lessons, reported that skills-learning (solving specific problems by using specific procedures), as opposed to mathematical thinking (exploration, development, and comprehension of mathematical concepts), was their priority (Stigler et al., 1999). Consequently, a standard mathematics lesson for most U.S. students consists mainly of practicing routine procedures (that are usually first demonstrated by the teacher) as opposed to such activities as applying concepts to new situations and thinking and reasoning about mathematics (Fuson, 2004; Kilpatrick et al., 2001; Stigler et al., 1999). Although procedural fluency is clearly one important aspect of student learning, it is incomplete if not accompanied by the related goal of understanding the ideas represented in mathematical symbols and the manipulations performed on them.

This is not to say that all instruction in standard mathematics classrooms focuses solely on the development of skills. There is evidence that some attention is given to mathematical concepts as well (Stigler et al., 1999). Looking more closely into how concepts are "taught" in elementary mathematics classrooms, however, is revealing, and results from international assessments show a disappointing picture. For example, although some instructional focus is indeed related to knowledge of mathematical concepts, in the majority of U.S. classrooms concepts are generally transmitted to students in a rote fashion (stated by the teacher or other students, but not explained) without any effort placed on the role of developing an understanding of them (Stigler et al., 1999). It is no wonder, then, that students display many difficulties in thinking and reasoning about mathematics at more complex and conceptual levels; they are rarely asked to do so.

There thus appears to be a discrepancy between teachers' professed goals for instruction and what actually occurs during the course of their practice. For example, Stigler and his colleagues (1999) found that U.S. teachers reported a strong identification with many of the ideas expressed in the NCTM reform documents, and furthermore, that they believed they were teaching according to those ideas. The data presented above, however, indicate that in practice, instructional ideals and instructional events are not aligned. A general assessment made in Stigler et al. (1999), for example, was that problem-solving – a major focus of the NCTM reform – was viewed by the U.S. teachers as an end goal and not as a vehicle by which understanding can and should be developed (Carpenter et al., 1989; Kilpatrick et al., 2001; NCTM, 2000). In other words, teachers teach the skills they believe their students will need to know in order to solve problems, as opposed to teaching them to problem-solve. It is not surprising, therefore, that students tend to view problem solving as little more than an exercise in proceeding directly from the problem statement to the solution, as opposed to an exercise in which the application of their thinking is necessary (Behr et al., 1983). This type of activity is clearly at odds with the goals of reform instruction and would suggest that teachers may be lacking clear and tangible ideas about how to transform their instructional goals into concrete teaching practices. Evidently, there is a need to know more about how to teach for mathematical understanding.

Meaningful instruction. In examining standard approaches to instruction, the focus of mathematics teaching can generally be characterized as that of developing students' skills for using written mathematical symbols and employing rules and procedures to perform symbolically-represented computations. Stated differently, the aim of most standard forms of instruction is the development of procedural competence. In their analysis of important mathematics for students to learn, Kilpatrick and his colleagues (2001) argued that procedural fluency is in fact a major component of mathematical proficiency. What then is the problem with standard instruction in mathematics? Among other things, two ideas are especially important in the context of this discussion.

First, Kilpatrick et al.'s (2001) conception of procedural fluency encompassed much more than the ability to simply use mathematical symbols to execute computations according to prescribed rules, rather, they conceived of procedural fluency as also involving the ability to use procedures flexibly. The earlier discussion on procedural and conceptual knowledge provides evidence that in order to use procedures flexibly, that is, to be able to apply them in a variety of mathematical situations, they must be related to conceptual understandings (Fuson, 2004; Hatano, 2003; Hiebert & Lefevre, 1986;

Hiebert & Wearne, 1996). Standard methods of instruction, although perhaps successful at teaching students to use symbols and procedures effectively in isolated contexts, do not allow learners to develop sufficient understandings so that procedures may be applied in novel ways.

Second, and related to the first point, mathematical competence involves both procedural and conceptual knowledge, the latter of the two which is largely ignored by standard methods of instruction. Though some instructional attention is given to mathematical concepts, these are not usually developed with an eye toward comprehension or building important mathematical relationships; on the contrary, they are taught in the same rote and isolated nature as procedures are (Stigler et al., 1999). Referring back to the earlier discussion on procedural and conceptual knowledge once more, the research suggests that key among all other goals for meaningful learning is the ability to make links or connections among the elements of one's knowledge (Fuson, 2004; Hatano, 2003; Hiebert & Wearne, 1996). Thus, the problem with standard instruction appears to be two-fold: (a) It tends to focus excessively on procedural fluency, but even at that, it does so inadequately, and (b) it does not effectively develop conceptual knowledge, with the related dilemma that it fails to link knowledge of procedures to knowledge of concepts.

Taken together, it appears that there is misalignment with the goals and practices of mathematics instruction. Moreover, this misalignment results in teaching methods that are in precise opposition to what is called for with respect to developing students' understandings. Meaningful instruction, therefore, with the larger goal of creating opportunities for meaningful understanding, should emphasize the development of relationships during learning. Although this idea has been discussed at length in the literature on mathematics learning, and although research such as that reported above has shed some light on it, there is still very little direction as to how instruction might be delivered in a way such that it takes advantage of how understanding and skill interact and develop together during learning.

Theoretical Framework

The theoretical foundation for this study was taken from Hiebert's (1984) proposed analysis regarding the potential points of contact that may be identified between conceptual and procedural knowledge during the learning process. In particular, Hiebert (1984) named three specific levels or *sites* where connections between conceptual and procedural knowledge might be especially productive for meaningful learning of mathematics, and thus targeted during instruction. The sites are specified as points in real time during the problem solving process, as three distinguishable phases during mathematical problem-solving activity (Hiebert & Wearne, 1986). Hiebert (1984) argued that the difficulties students encounter when learning mathematics result from an absence of connections between conceptual understandings and understandings of the symbols and procedures they are taught in school. He therefore recommended that mathematics instruction should focus on capitalizing on the sites where conceptual and procedural knowledge might theoretically interact to result in meaningful learning.

To be clear, it is essential to describe the way in which Hiebert (1984) has defined procedural and conceptual knowledge. Procedural knowledge, referred to by Hiebert as knowledge about form, includes symbols for numerals (for example, 8, 3.7, $\frac{1}{2}$), symbols for operations and relations (for example, +, =), and the resultant symbol phrases (for example, $8 + \frac{1}{2} = \square$). In addition, knowledge about form also includes knowledge about the rules, procedures, or algorithms used in mathematical tasks to manipulate symbols (for example, "to multiply two fractions, multiply across numerators and denominators"). Conceptual knowledge, referred to by Hiebert as understanding, involves intuitions and ideas about how mathematics works that make sense to students and that can be acquired through both formal and informal mathematical experiences. In his conceptualization of understanding, Hiebert placed a major emphasis on students' informal mathematical knowledge; that building relationships between new knowledge and existing intuitive knowledge is the most effective way to develop conceptual understandings.

Proposed Analysis of Conceptual-Procedural Links

Hiebert (1984) identified the following three sites as possible instances during the learning of mathematics where meaningful relationships may be formed between conceptual and procedural understandings: Site 1, called symbol interpretation, is the point at which the symbols of the problem or task are given meaning. Site 2, called procedural execution, is the point in the problem solving process when procedures are executed. And Site 3, called solution evaluation, is the point in the problem solving process when solutions are evaluated for reasonableness. I discuss each of these sites in further detail in turn below.

Site 1: Symbol interpretation. Conceptual and procedural understandings may be related if symbolic representations in a problem are linked with referents that give those representations meaning. The basic premise at this site is that mathematical symbols and notations, often internalized as arbitrary formalizations, must take on meaning for the learner if they are to be understood in conceptually meaningful ways. Hiebert proposed that this may be accomplished through instruction that connects symbols to their concrete or real-world referents. The following example illustrates this idea for the mathematical symbol "+". The "+" symbol, in its concrete or real-world application, represents the idea of joining. In the word problem, "Anna has three stickers and her sister gave her two more stickers. How many stickers does Anna have altogether?", the notion of joining two sets, as symbolized by the symbol "+", is encountered. The addition symbol, represented as "+", takes on meaning for the learner if it is connected to the idea of joining, such as that which is inherent in the word problem above. The basic idea in its most elementary sense is knowing the conceptual referent for the symbol.

Site 2: Procedural execution. Conceptual and procedural understandings may likewise be related if procedures, rules, or algorithms used to solve a problem are connected to their underlying rationales. Moreover, Site 2 understandings share a core with Site 1 understandings in that procedures and algorithms are frequently taught as, or at end point almost always become, written ones. Therefore, in addition to understanding the underlying principles that motivate procedures, Site 2 understandings also imply, again, that symbols used to represent procedures take on meaning for the learner. The following example provides an illustration. The written procedure that is normally taught for subtraction of multidigit numbers, symbolically noted in the standard vertical column format, may be linked with knowledge of the base-10 numeration system, such as trading between adjacent place values (one 10 for 10 ones) while maintaining equivalence of values. As such, the actions that are undertaken symbolically for a subtraction involving regrouping, such as that for the problem 33 - 16 = □, if linked to base-10 understandings, make the procedure, both its rationale and symbolism, more transparent to the student.

Site 3: Solution evaluation. Lastly, conceptual and procedural understandings may be related if symbolically represented problem solutions are linked to (a) ideas about how problems might be solved in concrete or real-world contexts, or (b) other knowledge of the number system. Site 3 understandings, then, are also fundamentally linked to Site 1 understandings in that the emphasis is on creating meaning for problem solutions that are represented symbolically. Hiebert (1984) characterized the activity at Site 3 as testing the reasonableness of an answer, checking to see that it makes sense from both a mathematical and real-world perspective. The notion of (a) above, concrete or real-world reasonableness, is illustrated by the following example. The answer to 2.25 ÷ 3 = □ may be connected to the real-world idea of sharing $2.25 equally among three people. Situating the problem in this type of context may allow students to better reflect on whether or not the symbolic outcome is sensible. The notion of (b) above, reasonableness as it relates to other knowledge of the number system, may be illustrated as follows. A sensible answer to the problem $\frac{12}{13} + \frac{7}{8} = \square$ may be derived from thinking about how the numbers in the problem relate to other number knowledge. In this particular example, both numbers in the problem are close to one, and because 1 + 1 = 2, a reasonable answer to this problem must be a number that is close to two (and not one that would be arrived at if the student erroneously added across numerators and denominators). Reasonableness as it relates to other knowledge of the number system relies heavily on the technique of estimation and the use of benchmarks as reference points.

In summary, according to Hiebert (1984), to result in meaningful learning, instruction in mathematics should emphasize the connections between conceptual understandings and understandings of symbols and procedures that may be made at each of the three sites. That is, Hiebert (1984) argued that students experience difficulties in learning school mathematics because it is taught in an abstract and formal nature that is very unlike the intuitive and informal mathematical knowledge they possess. According to Hiebert, the formalization of school mathematics focuses on representing ideas with symbols and on manipulating those symbols according to rules, with little emphasis on developing meaning for those formalizations. Hiebert argued that this leads to serious learning and instructional problems because many children do not connect those formalizations to the skills and concepts they already possess. Thus, Hiebert reasoned that it is the absence of connections between formalizations and intuitive understandings during instruction that moves the student from more meaningful to mechanical and meaningless mathematical activity. More precisely, Hiebert argued that meaningless understandings result from an absence of connections at the three specific sites during instruction. He thus suggested that his proposed sites approach to instruction, because it is designed to help students connect their conceptual and procedural knowledge with the symbols and procedures taught in school, may be successful at correcting some of these learning problems.

Empirical data exist to support elements of Hiebert's (1984) theory. For example, in the domains of place-value and multidigit addition and subtraction, there is evidence to show that connecting mathematical symbols and notations to their concrete or real-world referents, and connecting procedures or algorithms to their underlying conceptual meanings, can result in improved understanding and perfromance. In a series of studies, Fuson and Briars (1990) examined first- and second-graders' abilities to both perform addition and subtraction of multidigit numbers and their understandings of it. The researchers used a "learning/teaching" approach designed to make explicit the connections or relationships between spoken number words, the symbols used to represent those numbers, and the procedures used to manipulate the symbols when operating on the numbers. In brief, Fuson and Briars' approach involves the use of concrete materials that embody quantitative meanings for symbolically represented numbers and meanings for algorithmic procedures that are carried out on those numbers. In other words, their approach focuses on creating meaning for both the symbols used to represent a problem (Hiebert's Site 1) and the symbols used to represent the actions and outcomes of a procedure (Hiebert's Site 2). The students in Fuson and Briars' studies showed significant improvement from pretest to posttest in their understanding of place-value and their ability to perform multidigit additions and subtractions. In fact, they performed better than what is commonly reported for most children at one or two grade levels above. Moreover, the "learning/teaching" approach was also succesful at virtually eliminating students' tendency to perform computation "bugs" (Van Lehn, 1990) that are common in this domain (e.g., subtracting a larger digit from a smaller one).

In a similar endeavor, Hiebert, Wearne, and Taber (1991) examined low-achieving fourth graders' performance and understanding in the domain of decimal fractions. The instruction employed in the study centered on the use of external representations (i.e. physical models) to create meaning for, and relationships between, decimal fractions symbols, concepts (quantity and partitioning), and procedures (addition and subtraction). To develop quantitative understandings of decimal fractions, instruction focused on using different models to represent concretely the mathematical symbols that are used to denote decimal fractions (Hiebert's Site 1). In addition, instruction also focused on using the physical models to depict the concept of partitioning decimal fractions into smaller units and the actions performed on symbolic notations when adding and subtracting decimal fractions (Hiebert's Site 2). Hiebert et al. (1991) assessed student performance at three different times following three different instructional sessions. Assessment after the first instructional sequence, which focused on making connections between written decimal symbols and discrete physical representations (base-10 blocks) of decimals, indicated a significant increase in students' performance and understanding on tasks relating symbols and representations. Similar increases in performance and understanding were found on assessment tasks administered after the third instructional sequence, which also emphasized the use of discrete representations for modeling the procedures used to add and subtract decimal fractions symbolically. After the second instructional sequence however, in which continuous models (number lines and circular regions) were used to represent decimal fractions notations, Hiebert et al. (1991) reported that the participants did not show improved performance or understanding. They attributed this latter finding to the particular difficulty that children commonly experience with the continuous aspect of decimals and the potential fact that the models used to embody this aspect were not the most effective. Nevertheless, the overall findings reported by Hiebert and his colleagues (1991) do lend support to the idea that linking mathematical symbols to concrete referents, and linking symbolic manipulations to their underlying meanings, can improve students' understandings and thus help them build connections between informal and formal mathematical ideas.

Some literature also exists that supports the notion of using concrete and real-world referents to develop students' ability to test the reasonableness of solutions (Hiebert's Site 3). For example, Ball (1993) described the essential role teachers play in selecting and using models; either concrete, pictorial, or verbal (in the form of stories), to help students develop conceptual understandings and "mathematical power." She further noted the importance of using a variety of representations as a method for expanding students' "thinking spaces", so that for example, understandings do not become bound to specific representational contexts. From her work with the students in her third grade class, Ball reported anecdotal evidence describing how the use of pictorial models within the context of working a solution to a real-world problem[1] allowed her students to represent concretely the reasonableness of a solution to a division problem that involved the topic of fractions. In particular, some students' initial solutions to the problem were not reasonable from a mathematical point of view (fractional partitions of cookies were counted as wholes), but when prompted to create meaningful referents, that is, to represent and link their solutions to models, the students demonstrated shifts in their thinking that allowed them to arrive at more mathematically sensible solutions.

Saxe and his colleagues (2005) also examined instruction that focused on building relationships between conceptual and procedural knowledge in the domain of fractions, with particular emphasis placed on creating meaning for fractions symbols by relating

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[1] The problem Ball (1993) posed to her students was the following, "You have a dozen cookies and you want to share them with the other people in your family. If you want to share them equally, how many cookies will each person in your family get?" (p. 176). Knowing something about her students' family sizes, Ball anticipated divisions by five and seven, which would lead the students to encounter fractions.

them to concrete referents (Hiebert's Site 1). Their research contrasted three different teaching approaches – Traditional, Low Inquiry, and High Inquiry – and assessed the opportunities each afforded students to build their understandings of symbol-referent relationships. Saxe et al.'s (2005) results indicated greater overall gains for students in High Inquiry classrooms, in which instruction was distiguished from that in the other two classrooms by its ability to support conceptual relations among different representations for fractional quantities denoted symbolically. Although not all the students in the study were successful at transitioning between nonconventional (but conceptually relevant) and conventional notational use for fractions depicted by concrete referents, those from High Inquiry classrooms did so at a greater rate. Furthermore, students in Traditional and Low Inquiry classrooms, in which instruction did not emphasize the conceptual relations between fractions symbols and fraction representations, were more likely to use erroneous (and conceptually invalid) notational systems. It should be noted that in Saxe et al.'s (2005) research, all students in each instructional condition were taught via the same models to represent symbolic fractional quantities. What differed across instructional approaches was the extent to which connections between models and symbols were developed (to a larger extent in High Inquiry versus Traditional classrooms) and the degree to which those understandings were related to other knowledge the students possessed (knowledge was integrated in High Inquiry classrooms and not integrated in Traditional classrooms).

A similar line of research in the domain of fractions has been conducted by Mack (1990, 1995) and in particular lends support to all three of Hiebert's proposed sites. The overarching instructional goal in both of Mack's (1990, 1995) investigations was to build on the students' intuitive and informal understandings – to build relationships between their prior knowledge and new understandings. In addition, Mack made use of real-world problem contexts and concrete representations that drew on the participants' informal knowledge to aid them in making sense of fraction symbols and symbolically represented problem solutions. In both her 1990 and 1995 investigations, Mack found that when instruction encouraged students to consider concrete and real-world representations of symbols, they were able to repair misconceptions about quantitative notions of fractions denoted symbolically (Hiebert's Site 1) and build missing links between their intuitive knowledge and knowledge of symbolically represented fraction procedures (Hiebert's Site 2). Mack also taught the participants in her 1990 study to use estimation techniques based on their quantitative understandings (estimating relative sizes, sums, and differences of fractions close to one) to solve fraction problems involving equivalence, addition, and subtraction, and to verify the reasonableness of their solutions (Hiebert's Site 3). She reported that the use of estimation, in addition to aiding the students in determining the sensibleness of their answers, also allowed them to invent effective algorithms to solve order-equivalence and addition and subtraction problems involving fractions.

The Present Study

The research cited in the opening pages of this study presents evidence that students of varying ages and grade levels show unacceptable levels of mathematical achievement and performance across a wide range of topic areas. In addition, empirical data also indicate that students' understandings of mathematical concepts are fragile and disconnected, resulting in the mechanical performance of routine mathematical actions that are often devoid of any real meaning. Although in many instances students are quite successful at employing rules and procedures to perform computations and solve familiar tasks, the lack of relationships between their procedural and conceptual knowledge does not allow them to justify and make meaning of the mathematics they are learning (NAEP, 2005; OECD, 2004). Consequently, students are seldom able to apply their knowledge in meaningful ways – to think and reason mathematically – when faced with unfamiliar or novel mathematical situations.

Contrary to the impression that mathematics is inherently difficult and only within the purview of a few select students, researchers have turned to the notion of instructional practice as an important factor in students' learning and performance in mathematics. As a result, much effort has been expended in the goal of improving instructional practice and has culminated in reform ideas intended to move instruction from basic skills toward a focus on developing meaningful understandings (Hiebert et al., 1997). Unfortunately, these ideas have been expressed in generalities with few concrete descriptions available to teachers of what shape and form such ideas might take in actual practice. In addition to ideas expressed in reform curricula, other scholars in the mathematics education community have argued behind convincing evidence for increased instructional attention to the linking and relating of student knowledge and the important complementary idea that understanding and skill co-develop in an interactive process. Here too, however, few tangible ideas of what this actually means for teachers have been proposed, as evidenced by teachers' tendencies to teach in outdated modes even when they report agreement and identification with reform ideas (Stigler et al., 1999).

When taken together, the literature indicates that despite reform and research efforts, clear ideas about how to teach for mathematical understanding are still lacking. Though the picture is becoming somewhat more complete – parts of it have been articulated and examined quite thoroughly – there are still essential components that are missing. For instance, although scholars have gained some important insights, such as knowing that it is imperative to teach both skills and concepts and to do so in a manner that emphasizes the connecting relationships among them, they have yet to determine more precisely where those connections might lie during instruction. In other words, researchers know it is important for students to develop those connections and they likewise know of the importance of instruction that is focused on building them, but the question of how to link the two more precisely is still an open one. With so many facets to learning and instruction, it is hardly difficult to understand why scholars have been unable to isolate precise locations where procedural and conceptual knowledge interact during learning and instruction.

Although the evidence presented above indicates that researchers are indeed thinking about this question and actively seeking to answer it, findings to date, although insightful, still fall short of resolving it. That is, while research has allowed scholars to specify some ideas about teaching methodology that capitalizes on the places where skill and understandings interact during learning, little has been written in the way of a more complete and systematic instructional approach for doing so.

To my knowledge, Hiebert's (1984) proposed analysis of the three specific sites where conceptual-procedural links reside during learning and instruction is a unique one. First, as far as I am aware, no other theories of such a nature have been specified. Second, it represents a potential answer to the question of locating, more precisely of isolating, exactly where procedural and conceptual links reside during learning and instruction. (Recall from the above discussion that Hiebert's theory takes the learner all the way through the initial interpretation of the problem statement, to working on it, and finally, to its solution.) Although intuitively Hiebert's theory appears to make much sense, and although parts of it have been supported by research, the theory in its entirety, particularly as translated in actual practice, has not been tested empirically. Moreover, because it is both unique and comprehensive, it has the potential to inform and benefit researchers and practitioners who are interested in mathematics education. I used all of these reasons to support my view that Hiebert's theory merits further attention and investigation.

The objective of the study was to empirically investigate Hiebert's (1984) theory regarding the proposed sites where procedural and conceptual knowledge may be linked during instruction and learning. More specifically, the objective was to evaluate mathematics instruction that is based on Hiebert's model, referred to here as the *Sites* approach. Within the context of classroom-based instruction on the topic of fractions with fifth- and sixth-grade students, therefore, I compared two different teaching approaches: (a) Hiebert's Sites approach and (b) standard instruction. Standard instruction mirrored closely what has been reported in the research literature with respect to the way that the teaching of mathematics can typically be defined by its procedural focus and its disconnect between procedural and conceptual knowledge.

Based on Hiebert's theory and on my review of other relevant literature that has supported it in part, I predicted that students who would be instructed via Hiebert's proposed approach would develop improved meaningful understandings of fractions. In specific, I predicted that students who would be instructed via the Sites approach would demonstrate greater improvements in both (a) conceptual understandings of fractions and (b) the ability to link their conceptual understandings of fractions to their knowledge of fractions procedures at each of Hiebert's three sites. In addition, given the research evidence that has supported the notion that procedural and conceptual knowledge develop in an iterative fashion (e.g., Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2001), I further predicted that students who would be instructed via Hiebert's approach would also demonstrate improved procedural understandings of fractions.

The decision to use fractions as the topical area for this investigation was justified for two reasons. First, the domain of fractions has been recognized as key within the elementary mathematics curriculum as fractions are related in important ways to understandings of other number concepts (Saxe, Gearhart, & Nasir, 2001) and also serve as foundational or prerequisite understandings for the study of more advanced mathematical topics, such as algebra (Behr et al., 1983). Second, research has consistently demonstrated that the topic of fractions presents a major obstacle for many students along their learning paths (Behr et al., 1983; Brown, 1993) and that in addition, it is difficult to both teach and to learn fractions in a conceptually meaningful way (Ball, 1990, 1993; D'Ambrosio & Mewborn, 1994; Saxe et al., 2001; Tirosh, 2000).

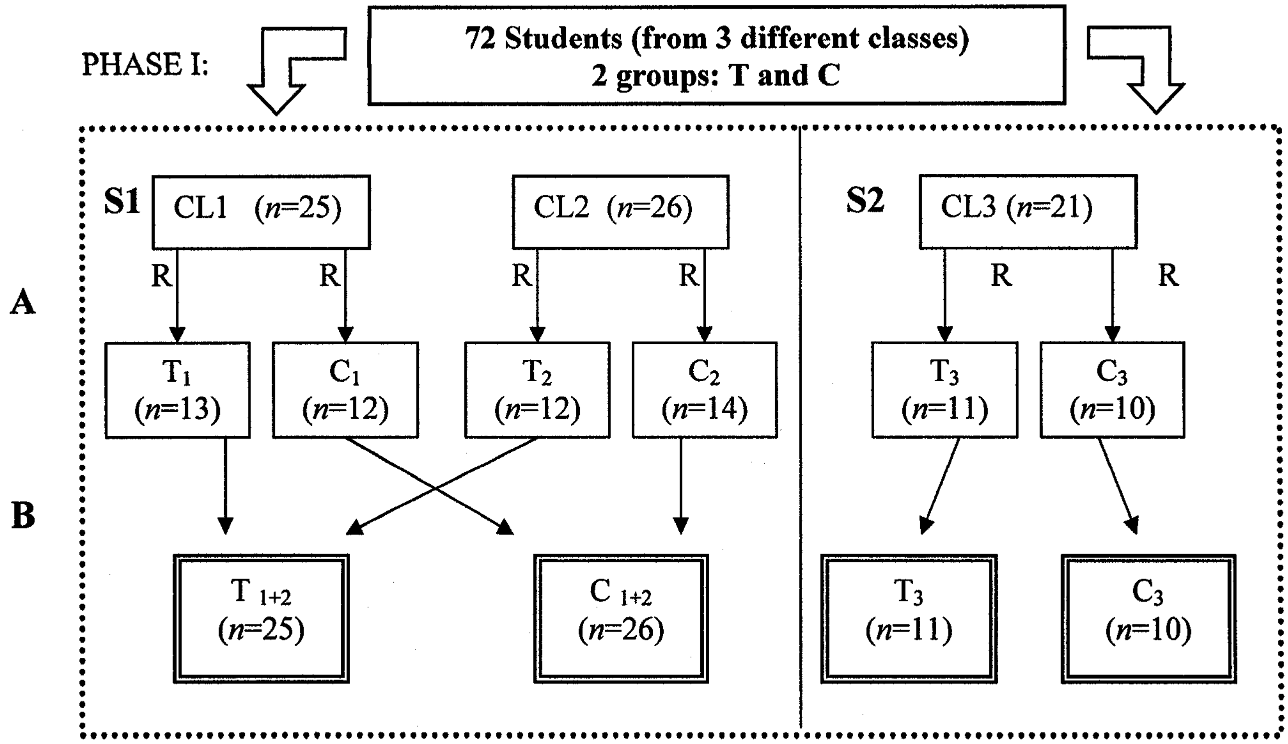
Method

The present study aimed to answer the following research questions: (1) Does instruction that explicitly connects procedures and concepts at the symbolic (Site 1), procedural (Site 2), and solution evaluation (Site 3) phases of learning result in improved conceptual and procedural understandings of fractions for fifth- and sixth-grade students? (2) Does the Sites instructional approach enable students to link their procedural and conceptual understandings of fractions at each of Hiebert's three sites? I investigated the research questions by comparing Hiebert's Sites approach to a standard instructional approach on the topic of fractions with fifth- and sixth-grade students. I delivered a three-week instructional intervention (Hiebert's Sites approach) to a group of randomly assigned students and compared that instructional method to a standard method of instruction, referred to here as the *Standard* approach, that I also delivered to a randomly assigned control group. The main difference between the two instructional methods was that in the Sites approach, I explicitly highlighted the connections between fractions procedures and concepts at each site during instruction. Conversely, I made no such connections in the Standard approach; standard instruction focused only on procedural aspects of fractions.

Design and Procedures

I used a pretest-posttest control group design to answer the research questions. Figure 1 illustrates the design and procedures of the study. I compared two instructional conditions: the Sites approach and the Standard approach (in Figure 1, Phase IV).

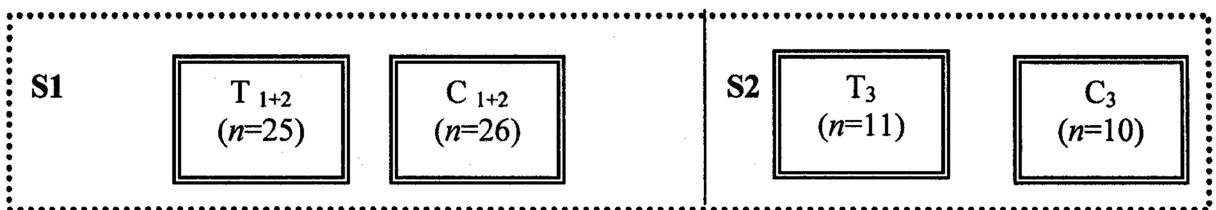
PHASE I:



PHASE II: PRE-TESTS: 72 Students
(all students from T_{1+2}, T_3 and C_{1+2}, C_3)

PHASE III: PRE-INTERVIEW: 30 Students
(5 randomly selected students from each of T_1, T_2, T_3, C_1, C_2, C_3)

PHASE IV: CLASSROOM INTERVENTION



PHASE V: POSTTEST: 72 Students
(all students from T_{1+2}, T_3 and C_{1+2}, C_3)

PHASE VI: POST-INTERVIEW: 30 Students
(same 30 students selected at Phase III)

<u>Legend</u>

T: Treatment Group
C: Control Group
S1: School 1
S2: School 2
CL1: Classroom 1
CL2: Classroom 2
CL3: Classroom 3
R: Random Assignment

Figure 1. Study design and procedures.

