On Observer Design for a Class of Impulsive Switched Systems

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A Thesis

in

The Department

of

Electrical and Computer Engineering

Presented in Partial Fulfillment of the Requirements

for the Degree of Master of Applied Science (Electrical and Computer Engineering) at

Concordia University

Montréal, Québec, Canada

November 2007

Your file Votre référence
ISBN: 978-0-494-40890-2
Our file Notre référence
ISBN: 978-0-494-40890-2

ABSTRACT

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In this thesis, the problem of state observation for a class of impulsive switched systems is addressed. Corresponding to each subsystem, an identity Luenberger observer is employed and a switching observer is constructed accordingly. The asymptotic stability property of the proposed switching observer is discussed and LMI-based algorithms are given which provide necessary conditions for the asymptotic stability of the switching observer for the switching signals with an average dwell time greater than a specific value. Since switched systems without impulse are a special case of impulsive switched systems, all the results in this work can be applied to design observers for switched systems without impulse. The design of finite time switching observers for a class of linear switched systems is another problem addressed in this work. The finite convergence time property of the proposed switching observer is discussed and the exponential stability of the observation error is investigated. An LMI-based algorithm is given which provides conditions for the exponential stability of the switching observer. Finally, the idea of finite time observers for linear continuous time systems is extended to linear time invariant discrete time systems. The main motivation for this extension is that unlike the famous dead-beat observers designed for discrete time systems, the proposed observer in this work need not place all the eigenvalues at the origin, which leads to a much more flexible design compared to the existing techniques.

To my parents
Shahin and Hossein

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Chapter 1

Introduction

1.1 Motivation and Related Works

Switched systems are a class of hybrid systems and have been at the center of increasing attention in recent years due to their wide applications in practical systems. In general, switched systems consist of several modes of operation and a switching rule that orchestrates the switching between them. Unlike the stability problem of switched systems that has been extensively studied in the literature [7], [8], [23], the observer design problem for switched systems has attracted less attention and few works are available in this area, (see [4], [5]). Switching can also be applied to control, in order to cope with highly uncertain systems [1], [2], [30], [35]. There are many examples of switched systems in power electronics [14], process control, biomedical and biochemical processes [18] and aerospace, to name only a few. In many practical switched systems, due to sudden changes in the states of the system at certain instants of switching, the system exhibits an impulsive dynamical behavior [9], [15], [20]. There are a few number of works dealing with the impulse factor and few results have been developed in this area, [37], [38]. In [37], impulsive phenomena are introduced into switched systems and necessary and sufficient conditions for controllability and observability of impulsive switched control systems are developed, while in [38] necessary and sufficient conditions for stability of impulsive switched systems under switching signals are obtained. Robust asymptotic stability of linear discrete impulsive systems and a class of uncertain nonlinear discrete impulsive systems are studied in [24], and input-to-state stability properties of impulsive systems are discussed in [16] and [17]. In conventional observer design techniques proposed in the literature, the estimation error tends to zero asymptotically. However, in many control problems, finite convergence time of the error is of great interest, especially in the context of observer-based control systems [33]. An example is the chemical batch processing mode where only a finite time period is available to perform the process satisfactorily, in comparison with the continuous processing mode [3]. The works [11] and [12] present sliding mode based observers which provide finite-time convergence by means of nonlinear dynamics. In [34] and [38], moving horizon based observers are studied which use on-line solutions of dynamic optimization problems and guarantee finite convergence time of the estimation error. The finite-time observer introduced in [13] consists of two identity Luenberger observers [26], two gain components and a delay element. This observer, compared with the ones designed by either of the two approaches mentioned above, has a simpler form. Further, it is shown in [13] that finite-time convergence of the observer is guaranteed if the poles of the two Luenberger observers are placed in specific regions of the s-plane.

1.2 Thesis Outline and Contribution

This thesis focuses on the problem of observer design for impulsive switched systems with either continuous or discrete linear subsystems, and LMI-based algorithms are developed to guarantee the stability of the proposed switching observer for a class of impulsive switched systems with constrained switching rule. Basic theories of linear observers are given and stability of switched systems is discussed in Chapter 2. Stability analysis for linear impulsive switched systems is discussed in Chapter 3. Chapter 3 focuses on the problem of switching observers for impulsive switched systems. In this chapter, LMI-based algorithms for continuous and discrete impulsive switched systems are developed to design switching observers to achieve the stability of error dynamics for constrained switching rules while the required average dwell time is minimized. To show the effectiveness of the proposed approach, two numerical examples are given. In Chapter 4, the problem of state observation in finite time for continuous-time LTI switched systems is addressed. Due to the simplicity of structure, the finite-time observer proposed in [13] is adopted as a basis for developing the main result in this chapter. Therefore, to develop a switching observer for the given switched system, a finite-time observer corresponding to each mode of operation is designed. It is then shown that the switching observer obtained from switching between these observers can observe the exact state of the switched system in finite time provided that the system stays in the same mode at least for the duration of an interval of length Δ. Nevertheless, it is still required that the error dynamics of the switching observer be stable. To address this issue, a linear matrix inequality (LMI) based approach is considered to design a finite-time observer for each mode to attain stability for switching with constrained average rate rules. It is shown later that the stability of finite-time switching observer can be translated into a stabilizability problem for an impulsive switched system. By pursuing this approach, an algorithm for observer design using LMIs is developed. To guarantee that the observer obtained using this algorithm for each mode operates as a finite-time observer, the eigenvalues of the Luenberger observers are placed in a specific region in the left half of the complex plane. Next, a common Lyapunov function is found to verify the stability of the switching observer while the lower bound for the average dwell time of the switched system is minimized. The main contributions of this work given in Chapter 3 and 4 of the thesis are submitted to *American Control Conference 2008* and accepted for publication in *European Journal of Control*, respectively. See [28].

Chapter 2

Background

2.1 Introduction

When we discuss about feedback control systems, usually it is reasonable to assume that the entire states of the system which are desired to be controlled are all available through measurements. For a linear time-invariant system given by

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(t_0) = x_0, \qquad t \geq t_0 \tag{2.1a}$$

$$y(t) = Cx(t) \tag{2.1b}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $u \in \mathbb{R}^m$ are the state, the output and the input of the system, respectively, we can design a feedback control of the form $u(t) = \phi(x(t), t)$ provided that all states of the system are available. Since in most practical systems the entire state vector can not be measured, the feedback control law mentioned before can not be implemented, and an approximation of states obtained by an observer will be replaced for the unavailable states. Like any other to dynamic systems, an observer is a system and its characteristics are free to be determined by its designer.

2.2 Basic Theory of Observers

Without loss of generality we consider the problem of observing the states of a *free* system S_1. A system is called *free* when there is no input present. If the available outputs of this system are used as inputs to drive another system S_2, the second system will be an observer for the first system, in other words, the states of S_2 may track a linear transformation of the states of S_1.

2.2.1 Observation of free systems

Theorem 1 *Assume that S_1 is a free system governed by $\dot{x}(t) = Ax(t)$ and is used to derive S_2 governed by $\dot{z}(t) = Fz(t) + Hx(t)$. Suppose there is a transformation T which satisfies $TA - FT = H$. Then*

$$z(t) = Tx(t) + e^{Ft}\left[z(t_0) - Tx(t_0)\right] \quad \forall t \geq t_0 \tag{2.2}$$

<u>Proof:</u> See [25]. ■

It is to be noted that to track the states of the first system, all eigenvalues of F should have negative real parts. Also it is clear that S_1 and S_2 need not have the same dimension.

Theorem 2 *If A and F have no common eigenvalues, there is a unique solution T for the equation $TA - FT = H$*

<u>Proof:</u> See [25]. ■

Thus, it is concluded that any system S_2 which has different eigenvalues from S_1 can serve as an observer for it. This result can be extended for a forced system (a system with input) by applying the input of the first system into the observer as well as to the original system. Assume that S_1 is defined by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.3}$$

and that S_2 is governed by

$$\dot{z}(t) = Fz(t) + Lx(t) + TBu(t) \tag{2.4}$$

It can be easily shown that $z(t)$ obtained in (2.4) will satisfy (2.2). Thus, to obtain an observer for a system, initially one can obtain an observer for the system assuming that system is free, and then consider the effect of input as in (2.4). This observer is often called Luenberger observer [26].

2.2.2 Identity Luenberger observer

As mentioned earlier, in general, a Luenberger observer observes a transformation of the states of the original system. An obviously convenient observer is the one in which the transformation matrix T, which relates the states of the observer to the states of the original system, is an identity matrix. Such an observer is often referred to as identity Luenberger observer [5]. This requires the identity observer to have the same dimensions as the original system and therefore $F = A - H$ (since $T = I$). However, the characteristics of this observer is the same as the one described earlier. The matrix H is determined both by the fixed output structure of the original system and by the input structure of the observer. If S_1 with the output vector $y \in \mathbb{R}^p$ is governed by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(t_0) = x_0, \qquad t \geq t_0 \\ y(t) &= Cx(t) \end{aligned}$$

and the observer S_2 which is governed by

$$\dot{z}(t) = Fz(t) + Ly(t)$$

then $H = LC$. In designing the identity Luenberger observer, the matrix C is a fixed matrix and L is the design parameter. Thus, the identity Luenberger observer is determined by selecting the matrix L and has the form

$$\dot{z}(t) = (A - LC)z(t) + Ly(t)$$

Different matrices for L will result in different observers, but the dynamic of all these observers is determined by the matrix $A - LC$.

Lemma 1 *When the dynamic of the system is given by (2.1) (real matrices A and C), by a suitable choice of the matrix L, the eigenvalues of $A - LC$ can be placed at any arbitrary location iff the pair (A, C) is completely observable.*

Proof: See [25]. ■

Theorem 3 *An identity Luenberger observer with arbitrary dynamic can be designed for a linear time-invariant system iff the system is completely observable*

Proof: See [25]. ■

The real parts of the eigenvalues of the observer are selected to be negative, so that the state of the observer will asymptotically converge to the states of the original system. In practice, these eigenvalues are chosen to be less than the eigenvalues of the original system so that the convergence is faster than the other system effects. Theoretically, the eigenvalues of the observer can be placed at minus infinity, which results in extremely fast convergence. However, in practice this observer becomes highly sensitive to noise; in other word this observer will behave like a differentiator.

Example 1 Consider a linear time-invariant system that has a state space representation as follows

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

An identity observer is obtained by selecting the observer gains

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

The resulting Luenberger identity observer is obtained as

$$A - LC = \begin{bmatrix} -2 - L_1 & 1 \\ 0 - L_2 & -1 \end{bmatrix}$$

which has the characteristic equation given by

$$\lambda^2 + (3 + L_1)\lambda + 2 + L_1 + L_2 = 0$$

For any arbitrary (negative) values of λ_1 and λ_2 one can obtain the desired values of L_1 and L_2. Assume that the designer would like to make the observer have two eigenvalues at -2 and -3. This would give the characteristic equation as $\lambda^2 + 5\lambda + 6 = 0$. Matching the coefficients will give us $L_1 = L_2 = 2$. Thus the observer is governed by

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

2.2.3 Reduced order observer

The identity Luenberger observer has a certain degree of redundancy. Although a subset of the system states are already available through direct measurement, this observer constructs an estimate for the entire state vector. This redundancy can be eliminated by use of an observer of a smaller dimension but still of arbitrary dynamics. Assume that in the system given by (2.1) $y(t)$ is of dimension p. An observer of order $n-p$ is constructed with state $z(t)$ that approximates $Tx(t)$ for some matrix T of order $p \times n$. Then an estimate of $x(t)$, namely, $\widehat{x}(t)$ can be obtained as

$$\widehat{x}(t) = \begin{bmatrix} T \\ C \end{bmatrix}^{-1} \begin{bmatrix} z(t) \\ y(t) \end{bmatrix}$$

provided that $\begin{bmatrix} T \\ C \end{bmatrix}$ is invertible. Consider again the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(t_0) = x_0, \qquad t \geq t_0 \\ y(t) &= Cx(t) \end{aligned}$$

Without loss of generality assume that p outputs of the system are linearly independent; in other words, C has rank p. Also assume that C can take the form $C = \begin{bmatrix} I_{p,p} & 0_{p,n-p} \end{bmatrix}$, i.e. C is partitioned into a $p \times p$ identity matrix and a $p \times (n-p)$ zero matrix. An appropriate change of coordinates is obtained by selecting the matrix D such that

$$M = \begin{bmatrix} C \\ D \end{bmatrix}$$

is invertible, in which case we write $\bar{x} = Mx$ to relate the new and old variables. It is convenient to partition the state vector as $x = \begin{bmatrix} y \\ w \end{bmatrix}$ and rewrite the system equations in the form

$$\dot{y}(t) = A_{11}y(t) + A_{12}w(t) + B_1u(t) \tag{2.6a}$$

$$\dot{w}(t) = A_{21}y(t) + A_{22}w(t) + B_2u(t) \tag{2.6b}$$

By this construction the vector $y(t)$, which is available through measurement, provides the measurement of $w(t)$ for the system (2.6b) by use of an identity observer of order $(n-m)$. In fact, $A_{21}y(t) + B_2u(t)$ is the the input for (2.6b) and $w(t)$ is the state to be measured.

Lemma 2 *If (A,C) is completely observable, so is (A_{12},A_{22}).*

Proof: See [27]. ■

To construct the reduced order observer for this system, define it in the form

$$\dot{\widehat{w}}(t) = (A_{22} - LA_{12})\widehat{w}(t) + A_{21}y(t) + B_2u(t) + L(\dot{y}(t) - A_{11}y(t))$$

The matrix L can be chosen such that $A_{22} - LA_{21}$ has arbitrary eigenvalues, or

$$\dot{z}(t) = (A_{22} - LA_{12})z(t) + (A_{22} - LA_{12})Ly(t) + (A_{21} - LA_{11})y(t) + (B_2 - LB_1)u(t)$$

where $z(t) = \widehat{w}(t) - Ly(t)$. By this construction we have the following theorem.

Theorem 4 *Corresponding to an n-th order linear time invariant system having m linearly independent outputs, an observer of order $(n-m)$ can be constructed having arbitrary eigenvalues.*

Proof: See [27]. ■

2.3 Nonlinear Observers

In the previous sections we have seen that to observe the states of a linear system, one can construct a linear observer that has the same structure as the system plus the driving feedback term whose role is to reduce the observation error to zero. Although in case of nonlinear systems the construction of state observer is much harder, one can use the same logic as for linear systems to construct a nonlinear observer for the system. Consider a nonlinear system defined by

$$\dot{x}(t) = f(x(t), u(t)) \tag{2.7a}$$

$$y(t) = g(x(t), u(t)) \tag{2.7b}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $u \in \mathbb{R}^m$, f and g are nonlinear vector functions, respectively, of dimensions n and p. Based on the knowledge of linear observers for linear systems, one can propose the following structure for a nonlinear observer

$$\dot{\widehat{x}}(t) = f(\widehat{x}(t), u(t)) + L(y(t) - \widehat{y}(t)) \tag{2.8a}$$

$$\widehat{y}(t) = g(\widehat{x}(t), u(t)) \tag{2.8b}$$

Thus the nonlinear observer is defined by

$$\dot{\widehat{x}}(t) = f(\widehat{x}(t), u(t)) + L(y(t) - g(\widehat{x}(t), u(t))) \tag{2.9}$$

In general the observer gain L is a nonlinear matrix function that depends on x and u, i.e. $L = L(x,u)$. Like the linear case it has to be chosen such that the observation error at steady state tends to zero. The observation error dynamics is determined by

$$\begin{aligned}\dot{e}(t) = \dot{x}(t) - \dot{\widehat{x}}(t) = f(x(t), u(t)) - f(\widehat{x}(t), u(t))\\ - L(g(x(t), u(t)) - g(\widehat{x}(t), u(t)))\end{aligned}$$

By eliminating $\widehat{x}(t)$ from the error equation, one can obtain

$$\dot{e}(t) = f(x(t), u(t)) - f(x(t) - e(t), u(t)) - L(g(x(t), u(t)) + Lg(x(t) - e(t), u(t))$$

In the steady state we have

$$0 = f(x(t),u(t)) - f(x(t)-e(t),u(t)) - L(g(x(t),u(t)) + Lg(x(t)-e(t),u(t))$$

It is clear that $e = 0$ is the solution of this equation which means that the constructed observer may have $e = 0$ at steady state. The gain $L = L(x,u)$ must be chosen such that the observer and error dynamics are asymptotically stable. The asymptotic stability is examined using the first stability method of Lyapunov. The Jacobian matrix for the error equation is given by

$$J_e = \frac{\partial f(x-e,e)}{\partial e} - L(x,u)\frac{\partial g(x-e,e)}{\partial (x-e)}$$

By the first stability method of Lyapunov, the Jacobian matrix must have all the eigenvalues in the left half plane for all working conditions, that is, for all $x \in X$ and $u \in U$, where X and U are the sets of admissible states and control inputs. The error dynamics asymptotic stability condition is

$$\mathrm{Re}\{\lambda_i(J_e, s.t: e = 0, x \in X, u \in U)\} < 0, \quad \forall \lambda_i$$

Similarly, for the observer we have

$$J_{\widehat{x}} = \frac{\partial f(\widehat{x},u)}{\partial \widehat{x}} - L(x,u)\frac{\partial g(\widehat{x},u)}{\partial \widehat{x}}$$

It is required that the observer is also asymptotically stable or equivalently

$$\mathrm{Re}\{\lambda_i(J_{\widehat{x}}, s.t: e = 0, x \in X, u \in U)\} < 0, \quad \forall \lambda_i$$

2.4 Switched Systems

Many practical systems and processes consist of several modes with different dynamical behaviors in each mode of operation. Such systems and processes are modeled by a class of systems known as switched systems. In general switched systems consist of several modes of operation and a switching rule orchestrating the switching between them, where the switching rule determines the active mode of operation (subsystem) at each instant of time. The class of continuous time switched systems considered in this work are described by

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \tag{2.10a}$$

$$y(t) = C_{\sigma(t)}x(t) \tag{2.10b}$$

The function σ maps the time axis into an index set $\bar{\mathrm{N}} = \{1, 2, \cdots, N\}$ whose elements represent the index of the active LTI system at any given time. For example, $\sigma(t) = i$ where $i \in W$ indicates that the i-th mode is active. Matrices A_i, B_i, $C_i i \in \{1, 2, \cdots, N\}$ are constant matrices and determine the dynamical behavior of the switched system when the i-th mode is active. Usually when we discuss about switched systems initially it is convenient to assume that $\sigma : [0, \infty) \to \bar{\mathrm{N}}$, $\bar{\mathrm{N}} = \{1, 2, ..., N\}$, is a piecewise constant function of time. $\sigma(t)$ is often called the switching signal. $\sigma(t^-) = j$ while $\sigma(t^+) = i$ indicates that t is a switching instant and the system switches from the j-th mode to the i-th mode at time t. Based on the nature of the switching signal $\sigma(t)$ there are two distinct classes of switching. If there is no restriction on the switching signal it is called *arbitrary switching*, otherwise we call it *constrained switching*. As we shall see in the coming chapters, based on the type of switching (arbitrary or constrained), we have to use different stability analysis methods. Moreover, if the state of the switched system is continuous for any control input, that is, the state does not jump at the switching instants, then the switched system is *Non-impulsive*, otherwise we call the system *Impulsive*. It is discussed in the next chapter that in many practical systems there are changes in the states of the system at certain instants of switching and the system exhibits impulsive dynamical behavior.

