The John-Nirenberg Inequality for $Q_\alpha(\mathbb{R}^n)$ and Related Problems

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ABSTRACT

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The John-Nirenberg inequality characterizes functions in the space BMO in terms of the decay of the distribution function of their oscillations over a cube. In the first part of this thesis, separate necessary and sufficient John-Nirenberg type inequalities are proved for functions in the space $Q_\alpha(\mathbb{R}^n)$. The results are a modified version of the conjecture made by Essén, Janson, Peng and Xiao, who introduce the space $Q_\alpha(\mathbb{R}^n)$. The necessity for this modification is shown by two counterexamples.

The counterexamples provide us with a borderline case function for $Q_\alpha(\mathbb{R}^n)$. In the second part, the discussion on the relation between the function and the space Q_α in a wider range is presented. Moreover, the analytic and fractal properties of the function are studied and the fractal dimensions of the graph of the function are determined. These properties and dimensions illustrate some form of regularity for functions in $Q_\alpha(\mathbb{R}^n)$.

Lastly, the relation between the tent spaces T_q^p and L^p, H^p, and BMO for $q \neq 2$ is discussed. By the theory of Triebel-Lizorkin spaces, a projection from T_q^p to L^p for $1 < p < \infty$, to H^p for $p \leq 1$, and to BMO for $p = \infty$, is shown to exist for $1 < q \leq 2$.

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Contribution of Authors

Chapter 2 and Section 3.1 of the thesis includes a joint paper with my supervisor Dr. Galia Dafni:

H. Yue and G. Dafni, *A version of the John-Nirenberg inequality for* $Q_\alpha(\mathbb{R}^n)$, submitted, 2006.

I did all proofs of the theorems concerning the inequality, and Dr. Dafni helped me with the counterexamples.

Chapter 3 and 4 include the paper:

H. Yue, *A Fractal Function Related to the John-Nirenberg Inequality for* $Q_\alpha(\mathbb{R}^n)$, Submitted, 2007.

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Chapter 1

Introduction

In 1961, John and Nirenberg [22] introduced the space BMO of functions of *bounded mean oscillation*. It consists of functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying

$$\|f\|_{\mathrm{BMO}} = \sup_I |I|^{-1} \int_I |f(x) - f(I)| dx < \infty, \tag{1.1}$$

where the supremum is taken over all cubes I in $\mathbb{R}^n$ with sides parallel to the coordinate axes, $|I|$ is the volume of I, and $f(I)$ denotes the mean value of f on the cube I: $f(I) = \frac{1}{|I|}\int_I f(x)dx$. Modulo constants, BMO becomes a Banach space under the norm defined in (1.1).

The space BMO is an extension of L^∞. From its introduction, BMO has been an important function space in the study of partial differential equations, in particular in endpoint results for Sobolev embeddings. Ten years later, C. Fefferman [17] connected BMO to harmonic analysis by identifying BMO with the dual of the real Hardy space H^1 in his famous work. This duality theorem provided a new direction for research in harmonic analysis and PDE. The year after, the subsequent paper by Fefferman and Stein [19] developed this subject, including the characterization of BMO in terms of Carleson measures. Since then, the space BMO has played a prominent role in harmonic analysis. In one dimension, it is also closely connected to function theory in the disk (see [21]). A more general version of the BMO-H^1 duality is provided by the theory of tent spaces given by Coifman, Meyer and Stein [3].

John and Nirenberg [22] characterized functions in BMO via an inequality which now bears their names. Let $\lambda_I(t)$ be the distribution function of $f - f(I)$ on the cube I:

$$\lambda_I(t) = |\{x \in I : |f(x) - f(I)| > t\}|, \tag{1.2}$$

where $|S|$ denotes the Lebesgue measure of a set S.

John-Nirenberg Inequality: *There exist two positive constants B and b (depending only on n) such that, if $f \in \mathrm{BMO}(\mathbb{R}^n)$, then for all cubes I in $\mathbb{R}^n$, and all $t > 0$,*

$$\lambda_I(t) \leq B|I| \exp(-bt/\|f\|_{\mathrm{BMO}}). \tag{1.3}$$

Conversely, if for any cube I and any $t > 0$, $\lambda_I(t) \leq Ce^{-ct}|I|$, for some positive constants C and c, then $f \in \mathrm{BMO}(\mathbb{R}^n)$ (see [23], [4]).

John and Nirenberg showed the inequality (1.3) by applying the Calderón-Zygmund decomposition [6] to $|f|$ on the cube $|I|$. Their proof was improved by Neri [23] in 1977, using the same decomposition, but in a more straightforward manner (based on an unpublished proof by Calderón).

The inequality (1.3) is equivalent to

$$|I|^{-1} \int_I e^{k|f(x)-f(I)|} dx \leq C$$

for some constants k and C.

Let $1 \leq p < \infty$, and set

$$\|f\|_{\mathrm{BMOp}} := \sup_I \left(|I|^{-1} \int_I |f(x) - f_I|^p dx \right)^{1/p}. \tag{1.4}$$

Then this defines an equivalent norm in BMO. By Jensen's inequality,

$$\|f\|_{\mathrm{BMO}} \leq \sup_I \left(|I|^{-1} \int_I |f(x) - f_I|^p dx \right)^{1/p}, \tag{1.5}$$

and conversely, the inequality

$$\|f\|_{\mathrm{BMOp}} \leq c_p \|f\|_{\mathrm{BMO}} \quad \forall\, p < \infty \tag{1.6}$$

is a corollary of the John-Nirenberg inequality (1.3). On the other hand, Fefferman and Stein (see [19]) showed that (1.6) with $c_p = cp$ implies the John-Nirenberg inequality (1.3). Moreover, they proved (1.6) via the duality of H^1 and BMO.

In recent years, a new family of function spaces, called Q spaces, was first introduced by Aulaskari, Xiao and Zhao [1] in the case of the unit disk D as the class of holomorphic functions f satisfying

$$\sup_{w \in D} \int \int_D |f'(z)|^2 [g(z, w)]^p dm(z) < \infty,$$

where $g(z, w)$ is the Green's function for D, m is the Lebesgue area measure, and $0 < p < 1$. They named the Banach spaces of such functions $Q_p(D)$. Note that these spaces are invariant under conformal mappings of the disk.

Essén and Xiao [15] showed a characterization of the boundary value of functions in Q_p:

$$\sup_I |I|^{-p} \int_I \int_I \frac{|f(e^{is}) - f(e^{it})|^2}{|e^{is} - e^{it}|^{2-p}} ds dt < \infty,$$

where the supremum is taken over all arcs $I \subset \partial D$, the unit circle.

Following their work (see [13] for more details), Essén, Janson, Peng and Xiao [14] introduced the spaces $Q_\alpha(\mathbb{R}^n)$, corresponding to a parameter $\alpha \in \mathbb{R}$:

$$Q_\alpha(\mathbb{R}^n) = \{f \in L^2_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{Q_\alpha} < \infty\},$$

where

$$\|f\|_{Q_\alpha} := \sup_I \left(\ell(I)^{2\alpha-n} \int_I \int_I \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}} dxdy \right)^{1/2}. \tag{1.7}$$

As above, the supremum is taken over all cubes I, and $\ell(I)$ denotes the side-length of I.

As is the case for BMO, (1.7) defines a norm modulo constants, and the resulting Banach spaces are in fact subspaces of BMO, which are proper and nontrivial for the range $0 \leq \alpha < 1$ (when $n \geq 2$), or $0 \leq \alpha \leq \frac{1}{2}$ (when $n = 1$) (see [14])). When $\alpha = \frac{1}{2}$ and $n = 1$, $Q_{\frac{1}{2}}(\mathbb{R}^1)$ is the same as the homogeneous fractional Sobolev space $L^2_{\frac{1}{2}}(\mathbb{R}^1)$. More generally, when $0 < \alpha < \min\{1, n/2\}$, Q_α contains the homogeneous Besov spaces $\Lambda^{n/\beta,q}_\alpha$, for all $\beta > \alpha$ and $q \leq \infty$. It was also shown by Wu and Xie [29] that Q_α spaces are closely related to Morrey spaces, which play an important role in PDE.

We will assume from now on that $0 < \alpha < \min\{1, n/2\}$. For α in this range, the spaces $Q_\alpha(\mathbb{R}^n)$ satisfy interesting analogues of the important properties of BMO, such as the relation with Carleson measures (see [14]), duality (see [8], [29]), and decomposition via wavelets or quasi-orthogonal "atoms" (see [14],[9]).

The spaces Q_α are characterized by the p-Carleson measures, with $p = 1 - 2\alpha/n$. As usual, we denote the upper half Euclidean space by $\mathbb{R}^{n+1}_+$ and points in it by (x,t) with $x \in \mathbb{R}^n$ and $t > 0$. A p-Carleson measure μ is a Borel measure defined on $\bar{\mathbb{R}}^{n+1}_+$ which satisfies $\mu(\hat{B}) \leq C|B|^p$ for some $p > 0$, where

$$\hat{B} = \{(x,t), \operatorname{dist}(x, \partial B) \geq t\} \tag{1.8}$$

is the tent over the ball B. When $p = 1$, these are the usual Carleson measures.

Dafni and Xiao [8] explored the Q_α spaces further. Using the relation between fractional Carleson measure and Hausdorff capacity, they defined a predual for Q_α spaces, called the Hardy-Hausdorff space, and showed that it contains the real Hardy space

H^1. In addition, they gave the corresponding atomic decomposition of distributions in the predual [8], and showed a decomposition of Q_α in terms of quasi-orthogonal functions [9].

While these spaces have been extensively studied recently, an important question remaining has been to find an analogue of the John-Nirenberg inequality. A conjecture proposed in [14] claims:

Conjecture 1 (EJPX). *There exist two positive constants B and b such that for any function $f \in Q_\alpha(\mathbb{R}^n)$, and any cube I in $\mathbb{R}^n$,*

$$\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} \leq Bt^{-1} \exp\left(\frac{-bt}{\|f\|_{Q_\alpha}}\right). \tag{1.9}$$

Here $\mathcal{D}_k(I)$, $k \geq 0$, denotes the collection of subcubes of I of sidelength $2^{-k}\ell(I)$, obtained by successively halving all edges of I, k times, and $\lambda_J(t)$ is the distribution function of $f - f(J)$ on the cube J, as defined in (1.2).

We prove modified, separate necessary and sufficient versions of this conjecture. First, we show that such an inequality is sufficient for a function to be in $Q_\alpha(\mathbb{R}^n)$. In fact, an even weaker version will suffice:

Theorem 1. *Let $0 \leq \alpha < 1$ $(n \geq 2)$, or $0 \leq \alpha \leq \frac{1}{2}$ $(n = 1)$, and $0 \leq p < 2$. If there exist positive constants B, C and c, such that, for all cubes $I \subset \mathbb{R}^n$, and any $t > 0$,*

$$\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} \leq B \max\{1, \left(\frac{C}{t}\right)^p\} \exp(-ct), \tag{1.10}$$

then f is a function in $Q_\alpha(\mathbb{R}^n)$.

For $f \in \mathrm{BMO}$, the John-Nirenberg inequality in the case $t \leq \|f\|_{\mathrm{BMO}}$ is trivial, since $\lambda_I(t) \leq |I|$. However, the version of the inequality we discuss involves a summation of the quotients $\frac{\lambda_J(t)}{|J|}$. If we use the trivial bound of 1 for these quotients as $t \to 0$, the sum diverges. Thus it is important to find the rate of divergence.

Our second theorem shows a necessary version of the inequality which has a higher rate of divergence as $t \to 0$:

Theorem 2. *Let $0 \le \alpha < 1$ ($n \ge 2$), or $0 \le \alpha \le \frac{1}{2}$ ($n = 1$). For any $f \in Q_\alpha(\mathbb{R}^n)$, there exist two positive constants B and b (depending only on n), such that, for all cubes $I \subset \mathbb{R}^n$,*

$$\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} \le B \max\left\{1, \left(\frac{\|f\|_{Q_\alpha}}{t}\right)^2\right\} \exp\left(\frac{-bt}{\|f\|_{Q_\alpha}}\right). \tag{1.11}$$

Note that in the case $t \le \|f\|_{Q_\alpha}$, there is a gap between Theorems 1 and 2. We could not include $p = 2$ in Theorem 1, since $p = 2$ in (2.5) is not sufficient for $f \in Q_\alpha$. On the other hand, it is impossible to sharpen $\left(\frac{\|f\|}{t}\right)^2$ to $\left(\frac{\|f\|}{t}\right)^p$ in (2.7) for any $0 \le p < 2$. This is shown by two counterexamples which give the following:

Theorem 3. *Let $0 < \alpha < \frac{1}{2}$.*
(a) *If $0 \le p < 2$, then there are no constants B and b such that the following inequality holds for all functions $f \in Q_\alpha(\mathbb{R}^n)$, all cubes I in $\mathbb{R}^n$, and all $t > 0$:*

$$\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} \le B \max\left\{1, \left(\frac{\|f\|_{Q_\alpha}}{t}\right)^p\right\} \exp\left(\frac{-bt}{\|f\|_{Q_\alpha}}\right). \tag{1.12}$$

(b) *There exists a function f satisfying an inequality of the form*

$$\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} \le B \max\left\{1, \left(\frac{C}{t}\right)^2\right\} \exp(-ct) \tag{1.13}$$

for all cubes I in $\mathbb{R}^n$, with some positive constants B, C, c, but which is not in $Q_\alpha(\mathbb{R}^n)$.

This theorem states that any version of the John-Nirenberg inequality in Theorem 2 with $p < 2$ cannot hold. In particular, this applies to Conjecture 1 from [14] (with $p = 1$). Moreover, it shows that putting $p = 2$ in the inequality in Theorem 1 will not be sufficient for membership in Q_α.