I used two dependent variables to measure the participants' procedural and conceptual understandings of fractions at pretest (in Figure 1, Phases II and III) and posttest (in Figure 1, Phases V and VI): (a) a paper-and-pencil test of fractions knowledge (Saxe et al., 2001), and (b) a fractions knowledge interview that I developed. I created and used two structurally isomorphic versions (Version A and Version B) of the fractions knowledge interview to counterbalance any order effects. In addition, I measured the participants' verbal ability at pretest (in Figure 1, Phase II) using a vocabulary checklist.

Before any data collection began, I randomly assigned each participant to one of the two instructional conditions: the Sites approach or the Standard approach (in Figure 1, Phase I). I performed the randomization procedure separately at each of the two research sites – that is, at both school 1 and school 2, I randomly assigned each student in each participating classroom to one of the two instructional conditions. In school 1, I first randomly assigned the students from classroom 1 to either the treatment or control group; I employed the same procedure in classroom 2 (in Figure 1, Phase IA). Following that, I grouped the students assigned to the treatment condition from classroom 1 with the students assigned to the treatment condition from classroom 2 (in Figure 1, Phase IB), and this newly combined class formed the treatment group at school 1. I performed the identical combination procedure at school 1 to form the control group. In school 2, I repeated Phase IA (see Figure 1) of the randomization procedure to form the treatment and control groups from classroom 3. Given that only one class from school 2 participated, however, I did not repeat Phase IB of the randomization procedure in school 2 (i.e., no recombining of classes because only one class participated).

The procedure in school 1 of regrouping the students from classroom 1 and classroom 2 into one treatment group and one control group (Phase IB) was performed so that subgroups (i.e., treatment and control) from each of classroom 1 and classroom 2 could be instructed together, as one larger group. The reason I employed this procedure related to issues of practicality with respect to time (the duration of the study) and the participating teachers' daily schedules. That is, because the students in classrooms 1 and 2 at school 1 had mathematics at the same time of the day, to instruct each subgroup of students separately would have required either that I doubled the duration of the study (instructing classroom 1 for a period of three weeks and then instructing classroom 2 for another three-week period), or that the classroom teachers made changes to their daily schedules so that the students in each participating classroom would have had mathematics at a different time of the day. Both options were unfeasible given the amount of disturbance each would have caused for the teachers and students alike with respect to the regular mathematics routine. Therefore, to keep to a minimum the time frame within which the research was conducted, and to limit as much as possible the imposition on the teachers' and students' regular classroom activities, regrouping the subgroups of students from each classroom into one larger treatment group and one larger control group allowed me to complete the study in a more timely fashion. By regrouping the students in this way, and instructing the treatment and control groups on alternate days (the schedule of instructional activities is discussed in further detail below), I only pulled the students from class for one 50-minute session per day (the equivalent amount of instructional time devoted to mathematics under normal conditions) as opposed to two 50-minute sessions per day (i.e., treatment and control) had I not regrouped the students to be instructed together. In addition, the duration of Phase IV of the study (see Figure 1) was limited to three weeks' time, as opposed to double the time had I administered the instruction separately in the two participating classrooms.

In school 2, where only one classroom participated, there was no need to regroup students into larger groups. To minimize the interruption to the participating teacher's regular daily schedule in school 2, however, I instructed the treatment and control groups on alternate days. That is, by instructing the treatment and control groups on alternate days, I only pulled students from the classroom for one 50-minute session per day (the equivalent amount of time for a mathematics lesson under normal instructional conditions in classroom 3), as opposed to two 50-minute sessions per day (i.e., treatment and control) had I instructed the groups on the same day but at alternate times.

At Phase II (see Figure 1), I administered the fractions knowledge test (Saxe et al., 2001) and the measure of verbal ability to all participants. At Phase III, I individually interviewed 15 randomly selected students from each of the treatment and control groups, represented in equal numbers across all classrooms (see Figure 1, Phase III). Half of the interviewed students from each instructional group completed Version A of the interview and the other half completed Version B. At Phase IV, I administered instruction on the topic of fractions to the students in each randomly assigned instructional condition. The schedule of instructional activities is presented in Table 1. In the Sites approach, I explicitly highlighted the connections between fractions procedures and concepts at three specific sites during the learning process: Site 1, symbol interpreation; Site 2, procedural execution; and Site 3, solution evaluation (Hiebert, 1984). In the Standard approach, I administered instruction covering the same topics, but from a purely procedural

Table 1

Schedule of Instructional Activities

Day	Lesson Content	School	Condition
	Week 1		
1	Part-whole meaning of fractions	S1	Treatment
		S2	Treatment
2	Part-whole meaning of fractions	S1	Control
		S2	Control
3	Finish part-whole lesson, start equivalent fractions	S1	Treatment
		S2	Treatment
4	Finish part-whole lesson, start equivalent fractions	S1	Control
		S2	Control
5	Equivalent fractions	S1	Treatment
		S2	Treatment
	Week 2		
1	Equivalent fractions	S1	Control
		S2	Control
2	Ordering and comparing fractions	S1	Treatment
		S2	Treatment
3	Ordering and comparing fractions	S1	Control
		S2	Control
4	Finish ordering/comparing lesson, start improper fractions and mixed numbers	S1	Treatment
		S2	Treatment
5	Finish ordering/comparing lesson, start improper fractions and mixed numbers	S1	Control
		S2	Control

Table 1 (*continued*)

Day	Lesson Content	School	Condition
	Week 3		
1	Improper fractions and mixed numbers	S1	Treatment
		S2	Treatment
2	Improper fractions and mixed numbers	S1	Control
		S2	Control
3	Adding and subtracting with fractions	S1	Treatment
		S2	Treatment
4	Adding and subtracting with fractions	S1	Control
		S2	Control

perspective. That is, except for the initial part-whole introductory lesson (see Table 1) during which I presented part-whole concepts for fractions alongside symbols and pictorial representations, I did not teach any concepts underlying symbols, procedures, or evaluations of solutions. The part-whole lesson is discussed in further detail below.

I instructed the treatment and control groups separately on the topic of fractions for a period of approximately three weeks, for a total of approximately six hours of instructional time per instructional condition, or seven 50-minute class periods per instructional condition. All instruction took place during regularly scheduled mathematics class time, either in the students' regular classrooms or in another quiet area of the school, such as the library or an unoccupied classroom. I presented mathematical content in identical sequence and time in both instructional conditions and the lessons comprised various activities and tasks related to: the part-whole meaning of fractions; the concept of equivalence; ordering fractions; comparing fractions; improper fractions and mixed numbers; and addition and subtraction with fractions (see Table 1). While I taught one instructional condition on the topic of fractions, the other instructional group remained with the classroom teacher. Classroom teachers did not administer any instruction on the topic of fractions to any participating students during the course of the research.

In summary, this design allowed me to test Hiebert's theory as it translated into practice, and particularly its effects on the development of the participants' procedural and conceptual understandings, while enabling me to control for instructional variables that have been shown to be related to learning outcomes, such as instructor effects, curriculum and content differences, lesson organization and structure, and time on task (Stigler et al., 1999).

At Phase V, I once again administered the fractions knowledge test to all participants in each instructional condition. At Phase VI, I interviewed the same 30 randomly selected participants who were interviewed at Phase III a second time. Students who completed Version A of the interview at pretest completed Version B at posttest, and students who completed Version B of the interview at pretest completed Version A at posttest.

Participants

I conducted the study with Cycle 3 students (grades 5 and 6) who ranged in age from 10 to 12 years. The participants were drawn from two public elementary schools in Montreal, Quebec. Both schools serve students from low-mid income families and both schools are rated on the lower end of the school board's socioeconomic index (S. Morgan, H. Silver[2], personal communication, March 20, 2007). Seventy-two students from three different classrooms participated in this study. Thirty-six students formed the

[2] Pseudonyms are used to protect the identity of school personnel.

treatment group (Sites condition) and 36 students formed the control group (Standard condition). No participating student had received formal instruction on the topic of fractions within the school year during which the research was conducted. I chose Cycle 3 students as the participants for this study because it is during the upper elementary grades that students in Quebec are typically engaged in the formal study of fractions (Québec Ministère de l'Éducation, 2001).

The participants were selected on the basis of the willingness of school principals and teachers to grant access to their classrooms and time for the purposes of the research. In addition, selection also depended upon the consent granted by the participants and their parents or guardians. Prior to data collection, I visited the three participating classrooms to inform the students about the goals and procedures of the study. Thereafter, I distributed parent information letters detailing the goals and procedures of the study, as well as parent and child consent forms. I distributed the child consent forms with the instructions that they be read and signed at home, in the company of parents or guardians. The parent information letter can be found in Appendix A. All consent forms are contained in Appendix B. The students and their parents or guardians, classroom teachers, and school principals were not compensated in any way for their participation in the study.

Measures and Instrument Administration

Fractions knowledge test. I measured the participants' procedural and conceptual understandings of fractions at pretest and posttest using the fractions knowledge test, a paper-and-pencil test developed by Saxe et al. (2001) to assess elementary students' computational skill and conceptual understanding of fractions. The test is included in

Appendix C. Computation items on the fractions knowledge test that involve operations are limited to those of addition and subtraction and are presented in both standard symbolic format (i.e., vertically and horizontally written equations) and word-problem format (Items 3.1-3.7; 10a). In addition, computation items also include converting fractional equivalencies (Items 4.1-4.4), computing with values in a circular region model (or a pie; Item 9), and missing value equivalence problems (Items 8.1-8.4). Conceptual items on the fractions knowledge test include problems of constructing fractions for unequal parts of wholes (Items 1.1-1.3), estimating fractional parts of areas (Items 2.1-2.2), fair share problems (Items 5-7), and graphical depiction of computational word problems (Item 10b).

Although Saxe and his colleagues (2001) acknowledged the difficulty in labeling procedural and conceptual understanding as distinct constructs, they selected and created test items on the basis of belonging more to one category than the other. For example, items included on the computation subscale are those that can clearly be solved by applying a routine procedure or memorized fact (Saxe et al., 2001). On the other hand, items included on the conceptual subscale are those that could not be easily solved via a routine procedure or common rule; to solve conceptual items, knowledge of mathematical relations involving fractions is needed (Saxe et al., 2001).

Saxe et al. (2001) conducted a three-factor model (general knowledge of fractions, computational skills, and conceptual understanding) confirmatory factor analysis to distinguish between conceptually orientated and computation test items. Results demonstrated strong support that the two subscales of the test are indices of independent areas of competence in children's mathematical performance (the fractions knowledge test as a whole – that is, the procedural and conceptual subscales taken together comprises the third factor of general knowledge of fractions). Specifically, for the posttest, the confirmatory factor analysis indicated that all fit indices were high (Bentler-Bonett NFI = .984, Bentler-Bonett NNFI = .985, CFI = .994). Similarly, for the pretest, the best fit indices were high (Bentler-Bonett NFI = .981, Bentler-Bonett NNFI = .979, CFI = .992). Cronbach's alpha indicated internal consistency for each scale. For the conceptually-orientated subscale, the indices were .73 (pretest) and .83 (posttest); for the computation subscale, the indices were .86 (pretest) and .87 (posttest).

Vocabulary test. At pretest, I assessed the participants' verbal ability using a vocabulary checklist that has been shown to be a reliable and valid measure of vocabulary (Anderson & Freebody, 1983; Stanovich et al., 1995). Verbal ability has also been shown to be a valid indicator of general cognitive ability (see Matarazzo, 1972). Therefore, I used verbal ability, as measured by the vocabulary checklist, as a covariate to ensure that any differences on this variable between the two groups were controlled. The vocabulary checklist can be found in Appendix D. The checklist was modeled after a similar measure used by Zimmerman, Broder, Shaughnessy, and Underwood (1977) to measure verbal ability in adults. Zimmerman et al.'s measure consisted of a 60-item checklist containing 40 real words and 20 pronounceable nonwords. Participants were instructed to identify, without guessing, all real words that were known to them.

The vocabulary checklist that I used for this study contained 43 items in total, consisting of both real words and pronounceable nonwords. Thirty words that appeared on the checklist were real words and 13 were pronounceable nonwords. All 30 real words were taken from Marzano, Kendall, and Paynter's (1991) list of vocabulary for fifth- and sixth-grade students. I created nonwords based on Marzano et al.'s vocabulary list by changing one or more letters in a real word to form a pronounceable nonword.

Fractions knowledge interview. At pretest and posttest, I measured the participants' procedural and conceptual understandings of fractions as they relate to Hiebert's sites using interview items that I developed. All interview items can be found in Appendix E. To counterbalance any order effects from pretest to posttest, I used two structurally isomorphic versions (Version A and Version B) of the interview. That is, I developed both versions of the interview to measure the same fractions concepts and procedures related to tasks involving Hiebert's Site 1, Site 2, and Site 3. The specific numerical values used, as well as superficial features of the tasks, however, varied between each version of the interview.

To illustrate, one goal of the interview was to measure the participants' conceptual and procedural understandings of the algorithm used to add fractions with common denominators. Figure 2 depicts the corresponding item, Item 7, from each of Version A and Version B of the interview. The goal of interview Item 7 was to assess the students' abilities to connect the algorithm to its underlying rationale and to represent the meaning of the symbolic manipulation involved in computing an answer to this type of problem (Site 2). As such, both versions of the interview contained items that were identical in their underlying structure. The difference between Item 7 in Version A and Item 7 in Version B, however, was that different fractional quantities were used and superficial features of the task (e.g., names used, food item) were also different (see Figure 2).

During the interviews, I asked the participants to respond orally and in writing (e.g., draw a picture, show your thinking) to questions about fractions concepts and procedures, and I also asked them to solve some mathematics problems involving fractions. For each interview item, I first read the question to the participant and then followed that with an oral explanation of the objective of the task. As I read each question aloud, I placed the typed question in front of the participant to allow the student to refer to the question at any time during which he or she was answering a particular item. I informed each participant that any question could be repeated or re-read at any time if they desired.

The interview items I designed to assess mathematical activity at Hiebert's Site 1 were intended to measure the participants' abilities to connect fractional symbols to mathematical meanings. Thus, I asked the students to produce both the symbolic and concrete representations of a given fractional quantity stated orally and in number words (Item 1) and to produce a representation for a given fractional quantity stated orally and symbolically by partitioning a specified model (Item 2). At Site 2, interview items were intended to measure the participants' abilities to connect fractional symbols used to perform computations, or employ rules and algorithms, to their underlying meanings. Therefore, for Site 2 items I asked the students to solve fraction problems involving (a) converting improper fractions to mixed numbers (Item 3) and vice versa (Item 4); (b) ordering and comparing fractions (Item 5 and Item 6, respectively); and (c) writing equations for and computing the addition and subtraction of fractions word problems (Item 7 and Item 8, respectively). In addition to solving each of the above problem types, at Site 2 I also asked the students to provide a rationale for or justification of the procedures they used. Site 3 items were intended to measure the participants' abilities to connect symbolically represented solutions to fractions problems with understandings used to evaluate the reasonableness of those solutions. Thus, for Site 3 items I asked the students to write, draw, or explain how a symbolically represented problem solution could be evaluated for its mathematical or real-world reasonableness (Item 9 and Item 10).

Interview Version A

(7). Marie ate three-fifths of a cheese pizza and Sandra ate one-fifth of a pepperoni pizza. How much of a pizza was eaten altogether?

(a). Write an equation to go with this problem.
(b). Solve your equation.
(c). Draw a picture to show how you solved the problem.

- -

Interview Version B

(7). Alexa ate two-sixths of a chocolate cake and Steven ate three-sixths of a strawberry cake. How much of a cake was eaten altogether?

(a). Write an equation to go with this problem.
(b). Solve your equation.
(c). Draw a picture to show how you solved the problem.

Figure 2. Illustration of the structural isomorphism of Versions A and B of the student interview for Item 7.

The reason I administered individual interviews in addition to Saxe et al.'s (2001) fractions knowledge test was to ensure that all topics addressed during instruction were reflected in the outcome measures. For example, instruction included activities designed to teach students how to assess the reasonableness of symbolically represented solutions to fractions problems. This concept is not assessed on Saxe et al.'s (2001) fractions knowledge test and yet is critical in the evaluation of Hiebert's theory.

Administration of pretest. Before instruction began, all students in each instructional condition independently completed the fractions knowledge test (Saxe et al., 2001) and the vocabulary test that I developed. I administered the fractions knowledge test to the students as a group, in their classrooms, during regularly scheduled class time.

The students were given 40 minutes to complete it. When every student had completed the fractions knowledge test, I then administered the vocabulary measure to the students as a group. The students were given approximately 10 minutes to complete the vocabulary test. Two students from each group (treatment and control) were absent on the day of testing. I adminstered the fractions knowledge test and the vocabulary checklist to the absent students on the following day, and the same procedure was repeated. Absent students completed the pretests either in the library or in a vacant nearby classroom.

I also individually interviewed 15 students from each of the treatment and control groups at pretest using the fractions knowledge interview that I developed. Half of the students from each group completed Version A of the interview and the other half completed Version B of the interview. The interviews were conducted outside of the students' classrooms, either in the library, a vacant nearby classroom, or at a free table in the hallway. Each interview lasted between 25 and 40 minutes. I audio-recorded all interviews for subsequent analyses.

Administration of posttest. At the end of the three-week instructional period, all students in each condition individually completed the fractions knowledge test (Saxe et al., 2001) for a second time. At posttest, I followed the identical administration procedure that I followed at pretest. That is, all students completed the fractions knowledge test as a group, in their classrooms, during regularly scheduled mathematics class time. Students were again given 40 minutes to complete the test. In addition, I once again individually interviewed the same 30 students who had participated in the pretest interview. Students who completed Version A of the interview at pretest completed Version B at posttest, and vice versa. The format and administration procedure of the posttest interview was identical to that which is described above for the pretest session.

Intervention

Instructional activities and content. Participants in each instructional condition received instruction that was identical in mathematical content and sequence, but the instruction differed in form for each of the treatment and control groups. Specifically, the form of instruction for the Sites approach (the treatment group) involved explicitly highlighting the relationships between fractions procedures and concepts during the learning process at each of Hiebert's (1984) three sites. For Site 1, symbol interpretation, instruction emphasized the meaning of the symbols used to represent mathematical problems or tasks involving fractions. For Site 2, procedural execution, instruction emphasized the underlying meaning or rationale for proceures, rules, or algorithms used to solve problems involving fractions. For Site 3, solution evaluation, instruction emphasized different methods for evaluating the reasonableness of symbolically represented solutions to problems involving fractions.

According to Hiebert (1984), children's learning problems in mathematics are a result of a disconnect between their informal conceptual understandings and the formal symbols and procedures they are taught in school. That is, the formalized abstract nature that is characteristic of most standard methods of mathematics instruction is at odds with children's intuitive, meaningful understandings. Hiebert further suggested that knowledge of the potential points of contact at which links between informal conceptual and formal procedural understandings may be made is necessary to teach mathematics in a way that supports students' abilities to forge those connections.

The three sites described above have been proposed as moments during the learning of mathematics when meaningful relationships may be formed between conceptual and procedural understandings, but only if those relationships are explictly highlighted during instruction (Hiebert, 1984). For example, Hiebert stated that meaningful understandings at Site 1 can be made by connecting mathematical symbols and notations to their concrete or real-world referents. Similarly, connecting symbolic procedures or algorithms to their underlying conceptual rationales can result in meaningful understandings at Site 2. And finally, meaningful understandings of symbolically represented problem solutions at Site 3 can be achieved by testing the reasonableness of answers – that is, by considering (a) how a given problem might have been solved in a real-world or concrete context, or (b) how symbolic solutions connect to other understandings of the number system.

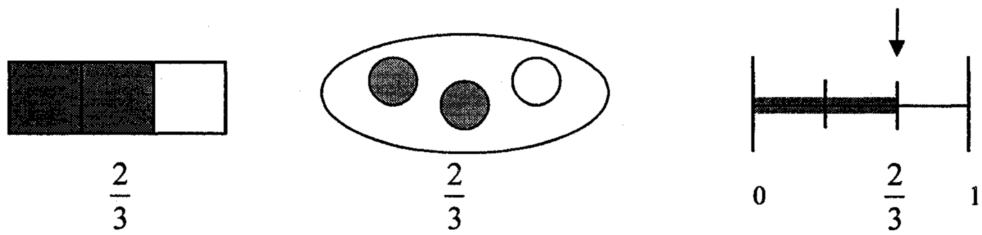
The three sites represent three distinguishable phases during mathematical problem-solving activity when connections between procedures and concepts are especially important to the development of meaningful understandings of mathematics (Hiebert & Wearne, 1986). Although connections at all three sites are necessary for meaningful problem-solving as a whole, Site 1 connections are the most critical because they serve as essential foundational understandings that allow for connections at Site 2 and Site 3 to be made (Hiebert, 1984; Hiebert & Wearne, 1986). In other words, the ability to connect procedures and concepts at Site 2 and Site 3 is largely dependent on meaningful understandings of symbols created at Site 1. For example, Site 2 connections relate to symbols used to carry out procedures, rules, or algorithms, and Site 3 connections relate to symbols used to represent problem solutions. Thus, with respect to the three sites, although Hiebert (1984) did not specify a particular instructional order for translating his proposed approach into practice, an inherent order does nonetheless reside in his theory. In particular, Site 1 connections foster the understandings that are foundational to the development of connections at the remaining two sites. As such, initial instructional activities for the Sites (treatment) group focused on creating connections at Site 1 – that is, on explicitly linking fraction symbols to their underlying meanings. The order of implementation of instructional activities designed to create connections at Site 2 and Site 3, however, was determined by the topic at hand during a particular instructional session.

Give the ideas outlined in Hiebert's theory, the instruction I administered in the treatment condition continuously and explicitly emphasized the connections between symbols and their conceptual underpinnings (Site 1) by creating meaning for the symbols and mathematical notations used to represent fractional quantities. For example, according to Hiebert (1984), two important concepts related to understanding a fractional symbol include (a) partioning: the notion that the entire whole must be partitioned into equal sized parts; and (b) equivalence: the fractional symbol itself is a representation of equivalent parts of a defined unit. Therefore, one component of instruction for the treatment group included activities designed to link fractional symbols to various models that embody both the concept of partitioning and the concept of equivalence (see Figure 3 for an example using the fraction $\frac{2}{3}$).

In addition, part-whole understandings are also necessary for students to develop meaningful understandings of fraction symbols (Ball, 1993; Mack, 1990, 1995, 2001; Saxe, Gearhart, & Seltzer, 1999; Saxe et al., 2005). Part-whole understandings refer to the notion that a fraction is a quantity derived by partitioning a whole into some number of equal parts, and then "taking" some number of those equal parts (Ball, 1993). With regards to symbolic representations and part-whole understandings, fractional symbols (e.g., $\frac{a}{b}$, where a and b are whole numbers, b $\neq$ 0) specify how many parts of the whole are being considered. Because students often misinterpret fractions and fractonal symbols in relation to their whole-number understandings – for example, a region divided into three equal parts with two of those parts shaded may be interpreted as a representation of 2 rather than of $\frac{2}{3}$ (Saxe et al., 1999) – instruction at Site 1 also explicitly highlighted the part-whole relations in fractional quantities by continuously drawing the participants' attention to the idea that fractions are quantities that represent parts of wholes. In specific, I did this by (a) using various models of fractions, both discrete and continuous, to represent concretely and pictorially the part-whole relations (Ball, 1993); and (b) relating fractional symbols to real-world referents, such as thinking about the fraction $\frac{2}{3}$ as two parts of a whole pie that is divided into three equal parts (Ball, 1993; Mack, 1990, 1995, 2001).

At Site 2, I made continuous and explicit connections between symbolic procedures and their conceptual underpinnings by creating meaning for (a) the rules and algorithms used to add and subtract fractions, (b) the procedures used to convert improper fractions to mixed numbers and vice versa, and (c) the rules employed to order and compare fractions. For example, a procedural algorithm for ordering unit fractions (fractions with a numerator of 1, e.g., $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{8}$) from smallest to largest, or largest to

What does the number $\frac{2}{3}$ mean? Here are a few examples:



What is similar about each example? They're all divided into 3 parts.

What do you notice about each part in each example? They're all the same size.

What would you call one part of something that is divided into three? A third.

How many thirds are there in each example? Three.

Of the three thirds in each example, how many have been shaded? Two.

So we're talking about two what? Thirds.

That's right. Each example shows two-thirds because the whole in each example is divided into three equal parts and two of those three parts have been shaded. The number two-thirds is written like this $\frac{2}{3}$, which means two of three equal parts. So the bottom number in the fraction tells you how many equal sized parts your whole is divided into and the top number tells you how many of those parts have been chosen. So, what do you know when you see a number like $\frac{2}{3}$? You know that each part – for these examples, thirds – is equal in size or length, and you know how many of those equal parts have been selected.

Figure 3. Instruction in the Sites approach designed to create connections at Site 1 for the meaningful interpretation of the fraction symbol $\frac{2}{3}$.

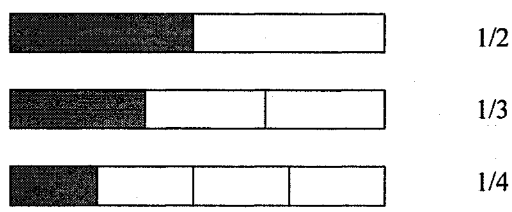
smallest, is the rule of the inverse relationship between the unit of measure of the fraction and the outcome of the measurement – that is, the rule that the larger the denominator, the smaller the fraction, or vice versa. In addition to stating this rule, I also included activities designed to create meaning for the concepts underlying the rule in the instruction for the treatment condition (see Figure 4 for an example of instruction highlighting the conceptual rationale for the rule used to order unit fractions). An important concept is the idea that the more parts the whole is divided into, the smaller each part will be (Ball, 1993; Mack, 1990, 1995, 2001). Linking this rule to models that represent it provides a conceptual rationale for the rule.

Part-whole understandings of fractions are also necessary for students' development of meaningful understandings of fractions procedures. For example, students often misapply whole-number understandings to the procedure for adding fractions (Mack, 1990, 1995, 2001), such as committing the common error of adding across both numerators and denominators (e.g., $\frac{2}{4} + \frac{1}{4} = \frac{3}{8}$). Part-whole understandings allow students to conceptualize the addends in a problem involving addition of fractions as parts of wholes to be added, and therefore can serve as meaningful referents for the procedure (Ball, 1993). As such, at Site 2 I also focused instruction on explictly highlighting the conceptual rationale for adding and subtracting fractions by focusing on part-whole relations of fractions through the use of concrete and pictorial models.

Moreover, as it relates specifically to the addition of fractions, the idea of unit is also a central one in students' understanding of the conceptual rationale for the procedure (Ball, 1993). For example, when adding two fractions, it is important to hold the unit constant, such that $\frac{2}{4} + \frac{1}{4}$ can be thought of as the problem, "I ate $\frac{2}{4}$ of a pizza for lunch and $\frac{1}{4}$ of a pizza for dinner. How much of a pizza did I eat?," which makes clear that the unit is one pizza and helps students recognize that an answer such as $\frac{3}{8}$, for a whole that is divided into fourths, does not make sense (Ball, 1993). Therefore, at Site 2 I focused on using story problems that emphasize the idea of a constant unit to contextualize symbolically represented problems and procedures.

Here is a set of fractions: {$\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$}. How could we put these numbers in order from smallest to largest?

You might think about what you already know about whole numbers and you might think that $\frac{1}{2}$ is smaller than $\frac{1}{3}$ and that $\frac{1}{3}$ is smaller than $\frac{1}{4}$, since 2 is smaller than 3, and 3 is smaller than 4. But we're not talking about whole numbers here; we're talking about fractions, so the ideas are a little different. When you want to compare and order fractions with a numerator of 1 you have to look at the denominators…remember, that bottom number tells you how many equal parts your whole is divided into. The numerator tells you how many of those equal parts you're comparing, which is 1 for each of these numbers. So here we know we're comparing one-half, one-third, and one-fourth of a same-sized whole. Let's draw this so we can really see how it works:



Now you see that with fraction numbers, if your wholes are the same size, the smaller the denominator is the bigger the parts will be, because you're dividing your whole into fewer parts. So any easy rule when ordering and comparing fractions with a numerator of 1 is to remember that a smaller denominator means a bigger fraction, or a bigger denominator means a smaller fraction. So if we list these numbers from smallest to largest following that rule, our list would look like this {$\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$}, and we can see from the picture that this is true.

Figure 4. Instruction in the Sites approach designed to create connections at Site 2: Highlighting the conceptual rationale for the rule used to order unit fractions.