2.5 Basic Definitions

Since Lyapunov stability theorem is used frequently in the next two chapters, basic facts and definitions related to this theory are reviewed in the next subsections.

2.5.1 Lipschitz functions

A real valued function f defined on a subset of real numbers $D \subseteq \mathbb{R}$

$$f : D \rightarrow \mathbb{R}$$

is called Lipschitz continuous if there exist a constant $B \geq 0$ such that for all x_1, x_2 in D we have

$$|f(x_1) - f(x_2)| \leq B|x_1 - x_2|$$

The function is called locally Lipschitz continuous if for every x in D there exists a neighborhood of x such that f restricted to this neighborhood is Lipschitz continuous.
In reviewing the definition of stability attention is restricted to time invariant systems described by

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n \tag{2.11}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is locally Lipschitz, also it is assumed that the origin is an equilibrium point of the system (2.11), i.e $f(0) = 0$.

2.5.2 Stability definitions

Without loss of generality one can assume the initial time to be $t_0 = 0$. The origin is said to be a *stable* equilibrium point of (2.11), in the sense of Lyapunov, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|x(0)| \leq \delta \quad \Rightarrow \quad |x(t)| \leq \varepsilon \quad \forall t \geq 0$$

If the origin is stable we simply say that the system (2.11) is *stable*. A similar convention applies to other stability concepts introduced below.
The system (2.11) is called *asymptotically stable* if it is stable and δ can be chosen such that

$$|x(0)| \leq \delta \quad \Rightarrow \quad x(t) \rightarrow 0 \quad as \quad t \rightarrow \infty$$

Region of attraction is the set of all initial states from which the system trajectories converge to the origin. If the region of attraction is the entire state space, then the system (2.11) is called *globally asymptotically stable*. If the system trajectories converge to the origin when the initial state is selected in some neighborhood of the origin the system is said to be *locally asymptotically stable*. Finally, the system (2.11) is called *exponentially stable* if there exist three positive constants σ, c and λ such that the inequality

$$|x(t)| \leq c\,|x(0)|\,e^{-\lambda t}, \quad \forall t > 0$$

holds for all solutions of the system with $|x(0)| \leq \sigma$. If the above condition holds for all σ then the system is said to be *globally exponentially stable*.

2.5.3 K, K_∞ and KL functions

A function $f(x) : [0,\infty) \rightarrow [0,\infty)$ is of class K if it is continuous, strictly increasing and $f(0) = 0$; moreover if f is unbounded it is said to be of class K_∞. A function $f(x,y)$: $[0,\infty) \times [0,\infty) \rightarrow [0,\infty)$ is said to be a class KL function if for each fixed $y \geq 0$ the function $f(x,y)$ is of class K and for each fixed $x \geq 0$ it is decreasing to zero as $y \rightarrow \infty$. Based on the definitions of different classes of functions the stability definitions mentioned earlier can be rewritten in a more compact way. The system (2.11) is stable if there exists a $\sigma > 0$ and a class K function f such that, provided $|x(0)| < \sigma$, we have

$$|x(t)| \leq f\,|x(0)|, \quad \forall t \geq 0$$

for all solutions of the system. The system (2.11) is asymptotically stable if there exists a $\sigma > 0$ and a class KL function such that, provided $|x(0)| < \sigma$, we have

$$|x(t)| \leq f(|x(0)|, t), \quad \forall t \geq 0$$

If this condition holds for all initial states then the system is globally asymptotically stable. Finally if the class KL function is of the form $f(x,y) = cxe^{-\lambda y}$ for some $c, \lambda > 0$ then the system is exponentially stable.

2.5.4 Continuous differentiability

A function $f(x,t)$ where $f : D \times [a,b] \to \mathbb{R}^n$ for a region $D \subseteq \mathbb{R}^n$ is said to be continuously differentiable over $D \times [a,b]$ if both $f(x;t)$ and $\frac{\partial f}{\partial x}$ are continuous over $D \times [a,b]$.

2.5.5 Positive definite functions and matrices

A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite in a region D of $\mathbb{R}^n$ that contains origin if

- $V(x) > 0 \quad \forall x \in D, \ x \neq 0$
- $V(x) = 0$ iff $x = 0$
- All sub-level sets of V are bounded

Furthermore, V is said to be positive semi-definite if $V(x) \geq 0, \forall x \in D, \ x \neq 0$. Conversely, if the first condition of the above definition is changed to $V(x) < 0$, then V is said to be negative definite, and if it is changed to $V(x) \leq 0$, V is called negative semi-definite.
A matrix $A_{n \times n}$ is said to be positive definite if for all nonzero vectors $x \in C^n$

$$Re\{x^* A x\} > 0$$

where x^* denotes the conjugate transpose of the vector x. When A is a real matrix the above condition reduces to

$$x^T A x > 0$$

If A is positive-definite, one writes $A > 0$. One can conclude that all eigenvalues of a positive definite matrix are positive. Definition of positive semi-definite and negative definite can be concluded in the same way as for semi-definite and negative functions.

2.5.6 Lyapunov stability theorem for systems with no input

Let $x = 0$ be an equilibrium point for a system described by:

$$\dot{x} = f(x)$$

where $f : D \longrightarrow \mathbb{R}^n$ is locally Lipschitz and $D \subseteq \mathbb{R}^n$ is a domain that contains the origin. Let $V : D \longrightarrow \mathbb{R}$ be a continuously differentiable, positive definite function in D.

- If $\dot{V}(x) = \frac{\partial V}{\partial x}$ is negative semi-definite, then $x = 0$ is a stable equilibrium point.
- If $\dot{V}(x)$ is negative definite, then $x = 0$ is an asymptotically stable equilibrium point.

In both cases above V is called a Lyapunov function. It is to be noted that these conditions are only sufficient. Failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not necessarily mean that the equilibrium is not stable or asymptotically stable.

An important class of systems is the class of linear time invariant systems. For this class of systems the above Lyapunov theory is simplified as follows.

Theorem 5 *A linear time invariant system described by*

$$\dot{x}(t) = Ax(t)$$

is asymptotically stable iff for any symmetric positive definite matrix $Q = Q^T > 0$ *there exists a unique positive definite matrix* $P = P^T > 0$ *such that*

$$A^T P + PA = -Q$$

Proof: See [19]. ■

2.6 Switching Observers

Consider again the switched system described by (2.10). To design an observer to obtain an approximation of the states of the switched system different methodologies can be employed, but in general to observe the states of a switched system one has to design a switching observer, i.e., for each mode, an observer O_i is designed and employed when the corresponding mode is active. This indicates that the observer is a switched system as well. A classic Luenberger switching observer is an observer which employs identity Luenberger observers in each mode of operation to observe the state of the switched system and obeys the following general formulation

$$\dot{\hat{x}} = A_{\sigma(t)}\hat{x}(t) + B_{\sigma(t)}u(t) + L_{\sigma(t)}\left(y(t) - C_{\sigma(t)}\hat{x}\right) \tag{2.12}$$

where $\hat{x}$ is the state estimate, $L_{\sigma(t)}$, $\sigma(t) \in \{1, 2, \cdots, N\}$ are Luenberger observers gains and $\hat{x}(0)$ is the initial condition of the observer and is chosen a priori. As mentioned earlier, it is clear from the structure of the observer that it is also a switched system. The most important issue when designing a classic switching observer, i.e. a switching observer consisting of Luenberger observers, is to choose the gains such that the stability of the estimation error, i.e., $x(t) - \hat{x}(t)$, is guaranteed. Similar to the well known fact that there is no guarantee for the stability of a switched system consisting of stable modes, it will be shown in the next chapter that employing a stable observer for each mode does not guarantee the stability of the switching observer in general.

Theorem 6 *Consider again the switched system described by (2.10), and assume that the pairs* (A_i, C_i), $i = 1, 2, \ldots, N$, *are all observable. If there exists a symmetric positive definite matrix P such that*

$$(A_i - L_iC_i)^T P + P(A_i - L_iC_i) < 0, \quad i = 1, 2, \ldots, N$$

then the estimation error in the switching observer (2.12) is exponentially convergent to zero.

Proof: See [4]. ■

Chapter 3

Observer Design for a Class of Impulsive Switched Systems

3.1 Introduction

In many practical switched systems, due to sudden changes in the states of the system at certain instants of switching, there is an impulsive dynamical exhibition behavior. There are a few number of works dealing with the impulse fact and few results have been developed in this area. In this chapter the problem of observer design for impulsive switched systems with either continuous or discrete linear subsystems is discussed and LMI-based algorithms are developed to guarantee the stability of the proposed switching observer for a class of impulsive switched system with constrained switching rule.

3.2 Stability of Impulsive Switched Systems Under Constrained Switching

In this section stability of impulsive switched systems with either continuous or discrete subsystems is investigated. In the first subsection sufficient conditions to guarantee the stability of impulsive switched systems with continuous subsystems is obtained while stability analysis for impulsive systems with discrete subsystems is discussed in the second subsection.

3.2.1 Stability of continuous impulsive switched systems

An impulsive switched system with N modes of operation obeys the following general formulation

$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t), u(t)), & t \neq t_l, l = 1, 2, \ldots \\ x(t) = g_{\sigma(t),\sigma(t^-)}(x(t^-)), & t = t_l, l = 1, 2, \ldots \\ x(t_0) = x_0 \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and the input of the system. Furthermore, $f_{\sigma(t)}$ and $g_{\sigma(t),\sigma(t^-)}$ are a $\mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ and $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ functions, respectively, and $\{t_1, t_2, \cdots\}$ is a sequence of increasing impulse times in $[t_0, \infty)$. The right continuous function $\sigma(t) : [t_0, \infty) \longrightarrow \bar{\mathrm{N}}$ is the switching rule, where $\bar{\mathrm{N}} = \{1, 2, \cdots, N\}$. By construction of this impulsive switched system the state of the system $x(t) : [t_0, \infty] \longrightarrow \mathbb{R}^n$ is right continuous. Furthermore, $(.)^-$ denotes the left-limit operator, and $\sigma(t) = i$ shows that the the i-th mode is active $\left(\forall i \in \bar{\mathrm{N}}\right)$. The relation $\sigma(t) = i$ while $\sigma(t^-) = j$ means that t is a switching instant, and that the system switches from the j-th mode to the i-th mode at time t. Without loss of generality, one can assume that the origin is the equilibrium point of this system when there is no input; i.e. $f_{\sigma(t)}(0, 0) = 0, \;\; \forall t > t_0$.

Consider a class of linear impulsive switched control system given by

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), & t \neq t_l, l = 1,2,... \\ x(t) = G_{\sigma(t),\sigma(t^-)}x(t^-), & t = t_l, l = 1,2,... \\ y(t) = C_{\sigma(t)}x(t) \\ x(t_0) = x_0 \end{cases} \tag{3.2}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $u \in \mathbb{R}^m$ are the state, the output and the input of the system respectively. $A_{\sigma(t)}$, $B_{\sigma(t)}$, $C_{\sigma(t)}$ and $G_{\sigma(t),\sigma(t^-)}$ are constant matrices with proper dimensions. Jumps in the state of the system at switching instants are represented by $N^2 - N$ matrices G_{ij}, $\forall i, j \in \bar{\text{N}}, i \neq j$.

The easiest way to represent constrained switching is to introduce a number $\tau_d > 0$, often called *dwell time* [23], and restrict the switching signal such that the time interval between every two consecutive switching instants is greater than τ_d. Since this can be a restrictive requirement in general, one can consider the *average dwell time* instead, which allows fast switchings in some instants, provided that their effect would be compensated by sufficiently slow switchings in some other instants [23].

<u>Definition 1 [23]:</u> Let the number of discontinuities of the switching signal $\sigma(t)$ on a given interval $[t_0, t)$ be denoted by $N(t, t_0)$. The signal $\sigma(t)$ is said to have an *average dwell time* τ_a if there exists two positive numbers N_0 and τ_a such that

$$N(t, t_0) \leq N_0 + \frac{t - t_0}{\tau_a} \qquad 0 \leq t_0 \leq t, \forall t \geq t_0 \tag{3.3}$$

In the sequel, sufficient conditions are derived for the stability of the impulsive switched system given in (3.1).

Lemma 3 *Assume that there exist a C^1 Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}$, two class K_∞*

functions α_1 *and* α_2 *[22] satisfying*

$$\alpha_1(\|x(t)\|) \leq V(x(t)) \leq \alpha_2(\|x(t)\|), \qquad \forall t \geq t_0 \tag{3.4}$$

where $\|.\|$ *denotes any induced norm. Further, assume that there exist a number* $\mu > 1$ *and a strictly negative number* λ_0 *for which the derivative of* $V(x)$ *for the system (3.1) satisfies the inequalities*

$$\dot{V}(x(t)) \leq 2\lambda_0 V(x(t)), \qquad \forall t \in (t_l, t_{l+1}),\ \forall l \in \{1, 2, \cdots\} \tag{3.5}$$

and

$$V(x(t_l{}^+)) \leq \mu V(x(t_l^-)), \qquad \forall l \in \{1, 2, \cdots\} \tag{3.6}$$

Then the equilibrium point $x = 0$ *of (3.1) when* $u(t) = 0$ *is asymptotically stable for every switching signal* $\sigma(t)$*, with the average dwell time satisfying*

$$\tau_a > \tau_{min} = \frac{\log \mu}{-2\lambda_0} \tag{3.7}$$

Proof: It can be deduced from (3.5) and (3.6) that

$$V(x(t)) \leq \mu^{N(t,t_0)} e^{2\lambda_0(t-t_0)} V(x(t_0))$$

(note that $N(t, t_0)$ is the number of switchings in the interval $[t_0, t))$. Using the definition of average dwell time (Definition 1) and replacing the minimum value of average dwell time given by (3.7), it follows that there must exist a positive number ε such that

$$\frac{1}{\tau_a} \leq \frac{-2\lambda_0}{\log \mu} - \varepsilon$$

and as a result

$$N(t, t_0) \leq (\frac{-2\lambda_0}{\log \mu} - \varepsilon)(t - t_0) + N_0$$

therefore

$$V(x(t)) \leq \mu^{N_0} \mu^{-\varepsilon(t-t_0)} V(x(t_0))$$

Now, it can be concluded from (3.4) that

$$\|x(t)\| \leq \alpha_1^{-1}(\mu^{N_0} \mu^{-\varepsilon(t-t_0)} \alpha_2(\|x(t_0)\|))$$

Let $\beta(\|x(t_0)\|,t) = \alpha_1^{-1}(\mu^{N_0}\mu^{-\varepsilon(t-t_0)}\alpha_2(\|x(t_0)\|))$. Since α_1 is a class K_∞ function so is α_1^{-1}. On the other hand, α_2 is also a class K_∞ function and ε is a positive number. This implies that β is a class KL function and hence completes the proof. ■

Remark 1 *If the conditions of Lemma 3 hold for a quadratic Lyapunov function, then the equilibrium point in the switched system (3.1) will be exponentially stable.*

Remark 2 *If the inequality (3.6) in Lemma 3 holds for some* $0 < \mu < 1$, *then it can be shown that the switched system given by (3.1) is globally uniformly asymptotically stable for every arbitrary switching signal [23].*

3.2.2 Stability of discrete impulsive switched systems

In this subsection, inspired by the concept of discrete impulsive systems described in [24] and previous works on continuous impulsive switched systems, a class of discrete impulsive switched systems consisting of M modes of operation (subsystems) are introduced as follows

$$\begin{cases} x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \; k \neq k_l - 1, \; l \in \{1,2,\cdots\} \\ x(k+1) = G_{\sigma(k+1),\sigma(k)}x(k), \qquad k = k_l - 1, \; l \in \{1,2,\cdots\} \\ y(k) = C_{\sigma(k)}x(k) \\ x(k_0) = x_0 \end{cases} \tag{3.8}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $u \in \mathbb{R}^m$ are the state, the output and the input of the system, respectively. Moreover, $\{k_1, k_2, \cdots\}$ is a sequence of increasing impulse times, for which the following assumptions are satisfied

Assumption 1 *The sequence* $\{k_l\}$ *has the property that* $k_l \in \mathbb{N}$ *and* $k_0 = 0$, $k_l < k_{l+1}, \forall l \in \mathbb{N}$.

Assumption 2 $k_{l+1} - k_l > 1$, $l \in \mathbb{N}$.

Define $\bar{\mathrm{M}} = \{1, 2, \cdots, M\}$. The function $\sigma(k) : \{1, 2, \cdots\} \rightarrow \bar{\mathrm{M}}$ is the switching rule and determines which of the M modes is active at each time. For instance, $\sigma(k) = i$ and $\sigma(k-1) = j$ indicates that k is a switching instant at which the system switches from the j-th mode to the i-th mode. Note that the state of this impulsive switched system $x(k) : \{0, 1, 2, \cdots\} \longrightarrow \mathbb{R}^n$ experiences impulses at the switching instants. Furthermore, $G_{\sigma(k_l),\sigma(k_l-1)}$, $l \in \mathbb{N}$ is a constant matrix which depends on the index of the active modes before and after that specific switching instant.

To represent constrained switching, a number $\tau_a > 0$ called average dwell time [23] is introduced here in a way similar to the continuous switched systems, which allows fast switchings in some instants, provided their effects are compensated by sufficiently slow switchings at other instants [23].

Definition 2: Let the number of switching instants of the switching signal $\sigma(k)$ on a given interval $[k_0, k)$ be denoted by $N(k, k_0)$. The signal $\sigma(k)$ is said to have an *average dwell time* k_a if there exists two positive numbers N_0 and k_a such that

$$N(k, k_0) \leq N_0 + \frac{k - k_0}{k_a} \qquad 0 \leq k_0 \leq k, \ \forall k \geq k_0 \tag{3.9}$$

(note that k_a is an integer). In the following, inspired by the works [11], [12], sufficient conditions for the stability of impulsive switched system (3.8) are provided.