As the counterexample for part (a) of Theorem 3, $\{f_l\}_{l\geq 0}$ is a sequence of multiples of Haar functions supported in the unit cube $I_0 = [0,1]^n$, with the coefficients depending on the parameter α of the space. When $0 < \alpha < \frac{1}{2}$, f_l are uniformly bounded in $Q_\alpha(\mathbb{R}^n)$. However, if we replace the power 2 by q with $q < 2$ in the norm of $Q_\alpha(\mathbb{R}^n)$, the new quantity goes to ∞ when l increases.

The role of the counterexample for part (b) is played by the sum function $f = \sum_{l=0}^{\infty} f_l$. This function is bounded and hence $f \in L^\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$. However, $f \notin Q_\alpha(\mathbb{R}^n)$ though it satisfies the John-Nirenberg type inequality for $Q_\alpha(\mathbb{R}^n)$. Therefore, f provides us with a sort of borderline case for $Q_\alpha(\mathbb{R}^n)$. Replacing α in the definition of f by another parameter β and denoting the new function by f_β, we have

Theorem 4. *Let* $0 < \alpha < \frac{1}{2}$.
(a) *If* $0 < \beta \leq \alpha$, $f_\beta \notin Q_\alpha(\mathbb{R}^n)$, *while if* $\alpha < \beta < \frac{n}{2}$, $f_\beta \in Q_\alpha(\mathbb{R}^n)$.
(b) *If* $\beta \geq \alpha$, *then* f_β *satisfies the John-Nirenberg type inequality (3.2).*

An equivalent expression of $f_\beta(x)$ for x on its support I_0 can be given by using the binary expansion of x as:

$$f_\beta(x) = \sum_{l=0}^{\infty} (-1)^{b_l^1 + \cdots + b_l^n} 2^{-\beta l},$$

where $0.b_0^i b_1^i b_2^i \cdots$ is the binary expansion of x_i, the ith coordinate of x. We call this expression the binary expression of f_β.

The proofs of Theorem 1 and 2 are given in Chapter 2, Section 2.2. Next, in Chapter 3, we construct the counterexamples which show Theorem 3 in Section 3.1. Then, we give the Haar wavelet decomposition and the binary expression of the function f_β in Section 3.2. We show by the binary expression that f_β maps I_0 onto $(-C_\beta, C_\beta]$ when $0 < \beta < 1$, where $C_\beta = (1 - 2^{-\beta})^{-1}$. After that, the proof of Theorem 4 is presented in Section 3.3.

The function f_β has some fractal properties on its support I_0, such as self-affinity and fine structure. Understanding these fractal properties will help us to grasp the nature of $Q_\alpha(\mathbb{R}^n)$. In order to measure its complexity, we explore the fractal properties and dimensions of the function in Chapter 4 for all values of $\beta > 0$. The analytic properties of the function are discussed in Section 4.1, for $n = 1$, and in Section 4.2, for $n > 1$. In $\mathbb{R}^1$, when $\beta \neq 1$, f is discontinuous at every dyadic point in $[0,1]$, and continuous elsewhere. In addition, f is not monotone in any subinterval of $[0,1]$. When $\beta = 1$, however, f is a linear function on $[0,1]$. Most of these properties can be generalized to $\mathbb{R}^n$ for $n > 1$, except in the case of $\beta = 1$, f is no longer continuous everywhere and its discontinuity set is also dense in $[0,1]^n$. Finally, following some preliminaries on fractal geometry in Section 4.3, Section 4.4 is devoted to the fractal properties and dimensions of the graph of f. The main result concerning the fractal properties of f_β is as follows:

Theorem 5. *Let $G_{f_\beta} = \{(x, f_\beta(x)), x \in I_0\}$, the graph of f_β over I_0. Then $\bar{G}_{f_\beta}$ is an affine invariant set on $I_0 \times [-C_\beta, C_\beta]$. Moreover,*

$$\dim_B G_{f_\beta} = \dim_F \bar{G}_{f_\beta} = \begin{cases} n+1-\beta & \text{if } 0 < \beta < 1, \\ n & \text{if } \beta \geq 1, \end{cases} \tag{1.14}$$

where $\dim_B G$ and $\dim_F G$ are, respectively, the Box dimension and the Falconer dimension of the set G, which will be defined in Section 4.3.

Another interesting consequence of the counterexamples provided in Chapter 3 is that power 2 in the definition of Q_α cannot be changed to q for $q \neq 2$. This is unlike the case of BMO, where the p-mean oscillation (1.4) defines an equivalent norm. In the paper of Coifman, Meyer and Stein [3], they introduced the tent spaces T^p_q for all $0 < p \leq \infty$, $1 \leq q \leq \infty$, and showed that the tent spaces are closely connected with Hardy spaces, L^p and BMO. In particular, for $q = 2$, they project T^p_2 to $L^p(\mathbb{R}^n)$, for $1 < p < \infty$, T^p_2 to $H^p(\mathbb{R}^n)$ for $0 < p \leq 1$, and T^∞_2 to BMO. As for Q_α, the power 2 is special compared to other powers for the tent spaces. Moreover, the space T_q^p varies when q changes. This motivated our interest in seeing what will happen if we change the power 2 to q for $q \neq 2$. In Chapter 5, we prove an analogue to Theorem 6 in [3], that is,

Theorem 6. *For $1 < q \leq 2$, there is a projection which maps*

(a). *T_q^p to $L_p(\mathbb{R}^n)$, if $1 < p < \infty$;*

(b). *T_q^1 to $H^1(\mathbb{R})$;*

(c). *T_q^∞ to BMO;*

(d). *T_q^p to $H^p(\mathbb{R}^n)$, for $p < 1$.*

The proof of this theorem involves the use of the homogeneous Triebel-Lizorkin spaces $\dot{F}_p^{0,q}$.

Remark on notation: Throughout the thesis, we will follow the notation used above. Unless otherwise stated, the letters C, c will denote arbitrary constants which may change from line to line.

Chapter 2

The John-Nirenberg Inequality for $Q_\alpha(\mathbb{R}^n)$

In this chapter, we prove, separately, the necessary and sufficient versions of the John-Nirenberg inequality for the space $Q_\alpha(\mathbb{R}^n)$.

2.1 Some background on $Q_\alpha(\mathbb{R}^n)$

Let I be any cube in $\mathbb{R}^n$ with edges parallel to the axes (this will be assumed in everything that follows). One can modify the definition of the mean oscillation of f over the cube I by considering the q-mean:

$$\Phi_f^q(I) = |I|^{-1}\int_I |f(x) - f(I)|^q dx. \tag{2.1}$$

The fact that for $f \in \mathrm{BMO}$, the supremum over all cubes of $\Phi_f^q(I)$ is bounded for any $q < \infty$ is a consequence of the John-Nirenberg inequality (and is in fact equivalent to it if the bound is given by $(cq)^q$ - see [26], Chapter 4, Section 1.3).

For $Q_\alpha(\mathbb{R}^n)$, we will be interested in the case $q = 2$. Following [14], Section 5, we write:

$$\Phi_f(I) = \Phi_f^2(I) = \frac{1}{|I|}\int_I |f(x) - f(I)|^2 dx = \frac{1}{2|I|^2}\int_I\int_I |f(x) - f(y)|^2 dxdy.$$

With $\mathcal{D}_k(I)$ as defined following Conjecture 1, namely $D_0(I) = I$, $\mathcal{D}_k(I)$, $k \geq 1$, is the collection of the subcubes of the kth generation of I obtained by dyadic subdivision of the sides, and $\mathcal{D}(I) = \cup_k \mathcal{D}_k(I)$, we let

$$\Psi_{f,\alpha}(I) = \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} \Phi_f(J) = \sum_{J\in\mathcal{D}(I)} \left(\frac{\ell(J)}{\ell(I)}\right)^{n-2\alpha} \Phi_f(J).$$

The following is Theorem 5.5 of [14]:

Theorem 7 (EJPX). $\sup_I(\Psi_{f,\alpha}(I))^{1/2}$ *is an equivalent norm in* $Q_\alpha(\mathbb{R}^n)$, *i.e.*

$$f \in Q_\alpha(\mathbb{R}^n) \Leftrightarrow \sup_I(\Psi_{f,\alpha}(I))^{1/2} < \infty. \tag{2.2}$$

In order to show the equivalence of $\Psi_{f,\alpha}(I)$ and the double integral defining $Q_\alpha(\mathbb{R}^n)$ in (1.7) for each individual cube I (without taking the supremum), it was necessary in [14] to restrict to the case $\alpha < \frac{1}{2}$ and use Lemma 5.6. A generalization of this lemma to the generic case in $\mathbb{R}^n$ is given in Lemma 2.6 of [9]. We will need a slight variation of that lemma in Section 3.1:

Lemma 1. *Assume* $\alpha < \frac{1}{2}$. *Let* $I^1, \cdots, I^l$ *be* l *cubes of the same size, that is,* $|I^1| = \cdots = |I^l| = V$, *for same* $V > 0$. *If a cube* $I \subset I^1 \cup \cdots \cup I^l$, *with* $V \leq |I| < 2^n V$, *then,*

$$\Phi_f(I) \leq \sum_{j=1}^{l} \Phi_f(I^j) + \frac{2(l-1)}{l^2} \sum_{1\leq i<j\leq l} |f(I^i) - f(I^j)|^2, \tag{2.3}$$

and

$$\Psi_{f,\alpha}(I) \leq C_l \left[\sum_{j=1}^{l} \Psi_{f,\alpha}(I^j) + \sum_{1\leq i<j\leq l} |f(I^i) - f(I^j)|^2\right]. \tag{2.4}$$

Note that in general we may assume $l \leq 2^n$. The proof of the lemma is the same as that in [9], except for replacing $|I|$ by $V = |I^j|$, $j = 1, \cdots, l$, in the proof of the first inequality.

2.2 Proofs of Theorems 1 and 2

We restate and prove Theorems 1 and 2 in this section.

Theorem 1. *Let $0 \leq \alpha < 1$ $(n \geq 2)$, or $0 \leq \alpha \leq \frac{1}{2}$ $(n = 1)$, and $0 \leq p < 2$. If there exist positive constants B, C and c, such that, for all cubes $I \subset \mathbb{R}^n$, and any $t > 0$,*

$$\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} \leq B \max\{1, \left(\frac{C}{t}\right)^p\} \exp(-ct), \tag{2.5}$$

then f is a function in $Q_\alpha(\mathbb{R}^n)$.

Proof: Using Theorem 7, it suffices to show that $\Psi_{f,\alpha}(I)$ is bounded independent of f or I. In fact, we will show, more generally, that for any $q > p$, we have (using the notation from (2.1))

$$\Psi^q_{f,\alpha}(I) := \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I)} \Phi^q_f(J) \leq qBK_{C,c,q,p}, \tag{2.6}$$

where B, C, c are the contstants appearing in (2.5), and $K_{C,c,q,p}$ is a constant depending only on C, c, p, and q. When $q = 2$, $\Psi^q_{f,\alpha}(I) = \Psi_{f,\alpha}(I)$, so this implies the theorem.

Fix a cube I. For any $J \in \mathcal{D}_k(I)$, write $\int_J |f(x) - f(J)|^q dx = q \int_0^\infty t^{q-1}\lambda_J(t)dt$, and proceed as follows, changing the order of integration and summation by the Monotone Convergence Theorem and using inequality (2.5):

$$
\begin{aligned}
\Psi^q_{f,\alpha}(I) &= \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{1}{|J|}\, q \int_0^{\infty} t^{q-1}\cdot \lambda_J(t)dt \\
&= q\int_0^{\infty} t^{q-1}\left(\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|}\right) dt \\
&\le q\int_0^{\infty} t^{q-1}\cdot B(1+\left(\frac{C}{t}\right)^p)e^{-ct}dt \\
&= qB\left(c^{-q}\int_0^{\infty} u^{q-1}e^{-u}du + C^p c^{-(q-p)}\int_0^{\infty} u^{q-p-1}e^{-u}du\right) \\
&= qB\left(c^{-q}\Gamma(q) + C^p c^{-(q-p)}\Gamma(q-p)\right)
\end{aligned}
$$

where $\Gamma(y) = \int_0^{\infty} u^{y-1}e^{-u}du$. Since $0 \le p < q$, the two integrals converge. We set $K_{C,c,p,q} = c^{-q}\Gamma(q) + C^p c^{-(q-p)}\Gamma(q-p)$.

Theorem 2. *Let $0 \le \alpha < 1$ ($n \ge 2$), or $0 \le \alpha \le \frac{1}{2}$ ($n = 1$). For any $f \in Q_\alpha(\mathbb{R}^n)$, there exist two positive constants B and b (depending only on n), such that, for all cubes $I \subset \mathbb{R}^n$,*

$$
\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} \le B \max\left\{1, \left(\frac{\|f\|_{Q_\alpha}}{t}\right)^2\right\} \exp\left(\frac{-bt}{\|f\|_{Q_\alpha}}\right). \tag{2.7}
$$

Proof: Assume f is a (nontrivial) element of $Q_\alpha(\mathbb{R}^n)$, and write $\gamma = \sup_I(\Psi_{f,\alpha}(I))^{1/2}$, which by Theorem 7 may be considered the norm of f in $Q_\alpha(\mathbb{R}^n)$. Note in particular that for all cubes I we thus have

$$
\frac{1}{|I|}\int_I |f(x) - f(I)|dx \le (\Phi_f(I))^{1/2} \le (\Psi_{f,\alpha}(I))^{1/2} \le \gamma. \tag{2.8}
$$

Fix a cube I. For each $J \in \mathcal{D}_k(I)$, we have by Chebychev's inequality, for $t > 0$,

$$
\lambda_J(t) \le t^{-2}\int_J |f(x) - f(J)|^2 dx,
$$

so it follows that

$$\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} \le t^{-2}\Psi_{f,\alpha}(I) \le t^{-2}\gamma^2. \tag{2.9}$$

Thus in the case $t \le \gamma$, (2.7) holds with $B = e$ and $b = 1$.