In addition, during instruction for the treatment group I explicitly emphasized a meaningful understanding of symbolically represented problem solutions (Site 3) by linking problem solutions to (a) real-world or concrete contexts, or (b) other knowledge of the number system. For example, as it relates to real-world or concrete contexts, I linked symbolically represented solutions to real-word story problems and to concrete and pictorial representations of the problems (Ball, 1993; Mack, 1990, 1995, 2001). As it relates to other knowledge of the number system, Hiebert (1984) noted that using the skill of estimation is particulary effective because estimation prompts the use of intuitive and nonroutine methods (which are usually more conceptually meaningful for the user). As such, during instruction in the treatment condition I emphasized the use of estimation and benchmark numbers as methods for verifying the reasonableness of problem solutions expressed symbolically (see Figure 5 for an example of instruction emphasizing the use of estimation to verify the reasonableness of symbolic solutions to problems involving fractions).

The form of instruction for the control group, the Standard approach, involved engaging the participants in the same tasks and activities as those administered to the treatment group, but in a purely procedural fashion. That is, except for the introductory lesson on the part-whole meaning of fractions (see Table 1), I made no explicit mention of any of the connections or relationships that exist between fractional symbols, procedures, and problem solutions at any time during instruction in the control condition. For example, control group instruction did not emphasize the underlying meaning and rationale for procedures, rules, or algorithms used to solve fractions problems. Moreover, no meaning was given to the symbolically represented solutions to fraction problems during instruction for the control group, nor was the control group instructed on how to verify the reasonableness of problem solutions other than by applying a standard procedure.

Problem: $\frac{3}{4} + \frac{5}{6} = \frac{8}{10}$

Okay, so we know that $\frac{8}{10}$ is the wrong answer because Mary [student] reminded us that when we're adding fractions, we don't add the denominators. Mary gave us the right answer, $\frac{19}{12}$,...but what could you have done to check if your answer made sense? Let's look at the numbers in the problem...a good way to check your work is to estimate. So how could we estimate an answer to this problem? Well, we know that $\frac{3}{4}$ is almost 1 and we also know that $\frac{5}{6}$ is close to 1, and if we added 1 + 1, the answer would be 2. So a good estimate to the problem $\frac{3}{4} + \frac{5}{6}$ is 2. Now, using that information, which answer makes more sense, $\frac{8}{10}$ or $\frac{19}{12}$? Which is closer to 2?

Figure 5. Instruction in the Sites approach designed to create connections at Site 3: Checking the reasonableness of symbolic solutions to problems involving fractions.

Because many different meanings can be used to interpret the concept of a fraction, understanding of the part-whole meaning as it relates to fractions is a matter of convention (Ball, 1993). Therefore, because the study focused on the part-whole meaning of fractions, my instruction for the control group during the introductory part-whole lesson also explicitly highlighted the part-whole relations in fractional quantities. That is, for the control group too, I also emphasized the idea that fractions are quantities that represent parts of wholes. Unlike the instruction I administered to the treatment group, however, I only addressed the part-whole meaning of fractions for the control through the use of the region model for fractions. According to Ball (1993), when multiple and varied representations are used during instruction (such as the set, length, and area models), students' "thinking spaces" and understandings are broadened. Because standard instruction for the part-whole meaning of fractions typically only makes use of the region model (Ball, 1993), however, I used no other models during part-whole instruction with the control group.

Figure 6 illustrates an example of the instructional activities in the Standard condition designed to teach the symbols and mathematical notations of fractional numbers. As illustrated, during Standard instruction I did not connect fraction symbols and mathematical notations to graphical representations of fractions nor did I give any conceptual explanations for fractional symbols. Only conventional knowledge, such as the words and symbols that are used to represent fractional parts (Ball, 1993), was used.

With respect to the procedures used to solve fractions problems, I made no connections between rules, algorithms, or procedures and their underlying conceptual rationales during Standard instruction for none of (a) computation problems involving fractions, (b) conversion problems involving improper fractions and mixed numbers, nor (c) order and compare problems involving fractions. Students in the control group were only taught about the procedural aspects of rules and algroithms (see Figure 7 for an example related to ordering fractions).

What does the number $\frac{2}{3}$ mean?

The number two-thirds is written like this $\frac{2}{3}$, and it means two of three parts. The bottom number in the fraction tells you how many equal sized parts your whole is divided into and the top number tells you how many of those parts have been chosen. So, what do you know when you see a number like $\frac{2}{3}$? You know that each part – in this number, thirds – is equal, and you know how many of those equal parts have been selected.

Figure 6. Instruction in the Standard approach designed to teach fractional symbols without links to concrete or real-world referents.

Finally, during Standard instruction I did not connect symbolically represented solutions of fractions problems to (a) real-world or concrete contexts, nor to (b) other knowledge of the number system. I used only standard symbolic procedures as a method for verifying symbolic solutions (see Figure 8 for an example involving the addition of fractions).

Here is a set of fractions: $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$. How could we put these numbers in order from smallest to largest?

When you want to compare and order fractions with a numerator of 1, you have to look at the denominators…remember, that bottom number tells you how many equal parts your whole is divided into. The numerator tells you how many of those equal parts you're comparing. So here we know we're comparing one-half, one-third, and one-fourth. An easy rule when ordering and comparing fractions with a numerator of 1 is to remember that a smaller denominator means a bigger fraction, or a bigger denominator means a smaller fraction. So if we list these numbers from smallest to largest following that rule, our list would look like this $\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\}$.

Figure 7. Instruction in the Standard approach designed to teach the rule used to order unit fractions without highlighting its conceptual rationale.

Organization of instruction. The organization of instruction for both the treatment and control groups was identical. That is, I structed the lessons in each instructional condition to follow the same format. The instructional format included a combination of (a) direct instruction within a whole-class organization, (b) teacher- and student-led discussion within a whole-class organization, and (c) individual and small-group seat-work activities. The organization of instruction was identical for both the Sites and Standard groups to control for instructional variables that have been shown to be related to learning outcomes, such as lesson organization and structure, and time on task (Stigler et al., 1999).

Problem: $\frac{3}{4} + \frac{5}{6} = \frac{8}{10}$

Okay, so we know that $\frac{8}{10}$ is the wrong answer because Mary [student] reminded us that when we're adding fractions we don't add the denominators. Mary gave us the right answer, $\frac{19}{12}$...she didn't add the denominators but she found one common denominator for fourths and sixths, which is twelfths. So $\frac{3}{4}$ is equal to $\frac{9}{12}$ and $\frac{5}{6}$ is equal to $\frac{10}{12}$, so her answer of $\frac{19}{12}$ is right because $\frac{9}{12} + \frac{10}{12} = \frac{19}{12}$ ...remember, we only add up the numerators.

Figure 8. Instruction in the Standard approach designed to teach students to use procedures, rules, or algorithms to verify their solutions to fractions problems.

Although the instructional organization was identical in both treatment conditions, the specific activities and tasks I used during instruction differed as a function of instructional group. That is, while both groups experienced direct instruction and whole-class discussions, the particular way I taught the mathematical content during each lesson differed according to the treatment condition. For example, whole-class discussions in the Sites approach focused on creating links between procedures and concepts; in the Standard approach, whole-class discussions focused strictly on procedures used to complete tasks involving fractions. Likewise, although both the treatment and control groups received mathematical activities of identical content to complete individually and in small groups, the format of the activities differed in accordance with the instructional goals outlined for each group. For example, activities in the Sites approach were designed to highlight the connections between procedures and concepts involving fractions; in the Standard approach, activities were designed to focus only on the procedural aspects of tasks involving fractions.

Lessons that are structured to include both whole-class and independent student activities, such as those described above, have been shown to be effective instructional formats with respect to specific classroom goals. For example, a whole-class instructional organization can be an excellent forum for student learning if a teacher is able to elicit and build upon students' thinking and address conceptual issues during problem-solving (NCTM, 2000; Gearhart et al. 1999; Hiebert et al. 1997; Saxe et al., 1999; Saxe et al., 2001; Saxe et al., 2005). These goals can be attained by posing the right types of questions – those that elicit students' understandings and misunderstandings (Ball, 1993; Gearhart et al., 1999; Hiebert et al. 1997) and by engaging in "public problem-solving" (Gearhart et al., 1999) – solving problems in a whole-group context intended for all students to hear (Ball, 1993; Gearhart et al., 1999; Hiebert et al., 1997; Stigler et al., 1999). Furthermore, conceptual understandings can also be enhanced during "public problem-solving" contexts if students are given opportunities to consider how problem-solving procedures link to concepts (NCTM, 2000; Gearhart et al., 1999).

Therefore, my decision to include whole-class instructional activities in both the Sites and Standard approaches was supported by evidence that has demonstrated the effectiveness of this type of lesson structure. In the Standard instructional group, however, I only posed questions that addressed the procedural aspects of fractions topics and I addressed only those same procedural aspects during any "public problem-solving" that took place. In other words, I posed no questions that addressed the conceptual issues of fractions and I made no links between problem-solving procedures and their underlying concepts or rationales during Standard instruction.

Finally, including a seat-work component to the organizational structure of a lesson can also be an effective way to support student learning (Ball, 1993; Gearhart et al., 1999; Hiebert et al., 1997; NCTM, 2000; Stigler et al., 1999). That is, independent student activities – those that are completed either on an individual basis or in small groups, but independent of the teacher – afford students opportunities to reflect upon and use their understandings in personally meaningful ways (Gearhart et al., 1999; Hiebert et al., 1997). As such, my decision to include an independent student work component as part of the instructional organization for both the Sites and Standard approach was also supported.

Although it was possible that students in the control group could have forged conceptual understandings independent of my instruction when working collaboratively with their classmates (Hiebert et al., 1997), given that the student activities in the Standard approach were designed to focus only on procedural aspects of fractions, it is likely that few opportunities to engage conceptual issues presented themselves. That is, research suggests that low-level tasks rarely incite students to engage in high-level mathematical thinking (Henningsen & Stein, 1997; Stein, Grover, & Henningsen, 1996).

Coding/Scoring

Scoring

Vocabulary test. The vocabulary test contained 43 items in total; 30 real words and 15 pronounceable nonwords. I calculate the scores on the vocabulary test by subtracting the number of selected nonwords from the number of correctly identified real words.

Fractions knowledge test. Saxe et al.'s (2001) fractions knowledge test contained 17 items that assessed the participants' procedural understanding of fractions and 12 items that assessed the participants' conceptual understanding of fractions. I scored the test according to a reliable scoring procedure that was established in previous research conducted by Rayner, Pitsolantis, and Osana (2006; for example, the percentage of agreement for inter-rater reliability was reported to be 97.06%, and the reported calculated Kappa coefficient was $r_k = .97$). Using the previously established scoring procedure, I generated two overall scores – one for each of the procedural and conceptual subscales of the test. I calculated subscale scores by summing the points received for each item within a subscale, resulting in a total score for each of the procedural and conceptual knowledge components of the test. Correct responses were given a score of 1 and incorrect responses were given a score of zero. A half-mark was permitted for Item 10b on the conceptual subscale[3]. Thus, the maximum total summed score for the procedural subscale was 17. For the conceptual subscale, the maximum total summed score was 12.

Fractions knowledge interview. The fractions knowledge interview contained a total of 17 items that were designed to assess the participants' abilities to connect procedures and concepts at each of Hiebert's (1984) three sites. Five interview items were included that assessed Site 1 understandings. At Site 2, a total of 10 items were included that assessed the participants' understandings. The final two items assessed the participants' Site 3 understandings. I generated three overall scores – one for each of the

[3] Item 10b involved the concrete representation of the addition of two fractions. The representation was considered complete only if the solution of the addition was also included in the representation. A half-mark, rather than a full mark, was assigned to a representation that included only the two addends in the problem and not the solution of the addition.

Site 1, Site 2, and Site 3 subscales of the interview. I calculated subscale scores by summing the points received for each item within a subscale, resulting in a total score for each of the Site 1, Site 2, and Site 3 components of the interview. Correct responses were given a score of 1 and incorrect responses were given a score of zero. Thus, the maximum total summed score for the Site 1 subscale was 5. For the Site 2 subscale, the maximum total summed score was 10. For the Site 3 subscale, the maximum total summed score was 2.

For the Site 1 subscale items, responses were considered correct only when both the symbolic notation and the pictorial representation of the fraction were accurate. For the Site 2 subscale items, responses were considered correct only if (a) a valid procedure was used to solve the problem and (b) the conceptual underpinning of the procedure, or its rationale, could be identified. For the Site 3 subscale items, responses were scored as correct only if the student could verify the reasonableness of the symbolically represented problem solution by either (a) providing a concrete representation; (b) framing the problem within a real-world context; or (c) using knowledge of the number system, such as estimation and benchmarks. The criteria used to determine whether or not the content of a student's response was correct are described in further detail below.

Qualitative Coding

In addition to the quantitative scoring procedure, I also analyzed the fractions knowledge interview qualitatively to provide a more detailed picture of the participants' abilities to link fractions procedures and concepts both within and across Hiebert's (1984) three sites. The qualitative analysis was employed in an effort to yield richer, more descriptive data vis-à-vis the participants' strengths and weaknesses in connecting procedures and concepts. For example, within each site, I analyzed the students' responses to assess precisely where difficulties in linking symbols or procedures with concepts occurred (e.g., the student was able to represent the symbol but not the conceptual referent for a fractional quantity).

The qualitative analysis involved describing the participants' responses, whether correct or incorrect, to each item within a subscale of the interview. The scoring rubric is presented in its entirety in Appendix G. I analyzed the items within each of the Site 1 and Site 2 subscales of the interview along two dimensions. Specifically, I analyzed Site 1 items, those that pertained to Hiebert's symbolic interpretation construct, for the participants' ability to represent (a) the symbolic notation of a given fraction, and (b) the concrete or pictorial representation of a given fraction. I analyzed Site 2 items, those that pertained to Hiebert's procedural execution construct, for the participants' ability to (a) use a valid and correct procedure to solve fraction problems involving conversions, ordering, comparing, and addition; and (b) either link a procedure to a concrete representation or provide an explanation of the underlying rationale for a procedure used. Lastly, I analyzed Site 3 items (Items 9 and 10), those that pertain to Hiebert's solution evaluation construct, along one dimension only, the participants' ability to use either a concrete, real-world, or knowledge of the number system approach to evaluate the reasonableness of symbolically represented problems involving fractions.

The criteria for a correct response to each item within a subscale are described in detail in Appendix G. Responses provided by participants that did not meet those criteria were coded as incorrect, followed by a detailed description of the incorrect form of the response. Appendix G also contains the descriptions of the different forms of incorrect responses.

After coding each interview, I trained a second individual to use the scoring rubric. The trained rater then separately coded a random selection of 20% of the sample data. Thereafter, I calculated inter-rater reliability between our scoring on that randomly selected sample. The percentage of agreement was 93.33% and the calculated Kappa was $r_k = .91$.

Results

I begin this chapter by presenting the relevant descriptive statistics with respect to the quantitative data that were gathered. Following that, I present the results of the quantitative analyses. I analyzed both the fractions knowledge test (Saxe et al., 2001) and the fractions knowledge interview quantitatively. I first report the results pertaining to the fractions knowledge test and then the results related to the fractions knowledge interview. In the third section of this chapter I present the results of the qualitative analyses. I analyzed each subscale of the fractions knowledge interview qualitatively along different dimensions, as noted above. I report the results in subscale order as enumerated in Hiebert's (1984) theory: Site 1, Site 2, and Site 3, along with a brief summary following the results of each subscale.

The original sample in the study ($N = 72$) was reduced by two, with the loss of two participants from the control group. That is, because of a high number of absences over the course of the instructional sessions, I did not include the data that were collected from the two participants in question in any of the analyses[4]. Final group sizes, therefore, were $n = 36$ and $n = 34$ for the treatment and control groups, respectively.

Descriptive Statistics

Vocabulary Test

The highest possible score for the vocabulary test was 30. The vocabulary test scores for the treatment group ranged from a low of 3 to a high of 22. The mean score for the treatment group was $M = 14.36$ ($SD = 4.62$). The vocabulary test scores for the

[4] Data for any participant who was absent for three or more lessons were excluded from all final analyses.

control group ranged from a low of 5 to a high of 23. The mean score for the control group was $M = 15.5$ ($SD = 3.95$).

Fractions Knowledge Test

The maximum score attainable for the procedural knowledge subscale of the test was 17. The highest possible score for the conceptual knowledge subscale of the test was 12. I converted each raw subscale score to a percentage score to obtain equally weighted scores. The pretest and posttest means, standard deviations, and range of scores for each subscale of the fractions knowledge test are presented in Table 2.

Fractions Knowledge Interview

The maximum score attainable on each subscale of the interview was 5, 9, and 2 for the Site 1 (symbol interpretation), Site 2 (procedural execution) [5], and Site 3 (solution evaluation) subscales, respectively. I converted the raw subscale scores to percentage scores to obtain equally weighted scores. The pretest and posttest means, standard deviations, and range of scores for each subscale of the fractions knowledge interview are presented in Table 3.

Quantitative Analyses

Fractions Knowledge Test

I addressed the first research question by performing a 2 x 2 x 2 (Group x Time x Test) repeated measures MANCOVA with Group (treatment, control) as the between groups factor and Time (pretest, posttest) and Test (procedural, conceptual) as within

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[5] Data pertaining to Item 8 of the Site 2 subscale were dropped from all analyses because of an error in the wording of the question.

Table 2

Means, Standard Deviations, and Range of Scores for the Procedural and Conceptual Subscales of the Fractions Knowledge Test at Pretest and Posttest (N = 70)

	Control (n = 34)						Treatment (n = 36)					
	Pretest			Posttest			Pretest			Posttest		
Subscale	*M*	*SD*	Range	*M*	*SD*	Range	*M*	*SD*	Range	*M*	*SD*	Range
Procedural	39.45	28.14	0 - 94.1	69.20	25.95	0 - 100	36.76	30.17	0 - 100	65.03	28.95	5.9 - 100
Conceptual	48.16	22.38	8.3 - 91.7	59.07	18.79	16.7 - 100	43.06	22.97	0 - 100	67.48	16.54	25 - 100

Note. Means, standard deviations, and ranges are presented in percent scores.

Table 3

Means, Standard Deviations, and Range of Scores for Each Subscale of the Fractions Knowledge Interview at Pretest and Posttest ($N = 29^a$)

	Control (n = 15)						Treatment (n = 14)					
	Pretest			Posttest			Pretest			Posttest		
Subscale	*M*	*SD*	Range	*M*	*SD*	Range	*M*	*SD*	Range	*M*	*SD*	Range
Site 1	76	27.46	0 - 100	84	17.24	60 - 100	58.57	40.36	0 - 100	91.43	17.03	40 - 100
Site 2	13.33	13.41	0 - 33.3	28.89	24.77	0 - 66.7	19.84	27.62	0 - 100	53.17	37.54	0 - 100
Site 3	20	31.62	0 - 100	23.33	32	0 - 100	21.43	32.31	0 - 100	32.14	37.25	0 - 100

Note. Means, standard deviations, and ranges are presented in percent scores.

[a]Pretest data missing for one participant in the treatment group (n = 14).

groups factors, and the measure of general cognitive ability (vocabulary test) as the covariate. More specifically, a doubly-multivariate analysis of covariance was performed on each measure of fractions knowledge (i.e., procedural knowledge subscale and conceptual knowledge subscale). Repeated measures ANOVA requires commensurate dependent variables (i.e., dependent variables that are all scaled in the same way; Tabachnick & Fidell, 2001), and as such, the analysis was performed using standardized z scores on each measure (i.e., procedural knowledge subscale scores, conceptual knowledge subscale scores, and vocabulary scores). For ease of interpretation and readability, however, all performance profiles in the graphs below are plotted using the original percent scores.

No data were missing and results of evaluation of assumptions of doubly-multivariate analysis of covariance were satisfactory. For example, with respect to sample size, doubly-multivariate ANOVA requires that there be more cases in each group than there are dependent variables (Tabachnick & Fidell, 2001). This assumption was satisfied given that there were four dependent variables (pretest procedural, posttest procedural, pretest conceptual, posttest conceptual) included in the analysis and 34 participants in the smallest group (the control group). Multivariate normality must also be satisfied for doubly-multivariate ANOVA, but deviation from normality is not expected when there are more cases than dependent variables in the smallest group and when group sizes are relatively equal (Tabachnick & Fidell); therefore, this assumption was satisfied as well. Finally, homogeneity of the variance-covariance matrices must also be met in doubly-multivariate ANOVA, but this evaluation is only necessary in the case of notably discrepant sample sizes (Tabachnick & Fidell).

Table 4 presents the results of the MANCOVA tests of statistical significance computed using an alpha level of .05. As noted in Table 4, The Group x Time x Test interaction was statistically reliable, multivariate $F(1, 67) = 7.85, p < .01, \eta^2 = .11$. The Group x Time interaction was also statistically significant, multivariate $F(1, 67) = 4.45$, $p < .05, \eta^2 = .06$. Figure 9 plots the profiles for the two groups on the measure of conceptual knowledge of fractions over the two testing sessions. The findings indicated that the treatment group's gains in conceptual knowledge of fractions was significantly greater than the control group's gains from pretest to posttest. Figure 10 plots the profiles for the two groups on the measure of procedural knowledge of fractions over the two testing sessions. The results revealed no main or interaction effects with respect to procedural knowledge of fractions. knowledge of fractions. Given these results, partial support was provided for the first research hypothesis that the Sites instructional approach would result in greater improvements in both procedural and conceptual understandings of fractions for students in the treatment condition.

Table 4

MANCOVA Results for Procedural and Conceptual knowledge of fractions (N = 70)

Source	*df*	*F*	η	*p*
	Between subjects			
Group	1	0.49	.01	.49
Vocabulary	1	21.11**	.24	.00
S within-group error	67	(2.21)		

Table 4 (*continued*)

Within subjects				
Time	1	0.004	.00	.95
Time x Group	1	4.45*	.06	.05
Test	1	0.002	.00	.96
Test x Group	1	2.69	.04	.11
Time x Group x Test	1	7.85**	.11	.01
Time x Test error	67	(.310)		

Note. Values enclosed in parentheses represent mean square errors. *S* = subjects.

$*p < .05$. $**p < .01$.

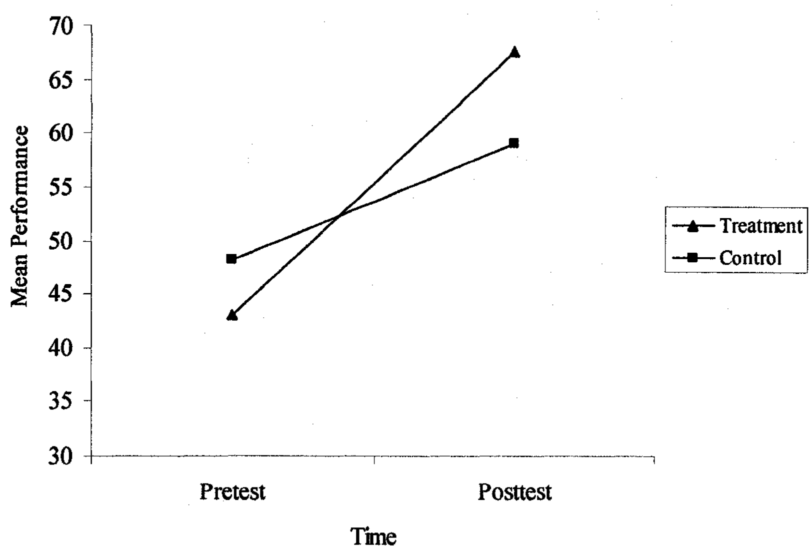

Figure 9. Profile plot of mean performance for the conceptual knowledge subscale of the fractions knowledge test ($N = 70$).

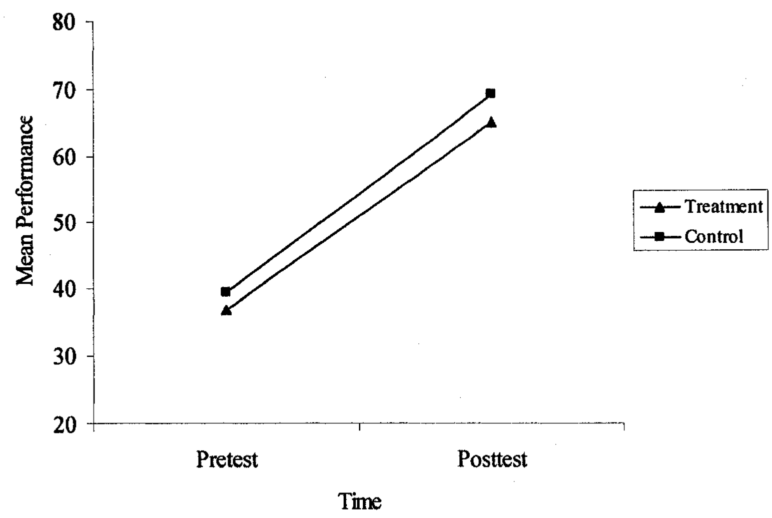

Figure 10. Profile plot of mean performance for the procedural knowledge subscale of the fractions knowledge test ($N = 70$).

Fractions Knowledge Interview

To address the second research question, I performed a 2 x 2 x 3 (Group x Time x Test) repeated measures MANCOVA with Group (treatment, control) as the between groups factor and Time (pretest, posttest) and Test (Site 1, Site 2, Site 3) as within groups factors, and the measure of general cognitive ability (vocabulary test) as the covariate. More specifically, a doubly-multivariate analysis of covariance was performed on each subscale of the fractions knowledge interview (i.e., Site 1 (symbol interpretation) subscale, Site 2 (procedural execution) subscale, and Site 3 (solution evaluation) subscale. Given the requirement for equally scaled dependent variable scores in repeated measures ANOVA (Tabachnick & Fidell, 2001), this analysis was also performed using standardized z scores on each measure (i.e., Site 1 subscale scores, Site 2 subscale scores, Site 3 subscale scores, and vocabulary scores); however, the related profile plot below is graphed using the original percent scores.

Pretest data for the Site 2 subscale for one participant in the treatment group were missing. Consequently, I did not include the data from this participant in the analysis and the following MANCOVA results are based on sample sizes of $n = 14$ and $n = 15$ for the treatment and control groups, respectively. Results of an evaluation of the assumptions of doubly-multivariate analysis of variance as described by Tabachnick and Fidell (2001) and noted above were satisfactory. For example, sample sizes were not notably discrepant and there were more cases in each group than the number of dependent variables included in the analysis (six dependent variables in total were included in the analysis: (a) Site 1 pretest; (b) Site 1 posttest; (c) Site 2 pretest; (d) Site 2 posttest; (e) Site 3 pretest; (f) Site 3 posttest). As such, assumptions for sample size, multivariate normality, and homogeneity of variance-covariance matrices were satisfied.

Table 5 presents the results of the MANCOVA tests of statistical significance computed using an alpha level of .05. As indicated in Table 5, The Group x Time interaction was statistically reliable, multivariate $F(1, 26) = 5.47, p < .05, \eta^2 = .17$. No other main or interaction effects were significant. Therefore, the results indicated that the treatment group's gains in ability to link concepts and procedures across Site 1, Site 2, and Site 3 was significantly greater than the control group's gains from pretest to posttest.

Figure 11 plots the profiles of the two groups over the two testing sessions, collapsing across test type. As can be seen in Figure 11, the groups' overall mean performance (i.e., across the three sites) across time differed with respect to the type of

Table 5

MANCOVA Results for Links between Procedural and Conceptual Understandings of Fractions at Site 1, Site 2, and Site 3 (N = 29)

Source	*df*	*F*	η	*p*
		Between subjects		
Group	1	1.60	.06	.22
Vocabulary	1	4.06	.14	.06
S within-group error	26	(0.49)		
		Within subjects		
Time	1	0.03	.00	.86
Time x Group	1	5.47*	.17	.03
Test	2	0.02	.00	.98
Test x Group	2	2.31	.16	.12
Time x Group x Test	2	1.84	.13	.18
Time x Test error	67	(.310)		

Note. Values enclosed in parentheses represent mean square errors. *S* = subjects.

*$p < .05$.

instruction each received, indicating that the gains in mean performance were dependent on the type of treatment (i.e., Sites instructional approach and Standard instructional approach). Both groups' overall mean performance at pretest was very similar and

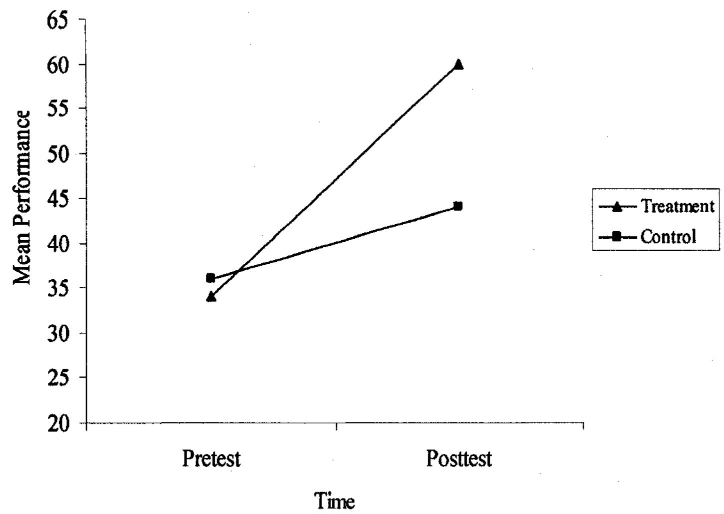

Figure 11. Profile plot of overall mean performance on the fractions knowledge interview ($N = 29$).

although both group's overall performance increased from pretest to posttest, the treatment group demonstrated a greater rate of increase. The results, therefore, provided support for the second research hypothesis that the Sites instructional approach would result in greater improvements in linking procedural and conceptual understandings of fractions at each of Hiebert's three sites for students in the treatment condition.