Lemma 4 *Consider the switched system (3.8) with the switching instants* $\{k_1, \ldots, k_l, k_{l+1}, \ldots\}$. *Suppose that there exists a* C^1 *function* $V : \mathbb{R}^n \rightarrow \mathbb{R}$, *and two class* K_∞ *functions* α_1 *and* α_2

for which the following inequality holds

$$\alpha_1(\|x(k)\|) \leq V(x(k)) \leq \alpha_2(\|x(k)\|), \qquad \forall k \geq k_0 \tag{3.10}$$

if there exists a number $0 < \beta < 1$ *for which* $V(x(k))$ *along the solution of the system (3.8) satisfies the inequality*

$$V(x(k+1)) \leq \beta V(x(k)) \qquad k_l \leq k < k_{l+1} - 1 \tag{3.11}$$

and a number $\mu > 1$ *such that*

$$V(x(k+1)) \leq \mu V(x(k)), \qquad k = k_l - 1 \tag{3.12}$$

then the impulsive switched system (3.8) is asymptotically stable for every switching signal $\sigma(k)$ *with the average dwell time*

$$k_a > 1 - \frac{\log \mu}{\log \beta} \tag{3.13}$$

Proof: It can be deduced from (3.11) and (3.12) that

$$V(x(k)) \leq \mu^{N(k,k_0)} \beta^{k-k_0-N(k,k_0)} V(x(k_0)), \quad \forall k \geq k_0$$

Using the definition of average dwell time for linear impulsive switched systems (see Definition 2) and replacing the minimum value of average dwell time given by (3.13), it follows there must exist a number ρ such that

$$\begin{aligned} V(x(k)) &\leq \mu^{N_0 + \frac{k-k_0}{k_a}} \beta^{k-k_0-N_0-\frac{k-k_0}{k_a}} V(x(k_0)) \\ &\leq \mu^{N_0} \beta^{-N_0} \rho^{k-k_0} V(x(k_0)) \end{aligned}$$

where $\rho = \mu^{\frac{1}{k_a}} \beta^{\frac{k_a - 1}{k_a}}$ (one can verify from (3.13) that $0 < \rho < 1$). Now it can be concluded from (3.10) that

$$\|x(k)\| \leq \alpha_1^{-1}(\mu^{N_0}\beta^{-N_0}\rho^{k-k_0}\alpha_2(\|x(k_0)\|))$$

Let $\beta(\|x(k_0)\|, k) = \alpha_1{}^{-1}(\mu^{N_0}\beta^{-N_0}\rho^{k-k_0}\alpha_2(\|x(k_0)\|))$. Since α_1 is a class K_∞ function, so is $\alpha_1{}^{-1}$. Moreover, α_2 is a class K_∞ function as well, and $0 < \rho < 1$. Hence it can be easily verified that β is a class KL function. This completes the proof. ■

3.3 Observer Design for Linear Impulsive Switched Systems

The stability results obtained in the previous section will now be used to develop LMI-based algorithms for designing a switching observer for the impulsive switched systems given by (3.2) and (3.8) such that the stability of the observation error dynamic under the constrained switching is guaranteed.

3.3.1 Observer design for continuous impulsive switched

Consider a switching observer O, for (3.2) as follows

$$\dot{\hat{x}}(t) = A_i\hat{x}(t) + B_iu(t) + L_i(y(t) - C_i\hat{x}(t)), \quad t_l < t < t_{l+1} \tag{3.14a}$$

$$\hat{x}(t^+) = H_{i,j}\hat{x}(t^-), \quad t = t_l \tag{3.14b}$$

where $\sigma(t_l^+) = i$, $\sigma(t_l^-) = j$, $l \in \mathbb{N}$ and $\forall i, j \in \bar{\mathrm{N}}$, $i \neq j$. For each mode, an identity Luenberger observer namely O_i is designed and is employed when the corresponding mode is active. It is to be noted that $N^2 - N$ constant matrices H_{ij} suggest that the proposed observer O is an impulsive switched system by its construction. In the remainder of this subsection, an LMI-based algorithm is introduced to design the proposed observer (i.e., to find the values of L_i and H_{ij}, $\forall i, j \in \bar{\mathrm{N}}$, $i \neq j$ such that the following properties hold:

- The eigenvalues of $A_i - L_iC_i$, $\forall i \in \bar{\mathrm{N}}$ are placed in the left of the line $\mathrm{Re}\{s\} = \lambda_1$ and in the right of the line $\mathrm{Re}\{s\} = \lambda_2$, where λ_1 and λ_2 are given and $\lambda_2 < \lambda_1 < 0$.

- The stability of the observation error dynamic in the proposed switching observer O under the constrained switching is guaranteed.

- The required average dwell time in the switching observer is minimized. This means that the proposed observer can be used to observe the states of a large class of impulsive switched systems, in the sense that the switchings are allowed to occur relatively fast.

Remark 3 *The first imposed property means that the error in the Luenberger observers* O_i, $\forall i \in \bar{\mathrm{N}}$ *converges exponentially to zero with rate of convergence greater than* λ_1. *However, if the poles are placed very far from the* $j\omega$ *axis, the resultant observer gains* L_i, $\forall i \in \bar{\mathrm{N}}$ *will be large. This can lead to a design highly sensitive to the numerical errors and prone to implementation difficulties.*

Algorithm 1: Consider the switched system described by (3.2). The following procedure is proposed to find the values of L_i and H_{ij}, $\forall i,j \in \bar{\mathrm{N}}$, $i \neq j$ such that all of the properties mentioned above hold.

Step1: Given λ_1, λ_2 where $\lambda_2 < \lambda_1 < 0$, find the minimum value of μ for which there exist $P_i > 0$, X_i and $H_{i,j}$, satisfying the following matrix inequalities

$$A_i{}^T P_i + P_iA_i - C_i{}^T X_i{}^T - X_iC_i - 2\lambda_1 P_i < 0, \quad \forall i \in \bar{\mathrm{N}} \tag{3.15}$$

$$2\lambda_2 P_i - A_i{}^T P_i - P_iA_i + C_i{}^T X_i{}^T + X_iC_i < 0, \quad \forall i \in \bar{\mathrm{N}} \tag{3.16}$$

$$\begin{bmatrix} \mu P_j - G_{i,j}^T P_i G_{i,j} & -\mu P_j + G_{i,j}^T P_i H_{i,j} & 0 \\ -\mu P_j + H_{i,j}^T P_i G_{i,j} & \mu P_j & H_{i,j}^T P_i \\ 0 & P_i H_{i,j} & P_i \end{bmatrix} \geq 0, \quad \forall i,j \in \bar{\mathrm{N}},\ i \neq j \tag{3.17}$$

It is to be noted that this minimization can be formulated as a BMI problem. PENBMI can solve this problem efficiently and can be used as a MATLAB function with PEN or YALMIP interface. Denote the optimum of the above non-convex optimization problem with μ^*.

Step 2: Using the matrices P_i and X_i $(\forall i \in \bar{N})$ obtained in Step 1, find L_i, the observer gains of O proposed in (3.14) as follows

$$L_i = P_i^{-1} X_i, \qquad \forall i \in \bar{N} \tag{3.18}$$

Step 3: If $\mu^* > 1$, compute the minimum allowable dwell time as

$$\tau_{min} = \frac{\log \mu^*}{-2\lambda_1} \tag{3.19}$$

The above procedure arrives at the minimum value of μ, namely μ^*, and gives the matrices P_i, X_i, L_i $(\forall i \in \bar{N})$ and H_{ij}, $\forall i, j \in \bar{N}$, $i \neq j$. The following result is obtained.

Theorem 7 *If there exists N symmetric positive definite matrices P_i, N matrices X_i and $N^2 - N$ matrices $H_{i,j}$, $\forall i, j \in \bar{N}$, $i \neq j$, which satisfy the LMIs (3.15), (3.16) and (3.17) then:*

i) The eigenvalues of $A_i - L_i C_i$ satisfy the inequality $\lambda_2 < eig(A_i - L_i C_i) < \lambda_1$, $\forall i \in \bar{N}$

ii) If $\mu^* > 1$, then the error dynamic in the switching observer O is globally uniformly exponentially stable for the switching signal $\sigma(t)$ with any average dwell time τ_a greater than τ_{min} given by (3.19). Otherwise $(0 < \mu^* \leq 1)$, the error dynamic in the switching observer O is globally uniformly exponentially stable for any arbitrary switching signal.

Proof: Since $L_i = P_i^{-1} X_i$ or equivalently $X_i = P_i L_i$, the inequalities (3.15) and (3.16) can be rewritten as

$$(A_i - L_i C_i)^T P_i + P_i (A_i - L_i C_i) - 2\lambda_1 P_i < 0, \quad \forall i \in \bar{N}$$

$$2\lambda_2 P_i - (A_i - L_i C_i)^T P_i - P_i(A_i - L_i C_i) < 0, \quad \forall i \in \bar{\mathbb{N}}$$

Now it can be easily concluded from the Lyapunov theory that these two LMIs are equivalent to this inequality $\lambda_2 < eig(A_i - L_i C_i) < \lambda_1$.

Assume that the active observer in the time intervals $[t_{l-1}, t_l)$ and $[t_l, t_{l+1})$ are j and i, respectively. The error dynamic in the proposed switching observer denoted by $\tilde{x}(t)$ can be described by

$$\dot{\tilde{x}}(t) = (A_i - L_i C_i)\tilde{x}(t), \qquad t_{l-1} < t < t_l,\ l \in \mathbb{N} \tag{3.20a}$$

$$\tilde{x}(t^+) = G_{i,j}x(t^-) - H_{i,j}\hat{x}(t^-), \qquad t = t_l,\ l \in \mathbb{N} \tag{3.20b}$$

Define a switched Lyapunov function as

$$V(\tilde{x}(t)) = \tilde{x}(t)^T P_i \tilde{x}(t),\ t_{l-1} < t < t_l,\ l \in \mathbb{N},\ \forall i \in \bar{\mathbb{N}} \tag{3.21}$$

where i is the index of active modes at each time and P_i, $\forall i \in \bar{\mathbb{N}}$ are obtained in Step 1. Since

$$\min_i \{\lambda_{min}(P_i)\} \|\tilde{x}\|^2 < V(\tilde{x}(t)) < \max_i \{\lambda_{max}(P_i)\} \|\tilde{x}\|^2, \forall i \in \bar{\mathbb{N}}$$

then the Lyapunov function V satisfies (3.4), where $\alpha_1(\|\tilde{x}\|)$ and $\alpha_2(\|\tilde{x}\|)$ are defined as

$$\alpha_1(\|\tilde{x}\|) = \min_i \{\lambda_{min}(P_i)\} \|\tilde{x}\|^2, \qquad \forall i \in \bar{\mathbb{N}}$$

$$\alpha_2(\|\tilde{x}\|) = \max_i \{\lambda_{max}(P_i)\} \|\tilde{x}\|^2, \qquad \forall i \in \bar{\mathbb{N}}$$

according to (3.18), since $X_i = P_i L_i$, (3.15) can be written as

$$(A_i - L_i C_i)^T P_i + P_i(A_i - L_i C_i) < 2\lambda_1 P_i$$

Considering the definition of switched Lyapunov function given by (3.21) and the equality (3.20a), the above inequality can be rewritten as

$$\dot{V}(\tilde{x}(t)) < 2\lambda_1 V(\tilde{x}(t)), \quad t_{l-1} < t < t_l, \ l \in \mathbb{N}$$

In other words (3.5) is satisfied by this choice of V. According to (3.20b), the last condition of Lemma 3 by this choice of V can be rewritten as

$$(x(t_l^-)^T G_{i,j}^T - \hat{x}(t_l^-)^T H_{i,j}^T) P_i (G_{i,j} x(t_l^-) - H_{i,j} \hat{x}(t_l^-)) \leq$$
$$\mu (x(t_l^-)^T - \hat{x}(t_l^-)^T) P_j (x(t_l^-) - \hat{x}(t_l^-))$$

or equivalently

$$X^T \begin{bmatrix} \mu P_j - G_{i,j}^T P_i G_{i,j} & -\mu P_j + G_{i,j}^T P_i H_{i,j} \\ -\mu P_j + H_{i,j}^T P_i G_{i,j} & \mu P_j - H_{i,j}^T P_i H_{i,j} \end{bmatrix} X \geq 0$$

where $X = \begin{bmatrix} x(t_l^-) \\ \hat{x}(t_l^-) \end{bmatrix}$. Again this inequality can be rewritten as

$$\begin{bmatrix} \mu P_j - G_{i,j}^T P_i G_{i,j} & -\mu P_j + G_{i,j}^T P_i H_{i,j} \\ -\mu P_j + H_{i,j}^T P_i G_{i,j} & \mu P_j \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & H_{i,j}^T P_i H_{i,j} \end{bmatrix} \geq 0 \qquad (3.22)$$

replacing the second term in the above inequality by $\begin{bmatrix} 0 \\ P_i H_{i,j} \end{bmatrix}^T P_i^{-1} \begin{bmatrix} 0 & P_i H_{i,j} \end{bmatrix}$ and using the schur complement one can verify that the above inequality becomes the same as (3.17). Thus all the conditions of Lemma 3 are satisfied by this choice of V. This completes the proof. ■

The optimization problem introduced above is non-convex, in general. In the following algorithm, some additional assumptions are made on the structure of the proposed observer, to turn the above non-convex optimization problem to a convex LMI problem.

Algorithm 2: Consider the switched system described by (3.2). The following procedure is followed to find the values of L_i and H_{ij}, $\forall i,j \in \bar{\mathrm{N}}$, $i \neq j$ in the structure of the proposed observer O.

Step1:Find the minimum value of μ for which there exist N matrices X_i, $\forall i \in \bar{\mathrm{N}}$, and N positive definite symmetric matrices P_i, $\forall i \in \bar{\mathrm{N}}$ satisfying the following LMIs

$$A_i^T P_i + P_i A_i - C_i^T X_i^T - X_i C_i - 2\lambda_1 P_i < 0, \quad \forall i \in \bar{\mathrm{N}} \tag{3.23}$$

$$2\lambda_2 P_i - A_i^T P_i + P_i A_i - C_i^T X_i^T - X_i C_i < 0, \quad \forall i \in \bar{\mathrm{N}} \tag{3.24}$$

$$\mu P_j \geq G_{ij}^T P_i G_{ij}, \ \forall i,j \in \bar{\mathrm{N}}, \ i \neq j \tag{3.25}$$

It is to be noted that this minimization can be formulated as a GEVP problem. (MATLAB can solve this problem efficiently). Moreover, denote the optimum of the above convex optimization problem with μ^*.

Step 2: Using the matrices P_i and X_i, $\forall i \in \bar{\mathrm{N}}$ obtained in Step 1, find L_i, the observer gains of O proposed in (3.14a) as follows

$$L_i = P_i^{-1} X_i, \qquad \forall i \in \bar{\mathrm{N}} \tag{3.26}$$

Step 3: If $\mu^* > 1$, compute the minimum allowable dwell time

$$\tau_{min} = \frac{\log \mu^*}{-2\lambda_1} \tag{3.27}$$

Theorem 8 *Using this algorithm to obtain the minimum value of μ namely μ^*, matrices P_i, X_i, L_i $\left(\forall i \in \bar{\mathrm{N}}\right)$ and H_{ij}, $\forall i,j \in \bar{\mathrm{N}}$, $i \neq j$, we have the following result. If there exists N symmetric positive definite matrices P_i and N matrices X_i, $\forall i \in \bar{\mathrm{N}}$ which satisfy the LMIs (3.23), (3.24) and (3.25), then:*

i) The eigenvalues of $A_i - L_i C_i$ satisfy the inequality $\lambda_2 < eig(A_i - L_i C_i) < \lambda_1$, $\forall i \in \bar{\mathrm{N}}$

ii) If $\mu^* > 1$, the error dynamic in the corresponding switching observer O is globally uniformly exponentially stable for the switching signal $\sigma(t)$ with any average dwell time τ_a greater than τ_{min} given by (3.27). Otherwise ($0 < \mu^* \leq 1$), the error dynamic in the switching observer O is globally uniformly exponentially stable for arbitrary switching signals.

Proof: Proof is similar to the proof of the previous theorem, in fact LMIs (3.23) and (3.24) guarantees that convergence rate of each Luenberger observer is in the desired region, also by substituting $H_{ij} = G_{ij}$, one can verify that LMI's (3.25) and (3.17) are the same. Thus if the same Lyapunov function V as in the previous theorem is considered all the conditions of Lemma 3 are satisfied by this choice of V and error in the observer is asymptotically stable for any impulsive switched system with the average dwell time greater than (3.27). ■

Remark 4 *Using this algorithm the proposed switching observer will have the same jumps at switching instants as in the impulsive switched system, in other words $H_{ij} = G_{ij}$ will be imposed on the structure of the observer.*

Remark 5 *The proposed two algorithms in this section can be applied to switched systems without impulse, in fact the systems without impulse are a special case of impulsive systems when $G_{ij} = I$.*

3.3.2 Observer design for discrete impulsive switched system

Consider a switching observer for (3.8) as follows

$$\hat{x}(k+1) = A_i\hat{x}(k) + B_iu(k) + L_i(y(k) - C_i\hat{x}(k)), \quad k \neq k_l - 1,\ l \in \mathbb{N} \tag{3.28a}$$

$$\hat{x}(k+1) = H_{i,j}\hat{x}(k), \quad k = k_l - 1, l \in \mathbb{N} \tag{3.28b}$$

$\sigma(k_l) = i$, $\sigma(k_{l-1}) = j$ where $l \in \mathbb{N}$ and $\forall i, j \in \bar{\mathrm{M}}$, $i \neq j$. For each mode, an identity Luenberger observer, namely O_i, is designed and is employed when the corresponding mode is active. It is to be noted that $M^2 - M$ constant matrices H_{ij} show that the proposed observer O is a discrete impulsive switched system by its construction.

In the remainder of this subsection, an LMI-based algorithm is introduced to design the proposed observer (i.e., to obtain the values of L_i and H_{ij}) such that the following properties in the observer are held

- The eigenvalues of $A_i - L_iC_i$ are placed inside the circle centered at the origin with the radius $r = \beta$ where $\beta \in (0,1)$ is given.
- The stability of the observation error dynamic in the proposed switching observer O under the constrained switching with a needed minimum average dwell time is guaranteed.

Remark 6 *Unlike the continuous dynamics that locating the eigenvalues of Luenberger observer far from the imaginary axis in LHP results in high gains, in the discrete dynamics locating the eigenvalues of Luenberger observer near the origin does not result in high gains. In fact, an observer with all of its eigenvalues located at the origin is desirable, and is referred to as a dead-beat observer. Unlike the continuous impulsive switching observer design, here the desired region for the eigenvalues of a discrete Luenberger observer is given only by one parameter β (which directly reflects the speed of convergence).*

Remark 7 *For a given constant matrix A and a given $0 < \eta < 1$, all the eigenvalues of A are placed inside a circle in the s-plane centered at the origin with the radius η iff there exists a positive symmetric matrix P such that $A^TPA - \eta^2 P < 0$.*

Algorithm 3: Consider the switched system described by (3.8). The following steps should be followed to obtain the values of L_i and H_{ij}, $\forall i, j \in \bar{\mathrm{M}}$, $i \neq j$ such that all the properties mentioned above are satisfied by the proposed switched observer O.

Step 1: For the given $0 < \beta < 1$, find the minimum value of μ for which there exist M matrices X_i, $\forall i \in \bar{\mathrm{M}}$, and M positive definite symmetric matrices P_i, $\forall i \in \bar{\mathrm{M}}$, and $M^2 - M$

matrices H_{ij} satisfying the following LMIs

$$\begin{bmatrix} \beta^2 P_j & A_i^T P_i - C_i^T X_i^T \\ P_i A_i - X_i C_i & P_i \end{bmatrix} > 0 \quad \forall i, j \in \bar{\mathrm{M}} \tag{3.29}$$

$$\begin{bmatrix} \mu P_i - G_{i,j}^T P_i G_{i,j} & -\mu P_i + G_{i,j}^T P_i H_{i,j} & 0 \\ -\mu P_i + H_{i,j}^T P_i G_{i,j} & \mu P_i & H_{i,j}^T P_i \\ 0 & P_i H_{i,j} & P_i \end{bmatrix} \geq 0, \ \forall i, j \in \bar{\mathrm{M}}, \ i \neq j \tag{3.30}$$

Denote the optimum value of the above non-convex optimization problem with μ^*.