It therefore remains to consider only the following

Case $t > \gamma$: Following the proof of the John-Nirenberg inequality for BMO in [23] (based on an unpublished proof by Calderón), we mainly use the Calderón-Zygmund decomposition [6] to demonstrate that f satisfies (1.3). However, unlike the BMO case in which the decomposition is applied to the given cube I, in the Q_α case we instead decompose the subcubes of I.

Recall the Calderón-Zygmund decomposition [6]:

Lemma 2 (CZ). *Assume that f is a nonnegative function in $L^1(\mathbb{R}^n)$ and ξ is a positive constant. There is a decomposition $\mathbb{R}^n = P \cup \Omega$, $P \cap \Omega = \emptyset$, such that*

(a) $\Omega = \bigcup_{k=1}^{\infty} I_k$, *where $\{I_k\}$ is a collection of cubes whose interiors are disjoint;*

(b) $f(x) \le \xi$ *for a.e.* $x \in P$;

(c) $\xi < \frac{1}{|I|}\int_I f(x)dx \le 2^n\,\xi$, *for all I in the collection $\{I_k\}$.*

As in [23], we may add the following consequence of properties (a) and (c):

(d) $\xi|\Delta| \le \int_\Delta f(x)dx \le 2^n\xi|\Delta|$, *if Δ is any union of cubes I from $\{I_k\}$.*

Fix a cube I, and consider a subcube $J \in \mathcal{D}_k(I)$. Applying the Calderón-Zygmund decomposition to $|f(x) - f(J)|$ on J with $\xi = t$ for some $t > 0$, we write $\Omega_J(t)$ instead of Ω and $J \setminus \Omega_J(t)$ instead of P in the Lemma above.

Again following [23], we note that estimate (c) can be sharpened as follows:

$$t < \frac{1}{|K|}\int_K |f(x) - f(J)|dx \le 2^n\gamma + t, \tag{2.10}$$

for all cubes K in the decomposition of $\Omega_J(t)$. To see this, note that if K is such a cube, then $K \neq J$, since by (2.8),

$$\frac{1}{|J|}\int_J |f(x) - f(J)|dx \leq \gamma \leq t,$$

which contradicts (c). From the proof of the Calderón-Zygmund decomposition (see, for example, [25], Section I.3.3), we thus know that K must have a "parent" cube $K^* \subset J$ with $\ell(K^*) = 2\ell(K)$ for which

$$|f(K^*) - f(J)| \leq |K^*|^{-1}\int_{K^*} |f(x) - f(J)|dx \leq t.$$

Again using (2.8), we have

$$\begin{aligned} t < \frac{1}{|K|}\int_K |f(x) - f(J)|dx &\leq \frac{1}{|K|}\int_K |f(x) - f(K^*)|dx + |f(K^*) - f(J)| \\ &\leq \frac{2^n}{|K^*|}\int_{K^*} |f(x) - f(K^*)|dx + t \\ &\leq 2^n\gamma + t. \end{aligned}$$

In order to deal with the sum in (1.3) we also need the following variant of property (d):

$$t^2|\Delta| \leq \int_\Delta |f(x) - f(J)|^2 dx \tag{2.11}$$

for an union Δ of the cubes K in the decomposition of $\Omega_J(t)$. This is simply obtained by squaring and applying the Cauchy-Schwarz inequality to the left-hand inequality in (d).

Continuing as in [23], we note that $\Omega_J(t') \subset \Omega_J(t)$ whenever $t < t'$. For if any cube K in the decomposition of $\Omega_J(t')$ is not contained in $\Omega_J(t)$, then K is in the complement $J \setminus \Omega_J(t)$, and by property (b), $t \geq \frac{1}{|K|}\int_K |f(x) - f(J)|dx > t'$, which is a contradiction.

Lastly, setting $t' = t + 2^{n+1}\gamma$, we want to prove

$$|\Omega_J(t')| \le 2^{-n}|\Omega_J(t)|. \tag{2.12}$$

To see this, take a cube K in the decomposition for $\Omega_J(t)$. From (2.10) we have

$$\frac{1}{|K|}\int_K |f(x) - f(J)|dx \le 2^n\gamma + t < t'.$$

This means that K is not a cube in the decomposition of $\Omega_J(t')$, and was further subdivided. Let $\Delta' = K \cap \Omega_J(t')$. If $\Delta' \neq \emptyset$, it must be a union of cubes from the decomposition of $\Omega_J(t')$. Therefore by (d), (2.8), and (2.10),

$$\begin{aligned}
t' &\le |\Delta'|^{-1}\int_{\Delta'} |f(x) - f(J)|dx \\
&\le |\Delta'|^{-1}\int_{\Delta'} |f(x) - f(K)|dx + |f(K) - f(J)| \\
&\le |\Delta'|^{-1}|K|\frac{1}{|K|}\int_K |f(x) - f(K)|dx + \frac{1}{|K|}\int_K |f(x) - f(J)|dx \\
&\le |\Delta'|^{-1}|K|\gamma + 2^n\gamma + t.
\end{aligned}$$

Replacing t' by $t + 2^{n+1}\gamma$, subtracting t and dividing by γ, we get

$$(2^{n+1} - 2^n) \le |\Delta'|^{-1}|K|,$$

that is,

$$|K \cap \Omega_J(t')| = |\Delta'| \le 2^{-n}|K|$$

for any cube K in the decomposition of $\Omega_J(t)$. Summing over all such K, and using the fact that $\Omega_J(t') = \Omega_J(t) \cap \Omega_J(t')$, gives (2.12).

As a result of (2.12), we get

$$\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in D_k(I)} \frac{|\Omega_J(t')|}{|J|} \le 2^{-n}\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in D_k(I)} \frac{|\Omega_J(t)|}{|J|}, \tag{2.13}$$

for $t' = t + 2^{n+1}\gamma$.

For every $J \in \mathcal{D}_k(I)$, property (b) of the decomposition for $|f - f(J)|$ gives

$$\lambda_J(t) = |\{x \in J : |f(x) - f(J)| > t\}| = |\Omega_J(t)|, \tag{2.14}$$

so we are now in a position to estimate the left-hand side of (2.7) for $t > \gamma$. Denoting by j the integer part of $\frac{t-\gamma}{2^{n+1}\gamma}$, and setting $s = (1 + j2^{n+1})\gamma$, we have $\gamma \le s \le t$. It follows that

$$\begin{aligned}
\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} &\le \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{\lambda_J(s)}{|J|} \\
&= \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{|\Omega_J((1+j2^{n+1})\gamma)|}{|J|} \\
&\le 2^{-n} \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in D_k(I)} \frac{|\Omega_J((1+(j-1)2^{n+1})\gamma)|}{|J|}
\end{aligned}$$

by (2.13). Iterating this j times and using (2.14), (2.9) with $t = \gamma$, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in D_k(I)} \frac{\lambda_J(t)}{|J|} &\le (2^{-n})^j \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in D_k(I)} \frac{|\Omega_J(\gamma)|}{|J|} \\
&\le (2^{-n})^j \gamma^2 \cdot \gamma^{-2} \\
&\le 2^{-n(\frac{t-\gamma}{\gamma 2^{n+1}}-1)} = 2^{-\frac{n}{2^{n+1}}(t/\gamma)} 2^{\frac{n}{2^{n+1}}+n}.
\end{aligned}$$

Letting $B = 2^{(n/2^{n+1})+n}$ and $b = \frac{n}{2^{n+1}} \ln 2$, we obtain the following:

$$\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} \le B\exp(-bt/\gamma),$$

which is (2.7), since we assumed $\gamma = \|f\|_{Q_\alpha}$.

Chapter 3

Counterexamples

In the first section of this chapter, we define a sequence of function $\{f_l\}$ to show part (a) of Theorem 3. Then, using the sum function $f = \sum f_l$, we show part (b). In the second section, we replace α in the definition of the sum function by another parameter β to get a function f_β, and discuss the relation between f_β and the space $Q_\alpha(\mathbb{R}^n)$.

3.1 Proof of Theorem 3

Again, we review Theorem 3, then prove it.

Theorem 3. *Let* $0 < \alpha < \frac{1}{2}$.

(a) *If* $0 \leq p < 2$, *then there are no constants* B *and* b *such that the following inequality holds for all functions* $f \in Q_\alpha(\mathbb{R}^n)$, *all cubes* I *in* $\mathbb{R}^n$, *and all* $t > 0$*:*

$$\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} \leq B \max\left\{1, \left(\frac{\|f\|_{Q_\alpha}}{t}\right)^p\right\} \exp\left(\frac{-bt}{\|f\|_{Q_\alpha}}\right). \tag{3.1}$$

(b) *There exists a function f satisfying an inequality of the form*

$$\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} \leq B \max\left\{1, \left(\frac{C}{t}\right)^2\right\} \exp(-ct) \tag{3.2}$$

for all cubes I in $\mathbb{R}^n$, with some positive constants B, C, c, but which is not in $Q_\alpha(\mathbb{R}^n)$.

Proof of Part (a): Let $0 < \alpha < 1/2$. We will prove this part by contradiction. Suppose (3.1) holds for some $0 \leq p < 2$, and take q with $p < q < 2$. Then, replacing C and c in inequality (2.6) in the proof of Theorem 1 by $\|f\|_{Q_\alpha}$ and $\frac{b}{\|f\|_{Q_\alpha}}$, respectively, we get

$$\begin{aligned}
\Psi^q_{f,\alpha}(I) &\leq qB\left[\left(\frac{\|f\|_{Q_\alpha}}{b}\right)^q \Gamma(q) + \|f\|^p_{Q_\alpha}\left(\frac{\|f\|_{Q_\alpha}}{b}\right)^{q-p} \Gamma(q-p)\right] &(3.3)\\
&= qB\|f\|^q_{Q_\alpha}\left(\frac{1}{b^q}\Gamma(q) + \frac{1}{b^{q-p}}\Gamma(q-p)\right) &(3.4)\\
&= qB\|f\|^q_{Q_\alpha} K_{q,p,b}, &(3.5)
\end{aligned}$$

where $K_{q,p,b} = \frac{1}{b^q}\Gamma(q) + \frac{1}{b^{q-p}}\Gamma(q-p)$. In particular, if f lies in a bounded subset of Q_α, then $\Psi^q_{f,\alpha}(I)$ will be uniformly bounded. We will see from the following example that this may fail to hold.

First, we construct a sequence of functions $H_l(x)$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, which are multiples of Haar functions. Let h be a function supported in the interval $[0, 1]$, with

$$h(x) = \begin{cases} 1 & \text{if } \ 0 \leq x < \frac{1}{2} \\ -1 & \text{if } \ \frac{1}{2} \leq x < 1 \\ 0 & \text{if } \ x \notin [0, 1). \end{cases}$$

Define

$$H(x) = \prod_{i=1}^{n} h(x_i),$$

where x_i is the ith coordinate of x. $H(x)$ is supported in the unit cube $I_0 = [0, 1]^n$

and takes the value 1 or -1 on each subcube in $\mathcal{D}_1(I_0)$, the first dyadic partition of I_0, with any two adjacent cubes having opposite values. Also note $\int H(x)dx = \prod_{i=1}^{n} \int h(x_i)dx_i = 0$.

Fix $l \geq 0$. For each dyadic cube $J \in \mathcal{D}_l(I_0)$, write $J = \prod_{i=1}^{n} [m_i 2^{-l}, (m_i+1)2^{-l}]$, where $m_i \in \{0, \cdots, 2^l - 1\}$. Let $H_{l,J}(x)$ be a function supported in J, such that:

$$H_{l,J}(x) = 2^{-\alpha l} \prod_{i=1}^{n} h(2^l x_i - m_i) = 2^{-\alpha l} H(2^l x - m), \tag{3.6}$$

with $m = (m_1, \cdots, m_n)$. Namely, $H_{l,J}$ takes the value $2^{-\alpha l}$ or $-2^{-\alpha l}$ on each subcube in $\mathcal{D}_1(J)$, and any two such neighboring cubes have opposite values. Again $\int H_{l,J}(x)dx = 0$.

Finally, define a sequence of functions $\{f_l\}_{l \geq 0}$ on $\mathbb{R}^n$ by

$$f_l(x) = \begin{cases} H_{l,J}(x) & \text{if } x \in J, J \in \mathcal{D}_l(I_0); \\ 0 & \text{if } x \in \mathbb{R}^n \backslash I_0. \end{cases} \tag{3.7}$$

Claim 1. *There exist a constant C, depending only on α, such that*

$$\sup_I \Psi_{f_l,\alpha}(I) = \sup_I \Psi^2_{f_l,\alpha}(I) \leq C \quad \forall l \geq 0,$$

that is (by Theorem 7), $\{f_l\}_{l \geq 0}$ is a bounded set of functions in $Q_\alpha(\mathbb{R}^n)$. On the other hand, for $q < 2$

$$\sup_I \Psi^q_{f_l,\alpha}(I) \to \infty \text{ as } l \to \infty.$$

This claim will provide the contradiction to (3.5).

Proof of Claim 1 : We first calculate $\Psi^q_{f_l,\alpha}(I_0)$. Let J be a dyadic subcube in $\mathcal{D}_k(I_0)$.

If $k \leq l$, then J is a union of cubes J' from $\mathcal{D}_l(I_0)$, so the mean

$$f_l(J) = \frac{1}{|J|} \sum_{J' \subset J} \int H_{l,J'}(x)dx = 0.$$

Thus, the oscillation

$$\Phi^q_{f_l}(J) = \frac{1}{|J|} \int_J |2^{-\alpha l}|^q dx = 2^{-q\alpha l}.$$

On the other hand, if $k > l$, then J is contained in a single subcube from $\mathcal{D}_1(J')$ for some $J' \in \mathcal{D}_l(I_0)$, so $f_l(x)$ is constant on J, and the oscillation $\Phi^q_{f_l}(J) = 0$.