Qualitative Analyses

In this section, I present the results of the qualitative analyses I performed on the fractions knowledge interview. Given the quantitative results reported above with respect to the interview – that is, the finding that treatment group demonstrated an overall greater ability to link fractions procedures and concepts at each of the three sites after instruction (see Figure 11) – I present the results in this section to augment the above findings. In this way, I provide a more detailed picture of the quality of each groups' understandings as well as misconceptions as they relate to Site 1, Site 2, and Site 3 knowledge of fractions both before and after instruction.

The qualitative analyses entailed describing the participants' responses to each item within a subscale. That is, based on the students' responses, I first generated descriptive codes that reflected the form that a correct or an incorrect response took. Thereafter, I calculated the frequency of participant responses coded at each descriptive category, as well as the percent of time that a particular response occurred out of all responses. I performed these calculations separately for each of the three subscales of the fractions knowledge interview (i.e., the Site 1, Site 2, and Site 3 subscales), and at each testing session (pretest and posttest). Below I present the results of the qualitative analyses organized in the following manner. I present the results for each subscale separately according to the various dimensions that I examined within each. Following the results of each subscale dimension is a summary of the general findings within that dimension.

Site 1: Symbol Interpretation

Site 1 understandings involve the ability to link mathematical symbolic notations to their underlying meanings (Hiebert, 1984). Therefore, I analyzed the Site 1 subscale along two dimensions: notation and representation. For the notation component, I examined the participants' ability to accurately represent the symbolic notation of a given fraction. For the representation component, I examined the participants' ability to accurately represent the concrete pictorial representation of a given fraction. The results for the notation component of the Site 1 subscale are presented first.

Site 1 notation, pretest. Table 6 presents the pretest and posttest results for the symbolic notation component of the Site 1 items. There were five items at Site 1 that assessed the participants' ability to correctly represent the symbolic notation for fractions. Therefore, frequencies for each group are reported as a total out of 75 (15 participants x 5 responses per participant).

As noted in Table 6, at pretest, approximately 95% of the responses coded for the symbolic notation of fractions both smaller than and larger than one were correct for participants in the control group. In comparison, 92% of responses that treatment group students provided at pretest entailed the correct symbolic notation for the same fractional quantities. The most frequent error made by participants in the control group at pretest was the compounded notation error, accounting for 4% of the incorrect responses. By contrast, this error only accounted for a little over 1% of the incorrect responses that treatment group students provided at pretest. The compounded notation error consisted of compounding the numbers in a fractional quantity to produce a wrong symbolic notation. For example, one student represented the number one and four-fifths symbolically as $\frac{1}{4/5}$. Similarly, as illustrated in Figure 14, another student represented the number one and three-eighths symbolically as $\frac{1}{3/8}$. The compounded notation error was only made for mixed numbers.

For treatment group students at pretest, the most frequent errors in symbolic notation for fractions were the no notation and irrelevant notation errors, each accounting for nearly 3% of the incorrect responses (see Table 6). By contrast, the no notation

Table 6

Frequency (f) and Percentage (%) of Distributions at Pretest and Posttest for the Notation Component of the Site 1 (Symbol Interpretation) Subscale of the Fractions Knowledge Interview (N = 30)

	Control (n = 15)				Treatment (n = 15)			
	Pretest		Posttest		Pretest		Posttest	
Response Category	*f*	%	*f*	%	*f*	%	*f*	%
Correct notation	71	94.67	73	97.33	69	92.00	75	100
Incorrect notation								
No notation	1	1.33	0	0	2	2.67	0	0
Irrelevant	0	0	0	0	2	2.67	0	0
Inverted	0	0	0	0	1	1.33	0	0
Compounded	3	4	0	0	1	1.33	0	0
Unit error	0	0	2	2.67	0	0	0	0

Note. Frequencies for each group are reported as a total out of 75.

response accounted for only 1.33% of incorrect responses for control group students at pretest, and no student in the control group made the irrelevant notation error at pretest. The no notation error was assigned when a student could not provide a symbolic notation for a given fractional quantity. In other words, the student left the item blank. For example, one student in the treatment group could not write the symbolic notation for the fraction two-eighths. The remaining no notation errors that were made were for a fractional quantity larger than one. For example, two students could not write the symbolic notation for the number one and three-eighths.

The irrelevant notation error was assigned when a student wrote a symbolic notation that was irrelevant in form with respect to common fractions. For example, one student represented the number one-half symbolically as $\frac{1/2}{0}$, and another student represented the number one-half symbolically as $\frac{1}{-}$. The results therefore indicated that both irrelevant notation errors were made for the fraction one-half.

Treatment group students also made the inverted notation error at pretest, accounting for just over 1% of the incorrect responses (see Table 6). On the other hand, no student in the control group made the inverted notation error at pretest. The inverted notation error referred to reversing or inverting the numerator and denominator in a given fractional quantity to produce an incorrect symbolic notation. For example, for the fraction five-ninths, one student inverted the numerator and denominator and wrote the answer $\frac{9}{5}$. The inverted notation error occurred only in the case of proper fractions (where the numerator is smaller than the denominator) and only for fractional quantities smaller than one.

Site 1 notation, posttest. As indicated in Table 6, at posttest, control group students responded with correct symbolic notations for fractional quantities both smaller than and larger than one 97% of the time. In comparison, treatment group students responded correctly 100% of the time at posttest. That is, every error that the students in either the treatment or the control group made at pretest in representing fractions symbolically was corrected at posttest. On the other hand, control group students made a new error at posttest, the unit error, accounting for nearly 3% of the incorrect responses.

The unit error code was assigned to a response that demonstrated a misinterpretation of the unit (or whole) for a given fractional quantity, resulting in an incorrect symbolic notation. Both unit errors were made on the same subscale item, for a fractional quantity smaller than one. That is, two students misinterpreted the unit for the number one-half to be larger than one, and both students wrote the symbolic notation $1\frac{1}{2}$ to represent the fraction one-half.

In summary, the participants in the control group at pretest showed a slightly better ability to correctly represent fractional quantities symbolically than the participants in the treatment group. At the posttest, on the other hand, the results indicated the reverse, with treatment group students demonstrating slightly more ability over control group students to correctly represent fractions symbolically. At the pretest, the errors made by the treatment group were more diverse than those made by students in the control group. That is, at the pretest, the errors made by the treatment group fell into one of four different categories (no notation, irrelevant notation, inverted notation, and compounded notation); the errors made by the control group fell into only one of two categories (no notation and compounded notation). Moreover, at pretest, treatment group students made errors in the symbolic notation for fractions both smaller than and larger than one. The pretest errors made by the control group students, on the other hand, were all related to the symbolic notation of fractional quantities that are larger than one. At the posttest, all pretest errors that the treatment group students made were corrected. Likewise, all pretest errors that the control group made were corrected, but the control group students made a new error at the second testing session (the unite error), one that was not made at pretest.

Site 1 representation, pretest. Table 7 presents the pretest and posttest results for the representation component of the Site 1 subscale items. There were five items at Site 1 that assessed the participants' ability to produce a correct pictorial representation or model for a given fractional quantity. Therefore, frequencies for each group are reported as a total out of 75 (15 participants x 5 responses per participant).

At pretest, participants in the control group drew a correct representation or model for fractions both smaller than and larger than one just over 77% of the time (see Table 7). By comparison, participants in the treatment group correctly represented fractions close to 63% of the time at pretest. Therefore, the results indicated that at the pretest, control group students responded with correct representations for fractions more frequently than treatment group students did.

The representational error that occurred most frequently at pretest for both groups was the no representation error, accounting for nearly 7% of incorrect responses for the control group and close to 11% of incorrect responses for the treatment group. The no representation error indicated that no pictorial representation or model for a given fraction could be provided. In other words, the student left the item blank. Although both groups of students made this error relatively frequently, the results indicated that participants in the treatment group responded more often than participants in the control group by leaving an item blank. Students in both groups made the no representation error when asked to represent fractional quantities both smaller than and larger than one. For

Table 7

Frequency (f) and Percentage (%) of Distributions at Pretest and Posttest for the Representation Component of the Site 1 (Symbol Interpretation) Subscale of the Fractions Knowledge Interview (N = 30)

	Control (*n* = 15)				Treatment (*n* = 15)			
	Pretest		Posttest		Pretest		Posttest	
Response Category	*f*	%	*f*	%	*f*	%	*f*	%
Correct representation	58	77.33	64	85.33	47	62.67	69	92.00
Incorrect								
No representation	5	6.67	2	2.67	8	10.67	1	1.33
Incorrect whole	1	1.33	2	2.67	8	10.67	2	2.67
Direct translation	2	2.67	0	0	3	4	2	2.67
Unequal partitions	2	2.67	1	1.33	4	5.33	0	0
Incorrect shading	3	4	2	2.67	3	4	1	1.33
Incorrect number of parts	2	2.67	2	2.67	2	2.67	0	0
Unequal units	2	2.67	2	2.67	0	0	0	0

Note. Frequencies for each group are reported as a total out of 75.

example, with respect to fractions that are smaller than one, several students were unable to draw a representation or model for the fraction two-sevenths. With respect to fractions that are larger than one, several students were unable to draw a representation or model for the fraction one and four-fifths.

At pretest, in addition to the inability to provide a pictorial representation for a fraction, representing fractions with incorrect wholes was the other major source of difficulty for students in the treatment group, accounting for nearly 11% of incorrect responses. For control group students, on the other hand, the incorrect whole error only accounted for 1.33% of responses at pretest. The incorrect whole error was assigned to a representation that failed to demonstrate a clearly delimited whole. Depending on the model that was used to represent the fraction, this error took on different forms. For example, one student attempted to use the set model to represent the fraction two-thirds but did not enclose the set, thus failing to clearly indicate what he considered the whole to be. Another student attempted to use the region model to represent the fraction two-thirds but drew a fractured whole – that is, the student partitioned the region into fourths and then crossed off one of the fourths to make thirds, and stated, "I'm forgetting about this part; pretend it's not there." The incorrect whole error was also assigned to a representation that was depicted by the wrong number of wholes. For example, Figure 12 illustrates one student's response that was coded by the incorrect whole error. The student represented the number one and three-eighths with only one whole and in addition, drew a fractured whole. That is, the student first partitioned the whole into fourths and then crossed out one of the fourths, explaining, "that one's not there, so this is 1 and 3...," the student then further partitioned one of the thirds into eight parts and explained, "and now I'm going to shade 8."

Another error that occurred frequently for students in the treatment group at pretest was the unequal partitions error, accounting for 5.33% of the incorrect responses. By comparison, students in the control group made this error only 2.67% of the time at

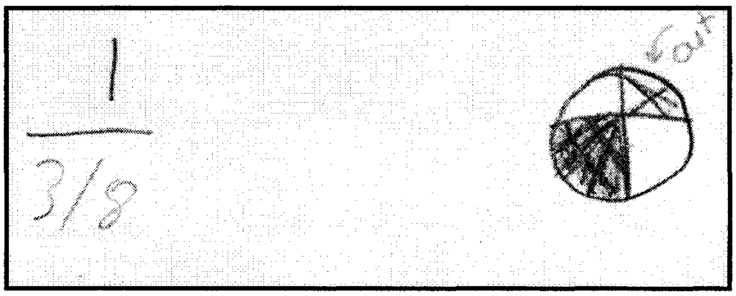

Figure 12. Illustration of the compounded notation error and the incorrect whole representational error.

pretest. The unequal partitions error was assigned to a representation that was depicted by a whole that was not equally partitioned. Every instance of this error involved the use of the region model for fractions. Therefore, the unequal partitions error referred to a whole in which the parts were represented by unequal areas. For example, one student, who used a circular region to represent the fraction two-thirds, first partitioned the circle in half and then further partitioned one of the halves in two. The resulting representation was a whole with three partitions, but the partitions did not have equal area. Another student, as illustrated in Figure 13, used a square region to represent the fraction one-sixth but partitioned the square into six parts of varying sizes and explained that unless it was clearly specified in the problem, the partitions did not have to have equal areas. Yet another example was a student who used a rectangular region to represent the fraction four-fifths, and who partitioned the rectangle into five parts of different sizes. All of the unequal partitions errors occurred for fractions of thirds, fifths, or sixths. That is, the results indicated that students had difficulty partitioning wholes for (a) fractions with an odd number in the denominator, and (b) fractions that are encountered less regularly (i.e., fifths and sixths) in most standard mathematics classrooms as compared to more familiar fractions, such as halves or quarters.

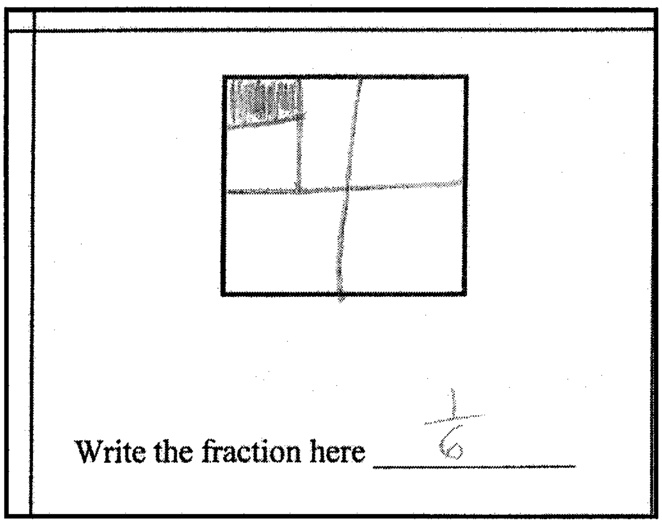

Figure 13. Illustration of the unequal partitions representational error.

The incorrect shading error occurred in equal frequency for both groups at pretest, accounting for 4% of the incorrect responses. The incorrect shading error referred to a representation in which the incorrect number of parts was shaded. One student made this error for a fractional quantity smaller than one. That is, for the fraction five-ninths, the student drew a rectangular region partitioned into ninths but did not shade any of the partitions to reflect the value of the fraction. All other occurrences of the incorrect shading error were made for fractional quantities larger than one. Moreover, in each of those instances, the error entailed failing to shade the part of the representation that corresponded to the whole number. For example, three students represented the fraction one and four-fifths by drawing two circular regions, one to represent the whole number 1 and a second to represent the fraction $\frac{4}{5}$. All three students correctly shaded four out of the five partitions in the region used to represent the fractional part of the quantity, but none of the students shaded the region that represented the whole number in the quantity. Similarly, two students made the same error in representing the fraction one and three-eighths. That is, neither student shaded the whole that was drawn to represent the whole number 1 in the quantity one and three-eighths.

Directly translating from symbol to pictorial representation was another source of difficulty in representing fractions concretely for participants in both groups at pretest, accounting for 4% of the incorrect responses by treatment group students and close to 3% of the responses by control group students. A direct translation error in representing a fraction reflected a misunderstanding of the part-whole meaning of fractions. That is, the numbers in a fractional quantity were not treated as parts of wholes but rather as independent whole numbers. Figure 14 illustrates this error. In this example, the student represented the fraction $\frac{1}{2}$ as a whole with three partitions, where one partition was shaded to reflect the numerator and the two partitions left unshaded reflected the denominator. Another student represented the fraction $\frac{1}{6}$ by partitioning a square region into six parts and shading all six partitions. The student explained that the numerator 1 was represented by the one whole and the denominator six was represented by the six shaded parts in the whole.

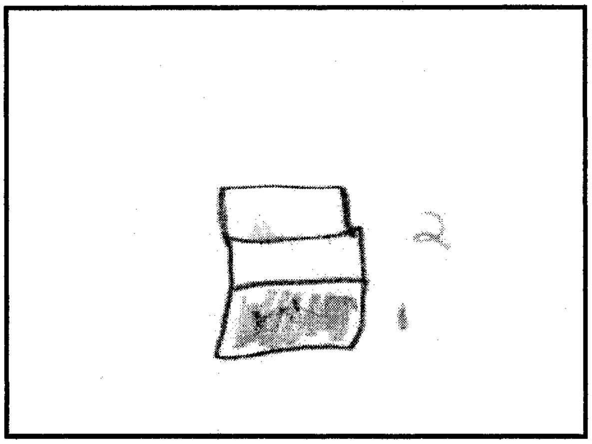

Figure 14. Illustration of the direct translation representational error.

Site 1 representation, posttest. As noted in Table 7, at posttest, just over 85% of responses by participants in the control group, compared to 92% of responses by participants in the treatment group, were correct with respect to generating pictorial representations of fractional quantities both smaller than and larger than one. The posttest results also indicated that some of the pretest errors that were made when representing fractions pictorially disappeared after the instruction for students in both instructional groups. For example, no control group responses were categorized by the direct translation error at posttest. With respect to the treatment group, both the unequal partitions error and the incorrect number of parts error disappeared at posttest. Furthermore, at posttest, the no representation error – one of the errors that occurred most frequently for both groups at pretest – was no longer the major source of error in representing fractions for either group. That is, treatment group students made this error only 1.33% of the time at posttest and control group students made this error only 2.67% of the time at posttest. As demonstrated by the results, the treatment group made greater improvements on correcting this error from pretest to posttest than the control group. Furthermore, at the pretest, students in both groups made the no representation error for fractional quantities both smaller than one and for mixed numbers. At the posttest, by contrast, every occurrence of the no representation error was for a mixed number.

Students in both groups continued to have difficulty in correctly representing the whole for fractions at posttest, accounting for close to 3% of the incorrect responses for both groups. Compared to pretest scores, however, treatment group students improved in this area of difficulty while control group responses indicated an increased difficulty in correctly representing the whole. Unlike the pretest results, no incorrect whole errors at posttest were made for fractions that were represented by a region model. That is, every occurrence of the incorrect whole error at posttest involved the set model for fractions. Therefore, as described above, the error entailed a failure to enclose the set when a group of objects was drawn to represent a fractional quantity.

No change in the frequency of the incorrect number of parts and unequal units errors occurred from pretest to posttest for students in the control group. That is, at the posttest, each of these errors continued to account for nearly 3% of the incorrect responses for the control group. The incorrect number of parts error was assigned to a representation in which the number of partitions in the whole did not match the denominator of the fraction. For example, one student, who used a circular region to represent the fraction five-ninths, partitioned the circle into eight parts rather than nine. Another student, who also used a circular region, represented the fraction two-eighths by partitioning the circle into nine parts rather than eight. With respect to the unequal units error, this code was only assigned when a pictorial representation with more than one whole was drawn. The error involved drawing each whole in the representation as a different size or shape. For example, as illustrated in Figure 15, to represent the fraction one and three-eighths, one student drew a rectangular region to represent the whole number 1 and a circular region to represent the fraction three-eighths. A second occurrence of this error was made by a student who represented the fraction one and four-fifths by drawing two circular regions, one to represent the whole number in the quantity and the other to represent the fraction four-fifths. The student in question, however, did not draw equivalent sized wholes but rather represented the whole number in the quantity by a smaller sized whole. At the posttest, every occurrence of this error was related to representations of mixed numbers.

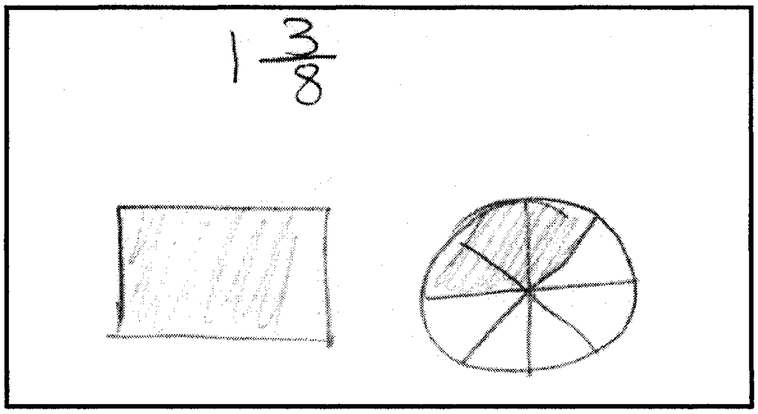

Figure 15. Illustration of the unequal units representational error.

In summary, with respect to correctly representing fractions pictorially, students in the control group at pretest demonstrated a greater ability than the treatment group to do so. At the posttest, however, the results indicated that the treatment group responded more frequently with correct pictorial representations or models for fractions. The results also indicated that at the pretest, the frequency of errors for the treatment group were more concentrated in type. That is, nearly 30% of the time, treatment group students made one of three errors at pretest: the no representation error, the incorrect whole error, or the unequal partitions error. By contrast, the frequency of errors for the control group at pretest was more dispersed over the various response categories. That is, although control group students made more errors of different types, with the exception of one, the frequency of incorrect responses at each category was lower than it was for the treatment group. Similar results were obtained at the posttest. That is, students in the treatment group made representational errors that fell into one of four categories at posttest. On the other hand, students in the control group made representational errors that were more diverse in type, falling into one of six different categories.

With respect to representational errors as they related to fractional quantities, at pretest, students in both groups had difficulty representing fractions that are both smaller than and larger than one. At posttest, on the other hand, representational errors were only made for mixed numbers. In other words, from pretest to posttest, both groups of students improved on their ability to represent fractions that are smaller than one. With respect to representational errors as they related to the different models used to represent fractions, at the pretest, students in both groups made errors when they used either the set model or the region model to represent fractions pictorially. At posttest, by contrast, representational errors, as they related strictly to the different models used to represent fractions, were only made when the set model was used. That is, from pretest to posttest, both groups of students improved on their ability to represent fractions correctly using the region model.

Site 2: Procedural Execution

Site 2 understandings involve the ability to link procedures, rules, or algorithms used to solve problems with their underlying conceptual meanings or rationales (Hiebert, 1984). Therefore, I analyzed the Site 2 subscale along two dimensions: procedural application and procedural link. For the procedural application component, I examined the participants' ability to accurately use a valid procedure, rule, or algorithm to solve fractions problems involving (a) converting improper fractions to mixed numbers, and vice versa, (b) ordering and comparing fractions, and (c) adding fractions. For the procedural link component, I examined the participants' ability to link the procedure, rule, or algorithm that was used to solve a problem to its underlying conceptual meaning or rationale. The results for the procedural application component of the Site 2 subscale are presented first. For ease of discussion, I use the term "procedure" in this section to refer to standard procedures, rules, and algorithms alike.

Site 2 procedural application, pretest. Table 8 presents the pretest and posttest results for the procedural application component of the Site 2 subscale. There were nine items at Site 2 that assessed the participants' ability to use a correct procedure, rule, or algorithm to solve a problem involving fractions. Therefore, frequencies are reported as a total out of 135 (15 participants x 9 responses per participant).

At pretest, control group students correctly applied a valid procedure to solve a problem involving fractions a combined total of 38.52% of the time (see Table 8). By

Table 8

Frequency (f) and Percentage (%) of Distributions at Pretest and Posttest for the Procedural Application Component of the Site 2 (Procedural Execution) Subscale of the Fractions Knowledge Interview (N = 30)

	Control (n = 15)				Treatment (n = 15)			
	Pretest		Posttest		Pretest		Posttest	
Response Category	*f*	%	*f*	%	*f*	%	*f*	%
Correct								
Standard	31	22.96	82	60.74	27	20	42	31.11
Concrete model	4	2.96	9	6.67	9	6.67	36	26.67
Number knowledge	17	12.59	9	6.67	7	5.19	15	11.11
Total Correct	52	38.52	100	74.01	43	31.85	93	68.89
Incorrect								
Procedural error[a]	55	40.74	24	17.78	68	50.37	15	11.11
No procedure	16	11.85	0.00	0.00	21	15.56	10	7.41
Wrong procedure	8	5.93	6	4.44	0.00	0.00	0.00	0.00
Concrete model error	1	0.74	4	2.96	0.00	0.00	12	8.89
Number knowledge error	3	2.22	1	0.74	1	0.74	4	2.96
Guess/no explanation	0.00	0.00	0.00	0.00	2	1.48	1	0.74

Note. Frequencies are reported as a total out of 135.

[a]Errors included in this category were: incomplete procedures, incorrect invented procedures, and whole number misconceptions.

contrast, treatment group students correctly applied a valid procedure to solve a problem involving fractions a combined total of 31.86% of the time. Correct procedures consisted of three types: (a) standard procedures, (b) the concrete model procedure, and (c) the number knowledge procedure. The breakdown of the frequency of correct procedures by type (see Table 8) indicated that at the pretest, control group students used a correct standard and number knowledge procedure more frequently than treatment group students. On the other hand, at pretest, the treatment group used the correct concrete model procedure more frequently than the control group.

The correct standard code was assigned to a response that entailed the accurate application of a standard symbolic procedure (or step-by-step algorithm) to solve a problem involving fractions. For example, one item required the students to convert the mixed number $3\frac{4}{6}$ to an improper fraction. Students who used the standard procedure to solve the problem performed the following symbolic manipulations on the numbers: (1) the whole number in the quantity was first multiplied by the denominator of the fraction (i.e., $3 \times 6 = 18$); (2) the value of the numerator was then added to the obtained product from step 1 (i.e., $18 + 4 = 22$); and (3) the value obtained in step 2 was noted as the new numerator of the fraction, and the original denominator was repeated to produce the correct improper fraction (i.e., $\frac{22}{6}$). Another example of a response coded as correct standard occurred for ordering problems. For example, one item required the students to order the fractions $\frac{5}{6}$, $\frac{3}{4}$, and $\frac{2}{3}$ from smallest to largest. A standard procedure in the context of this problem was the common denominator method. That is, several students solved the problem by finding a common denominator for each fraction in the problem and converting each fraction to an equivalent fraction using the new common denominator. The students then ordered the fractions from smallest to largest by comparing the numerator.

The correct concrete model code was assigned to a response that entailed drawing a pictorial representation or model as a method for solving a problem. For example, one item required the students to compare the fractions $\frac{5}{11}$ and $\frac{1}{2}$ to determine which is the larger fraction. Students who used the concrete model procedure to solve the problem drew a representation of each of the fractions and then compared the shaded portions in each representation to determine which fraction is the larger. A different problem type required the students to convert the improper fraction six-fourths to a mixed number. Students who used the concrete model procedure in this context drew a pictorial representation of the number six-fourths and were able to determine that the mixed number was one and two-fourths by counting the number of wholes that were shaded and the remaining fractional quantity that was shaded.

The correct number knowledge code referred to a response that involved the use of number sense, such as estimation or benchmarks, as a method for solving a problem. Students in both groups only used the correct number knowledge procedure in the context of order and compare problems. That is, no student used estimation, benchmarks, or other number knowledge to solve conversion or addition of fractions problems. An example of the number knowledge procedure applied to an order problem occurred for the item that required the students to order the fractions $\frac{3}{4}$, $\frac{2}{3}$, and $\frac{2}{6}$ from smallest to largest. For this problem, one student estimated the value of each fraction in terms of its percentage equivalency to solve the problem. The following excerpt details the student's response:

> I know that three-fourths is like 75%, so that's actually the largest fraction, and two-thirds, that's more than half, so it's more than 50%, so that makes it the second largest, and two-sixths, well that's definitely less than 50% because it's not half, maybe it's like 25% or something. I'm not sure exactly but I know it's the smallest.

An example of the number knowledge procedure applied to a compare problem occurred for the item that required students to compare the fractions $\frac{2}{4}$ and $\frac{3}{8}$ to determine which is the larger fraction. For this problem, one student used benchmark numbers to figure out the answer. The following example details the student's response: "Two-fourths is larger because it's the same as a half. Three-eights would have to be four to be half and it's not four, it's three, so that makes it less than half."

At pretest, the most notable error in applying a procedure to solve a fractions problem involved making some form of procedural error. That is, the procedural error mistake accounted for incorrect responses close to 41% of the time for the control group, and just over 50% of the time for the treatment group. As noted in Table 8, the procedural error code was assigned to any incorrect form of response that entailed either (a) the incomplete application of a procedure, (b) the use of an incorrect invented procedure, or (c) whole number concepts misapplied to fractions. The incomplete application of procedures, as well as the use of incorrect invented procedures, occurred for the most part in relation to fractions problems involving conversions. That is, with the exception of just three responses, these forms of procedural error were made typically for items that required students to convert improper fractions to mixed numbers, and vice versa, but not for items that entailed ordering, comparing, or adding fractions. An example of the incomplete application of a procedure was observed in one student's response when she was asked to convert the improper fraction seven-fifths to a mixed number. The student in question correctly applied the standard symbolic procedure (i.e., divide the numerator by the denominator to obtain the number of wholes and the remaining fractional quantity) and obtained the answer 1 with a remainder of 2; the student did not, however, complete the procedure because she was unable to translate the result of the division into its corresponding fractional form (i.e., to express the solution in terms of a mixed number, she had to translate her answer of 1 remainder 2 to the mixed number $1\frac{2}{5}$).