Step 2: Using the matrices P_i and X_i^1, $\forall i \in \bar{\mathrm{M}}$ obtained in Step 1, find L_i, the observer gain of O_i (given in (3.28) for each mode) as follows

$$L_i = P_i^{-1} X_i, \qquad \forall i \in \bar{\mathrm{M}} \tag{3.31}$$

Step 3: If $\mu^* > 1$, compute the minimum allowable dwell time

$$\tau_{min} = 1 - \frac{\log \mu^*}{\log \beta^2} \tag{3.32}$$

Using this algorithm to obtain the minimum value of μ namely μ^*, matrices P_i, X_i, L_i $(\forall i \in \bar{\mathrm{M}})$ and H_{ij}, $\forall i, j \in \bar{\mathrm{M}}$, $i \neq j$, we have the following result.

Theorem 9 *If there exists M symmetric positive definite matrices $P_i > 0$, and M matrices X_i, $\forall i \in \bar{\mathrm{M}}$ and $M^2 - M$ matrices H_{ij}, $\forall i, j \in \bar{\mathrm{M}}$, $i \neq j$ which satisfy the LMIs (3.29), (3.30), then:*

- *i) The eigenvalues of $A_i - L_i C_i$ satisfy the inequality $|eig(A_i - L_i C_i)| < \beta$, $\forall i \in \bar{\mathrm{M}}$.*
- *ii) If the $\mu^* > 1$, the error dynamic in the switching observer O is globally uniformly exponentially stable for the switching signal $\sigma(k)$ with any average dwell time k_a greater than k_{min} given by (3.32). Otherwise $(0 < \mu^* \leq 1)$, the error dynamic in the*

switching observer O is globally uniformly exponentially stable for arbitrary switching signals.

Proof: According to (3.31), since $X_i = P_i L_i$, (3.29) can be written as

$$\begin{bmatrix} \beta^2 P_j & (A_i - L_i C_i)^T P_i \\ P_i(A_i - L_i C_i) & P_i \end{bmatrix} > 0 \quad \forall i,j \in \bar{\mathbf{M}}$$

Using the schur complement the above inequality can be rewritten as

$$\beta^2 P_j - (A_i - L_i C_i)^T P_i P_i^{-1} P_i (A_i - L_i C_i) > 0, \quad \forall i,j \in \bar{\mathbf{M}}$$

or

$$\beta^2 P_j - (A_i - L_i C_i)^T P_i (A_i - L_i C_i) > 0, \quad \forall i,j \in \bar{\mathbf{M}} \tag{3.33}$$

Since (3.33) is valid for $\forall i,j \in \bar{\mathbf{M}}$ one can conclude the following:

$$(A_i - L_i C_i)^T P_i (A_i - L_i C_i) - \beta^2 P_i < 0, \quad \forall i \in \bar{\mathbf{M}}$$

which means $|eig(A_i - L_i C_i)| < \beta$. Moreover the error dynamic in the proposed switching observer can be described by

$$\tilde{x}(k+1) = (A_i - L_i C_i)\tilde{x}(k), \quad k \neq k_l - 1 \tag{3.34a}$$

$$\tilde{x}(k+1) = G_{i,j} x(k) - H_{i,j} \hat{x}(k), \quad k = k_l - 1 \tag{3.34b}$$

define a switched Lyapunov function as

$$V(\tilde{x}(k)) = \tilde{x}(k)^T P_i \tilde{x}(k), \quad k_l < k \leq k_{l+1} \tag{3.35}$$

where i is the index of active modes at each time and P_i, $\forall i \in \bar{\mathbf{M}}$ are obtained in Step 1.

Since

$$\min_i \{\lambda_{min}(P_i)\} \|\tilde{x}\|^2 < V(\tilde{x}(t)) < \max_i \{\lambda_{max}(P_i)\} \|\tilde{x}\|^2, \forall i \in \bar{\mathrm{M}}$$

then the Lyapunov function V satisfies (3.10), where $\alpha_1(\|\tilde{x}\|)$ and $\alpha_2(\|\tilde{x}\|)$ are defined as

$$\alpha_1(\|\tilde{x}\|) = \min_i \{\lambda_{min}(P_i)\} \|\tilde{x}\|^2, \qquad \forall i \in \bar{\mathrm{M}}$$

$$\alpha_2(\|\tilde{x}\|) = \max_i \{\lambda_{max}(P_i)\} \|\tilde{x}\|^2, \qquad \forall i \in \bar{\mathrm{M}}$$

Considering the definition of the switched Lyapunov function given by (3.35), the inequality (3.33)can be written as

$$V(\tilde{x}(k+1)) < \beta^2 V(\tilde{x}(k)), \quad k_l \leq k < k_{l+1}$$

which means (3.11) is satisfied. To check if the last condition of Lemma 4 ($V(\tilde{x}(k_l) \leq \mu V(\tilde{x}(k_l - 1))$ holds by this choice of V, assume the active observer in the time intervals $[k_{l-1}, k_l)$ is i. According to (3.34b) this condition can be rewritten as

$$\begin{aligned}
&(x(k)^T G_{i,j}^T - \widehat{x}(k)^T H_{i,j}^T) P_j (G_{i,j} x(k) - H_{i,j} \widehat{(k)}) \leq \\
&\mu (x(k)^T - \widehat{x}(k)^T) P_j (x(k) - \widehat{x}(k))
\end{aligned}$$

or equivalently

$$X^T \begin{bmatrix} \mu P_j - G_{i,j}^T P_j G_{i,j} & -\mu P_j + G_{i,j}^T P_j H_{i,j} \\ -\mu P_j + H_{i,j}^T P_j G_{i,j} & \mu P_j - H_{i,j}^T P_j H_{i,j} \end{bmatrix} X \geq 0$$

where $X = \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}$. Again this inequality can be rewritten as

$$\begin{bmatrix} \mu P_j - G_{i,j}^T P_j G_{i,j} & -\mu P_j + G_{i,j}^T P_j H_{i,j} \\ -\mu P_j + H_{i,j}^T P_j G_{i,j} & \mu P_j \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & H_{i,j}^T P_j H_{i,j} \end{bmatrix} \geq 0 \tag{3.36}$$

replacing the second term in the above inequality by $\begin{bmatrix} 0 \\ P_j H_{i,j} \end{bmatrix}^T P_j^{-1} \begin{bmatrix} 0 & P_j H_{i,j} \end{bmatrix}$ and using the schur complement one can verify that the above inequality is the same as (3.30). same. ■

Similar to the previous section, to relax the above non-convex problem to an LMI problem in the next algorithm the same restriction on the structure of the proposed observer is made and H_{ij} in the structure of the observer are assumed to be the same as G_{ij} in the system.

Algorithm 4: Consider the switched system described by (3.8). Similar to the previous algorithm the following step are followed to obtain the values of gains L_i and H_{ij}, $\forall i, j \in \bar{\mathrm{M}}$, $i \neq j$.

Step 1: For the given $0 < \beta < 1$, find the minimum value of μ for which there exist M matrices X_i, $\forall i \in M$, and M positive definite symmetric matrices P_i, $\forall i \in M$, satisfying the following LMIs

$$\begin{bmatrix} \beta^2 P_j & A_i^T P_i - C_i^T X_i^T \\ P_i A_i - X_i C_i & P_i \end{bmatrix} > 0 \quad \forall i, j \in \bar{\mathrm{M}} \tag{3.37}$$

$$\mu P_i \geq G_{ij}^T P_i G_{ij} \quad \forall i, j \in \bar{\mathrm{M}},\ i \neq j \tag{3.38}$$

Denote the optimum of the above convex optimization problem with μ^*.

Step 2: Using the matrices P_i and X_i^1, $\forall i \in M$ obtained in Step 1, find L_i, the observer gain of O_i proposed in (3.28) as follows

$$L_i = P_i^{-1} X_i, \qquad \forall i \in \bar{\mathrm{M}} \tag{3.39}$$

Step 3: If $\mu^* > 1$, compute the minimum allowable dwell time

$$\tau_{min} = 1 - \frac{\log \mu^*}{\log \beta^2} \tag{3.40}$$

Theorem 10 *If there exists M symmetric positive definite matrices $P_i > 0$, and M matrices X_i, $\forall i \in \bar{\mathrm{M}}$ which satisfy the LMIs (3.37), (3.38), then if the $\mu^* > 1$, the error dynamic in the switching observer O is globally uniformly exponentially stable and for the switching signal $\sigma(k)$ with any average dwell time k_a greater than k_{min} given by (3.40). Otherwise ($0 < \mu^* \leq 1$), the error dynamic in the switching observer O is globally uniformly exponentially stable for arbitrary switching signals. Moreover, $|eig(A_i - L_iC_i)| < \beta, \ \forall i \in \bar{\mathrm{M}}$.*

Proof: See the proof of Theorem 9. ■

One can verify that if instead of the switched Lyapunov function given in (3.35) an alternative form of this function as $V(\tilde{x}(k)) = \tilde{x}(k)^T P_i \tilde{x}(k), \quad k_l \leq k < k_{l+1}$ is considered, Algorithms 3 and 4 will slightly change. In the alternative form of Algorithm 3, LMI conditions in Step 1 should be replaced by

$$\begin{bmatrix} \beta^2 P_i & A_i^T P_i - C_i^T X_i^T \\ P_i A_i - X_i C_i & P_i \end{bmatrix} > 0 \quad \forall i \in \bar{\mathrm{M}} \tag{3.41}$$

$$\begin{bmatrix} \mu P_j - G_{i,j}^T P_i G_{i,j} & -\mu P_j + G_{i,j}^T P_i H_{i,j} & 0 \\ -\mu P_j + H_{i,j}^T P_i G_{i,j} & \mu P_j & H_{i,j}^T P_i \\ 0 & P_i H_{i,j} & P_i \end{bmatrix} \geq 0 \tag{3.42}$$

$$\forall i, j \in \bar{\mathrm{M}}, \ \mathrm{i} \neq \mathrm{j}$$

Similarly, in the new Algorithm 4, LMI conditions in Step 1 should be changed to

$$\begin{bmatrix} \beta^2 P_i & A_i^T P_i - C_i^T X_i^T \\ P_i A_i - X_i C_i & P_i \end{bmatrix} > 0 \quad \forall i \in \bar{\mathrm{M}} \tag{3.43}$$

$$\mu P_j \geq G_{ij}^T P_i G_{ij} \quad \forall i, j \in \bar{\mathrm{M}}, \ i \neq j \tag{3.44}$$

In the following, inspired by the work given in [7], alternate sufficient conditions for the LMI (3.29) are proposed.

Proposition 1 *Assume C_i, $\forall i \in \bar{\mathrm{M}}$, in (3.29) are full-row rank. Given $\beta \in (0,1)$, if there exists M positive definite matrices $S_1, \cdots, S_M$, M matrices $U_1, \cdots, U_M$, M matrices $G_1, \cdots, G_M$, and M matrices $V_1, \cdots, V_M$ satisfying*

$$\begin{bmatrix} \beta^2 G_i + \beta^2 G_i^T - \beta^2 S_j & G_i^T A_i^T - C_i^T U_i^T \\ A_i G_i - U_i C_i & S_i \end{bmatrix} > 0, \forall i, j \in \bar{\mathrm{M}} \tag{3.45}$$

and

$$V_i C_i = C_i G_i, \quad \forall i \in \bar{\mathrm{M}} \tag{3.46}$$

then there exist M positive definite matrices $P_1, \cdots, P_M$ and M matrices $L_1, \cdots, L_M$ satisfying

$$\begin{bmatrix} \beta^2 P_j & (A_i - L_i C_i)^T P_i \\ P_i (A_i - L_i C_i) & P_i \end{bmatrix} > 0 \quad \forall i, j \in \bar{\mathrm{M}} \tag{3.47}$$

Proof: From (3.45), it can be concluded that for all $\forall i, j \in \bar{\mathrm{M}}$

$$\beta^2 G_i + \beta^2 G_i^T > \beta^2 S_j \tag{3.48}$$

and thus the matrix G_i, $\forall i \in \bar{\mathrm{M}}$, is full-rank. Furthermore, since C_i, $\forall i \in \bar{\mathrm{M}}$, is assumed to be full-row rank, the matrix V_i satisfying (3.46) is nonsingular, $\forall i \in \bar{\mathrm{M}}$ matrices. Define $L_i = U_i V_i^{-1}$ and hence, rewrite (3.45) as

$$\begin{bmatrix} \beta^2 G_i + \beta^2 G_i^T - \beta^2 S_j & G_i^T (A_i^T - C_i^T L_i^T) \\ (A_i - L_i C_i) G_i & S_i \end{bmatrix} > 0 \tag{3.49}$$

It follows from positive definiteness of S_i, $\forall i \in \bar{\mathrm{M}}$ that

$$\beta^2 (S_j - G_i)^T S_j^{-1} (S_j - G_i) \geq 0 \quad \forall i, j \in \bar{\mathrm{M}}$$

or, equivalently

$$\beta^2 G_i^T S_j^{-1} G_i \geq \beta^2 G_i^T + \beta^2 G_i - \beta^2 S_j \tag{3.50}$$

It follows from (3.49) and (3.50) that

$$\begin{bmatrix} \beta^2 G_i^T S_j^{-1} G_i & G_i^T (A_i^T - C_i^T L_i^T) \\ (A_i - L_i C_i) G_i & S_i \end{bmatrix} > 0$$

Therefore,

$$\begin{bmatrix} G_i^T & 0 \\ 0 & S_i \end{bmatrix} \begin{bmatrix} \beta^2 S_j^{-1} & (A_i^T - C_i^T L_i^T) S_i^- 1 \\ S_i^{-1}(A_i - L_i C_i) & S_i^{-1} \end{bmatrix} \begin{bmatrix} G_i & 0 \\ 0 & S_i \end{bmatrix} > 0$$

which is equivalent to

$$\begin{bmatrix} \beta^2 S_j^{-1} & (A_i^T - C_i^T L_i^T) S_i^- 1 \\ S_i^{-1}(A_i - L_i C_i) & S_i^{-1} \end{bmatrix} > 0$$

Let S_i^{-1} be denoted by P_i, $\forall i \in \bar{\mathrm{M}}$. Then, one can obtain the following matrix inequality

$$\begin{bmatrix} \beta^2 P_j & (A_i - L_i C_i)^T P_i \\ P_i (A_i - L_i C_i) & P_i \end{bmatrix} > 0 \quad \forall i, j \in \bar{\mathrm{M}}$$

This completes the proof. ■

It is to be noted that in the inequality given by (3.29), X_i is equal to $P_i L_i$. As a result, the LMI conditions in (3.29) and (3.47) are the same.

Remark 8 *According to Proposition 1, if the LMI (3.47) does not hold, the proposed alternate conditions (3.45) and (3.46) are also infeasible. To specify the advantage of using the proposed alternate LMI conditions, consider an impulsive parameter varying switched system. In this case, it is aimed to search for a parameter dependent Lyapunov function corresponding to each uncertain modes in the switched system. In this context, introducing* *slack variables G_i in (3.45) and (3.46) is of great importance which leads to a less degree of conservatism [6].*

Similarly, alternate sufficient conditions for LMI (3.37) can be obtained.

Proposition 2 *Assume C_i, $\forall i \in \bar{\mathrm{M}}$, in (3.37) are full-row rank. For $0 < \beta < 1$, if there exist M symmetric matrices $S_1, \cdots, S_M$ and M matrices $U_1, \cdots, U_M$, M matrices $G_1, \cdots, G_M$, and M matrices $V_1, \cdots, V_M$ satisfying*

$$\begin{bmatrix} \beta^2 G_i + \beta^2 G_i^T - \beta^2 S_i & G_i^T A_i^T - C_i^T U_i^T \\ A_i G_i - U_i C_i & S_i \end{bmatrix} > 0 \quad \forall i \in \bar{\mathrm{M}} \tag{3.51}$$

and

$$V_i C_i = C_i G_i, \quad \forall i \in \bar{\mathrm{M}} \tag{3.52}$$

Then there exist M symmetric matrices $P_1, \cdots, P_M$ and M matrices $L_1, \cdots, L_M$ satisfying

$$\begin{bmatrix} \beta^2 P_i & (A_i - L_i C_i)^T P_i \\ P_i (A_i - L_i C_i) & P_i \end{bmatrix} > 0 \quad \forall i \in \bar{\mathrm{M}} \tag{3.53}$$

Proof: Proposition 2 can be regarded as a special case of Proposition 1 when all the indexes are the same. ■

3.4 Numerical Example

In this section, two numerical examples are given to show the effectiveness of the proposed algorithms.

Example 1 Consider a continuous impulsive switched system given by (3.2) where the switching signal $\sigma(t)$ is a piecewise constant function with the set of images equal to $\{1, 2\}$

and the system in different modes is represented by

$$A_1 = \begin{bmatrix} -2 & 0 \\ 2 & -3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -1 & -4 \\ 3 & -8 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 2 & -1 \end{bmatrix}$$

and

$$G_{12} = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}, \quad G_{21} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The desired region for both Luenberger observers' rate of convergence is assumed to be $-10 < \lambda_0 < -4$. Using Algorithm 2, the observer gains are obtained as follows

$$L_1 = \begin{bmatrix} -16.6293 \\ -22.6511 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 2.6276 \\ 1.6847 \end{bmatrix}$$

The minimum value of μ using GEVP (generalized eigenvalue problem) is obtained as $\mu^* = 36.0086$, which implies that the proposed observer is stable for the given impulsive switched system for any switching signal with the average dwell time greater than $\tau_{min} = \frac{\log(36.0086)}{(-2)(-4)} = 0.4480$.

<u>Example 2</u> Consider a discrete impulsive switched system given by (3.8) consisting of two modes represented by the state-space matrices

$$A_1 = \begin{bmatrix} -0.2 & 0 & 0.1 \\ 0.2 & -0.3 & 0.8 \\ 0.5 & -0.3 & 0.6 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \\ 0.5 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.1 & -0.4 & 0 \\ 0.3 & -0.8 & 0.5 \\ 0.1 & 0 & 0.7 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & -1 & -2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}$$

and

$$G_{12} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad G_{21} = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix}$$

Assume that it is desired to have the eigenvalues of each Luenberger observer O_i designed for each mode inside the circle in the s-plane centered at the origin and with the radius equal to 0.3 that is, $\beta = 0.3$. The minimum value of μ using GEVP is obtained as $\mu^* = 206.7415$. Since $\tau_{min} = 1 - \frac{\log(206.7415)}{\log(0.09)} = 3.2141$, the proposed observer is stable for the given impulsive switched system for any switching signal with the average dwell time greater than or equal to 4. The two Luenberger observer gains are

$$L_1 = \begin{bmatrix} -0.0292 \\ -0.2206 \\ -0.1575 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.6840 \\ -0.5613 \\ 0.6031 \end{bmatrix}$$

Using the alternative form of Algorithm 4 which has fewer LMI conditions will result in $\mu^* = 9.1325$, and the corresponding minimum required average dwell time will be greater than or equal to 3.