It follows that

$$\Psi^q_{f_l,\alpha}(I_0) = \sum_{k=0}^{l} 2^{2\alpha k - nk} \sum_{J \in \mathcal{D}_k(I)} 2^{-\alpha q l} = \sum_{k=0}^{l} 2^{2\alpha k} \cdot 2^{-\alpha q l} \tag{3.8}$$

$$= \frac{2^{2\alpha(l+1)} - 1}{2^{2\alpha} - 1} \cdot 2^{-\alpha q l} = \frac{2^{(2-q)\alpha l} - 2^{-2\alpha - q\alpha l}}{1 - 2^{-2\alpha}}. \tag{3.9}$$

Hence, if $q < 2$,

$$\Psi^q_{f_l,\alpha}(I_0) \to \infty \text{ as } l \to \infty,$$

proving the second part of the claim, while

$$\Psi_{f_l,\alpha}(I_0) = \Psi^2_{f_l,\alpha}(I_0) \leq \frac{1}{1 - 2^{-2\alpha}} \quad \forall l \geq 0.$$

We still need to show $\Psi_{f_l,\alpha}(I)$ is bounded for all cubes $I \subset \mathbb{R}^n$. We now estimate $\Psi_{f_l,\alpha}(I)$ for a dyadic cube. Since $\mathrm{supp} f_l = I_0$, $\Psi_{f_l,\alpha}(I) = 0$ if $I \cap I_0 = \emptyset$. So for smaller cubes, we need only consider the dyadic subcubes of I_0, that is, $I \in \mathcal{D}_j(I_0)$, $j > 0$, while for bigger ones, we consider only dyadic cubes containing I_0, namely $I = [0, 2^j]^n$, $j > 0$.

Case 1: $I \in \mathcal{D}_j(I_0)$, $j > 0$. Write $I = \prod_{i=1}^{n} [m_i 2^{-j}, (m_i + 1)2^{-j}]$, $m_i \in \{0, \cdots, 2^j - 1\}$. We can further assume $j \leq l$, otherwise $\Psi_{f_l,\alpha}(I) = 0$ since f_l will be constant on I.

As in the case of I_0, $f_l(J) = 0$ if $J \in \mathcal{D}_k(I)$ for $k \leq l-j$, resulting in $\Phi_{f_l}(J) = 2^{-2\alpha l}$, while $f_l(x) =$ constant on J if $k > l-j$, resulting in $\Phi_{f_l}(J) = 0$. Therefore we again get

$$\begin{aligned}
\Psi_{f_l,\alpha}(I) &= \sum_{k=0}^{l-j} 2^{2\alpha k - nk} \sum_{J \in \mathcal{D}_k(I)} 2^{-2\alpha l} \\
&= \frac{2^{2\alpha(l-j+1)} - 1}{2^{2\alpha} - 1} \cdot 2^{-2\alpha l} = \frac{2^{-2\alpha j} - 2^{-2\alpha(l+1)}}{1 - 2^{-2\alpha}} \\
&\leq \frac{1}{1 - 2^{-2\alpha}}.
\end{aligned}$$

<u>Case 2, $I = [0, 2^j]^n$, $j > 0$.</u> Consider $J \in \mathcal{D}_k(I)$. If $k \leq j$, then either J is disjoint from I_0, in which case $\Phi_{f_l}(J) = 0$, or it contains it, which occurs only in one case, namely $J = [0, 2^{j-k}]^n$, in which case $f_l(J) = 0$ and $\Phi_{f_l}(J) = |J|^{-1} \int_{I_0} |f_l(x)|^2 dx = 2^{n(k-j)-2\alpha l}$.

If $j < k \leq l+j$, then again either J is disjoint from I_0, in which case $\Phi_{f_l}(J) = 0$, or it is contained in it, i.e. $J \in \mathcal{D}_{k-j}(I_0)$, and again, since $k - j \leq l$, $f_l(J) = 0$ and $\Phi_{f_l}(J) = 2^{-2\alpha l}$.

Finally, if $k > l+j$, we also have that either J is disjoint from I_0 and $\Phi_{f_l}(J) = 0$, or $J \in \mathcal{D}_{k-j}(I_0)$, but now since $k - j > l$, f_l is constant on J, resulting in $\Phi_{f_l}(J) = 0$. Consequently (using the fact that $j \geq 0$ and $\alpha \leq n/2$) we get

$$\begin{aligned}
\Psi_{f_l,\alpha}(I) &= \sum_{k=0}^{j} 2^{2\alpha k - nk} 2^{n(k-j)-2\alpha l} + \sum_{k=j+1}^{l+j} 2^{2\alpha k - nk} \sum_{J \in \mathcal{D}_{k-j}(I_0)} 2^{-2\alpha l} \\
&= 2^{-nj-2\alpha l} \sum_{k=0}^{l+j} 2^{2\alpha k} = 2^{-nj-2\alpha l} \frac{2^{2\alpha(l+j+1)} - 1}{2^{2\alpha} - 1} \\
&= \frac{2^{(2\alpha-n)j} - 2^{-2\alpha(l+1)-nj}}{1 - 2^{-2\alpha}} \leq \frac{1}{1 - 2^{-2\alpha}}.
\end{aligned}$$

Lastly, we use Lemma 1 to estimate $\Psi_{f_l,\alpha}(I)$ for any cube $I \subset \mathbb{R}^n$. This is the only place where the restriction $\alpha < 1/2$ is used.

For any given I, there exists an integer j such that the sidelength of I satisfies

$2^j \leq \ell(I) < 2^{j+1}$. Moreover, there exist 2^n adjacent dyadic cubes $I^1, \cdots, I^{2^n}$, with sidelength 2^j, such that $I \subset I^1 \cup \cdots \cup I^{2^n}$. Since the mean of f_l on each of these dyadic cubes, $f_l(I^i)$, is either zero or $\pm 2^{-\alpha l}$, we have $|f_l(I^i) - f_l(I^j)| \leq 2 \cdot 2^{-\alpha l}$. So, by Lemma 1, and the estimates above of $\Psi_{f_l,\alpha}(I)$ for dyadic cubes I, we have

$$\begin{aligned}
\Psi_{f_l,\alpha}(I) &\leq C_n \left[\sum_{j=1}^{2^n} \Psi_{f_l,\alpha}(I^j) + \sum_{1 \leq i < j \leq 2^n} |f_l(I^i) - f_l(I^j)|^2 \right] \\
&\leq C_n \left(2^n \cdot \frac{1}{1 - 2^{-2\alpha}} + \frac{2^{2n} - 2^n}{2} \cdot 4 \cdot 2^{-2\alpha l} \right) \\
&= C_{n,\alpha}.
\end{aligned}$$

This completes the proof of the Claim 1 .

Proof of Part (b): We will now show that the form of the John-Nirenberg-type inequality (2.5) with $p = 2$ is not sufficient.

Consider the function $f = \sum_l f_l$. Note that the sum converges absolutely since

$$\sum_l |f_l| \leq \sum_{l=0}^{\infty} 2^{-\alpha l} = \frac{1}{1 - 2^{-\alpha}} =: C_\alpha < \infty.$$

In what follows we will continue to use the notation C_α for this constant, and C for any other constant, which may also depend on α.

Claim 2. *The function f is not in $Q_\alpha(\mathbb{R}^n)$ for any $\alpha > 0$ (it is in fact in $L^\infty \setminus Q_\alpha$). However, when $\alpha < 1/2$, f satisfies the John-Nirenberg-type inequality (3.2), i.e. there exist positive constants B, C and c, such that, for all cubes $I \subset \mathbb{R}^n$, and any $t > 0$,*

$$\sum_{k=0}^{\infty} 2^{(2\alpha - n)k} \sum_{J \in \mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} \leq B \max\{1, \left(\frac{C}{t}\right)^2\} \exp(-ct). \tag{3.10}$$

As a consequence, for all $q > 2$, $0 < \alpha < 1/2$, $\sup_I \Psi^q_{f,\alpha}(I) < \infty$. Moreover, we can *show, even for $\alpha \geq 1/2$, that for all $q > 2$,*

$$\sup_{\text{dyadic cubes } I} \Psi^q_{f,\alpha}(I) < \infty.$$

Proof of Claim 2: We have already shown that f is bounded by C_α. Now we will show $\Psi_{f,\alpha}(I_0) = \infty$, hence $f \notin Q_\alpha(\mathbb{R}^n)$.

Given $k \geq 0$, let J be a dyadic cube in $\mathcal{D}_k(I_0)$. Since

$$\sum_{l \geq 0} \int_J |f_l(x)| dx = \sum_{l \geq 0} \int_J 2^{-\alpha l} dx = C_\alpha |J| < \infty,$$

the average of f on J can be written as $f(J) = \sum_{l=0}^{\infty} f_l(J)$. Recalling that $f_l(J) = 0$ when $k \leq l$ and f_l is a constant $(= \pm 2^{-\alpha l})$ on J when $k > l$, we have

$$|f(x) - f(J)| = \left| \sum_{l=0}^{\infty} f_l(x) - \sum_{l=0}^{k-1} f_l(J) \right| = \left| \sum_{l=k}^{\infty} f_l(x) \right|.$$

By the orthogonality of $\{f_l\}_{l \geq k}$ on J, we have

$$\begin{aligned} \Phi_f(J) = \frac{1}{|J|} \int_J |\sum_{l=k}^{\infty} f_l(x)|^2 dx &= \frac{1}{|J|} \sum_{l=k}^{\infty} \int_J |f_l(x)|^2 dx \\ = \sum_{l=k}^{\infty} 2^{-2\alpha l} = C_{2\alpha} 2^{-2\alpha k}. \end{aligned}$$

Consequently,

$$\Psi_{f,\alpha}(I_0) = \sum_{k=0}^{\infty} 2^{(2\alpha - n)k} \sum_{J \in \mathcal{D}_k(I_0)} \Phi_f(J) = C_{2\alpha} \sum_{k=0}^{\infty} 2^{(2\alpha - n)k + nk - 2\alpha k} = \infty.$$

This shows $f \notin Q_\alpha(\mathbb{R}^n)$ and proves the first part of the claim.

To show the second part, namely that f satisfies a John-Nirenberg-type inequality of the form (3.2), we first deal with the case of t large, where the inequality follows from the fact that $f \in L^\infty$. Namely, since f is bounded by C_α, we have, for $t \geq 2C_\alpha$, $\lambda_J(t) = 0$ for all cubes J, and therefore

$$\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} = 0$$

for all cubes I, so for such t the inequality holds with any $B > 0$, $C = 2C_\alpha$, and any real number c.

So it remains to consider t small, i.e. $t < 2C_\alpha$.

We will deal with the summation in k depending on the size of k relative to t.

Case 1 : k small. When $2^{\alpha k} < 2C_\alpha/t$, we use the trivial estimate 2^{nk} on the number of cubes J in $\mathcal{D}_k(I)$. This gives

$$\sum_{\{k:1\leq 2^{\alpha k}<2C_\alpha/t\}} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} \leq \sum_{\{k:1\leq 2^{\alpha k}<2C_\alpha/t\}} 2^{2\alpha k} \leq C_1\left(\frac{2C_\alpha}{t}\right)^2. \tag{3.11}$$

Case 2 : k large. When $2^{\alpha k} \geq 2C_\alpha/t$, we want to estimate more carefully the number of of cubes J in $\mathcal{D}_k(I)$ which contribute a nonzero amount to the sum. Given any subcube J, we have, as above, $f(J) = \sum_l f_l(J)$ and therefore

$$|f(x) - f(J)| \leq \sum_l |f_l(x) - f_l(J)| \leq \sum_{\{l\,:\, f_l \neq \text{constant on } J\}} 2 \cdot 2^{-\alpha l}.$$

For a fixed $l \geq 0$, f_l is constant on J when either J is disjoint from I_0 or J is contained in some dyadic cube J' belonging to $\mathcal{D}_{l+1}(I_0)$. Varying l and J, we will denote by G the set of all "good" pairs (J, l), namely those for which f_l is constant on J.

If $(J, l) \notin G$ for some l, i.e. J intersects I_0 and is not contained entirely in any element of $\mathcal{D}_{l+1}(I_0)$, then it cannot, a fortiori, be contained in any dyadic cube in $\mathcal{D}_{l+2}(I_0)$, hence we also have $(J, l+1) \notin G$. For such J we will denote by l_J the first value of l for which $(J, l) \notin G$.