One example of the application of an incorrect invented procedure as it related to conversion problems was also observed for the item requiring students to convert the improper fraction seven-fifths to a mixed number. That is, one student changed the fraction $\frac{7}{5}$ to the fraction $\frac{2}{10}$ by subtracting five from the numerator and adding it to the denominator. The student's explanation revealed that she believed the answer two-tenths to be correct because the fraction was no longer improper.

The procedural error of misapplying whole number concepts to problems involving fractions occurred either in the context of order and compare problems, or for problems involving the addition of fractions. That is, students in both instructional groups misapplied whole number concepts to fractions when asked to order, compare, or add fractions, but not when they were asked to convert improper fractions to mixed numbers (and vice versa). In general, in the context of order and compare problems, misapplying whole number concepts to fractions referred to treating the numbers in a fractional quantity as independent whole numbers, as opposed to thinking about how the numbers in the numerator and the denominator together represent one value (i.e., the numbers 2 and 3, when combined to form the fraction $\frac{2}{3}$, represent one quantity). More specifically, this error took on one of three different forms: (a) the denominator of the fraction was treated as an independent whole number; (b) the numerator of the fraction was treated as an independent whole number, or (c) both the numerator and the denominator were treated as independent whole numbers. For example, when asked to order the fractions $\frac{3}{4}$, $\frac{2}{3}$, and $\frac{2}{6}$ from smallest to largest, one student treated the denominators as independent whole numbers and put the fractions in the following incorrect order: $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{2}{6}$. That is, the student in question solved the problem by treating the denominators as independent whole numbers, and simply ordered those numbers from smallest to largest. For a different problem involving ordering the fractions $\frac{5}{6}$, $\frac{3}{4}$, and $\frac{2}{3}$ from smallest to largest, one student treated the numerators as independent whole numbers and put the fractions in the following order: $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{5}{6}$. Although the student's (incorrect) strategy resulted in the right answer (given the numbers in the problem), her explanation revealed that her thinking was incorrect because she treated the numerators in the fractions as independent whole numbers, and simply ordered those numbers from smallest to largest. An example of treating both the numerator and the denominator as independent whole numbers in the context of compare problems was observed for the item that required students to compare the fractions $\frac{1}{2}$ and $\frac{2}{6}$ to determine which is the larger. For example, one student responded that the fraction two-sixths is larger, explaining that, "two is bigger than one, and six is bigger than two."

Whole number concepts misapplied to problems involving the addition of fractions took on one of two different forms: (a) students added across denominators, and (b) students added numerators and denominators together as if each represented an independent whole number. At pretest, students in both the treatment and control groups made the error of adding across denominators. On the other hand, at pretest, only treatment group students made the error of adding numerators and denominators together as if each represented an independent whole number. An example of adding across denominators was observed for the item that required the students to add the fractions $\frac{2}{6}$ and $\frac{3}{6}$. Several students made this error and in each instance, the error entailed adding the denominators together to obtain the incorrect solution $\frac{5}{12}$. An example of adding the numerators and denominators together as if each represented an independent whole number was observed for the same item. That is, students who made this error solved the problem by adding 2 + 6 + 3 + 6, to obtain the incorrect solution 17.

As indicated in Table 8, in addition to the various forms of procedural errors that were made, the inability to apply a procedure to solve a problem involving fractions was the second largest source of difficulty for students in both groups at pretest. That is, the no procedure error accounted for incorrect responses close to 12% of the time for students in the control group. For students in the treatment group, the no procedure error accounted for incorrect responses almost 16% of the time. The no procedure code was assigned when a response indicated that the student could not solve the problem in any way. In other words, the student left the item blank.

With the exception of one occurrence, the no procedure error was made almost exclusively in the context of conversion problems. That is, students in both the treatment and control groups had difficulty applying any form of procedure to solve problems that required them to convert improper fractions to mixed numbers, and vice versa. The one exception was observed in the context of an addition problem. That is, one student in the treatment group could not solve the problem $\frac{1}{5} + \frac{3}{5}$. Overall then, procedural errors notwithstanding, students in both groups had little difficulty solving order, compare, and addition problems involving fractions; rather, they had more difficulty solving conversion problems as evidenced by the tendency to leave those items blank.

At pretest, the results of the wrong procedure error were also noteworthy because although control group students made this error, no student in the treatment group made this error (see Table 8). That is, the wrong procedure error accounted for incorrect responses close to 6% of the time for students in the control group. The wrong procedure error was assigned to a response in which a standard symbolic procedure was used correctly, but was invalid with respect to the type of problem it was applied to. For example, when asked to convert the improper fraction $\frac{4}{3}$ to a mixed number, one student used the procedure of finding an equivalent fraction and gave the answer $\frac{8}{6}$. As illustrated in Figure 16, the student used the procedure correctly (i.e., four-thirds is indeed equivalent to eight-sixths), but it was invalid in the context of the problem because the problem required that the student produce a mixed number and not an equivalent fraction.

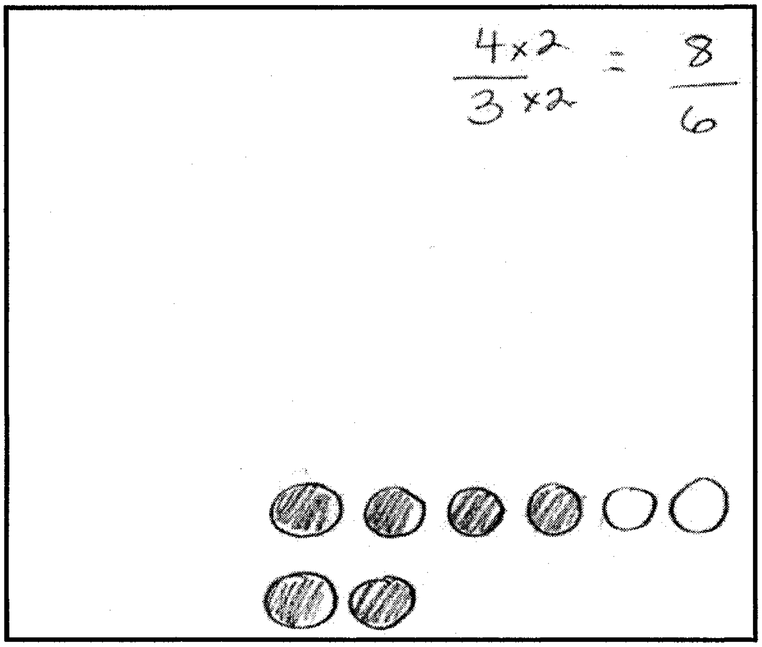

Figure 16. Illustration of the use of a wrong procedure to solve a conversion problem, along with a pictorial representation of the solution.

Site 2 procedural application, posttest. As noted in Table 8, the frequency of responses with respect to applying a correct procedure to solve problems involving fractions nearly doubled from pretest to posttest for the control group. That is, at posttest, control group students responded with correct procedures approximately 74% of the time. For the treatment group, the frequency of correct responses more than doubled from pretest to posttest. Treatment group students responded with correct procedures almost 69% of the time.

The breakdown of the frequency of correct procedures by type at posttest indicated that control group students made more frequent use of standard procedures at posttest (close to 61% of the time), compared to the correct concrete model procedure (nearly 7% of the time) and the number knowledge procedure (nearly 7% of the time). Treatment group students, on the other hand, used both standard procedures and the concrete model procedure in almost equal frequency (approximately 31% and 27% of the time, respectively) at the posttest, and also used the number knowledge procedure just over 11% of the time. For the control group, therefore, the majority of increase in the frequency of correct responses was related to the use of standard procedures, rules, or algorithms to solve fractions problems. Moreover, from pretest to posttest, control group responses entailed fewer occurrences of the number knowledge procedure (12.59% of the time at pretest and 6.67% of the time at posttest). For the treatment group, the majority of increase in the frequency of correct responses at posttest was related to the use of a concrete pictorial representation or model as a method for solving problems involving fractions (6.67% of the time at pretest compared to 26.67% of the time at posttest). The frequency of responses coded at each of the correct standard and correct number knowledge categories also increased from pretest to posttest for the treatment group (20% of the time at pretest compared to 31.11% of the time at posttest, and 5.19% of the time at pretest compared to 11.11% of the time at posttest, respectively).

The majority of incorrect responses at pretest for both groups fell into the procedural error category. At posttest, although this was still the error that occurred most often for each group, its frequency was not as high as it was at the pretest. That is, at posttest, control group students responded with procedural errors nearly 18% of the time (compared to nearly 41% of the time at pretest). Treatment group students responded with procedural errors just over 11% of the time at posttest, compared to almost 51% of the time at pretest. With respect to making procedural errors after instruction, therefore, the results indicated that the treatment group did so in less frequently.

Procedural errors at posttest consisted mainly of one of two types: (a) incorrect invented procedures, and (b) whole number concepts misapplied to fractions. That is, at posttest, only one procedural error was in the form of an incomplete procedure. Similar to the pretest results, the procedural error of using an incorrect invented procedure continued to occur mainly in the context of conversion problems at posttest. That is, students in both the treatment and control groups used incorrect invented procedures most often when solving fractions that required them to convert improper fractions to mixed numbers, and vice versa. For example, one student, when asked to convert the improper fraction $\frac{12}{8}$ to a mixed number, responded in the following manner, "if I do 8 x 3, that's 24 for the denominator… so then to get 24, I multiply 12 x 2…so it's $\frac{24}{24}$." Another student, when asked to convert the fraction six-fourths to a mixed number, responded in the following manner:

> Well, first you need to find a common multiple, so that's 6…and it's $\frac{6}{4}$, so I guess 6 becomes the denominator and four becomes the numerator, so the answer is $\frac{4}{6}$…but it has to be a mixed number so then you have to put a big one in front.

Also similar to the pretest results, the procedural error of misapplying whole number concepts to fractions continued to occur only in the context of order, compare, and addition problems involving fractions. That is, no student made this procedural error in the context of conversion problems. With respect to order and compare problems, students in both groups continued to treat both the numerators and the denominators of fractions as independent whole numbers. For example, one student explained, "three-eighths is bigger than two-fourths because three is bigger than two and eight is bigger than four." With respect to addition problems, students in both groups continued to make the add across denominators error at posttest (i.e., $\frac{1}{5} + \frac{3}{5} = \frac{4}{10}$), but no student made the error of adding all the numerators and denominators together as if each represented an independent whole number (e.g., $\frac{1}{5} + \frac{3}{5} = 1 + 5 + 3 + 5 = 14$).

At pretest, the no procedure error was also one of the most frequent errors made by students in both instructional groups. At the posttest, however, this error disappeared completely for control group students. That is, no student in the control group left an item blank with respect to solving problems involving fractions. Treatment group students, by contrast, continued to make the no procedure error at posttest, but did so approximately half as often as they did at the pretest. That is, at posttest, treatment group students responded with the no procedure error 7.41% of the time, compared to 15.56% of the time at pretest. Similar to the pretest results, all of the occurrences of the no procedure error at posttest were made in the context of conversion problems. In other words, treatment group students continued to have difficulty solving problems that required them to convert improper fractions to mixed numbers, and vice versa.

A new error for the treatment group at posttest was the concrete model error. This error accounted for incorrect responses close to 9% of the time. Control group students also responded with the concrete model error at posttest, but less frequently than the treatment group did (2.96 % of the time). The concrete model error code was assigned to any response that entailed drawing a pictorial representation or model as a method for solving the problem, and in which a representational error was made. This error only occurred in the context of order, compare, and conversion problems. That is, no student made the concrete model error in the context of addition problems. One instance of the concrete model error made in the context of compare problems occurred for the item that required the students to compare the fractions $\frac{5}{11}$ and $\frac{1}{2}$ to determine which is the larger fraction. For example, as illustrated in Figure 17, one student used the region model to represent each fraction pictorially, but drew each whole as a different size. Consequently,

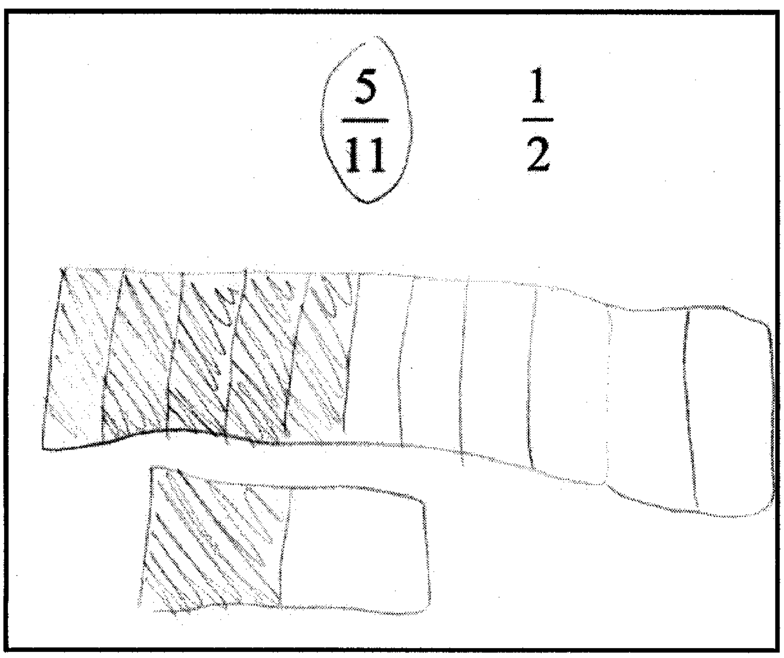

Figure 17. Illustration of an incorrect concrete representation to solve a compare problem involving fractions.

the fraction $\frac{5}{11}$ appeared to be larger because a larger area was shaded. An example of the concrete model error made in the context of conversion problems occurred for the item that required the students to convert the mixed number $3\frac{4}{6}$ to an improper fraction. For example, as illustrated in Figure 18, one student, who used the region model, did not draw the correct number of wholes but rather drew two rectangular regions, each partitioned into sixths, with three parts shaded in one region (to represent the whole number 3) and four parts shaded in the other region (to represent the fraction four-sixths).

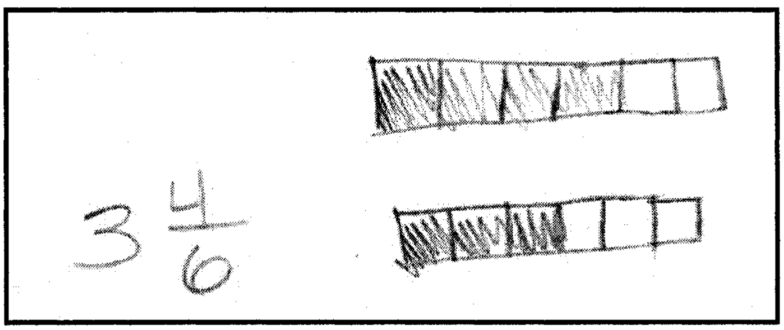

Figure 18. Illustration of an incorrect concrete representation to solve a conversion problem involving fractions.

In summary, with respect to demonstrating the ability to accurately use a valid procedure, rule, or algorithm to solve fractions problems involving (a) converting improper fractions to mixed numbers (and vice versa), (b) ordering and comparing fractions, and (c) adding fractions, students in the control group responded more frequently with correct procedures at pretest and at posttest. Although control group students responded correctly more frequently at each testing session, this did not represent differences that were very large in number. At pretest, the majority of correct responses provided by the control group were categorized as a standard symbolic procedure, and the occurrence of this response increased nearly threefold at posttest. For the treatment group, the majority of correct responses at pretest were also in the form of a standard symbolic procedure. Although the frequency of this response also increased for treatment group students at posttest, the largest increase in correct responses from pretest to posttest for the treatment group occurred for the correct concrete model procedure.

The most frequently occurring error for both groups at pretest was the procedural error, which entailed either (a) incomplete procedures, (b) incorrect invented procedures, or (c) errors made by applying whole number concepts to problems involving fractions. Although this continued to be the most frequently occurring error for both groups at posttest, compared to the pretest results, both groups made fewer of these errors at the second testing session. With respect to procedural errors and problem type, pretest and posttest results were similar for both groups of students. That is, at the pretest, particular procedural errors occurred more often within the context of some problem types and not others (for example, the incorrect invented procedure occurred mainly in the context of conversion problems), and the same patterns were observed at the posttest. One procedural error, that of adding the numbers in numerators and denominators by treating each as an independent whole number, was made only by treatment group students at the pretest. At posttest, on the other hand, treatment group students no longer made this error.

Another error that occurred in high frequency at the pretest for both groups was the no procedure error. At the posttest, on the other hand, no student in the control group made this error. With respect to students in the treatment group, the frequency of the no procedure error decreased by more than half at posttest. Finally, control group students applied the wrong procedure when solving fractions problems at each testing session, but did so slightly less often at posttest. No treatment group student made the wrong procedure error at either testing session but some made the concrete model error at posttest and this represented a new type of error for treatment group students at the second testing session.

Site 2 procedural link, pretest. Table 9 presents the pretest and posttest results for the procedural link component of the Site 2 subscale. That is, in addition to examining the participants' ability to apply correct procedures, I also analyzed the Site 2 subscale with respect to the students' ability to link those procedures to their underlying meanings or rationales. There were nine items at Site 2 that assessed the participants' ability to link procedures to their meanings. Frequencies for the control group are therefore reported as a total out of 135 (15 participants x 9 response per participant). With respect to the treatment group, however, data from one participant were missing. Therefore, frequencies for the treatment group are reported as a total out of 126 (14 participants x 9 responses per participant).

At pretest, control group students provided a correct rationale or link to the underlying conceptual meanings for the procedures they used to solve problems nearly 18% of the time (see Table 9). In comparison, treatment group students provided a correct rationale or link to the underlying conceptual meanings for the procedures they used just over 26% of the time. Evidence of the ability to provide a link between a procedure and its underlying conceptual meaning differed according to the type of procedure that was used. As noted above, correct at Site 2 procedures entailed one of

Table 9

Frequency (f) and Percentage (%) of Distributions at Pretest and Posttest for the Procedural Link Component of the Site 2 (Procedural Execution) Subscale of the Fractions Knowledge Interview (N = 29[a])

	Control (n = 15)				Treatment (n = 14)			
	Pretest		Posttest		Pretest		Posttest	
Response Category	f	%	f	%	f	%	f	%
Correct link/rationale	24	17.78	47	34.81	33	26.19	79	62.70
No link/rationale	111	82.22	88	65.12	102	80.96	47	37.3

Note. Frequencies for the control group are reported as a total out of 135. Frequencies for the treatment group are reported as a total out of 126.

[a]Data missing for one participant in the treatment group (n = 14).

three forms: (a) standard symbolic procedures; (b) the concrete model procedure; and (c) the number knowledge procedure. With respect to standard symbolic procedures, the correct link/rationale code was assigned to a response in which the student could justify the conceptual meaning of the symbolic manipulations performed on the numbers in the problem. In the context of a conversion problem, the following example provides an illustration. One student, who used the standard symbolic procedure to convert the mixed number $2\frac{1}{4}$ to the improper fraction $\frac{9}{4}$ (i.e., multiply the whole number by the denominator, add the value of the numerator to the obtained product, and write the new sum over the original denominator) explained the underlying conceptual meaning of the algorithm in the following manner:

> I did 2 x 4 because it's divided into four and there are two of them [i.e., two wholes], and then there's another one but it's not a whole, it's just one fourth, so you have to add it to the other fourths, so it's like nine fourths altogether.

The student's explanation revealed an understanding of (1) the equivalency of the whole in terms of its fractional parts (i.e., one whole is equivalent to four-fourths), and (2) that the goal was to figure out how many fourths there are all together in two and one-fourth. Furthermore, the student was able to link these understandings to the symbolic manipulations he performed on the numbers.

Similarly, the following example illustrates the ability to justify the meaning of a standard symbolic procedure in the context of a problem involving the addition of two fractions. That is, one student, who applied the procedural rule add up the numerators but not the denominators to solve the problem $\frac{2}{6} + \frac{3}{6} = \frac{5}{6}$, explained the underlying meaning of the rule in the following manner: "you don't add the denominators because it's just one cake, it's the same cake...because then it would be $\frac{5}{12}$, and then it would be like a different fraction because it's not twelfths, it's sixths." This student's explanation revealed an understanding that (1) she was adding parts of a whole (i.e., one cake), and (2) that changing the unit of the fraction would change the value of its quantity.

With respect to the concrete model procedure, the correct link/rationale code was assigned when a response indicated that the student could provide a link between the pictorial representation or model that was used to solve the problem and the symbolic solution that was derived from the representation. For instance, within the context of conversion problems, several students modeled an improper fraction to figure out its equivalency in terms of a mixed number. For example, one student drew a model of the improper fraction $\frac{7}{5}$ and from that model, figured out that the mixed number was $1\frac{2}{5}$. When asked to link the representation to the symbolic solution that was obtained, the student explained the following:

> First I drew two rectangles with five in each, and in this one I shaded all five and in this one I shaded two from the five. Then I saw that this one was shaded full, so I knew it was one [i.e., one whole] and then this one is two-fifths.

The student's explanation revealed an understanding that the symbol 1 in the number one and two-fifths represents one whole, and an understanding of the equivalency of the whole in terms of its fractional parts (i.e., that the whole number 1 in the symbolic notation $1\frac{2}{5}$ is equivalent to five-fifths).

Several students also used the concrete model procedure within the context of order problems. That is, the students drew a pictorial representation of each fraction in the problem and then used the representation to figure out the order of the fractions from smallest to largest. For example, one student, who used this strategy to order the fractions $\frac{3}{4}$, $\frac{2}{3}$, and $\frac{2}{6}$, explained the link between her representation and the symbolic solution she derived from the representation in the following way:

> I drew each fraction and they're all the same size [i.e., the size of the wholes] and in this one [three-fourths] I shaded three parts, in this one [two-thirds] I shaded two parts, and in this one [two-sixths] I also shaded two parts. But this one [two-

sixths], is the smallest because its shaded the least, and this one [three-fourths] is the largest because it has the most shaded, and this one [two-thirds] is in the middle because its more shaded than this one [two-sixths] but it's not as much shaded as this one [three-fourths].

This student's explanation revealed an understanding of (1) the importance of holding the unit constant with respect to comparing fractional values, and (2) that the shaded portion of the fraction determines its value.

With respect to the number knowledge procedure, the correct link/rationale code was assigned to a response that indicated the student's ability to justify why a particular estimation, use of a benchmark number, or use of other number knowledge made sense it the context of the given problem. For example, regarding order problems, one student estimated the value of each fraction in terms of its percentage equivalency and used that knowledge to correctly order the fractions. The student further explained that it made sense to think of the fractions as percentages because, "it's the same thing, it's just another way to write the same number." The student's explanation, therefore, revealed an understanding of why it made sense to use the particular strategy that he used. Another example was a student who used the benchmark number one-half to determine which of two fractions, $\frac{2}{5}$ or $\frac{1}{3}$, is the larger. The student explained, "if you think of a half, it's easier to know because two is almost half of five, so it' almost a half, but one is not half of three so that makes it [one-third] smaller." The student was able to justify the use of a familiar benchmark number (one-half) and its meaning in the context of the problem. She was also able to link her knowledge of the number one-half to her knowledge of the other numbers in the problem to correctly solve it.

At pretest, on the other hand, the results indicated that just over 80% of the time, control group students could not provide a link between the procedures they used and the underlying meanings or rationales for those procedures (see Table 9). Similarly, the pretest results indicated that close to 81% of the time, students in the treatment group could not provide a link between the procedures they used and the underlying meanings or rationales for those procedures. For both groups of students, the inability to provide a link occurred mainly with respect to conversion and addition problems. That is, at pretest, students in both instructional groups demonstrated more ability to provide a link or rationale for the procedures they used to solve order and compare problems. As noted above, at pretest, students in both groups solved order and compare problems for the most part by applying either the concrete model procedure or the number knowledge procedure. On the other hand, at pretest, participants in both groups mainly used standard symbolic procedures to solve conversion and addition problems. The results therefore indicated that students in both groups had more difficulty providing a rationale or link between procedures and their underlying meanings when they used standard symbolic procedures, and less so when they either modeled a problem or used number knowledge to figure out its solution.

Site 2 procedural link, posttest. As noted in Table 9, at posttest, control group students provided a correct rationale or link to the underlying conceptual meanings of the procedures they used to solve problems nearly 35% of the time. This means that just over 65% of the time, students in the control group could still not link the procedures they used to their underlying meanings or rationales. In comparison, treatment group students provided a correct rationale or link to the underlying conceptual meanings for the procedures they used close to 63% of the time. Conversely, just over 37% of the time, students in the treatment group could not link the procedures they used to their underlying meanings or rationales. Overall, therefore, control group students, in comparison to treatment group students, responded less frequently with correct links or rationales.

At posttest, correct procedures continued to entail either (a) standard symbolic procedures, (b) the concrete model procedure, or (c) the number knowledge procedure. As noted in Table 8, however, at posttest, control group students responded more frequently with standard symbolic procedures as opposed to either the concrete model or number knowledge procedures. The results therefore indicated that the high frequency of the no link/rationale response by control group students at posttest occurred mainly when standard symbolic procedures were used to solve problems. At posttest, treatment group students responded more frequently with correct links or rationales, and they also made more use of the concrete model and number knowledge procedures at posttest (see Table 9). The results therefore indicated that similar to the control group, the frequency of the no link/rationale response for treatment group students at posttest occurred mainly when standard symbolic procedures were used to solve problems. This occurred nearly half as often for treatment group students (37.3% of the time), however, as compared to their counterparts in the control group (65.12% of the time).

In summary, participants in the treatment group responded more frequently with correct links or rationales at both testing sessions. At pretest, this represented only a small difference in number (33 versus 24 responses for the treatment and control group, respectively) but at posttest, the difference was larger (79 versus 47 responses, for the treatment and control group, respectively). At each testing session, students in both groups were able to provide a link or rationale between the procedures they used and their underlying meanings for every type of correct procedure that was employed. At each testing session, however, students in both groups demonstrated that this was easier with respect to the concrete model and number knowledge procedures than it was for standard symbolic procedures. That is, at both the pretest and the posttest, the frequency of the no link/rationale response for each group was related mainly to the use of standard symbolic procedures. At the pretest, this occurred in almost equal frequency for the two groups, but at the posttest, the distinction was more pronounced with control group students using more standard procedures and demonstrating more difficulty in providing links or rationales.

Site 3: Solution Evaluation

Site 3 understandings involve the ability to verify the reasonableness of symbolically stated problem solutions (Hiebert, 1984). Furthermore, as it was conceptualized by Hiebert, in evidence of Site 3 understandings, the verification of symbolic solutions is achieved by some method other than applying a symbolic procedure (such as estimating, using benchmark numbers, or thinking about a problem in a real-world context). That is, Site 3 understandings refer to the ability to create meaning for the symbols that are used to represent problem solutions and to use those meanings to evaluate the reasonableness of symbolic mathematical statements. Therefore, I analyzed the Site 3 subscale along one dimension only: the participants' ability to evaluate the reasonableness of problem solutions stated symbolically, by using a method other than a symbolic procedure to do so. There were two items at Site 3 that assessed the participants' ability to correctly evaluate the reasonableness of problem solutions stated symbolically. Therefore, frequencies for each group are reported as a total out of 30 (15 participants x 2 responses per participant).

Site 3 solution evaluation, pretest. Table 10 presents the pretest and posttest results for the Site 3 subscale of the interview. At pretest, responses provided by the control group demonstrated that students could correctly evaluate the reasonableness of problem solutions stated symbolically just over 13% of the time. By contrast, treatment group students responded more frequently with correct evaluations at pretest (20% of the time). The results also indicated that with respect to valid verification methods (i.e., not a standard symbolic procedure), students in both groups used only the concrete model method to verify symbolically stated problem solutions at pretest. That is, no student from either instructional group used the number knowledge verification method at pretest.

The correct concrete model code was assigned to a response in which the student used a pictorial representation or model of the problem to verify its reasonableness. That is, the use of this method at Site 3 entailed producing an accurate pictorial model of a problem as a method for evaluating a symbolical mathematical statement. There were two symbolical mathematical statements at Site 3 to be evaluated. The first required students to verify the reasonableness of an addition of fractions statement noted symbolically (i.e., $\frac{1}{4} + \frac{2}{4} = \frac{3}{8}$ in version A of the interview, and $\frac{1}{3} + \frac{1}{3} = \frac{2}{6}$ in version B of the interview). The second required the students to verify the reasonableness of a compare problem noted symbolically (i.e., $\frac{3}{4} > \frac{5}{7}$ in version A of the interview and,

Table 10

Frequency (f) and Percentage (%) of Distributions at Pretest and Posttest for the Site 3 (Solution Evaluation) Subscale of the Fractions Knowledge Interview (N = 30).

	Control (n = 15)				Treatment (n = 15)			
	Pretest		Posttest		Pretest		Posttest	
Response Category	*f*	%	*f*	%	*f*	%	*f*	%
Correct, concrete model	4	13.33	5	16.67	6	20	8	26.66
Correct, number knowledge	0	0	2	6.67	0	0	2	6.67
Total correct	4	13.33	7	23.34	6	20.00	10	33.33
Incorrect								
Applied Procedure Total	17	56.67	21	70.00	17	56.67	14	46.66
Correct Procedure	5	16.67	15	50.00	4	13.33	8	26.66
Procedural Error[a]	12	40.00	6	20.00	13	43.33	6	20.00
Concrete model error	7	23.33	1	3.33	5	16.67	5	16.67
Number knowledge error	2	6.67	1	3.33	1	3.33	0.00	0.00
Guess/Unable to verify	0.00	0.00	0.00	0.00	1	3.33	1	3.33

Note. Frequencies are reported as a total out of 30.