Chapter 4

A Design Methodology to Observe the States of Switched Systems in Finite Time

4.1 Introduction

In this chapter of the thesis, the problem of state observation for a continuous-time LTI switched system is addressed. Corresponding to each subsystem, a finite-time observer (FTO) is employed and a switching observer is constructed accordingly. The finite convergence time property of the proposed switching observer is discussed and the exponential stability of the observation error is investigated. An LMI-based algorithm is given which provides conditions for the exponential stability of the switching observer for the switching signals with an average dwell time greater than a specific value. A numerical example is given to show the effectiveness of the proposed algorithm.

4.2 Continuous Finite-Time Observers (CFTO)

Consider the following linear time-invariant (LTI) continuous system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(t_0) = x_0, \qquad t \geq t_0 \tag{4.1a}$$

$$y(t) = Cx(t) \tag{4.1b}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $u \in \mathbb{R}^m$ are the state, the output and the input of the system, respectively. The observer design theory for system (4.1), often known as Luenberger observer, is well documented in the literature. In this type of observers, by a proper choice of gains, the error in the observation of states approaches to zero exponentially with an arbitrarily fast rate of convergence. Unlike the continuous-time Luenberger observers where the convergence is always asymptotic with time, discrete-time Luenberger observers can achieve finite convergence time by placing all the eigenvalues of observers at the origin. This type of discrete-time observers is often referred to as dead-beat. Nevertheless, a methodology to observe the states in finite time using purely continuous observers was recently introduced in [13]. The corresponding observer consists of two identity Luenberger observers and a delay Δ (see Fig. 4.1). It will later be shown that the finite convergence time is equal to Δ. For the system (4.1),

$$\dot{z}^i = F^i z^i + L^i y + Bu, \qquad (i = 1, 2)$$

represents two identity Luenberger observers, where $F^i := A - L^i C$, $i = 1, 2$. Define

$$F = \begin{bmatrix} F^1 & 0 \\ 0 & F^2 \end{bmatrix} \quad H = \begin{bmatrix} L^1 \\ L^2 \end{bmatrix}$$

$$G = \begin{bmatrix} B \\ B \end{bmatrix} \quad T = \begin{bmatrix} I_{n\times n} \\ I_{n\times n} \end{bmatrix} \quad z = \begin{bmatrix} z^1 \\ z^2 \end{bmatrix}$$

By combining these two identity Luenberger observers and introducing a delay $\Delta \in \mathbb{R}^+$ in the structure of the observer, a new state estimate $\widehat{x}$ can be generated as follows

$$\dot{z} = Fz + Hy + Gu, \qquad t \geq t_0 \tag{4.2a}$$

$$\widehat{x}(t) = K[z(t) - e^{F\Delta}z(t-\Delta)] \tag{4.2b}$$

Structure of this observer is shown in Fig. 4.1.

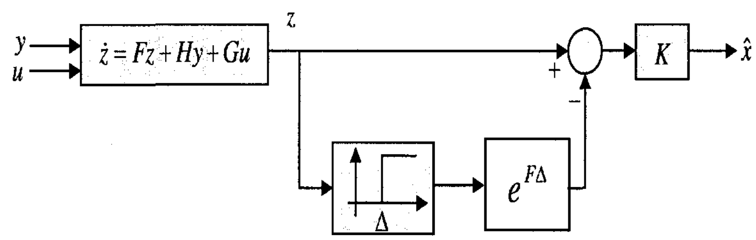

Figure 4.1: The structure of a finite-time observer [13].

Theorem 11 *Let L and Δ be chosen such that*

i. F is Hurwitz; i.e. all eigenvalues of F have strictly negative real parts;

ii. $\det[T \quad e^{F\Delta}T] \neq 0$,

then the observer given by (4.2) with $K := [I_{n,n} \quad 0_{n,n}][T \quad e^{F\Delta}T]^{-1}$ observes the states of the system (4.1) exactly within the finite time Δ, which implies that $\widehat{x}(t) = x(t)$ for $t \geq \Delta$.

Proof: See [13]. ∎

Remark 9 *From the definition of K, it can be easily shown that $KT = I_{n,n}$ and $Ke^{F\Delta}T = 0_{n,n}$.*

It is not difficult to show that for any given Δ, if the pair (A,C) is observable, L can be chosen such that the two conditions in Theorem 11 are satisfied. In fact, as a direct result of observability, the first condition can be satisfied by choosing two gains L^1 and L^2 such that the matrices F^1 and F^2 are both Hurwitz. The following lemma is borrowed from [13] to address the second condition.

Lemma 5 *Let L be chosen such that*

$$Re\{\lambda_j(F^2)\} < \gamma < Re\{\lambda_j(F^1)\} < 0, \quad j = 1,2,...,n$$

for some $\gamma < 0$. *Then* $det[T \quad e^{F\Delta}T] \neq 0$ *for almost all* $\Delta \in \mathbb{R}^+$.

Lemma 5 states that for almost all arbitrary positive values of Δ, one can guarantee finite convergence time by a suitable choice of L. To this end, it suffices to choose L^1 and L^2 such that the eigenvalues of F^1 and F^2 have strictly negative real parts and are ordered according to the condition of Lemma 5.

Now, let $z(t) = T\hat{x}(t_0)$, where $t \in [-\Delta + t_0, t_0]$ and $\hat{x}(t_0)$ is the initial estimate of $x(t)$. Then the observation error defined as $\tilde{x} = x - \hat{x}$ is given by [13]:

$$\tilde{x}(t) = \begin{cases} Ke^{F(t-t_0)}T\tilde{x}(t_0), & t_0 \leq t < t_0 + \Delta \\ 0, & t \geq t_0 + \Delta \end{cases} \tag{4.3}$$

In the following section, the finite-time observers theory will be employed to design an observer for linear switched systems.

4.3 Finite-Time Observers for Switched Systems

Consider a class of switched linear continuous-time systems with N modes of operation described by

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \tag{4.4a}$$

$$y(t) = C_{\sigma(t)}x(t) \tag{4.4b}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $u \in \mathbb{R}^m$ denoting the state, the output and the input of the system, respectively. $\sigma : [0,\infty) \rightarrow \bar{\text{N}}$, where $\bar{\text{N}} = \{1,2,...,\text{N}\}$, is a piecewise constant function of time called switching signal. The function σ maps the time axis into the index set $\{1,2,...,N\}$ whose elements represent the index of the active LTI system at any given time. The switching instants are determined by a sequence $\{t_1,...,t_l,t_{l+1},...\}$ which can have infinitely many members. The matrices A_i, B_i and C_i, $i \in \bar{\mathrm{N}}$, are constant matrices with proper dimensions.

Assumption 3 *The state in (4.4) is continuous for any control input; i.e. there is no jump in the state of the system at the switching instants.*

4.3.1 Proposed finite-time switching observer

In the sequel, an observer is introduced for the switched system (4.4) which under certain conditions observes the states of the system with finite convergence time.

Assume that in an arbitrary $[t_l,t_{l+1})$ the i-th mode is active; i.e. $\sigma(t) = i$, $t \in [t_l,t_{l+1})$. The switched system in this interval can be described by

$$\dot{x}(t) = A_i x(t) + B_i u(t) \tag{4.5a}$$

$$y(t) = C_i x(t) \tag{4.5b}$$

A finite-time observer for the i-th mode, denoted by O_i, $i \in \bar{\mathrm{N}}$, can be constructed as follows

$$\dot{z}_i = F_i z_i + L_i y + G_i u, \qquad t_l \leq t < t_{l+1} \tag{4.6a}$$

$$\widehat{x}(t) = K_i[z_i(t) - e^{F_i \Delta_i} z_i(t-\Delta_i)] \tag{4.6b}$$

where

$$F_i = \begin{bmatrix} F_i^1 & 0 \\ 0 & F_i^2 \end{bmatrix}, \quad L_i = \begin{bmatrix} L_i^1 \\ L_i^2 \end{bmatrix}$$

$$G_i = \begin{bmatrix} B_i \\ B_i \end{bmatrix}, \quad T = \begin{bmatrix} I_{n\times n} \\ I_{n\times n} \end{bmatrix}, \quad z_i = \begin{bmatrix} z_i^1 \\ z_i^2 \end{bmatrix}$$

and

$$F_i^j = A_i - L_i^j C_i, \qquad j = 1,2 \tag{4.7a}$$

$$\dot{z}_i^j(t) = F_i^j z_i^j(t) + L_i^j y + B_i u, \qquad j = 1,2 \tag{4.7b}$$

$$K_i = [I_{n,n} \;\; 0_{n,n}][T \;\; e^{F_i \Delta_i} T]^{-1} \tag{4.7c}$$

Assumption 4 *The i-th mode is assumed to be observable for all $i \in \bar{\mathrm{N}}$.*

Remark 10 *It is to be noted that by the above assumption, the condition of Lemma 5 holds and as a result of Theorem 11, the existence of the proposed finite-time observer for each mode is guaranteed.*

Combining these N finite-time observers, it is possible to construct an observer, denoted by O, for the switched system. For each mode, a finite-time observer O_i is employed and a switching observer is constructed accordingly. The following initial conditions for the i-th Luenberger observers at the switching instants t_l are considered

$$z_i(t) = T\widehat{x}(t_l^-), \qquad t \in [t_l - \Delta_i, t_l] \tag{4.8}$$

where the index of the active mode is equal to i for $t \in [t_l, t_{l+1})$, or equivalently $i = \sigma(t_l^+)$.

Lemma 6 *The state estimation $\widehat{x}(t)$ in the proposed observer with the initial condition (4.8) is continuous; i.e. $\widehat{x}(t_l^+) = \widehat{x}(t_l^-)$*

Proof: From (4.6b)

$$\widehat{x}(t_l^+) = K_i[z_i(t_l) - e^{F_i\Delta_i}z_i(t_l - \Delta_i)] \tag{4.9}$$

From (4.8), by substituting $z_i(t_l) = z_i(t_l - \Delta_i) = T\widehat{x}(t_l^-)$ in (4.9), it follows that

$$\widehat{x}(t_l^+) = K_iT\widehat{x}(t_l^-) - K_ie^{F_i\Delta_i}T\widehat{x}(t_l^-) \tag{4.10}$$

Since $K_iT = I$ and $K_ie^{F_i\Delta_i}T = 0$, it follows that $\widehat{x}(t_l^+) = \widehat{x}(t_l^-)$.

It can be concluded from the above lemma that the observation error of the switching observer, denoted by $\tilde{x} = x - \widehat{x}$, is continuous as well. Furthermore, if $t_{l+1} - t_l \leq \Delta_i$, the observation error is

$$\tilde{x}(t) = K_ie^{F_i(t-t_l)}T\tilde{x}(t_l), \qquad t_l \leq t < t_{l+1} \tag{4.11}$$

On the other hand, if $t_{l+1} - t_l > \Delta_i$, the observation error can be described as follows

$$\tilde{x}(t) = \begin{cases} K_ie^{F_i(t-t_l)}T\tilde{x}(t_l), & t_l \leq t < t_l + \Delta_i \\ 0, & t_l + \Delta_i \leq t < t_{l+1} \end{cases} \tag{4.12}$$

This completes the proof. ■

In the forthcoming theorem, it is shown that if one of the modes of the switched system is active for an interval longer than Δ_i, where i is the index of the active modes of operation, the exact value of the states of the system is extracted by the proposed observer O. It is also shown that regardless of the future switches, the state observation error will stay at zero.

Theorem 12 *The proposed switching observer O observes the states of the system described by (4.4) in finite time with no observation error, provided that there exist two consecutive switching instants t_l and t_{l+1} such that*

$$t_{l+1} - t_l \geq \Delta_i \tag{4.13}$$

the index i is equal to $\sigma(t_l^+)$, as noted before.

Proof: It follows from (4.13) and (4.12) that

$$\tilde{x}(t_{l+1}^-) = 0$$

On the other hand, the continuity of $\tilde{x}(t)$ yields $\tilde{x}(t_{l+1}) = 0$.

Consider now the next switching interval, i.e. $[t_{l+1}, t_{l+2})$, and let $\sigma(t_{l+1}^+)$ be denoted by j, where $j \in W$. If $t_{l+2} - t_{l+1} \leq \Delta_j$, the observation error is

$$\tilde{x}(t) = K_j e^{F_j(t-t_{l+1})} T \tilde{x}(t_{l+1}), \qquad t_{l+1} \leq t < t_{l+2} \tag{4.14}$$

while if $t_{l+2} - t_{l+1} > \Delta_j$, the observation error is

$$\tilde{x}(t) = \begin{cases} K_j e^{F_j(t-t_{l+1})} T \tilde{x}(t_{l+1}), & t_{l+1} \leq t < t_{l+1} + \Delta_j \\ 0, & t_{l+1} + \Delta_j \leq t < t_{l+2} \end{cases} \tag{4.15}$$

In either case, since $\tilde{x}(t_{l+1})$ is zero, it can be deduced from (4.14) and (4.15) that $\tilde{x}(t) = 0$ for all $t \in [t_{l+1}, t_{l+2})$, and by continuity of $\tilde{x}$, $\tilde{x}(t_{l+2}) = 0$. It follows by induction that $\tilde{x}(t)$ remains zero for all subsequent switching intervals, i.e. $\tilde{x}(t) = 0$ for all $t \geq t_{l+1}$. This completes the proof. ■

Remark 11 *If the condition (4.13) holds, then the finite convergence time in the proposed switching observer O is $t_l + \Delta_{\sigma(t_l^+)}$, where $[t_l, t_{l+1})$ is the first interval which satisfies $t_{l+1} - t_l \geq \Delta_{\sigma(t_l^+)}$.*

Theorem 12 implies that after having a switching interval longer than Δ_i the observation error in the switched observer O remains zero, and thus the observation error is bounded. This in turn means that if (4.13) is satisfied no other conditions are needed to be imposed on the design parameters to achieve a bounded observation error. However, this is only true if no uncertainties or source of disturbance exist in the system. Thus, regardless of the fact that the condition of Theorem 12 given in (4.13) holds or not, it is desired to obtain each of the finite-time observers O_i such that the error dynamics of O is stable. In the following subsection, an algorithm is proposed which results in the stability of the error dynamics of O under constrained switching.

4.3.2 Stability of state observation error dynamics

Definition 1: Corresponding to the switching observer O, consider a switched system $\mathscr{E}$ described by

$$\begin{aligned} \dot{\theta}(t) &= F_i\theta(t), \qquad t_l \leq t < t_{l+1} \\ \theta(t_l{}^+) &= T\tilde{x}(t_l^-) \end{aligned} \tag{4.16}$$

where $\sigma(t_l^+) = i$ and $\tilde{x}(t)$ is the observation error in O.

Remark 12 *$\theta(t)$ in (4.16) is not necessarily continuous and might jump at the switching instants. In other words, the switched system $\mathscr{E}$ is impulsive.*

Definition 2: Assume a class of switching signals denoted by $\mathscr{X}$. The switched system $\mathscr{E}$ is globally uniformly asymptotically stable over $\mathscr{X}$ if there exists a class KL function β [23], such that for all switching signals belonging to $\mathscr{X}$, θ in (4.16) satisfies the following [23]

$$\|\theta(t)\| \leq \beta(\|\theta(t_0)\|,t), \qquad \forall t \geq t_0 \tag{4.17}$$

On the other hand, the switched system $\mathscr{E}$ is globally uniformly exponentially stable over $\mathscr{X}$, if there exist two scalars $\beta_1 > 0$ and $\beta_2 > 0$ such that for all switching signals belonging to $\mathscr{X}$, θ satisfies [23]

$$\|\theta(t)\| \leq \beta_1\|\theta(t_0)\|e^{-\beta_2(t-t_0)}, \qquad \forall t \geq t_0$$

Remark 13 *Similar definitions of asymptotic and exponential stability can be considered for the observation error dynamics of the switching observer O.*

Lemma 7 *State observation error dynamics of O is globally uniformly asymptotically stable if the switched dynamic system $\mathscr{E}$ is globally uniformly asymptotically stable.*

Proof:It follows from (4.16) that

$$\theta(t) = e^{F_i(t-t_l)} T\tilde{x}(t_l), \qquad t_l \le t < t_{l+1}$$

Note that $i = \sigma(t_l^+)$, compare the above equation with (4.11) and (4.12). If $t_{l+1} - t_l \le \Delta_i$, the observation error is

$$\tilde{x}(t) = K_i\theta(t), \qquad t_l \le t < t_{l+1} \tag{4.18}$$

On the other hand, if $t_{l+1} - t_l > \Delta_i$, the observation error can be obtained as following

$$\tilde{x}(t) = \begin{cases} K_i\theta(t), & t_l \le t < t_l + \Delta_i \\ 0, & t_l + \Delta_i \le t < t_{l+1} \end{cases} \tag{4.19}$$

Regardless of whether or not $t_{l+1} - t_l \le \Delta_i$, the following inequality can be deduced from (4.18) and (4.19):

$$\|\tilde{x}(t)\| \le K_M \|\theta(t)\|, \qquad \forall t \ge t_0 \tag{4.20}$$

where $K_M = \max \|K_i\|$, $i \in W$. Since θ in (4.16) is assumed to be globally uniformly asymptotically stable, there exists a class KL function β for which (4.17) holds. It can be concluded from (4.17) and (4.20) that

$$\|\tilde{x}(t)\| \le K_M \beta(\|\theta(t_0)\|, t), \qquad \forall t \ge t_0 \tag{4.21}$$

On the other hand, $\theta(t_0) = T\tilde{x}(t_0)$. Therefore,

$$\|\tilde{x}(t)\| \le K_M \beta(\|T\tilde{x}(t_0)\|, t), \qquad \forall t \ge t_0$$

function as well. Then, the above inequality can be written as

$$\|\tilde{x}(t)\| \leq \bar{\beta}(\|\tilde{x}(t_0)\|, t), \qquad \forall t \geq t_0 \tag{4.23}$$

This implies that the error dynamics in the proposed observer is globally uniformly asymptotically stable. ■

Remark 14 *It can be similarly shown that the state observation error dynamics of O is globally uniformly exponentially stable if the switched system $\mathscr{E}$ is globally uniformly exponentially stable.*

Consider again the switched system $\mathscr{E}$ given by (4.16). Denote the active observer immediately before and after the switching instant t_l with j and i, respectively. According to (4.18), if $t_l - t_{l-1} \leq \Delta_j$, then

$$\tilde{x}(t_l) = \tilde{x}(t_l^-) = K_j\theta(t_l^-) \tag{4.24}$$

and according to (4.19), if $t_l - t_{l-1} > \Delta_j$, then $\tilde{x}(t_l) = \tilde{x}(t_l^-) = 0$. Therefore, from (4.24), the switched system $\mathscr{E}$ can be rewritten as

$$\begin{aligned} \dot{\theta}(t) &= F_i\theta(t), \qquad t_l \leq t < t_{l+1} \\ \theta(t_l{}^+) &= \begin{cases} TK_j\theta(t_l^-), & \text{if } t_l - t_{l-1} \leq \Delta_j \\ 0, & \text{if } t_l - t_{l-1} > \Delta_j \end{cases} \end{aligned} \tag{4.25}$$

The easiest way to represent slow switching is to introduce a number $\tau_d > 0$, often called *dwell time* [23], and restrict the switching signal such that the time interval between every two consecutive switching instants is greater than τ_d. Since this can be a restrictive requirement in general, one can consider the *average dwell time* instead, which allows fast switchings in some instants, provided that their effect would be compensated by sufficiently slow switchings in some other instants [23].

every two consecutive switching instants is greater than τ_d. Since this can be a restrictive requirement in general, one can consider the *average dwell time* instead, which allows fast switchings in some instants, provided that their effect would be compensated by sufficiently slow switchings in some other instants [23].