If J is disjoint from I_0, we use the convention $l_J = \infty$. Thus we can write, for all J,

$$|f(x) - f(J)| \leq \sum_{\{l\,:\,(J,l)\notin G\}} 2 \cdot 2^{-\alpha l} = 2\sum_{l_J}^{\infty} 2^{-\alpha l} = 2C_\alpha 2^{-\alpha l_J}$$

and hence $|f(x) - f(J)| > t$ only if $t < 2C_\alpha 2^{-\alpha l_J}$, in which case we estimate $\lambda_f(J) \leq |J|$. Otherwise, we have $\lambda_f(J) = 0$. Summing, we get

$$\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} \frac{\lambda_J(t)}{|J|} \leq \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{\{J\in\mathcal{D}_k(I):2C_\alpha 2^{-\alpha l_J}>t\}} 1.$$

We estimate the right-most sum, namely the number of J for which $2C_\alpha 2^{-\alpha l_J} > t$, as follows:

$$\begin{aligned}
\#\{J \in \mathcal{D}_k(I) : 2C_\alpha 2^{-\alpha l_J} > t\} &= \sum_{\{l\,:\,2C_\alpha 2^{-\alpha l}>t\}} \#\{J \in \mathcal{D}_k(I) : l_J = l\} \\
&\leq \sum_{\{l\,:\,2^{\alpha l}<2C_\alpha/t\}} \#\{J \in \mathcal{D}_k(I) : l_J \leq l\} \\
&= \sum_{\{l\,:\,2^{\alpha l}<2C_\alpha/t\}} \#\{J \in \mathcal{D}_k(I) : (J,l) \notin G\}.
\end{aligned}$$

Now in order to satisfy $(J,l) \notin G$, the cube J must intersect the boundary of one of the cubes in $\mathcal{D}_{l+1}(I_0)$, i.e. we must have $J \bigcap (\bigcup\{\partial J' : J' \in \mathcal{D}_{l+1}(I_0)\}) \neq \emptyset$. Note that this intersection is contained in $I \cap \overline{I_0}$. We can estimate the $(n-1)$-dimensional volume of $I \bigcap (\bigcup\{\partial J' : J' \in \mathcal{D}_{l+1}(I_0)\})$ by

$$n\Big(\frac{\min(\ell(I),1)}{2^{-(l+1)}} + 1\Big) \min(\ell(I),1)^{n-1}.$$

Moreover, since $J \in \mathcal{D}_k(I)$, J must be contained in the set of points of distance no more than $2^{-k}\ell(I)$ from this $(n-1)$-dimensional set, measured along one of the coordinate directions. This means that J is contained in a set of n-dimensional volume at most

$$n\Big(\frac{\min(\ell(I),1)}{2^{-(l+1)}}+1\Big)\min(\ell(I),1)^{n-1}2^{-k}\ell(I). \tag{3.12}$$

When $\ell(I)\le 1$, we get an upper bound of $n\ell(I)^n 2^{-k}(\ell(I)2^{l+1}+1)$. Dividing this by $|J|=(2^{-k}\ell(I))^n$, and recalling that the cubes in $\mathcal{D}_k(I)$ have pairwise disjoint interiors, we get an estimate on the maximum number of "bad" cubes:

$$\#\{J\in\mathcal{D}_k(I):(J,l)\notin G\}\le n2^{(n-1)k}(\ell(I)2^{l+1}+1)\le n2^{(n-1)k+l+2}.$$

When $\ell(I)\ge 1$, estimate (3.12) gives

$$\#\{J\in\mathcal{D}_k(I):(J,l)\notin G\}\le\frac{n(2^{l+1}+1)2^{-k}\ell(I)}{(2^{-k}\ell(I))^n}=n2^{(n-1)k}\ell(I)^{1-n}(2^{l+1}+1),$$

so we again get an upper bound of $n2^{(n-1)k+l+2}$. Plugging this into the sum gives

$$\begin{aligned}&\sum_{\{l:2^{\alpha l}<2C_\alpha/t\}}\#\{J\in\mathcal{D}_k(I):(J,l)\notin G\}\\ \le\ & n2^{(n-1)k}\sum_{\{l:2^{\alpha l}<2C_\alpha/t\}}2^{l+2}\ \le\ C2^{(n-1)k}(2C_\alpha/t)^{1/\alpha}.\end{aligned}$$

Finally, summing over k, and using the assumption $\alpha<1/2$, we get

$$\sum_{\{k\ge0:2^{\alpha k}\ge(2C_\alpha)/t\}}2^{(2\alpha-n)k}\sum_{J\in\mathcal{D}_k(I)}\frac{\lambda_J(t)}{|J|}\ \le\ C\Big(\frac{2C_\alpha}{t}\Big)^{1/\alpha}\sum_{\{k:2^{\alpha k}\ge(2C_\alpha)/t\}}2^{(2\alpha-1)k}$$

$$\le\ C\Big(\frac{2C_\alpha}{t}\Big)^{1/\alpha}\Big(\frac{2C_\alpha}{t}\Big)^{(2\alpha-1)/\alpha}\ =\ C_2\Big(\frac{2C_\alpha}{t}\Big)^2. \tag{3.13}$$

Combining the two sums (3.11) and (3.13) gives the desired inequality for $t<2C_\alpha$ with $B=e\max(C_1,C_2)$, $C=2C_\alpha$ and $c=(2C_\alpha)^{-1}$. This completes the proof of the second part of the claim.

The uniform bound on $\Psi^q_{f,\alpha}(I)$ when $q > 2, \alpha < 1/2$, follows as a corollary to f satisfying (3.2) in the same way as (2.6) (see the beginning of the proof of Theorem 1).

Now we will show that for α, $0 < \alpha < \min(1, n/2)$, and any $q > 2$, $\Psi^q_{f,\alpha}(I)$ is uniformly bounded over dyadic cubes I.

Note that for a fixed cube I, we can use subadditivity of the L^q, ℓ^q norms to get:

$$\begin{aligned}
(\Psi^q_{f,\alpha}(I))^{1/q} &= \Big(\sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J\in\mathcal{D}_k(I)} \Phi^q_f(J)\Big)^{1/q} \\
&= \ell(I)^{(2\alpha-n)/q}\Big(\sum_{J\in\mathcal{D}(I)} \|\sum_{l=0}^{\infty} \frac{\chi_J(\cdot)}{\ell(J)^{2\alpha/q}}[f_l(\cdot)-f_l(J)]\|^q_{L^q(I)}\Big)^{1/q} \\
&\leq \ell(I)^{(2\alpha-n)/q}\Big(\sum_{J\in\mathcal{D}(I)} \Big(\sum_{l=0}^{\infty} \|\frac{\chi_J(\cdot)}{\ell(J)^{2\alpha/q}}[f_l(\cdot)-f_l(J)]\|_{L^q(I)}\Big)^q\Big)^{1/q} \\
&\leq \ell(I)^{(2\alpha-n)/q}\sum_{l=0}^{\infty}\Big(\sum_{J\in\mathcal{D}(I)} \|\frac{\chi_J(\cdot)}{\ell(J)^{2\alpha/q}}[f_l(\cdot)-f_l(J)]\|^q_{L^q(I)}\Big)^{1/q} \\
&= \sum_{l=0}^{\infty}\Big(\sum_{J\in\mathcal{D}(I)}\Big(\frac{\ell(J)}{\ell(I)}\Big)^{n-2\alpha}\Phi^q_{f_l}(J)\Big)^{1/q} = \sum_{l=0}^{\infty}(\Psi^q_{f_l,\alpha}(I))^{1/q}.
\end{aligned}$$

Therefore if we can show that for each l, and each dyadic cube I,

$$\Psi^q_{f_l,\alpha}(I) \leq C2^{(2-q)\alpha l} \tag{3.14}$$

with a constant $C < \infty$ independent of I, then we will get

$$\sup_{\text{dyadic cubes } I}(\Psi^q_{f,\alpha}(I))^{1/q} \leq C\sum_{l=0}^{\infty} 2^{(2/q-1)\alpha l},$$

and this sum is convergent if and only if q is strictly larger than 2.

For I_0, estimate (3.14) was computed in the beginning of the proof of Claim 1 (see (3.9)).

For a dyadic subcube $I \in \mathcal{D}_j(I_0)$, $j > 0$, we repeat the estimates in Case 1 of the proof of Claim 1 with q instead of 2 to get

$$\Psi^q_{f_l,\alpha}(I) = \sum_{k=0}^{l-j} 2^{2\alpha k - nk} \sum_{J \in \mathcal{D}_k(I)} 2^{-q\alpha l} = \frac{2^{2\alpha(l-j+1)} - 1}{2^{2\alpha} - 1} \cdot 2^{-q\alpha l} \leq C_{2\alpha} 2^{(2-q)\alpha l}.$$

Similarly, when $I = [0, 2^j]^n$, $j > 0$, we can replace 2 by q in Case 2 the proof of Claim 1 to obtain

$$\begin{aligned}
\Psi^q_{f,\alpha}(I) &= \sum_{k=0}^{j} 2^{2\alpha k - nk} 2^{n(k-j) - q\alpha l} + \sum_{k=j+1}^{l+j} 2^{2\alpha k - nk} \sum_{J \in \mathcal{D}_{k-j}(I_0)} 2^{-q\alpha l} \\
&= 2^{-nj - q\alpha l} \sum_{k=0}^{l+j} 2^{2\alpha k} = \frac{2^{(2-q)l + (2\alpha - n)j} - 2^{-q\alpha l - 2\alpha - nj}}{1 - 2^{-2\alpha}} \leq C_{2\alpha} 2^{(2-q)l}.
\end{aligned}$$

Note that here we only used the fact that $\alpha \leq n/2$. As in [14], [9], it seems that in order to go from the dyadic to the general case one needs the restriction $\alpha < 1/2$ in dimensions $n > 1$, perhaps because one is dealing with sets of dimension $n - 1$.

3.2 The wavelet decomposition and binary expression of the function f_β

The sum function f above can be viewed as a wavelet decomposition in the Haar basis (see [26]). In order to explore the relation between the function and the spaces $Q_\alpha(\mathbb{R}^n)$ in a wider range, in what follows we replace α in the coefficients of the wavelet decomposition by a new parameter β with $\beta > 0$. The new function is denoted by f_β.

More explicitly, let us define f_β using the orthonormal Haar wavelet basis in $\mathbb{R}^n$. Let $\mathcal{D}$ be the collection of all dyadic cubes in $\mathbb{R}^n$:

$$\mathcal{D} = \left\{ J = \prod_{i=1}^{n} [m_i 2^{-l}, (m_i + 1) 2^{-l}] \right\}, \quad l, m_1, \cdots, m_n \in \mathbb{Z}. \tag{3.15}$$

Recall $h(x)$ and $H(x)$, the Haar function in $\mathbb{R}^1$ and $\mathbb{R}^n$, respectively. Consider the normalized Haar functions in $L^2(\mathbb{R}^n)$, which we denote by $\tilde{H}_{l,J}$. Namely,

$$\tilde{H}_{l,J}(x) = 2^{nl/2}H(2^l x - m) = 2^{nl/2}\prod_{i=1}^{n} h(2^l x_i - m_i). \tag{3.16}$$

$\{\tilde{H}_{l,J}\}_{J\in\mathcal{D}}$ is the so-called orthonormal Haar wavelet basis.

The Haar wavelet decomposition of f_β is given by

$$f_\beta(x) = \sum_{l,J} a_{l,J}\tilde{H}_{l,J}$$

where $a_{l,J} = 2^{-(\beta+\frac{n}{2})l}$ if in (3.15) $l = 0, 1, \cdots$, and $J \in D_l(I_0)$, otherwise $a_{l,J} = 0$.

We have

$$|f_\beta(x)| \le \sum_{l=0}^{\infty} 2^{-\beta l} = \frac{1}{1-2^{-\beta}} := C_\beta < \infty.$$

For $x \in I_0 = [0, 1]^n$, we can also express $f_\beta(x)$ in terms of the binary expansions of x_i ($i \in \{1, \cdots, n\}$), where x_i is the ith coordinate of x. This equivalent expression is more straightforward and more convenient to use in some cases.

Consider the simplest case $n = 1$. Let $x = \sum_{l=0}^{\infty} b_l 2^{-(l+1)}$, corresponding to the expansion $0.b_0b_1b_2\cdots$, where $b_l = 0$ or 1. We will use the expansions ending in infinitely many zeros rather than infinitely many 1's. Thus, the function f_β can be written as

$$f_\beta(x) = \sum_{l=0}^{\infty}(-1)^{b_l}2^{-\beta l} = C_\beta - 2\sum_{l=0}^{\infty} b_l 2^{-\beta l}. \tag{3.17}$$

Again, we reserve C_β for $(1-2^{-\beta})^{-1}$.

In general, let $x = (x_1, x_2, \cdots, x_n) \in I_0$ and let $0.b_0^i b_1^i b_2^i \cdots$ be the binary expansion of x_i, i.e. $x_i = \sum_{l=0}^{\infty} b_l^i 2^{-(l+1)}$, for $i \in \{1, \cdots, n\}$. We can write

$$f_\beta(x) = \sum_{l=0}^{\infty}(-1)^{b_l^1+\cdots+b_l^n}2^{-\beta l}. \tag{3.18}$$

Lemma 3. *For $0 < \beta < 1$, $f = f_\beta$ defined by (3.18) is onto from I_0 to $(-C_\beta, C_\beta]$.*

Remark: $-C_\beta$ is excluded in the range of the function f. This is clear from its binary expression (3.17) when $n = 1$. The value $f(x) = -C_\beta$ corresponds to $b_l = 1$ $\forall l$, that is, the binary expansion of x is $0.111\cdots = 1$. However, $f(1) = 0$ from the definition of the function in Section 3.1.

Proof. We prove the lemma for the one dimensional case (3.17), then for (3.18):

By the second equality in (3.17), we just need to show $\forall y \in [0, C_\beta)$, y can be expanded as

$$y = \sum_{l=0}^{\infty} b_l(y) 2^{-\beta l}, \tag{3.19}$$

where $b_l(y) = 0$ or 1, $\forall l \geq 0$.

If $0 < \beta < 1$, then $1 < q = 2^\beta < 2$. Using the "greedy algorithm" in [12] (see also [11], [10]), the expansion (3.19) is obtained by putting digits $b_l(y)$ inductively as follows:

$b_0(y) = 1$ if $1 \leq y$, or $b_0(y) = 0$ if $1 > y$. For $l \geq 1$,

$$b_l(y) := \begin{cases} 1 & \text{if } \sum_{j=0}^{l-1} b_j(y) 2^{-\beta j} + 2^{-\beta l} \leq y, \\ 0 & \text{if } \sum_{j=0}^{l-1} b_j(y) 2^{-\beta j} + 2^{-\beta l} > y. \end{cases} \tag{3.20}$$

Claim 3. *With such $b_l(y)$, $\sum_{l=0}^{\infty} b_l(y) 2^{-\beta l}$ converges to $y \in [0, C_\beta)$.*

While this result may be found in the literature on base q expansion (see [7]), we give our own proof below.

Proof of the claim:

Note that for any $y \in [0, C_\beta)$, (3.20) guarantees $\sum_{j=0}^{l} b_j(y) 2^{-\beta j} \leq y$, $\forall l$. Since $y \neq C_\beta = \sum_{l=1}^{\infty} 2^{-\beta l}$, $\exists l$ such that $b_l(y) = 0$. If $\exists l_0$ such that $b_l(y) = 0$, $\forall l > l_0$, then $y = \sum_{j=0}^{l_0} b_l(y) 2^{-\beta l}$ and we are done.