[a]Errors included in this category were incorrect invented procedures and whole number procedures.

and $\frac{3}{6} > \frac{2}{9}$ in version B of the interview). At pretest, students from both groups used the concrete model evaluation method in the context of each problem type.

With respect to applying the concrete model verification method in the context of an addition problem, the following provides an illustration. One student used the region model to represent the fractions one-fourth and two-fourths, and explained that the answer three-eights was wrong because the picture showed that, "there's still more room in this one [in one-fourth] so you can fill up the empty space with these [two-fourths] in there, so it doesn't change what [the fraction] is out of." In other words, based on her drawing, the student was able to verify that the symbolic solution $\frac{3}{8}$ was not reasonable because the unit did not match that of the representation (fourths). Similarly, the following illustrates how the concrete model verification method was used in the context of compare problems. One student used a circular region to represent the fractions three-sixths and two-ninths. Based on his drawing, he concluded that the symbolic statement $\frac{3}{6} > \frac{2}{9}$ was correct because, "more of the circle is shaded in three-sixths."

As noted in Table 10, at pretest, the most frequent form of incorrect response for students in both groups was the procedural error response. Furthermore, the results indicated that this incorrect form of response occurred in similar frequency for both groups (40% of the time for the control group and 3.33% of the time for the treatment group). Procedural errors at Site 3 were similar to those made at Site 2. That is, procedural errors at Site 3 entailed either (a) incorrect invented procedures, or (b) whole number concepts misapplied to fractions. With the exception of one response from a participant in the control group, however, all of the procedural errors made with respect to evaluating symbolic statements at Site 3 were in the form of whole number concepts misapplied to fractions. In other words, the pretest results indicated that both groups of students misapplied whole number concepts to fractions in high frequency when evaluating the reasonableness of addition and compare problems.

In the context of addition problems, the following illustrates one student's incorrect use of whole number concepts. For the symbolic statement $\frac{1}{3} + \frac{1}{3} = \frac{2}{6}$, the student in question made the add across denominators error and agreed that two-sixths was the correct answer because, "1 + 1 = 2 and 3 + 3 = 6." Another student incorrectly evaluated the reasonableness of the same symbolic statement by applying whole number concepts of subtraction to the problem. The student in question explained, "two-sixths is right because for addition problems, you can check by subtracting, and $\frac{2}{6} - \frac{1}{3} = \frac{1}{3}$; you just do 2 - 1 and 6 - 3."

In the context of compare problems, misapplying whole number concepts to fractions entailed treating the numbers in a fractional quantity as independent whole numbers. For example, one student (incorrectly) evaluated the reasonableness of the statement $\frac{3}{4} > \frac{5}{7}$ in the following manner, "it's wrong because 5 is bigger than 3 and 7 is bigger than 4, so this one [five-sevenths] is bigger." Another whole number misconception that several students made in the context of compare problems was determining the value of a fraction based solely on its number of parts. For example, one student (incorrectly) evaluated the symbolic statement $\frac{3}{6} > \frac{2}{9}$ in the following manner, “it’s wrong because that one [two-ninths] has nine parts and this one [three-sixths] has three parts, so two-ninths is bigger because nine is bigger than six.” The student treated the denominators as independent whole numbers and also failed to take into consideration the role of the numerator.

For students in the control group at pretest, another error that occurred in high frequency with respect to evaluating symbolic solutions was the concrete model error (23.33% of the time). By comparison, treatment groups students made this error 16.67% of the time at pretest. The concrete model error code was assigned to a response in which a student attempted to draw a pictorial representation or model of a problem to evaluate its reasonableness, but in which the student made a representational error. In the context of symbolic addition statements, every occurrence of this error was related to drawing an incorrect representation of the solution of the addition. For example, for the problem $\frac{1}{4}$ + $\frac{2}{4} = \frac{3}{8}$, one student correctly represented the fractions one-fourth and two-fourths, but drew a model of three-eighths to represent the solution of the addition. The student further explained, “it’s right because when you draw it, you see the answer is three-eights.” In other words, believing that the symbolic statement was accurate, the student drew a representation to match it. In the context of compare problems, the concrete model error was similar to that which was reported above for Site 2. That is, students made the same types of errors (such as drawing unequal units or wholes, taking the unshaded part of the picture to represent the value of the fraction, or failing to partition a whole equally), and because of faulty representations, were unable to correctly verify the reasonableness of the compare statements.

Finally, at pretest, the applied procedure error occurred in relatively high frequency for both groups as well (16.67% of the time for the control group and 13.33% of the time for the treatment group). A response that was coded as applied procedure referred to a response in which the student correctly applied a standard symbolic procedure, rule, or algorithm to verify the reasonableness of a mathematical statement. That is, the student correctly applied a valid procedure, and correctly evaluated the symbolic statement. This method was coded as incorrect, however, because as noted above, Site 3 understandings refer to the ability to apply some method other than a standard procedure to evaluate a symbolically stated problem solution. Therefore, the use of a standard procedure as the verification method indicated that other than procedurally, the student could not explain why a symbolic statement was correct or incorrect. For example, one student evaluated the symbolic statement $\frac{1}{4}+\frac{2}{4}=\frac{3}{8}$ in the following manner, "it's wrong because in fractions, you never add the denominators, so the answer is actually three-fourths." When asked to explain the underlying rationale for the rule, the student responded, "I'm not sure...you just never add the denominators in fractions." A further example in the context of compare problems is the following. One student evaluated the statement $\frac{3}{6}>\frac{2}{9}$ in the following manner, "that's right because when the denominator is smaller, it means a bigger fraction." When asked to justify the rule, the student responded, "I don't know, I just know it's the opposite with fractions [i.e., the opposite of whole numbers]."

Site 3 solution evaluation, posttest. As noted in Table 10, at the posttest, the frequency of correct responses for both groups of students increased with respect to the concrete model method of solution evaluation (16.67% of the time for the control group and 26.66% of the time for the treatment group), but the increase for each group was small in number. Furthermore, at the posttest, both groups of students used the correct number knowledge method of solution evaluation in equal frequency (6.67% of the time for each group). Compared to the pretest results, this represented a new verification method for both groups. That is, no student in either group used the number knowledge verification method at pretest.

The correct number knowledge code was assigned to a response in which the student used estimation, benchmark numbers, or other number knowledge to verify the reasonableness of the mathematical statement. Students in both groups only used this method in the context of compare problems. That is, no student used estimation, benchmark numbers, or other number knowledge to evaluate addition problems that were stated symbolically. The following is an example of a response in which the number knowledge verification method was applied to a compare problem stated symbolically. For the problem $\frac{3}{6} > \frac{2}{9}$, one student explained, "it's right because three-sixths is the same as a half, and two-ninths is much less than half; it would have to be four or something [i.e., the numerator] to be a half."

With respect to incorrect verification methods at posttest, the most notable result in Table 14 is the high frequency of the applied procedure method for the control group. The results indicated that at the posttest, 50% of the time, students in the control group verified the reasonableness of a symbolic mathematical statement by applying a standard symbolic procedure for which, as explained above, they could not justify the underlying conceptual meaning. Stated differently, although control group students were able to correctly evaluate symbolic solutions 50% of the time by applying a standard procedure, they could not explain why the solution was correct or incorrect. In comparison, students in the treatment group responded with this verification method nearly half as frequently (26.66% of the time).

Another notable finding at the posttest was related to the incorrect verification method coded as procedural error (see Table 10). That is, at posttest, the frequency of this error decreased by half for the control group (40% of the time at pretest compared to 20% of the time at posttest), and decreased by more than half for the treatment group (43.33% of the time at pretest compared to 20% of the time at posttest). As described above, this incorrect verification method referred to a response in which the student evaluated a symbolic mathematical statement either by using an incorrect invented procedure, or by misapplying whole number concepts to fractions. The results therefore indicated that with respect to evaluating problem solutions, both groups of students made fewer of these types of errors at the posttest.

Finally, with respect to the concrete model error at posttest, the results indicated that students in the control group made this error much less frequently than they did at pretest (3.33% of the time at posttest compared to 23.33% of the time at pretest). On the other hand, they also applied the correct concrete model method less frequently at posttest but used the applied procedure method nearly three times as often as they did at pretest. Therefore, the results indicated that the decrease in the concrete model error was due mainly to the fact that control group students used this method less frequently at the posttest, and not to the fact that they improved with respect to using correct problem representations as a method of verification.

In summary, with respect to Site 3 understandings – that is, the ability to verify the reasonableness of a symbolically represented problem solution via a method other than a standard symbolic procedure – treatment group students responded with valid methods more frequently at both pretest and posttest. these differences, however, were not very substantial. In general, the responses provided by each group indicated that across instructional conditions, students responded in higher frequency with the concrete model method as opposed to the number knowledge method. With regards to incorrect evaluation methods, at the pretest, both groups of students responded in high frequency with the procedural error method. That is, before instruction, students in both groups used incomplete, incorrect invented, or whole number procedures in an attempt to evaluate the reasonableness of symbolic solutions for problems involving fractions. After instruction, although students in both groups continued to make these procedural errors when evaluating solutions, the error that occurred most often was the applied procedure error. That is, at posttest, when students made a mistake in verifying the reasonableness of solutions, the majority of the time, these mistakes referred to applying a standard symbolic procedure, rule, or algorithm as a method of verification. This incorrect method occurred almost twice as often for control group students than it did for treatment group students. In other words, nearly twice as often as the treatment group, control group students could not explain why a solution was correct or incorrect other than procedurally.

Discussion

The objective of the present study was to investigate empirically the effectiveness of Hiebert's (1984) proposed Sites instructional approach as it relates to improving students' meaningful understandings of mathematics. To test Hiebert's theory, I instructed two separate groups of students on the topic of fractions via two different instructional approaches: (1) the Sites approach, and (2) the Standard approach, an instructional approach that mirrored closely what has been described in the research literature as typical, procedurally-oriented mathematics instruction (Kilpatrick et al., 2001; Stigler et al., 1999). Based on that which Hiebert put forth in his theory, in addition to the other relevant research findings cited in the opening chapters above, I predicted that students who would be taught via the Sites approach would demonstrate greater gains in (a) conceptual knowledge of fractions, (b) procedural knowledge of fractions, and (c) in their ability to link their knowledge of fractions concepts to their knowledge of fractions procedures at each of Hiebert's three sites.

The overall results of the present study indicated partial support for the research hypotheses. In this section, I turn to discussing and interpreting those findings. The following discussion is organized by research hypothesis, in the order that I put each forth earlier in this thesis. In addition, in this section I also briefly review the theories on which each research hypothesis was based.

Research Hypothesis 1: Improved Conceptual and Procedural Knowledge of Fractions

With the first research question in the present study I asked, Does instruction that explicitly connects procedures and concepts at the symbolic (Site 1), procedural (Site 2), and solution evaluation (Site 3) phases of learning result in improved conceptual and procedural understandings of fractions for fifth- and sixth-grade students? This question was answered with the quantitative analysis reported above that was performed on the measure of conceptual and procedural knowledge of fractions (i.e., the fractions knowledge test). The results of the analysis provided partial support for the first research hypothesis that the Sites instructional approach would result in greater improvements in fifth- and sixth-grade students' conceptual and procedural understandings of fractions. Specifically, the findings indicated that students who were taught fractions via the Sites instructional approach demonstrated greater improvements in their conceptual knowledge of fractions, but not in their procedural knowledge of fractions. With respect to procedural knowledge of fractions, students in both the Sites and Standards instructional groups demonstrated comparable improvement over the course of instruction.

I based my first research hypothesis on two theories regarding the development of conceptual and procedural knowledge in mathematics. Therefore, before turning to questions that arise from the first analysis in this study, a brief review of those theories is in order.

Hiebert's theory regarding the sites instructional method is a theory about the teaching and learning of mathematics for conceptual understanding. That is, the goal of the sites approach is to teach mathematics in ways that allow the mathematics to become meaningful to the learner. Meaningful, in the context of Hiebert's theory, refers to knowledge of mathematical concepts and knowledge of mathematical procedures that are linked. That is, Hiebert argued that in a given mathematical domain, some children possess mathematical concepts and mathematical procedures, but they lack links between the two and the absence of those links is what makes much of mathematics meaningless to them. Hiebert proposed his Sites instructional approach as a way to help students connect their conceptual knowledge to their knowledge about mathematical symbols and procedures, therefore fostering meaningful mathematical understandings. As such, Hiebert's theory did not speak to increasing conceptual knowledge per se, but rather to increasing understanding of one's understandings, in coordination with knowledge of procedures.

Implicit in Hiebert's theory, therefore, is that with improved conceptual understanding, students should develop improved procedural skill. That is, he argued that because students lack connections between what they understand about mathematics and what they do procedurally, they cannot bring to bear their conceptual understandings on their problem-solving actions – rather, they act without meaning when employing procedures. Hiebert believed that these meaningless actions often lead students to perform procedures in rote, mechanical ways, resulting in performance that is fraught with errors. On the other hand, Hiebert argued that if those connections are present, if procedures are anchored to concepts, than conceptual knowledge can be brought to bear on problem-solving activity. When procedures are anchored to concepts, procedural actions are better understood and take on deeper meaning for the student. In these ways then, implicit in Hiebert's theory is the notion that improvements in conceptual understandings lead to improvements in procedural skill.

In a separate but related theory, the iterative model of the development of conceptual and procedural knowledge (Rittle-Johnson & Siegler, 1998) states that the two types of knowledge develop in a hand-over-hand process, so that gains in conceptual knowledge lead to gains in procedural knowledge (or vice versa), that in turn lead to gains in the first. The iterative model, therefore, directly emphasizes that which is implicit in Hiebert's theory: with improvements in conceptual understandings come improvements in procedural skill.

In line with Hiebert's theory, therefore, and taking into account the iterative model of the development of conceptual and procedural knowledge, in the context of the present study, the findings from the first analysis require explanation. That is, given that the students in the Sites instructional group demonstrated greater gains over the students in the Standard instructional group with respect to conceptual knowledge of fractions, why did they not, then, demonstrate greater gains in procedural knowledge of fractions?

This question may begin to be answered by examining possible explanations for the control group's performance. That is, the results demonstrated that the students in the Standard instructional group showed comparable gains in procedural knowledge to those who were instructed via the Sites approach. In other words, even with instruction that was strictly procedural in nature, that did not expose the students to any connections between fractions concepts and procedures, the students learned just as much as those who were instructed via a more conceptually oriented approach that explicitly highlighted those connections. Although this seems counterintuitive at first, it is not an altogether surprising finding. Much research exists that has documented the fact that instruction on procedures alone can be effective with regards to increasing students' procedural skill. For example, in the domain of decimal fractions, Rittle-Johnson, Siegler and Alibali (2001; experiment 1) found that instruction that focused on either (a) conceptual knowledge, (b) procedural knowledge, (c) both, or (d) neither, was effective at improving students' procedural knowledge. Similarly, Byrnes and Wasik (1991) reported that students who were taught to add fractions via only a procedural rule demonstrated increases in their procedural performance that were comparable to those demonstrated by students who were taught the underlying concepts in addition to the rule. Likewise, in the context of multidigit addition and subtraction, Hiebert and Wearne (1996) found that traditional, standard instruction was as effective as conceptually oriented instruction in increasing students' accurate performance of procedures. These findings shed some light on the present study's results, and offer some explanation as to why the students who were instructed via a traditional, standard approach improved equally as well with respect to their procedural knowledge of fractions as those who were instructed via the Sites method.

An important point, however, is that in each of these reported findings, as well as in the findings of the present study, improvement in procedural performance is defined strictly as improvement in procedural skill, and not as improvement in the quality of students' procedural understandings. That is, although the results of the present study indicated that the students who were taught fractions by the Standard method demonstrated gains in their procedural skill, the results provide no direct evidence that they learned those procedures meaningfully. The measure that was used to assess the students' procedural knowledge in the present study could not speak to whether or not the students applied their procedural knowledge with understanding. This point is an important one with respect to interpreting the results of the findings; if evidence of this nature is taken to index students' acquisition of procedural knowledge, no claims can be made about the quality of those improvements. In fact, the findings that Hiebert and

Wearne (1996) reported illustrate this notion well. Although students in their study who received standard instruction improved with respect to procedural skill, these improvements were made only in the way of applying those procedures and not with respect to understanding the procedures. That is, the researchers reported that although the students could use the procedures accurately, none could demonstrate their underlying rationale. In the present study too, therefore, although the results indicated that the students in the Standard instruction group demonstrated gains in their procedural knowledge, these results should be interpreted cautiously. At the very least, the results must be interpreted with the understanding that the findings only provide evidence of the degree of the students' success in applying procedures correctly, and not necessarily to the quality of their understandings when doing so.

Unfortunately, given that the treatment group students in the present study were assessed in the same way, no claims can be made about whether or not they used procedures with meaning either. Stated differently, no claims can be made as to the effectiveness of the Sites method in supporting the students in the present study to foster meaningful understandings of their procedural knowledge of fractions. Although it may seem plausible to assume so, given the improvements they demonstrated in their conceptual knowledge, the data do not directly address this issue.

Previous research has indeed documented that instruction on procedures coupled with instruction on the conceptual meanings of those procedures does not always lead to improved procedural knowledge, and this introduces another important point regarding the reason the Sites instructional approach made no difference with respect to the students' procedural knowledge of fractions. For example, in a study conducted by

Hiebert, Wearne, and Taber (1991), students received conceptually based instruction on decimal fractions that consisted of highlighting the links between different models for decimal fractions and their symbolic notations, and the links between representations of decimal fractions and symbolic procedures used solve problems involving decimal fractions. The researchers found that although the students in their study demonstrated the acquisition of some important concepts related to decimal fraction knowledge (such as the idea of partitioning by tenths), they could not always apply those understandings in the context of solving written problems. Hiebert et al.'s (1991) results provide evidence that even when gains are made in conceptual understandings, commensurate gains in procedural knowledge do not always follow. Applied to the present study, this may cast some further light on the finding that the Sites instructional approach resulted in no significant differences where improvements in procedural knowledge was concerned.

Resnick and Omanson (1987) found similar results and in the context of their work, another noteworthy point is raised that is also paralleled in Hiebert's (1984) theory. That is, the researchers examined the effects of conceptually-based instruction on correcting students' use of faulty subtraction procedures in the domain of multidigit whole numbers. They found that instruction that concretely highlighted the conceptual rationale for symbolic subtraction procedures did not result in students' improved performance of those procedures. Resnick and Omanson conjectured that because the students, who were in the upper elementary grades at the time of the study, were already using faulty algorithms for an extended period of time (traditionally, instruction on whole number operations begins in the second grade), it was very difficult to correct those misconceptions even when instruction was carefully designed to do so. These findings appear to suggest that the effectiveness of instruction on mathematical concepts alongside that for procedures is dependent on students' prior learning experiences, and more specifically still, on the level of disconnectedness of their knowledge.

This assumption is especially plausible when considered alongside Hiebert's theory. Hiebert argued the viability of the Sites instructional method when it is implemented at the very beginning of learning about a mathematical topic (or when new procedures are first introduced) rather than later in students' development when they have already experienced years of learning procedures in the absence of connections to concepts. That is, Hiebert argued that the culture of school mathematics – the way it is taught and the perceptions students extract from those experiences – is such that it often causes students to develop more of a concern about the form of mathematics than its meaning. Mathematics becomes for many students simply an activity in following rules with little attention paid to what those rules mean. Consequently, the understandings students develop are not well related, nor are students encouraged to view creating meaning as a goal of learning in mathematics. This becomes compounded over the years with repeated experiences of this nature, which is why Hiebert argued that it is difficult to establish the connections that are at the heart of his theory so late in development.

With respect to the present study then, the findings that Resnick and Omanson (1987) reported, and the arguments Hiebert (1984) put forth, may provide some explanation as to why the Sites instructional method was only partially successful. The children who participated in this study were in their third cycle (grades 5 and 6) of schooling. Although there was no direct evidence to qualify the type of fractions instruction they received from teachers past, research has documented that typically, mathematics instruction can be characterized by its procedural focus (Kilpatrick et al., 2001; Stigler et al., 1999). As such, it seems reasonable to assume that the students in the present study came to instruction with understandings that were not well related and, particularly where procedural knowledge is concerned, perhaps even grounded in many misconceptions. After only a relatively short instructional intervention, it would be difficult to eradicate those misconceptions.

This leads to another important and related consideration with respect to explaining why the Sites instructional approach was no different than the Standard instructional approach where the students' procedural knowledge was concerned. As stated above, Hiebert (1984) believed that the Sites approach would be most effective when implemented at the outset of students' learning about a mathematical topic. According to Hiebert, the difficulty in predicting the effectiveness of his instructional method stems from the myriad possible learning experiences students may encounter during formal mathematics education and the different ways those may have influenced their learning. Stated differently, Hiebert argued that because the success of his theory is subject to children's past learning experiences, it cannot always be said with certainty that the Sites approach will be effective. With regards to the present study then, it is possible that the students who were instructed via the Sites approach may have required other teaching methods in conjunction with those involved in Hiebert's approach in order for its influence on their procedural knowledge of fractions to have been detected.

Hiebert argued that his theory has increased potential for success if it is implemented before students develop deeply rooted misconceptions and before they have had too many experiences with instruction lacking in important links between conceptual and procedural understandings. If this holds true, it can also serve to explain, at least in part, why the students in this study who were instructed via the Sites approach demonstrated greater gains than those who received standard instruction only where conceptual, but not procedural knowledge, was concerned. That is, assuming again that the students in the treatment group came to instruction with experiences grounded mostly in procedurally based instruction, it may be reasonable to also assume the reverse: that they came to instruction with less rigid conceptual understandings. That is, because mathematics instruction tends to largely ignore conceptual knowledge (Kilpatrick et al., 2001; Stigler et al., 1999), their conceptual understandings may have been more flexible and malleable, or at the very least, more so than their procedural understandings were. In this way then, they would have been coming to instruction "early" in their conceptual knowledge development, in the sense that Hiebert described.

Turning still to Hiebert's theory, another important aspect of it can provide some further explanation as to why the students who were instructed via this approach outperformed the students who were instructed via the Standard approach only with respect to their conceptual knowledge of fractions. Hiebert's theory was premised on creating links between concepts and procedures at three specific sites during instruction and learning: symbol interpretation (Site 1), procedural execution (Site 2), and solution evaluation (Site 3). Of these three sites, Hiebert maintained that establishing links between conceptual and procedural understandings is most difficult at the procedural execution site, the point in the problem-solving process where students actually put procedures to use. Hiebert attributed the difficulty (for learners and instructors alike) in establishing those connections at Site 2 to the uncertainty of how those can best be achieved given the multifaceted nature of students' existing knowledge and the complex nature that characterizes many symbolic procedures. He argued, for instance, that as procedures grow in complexity, there are not always clear and transparent maps between those procedures and models or concrete representations of them. Further still, even when those clear maps do exist, for many students, linking the ideas represented in those maps to conceptual understandings requires frequent and repeated exposure. As such, given that the instructional intervention in this study consisted of only a relatively short period of time, coupled with the fact that fractions represent one of the most complex mathematical domains in the elementary school curriculum (Ball, 1990, 1993; D'Ambrosio & Mewborn, 1994; Saxe et al., 2001; Tirosh, 2000), the students may have needed additional time and exposure to the facets of instruction entailed in the Sites method for it to have effected a substantial change in their procedural understandings.

Finally, the prediction that the treatment group would demonstrate greater gains in procedural knowledge of fractions in addition to greater gains in conceptual knowledge of fractions was based in large part on the iterative model (Rittle-Johnson & Siegler, 1998) of the development of mathematical understandings. The iterative model states that procedural and conceptual knowledge develop iteratively, in a hand-over-hand process, with increases in one form of knowledge leading to increases in the other. The iterative model also makes clear, however, that this back-and-forth process in knowledge development is a gradual one that (a) is not necessarily symmetrical, and (b) is largely dependent on students' prior experiences (Rittle-Johnson & Alibali, 1999).

For example, Rittle-Johnson and Alibali (1999) examined the iterative development of procedural and conceptual knowledge in the context of students' understandings of mathematical equivalence. In their study, students who received conceptually-based instruction on the topic of equivalence increased their conceptual understandings and also generated new problem-solving procedures. The students who were instructed via a procedurally-oriented approach also learned new procedures and in addition, increased their conceptual understandings too. In this way, Rittle-Johnson and Alibali (1999) found support for a bidirectional relationship where the development of procedural and conceptual knowledge are concerned. On the other hand, when compared to a third control group who did not receive instruction on the topic of equivalence, the researchers found that the procedurally-instructed group made only small gains in their conceptual understandings of mathematical equivalence. Rittle-Johnson and Alibali (1999) thus concluded that although their findings indicated evidence for the bidirectional nature of procedural and conceptual knowledge development, for the procedurally-instructed students in their study, the influence of this relationship was limited.

Considering Rittle-Johnson and Alibali's (1999) findings alongside Rittle-Johnson and Siegler's (1998) description of the iterative model of the development of procedural and conceptual understanding may help explain the results reported in the above mentioned study. That is, Rittle-Johnson and Siegler (1998) have argued that at a particular point in type, either type of knowledge may be better developed than the other and that this is largely dependent upon students' prior experiences in mathematics. Thus, the procedurally-instructed students in Rittle-Johnson and Alibali's (1999) study may not have made significant gains in their conceptual understandings because their procedural knowledge was better developed at that particular point in time. Applied to the present study, a similar line of reasoning may serve to explain the treatment group students' procedural performance. In other words, the treatment group students in this study may have been at a particular point in their development where their conceptual understandings of fractions were better formed than their procedural understandings were. Moreover, given the gradual iteration in knowledge development that is specified in the model, along with the finding that gains in procedural and conceptual understandings are not always symmetrical (Rittle-Johnson & Alibali, 1999), the treatment group may have been in a better position to benefit more from the conceptual rather than the procedural elements of the instruction they received.

Research Hypothesis 2: Improved Ability to Link Conceptual and Procedural Understandings of Fractions at Site 1, Site 2, and Site 3

With the second research question in the present study I asked, Does the Sites instructional approach enable students to link their procedural and conceptual understandings of fractions at each of Hiebert's (1984) three sites? This question was answered with the quantitative analysis that was performed on the fractions knowledge interview and with the qualitative analyses that examined the students' responses to the interview items designed to measure Site 1, Site 2, and Site 3 understandings. With respect to the second research hypothesis, the results of the quantitative analysis indicated support for the prediction that students who were instructed via the Sites approach would demonstrate greater improvements in their ability to link procedural and conceptual understandings of fractions at each of Hiebert's three sites.

The qualitative analyses revealed similar findings, and also pointed to other noteworthy ones as well. For example, these analyses revealed that the students' abilities to make those links differed along several dimensions across the different sites. In addition, they also revealed that the control group's ability to make connections at Hiebert's three sites changed over the course of instruction, and similar to the treatment group, changed in different ways at each site. Each of these issues will be discussed in more detail below, but I first begin with a brief review of the theory on which the second research hypothesis was based.

Hiebert (1984) identified the following three sites as possible instances during the learning of mathematics where meaningful relationships may be formed between conceptual and procedural understandings: Site 1, called symbol interpretation, is the point at which the symbols of the problem or task are given meaning. Hiebert posited that at Site 1, conceptual and procedural understandings may be linked if symbolic representations in a problem are linked with referents that give those representations meaning. Site 2, called procedural execution, is the point in the problem solving process when procedures are executed. With respect to Site 2, Hiebert speculated that conceptual and procedural understandings may be related if procedures, rules, or algorithms used to solve a problem are connected to their underlying rationales, and if symbols used to represent procedures take on meaning for the learner. This latter statement points to the importance of Site 1 understandings for Site 2 understandings. Lastly, Site 3, called solution evaluation, is the point in the problem solving process when solutions are evaluated for reasonableness. At Site 3, Hiebert's conjecture was that conceptual and procedural knowledge may be linked if symbolically represented problem solutions are linked to (a) ideas about how problems might be solved in concrete or real-world contexts, or (b) other knowledge of the number system. Site 3 understandings, then, are also fundamentally linked to Site 1 understandings in that the emphasis is on creating meaning for problem solutions that are represented symbolically.

Site 1: Linking Symbols to their Meanings

To begin, the findings from this study indicated that students who were taught via the Sites approach demonstrated grater gains over those who were instructed by the Standard method in their ability to link conceptual and procedural understandings of the symbols that are used to represent fractional quantities. This ability was assessed by asking students to represent both the symbolic notations that stand for fractional values and the concrete pictorial referents that lie behind those notations. If students could represent both correctly for a given fraction, they demonstrated evidence of a link between the concept of that given number (the value it represents) and its expression in symbolic notational format. On the other hand, if only a symbolic notation but not a concrete (pictorial) referent could be provided, or vice versa, the students could not demonstrate that link.