Definition 3 [23]: Let the number of discontinuities of the switching signal $\sigma(t)$ on a given interval $[t_0, t)$ be denoted by $N(t, t_0)$. The signal $\sigma(t)$ is said to have an *average dwell time* τ_a if there exists two positive numbers τ_a and N_0 such that

$$N(t, t_0) \leq N_0 + \frac{t - t_0}{\tau_a} \qquad 0 \leq t_0 \leq \forall t \tag{4.26}$$

In the following, inspired by the works [16], [17], a sufficient condition for the stability of the impulsive switched system $\mathscr{E}$ is given.

Lemma 8 *Consider the switched system given by (4.25) which switches at the time instants* $\{t_1, \ldots, t_l, t_{l+1}, \ldots\}$. *Suppose that there exist a* C^1 *function* $V : \mathbb{R}^n \to \mathbb{R}$, *and two class* K_∞ *functions* α_1 *and* α_2 *for which the following inequalities hold*

$$\alpha_1(\|\theta(t)\|) \leq V(\theta(t)) \leq \alpha_2(\|\theta(t)\|), \qquad \forall t \geq t_0 \tag{4.27}$$

Assume also that there exist a number $\mu > 1$ *and a strictly negative number* λ_0 *for which the derivative of* $V(\theta(t))$ *along the solutions of the system (4.25) satisfies the inequality*

$$\dot{V}(\theta(t)) \leq 2\lambda_0 V(\theta(t)), \qquad \forall t \in (t_l, t_{l+1}), \; \forall l \in \mathbb{N}^+ \tag{4.28}$$

and

$$V(\theta(t_l{}^+)) \leq \mu V(\theta(t_l^-)), \qquad \forall l \in \mathbb{N}^+ \tag{4.29}$$

Then the switched system (4.25) is asymptotically stable for every switching signal $\sigma(t)$, *with average dwell time*

$$\tau_a > \tau_{min} = \frac{\log \mu}{-2\lambda_0} \tag{4.30}$$

Proof: It can be deduced from (4.28) and (4.29) that (see [21])

$$V(\theta(t)) \leq \mu^{N(t,t_0)} e^{2\lambda_0(t-t_0)} V(\theta(t_0))$$

(Note that $N(t,t_0)$ is the number of switchings in the interval $[t_0,t)$). Using the definition of average dwell time in (4.26) and replacing the minimum value of average dwell time given by (4.30), it follows that there must exist a positive number ε such that

$$N(t,t_0) \leq (\frac{-2\lambda_0}{\log\mu} - \varepsilon)(t-t_0) + N_0$$

Therefore

$$V(\theta(t)) \leq \mu^{N_0} \mu^{-\varepsilon(t-t_0)} V(\theta(t_0))$$

Now it can be concluded from (4.27) that

$$\|\theta(t)\| \leq \alpha_1{}^{-1}(\mu^{N_0}\mu^{-\varepsilon(t-t_0)}\alpha_2(\|\theta(t_0)\|))$$

Let $\beta(\|\theta(t_0)\|,t)$ be equal to $\alpha_1{}^{-1}(\mu^{N_0}\mu^{-\varepsilon(t-t_0)}\alpha_2(\|\theta(t_0)\|))$. Since α_1 and as a result $\alpha_1{}^{-1}$ and α_2 are all class K_∞ functions and ε is positive, one can verify that β is a class KL function. This completes the proof. ■

Remark 15 *If the conditions of the above lemma hold for a quadratic Lyapunov function, the switched system (4.25) will be exponentially stable.*

Remark 16 *If the inequality (4.29) holds for some $0 < \mu \leq 1$ in Lemma 8, then it can be shown that the switched system given by (4.25) is globally uniformly asymptotically stable for every arbitrary switching signal [17].*

In the remainder of this subsection, a LMI-based algorithm is introduced to design the proposed finite-time observer O such that the stability of the observation error dynamic under the is guaranteed. It can be inferred from (4.11) and (4.12) that the dynamics of the error in each observer O_i is determined by eigenvalues of F_i. In this algorithm, the following assumptions on the location of eigenvalues of matrices F_i in s-plane are imposed

- The eigenvalues of F_i are placed in the left of the line $\mathrm{Re}\{s\} = \lambda_0$, where λ_0 is a given strictly negative value. This means that the error in the Luenberger observers converges exponentially to zero with rate of convergence greater than λ_0.

- The eigenvalues of F_i are placed in the right of the line $\mathrm{Re}\{s\} = \gamma$, where $\gamma < \lambda_0 < 0$ is given. It is worth mentioning that if the poles are placed very far from the $j\omega$ axis, the resultant gains L_i will be large. This can lead to a design highly sensitive to the numerical errors.

- For any given $\rho \in (\gamma, \lambda_0)$, the eigenvalues of F_i^1 are placed between the lines $\mathrm{Re}\{s\} = \rho$ and $\mathrm{Re}\{s\} = \lambda_0$, while the eigenvalues of F_i^2 are placed between the lines $\mathrm{Re}\{s\} = \gamma$ and $\mathrm{Re}\{s\} = \rho$. Without loss of generality, assume that $\rho = \frac{\lambda_0+\gamma}{2}$. This implies that the available space between the lines $\mathrm{Re}\{s\} = \gamma$ and $\mathrm{Re}\{s\} = \lambda_0$ is assumed to be shared equally by F_i^1 and F_i^2.

On defining the regions R_1 and R_2 as follows

$$
\begin{aligned}
R_1 &= \{s|\rho < \mathrm{Re}\ \{s\} < \lambda_0\} \\
R_2 &= \{s|\gamma < \mathrm{Re}\ \{s\} < \rho\}
\end{aligned}
\tag{4.31}
$$

one can conclude from the above assumption that eigenvalues of F_i^1 and F_i^2 lie in regions R_1 and R_2, respectively. Regions R_1 and R_2 are depicted in Fig. 4.2.

Algorithm 1: Consider the switched system described by (4.4).

Step 1: Find $2N$ matrices X_i^1 and X_i^2, $i \in \bar{\mathrm{N}}$, and two positive definite symmetric matrices

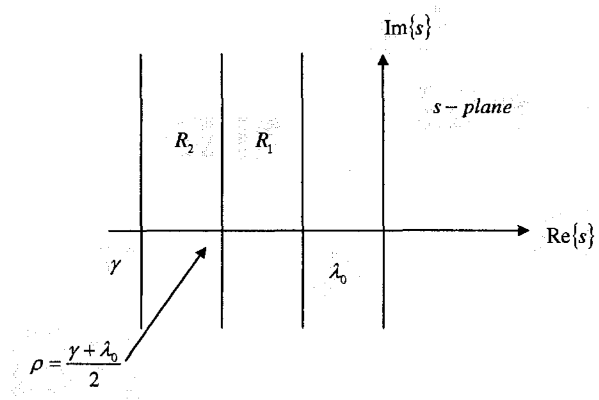

Figure 4.2: The location of the eigenvalues of the matrices F_i^1 and F_i^2.

P_1 and P_2 which satisfy the following LMIs

$$A_i^T P_1 + P_1 A_i - C_i^T X_i^{1^T} - X_i^1 C_i - 2\lambda_0 P_1 < 0 \tag{4.32a}$$

$$-A_i^T P_1 - P_1 A_i + C_i^T X_i^{1^T} + X_i^1 C_i + (\lambda_0 + \gamma) P_1 < 0 \tag{4.32b}$$

$$A_i^T P_2 + P_2 A_i - C_i^T X_i^{2^T} - X_i^2 C_i - (\lambda_0 + \gamma) P_2 < 0 \tag{4.32c}$$

$$-A_i^T P_2 - P_2 A_i + C_i^T X_i^{2^T} + X_i^2 C_i + 2\gamma P_2 < 0 \tag{4.32d}$$

To find the LMI variables X_i^1, X_i^2, P_1 and P_2, one can use the LMI toolbox of MATLAB.

Step 2: Using the matrices P_1, P_2, X_i^1 and X_i^2, $i \in \bar{\mathrm{N}}$ obtained in Step 1, find L_i, the observer gain of O_i proposed in (4.6) as follows

$$L_i = \begin{bmatrix} L_i^1 \\ L_i^2 \end{bmatrix} = \begin{bmatrix} P_1^{-1} X_i^1 \\ P_2^{-1} X_i^2 \end{bmatrix}, \qquad i \in \bar{\mathrm{N}} \tag{4.33}$$

Define P and Y_i as

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \tag{4.34}$$

$$Y_i = T K_i, \qquad i \in \bar{\mathrm{N}} \tag{4.35}$$

where K_i is given in (4.7c).

Step 3: Find the minimum value of μ subject to

$$\mu P - Y_i^T P Y_i > 0, \qquad \forall i \in \bar{\mathrm{N}} \tag{4.36}$$

It is to be noted that this minimization can be formulated as a mincx problem. (MATLAB can solve this problem efficiently). Moreover, denote the optimum of the above convex optimization problem with μ^*.

Step 4: If $\mu^* > 1$, compute the minimum allowable dwell time

$$\tau_{min} = \frac{\log \mu^*}{-2\lambda_0} \tag{4.37}$$

Remark 17 *It can be shown that for any μ satisfying*

$$\mu \geq \frac{\lambda_{max}(Y_i^T P Y_i)}{\lambda_{min}(P)}, \qquad \forall i \in \bar{\mathrm{N}}$$

the inequality (4.36) holds. In other words, the optimization problem given in Step 3 of Algorithm 1 always has a feasible solution.

Regarding to this algorithm we have the following result.

Theorem 13 *If there exists a symmetric positive definite matrix $P > 0$, and $2N$ matrices X_i^1, X_i^2, $i \in \bar{\mathrm{N}}$, which satisfy the LMIs (4.32a)-(4.32d), then the finite-time switching observer O obtained from Algorithm 1 possesses the following properties:*

i) Each observer O_i, $i \in \bar{\mathrm{N}}$, obtained from Algorithm 1 observes the state of the i-th mode in finite time.

ii) For each observer O_i, $i \in \bar{\mathrm{N}}$, obtained from Algorithm 1, the eigenvalues of the matrices F_i^1 and F_i^2 are placed in the regions R_1 and R_2, respectively (Fig. 4.2).

iii) If $\mu^* > 1$, the error dynamic in the switching observer O is globally uniformly exponentially stable for the switching signal $\sigma(t)$ with any average dwell time τ_a greater *than τ_{min} given by (4.37). Otherwise ($0 < \mu^* \leq 1$), the error dynamic in the switching observer O is globally uniformly exponentially stable for arbitrary switching signals.*

Proof: According to (4.33), since $X_i^1 = P_1 L_i^1$ and $X_i^2 = P_2 L_i^2$, (4.32a)-(4.32d) can be written as

$$A_i^T P_1 + P_1 A_i - C_i^T L_i^{1T} P_1 - P_1 L_i^1 C_i - 2\lambda_0 P_1 < 0 \tag{4.38a}$$

$$-A_i^T P_1 - P_1 A_i + C_i^T L_i^{1T} P_1 + P_1 L_i^1 C_i + (\lambda_0 + \gamma) P_1 < 0 \tag{4.38b}$$

$$A_i^T P_2 + P_2 A_i - C_i^T L_i^{2T} P_2 - P_2 L_i^2 C_i - (\lambda_0 + \gamma) P_2 < 0 \tag{4.38c}$$

$$-A_i^T P_2 - P_2 A_i + C_i^T L_i^{2T} P_2 + P_2 L_i^2 C_i + 2\gamma P_2 < 0 \tag{4.38d}$$

From the definition of F_i^1 and F_i^2 given in (4.7a), the inequalities (4.38a)-(4.38d) can be written as

$$(F_i^1 - \lambda_0 I)^T P_1 + P_1 (F_i^1 - \lambda_0 I) < 0 \tag{4.39a}$$

$$(\frac{\gamma + \lambda_0}{2} I - F_i^1)^T P_1 + P_1 (\frac{\gamma + \lambda_0}{2} I - F_i^1) < 0 \tag{4.39b}$$

$$(F_i^2 - \frac{\gamma + \lambda_0}{2} I)^T P_2 + P_2 (F_i^2 - \frac{\gamma + \lambda_0}{2} I) < 0 \tag{4.39c}$$

$$(\gamma I - F_i^2)^T P_2 + P_2 (\gamma I - F_i^2) < 0 \tag{4.39d}$$

It can be concluded from Lyapunov theory that if (4.39a)-(4.39d) hold, then the matrices

$$F_i^1 - \lambda_0 I, (\frac{\gamma + \lambda_0}{2} I - F_i^1), F_i^2 - \frac{\gamma + \lambda_0}{2} I, (\gamma I - F_i^2)$$

are all Hurwitz. On the other hand, it can be easily shown that for any matrix $Q \in \mathbb{R}^n$ and any scalar α, $\alpha I - Q$ is Hurwitz iff $\mathrm{Re}\{\lambda_i(Q)\} > \alpha$ and $(Q - \alpha I)$ is Hurwitz iff $\mathrm{Re}\{\lambda_i(Q)\} < \alpha$. Thus, it follows that

$$\gamma < \mathrm{Re}\{\lambda_j(F_i^2)\} < \frac{\gamma + \lambda_0}{2} < \mathrm{Re}\{\lambda_j(F_i^1)\} < \lambda_0,\ j = 1, 2, ..., n$$

Therefore, conditions of Lemma 5 hold and the first two properties of the proposed observer, (i) and (ii), are satisfied. Due to Lemma 7, to guarantee the stability of the proposed observer given by Algorithm 1, it suffices to show that the switched system $\mathscr{E}$ in (4.25) is globally uniformly asymptotically stable. Thus, it is desired to prove that for all switching signals satisfying the average dwell time $\tau_a > \frac{\log \mu}{-2\lambda_0}$, $\mathscr{E}$ is exponentially stable. To this end, according to Lemma 8, it suffices to show that there exists a Lyapunov function such that (4.27)-(4.29) are satisfied.

Consider the following quadratic Lyapunov function

$$V(\theta) = \theta^T P \theta \tag{4.40}$$

where P is obtained from (4.34) in Step 2. Since

$$\lambda_{min}(P)\,\|\theta\|^2 < V(\theta) < \lambda_{max}(P)\,\|\theta\|^2$$

then the Lyapunov function V satisfies (4.27), where $\alpha_1(s)$ and $\alpha_2(s)$ are defined as

$$\alpha_1(s) = \lambda_{min}(P)s^2, \qquad \alpha_2(s) = \lambda_{max}(P)s^2$$

From (4.40), it follows that $\dot{V}(\theta(t)) = \dot{\theta}(t)^T P\theta(t) + \theta(t)^T P\dot{\theta}(t)$. Since $\dot{\theta}(t) = F_i\theta(t)$, for $t \neq t_l$, and $F_i^j = A_i - L_i^j C_i$, $j = 1,2$, it can be concluded from (4.38a) and (4.38c) that

$$\dot{V}(\theta(t)) < \theta(t)^T \begin{bmatrix} 2\lambda_0 P_1 & 0 \\ 0 & (\gamma+\lambda_0)P_2 \end{bmatrix} \theta(t), \qquad t \neq t_l$$

It is to be noted that t_l denotes the switching instants. On the other hand since $\frac{\gamma+\lambda_0}{2} < \lambda_0$, it follows that

$$\dot{V}(\theta(t)) < \theta(t)^T \begin{bmatrix} 2\lambda_0 P_1 & 0 \\ 0 & 2\lambda_0 P_2 \end{bmatrix} \theta(t) = 2\lambda_0 \theta(t)^T P \theta(t)$$

This means that $\dot{V}(\theta(t)) < 2\lambda_0 V(\theta(t))$ for all times except the switching instants, and consequently (4.28) holds.

To check if the last condition of Lemma 8 is satisfied by this choice of V, assume the active observer in the time intervals $[t_{l-1}, t_l)$ and $[t_l, t_{l+1})$ are j and i, respectively. According to (4.25), $V(\theta(t_l^+))$ can be obtained as

$$V(\theta({t_l}^+)) = \begin{cases} \theta(t_l^-)^T {K_j}^T T^T P T K_j \theta(t_l^-), & \text{if } t_l - t_{l-1} \leq \Delta_j \\ 0, & \text{if } t_l - t_{l-1} > \Delta_j \end{cases}$$

For the case when $V(\theta({t_l}^+)) = 0$, the inequality (4.29) is clearly satisfied. For the other case given above, $V(\theta({t_l}^+))$ can be rewritten as

$$V(\theta({t_l}^+)) = \theta(t_l^-)^T {Y_j}^T P Y_j \theta(t_l^-)$$

where Y_i is obtained in Step 2 (equation (4.35)). One can conclude from (4.36) that

$$V(\theta({t_l}^+)) = \theta(t_l^-)^T {Y_j}^T P Y_j \theta(t_l^-) \leq$$
$$\theta(t_l^-)^T \mu^* P \theta(t_l^-) = \mu^* V(\theta({t_l}^-))$$

Thus, all the conditions of Lemma 8 hold for the switched system $\mathscr{E}$ given by (4.25). This completes the proof. ■

4.4 Numerical Examples

To show the validity of the Theorem 12 and Algorithm 1 introduced in the previous sections, three examples with simulation results are supplied in this section.

Example 1 Consider a switched system described by

$$A_1 = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -5 \\ 0 & 1 & -2 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 10 \\ -9 \\ 10 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & -4 \\ 0 & 1 & -4 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 8 \\ -7 \\ 10 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

where the switching signal $\sigma(t) \in \{1,2\}$ is defined as

$$\begin{aligned} \sigma(t) =& 1 \quad if \quad t \in [2KT \quad 2KT+T) \\ \sigma(t) =& 2 \quad if \quad t \in [2KT+T \quad 2(K+1)T) \end{aligned}$$

where $K \in \mathbb{N}$ and T is a constant determining the speed of switching. With this assumption, the switching instances are KT, $K \in \mathbb{N}$ and thus $t_{l+1} - t_l = T$. The proposed observer with

$\gamma_2 = -6$, $\gamma_1 = -2$ and Δ_1=Δ_2=0.5 using Algorithm 1 is obtained as

$$L_{11} = \begin{bmatrix} 0.7792 \\ 0.4039 \\ -0.9717 \end{bmatrix}, \quad L_{12} = \begin{bmatrix} 17.2379 \\ -10.1714 \\ -10.3686 \end{bmatrix}$$

$$L_{21} = \begin{bmatrix} 1.1016 \\ 0.6765 \\ -2.2522 \end{bmatrix}, \quad L_{22} = \begin{bmatrix} 21.1245 \\ -22.8602 \\ -12.9180 \end{bmatrix}$$

where L_{11} and L_{12} are identity observers gains for the finite time observer designed for the first mode and L_{21}, L_{22} are the ones for the second mode.

Simulation result for $T = 0.8$ is given in Fig. 4.3, which verifies the results obtained from Theorem 12. By Theorem 12 the time of convergence should be 0.5. The simulations are repeated for $T = 0.1$ and the results are shown in Fig. 4.4. Since the dwell time is much smaller than Δ_i, $i = 1, 2$, finite time convergence is not expected but due to Theorem 13 the proposed observer is exponentially stable, which is verified by the simulation results.