Otherwise, there are infinitely many l's such that $b_l(y) = 0$ and $b_{l+1}(y) = 1$.

Namely,

$$\sum_{l=0}^{l-1} b_l(y)2^{-\beta l} + 2^{-\beta(l+1)} < y < \sum_{l=0}^{l-1} b_l(y)2^{-\beta l} + 2^{-\beta l}. \tag{3.21}$$

To see this, suppose there were only finite many such l, and denote by l_0 the greatest one of them. Then we have $b_l(y) = 1 \; \forall l \geq l_0 + 1$, that is,

$$\sum_{l=0}^{l_0-1} b_l(y)2^{-\beta l} + \sum_{j=l_0+1}^{\infty} 2^{-\beta j} \leq y. \tag{3.22}$$

Comparing (3.22) with the right inequality of (3.21) for $l = l_0$, we get

$$2^{-\beta l_0} > \sum_{j=l_0+1}^{\infty} 2^{-\beta j} = \frac{1}{2^\beta - 1} \cdot 2^{-\beta l_0}.$$

This is a contradiction since $\frac{1}{2^\beta-1} > 1$ when $0 < \beta < 1$.

So with $b_l(y)$ given by (3.20), $\sum_{l=0}^{\infty} b_l(y)2^{-\beta l}$ converges to y since its partial sums are positive, monotone increasing, and the subsequence given in the left hand side of (3.21) clearly converges to y.

Setting $x = \sum_{l=0}^{\infty} b_l(y)2^{-(l+1)}$, we have $f(x) = C_\beta - 2y$. In addition, this expression cannot end with infinitely many 1's. This proves Lemma 3 for $n = 1$.

For $n \geq 2$, (3.18) is onto since for each i, $f(0, \cdots, x_i, \cdots, 0)$ is onto from $[0, 1)$ to $(-C_\beta, C_\beta]$ based on the case $n = 1$.

3.3 Relation of f_β to $Q_\alpha(\mathbb{R}^n)$

Recall the relation between f_β and $Q_\alpha(\mathbb{R}^n)$ is given by

Theorem 4. *Let* $0 < \alpha < \frac{1}{2}$.

(a) *If* $0 < \beta \leq \alpha$, $f_\beta \notin Q_\alpha(\mathbb{R}^n)$, *while if* $\alpha < \beta < \frac{n}{2}$, $f_\beta \in Q_\alpha(\mathbb{R}^n)$.

(b) *If* $\beta \geq \alpha$, *then* f_β *satisfies the John-Nirenberg type inequality (3.2).*

Proof:

Part (b) is a combination of Theorem 3 for the case $\beta = \alpha$, and a corollary of part (a) for the case $\beta > \alpha$.

The proof of part (a) is analogous to that of Theorem 3; for simplicity, we use f instead of f_β:

Let J be a subcube in $\mathcal{D}_k(I_0)$ for a fix integer $k \geq 0$. Recall that

$$f_l(x) = \sum_{J \in \mathcal{D}_l(I_0)} 2^{-(\beta+\frac{n}{2})l} \tilde{H}_{l,J}(x).$$

We have $f_l(J) = 0$ when $k \leq l$ and $f_l \equiv \pm 2^{-\beta l}$ on J when $k > l$. Hence

$$\begin{aligned} |f(x) - f(J)| &= |\textstyle\sum_{l=0}^{\infty} f_l(x) - \sum_{l=0}^{k-1} f_l(J)| \\ &= |\textstyle\sum_{l=k}^{\infty} f_l(x)|. \end{aligned}$$

In addition, by the orthogonality of the sequence $\{f_l\}_{l \geq k}$ on J, we have that

$$\begin{aligned} \Phi_f(J) &= \tfrac{1}{|J|} \textstyle\int_J |\sum_{l=k}^{\infty} f_l(x)|^2 dx \\ &= \tfrac{1}{|J|} \textstyle\sum_{l=k}^{\infty} \int_J |f_l(x)|^2 dx \\ &= \textstyle\sum_{l=k}^{\infty} 2^{-2\beta l} = C_{2\beta} 2^{-2\beta k}. \end{aligned}$$

where $C_{2\beta} = \frac{1}{1-2^{-2\beta}}$. Consequently,

$$\Psi_{f,\alpha}(I_0) = \sum_{k=0}^{\infty} 2^{(2\alpha-n)k} \sum_{J \in \mathcal{D}_k(I_0)} \Phi_f(J) = \sum_{k=0}^{\infty} 2^{(2\alpha-n)k+nk} \cdot C_{2\beta} 2^{-2\beta k} = C_{2\beta} \sum_{k=0}^{\infty} 2^{2(\alpha-\beta)k}.$$

It converges iff $\beta > \alpha$. This proves the first part of the claim.

To prove the second part, we will show that, for all cubes $I \subset \mathbb{R}^n$, $\Psi_{f,\alpha}(I)$ is bounded provided $\alpha < \beta \leq \frac{n}{2}$.

We estimate $\Psi_{f,\alpha}(I)$ for a dyadic cube, as in the proof of Claim 1 in Section 3.1.

Case 1: $I \in \mathcal{D}_j(I_0)$, $j > 0$. As in the proof of Claim 2 in Section 3.1, by the orthogonality of the sequence $\{f_l\}_{l\geq k+j}$ on $J \in \mathcal{D}_k(I)$, we have that

$$\begin{aligned}
\Phi_f(J) &= \tfrac{1}{|J|}\int_J |\sum_{l=k+j}^{\infty} f_l(x)|^2 dx \\
&= \tfrac{1}{|J|}\sum_{l=k+j}^{\infty}\int_J |f_l(x)|^2 dx \\
&= \sum_{l=k+j}^{\infty} 2^{-2\beta l} = C_{2\beta}2^{-2\beta(k+j)}.
\end{aligned}$$

So,

$$\begin{aligned}
\Psi_{f,\alpha}(I) &= \sum_{k=0}^{\infty} 2^{(2\alpha-n)k}\sum_{J\in\mathcal{D}_k(I_0)}\Phi_f(J) \\
&= \sum_{k=0}^{\infty} 2^{(2\alpha-n)k+nk}\cdot C_{2\beta}2^{-2\beta(k+j)} \\
&= C_{2\beta}2^{-2\beta j}\sum_{k=0}^{\infty} 2^{2(\alpha-\beta)k} \leq \tfrac{C_{2\beta}}{1-2^{\alpha-\beta}}.
\end{aligned}$$

Case 2, $I = [0,2^j]^n$, $j > 0$. Consider $J \in \mathcal{D}_k(I)$. If $k \leq j$, then for any l, either J is disjoint from I_0, in which case $f_l(x) \equiv 0$, or it contains it, which occurs only in one case, namely $J = [0, 2^{j-k}]^n$, in which case $f_l(J) = 0$. Hence, $f_l(x) - f_l(J) = f_l(x)$ if $x \in I_0$ and $f_l(x) = 0$ if $x \in J \setminus I_0$. Thus,

$$|f(x) - f(J)| = \begin{cases} |\sum_{l=0}^{\infty} f_l(x)| & \text{if } \ x \in I_0 \\ 0 & \text{if } \ x \in J\setminus I_0. \end{cases}$$

and by the orthogonality of $\{f_l\}_{l\geq 0}$ on I_0,

$$\begin{aligned}
\Phi_f(J) &= \tfrac{1}{|J|}\int_{I_0} |\sum_{l=0}^{\infty} f_l(x)|^2 dx \\
&= \tfrac{1}{|J|}\sum_{l=0}^{\infty}\int_{I_0} |f_l(x)|^2 dx \\
&= \tfrac{|I_0|}{|J|}\sum_{l=0}^{\infty} 2^{-2\beta l} = C_{2\beta}2^{-n(j-k)}.
\end{aligned}$$

If $k > j$, then for $0 < k - j \leq l$, again either J is disjoint from I_0, in which case $f_l(J) = 0$, or it is contained in it, i.e. $J \in \mathcal{D}_{k-j}(I_0)$, and again, $f_l(J) = 0$ and $f_l(x) - f_l(J) = f_l(x)$.

As for $k-j > l$, we also have that either J is disjoint from I_0 and $|f_l(x) - f_l(J)| = 0$, or $J \in \mathcal{D}_{k-j}(I_0)$, but now f_l is constant on J, resulting in $f_l(x) - f_l(J) = 0$.

It follows that

$$
\begin{aligned}
|f(x) - f(J)| &= |\sum_{l=0}^{\infty}[f_l(x) - f_l(J)]| \\
&= |\sum_{l=k-j}^{\infty} f_l(x)|.
\end{aligned}
$$

Once more, by the orthogonality of the sequence $\{f_l\}_{l\geq k-j}$ on J, we have that

$$
\begin{aligned}
\Phi_f(J) &= \tfrac{1}{|J|}\int_J |\sum_{l=k-j}^{\infty} f_l(x)|^2 dx \\
&= \tfrac{1}{|J|}\sum_{l=k-j}^{\infty}\int_J |f_l(x)|^2 dx \\
&= \sum_{l=k-j}^{\infty} 2^{-2\beta l} = C_{2\beta}2^{-2\beta(k-j)}.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\Psi_{f,\alpha}(I) &= C_{2\beta}\left(\sum_{k=0}^{j} 2^{(2\alpha-n)k}\cdot 2^{-n(j-k)} + \sum_{k=j+1}^{\infty} 2^{(2\alpha-n)k}\sum_{J\in\mathcal{D}_{k-j}(I_0)} 2^{-2\beta(k-j)}\right) \\
&= C_{2\beta}\left(\sum_{k=0}^{j} 2^{2(\alpha-\beta)k+2\beta(k-j)+(2\beta-n)j} + \sum_{k=j+1}^{\infty} 2^{(2\alpha-n)k+(k-j)n-2\beta(k-j)}\right) \\
&\leq C_{2\beta}\sum_{k=0}^{\infty} 2^{2(\alpha-\beta)k+(2\beta-n)j} = C_{2\beta}2^{(2\beta-n)j}\sum_{k=0}^{\infty} 2^{2(\alpha-\beta)k} \\
&= \tfrac{C_{2\beta}\ell(I)^{2\beta-n}}{1-2^{\alpha-\beta}}.
\end{aligned}
$$

provided $\alpha < \beta$.

As a result, we have, for all dyadic cubes I in $\mathbb{R}^n$,

$$
\Psi_{f,\alpha}(I) \leq \frac{C_{2\beta}\max\{1,\ \ |I|\}^{\frac{2\beta}{n}-1}}{1-2^{\alpha-\beta}} \leq C_{\alpha,\beta} < \infty \tag{3.23}
$$

when $\beta \leq \frac{n}{2}$.

Lastly, we use Lemma 1 to estimate $\Psi_{f,\alpha}(I)$ for any cube $I \subset \mathbb{R}^n$. Here we have to restrict to $\alpha < 1/2$.

$$\begin{aligned}
\Psi_{f,\alpha}(I) &\leq C_n \left[\sum_{j=1}^{2^n} \Psi_{f,\alpha}(I^j) + \sum_{1\leq i<j\leq 2^n} |f(I^i) - f(I^j)|^2\right] \\
&\leq C_n \left(C_{\alpha,\beta} + \frac{2^{2n} - 2^n}{2} \cdot 4C_\beta^2\right) \\
&= C_{n,\alpha,\beta}.
\end{aligned}$$

We are done.

Theorem 4 states that a borderline case occurs when $\beta = \alpha$. As one can see from the definition of f_β, it has intricate detailed structure on any small scale. However, it will be clearer later that the complexity of the function, in particular, the oscillation on a small scale reduces as β increases from 0 to 1, which is consistent with Theorem 4. This illustrates some form of regularity for functions in $Q_\alpha(\mathbb{R}^n)$.

In addition, by [14], Theorem 2.3, $Q_{\alpha'} \subset Q_\alpha$ for $\alpha < \alpha'$. f_β provides another example showing that this inclusion is strict.

Chapter 4

Fractal properties and dimensions

From its definition, f_β can be defined in a simple recursive way. Moreover, when we look at any small subcube of I_0, the complexity of the function on it is the same as that on I_0. We say that the function has a fine structure. In order to measure the complexity of the function, we discuss the analytic and the fractal properties of f_β on I_0 for all values of $\beta > 0$. We will show, for $0 < \beta < 1$, that the graph of f_β has a non-integer fractal dimension.

4.1 Analytic properties of f_β for $n = 1$

In what follows $f = f_\beta$ defined on I_0, unless stated otherwise. We mainly discuss the case $\mathbb{R}^1$, then we generalize those results to higher dimensions.

Proposition 1. *Let $n = 1$. When $\beta \neq 1$, The function f is a right continuous function on $I_0 = [0, 1]$. It is discontinuous at all dyadic points in I_0, and continuous elsewhere. In other words, f is continuous almost everywhere in the sense of 1-dimensional Lebesgue measure, whereas its set of discontinuity points is dense in I_0. In addition, f is not monotone in any subinterval of I_0.*

When $\beta = 1$, f is a linear function: $f(x) = 2 - 4x$, $\forall x \in [0, 1)$.

Proof:

Recall that the dyadic points are the endpoints of the dyadic intervals in $[0, 1]$, or the points with finite binary expansion. Denote by E_k the set of the end points of all dyadic intervals in $\mathcal{D}_k(I_0)$, and let $E = \bigcup_{k=0}^{\infty} E_k$, the set of dyadic points.

<u>Part 1 of the proof, f is continuous on $I_0 \setminus E$:</u>

By the convergence of $\sum_{l\geq 0} |f_l(y) - f_l(x)|$, we have

$$f(y) - f(x) = \sum_{l\geq 0} (f_l(y) - f_l(x)).$$

Let $x \in I_0 \setminus E$. Since $x \notin E$, $\forall k \geq 0$, $x \notin E_k$, so there exists a unique subinterval in $\mathcal{D}_k(I_0)$, denote it by $J_k(x)$, such that x is an interior point of $J_k(x)$. Moreover, we have

$$J_0(x) \supset J_1(x) \supset \cdots \supset J_k(x) \supset \cdots$$

and f_l $(l < k)$, as well as the partial sum $\sum_{l=0}^{k-1} f_l$, are constant on $J_k(x)$.