The qualitative analysis revealed that prior to instruction, the students in both groups were more successful at writing the symbolic notation for a given fraction than they were at representing the meaning behind that notation. For example, of the instances in which the treatment group students expressed fractions correctly using numbers, only 68% of that time could they also represent the ideas behind those symbols. Although noticeably higher, the findings were similar for the control group. That is, of the instances in which students in the control group expressed fractions correctly using numbers (95% of the time), they only correctly represented those fractions pictorially 82% of that time. After instruction that emphasized the links between symbolic notations for fractions and their underlying meanings, however, the students in the treatment group demonstrated accurate performance 92% of the time in representing fractions pictorially when they also wrote correct symbolic notations for fractions. After instruction in the Standard approach, on the other hand, the control group students only represented fractions correctly 87% of the time in which they could correctly write a given fractional quantity using numbers. Although this still indicated fairly accurate performance for the control group, it did not represent much of an improvement from their pre-instruction performance.

The fact that the students in this study came to instruction with a better developed ability to write fractions than to represent their meanings is not surprising. This is yet another instance of students demonstrating understandings that are not well connected. For example, previous investigations have documented that although students can often demonstrate their understandings of the meaning of fractional quantities outside of the context of symbols, when symbols are introduced, they make errors that are inconsistent with the understandings they are known to possess (Ball, 1993; Hiebert et al. 1991; Mack 1990, 1995). These errors, moreover, are often times very striking, as illustrated in the following example taken from Mack's (1995) work. In her study, students were asked to solve subtraction problems that entailed subtracting a fraction from a whole number. In the context of these problems, Mack (1995) reported two noteworthy findings. First, when students were asked to solve these in a real-world context (such as thinking about subtracting one-eighth of a pizza from three whole pizzas), they consistently demonstrated accurate understandings. On the other hand, when similar problems were posed in symbolic format, the students consistently made mistakes that were at odds with the understandings they demonstrated when solving the real-world problems. That is, although the students could write subtraction statements for fractions correctly using mathematical symbols, they interpreted a symbolic problem such as 2 - $\frac{3}{8}$ to mean "two-eighths minus three-eighths." In other words, Mack's findings indicated that while the students could demonstrate accurate representations of fractions in real-world contexts, they could not link those understandings to fractions they represented symbolically.

Ball's (1993) research on fractions in particular suggests that in large part, this is due to the fact that students seldom see representations for fractions during instruction. Further still, even when they do, those representations are rarely varied nor used by teachers in ways that would support students' development of meanings for fraction symbols. That is, when instruction includes representations for fractions, these consist mainly of the "standard" region model (a closed geometric shape such as a circle or rectangle) as opposed to other models for fractions such as the set and length models. Moreover, most student activities involving representations for fractions require students to shade pre-divided regions where the number of partitions in the model match the numeral in the denominator of the symbolic notation (Ball, 1993). Students are rarely asked to generate their own representations, or at the very least, to partition a model themselves. Consequently, while students see the symbols for fractions in varied and repeated contexts, typically, the same cannot be said for representations of those symbols.

In the present study, the students in both groups came to instruction with a better sense of how to represent fractions with symbols as opposed to how to represent their meanings in a concrete (pictorial) manner. After instruction, those who were instructed via the Sites approach demonstrated 100% accuracy in writing fractions using numbers and 92% accuracy in representing the meanings behind those symbolic notations. As such, the gains these students made with respect to Site 1 understandings were related mainly to improvements in their ability to represent fractions concretely (i.e., pictorially) rather than to their ability to write the symbolic notations for fractions. In comparison, after instruction, the students who were instructed via the Standard approach demonstrated fairly equal gains in both representing fractions with symbols and in representing fractions pictorially, but these gains were not very large in number compared to the performance they demonstrated before instruction. As such, the students in the control group were less successful at linking their knowledge of symbolic notations for fractions to their knowledge of the ideas that lie behind those symbols after instruction.

Keeping these findings in mind, the intervention in this study attempted to get at some of the instructional shortcomings that Ball (1993) has identified in her work. That is, the students in this study who were instructed via Hiebert's approach saw many different models for fractions (i.e., region, set, length, and area) and were also instructed explicitly as to how those models link to symbolic notations for fractions. Conversely, the control group students in this study received no instruction with respect to linking fractional symbols to representations of the meanings behind those symbols. The results of this study, therefore, fall in line with and support Ball's (1993) arguments. That is, they provide evidence that seeing fractions in many different representational contexts is important with respect to bringing meaning to symbols that are used to represent fractions.

Site 2: Linking Procedures to their Meanings

The results of the study also indicated that the students who were instructed via the Sites approach demonstrated greater gains over those who were instructed via the Standard approach in linking their knowledge of procedures used to solve problems involving fractions to their conceptual understandings of the meanings underlying those procedures. With respect to summarizing the overall picture of each group's Site 2 knowledge before and after instruction, several findings from the qualitative analyses are noteworthy.

To begin, the qualitative analyses revealed that the students in both groups were more successful at solving fractions problems than they were at linking procedures to concepts both prior to and after instruction. The control group, however, demonstrated more skill than the treatment group in doing so. Conversely, the treatment group demonstrated more ability than the control group to link their understandings of fractions procedures to their conceptual meanings. Furthermore, although the control group improved on both fronts over the course of instruction, at the end of the study, they were still more successful at solving problems than they were at linking their Site 2 understandings. That is, of the instances in which students in the control group correctly solved a problem involving fractions after instruction, they could only link the procedure they used to its underlying meaning less than 50% of that time. Similar to the control group, the students in the treatment group also made gains in both using procedures with accuracy and in linking their understandings at Site 2 over the course of instruction. Conversely to the control group, however, they demonstrated larger increases in their ability to do both in tandem at the end of the study. Taken together, these findings indicate that the Standard instruction was more effective at fostering the students' development of procedural skill than it was at fostering their Site 2 understandings, whereas the Sites instructional method was effective at fostering the students' abilities to both use procedures with accuracy and link the knowledge behind that to their conceptual understandings.

Looking at the types of procedures the students in both groups used to solve fractions problems adds to these findings. That is, the results of the qualitative analyses indicated that from pretest to posttest, the largest increase in frequency in with respect to the type of procedure that students in the control group employed to solve a problem was for a standard symbolic procedure. More specifically, the use of a standard symbolic procedure by students in the control group increased nearly threefold after instruction. Further still, for some students in the control group, the use of number knowledge as a method for solving fractions problems before instruction began was replaced by the use of standard symbolic procedures after instruction. In comparison, after instruction, the combined use of both the concrete model and number knowledge problem-solving methods by students in the treatment group represented the largest increase in frequency from pretest to posttest in the types of procedures those students used to solve fractions problems.

These findings provide evidence to support the effectiveness of the Sites instructional approach at fostering the students' development of meaningful procedures used to solve fractions problems. In particular, this was evidenced by the fact that treatment group students used the concrete model and number knowledge methods more frequently after instruction than they did prior to instruction, and by the fact that they used those two methods more often than they used standard symbolic procedures. On the other hand, the qualitative analyses also indicated that the frequency of errors in the concrete model and number knowledge procedure for students in the treatment group also increased after instruction. Although this particular finding seems somewhat surprising, it can be explained. For example, although the students in the treatment group could not demonstrate complete accuracy in using those procedures after instruction, they did nonetheless demonstrate an increased attempt to do so, suggesting that, at the very least, they developed some partial understandings of the concepts involved in those procedures. This is noteworthy because it provides evidence to support the argument that prior to instruction, they had less of an understanding of how to use problem representations and their knowledge of the number system as vehicles for solving problems whereas after instruction, they had a better, albeit partial sense, of how to do so.

The results of the qualitative analyses also indicated, however, that although the treatment group students increased their use of the concrete model and number knowledge problem-solving procedures after instruction, they also increased their use of standard symbolic procedures as well. This finding is equally noteworthy because the qualitative analyses also revealed that for both groups of students, the difficulties involved in making links at Site 2 were in large part related to the use of standard symbolic procedures. Stated differently, both groups of students demonstrated that they had weak links between their conceptual understandings and their knowledge of symbolic procedures even after instruction.

This raises an important question with respect to the treatment group's Site 2 understandings. That is, given Hiebert's theory and the central role that Site 1 understandings play in Site 2 understandings, alongside the finding that the treatment group made significant gains in their symbolic understandings at Site 1, the finding that they had weak conceptual understandings of symbolic procedures is somewhat puzzling. What might explain why the students in the treatment group had difficulties linking their conceptual understandings to their knowledge of standard symbolic procedures given that they demonstrated large improvements in their Site 1 understandings? Turning to the qualitative analyses once more, two explanations seem especially plausible.

The first relates to the particular shape the instructional intervention in this study took. That is, by virtue of what Site 1 entails – creating meaning for the symbolic notations used to represent fractions – Site 1 understandings were woven into instruction throughout the length of the intervention. In other words, during every mathematics lesson, regardless of its topic, Site 1 understandings were addressed in some way. For example, in every instance that a fraction symbol was used – whether this was when part-whole meanings were the topic of the lesson, or when equivalence was the topic of the lesson, or when addition was the topic of the lesson, and so on – it was coupled with a representation to support its meaning as well as explicit reference to the links between the two. In this sense then, the students received instruction on Site 1 knowledge throughout the course of the intervention.

With respect to Site 1 understandings as they relate to Site 2 understandings, on the other hand, meanings for fraction symbols as they linked to a given procedure were only seen when that particular procedure (or mathematical topic) was the focus of the lesson. Taking the topic of addition of fractions as an example can clarify. Students only saw the links between symbols and their meanings *in relation to* procedures for adding fractions when, and only when, the addition of fractions was the topic of the lesson. Fraction topics, moreover, changed at each instructional lesson, so that a given topic was addressed in only one 50-minute instructional session. As such, Site 1 understandings, as they related to Site 2, were seen in much less frequency than were Site 1 understandings in their strict sense. Overall then, the shape of instruction was such that the students had many more experiences in linking symbols to their meanings as they relate strictly to Site 1 understandings than they did as they relate to Site 2 understandings. This may in part explain what appears to be a somewhat ambiguous result with respect to the treatment group students' performance at Site 2.

The second explanation relates to Hiebert's argument regarding the complexity involved in establishing links at Site 2. As noted earlier, Hiebert made explicit mention of the fact that links at Site 2 are the most difficult to establish, particularly as procedures become more complex. Also noted earlier, research has documented that fractions in particular are difficult to teach and to learn with understanding (Ball, 1990, 1993; D'Ambrosio & Mewborn, 1994; Saxe et al., 2001; Tirosh, 2000). Hiebert argued, and others have demonstrated (Ball, 1993; Hiebert & Wearne, 1996; Mack, 1995, 1999) that it is not always easy to establish meanings for symbolic procedures or to link those procedural understandings to understandings of mathematical concepts. This requires substantial time and repeated experiences, which, unfortunately, the students in this study did not receive. Further still, Hiebert's argument that the effectiveness of his approach is best achieved when implemented early in the course of mathematics instruction can also be applied here. That is, as described earlier, it seems reasonable to assume that the students in this study came to instruction with years of typical standard instruction behind them. As such, in addition to the complexity involved in Site 2 understandings, the students' past experiences may have further constrained their opportunity to develop Site 2 understandings, particularly in the context of a relatively short instructional period.

These latter conjectures seem especially plausible when problem type, and the particular procedures the students used, are taken into consideration. That is, the results of the qualitative analysis revealed that most of the errors the students in the treatment (and control) group made when linking procedures and concepts at Site 2 were made on items that required them to convert improper fractions to mixed numbers (and vice versa). The qualitative analyses also revealed that they used more standard procedures to solve these problems, procedures for which they had more difficulty providing rationales. Taken together then, the students used standard procedures more frequently to solve problems they did not understand as well – problems that were more complex – and they often did so without understanding. In line with Hiebert's theory therefore, the results demonstrated that the difficulties the students evidenced in linking concepts and procedures at Site 2 were due in large part to the complexity of the algorithms they used.

Site 3: Evaluating Symbolic Mathematical Statements with Meaning

The results of the quantitative analysis of the fractions knowledge interview indicated that the students who were instructed via the Sites approach demonstrated greater gains over those who were instructed via the Standard approach with respect to their ability to link procedural and conceptual understandings of fractions in the context of evaluating the reasonableness of mathematical statements represented symbolically. On the other hand, the qualitative analyses indicated that of the three sites that were examined in this study, both groups of students made the least gains at Site 3. That is, when the percent of correct responses provided by each group prior to instruction was compared to the percent of correct responses each group made after instruction, the findings revealed that the increases for each group were, comparatively speaking, smaller at Site 3. This finding is of particular interest given that Hiebert (1984) hypothesized that Site 3 understandings are some of the easiest to facilitate.

With respect to the treatment group's performance at Site 3, then, the question raised by the results of the qualitative analyses is why did the students who were instructed via the Sites approach demonstrate the least gains at Site 3? Answers to this question may be provided in two ways: (1) with respect to considering some of the main ideas that lie in Hiebert's (1984) theory, and (2) by examining the treatment group's performance at Site 3 in relation to their performance at each of the other two sites.

To begin, Hiebert argued that the ability to link knowledge of concepts to knowledge of procedures at Site 1 is fundamentally related to the ability to do so at Site 3. That is, to evaluate the reasonableness of a problem solution that is stated in symbols, students need to have understandings of symbols that are linked to conceptual referents that give those symbols meaning. Although Hiebert maintained that Site 1 knowledge is critical to Site 3 performance, Site 3 understandings may not necessarily follow from Site 1 understandings. That is, the idea behind Site 3 understandings, and the links between those understandings and understandings at Site 1, entails much more than simply having well developed notions of symbols and their underlying meanings.

For example, Hiebert's notion of solution evaluation referred to using number knowledge and contextualizing problems to evaluate the reasonableness of symbolic problem solutions. Where the Site 1 component of his theory plays a role is in the presentation of problem solutions. That is, the goal of Site 3 is to be able to evaluate whether or not a solution to a problem that is stated in symbolic terms makes sense. To do so, therefore, students must combine knowledge of problems and how they are solved (which includes knowledge of procedures), knowledge of real-world ideas, knowledge of the number system, and knowledge of symbols and their conceptual referents – all in coordination, furthermore, with the particular mathematical topic at hand. Taken together, Site 3, which in and of itself is its own specialized kind of knowledge, also involves aspects of Site 1 and at least some aspects of Site 2 understandings. It appears to be quite clear, then, that Site 3 understandings are quite complex, and perhaps even more so than Hiebert may have believed. In the context of the present study, what is of import is that the seemingly inconsistent finding in the treatment group's performance at Site 3, when it is compared to their performance at Site 1, may not be that inconsistent after all.

Moreover, then, because at least some aspects of Site 2 understandings are also relevant to performance at Site 3, the findings with respect to the treatment group's performance at Site 2 can speak to their performance at Site 3. For example, the qualitative analyses revealed that after instruction, the students in the treatment group responded to Site 3 items in very similar ways that they did to Site 2 items. That is, at Site 2, the students used either a standard symbolic, concrete model, or number knowledge procedure to solve fractions problems correctly. At Site 3, the students used the same procedures (in the context of evaluating the reasonableness of symbolic solutions, in this case) and also used them in near equal frequency as they did at Site 2. The difference, however, was that at Site 3 the use of a standard symbolic procedure as a verification method entailed an error. That is, as described above, the goal at Site 3 is to use a method other than a procedural one to evaluate symbolic mathematical statements.

Given that the students in the treatment group used standard symbolic procedures in almost equal frequency to correct verification methods at Site 3, and because the results at Site 2 indicated that the instructional intervention made little difference in the students' learning gains where standard symbolic procedures were the case, it follows then that when the students used standard symbolic procedures at Site 3, they also did so with little meaning. As described above, because Site 3 entails combining knowledge of procedures and problem-solving in addition to its own specialized features, the fact that the Sites instructional approach was not entirely effective at Site 2 appears to explain, at least in part, why it was not entirely effective at Site 3.

Drawing on the qualitative results once again, further evidence exists to support this argument. That is, the incorrect verification methods at Site 3 that entailed a symbolic component were either (a) applying a (correct) standard procedure, or (b) attempting to apply a standard procedure but making a procedural error. The results indicated, then, that for the most part, the students in this study relied heavily on their procedural understandings to verify problem solutions. Once again, these were often not well connected to conceptual understandings, particularly with respect to the more complex problem types. Taken together, this provides some evidence to support the conclusion that the understandings the students showed difficulties with at Site 2 appeared to play a large role in the difficulties they evidence at Site 3.

The fact that the students often relied on their procedural knowledge to evaluate symbolic problem solutions raises another important point that may further clarify the findings. The students may not have seen the need to evoke other, related knowledge of the type Hiebert described to verify the symbolic solutions they evaluated. For example, when students used a correct standard procedure as the verification method, they did so without understanding. In other words, the use of this verification method at Site 3 indicated that the students could not explain, other than procedurally, why a symbolic solution was correct or incorrect. Further to that, when they used a faulty symbolic procedure (coded as procedural error), that also indicated a lack of understanding of the procedure. For all intents and purposes, however, to the student who believed that his faulty procedure was correct, making a procedural error was no different than applying a correct procedure. Thus, believing that the employed procedure (even when it was a faulty one) produced the correct symbolic solution, the students may not have seen the need to use other, more meaningful knowledge (such as estimation) to verify the symbolic solutions they evaluated. This seems especially plausible in light of research that has shown that for most students, mathematics is about following rules and getting the right answers; if it looks correct on paper, students rarely ask themselves questions (Ball, 1993; Hiebert, 1984; Hiebert & Wearne, 1996; Mack, 1995, 1999). This certainly appeared to be the case for the treatment group students in this study, as evidenced by the fact that although they used number knowledge to solve fractions problems at Site 2, they applied those understandings less frequently when evaluating symbolic solutions to fractions problems at Site 3.

Finally, the earlier point regarding the shape of the instructional intervention also applies to the results that were obtained at Site 3. That is, similar to Site 1, Site 3 understandings were also interwoven throughout instruction for the duration of the intervention. On the other hand, also similar to Site 2, with respect to a given problem type and its related procedures, this only occurred when those were the focus of the instructional lesson. Stated differently, although the instruction did explicitly highlight alternative methods for verifying symbolic solutions (such as using estimation, benchmark numbers, and other related number knowledge) throughout the course of the intervention, this occurred in less concentration where specific problem types and their related procedures were concerned. Therefore, like at Site 2, given the relatively short period of the instructional intervention, and specifically, the relatively short amount of time that a particular fractions topic was the focus of a lesson, the students may not have had ample time to strengthen their ability to link concepts and procedures with respect to verifying the reasonableness of symbolically represented solutions to problems involving fractions.

Educational Implications, Limitations, and Open Questions

The findings from this study build on prior research that has demonstrated that when presented with almost any form of instruction, students will learn (Kilpatrick et al., 2001). On the other hand, this study also demonstrated that the quality of that learning is often questionable. This raises important considerations with respect to the way mathematics is taught and the way student performance is measured in the classroom. The results from this study indicated that when assessed strictly on their ability to use mathematical procedures with accuracy to solve fractions problems, standard instruction, as well as a more conceptually-oriented instructional approach, were both relatively successful at improving the students' performance. On the other hand, when assessed on their ability to use procedures with understanding, the effectiveness of each of these instructional methods was less clear. Given that conventional, traditional forms of mathematics instruction often characterize much of students' classroom learning experiences, this study lends further support to the argument that formal mathematics education is in need of some revision. Teachers must carefully consider their goals for student learning when outlining instructional objectives and the ways they assess whether or not those goals have been achieved.

The objective of this research grew out of the need to better describe concrete teaching practices that can improve students' meaningful understandings of the mathematics they learn. The findings provide some important insights as to what those teaching practices may entail, and some initial answers to the more specific question of where exactly teachers need to focus instructional attention when meaningful learning of mathematics is the goal. In particular, the findings indicate that the ability to problem-solve, with respect to executing procedures with understanding, is difficult for students to develop. Furthermore, this study provides some evidence that the difficulty in doing so may relate in part to the specific types of problems students encounter in mathematics as well as to the specific problem-solving procedures they use. Each of these issues are important ones that teachers need to keep in mind if their instructional goals include facilitating students' meaningful understandings of problem-solving in mathematics.

Furthermore, this study demonstrates that attention must be paid to the quality of students' existing knowledge when learning mathematical procedures with understanding is an instructional focus. Some of the difficulties the students in the present study demonstrated in using standard procedures with understanding and in adopting more meaningful ones may have been related, at least in part, to the misconceptions they brought with them to instruction. As such, this should be taken into account when instructional programs in mathematics are developed, and when those are translated into classroom teaching practices.

The present findings also indicate that instructional attention needs to be directed toward fostering students' abilities to evaluate the solutions they arrive at when they use symbolic procedures to solve problems in mathematics. In particular, this research demonstrates that in addition to the need to learn more meaningful methods that can be applied to verifying the reasonableness of the symbolic solutions they generate, students may also need to develop an increased concern to do so. Therefore, in addition to teaching students how to relate the symbolic solutions they produce to their understandings of what those solutions mean, instruction must also be aimed at developing students' mindfulness to do so.

In line with each of these ideas, another important consideration drawn from the present research relates to the perceptions students tend to develop following from the way teachers teach mathematics. For at least some of the students in this study, difficulties in linking procedural and conceptual understandings when solving problems stemmed from the preference to use standard procedures to solve those problems as opposed to methods that were more meaningful to the students. Moreover, many of the students also appeared to show a more pronounced concern about the form of mathematics (the way mathematics looks) than they did about its meaning. These findings lend further support to the argument that when making decisions about the shape their instruction will take, teachers must remain cognizant of the views and ideas they may unwittingly transmit to students with respect to what it means to "do" mathematics and what important learning goals in mathematics are.

A major limitation in the present research with respect to determining the effectiveness of the tested instructional method was that it was implemented at a time in the students' development where they already had several years of mathematics instruction behind them. This made it difficult to determine on many fronts whether or not the approach would have been more successful had it been used earlier in the course of their learning about fractions. Future research replicating this study with younger students needs to be conducted to draw any conclusions of a more solid nature. Moreover, drawing firm conclusions about the effectiveness of the tested instructional method was also compromised by the fact that the intervention period was relatively short in time. Consequently, the students in this study may not have had ample opportunities to build links between the different forms of their understandings, particularly where specific problem types and specific problem-solving procedures were concerned. In this way too, it would be instructive if the study were replicated over a longer period of time.

With respect to the specific theory that this study examined, the present results indicate that several open questions remain to be resolved. For one, in this study, it was posited that the Sites instructional approach may have been more effective had it been employed in conjunction with other teaching practices. What those teaching practices may entail, however, also remains unresolved given the notion that these relate in large part to students' prior learning experiences and the existing knowledge they bring with them to instruction – both of which are often multifaceted and quite complex to unravel from an instructional point of view. These issues, therefore, remain to be examined with respect to determining the effectiveness of the approach.

Second, Hiebert conjectured that instruction in mathematics may be in a better position to foster students' development of Site 3 understandings than it may be to support their understandings at Site 2. The present findings, however, present some contradictions to this argument. That is, for the particular students in this study, creating links at Site 3 was just as difficult as it was at Site 2. Future research that directly examines this issue, in addition to investigations of how Site 3 understandings may best be supported, would be helpful with respect to providing teachers and researchers with more accurate descriptions of where instructional attention should be paid when fostering students' development of those critical understandings is a goal.

Finally, although the present research provides evidence that the Sites instructional approach is in large part an effective teaching method for developing students' meaningful understandings of mathematics, one draw back to these finding is that no assessments were made of the students' long-range understandings. That is, determining the effectiveness of the approach with respect to (1) maintenance of understandings over time, and (2) transfer of understandings to other mathematical domains, were not a part of this study's goals. These remain important open questions that should be investigated to determine more definitively the practicality and value of the Sites instructional approach.

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Appendix A

Parent Information Letter

Parent/Guardian Information Letter

Dear Parent(s) and/or Guardian(s),

<Insert Name of School here> is working in partnership with Concordia University to help children learn math. In order to learn about the best ways to teach fractions to Cycle 3 students, I would like to invite your child to participate in my math project.

I am a certified elementary school teacher, who is trained to teach mathematics, and a student in the Master of Arts Child Study programme at Concordia University. To learn about effective ways of teaching math for understanding, I would like to teach your child about fractions during his/her regularly scheduled mathematics class time for a few weeks this year. During my teaching activities in your child's classroom, your child will be asked to complete several fractions activities on paper. I would also like to work one-on-one with your child on fractions twice during the few weeks that I am teaching in your child's classroom. These individual sessions will be conducted in your child's classroom or in a quiet place such as the school library and will last a maximum of 20 minutes each. I would also like to audiotape the conversations I have with your child about math only. This will help me better understand how children think about fractions and how they solve fractions problems, and it will also help me understand how to best teach fractions to children.

None of your child's written work or the conversations that I have with him/her will be shared with anyone except with my university supervisor, Mrs. Helena Osana, professor at Concordia University. However, Mrs. Osana will never know your child's name; only numeric codes will be used in all discussions that I have with Mrs. Osana about your child's work or math conversations. For teaching and classroom planning purposes, your child's teacher will have access to all of your child's work during the time that I am teaching him/her about fractions.

All children are free to participate in the project or not. Neither you nor your child will be placed at any risk should you choose to be a part of this project. You are free to discontinue your participation or your child's participation at any time, with no negative consequences to you or your child. To do so, you will have to contact me by telephone or via email, or contact your child's classroom teacher.

Your child's school principal and classroom teacher are fully aware of this project and are looking forward to being a part of it. If you agree to allow your child to be part of our school's project too, please sign the attached consent form and return it with your child no later than <Insert date here>.

Sincerely,

Nicole Pitsolantis, MA Child Study
Concordia University
1455 deMaisonneuve Blvd. Ouest, LB-568-5
Montreal, QC H3G 1M8
Phone: (450) 505-3011 or Email: n_pitsol@education.concordia.ca

If you have any questions you'd like to ask my professor, Mrs. Helena Osana, she can be reached at:1455 deMaisonneuve Blvd. Ouest, LB-568-10, Montreal, QC H3G 1M8.
Telephone: 514-848-2424 ext. 2543, email: osana@education.concordia.ca

Appendix B

Consent Forms: Child and Parent(s)/Guardian(s)

MY AGREEMENT

You want to teach me about fractions in my math class for a few weeks this year. You also want to talk to me alone about fractions a few times and ask me to solve some fractions problems while you tape record my voice.

I want to talk to you about fractions. My teacher will know what I write down or what I say about fractions. One of your friends at your school will also know what I say about fractions but nobody except you will know my name.

I know that talking to you won't hurt me. If I want, I don't have to talk to you any more. That's fine with you. I just need to tell you or my teacher.

I know why you want to talk to me about math. I say it's OK.

NAME (please print) ______________________________

SIGNATURE ___________________________________

DATE ____

CONSENT FORM TO PARTICIPATE IN RESEARCH

This is to state that I agree to allow my child to participate in a research project being conducted by Nicole Pitsolantis, licensed teacher in the province of Quebec and student in the MA Child Study programme at Concordia University.

A. PURPOSE

I have been informed that the purpose of the project is to examine how to best teach fractions to Cycle 3 students in order to foster meaningful understanding of fractions. This information will be very useful in identifying effective instructional activities and techniques for use in teaching fractions to sixth grade students.

B. PROCEDURES

The project consists of daily teaching lessons in my child's math class, for a period of approximately three weeks, on various concepts related to the topic of fractions, to be conducted by Nicole Pitsolantis, certified teacher in the province of Quebec and student in the MA Child Study programme at Concordia University. My child and his/her classmates will do various math activities on the topic of fractions and will also be asked to solve fractions problems in writing. My child may also be asked to participate in a one-on-one, audio-recorded interview with Nicole Pitsolantis, during which she/he will be asked to answer questions about fractions concepts and to solve some fractions problems on paper. The interview will not last more than 20 minutes and will take place in my child's classroom or in another quiet area in the school, such as the library. It is possible that Nicole Pitsolantis may return to my child's classroom at a later date to teach him/her and his/her classmates some more activities on the topic of fractions.

I have been informed that there are no risks involved to myself or my child, nor will the interview cause my child any discomfort. I also understand that I am free to withdraw my consent at any time (or to withdraw the consent of my child) and discontinue my participation at no penalty to me or my child. The information provided by my child that will have his/her name attached to it will be stored in a locked cabinet in the home of Nicole Pitsolantis. Only Nicole Pitsolantis will have access to this cabinet. Only she and her university supervisor, Mrs. Helena Osana, will have access to the information provided by my child, but Mrs. Helena Osana will never know my child's name. Nicole Pitsolantis will refer to my child's work using a numeric code whenever she shares any information with Mrs. Osana. Nicole Pitsolantis and Mrs. Osana will not share any information concerning my child with anyone else. I understand that I have the right to have the information provided by my child withdrawn from the project at any time. For classroom teaching and planning purposes, my child's classroom teacher will have access to all of my child's work during the time that Nicole Pitsolantis is teaching him/her about fractions.

C. CONDITIONS OF PARTICIPATION

- I understand that I am free to withdraw my consent (or the consent of my child) and discontinue my participation at anytime without negative consequences. To do so, I must contact Nicole Pitsolantis or my child's classroom teacher.
- I understand that my child's participation in this project is CONFIDENTIAL (i.e., Nicole Pitsolantis will know, but will not disclose my child's identity to anyone).
- I understand that the data from this project may be published. No information disclosing my child's identity or my child's school's identity will be reported in any presentation or publication (only fake names will be used).