Example 2 Consider the system described by

$$A_1 = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -0.45 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 6 \\ -7 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

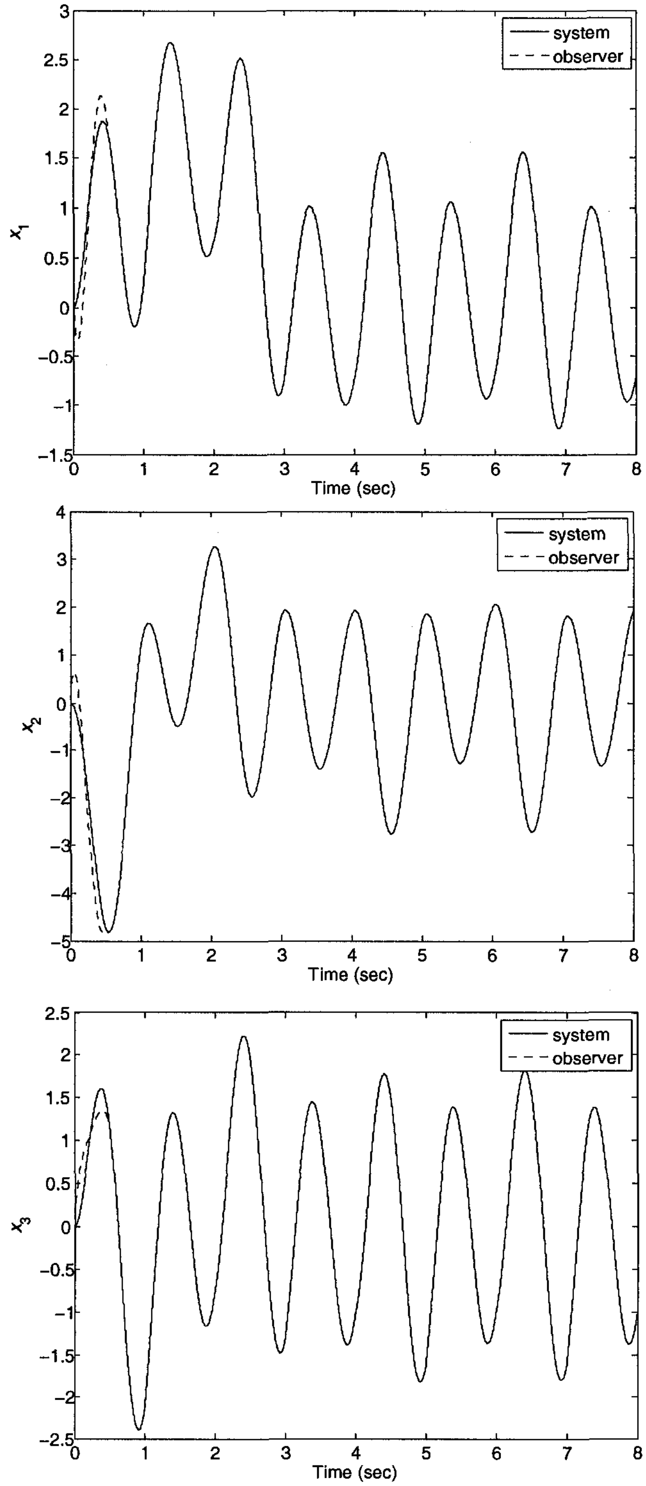

Figure 4.3: Finite time observer with T=0.8

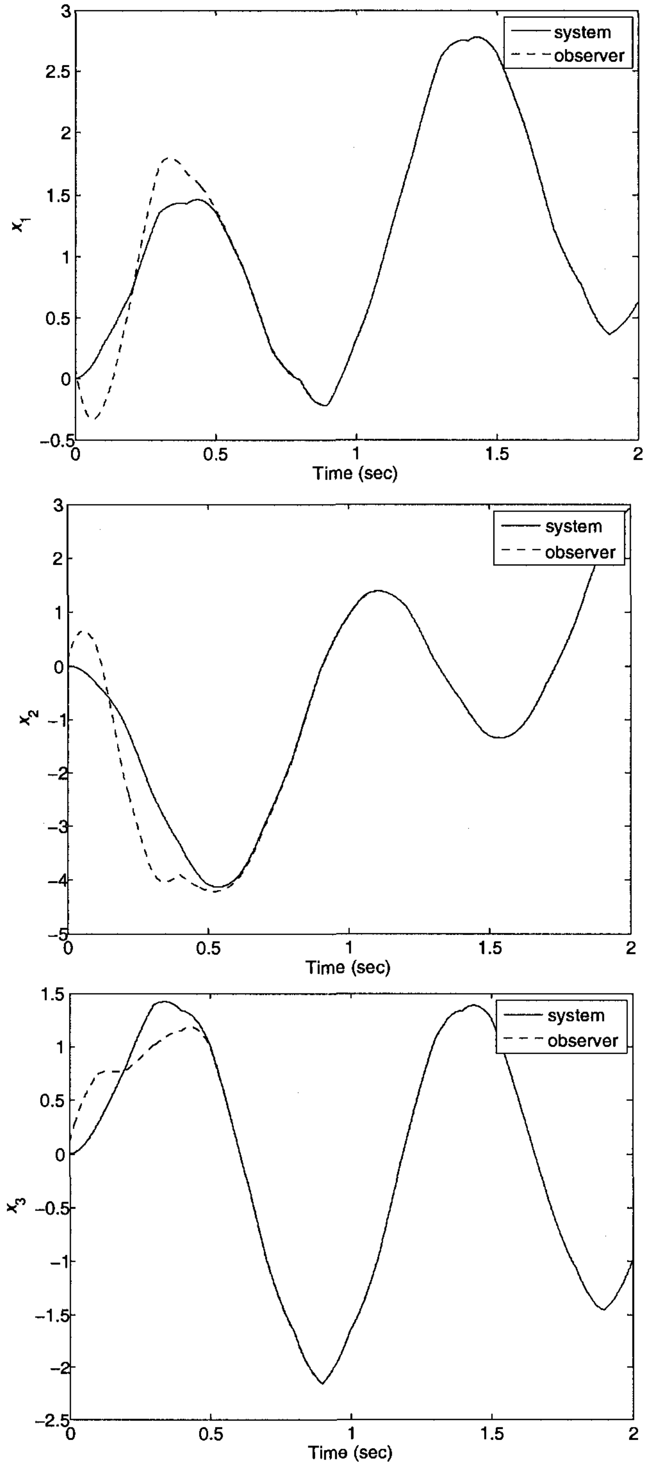

Figure 4.4: Finite time observer with T=0.1

and

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ -10 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

The proposed observer with $\gamma_2 = 2\gamma_1 = -10$, and Δ_1=Δ_2=0.5 is obtained using Algorithm 1 as follows:

$$L_{11} = \begin{bmatrix} -17.7417 \\ 28.1351 \\ 6.0227 \end{bmatrix}, \quad L_{12} = \begin{bmatrix} -27.9206 \\ 38.3612 \\ 7.4192 \end{bmatrix}$$

$$L_{21} = \begin{bmatrix} -14.4730 \\ 91.5330 \\ 6.4687 \end{bmatrix}, \quad L_{22} = \begin{bmatrix} -24.6474 \\ 107.6301 \\ 8.7014 \end{bmatrix}$$

$$\tau_{min} = 0.28$$

From Theorem 13, the minimum acceptable value for the average dwell time in the switched system to guarantee exponential stability of the proposed observer is obtained from Algorithm 1 to be $\tau_{min} = 0.28$. Simulation results for $T = 0.8 > 0.5$ are shown in Fig. 4.5. Like the previous example, finite convergence time equal to 0.5 is expected which is verified by the simulation result.

Simulations are repeated for $0.5 > T = 0.3 > \tau_{min}$, and exponential stability of the proposed observer guaranteed by Theorem 13 is verified in Fig. 4.6. In the next example

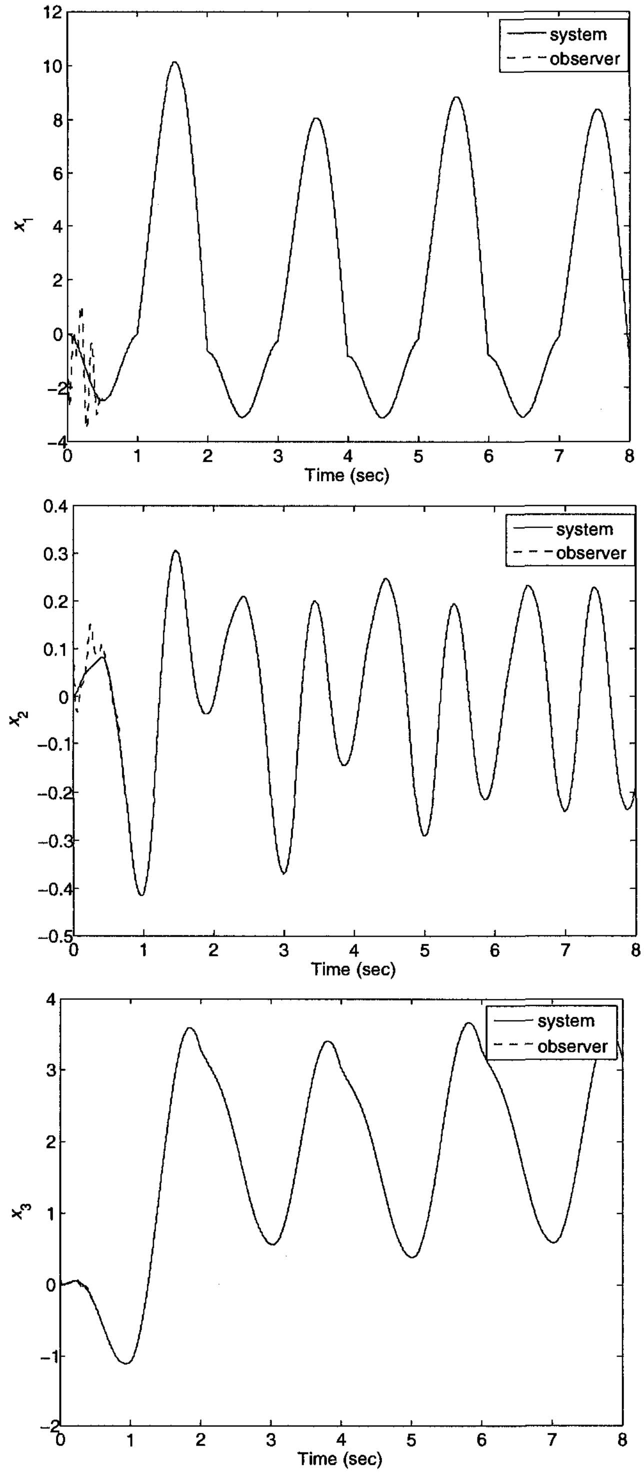

Figure 4.5: Finite time observer with T=0.8

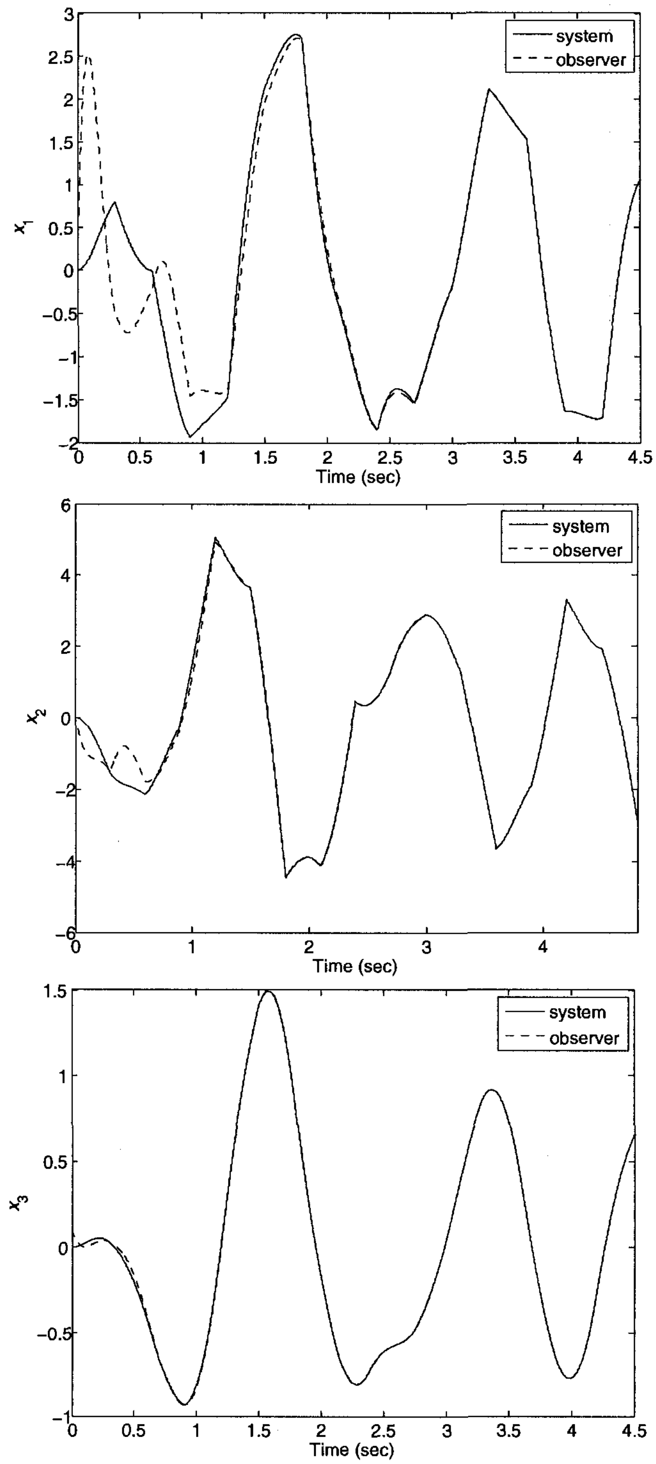

Figure 4.6: Finite time observer with T=0.3

the application of this methodology to observe the states of a switched system is applied to an industrial water tank system.

Example 3 Consider a liquid level control system shown in Fig. 4.7. The system consists of two tanks, one flow source, two outlet pipes, and one connecting pipe. The pipes contain valves that can be opened or closed by an external controller. Based on the status of each valve (opened or closed), there are eight different system modes. Consider the following three valve configurations

$$\begin{array}{lll} Mode\ 1 & R_2 : ON & R_1, R_3 : OFF \\ Mode\ 2 & R_1, R_2 : ON & R_3 : OFF \\ Mode\ 3 & R_2, R_3 : ON & R_1 : OFF \end{array}$$

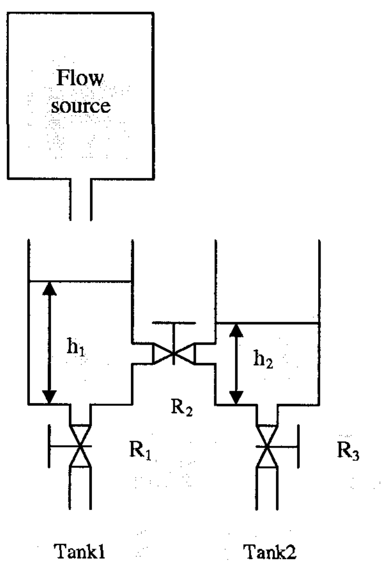

Figure 4.7: The two-tank system of Example 1.

It is assumed that the flow through the valves is laminar, which implies that the relation between the flow rate in the valves and the height of the liquid is linear [31]. Depending on the value of the tank capacity C_T and the pipe resistance R in each mode the behavior of the system is governed by a different differential equation. The state space representation of the system is given by

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \tag{4.43a}$$

$$y(t) = C_{\sigma(t)}x(t) \tag{4.43b}$$

where the state $x(t) = [h_1(t) \ \ h_2(t)]^T$ contains the heights of liquid in the tanks, $C_{\sigma(t)} = [0 \ \ 1]$, and $u(t) = 5e^{-0.5t}(1 + sin5\pi t)$ is the input flow from the flow source to tank 1. The switching signal $\sigma(t)$ in this example is a piecewise constant function with the set of images equal to $\{1,2,3\}$. Consider the following values for the system parameters:

$$C_{T1} = 5\ m^2, \quad C_{T2} = 3\ m^2, \quad R_1 = R_2 = 300\ \frac{s}{m^2}, \quad R_3 = 100\ \frac{s}{m^2}$$

For the three modes defined above, one can obtain

$$A_1 = \begin{bmatrix} -0.0007 & 0.0007 \\ 0.0011 & -0.0011 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.0013 & 0.0007 \\ 0.0011 & -0.0011 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.0007 & 0.0007 \\ 0.0011 & -0.0044 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$$

Algorithm 1 is now applied to this problem to observe the height of the liquid in each tank while the valve configuration can jump between the three given modes. Set $\lambda_0 = -2$, $\gamma = -8$, $\rho = -5$, and let the desired finite convergence time for each observer O_i be $\Delta_1 = \Delta_2 = \Delta_3 = 0.7$. On solving the set of LMIs in Step 1, the observer gains L_i, $i = 1,\ 2,\ 3$, are calculated in Step 2 of Algorithm 1. One can verify that the minimum lower bound of the average dwell time required for the stability of the estimation error dynamic in the proposed observer O is equal to 0.4010.

Now, let the switching between three modes be governed by the switching signal $\sigma_1(t)$ in Fig. 4.8(a). Applying the observer O to the system, the estimates of h_1 and h_2 are obtained and depicted in Fig. 4.8(b) and (c) which comply with the result of Theorem 12. Since the condition of Theorem 12 holds in the interval $[0,1]$ (as the time interval is greater than $\Delta = 0.7$), the state estimation error becomes zero at $t = 0.7$ and stays at zero for $t \geq 0.7$. This means the finite convergence time is 0.7. The simulations are repeated for the case when the switching signal is $\sigma_2(t)$. Since this switching signal does not satisfy the condition of Theorem 12, finite convergence time is not achieved, but as expected from Lemma 8, the estimation error is exponentially stable. The results obtained in this case are depicted in Fig. 4.9.

As mentioned earlier, there are two Luenberger observers in the structure of the proposed finite-time observer O. To compare the performance of the proposed observer O and a single Luenberger observer, assume that only mode 1 (associated with the configuration of the valves) is active. In this case estimates of h_1 for the proposed finite-time observer and the faster Luenberger observer within the structure of the observer O are compared in Fig. 4.10. Moreover, assume the switching signal is the same as before (Fig. 4.8a), the state estimate observed by the classic Luenberger switching observer and the proposed finite time switching observer are given in Fig. 4.11. It is perceived from this figure that the transient response of the proposed finite-time observer is superior. It is perceived from this figure that the transient response of the proposed finite-time observer is superior.