Let $\epsilon > 0$. For any $y \in J_k(x)$,

$$|f(y) - f(x)| \leq \sum_{l=k}^{\infty} |f_l(y) - f_l(x)| \leq 2\sum_{l=k}^{\infty} 2^{-\beta l} = 2 \cdot 2^{-\beta k} \sum_{l=0}^{\infty} 2^{-\beta l} = 2C_\beta 2^{-\beta k} < \epsilon, \tag{4.1}$$

for k sufficiently large. So, f is continuous at x.

<u>Part 2 of the proof, f is discontinuous on E:</u>

First, we have that $f(x)$ is discontinuous at $x = 0$ and $x = 1$, since

$$f(0^+) = \textstyle\sum_{l=0}^{\infty} 2^{-\beta l} = C_\beta \neq 0 = f(0^-)$$
$$f(1^-) = -\textstyle\sum_{l=0}^{\infty} 2^{-\beta l} = -C_\beta \neq 0 = f(1^+).$$

Now, consider the unique point $x = \frac{1}{2} \in E_1 \setminus E_0$.

Claim: $f(x)$ is right continuous at $\frac{1}{2}$ with

$$f\left(\frac{1}{2}\right)=f\left(\frac{1}{2}^{+}\right)=-1+\sum_{l=1}^{\infty}2^{-\beta l}=C_\beta-2,$$

while the left limit

$$f\left(\frac{1}{2}^{-}\right)=1-\sum_{l=1}^{\infty}2^{-\beta l}=-(C_\beta-2).$$

Since $C_\beta=2$ iff $\beta=1$, if $\beta\neq 1$, $f(x)$ is discontinuous at $\frac{1}{2}$ with a jump $2(C_\beta-2)$.

Proof of the claim:

Let $k\geq 1$, $a_k=\frac{1}{2}-2^{-k}$ and $d_k=\frac{1}{2}+2^{-k}$. Let $J_k=[a_k,\frac{1}{2}]$ and $J'_k=[\frac{1}{2},d_k]$, which are two adjacent subintervals in $\mathcal{D}_k(I_0)$ touching at $\frac{1}{2}$, such that,

$$J_k\supset J_{k+1},\quad \text{and}\quad J'_k\supset J'_{k+1},\quad \forall k\geq 0.$$

We have

1. $f_0(x)=\begin{cases}1 & x\in[a_k,\frac{1}{2});\\ -1 & x\in[\frac{1}{2},d_k).\end{cases}\qquad \forall k\geq 1;$

2. $f_l(x)=\begin{cases}-2^{-\beta l} & x\in[a_k,\frac{1}{2});\\ 2^{-\beta l} & x\in[\frac{1}{2},d_k).\end{cases}\qquad \forall l\geq 1\quad \&\quad k\geq l+1.$

Let $A=-1+\sum_{l=1}^{\infty}2^{-\beta l}=C_\beta-2$.

If $x\in[a_k,\frac{1}{2})$, $f(x)=1-2^{-\beta}-\cdots-2^{-\beta(k-1)}+\sum_{l=k}^{\infty}f_l(x)$, then,

$$|f(x)-(-A)|\leq\sum_{l=k}^{\infty}|f_l(x)+2^{-\beta l}|\leq 2\sum_{l=k}^{\infty}2^{-\beta l}=2C_\beta 2^{-\beta k}\to 0,\quad \text{as}\quad k\to\infty.$$

If $x\in[\frac{1}{2},d_k)$, $f(x)=-1+2^{-\beta}+\cdots+2^{-\beta(k-1)}+\sum_{l=k}^{\infty}f_l(x)$, then

$$|f(x)-A)|\leq\sum_{l=k}^{\infty}|f_l(x)-2^{-\beta l}|\leq 2\sum_{l=k}^{\infty}2^{-\beta l}=2C_\beta 2^{-\beta k}\to 0,\quad \text{as}\quad k\to\infty.$$

So, $f\left(\frac{1}{2}^+\right) - f\left(\frac{1}{2}^-\right) = 2A$. In addition, $f\left(\frac{1}{2}\right) = A = C_\beta - 2$ since $\frac{1}{2} \in [\frac{1}{2}, d_k)$, $\forall k$.

This proves the claim.

Comparing $f(0) = f(0^+) = C_\beta$ with $f\left(\frac{1}{2}-\right) = -(C_\beta - 2)$, as well as $f\left(\frac{1}{2}\right) = f\left(\frac{1}{2}+\right) = C_\beta - 2$ with $f(1^-) = -C_\beta$, and using the upper bound in (4.1), we have that $\forall J \in \mathcal{D}_1(I_0)$,

$$\sup_{x\in J} f(x) - \inf_{x\in J} f(x) = 2C_\beta 2^{-\beta}. \tag{4.2}$$

Now, for any $x \in E \setminus \{0, \frac{1}{2}, 1\}$, there exists a $k \geq 1$, such that $x \in E_{k+1} \setminus E_k$. Then, there exist an subinterval $J_k(x) \in \mathcal{D}_k(I_0)$, such that x is the middle point of $J_k(x)$. Furthermore, $f_0, \cdots, f_{k-1}$, as well as the partial sum $\sum_{l=0}^{k-1} f_l$ are constants on $J_k(x)$. Denote by $a = \sum_{l=0}^{k-1} f_l(x)$ for $x \in J_k(x)$. Similarly to the case $x = \frac{1}{2}$, we have the left limit,

$$f(x^-) = a + 2^{-\beta k} - \sum_{l=k+1}^{\infty} 2^{-\beta l} = a + 2^{-\beta k}(2 - C_\beta), \tag{4.3}$$

and the right limit,

$$f(x^+) = a - 2^{-\beta k} + \sum_{l=k+1}^{\infty} 2^{-\beta l} = a + 2^{-\beta k}(C_\beta - 2). \tag{4.4}$$

Thus,

$$f(x^+) - f(x^-) = 2(C_\beta - 2)2^{-\beta k}. \tag{4.5}$$

Therefore, there is a jump: $2(C_\beta - 2)2^{-\beta k}$ at x for $\beta \neq 1$.

A corollary here will be useful later in the estimate of the Box dimension of the graph of f:

Corollary 1. *Given a $k \geq 0$, and a dyadic subinterval $J \in \mathcal{D}_k(I_0)$, we have,*

$$\sup_{x\in J} f(x) - \inf_{x\in J} f(x) = 2C_\beta 2^{-\beta k} \tag{4.6}$$

Proof: Similar to the proof of (4.2). The lower bound is obtained by comparing the value of the left end point which is right continuous and the left limit to the right end point.

Part 3 of the proof, f is not monotone in any subinterval of I_0:

We just give a proof for the case $0 < \beta < 1$. The proof for the case $\beta > 1$ is similar.

First, we show that $f(x)$ is not monotone in $I_0 = [0, 1]$:

On the one hand, $f(x)$ is not monotone increasing in I_0 since for $\frac{3}{4} > \frac{1}{2}$, we have

$$f\left(\frac{3}{4}\right) = -1 - 2^{-\beta} + \sum_{l=2}^{\infty} 2^{-\beta l} < A = f\left(\frac{1}{2}\right).$$

On the other hand, we know, from the proof of part 2, that $\lim_{k\to\infty} a_k = -A$. So there exists a $k_0 > 0$, such that, $f(a_{k_0}) < -A/2 < f\left(\frac{1}{2}\right)$. Note that $a_{k_0} = \frac{1}{2} - 2^{-k_0} < \frac{1}{2}$, so $f(x)$ is not monotone decreasing in I_0.

Next, let $J \in \mathcal{D}_k(I_0)$. Similarly to the case I_0, we look at x_J, the middle point of J. Also from the proof of part 2, there exist two points x' and x'' in J, such that, $x' < x_J < x''$, while $f(x') < f(x_J)$ and $f(x'') < f(x_J)$. So, f is not monotone in any dyadic subinterval of I_0.

Finally, we conclude that f is not monotone on any subinterval in I_0 since any interval contains a dyadic interval.

Part 4 of the proof, $f_{\beta=1}(x) = 2 - 4x$, $(x \in (0, 1))$:

Recall the dyadic expression (3.17) and set $\beta = 1$, then

$$f(x) = 2 - 4\sum_{l=0}^{\infty} b_l 2^{-(l+1)} = 2 - 4x.$$

So, $f_{\beta=1}$ is linear on $[0, 1)$ and 0 and 1 are the only two discontinuous points of

$f_{\beta=1}(x)$ for $x \in \mathbb{R}$, since

$$f_{\beta=1}(0) = 2 \neq 0 = f_{\beta=1}(0^-), \text{ and } f_{\beta=1}(1^-) = -2 \neq 0 = f_{\beta=1}(1).$$

4.2 Analytic properties of f_β for $n > 1$

For the case $n > 1$, we get a parallel theorem to Theorem 1.

Proposition 2. *Let $n > 1$. For $\beta \neq 1$. The function f is continuous at every point which is not on the surface of any dyadic cube in I_0, and discontinuous at all dyadic points in $[0,1)^n$, i.e., points whose coordinates are dyadic points in $[0,1)$. Therefore, f is continuous in I_0 almost everywhere in the sense of n-dimensional Lebesgue measure, and its discontinuous points are dense in I_0. Moreover, f is not monotone along any coordinate direction in any subcube of I_0.*

For $\beta = 1$, f is discontinuous at some dyadic points and the set of those points is still dense in I_0.

Proof:

Denote by ∂J the surface of the cube J. The same proof as that in the case $n = 1$ will give the continuity of f in $I_0 \setminus (\bigcup_{J \in \mathcal{D}(I_0)} \partial J)$. In particular, (4.1) remains true for the case $n > 1$. We only need to show the discontinuity.

Recall $x_i = \sum_{l=0}^{\infty} b_l^i 2^{-(l+1)}$ $(i = 1, \cdots, n)$, with $0.b_0^i b_1^i b_2^i \cdots$ being the binary extension of the coordinate x_i. Rewrite the binary expression (3.18)

$$f(x) = \sum_{l=0}^{\infty} (-1)^{b_l^1} (-1)^{\sum_{i=2}^{n} b_l^i} 2^{-\beta l}.$$

Without lose of generality, we just look at those points of which the first coordinate x_1 is dyadic. Moreover, we can simplify the proof further by just checking the discontinuity of $f(x)$ at $(\frac{1}{2}, x_2, \cdots, x_n)$, a similar approach will work for other such points, as we have shown for $n = 1$.

Again, let $a_k = \frac{1}{2} - 2^{-k}$ and $d_k = \frac{1}{2} + 2^{-k}$, for $k \geq 1$. Fix $x_2, \cdots, x_n$, we have

$$1.\ f_0(x) = \begin{cases} (-1)^{\sum_{i=2}^n b_0^i} & x_1 \in [a_k, \frac{1}{2}); \\ -(-1)^{\sum_{i=2}^n b_0^i} & x_1 \in [\frac{1}{2}, d_k). \end{cases} \qquad \forall k \geq 1;$$

$$2.\ f_l(x) = \begin{cases} -(-1)^{\sum_{i=2}^n b_l^i} 2^{-\beta l} & x_1 \in [a_k, \frac{1}{2}); \\ (-1)^{\sum_{i=2}^n b_l^i} 2^{-\beta l} & x_1 \in [\frac{1}{2}, d_k). \end{cases} \qquad \forall l \geq 1 \quad \& \quad k \geq l+1.$$

Let $A = -(-1)^{\sum_{i=2}^n b_0^i} + \sum_{l=1}^{\infty} (-1)^{\sum_{i=2}^n b_l^i} 2^{-\beta l}$.

If $x_1 \in [a_k, \frac{1}{2})$,

$f(x) = (-1)^{\sum_2^n b_0^i} - (-1)^{\sum_2^n b_1^i} 2^{-\beta} - \cdots - (-1)^{\sum_2^n b_{k-1}^i} 2^{-\beta(k-1)} + \sum_{l=k}^{\infty} f_l(x)$, then,

$$|f(x) - (-A)| \leq \sum_{l=k}^{\infty} |f_l(x) + 2^{-\beta l}| \leq 2 \sum_{l=k}^{\infty} 2^{-\beta l} = 2C_\beta 2^{-\beta k} \to 0, k \to \infty.$$

If $x_1 \in [\frac{1}{2}, d_k)$,

$f(x) = -(-1)^{\sum_2^n b_0^i} + (-1)^{\sum_2^n b_1^i} 2^{-\beta} + \cdots + (-1)^{\sum_2^n b_{k-1}^i} 2^{-\beta(k-1)} + \sum_{l=k}^{\infty} f_l(x)$, then

$$|f(x) - A| \leq \sum_{l=k}^{\infty} |f_l(x) - 2^{-\beta l}| \leq 2 \sum_{l=k}^{\infty} 2^{-\beta l} = 2C_\beta 2^{-\beta k} \to 0, k \to \infty.$$

So, $f\left(\frac{1}{2}^+, x_2, \cdots, x_n\right) - f\left(\frac{1}{2}^-, x_2, \cdots, x_n\right) = 2A$, and $f\left(\frac{1}{2}, x_2, \cdots, x_n\right) = A$.

Generally, if $\sum_{i=2}^n b_l^i$ =even $\forall l$, $A = -1 + \sum_{l=1}^{\infty} 2^{-\beta l} = C_\beta - 2$; if $\sum_{i=2}^n b_l^i$ =odd $\forall l$, $A = 1 - \sum_{l=1}^{\infty} 2^{-\beta l} = -(C_\beta - 2)$. In both cases, $A = 0 \Leftrightarrow \beta = 1$.