I HAVE CAREFULLY STUDIED THE ABOVE AND UNDERSTAND THIS AGREEMENT. I FREELY CONSENT AND VOLUNTARILY AGREE TO ALLOW MY CHILD TO PARTICIPATE IN THIS PROJECT.

MY NAME (please print) ____

SIGNATURE ____

CHILD'S NAME (please print) ____

DATE ____

☐ I would like a report of the results of this project.

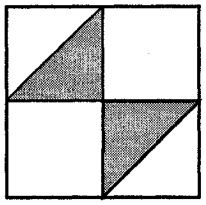
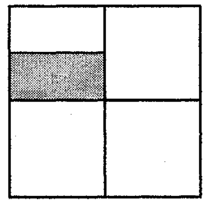
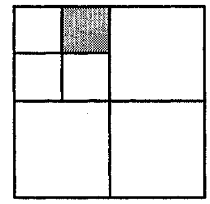
If at any time you have questions about this project, please contact Nicole Pitsolantis at:
Address: 1455 de Maisonneuve Blvd. Ouest, LB-568-5, H3G 1M8
Telephone: 450-505-3011;
Email: n_pitsol@education.concordia.ca

You may also contact my supervising professor, Mrs. Helena Osana at:
Address: 1455 de Maisonneuve Blvd. Ouest, LB-568-10, H3G 1M8
Telephone: 514-8482-2424 extension 2543
Email: osana@education.concordia.ca

Appendix C

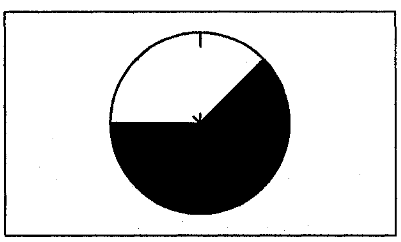
Fractions Knowledge Test (Saxe et al., 2001)

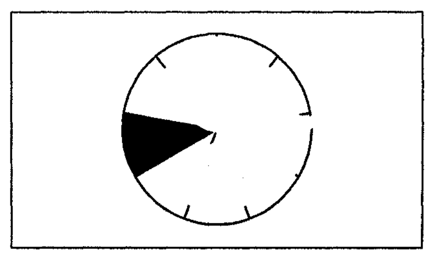
1) For each picture below, write a fraction to show what part is gray:

1.1)  1.2)  1.3) 

1.1) ____________ 1.2) ____________ 1.3) ____________

2) Circle the fractions that show what part of each circle below is gray:





2.1) $\frac{1}{4}$ $\frac{3}{5}$ $\frac{9}{10}$

2.2) $\frac{1}{9}$ $\frac{1}{3}$ $\frac{2}{5}$

3) Compute the following equations:

3.1) $\begin{array}{r} \frac{3}{5} \\ +\frac{1}{5} \\ \hline \end{array}$

3.2) $\begin{array}{r} \frac{2}{10} \\ +\frac{2}{5} \\ \hline \end{array}$

3.3) $\begin{array}{r} \frac{1}{3} \\ +\frac{1}{2} \\ \hline \end{array}$

3.4) $\begin{array}{r} 7\frac{5}{8} \\ +4\frac{1}{2} \\ \hline \end{array}$

3.5) $\begin{array}{r} \frac{7}{10} \\ -\frac{1}{10} \\ \hline \end{array}$

3.6) $\begin{array}{r} \frac{5}{6} \\ -\frac{1}{3} \\ \hline \end{array}$

3.7) $\begin{array}{r} \frac{2}{3} \\ -\frac{1}{2} \\ \hline \end{array}$

4) Write one fraction that is the same as each fraction below, for example: $\frac{1}{2}=\frac{2}{4}$

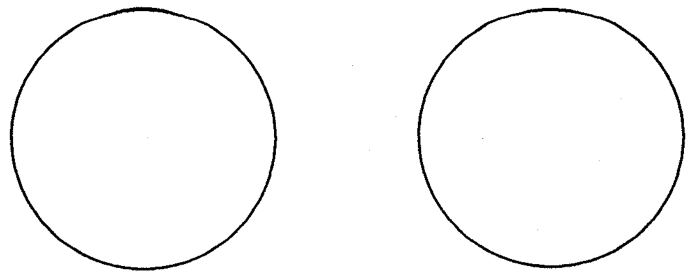
4.1) $\frac{2}{6}=$

4.2) $\frac{1}{5}=$

4.3) $\frac{12}{16}=$

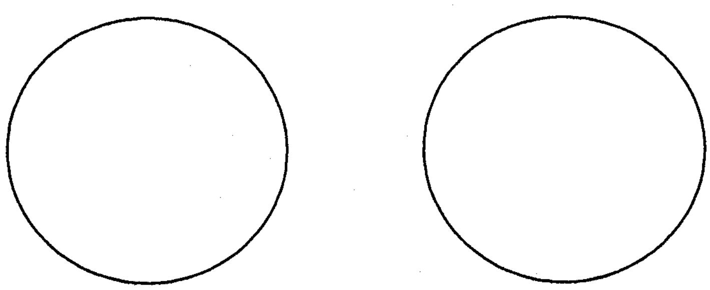
4.4) $\frac{7}{6}=$

5) a. Four people are going to share these two pizzas equally. Colour in one person's part.



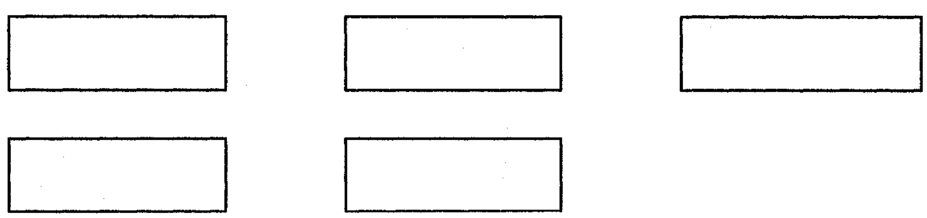
b. Write a fraction that shows how much one person gets _____________

6) a. Three people are going to share these pizzas equally. Colour in one person's part.



b. Write a fraction that shows how much one person gets _____________

7) a. Six people are going to share these five chocolate bars equally. Colour in one person's part.



b. Write a fraction that shows how much one person gets _____________

8) Fill in the missing numbers:

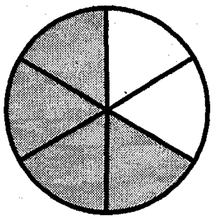
8.1) $\frac{1}{5}=\frac{\square}{10}$

8.2) $\frac{3}{4}=\frac{\square}{8}$

8.3) $2\frac{1}{2}=\frac{\square}{2}$

8.4) $3\frac{1}{4}=\frac{\square}{8}$

9) Circle a, b, c, or d below to show what part of this circle is gray:



a) $\frac{1}{2}+\frac{1}{3}$ b) $\frac{3}{6}+\frac{1}{6}$ c) $1+\frac{1}{3}$ d) 4

10a) John ran $\frac{2}{5}$ of a mile on Thursday and $\frac{3}{5}$ of a mile on Friday. How far did he run altogether on the two days? ______________________________

10 b.) Draw a picture to show your work.

Appendix D

Vocabulary Test

NAME ____

Circle all the real words from the list below. Remember, don't guess because some of the words are made-up. Circle only the words that you know are real.

applease	fean	outlandish
arbor	flue	outpost
brittle	formater	ply
burroll	fuse	pounch
ceap	garlic	prance
chasm	grave	remeer
compartment	honeysuckle	remorse
cowlick	illustrate	rigod
crescent	introle	stationary
dagger	jockey	stencil
darling	loober	sunspot
deeb	marathon	supplane
disgrace	maroon	torrip
dispatch	naive	
draftsperson	opaque	

Appendix E

Student Interview: Version A and Version B

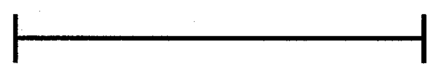
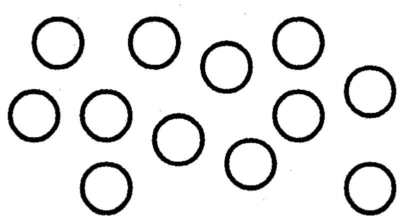
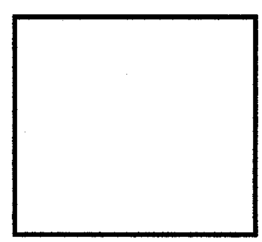
<u>Student Interview Version A</u>

(1). Write a fraction for each number below and draw a picture of the fraction:

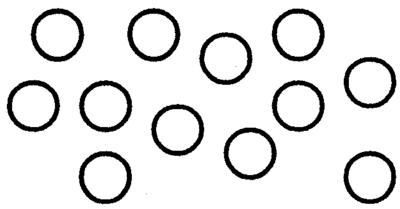
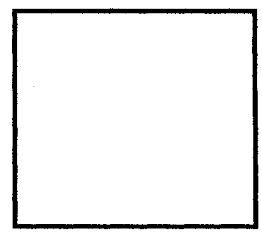
(a) two-thirds
(b) five-ninths
(c) one and three-eighths

(2). For each of the following fractions, use one of the models to show it:

(a). $\frac{4}{5}$

 This is a picture of the fraction _____	 This is a picture of the fraction _____
 This is a picture of the fraction _____	 This is a picture of the fraction _____

2 b). $\frac{1}{6}$

 This is a picture of the fraction _____	 This is a picture of the fraction _____
 This is a picture of the fraction _____	 This is a picture of the fraction _____

(3). Convert the improper fractions to mixed numbers. Draw a picture to show how you did it.

(a) nine-thirds

(b) six-fourths

(c) twelve-eighths

(4). Convert the mixed numbers to improper fractions. Draw a picture to show how you did it.

(a) two and two-thirds

(b) four and two-fifths

(5). Order these fractions from smallest to largest. Draw a picture to show that it's true.

$\frac{3}{4}, \frac{2}{3}, \frac{2}{6}$

(6). For each example, circle the larger fraction and draw a picture to show it's true:

(a) $\frac{5}{11}$ $\quad$ $\frac{1}{2}$

(b) $\frac{2}{4}$ $\quad$ $\frac{3}{8}$

(7). Marie ate three-fifths of a cheese pizza and Sandra ate one-fifth of a pepperoni pizza. How much of a pizza was eaten altogether?

(a). Write an equation to go with this problem.

(b). Solve your equation.

(c). Draw a picture to show how you solved the problem.

(8). Mark had five-eighths of a whole chocolate bar. He gave half to Steven. How much of a chocolate bar did Mark have left?

(a). Write an equation to go with this problem.

(b). Solve your equation.

(c). Draw a picture to show how you solved the problem.

(9). Tammy, a student in another school, wrote this answer for the following problem: $\frac{1}{4}+\frac{2}{4}=\frac{3}{8}$

What can you do to check if Tammy's answer is right?

(10). Jordan, a student in another school, said that the fraction $\frac{3}{4}$ is bigger than the fraction $\frac{5}{7}$.

What can you do to check if Jordan's answer is right?

Student Interview Version B

(1). Write a fraction for each number below and draw a picture of the fraction:

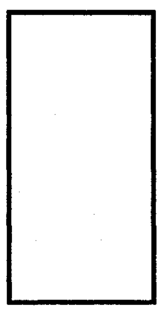
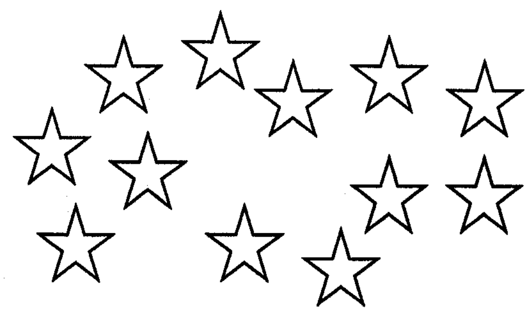
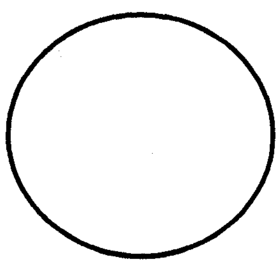
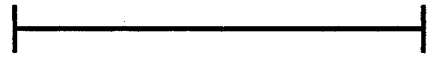
(a) two-sevenths

(b) one and four-fifths

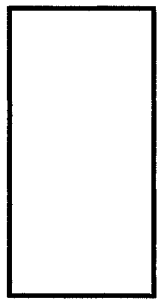
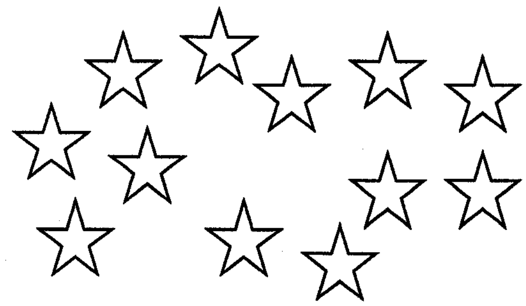
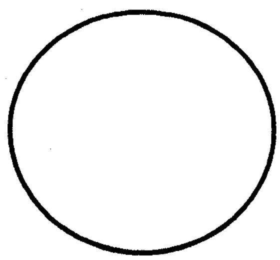
(c) one-half

(2). For each of the following fractions, use one of the models to show it:

(a). $\frac{2}{8}$

 This is a picture of the fraction ______	 This is a picture of the fraction ______
 This is a picture of the fraction ______	 This is a picture of the fraction ______

2b). $\frac{1}{5}$

 This is a picture of the fraction ______	 This is a picture of the fraction ______
 This is a picture of the fraction ______	 This is a picture of the fraction ______

(3). Convert the improper fractions to mixed numbers. Draw a picture to show how you did it.

(a) seven-fifths

(b) four-thirds

(c) ten-eighths

(4). Convert the mixed numbers to improper fractions. Draw a picture to show how you did it.

(a) three and four-sixths

(b) two and one-fourth

(5). Order these fractions from smallest to largest. Draw a picture to show that it's true.

$\frac{5}{6}, \frac{3}{4}, \frac{2}{3}$

(6). For each example, circle the larger fraction and draw a picture to show it's true:

(a) $\frac{2}{5}$ $\quad \frac{1}{3}$

(b) $\frac{1}{2}$ $\quad \frac{2}{6}$

(7). Alexa ate two-sixths of the chocolate cake and Steven ate three-sixths of the strawberry cake. How much of a cake was eaten altogether?

(a). Write an equation to go with this problem.

(b). Solve your equation.

(c). Draw a picture to show how you solved the problem.

(8). Angela had five-ninths of a whole cookie. She gave half to Melanie. How much of a cookie did Angela have left?

(a). Write an equation to go with this problem.

(b). Solve your equation.

(c). Draw a picture to show how you solved the problem.

(9). Jim, a student in another school, wrote this answer for the following problem:

$$\frac{1}{3}+\frac{1}{3}=\frac{2}{6}$$

What can you do to check if Jim's answer is right?

(10). Melissa, a student in another school, said that the fraction $\frac{3}{6}$ is bigger than the fraction $\frac{2}{9}$.

What can you do to check if Melissa's answer is right?

Appendix F

Principal and Teacher Information Letter

Principal and Teacher Information Letter

Dear <Insert Principal's or Teacher's Name>,

My name is Nicole Pitsolantis. I am a certified elementary school teacher and a student in the MA Child Study programme at Concordia University. I would like to invite the sixth-grade students in your school to participate in my Master's thesis project on math.

I would like to teach your sixth-grade students about fractions. The topic of fractions is one of the most difficult of the elementary mathematics topics and many students have difficulty learning about fractions with understanding. To better understand how students think about fractions, and to learn how to best teach the topic of fractions, I would like to compare two different instructional methods. Therefore, I would like to assign your sixth-grade students to one of two instructional groups and teach them about fractions. All students in each group will be taught the same math content, but the teaching method will vary from one group to the next. If the results of this teaching project indicate that one instructional method is better than the other, the group of students who did not receive the instruction in question will receive it when the project is completed.

I would like to teach your students on a daily basis for a period of approximately three weeks, during regularly scheduled class time. While I am teaching one group of students from your sixth-grade class about fractions, the classroom teacher will teach the second group of students about other topics in mathematics. In addition, should any child not agree to participate in this project, he/she will remain in the charge of the classroom teacher throughout the duration of my teaching activities in your school and it will remain the classroom teacher's responsibility for covering all related content on the topic of fractions with that student.

The mathematical content related to fractions that I would like to teach your students includes: the concept of a fraction; the concept of equivalence; fractions of an area, fractions of a set, fractions of a length; ordering fractions; comparing fractions; improper fractions and mixed numbers; and operations with fractions. As you well know, each of these fractions concepts is a part of the existing math curriculum content that sixth-grade students are expected to be familiar with. In other words, all of the instruction that your sixth-grade students will receive as part of this project will accord with the regular curriculum content.

Your students will be asked to participate in various math activities involving the above-mentioned fractions concepts and will also be asked to solve fractions problems on paper. In addition, some of the students who take part in this project will also be asked to participate in some one-on-one work with me, during which he/she will be asked to answer, both verbally and in writing, questions about fractions. These individual sessions will be audio-recorded. The classroom teacher will have access to all student work and any other relevant information regarding this project. My goal is to learn how to effectively teach math to children and I am very eager to share my insights from this project with all interested teachers/educators.

I would also like to inform you that my thesis supervisor, Dr. Helena Osana, Professor at Concordia University, will provide me with on-going collaboration and guidance throughout my work on this project. As such, I will share a great deal of information with her about my teaching activities in your school and about your students' work. Please rest assured, however, that I will strictly follow all ethical procedures in conducting this project. That is, all information regarding your school, teacher, and students' names will remain confidential. Only I will know the identity of the schools and students who take part in this project.

It would be a wonderful privilege to be invited into your school and classroom to see this project through. If you have any further questions, please do not hesitate to contact me. You can reach me by telephone at 450-505-3011 or via email at n_pitsol@education.concordia.ca. You can also reach my thesis supervisor, Dr. Helena Osana, at 1455 deMaisonneuve Blvd. Ouest, LB-568-10, Montreal, QC H3G 1M8; by telephone: 514-848-2424, extension 2543; or via email: osana@education.concordia.ca.

Sincerely,

Nicole Pitsolantis, MA Child Study
Concordia University
1455 deMaisonneuve Blvd. Ouest, LB-568-5
Montreal, QC H3G 1M8
Phone: (450) 505-3011
Email: n_pitsol@education.concordia.ca

Appendix G

Scoring Rubric: Fractions Knowledge Interview

Scoring Rubric: Fractions Knowledge Interview

Item		Site
1a, 1b, 1c 2a, 2b		Site 1 achieved if: Correct notation AND Correct representation
Correct notation: ☐ a/b or $\frac{a}{b}$ or N $\frac{a}{b}$	AND	Correct representation: ☑ clearly delimited whole Set model: must be enclosed; Length mode: start and end clearly indicated; Region model: complete region) N/A for all 2a, 2b ☑ correct number of parts (b) ☑ equal partitions ☑ correct shading (a) (VA 1c and VB 1b, entire whole must be shaded to signify whole number)
Incorrect notation: ▪ Inverted: notation is inverted, i.e., denominator over numerator (e.g., three-fourths written as $\frac{4}{3}$) ▪ Compounded: numbers are compounded together (e.g., one and four-fifths written as $\frac{1}{4/5}$) ▪ Irrelevant: notation is irrelevant, such as the inclusion of decimals or whole numbers where not applicable. Any notational form that does not make sense for fractions or notation does not make sense in context of problem. (e.g. Irrelevant form: one-half written as $\frac{1}{0.5}$. Irrelevant with respect to problem: writing a fraction when a mixed number is required or writing a mixed number when a fraction is required)		

▪ No notation: student could not write the symbolic notation; the item was left blank ▪ Unit Error: Notation error based on a misunderstanding of the unit. For fractions larger than 1 (such as $3\frac{4}{6}$), student believes the unit is 24 because there are four wholes in the drawing, each with six parts ($3\frac{4}{6} = \frac{22}{24}$ instead of $\frac{22}{6}$). For fractions smaller than 1 (such as $\frac{1}{2}$) student misinterprets fraction to mean $1\frac{1}{2}$.
Incorrect Representation: ▪ Incorrect whole: ☒ Incorrect whole -N/A for all 2a, 2b Set model: no enclosure; Length model: start and end not clearly indicated; Region model: fractured whole. Or wrong number of wholes (e.g., for mixed numbers greater than 1, only 1 whole drawn; for fractions less than 1, more than 1 whole drawn) ☑ correct number of parts (b) ☑ equal partitions ☑ correct shading (as per above) ▪ Incorrect number of parts ☑ clearly delimited whole (as per above) N/A for all 2a, 2b

☒ incorrect number of parts (e.g., number of parts do not conform to denominator or an equivalent of)
☑ equal partitions
☑ correct shading (as per above)

- Unequal partitions

☑ clearly delimited whole (set model, enclosed; length model, start & end clear)
N/A for all 2a, 2b

☑ correct number of parts (b)
☒ no equal partitions
☑ correct shading (as per above)

- Incorrect shading

☑ clearly delimited whole (as per above)
N/A for all 2a, 2b

☑ correct number of parts
☑ equal partitions
☒ incorrect shading (as per above)

- No representation

Unable to draw a representation. Item was left blank

- Unequal Units

More than one unit or whole drawn to compare fractions, but units/wholes drawn as different sizes, making it difficult to compare value of fraction. Or, two different size/shape wholes drawn to represent one number (e.g., 1 3/8 represented with 2 different wholes – one circular region for the 1 and one rectangular region for the fraction).

- Direct Translation

Representation reflects a misunderstanding of part-whole concept and a direct translation from symbol to representation. I.e.. fraction interpreted as two compounded whole numbers and represented as such. (e.g., 1/2 represented as a whole with 3 partitions, 1 shaded to represent the numerator and 2 unshaded to represent the denominator. Or, 1/5 represented as a whole with 5 partitions where the 1 in the symbol is represented by 1 whole and the 5 in the symbol is represented as the 5 partitions)

Differs from incorrect number of parts error in that incorrect number of parts error stems from miscount or wrong procedure such as adding denominators, whereas direct translation error stems from not understanding P-W concept and treating the in whole numbers terms.

Item	Site 2	
3a, 3b, 3c 4a, 4b	Site 2 achieved if: Correct procedure AND Correct link or rationale.	
Correct procedure: ☑ Standard (mixed number to improper fraction): divide numerator by denominator to find whole number; subtract value of denominator to find remain number of parts and record over original denominator ☑ Standard (improper fraction to mixed number): multiply whole number by denominator of fraction, add value of numerator, record over original denominator. ☑ Concrete model: Student correctly modeled the problem and figured out the answer ☑ Number knowledge: Student used estimation, benchmark numbers or other number knowledge to solve the problem correctly.	AND	Correct link / rationale ☑ Standard procedure: student is able to justify the conceptual reasoning/meaning underlying the procedure that was used. (e.g., "I multiplied the whole number and the denominator because it's 1 whole that is divided into six parts, so now I know it's six sixths, and then I added the numerator because that tells me there are two more sixths, so it's eight sixths in all.") ☑ Concrete model: student can identify both the improper fraction and the mixed number in the representation ☑ Number knowledge: student can justify why it makes sense to use the particular thinking strategy that was used (e.g., I know that seven-thirds is equal to two and one-third because if it were nine-thirds it would be three, but it's two less so there has to be two less, so its two and one-third.")

Item 5, 6a, 6b	Site 2 achieved if: Correct procedure AND correct link or rationale	
Correct procedure: ☑ Standard: common denominator method (e.g., student converts each fraction to an equivalent fraction using a common denominator and then compares value of numerators to order the fractions) ☑ Concrete model: Student correctly modeled the problem and figured out the answer ☑ Number knowledge: Student used estimation, benchmark numbers or other number knowledge to solve the problem correctly (e.g., two-thirds is less than three-fourths because two-thirds is like 66% and three-fourths is like 75%.")	AND	Correct rationale or link: ☑ Standard procedure: student is able to justify the conceptual reasoning/meaning underlying the procedure that was used. (e.g., "I found a common denominator because now they all have the same number of parts in them and now I can compare the numerators to see which is largest and which is smallest, because I know I am comparing the same thing.") ☑ Concrete model: if wholes are all drawn the same size, student must compare the shaded value in each. If wholes are drawn differently, student must be able to explain order or comparison in terms of amount of each value in relation to its whole and in relation to other fractions in the problem (e.g., "Each one is almost a whole because they're each missing one part each, but 5/6 is the largest because sixths means smaller parts so this one unshaded part is smaller than the 1/3 that is not shaded in 2/3…") ☑ Number knowledge: student can justify why it makes sense to use the particular thinking strategy that was used (e.g., "I turned all the fractions to percents and then I knew which was the smallest and which was the biggest…you can do percents because it's the same thing, it's just another way to say the same number.")

Item 7	Site 2 achieved if: Correct symbolic notation of the problem AND correct procedure AND correct rationale or link			
Correct symbolic notation of the problem: Both fractions written in correct form and operator & equal sign noted ☐ a/b or $\frac{a}{b}$ ☐ correct number sentence (right operator and equal sign)	AND	Correct Procedure: ☑ Standard: symbolic addition carried out correctly ☑ Concrete model: student correctly modeled the problem and figured out the answer ☑ Number knowledge: Student used estimation, benchmark numbers or other number knowledge to solve the problem correctly (e.g., "one-fifth plus three-fifths is four fifths because for three-fifths you need two more to make a whole, so if you add one more fifth, then it's almost a whole, so four-fifths is right.")	AND	Rationale: Show/tell why numerators are added and why denominators <u>are not</u> (e.g., "adding the denominators is wrong because it would change the fraction, it would change what it is out of, so then you wouldn't be adding the same thing.") Or, if student used concrete model procedure, must be able to explain using picture why denominators are not added (e.g., "because this shows that there is still more room in this whole, so these three fifths can be added to this whole because there is only one-fifth in that one.")
Incorrect Procedure: ▪ Whole number misconceptions: student ordered or compared fractions either by treating denominators as whole numbers, numerators as whole numbers, or both (e.g., "three is smaller than four so two-thirds is smaller than three-fourths). Or, for addition problems, student added across the denominators or added all numerators and denominators together as if each represents a separate whole number (e.g., 3/5 + 1/5 = 3 + 5 + 1 + 5 = 14) ▪ Concrete model error: attempted to draw a model of the problem as procedure for solving the problem but made a representational error (e.g., two-ninths is bigger than one-half because two parts are shaded in two-ninths but only one part is shaded in one-half). Or, for addition problem, no whole drawn to represent the result of the addition. ▪ Procedural error: student attempted to used common denominator procedure but made an error when converting the fractions				

- Number knowledge error: student attempted to use number knowledge but made an error/faulty thinking (e.g., two-thirds is almost a whole so it's like 90%, and three-fourths is almost a whole too, but no as much so it's like 80% or something…so two-thirds is more.")

- Guess/no explanation: No procedure was evident (e.g., student just circled an answer and said, "I just know it's that.") and/or student could not explain any type of procedure or method used to arrive at an answer other than stating that it was a guess or known fact.

- No procedure: Student could not solve the problem in any way. The item was left blank.

- Incorrect invented procedure: student invented a procedure that was incorrect.

- Wrong procedure: student used a valid procedure, but one that does not apply to the question (e.g., applied the procedure that is used to find an equivalent fraction when the item pertained to converting an improper fraction to a mixed number was to be found)

- Incomplete procedure: student used the correct procedure but could not transfer the obtained solution into an appropriate answer (e.g., did long division to convert improper fraction to mixed number but could not transfer result of division into a form appropriate for mixed numbers, "I got 1 remainder 2, but I don't know what to do next."). Or, student only added the numerators but did not record the denominator for an addition problem.

Item	Site 3
9, 10	Site 3 achieved if: student is able to correctly verify reasonableness of the symbolic statement in a manner other than by applying a standard symbolic procedure
Correct verification method: ☑ Correct, concrete model: student correctly models the problem to verify the reasonableness (see Site 2 for examples) ☑ Correct, number knowledge: student uses estimation, benchmarks numbers, or other number knowledge to verify the reasonableness (see Site 2 for examples)	
Incorrect verification method: ▪ Applied procedure: student used a correct standard procedure to verify but could not explain why the symbolic statement was correct or incorrect other than procedurally (e.g., "one-fifth plus three-fifths is four-fifths because you never add the denominators in fractions. I'm not sure why, but I just know that.") ▪ Procedural error: student applied a whole number misconception to verify the reasonableness (e.g., adding across denominators) or, student used an incorrect invented procedure to verify (e.g., three-sixths is larger than two-ninths because you can do 3 x 2 to get to six, but you can't do two times anything to get to nine.") ▪ Concrete model error: for example, the student agreed that 1/3 + 1/3 = 2/6 and drew a representation that reflected that misunderstanding (i.e., drew a representation of two-sixths as the solution) ▪ Unable to verify: students attempted some form of verification method (either correct or incorrect) but could not give a definitive answer (e.g., "I'm not sure because it could be 2/9 since there are more parts, but in 3/6 more are shaded…I'm really not sure."	