4.5 Discrete Finite Time Observer (DFTO)

In this section, the idea of finite time observers for linear continuous time systems is extended to linear time invariant discrete time systems. The main motivation for this extension

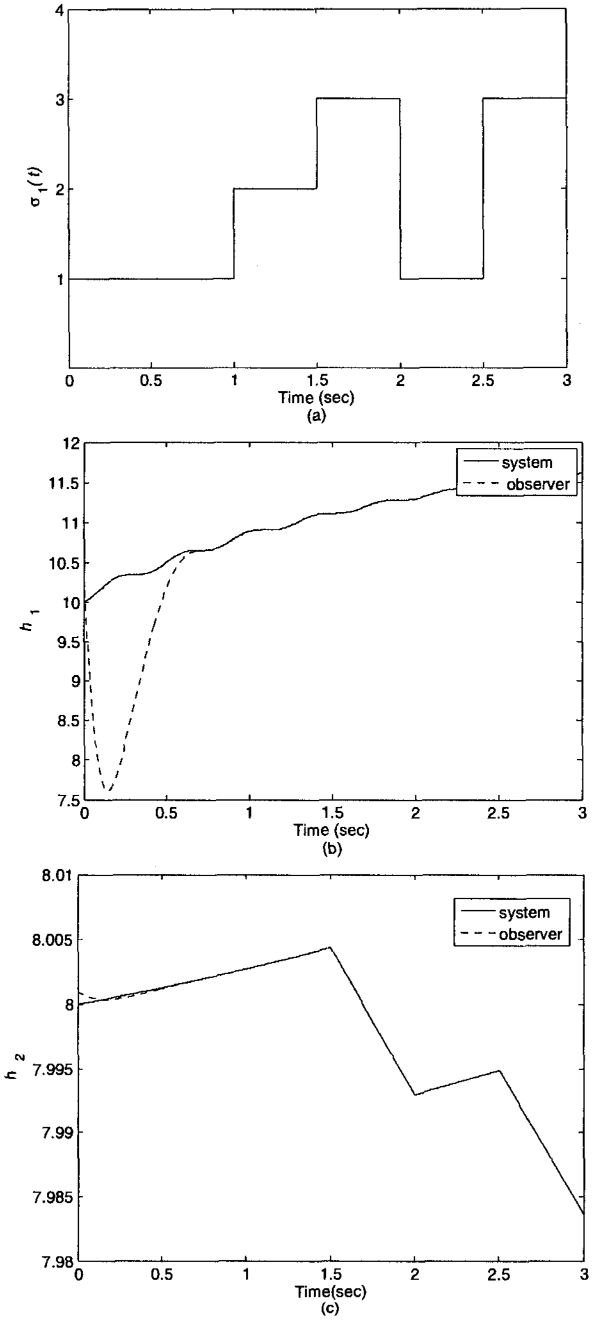

Figure 4.8: (a) Switching signal $\sigma_1(t)$; (b) finite convergence time to observe h_1; (c) finite convergence time to observe h_2.

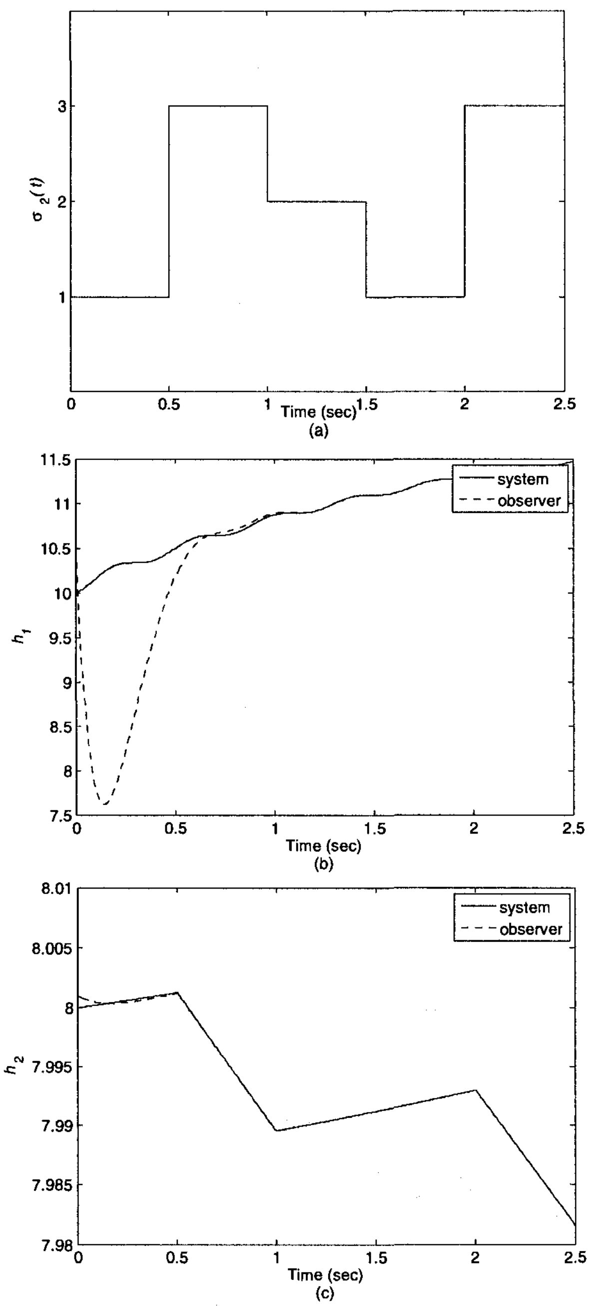

Figure 4.9: (a) Switching signal $\sigma_2(t)$; (b) no finite convergence time to observe h_1; (c) no finite convergence time to observe h_2.

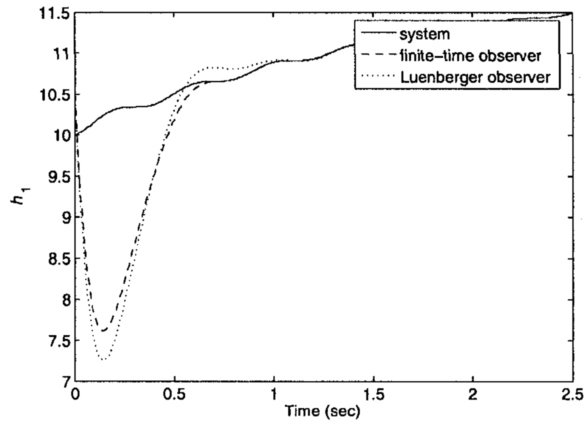

Figure 4.10: Comparison between the proposed finite-time observer and Luenberger observer.

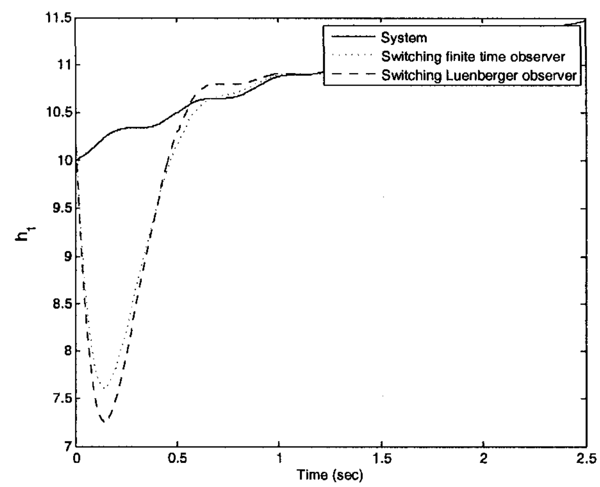

Figure 4.11: Comparison between the proposed switching finite-time observer and classic switching observer.

is that unlike dead-beat observers designed for discrete time systems, the proposed observer in this section need not place all the eigenvalues at the origin. This leads to a much more flexible design compared to the existing techniques.

Consider an observable linear time invariant (LTI) discrete time system

$$x(k+1) = Ax(k) + Bu(k), \qquad x(k_0) = k_0, \qquad k \geq k_0 \tag{4.44a}$$

$$y(k) = Cx(k) \tag{4.44b}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $u \in \mathbb{R}^m$ are the state, the output and the input of the system, respectively. The theory of dead-beat observers for such linear discrete time systems is well studied in the literature see [32], [35]. In such observers, by suitable choice of gains the error in the observation converges to zero in finite time $k \geq n$, where n is the order of the system. Unlike a dead-beat observer which is in essence a Luenberger observer with all the eigenvalues placed at the origin, in the developed observer in this section there is no necessity to place the eigenvalues at the origin, and they can be placed almost anywhere inside the unity circle. Consider a linear time invariant discrete time system defined by (4.44). Then

$$z^i(k+1) = F^i z^i(k) + L^i y(k) + Gu(k), \qquad (i = 1, 2)$$

represents two identity Luenberger observers for the system, where $F^i := A - L^i C$, $i = 1, 2$. Let

$$F = \begin{bmatrix} F^1 & 0 \\ 0 & F^2 \end{bmatrix} \quad H = \begin{bmatrix} L^1 \\ L^2 \end{bmatrix}$$

$$G = \begin{bmatrix} B \\ B \end{bmatrix} \quad T = \begin{bmatrix} I_{n\times n} \\ I_{n\times n} \end{bmatrix} \quad z = \begin{bmatrix} z^1 \\ z^2 \end{bmatrix}$$

By combining these two identity Luenberger observers and introducing a delay $\Delta \in \mathbb{Z}^+$ in the structure of the observer, a new state estimate $\hat{x}$ can be generated as follows

$$z(k+1) = Fz(k) + Hy(k) + Gu(k), \qquad k \geq k_0 \tag{4.45a}$$

$$\hat{x}(k+1) = K[z(k+1) - F^{\Delta}z(k+1-\Delta)] \tag{4.45b}$$

The structure of this observer is shown in Fig. 4.12.

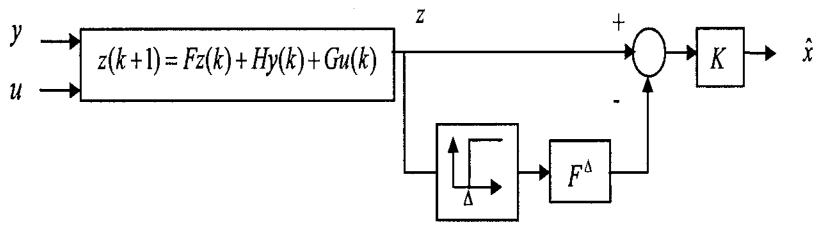

Figure 4.12: Structure of discrete finite-time observer.

Theorem 14 *If L and Δ are chosen such that*

i. F is Hurwitz, i.e., all eigenvalues of F are inside the unity circle;

ii. $\det[T \quad F^{\Delta}T] \neq 0$,

then the observer given by (4.45a) and (4.45b) with $H := [I_{n,n} \quad 0_{n,n}][T \quad F^{\Delta}T]^{-1}$ *observes the states of the system in (4.44) exactly within the finite time Δ, i.e. that* $\hat{x}(k) = x(k)$ *for* $k \geq k_0 + \Delta$.

Proof: for $k \geq k_0$, since $G = TB$ and $FT = TA - HC$, from (4.44) and (4.45) one can obtain

$$\begin{aligned} z(k+1) - Tx(k+1) &= Fz(k) + Hy(k) + Gu(k) - T(Ax(k) + Bu(k)) \\ &= F[z(k) - Tx(k)] + [FT - TA + HC]x(k) \\ &+ [G - TB]u(k) \\ &= F[z(k) - Tx(k)] \end{aligned} \tag{4.46}$$

and therefore

$$z(k) - Tx(k) = F^{\Delta}[z(k-\Delta) - Tx(k-\Delta)] \qquad k \geq k_0 + \Delta \tag{4.47}$$

By the second condition of the theorem, $[T \quad F^{\Delta}T]$ is assumed to be invertible, therefore by the definition of K, it is concluded that $K[T \quad F^{\Delta}T] = [I_{n,n} \quad 0_{n,n}]$. Using the fact that $KT = I_{n,n}$ and $KF^{\Delta}T = 0_{n,n}$ the equation of state observation can be rewritten as

$$\begin{aligned} \widehat{x}(k) &= K[z(k) - F^{\Delta}z(k-\Delta)] \\ &= x(k) + K[z(k) - Tx(k)] \\ &- KF^{\Delta}[z(k-\Delta) - Tx(k-\Delta)] \quad k > k_0 \end{aligned} \tag{4.48}$$

Using (4.47) implies that $\widehat{x}(k) = x(k)$ for $k \geq k_0 + \Delta$. ■

Although it is shown that by suitable choice of L and Δ the proposed observer, estimates the states of the system with no error in finite time $k = k_0 + \Delta$, it remains to show that such suitable choices exist.

Remark 18 *It is to be noted that a necessary condition for Condition i in theorem 14 to hold is the observability of the pair (A,C) in system (4.44).*

Lemma 9 *If H and Δ are chosen such that* $\det[F^{1\Delta} - F^{2\Delta}] \neq 0$ *then* $\det[T \;\; F^{\Delta}T] \neq 0$.

proof: since

$$[T \;\; F^{\Delta}T] = \begin{bmatrix} I_{n,n} & F^{1\Delta} \\ I_{n,n} & F^{2\Delta} \end{bmatrix} = \begin{bmatrix} I_{n,n} & 0_{n,n} \\ I_{n,n} & -I_{n,n} \end{bmatrix} \begin{bmatrix} I_{n,n} & F^{1\Delta} \\ 0_{n,n} & F^{1\Delta} - F^{2\Delta} \end{bmatrix}$$

it is concluded that $\det[\mathrm{T} \;\; F^{\Delta}T] = (1)(-1)^n(1)\det[F^{1\Delta} - F^{2\Delta}]$. ■

Theorem 15 *For almost all values of L^1 and L^2, $\det[T \;\; F^{\Delta}T] \neq 0$ iff $\Delta \geq n$, where n is the order of the system (4.44).*

proof: let $\Delta \geq n$, and assume L^1 and L^2 are chosen such that the matrix F^2 is nilpotent (i.e. all eigenvalues of F^2 are at the origin) while eigenvalues of F^1 are all nonzero. Then $\det[F^{1^{\Delta}} - F^{2^{\Delta}}] = \det[F^{1^{\Delta}} - 0] = \det F^{1^{\Delta}} \neq 0$. Since $\det[F^{1^{\Delta}} - F^{2^{\Delta}}]$ is an analytical function (polynomial function) of elements of the matrices L^1 and L^2, and is non zero for this specific choice of L^1 and L^2, due to the principle of isolated zeros [10] is non zero for almost all matrices L^1 and L^2.

Now let $\Delta < n$. Since the system is assumed to be observable there is a transformation matrix S such that

$$S^{-1}AS = \bar{A} = \begin{bmatrix} 0 & 0 & 0 & \cdots & -a_0 \\ 1 & 0 & 0 & \cdots & -a_1 \\ 0 & 1 & 0 & \cdots & -a_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

$$CS = \bar{C} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

For the given L^1 and L^2 define $\bar{L^1} = S^{-1}L^1$ and $\bar{L^2} = S^{-1}L^2$. It is not difficult to show that the matrices $\bar{F_1} = (\bar{A} - \bar{L^1}\bar{C})^{\Delta}$ and $\bar{F_2} = (\bar{A} - \bar{L^2}\bar{C})^{\Delta}$, depending on the value of Δ have at least one identical column; in fact one can obtain

$$\Delta + \text{number of identical columns in } \bar{F_1} \text{ and } \bar{F_2} = n$$

which results in the determinant of $[\bar{F_1} - \bar{F_2}]$ being equal to zero for all values of $\Delta < n$. For the general case one can obtain

$$\det[(A - L^1C)^{\Delta} - (A - L^2C)^{\Delta}] =$$

$$\det(S[(\bar{A} - \bar{L^1}\bar{C})^{\Delta} - (\bar{A} - \bar{L^2}\bar{C})^{\Delta}]S^{-1}) =$$

$$\det(S)\det[(\bar{A}-\bar{L}^1\bar{C})^{\Delta}-(\bar{A}-\bar{L}^2\bar{C})^{\Delta}]\det(S^{-1})$$

■

4.5.1 Error in observation

To have a clear idea about what is happening in the observer before it observes the states of the system with no error, in this subsection the value of error in the time interval $k_0 \leq k < k_0 + \Delta$ is obtained.

Assume that $z(k) = T\widehat{x}(k_0)$ for $k \in \{k_0 - \Delta + 1, ..., k_0\}$, then (4.46) gives

$$\begin{aligned} z(k) - Tx(k) &= F^{k-k_0}[z(k_0) - Tx(k_0)] \\ &= F^{k-k_0}[T\widehat{x}(k_0) - Tx(k_0)] \\ &= F^{k-k_0}T[\widehat{x}(k_0) - x(k_0)] \qquad k \geq k_0 \end{aligned}$$

and from (4.48) it follows for $k_0 < k < k_0 + \Delta$ that

$$\begin{aligned} \widehat{x}(k) &= K\left[z(k) - F^{\Delta}z(k-\Delta)\right] \\ &= x(k) + KF^{k-k_0}T\left(\widehat{x}(k_0) - x(k_0)\right) \end{aligned}$$

Therefore

$$\tilde{x}(k) = \widehat{x}(k) - x(k) = KF^{k-k_0}T\left(\widehat{x}(k_0) - x(k_0)\right) \quad k_0 < k < k_0 + \Delta$$

The error over the entire time axis is obtained as

$$\tilde{x}(t) = \begin{cases} KF^{k-k_0}T(\widehat{x}(k_0) - x(k_0)), & k_0 < k < k_0 + \Delta \\ 0, & k_0 + \Delta \leq k \end{cases} \tag{4.49}$$

Chapter 5

Conclusions

5.1 Overall Summary

In this work, necessary conditions for the stability of continuous and discrete impulsive switched systems are presented. LMI-based algorithms are developed subsequently to design observers for impulsive switched systems. These algorithms guarantee asymptotic (exponential) stability of the error dynamics in switching observers for a special class of impulsive switched systems under constrained switching. Moreover, a finite-time switching observer for a linear continuous switched system is presented. The proposed observer switches between finite-time observers, each designed for a subsystem, and observes the states of the switched system. The observation error vanishes in finite time provided that there are two consecutive switching instants with a time-gap larger than the finite convergence time of the active observer between the two instants. Regardless of this property, the observer under constrained switching will be stable if the proposed algorithm in this thesis is utilized to design the switching finite-time observer. The simulation results show the efficiency of the proposed technique in reducing the observation error to zero in finite time for different switching signals.

5.2 Future Work

Generally, regarding to the structure of switched systems and switching signals, there are two main areas that can be further investigated. The design of switching observers for impulsive switched systems when the switching signal is not necessarily available a priori is an open problem for future research. To be more specific, as in many practical systems in industry, the index of the active mode in a switched system may be known only a short time after the system has switched to the corresponding mode. Design of switching observers for impulsive switched systems while the switching signals are known after a delay is an open area of research to extend the current work. Design of switching observers for nonlinear uncertain impulsive switched systems is another interesting area for future research.

As mentioned in Chapter 4, unlike the famous dead-beat observers, the proposed discrete finite time observer in this work need not place the eigenvalues at the origin which leads to a more flexible design compared to the existing techniques. Specially in the case of discrete time switched systems if there is an interest to observe the states of the system in finite time, using the famous dead-beat observer to observe the state of each mode will restrict the designer to place all the eigenvalues of each observer on the origin and there is no guarantee for the stability of the switching observer after designing each dead-beat observer, while employing the proposed finite time observer will allow the designer to place the eigenvalues anywhere inside the unity circle. Designing discrete finite-time switching observer to observer the states of discrete switched systems is another open problem for future work.

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