Specifically, we consider $\sum_{i=2}^n b_l^i = 0$ $\forall l \geq 1$, that is, $x_i = 0$ or $\frac{1}{2}$ $\forall i = 2, \cdots, n$. In this case, x is a dyadic point in $\mathbb{R}^n$. We have

$$A = -(-1)^{\sum_{i=2}^n b_0^i} + \sum_{l=1}^{\infty} 2^{-\beta l}.$$

There are again two typical cases:

1. $\underline{\sum_{i=2}^n b_0^i \text{ =even}}$, for instance, $b_0^2 = \cdots = b_0^n = 0$, $A = C_\beta - 2 = 0 \Leftrightarrow \beta = 1$.

2. $\sum_{i=2}^n b_0^i$ =odd, for instance, $b_0^2 = 1, b_0^3 = \cdots = b_0^n = 0$, $A = C_\beta > 0$, $\forall \beta > 0$

In general, let x be a dyadic point in $[0,1)^n$, i.e., $x_i \neq 1$ for $i = 1, \cdots, n$. There exists an integer $k \geq 0$, such that, $\forall l > k$, $b_l^i = 0$ for $i = 1, \cdots, n$, and $\exists i \in \{1, \cdots, n\}$, such that, $b_k^i = 1$. Without lose of generality, we assume $b_k^1 = 1$. Similarly, we have

(a). $\sum_{i=2}^n b_k^i$ =even, $\quad f(x_1^+, x_2, \cdots, x_n) - f(x_1^-, x_2, \cdots, x_n) = 2(C_\beta - 2)2^{-\beta k}$;

(b). $\sum_{i=2}^n b_k^i$ =odd, $\quad f(x_1^+, x_2, \cdots, x_n) - f(x_1^-, x_2, \cdots, x_n) = 2C_\beta 2^{-\beta k}$.

In conclusion, when $\beta \neq 1$, f is discontinuous at all dyadic points inside I_0. When $\beta = 1$, on each subcube $J \in \mathcal{D}_k(I_0)$, there exists at least one point satisfying (b). So unlike in $\mathbb{R}^1$, f is no longer continuous everywhere, but has a discontinuous set dense in I_0 too. In particular, let x_J be the center point of the cube J. Then $\forall J \in \mathcal{D}_k(I_0)$, when n is even, f is discontinuous at x_J, while when n is odd, f is continuous at x_J, with limit $\sum_{l=0}^{k-1} f_l(x_J)$.

Following from the proof, we have

Corollary 2. *Let $n \geq 2$. Given $k \geq 0$ and a dyadic subcube $J \in \mathcal{D}_k(I_0)$, we have,*

$$\sup_{x \in J} f(x) - \inf_{x \in J} f(x) = 2C_\beta 2^{-\beta k}. \tag{4.7}$$

Finally, based on the above proof for the discontinuity at dyadic points, for $\beta \neq 1$, we can get that f is not monotone along any coordinate direction in any subcube of I_0 by an analogous approach to that of part 3, Section 4.1. Hence, f oscillates on all small scales.

4.3 Preliminaries of fractal geometry

In the following section, we will show that the graph of f is a self-affine set in $\mathbb{R}^{n+1}$ with a non-integer fractal dimension for $0 < \beta < 1$. So, f is a fractal function for β in this range (see [16], Introduction for the definition of a fractal).

We will need the following concepts (see [16], Section 9.4):

Definition 1. *Self-affine set: Let D be a closed subset of $\mathbb{R}^n$. Let $\tau_1, \cdots, \tau_m$ be affine, contractive transformations from D to D. A non-empty compact set F is called self-affine with $\tau_1, \cdots, \tau_m$ if F is invariant for the τ_i, i.e. F satisfies $F = \bigcup_{i=1}^m \tau_i(F)$.*

Remark: The existence and uniqueness of such an invariant set is guaranteed by Theorem 9.1 in [16].

Definition 2. *Singular values of a contracting and non-singular mapping: Assume that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a contracting and non-singular linear mapping. The singular values $1 > \eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$ are defined as the positive square roots of the eigenvalues of T^*T, where T^* is the adjoint of T.*

Definition 3. *Singular value function of T: Let $0 \leq s \leq n$. The singular function of T is given by*

$$\phi^s(T) = \eta_1 \eta_2 \cdots \eta_r^{s-r+1}, \tag{4.8}$$

where r is the integer for which $r - 1 < s \leq r$.

We will use the following notion of dimension, due to Falconer (see [16], Theorem 9.12):

$$d(T_1, \cdots, T_m) = \inf \left\{ s : \sum_{k=1}^{\infty} \sum_{S_k} \phi^s(T_{i_1} \circ \cdots \circ T_{i_k}) < \infty \right\} \tag{4.9}$$

where S_k denotes the set of all k-term sequences $\{i_1, \cdots, i_k\}$ with $1 \leq i_j \leq m$.

Following [20], we call (4.9) the "Falconer dimension" of the collection $\{T_1, \cdots, T_m\}$. It is related to the Hausdorff dimension ($\dim_H F$) and Box dimension ($\dim_B F$) as follows:

Theorem 8 (Falconer). *Let $\tau_i = T_i + b_i$, $(i = 1, \cdots, m)$ be affine, contractive transformations on $\mathbb{R}^n$, where T_i are linear contractive mappings and b_i are vectors in $\mathbb{R}^n$.*

If G is the affine invariant set satisfying

$$G = \bigcup_{i=1}^{m} (T_i(G) + b_i), \tag{4.10}$$

then $\dim_H G = \dim_B G = d(T_1, \cdots, T_m)$ *for almost all* $\{b_1, \cdots, b_m\} \in \mathbb{R}^{nm}$ *with respect to the nm-dimensional Lebesgue measure.*

Based on this theorem, for G satisfying (4.10), we also call $d(T_1, \cdots, T_m)$ the Falconer dimension of the set G and denote it by $\dim_F G$.

Box dimension is also called fractal dimension [2]. There are several equivalent definitions of this dimension. We adopt the following one which is convenient for our purpose (see [16], Section 3.1):

Definition 4. *Box dimension of a set $F \subset \mathbb{R}^n$: Let $\delta > 0$. A collection of cubes of the form*

$$[m_1\delta, (m_1 + 1)\delta] \times \cdots \times [m_n\delta, (m_n + 1)\delta]$$

where $m_1, \cdots, m_n$ are integers, is referred to as a δ-mesh of $\mathbb{R}^n$.

Let F be a non-empty bounded subset of $\mathbb{R}^n$ and let $N_\delta(F)$ be the number of δ-mesh cubes that intersect F. The upper and lower Box dimensions of F are defined, respectively, as $\overline{\dim}_B(F) = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$ *and* $\underline{\dim}_B(F) = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$. *If* $\overline{\dim}_B(F) = \underline{\dim}_B(F)$, *the common value is called the Box dimension of F and denoted by* $\dim_B(F)$.

In particular, the limit as δ tends to zero can be taken through $\delta_k = c^k$ with $0 < c < 1$.

The upper and lower Box dimension, $\overline{\dim}_B(F)$ and $\underline{\dim}_B(F)$, are both monotone. In addition, $\overline{\dim}_B(F)$ is finite stable, that is, $\overline{\dim}_B(E \cup F) = \max\{\overline{\dim}_B(E), \overline{\dim}_B(F)\}$, but there is no analogous result for $\underline{\dim}_B(F)$. (see [16], section 3.2)

4.4 Fractal dimension of f_β

Recall $G_f = \{(x, f(x)), x \in I_0\}$, the graph of the function f over the cube I_0. We will apply Theorem 8 to the closure $\bar{G}_f$ instead of G_f because of the compactness of the former. Since the discontinuous points of $f(x)$ are on the boundary of the dyadic subcubes of I_0, we have $\dim_B(\bar{G}_f \setminus G_f) \le n$. On the other hand, $\dim_B G_f \ge (\mathrm{Proj}_{\mathbb{R}^n} G_f) = \dim_B I_0 = n$. By the finite stableness of $\overline{\dim}_B$, one can expect that $\dim_B G_f = \dim_B \bar{G}_f$. Later, this is confirmed by calculating $\dim_B G_f$ from its definition directly.

In what follows we show that $\bar{G}_f$ is self-affine on $I_0 \times [-C_\beta, C_\beta]$, and compute its Falconer dimension and Box dimension. We discuss the self-affinity of $\bar{G}_f$ for the cases $n = 1$ and $n > 1$, respectively. Again, we consider $n = 1$ first.

Proposition 3. *Let τ_1 and τ_2 be affine transformations defined as follows:*

$$\tau_i(x, y) = T_i(x, y) + b_i, \quad i = 1, 2, \tag{4.11}$$

where

$$T_1 = T_2 = T = \begin{pmatrix} 2^{-1} & 0 \\ 0 & 2^{-\beta} \end{pmatrix}, \ b_1 = (0, 1) \textit{ and } b_2 = (\frac{1}{2}, -1). \tag{4.12}$$

For $n = 1$, we have that $\bar{G}_f$ is affine invariant of $\{\tau_1, \tau_2\}$, that is, it satisfies

$$G = \tau_1(G) \cup \tau_2(G) = (T_1(G) + b_1) \cup (T_2(G) + b_2). \tag{4.13}$$

In addition,

$$\dim_F \bar{G}_f = d(T_1, T_2) = \begin{cases} 2 - \beta & \textit{if } \ 0 < \beta < 1, \\ 1 & \textit{if } \ \beta \ge 1 \end{cases} \tag{4.14}$$

(See Figures 1, 2 and 3 on page 48).

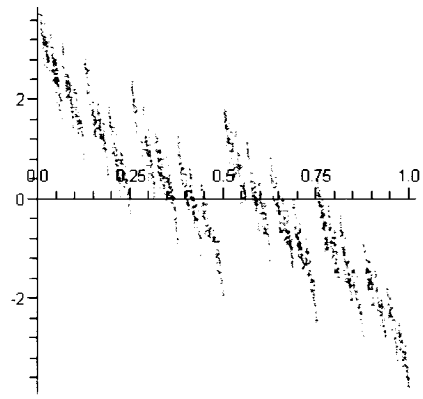

Figure 4.1: The affine invariant set satisfying (4.13) with $\beta = -\ln 0.77/\ln 2$.

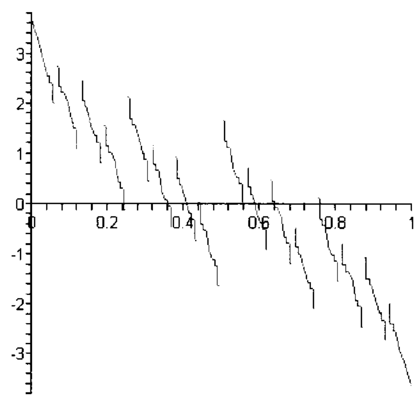

Figure 4.2: The partial sum of the first 7 terms of $f_\beta(x)$ with $\beta = -\ln 0.77/\ln 2$.

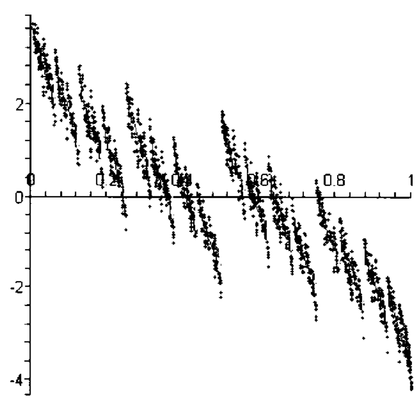

Figure 4.3: The two graphs above coincide well.

Proof of Proposition 3:

Note that for $x \in I_0 = [0,1]$, $f(x) = \sum_{l=0}^{\infty} f_l(x)$ can be written as:

$$f(x) = \begin{cases} f_0(x) + 2^{-\beta} f(2x) & = & 1 + 2^{-\beta} f(2x) & \text{if } \ 0 \le x < \frac{1}{2}; \\ f_0(x) + 2^{-\beta} f(2(x - \frac{1}{2})) & = & -1 + 2^{-\beta} f(2(x - \frac{1}{2})) & \text{if } \ \frac{1}{2} \le x < 1. \end{cases} \tag{4.15}$$

Hence, $\bar{G}_f$ is affine invariant under the following two contracting mappings in $\mathbb{R}^2$:

$$\begin{aligned} \tau_1 : (x, y) &\longrightarrow (2^{-1}x, 2^{-\beta}y) + (0, 1), \\ \tau_2 : (x, y) &\longrightarrow (2^{-1}x, 2^{-\beta}y) + (\tfrac{1}{2}, -1). \end{aligned} \tag{4.16}$$

The matrix forms of τ_i $(i = 1, 2)$ are given by (4.11).

To prove (4.14), we consider $0 < \beta < 1$ and $\beta \ge 1$, separately.

First, for $\underline{0 < \beta < 1}$, the singular values of T in (4.12) are $\eta_1 = 2^{-\beta} > \eta_2 = 2^{-1}$. Correspondingly, the singular values of $T^k = \underbrace{T \circ \cdots \circ T}_{k \text{ times}}$ are $\eta_1^k = 2^{-\beta k} > \eta_2^k = 2^{-k}$.

We consider $1 \le s \le 2$ instead of $0 \le s \le 2$ in Definition 3 since as a graph of a function, the dimension of G_f is greater than or equal to that of I_0, its projection on $\mathbb{R}^1$. (See [16], Chapter 6).

We calculate the singular function $\phi^s(T^k)$ given by (4.8) in three cases as follows:

Case 1, $s = 2$: then $r = 2$ and

$$\phi^s(T^k) = \eta_1^k (\eta_2^k)^{s-r+1} = 2^{-\beta k} 2^{-k} = 2^{-(1+\beta)k}.$$

Case 2, $1 < s < 2$: also $r = 2$, and

$$\phi^s(T^k) = \eta_1^k (\eta_2^k)^{s-r+1} = 2^{-\beta k} 2^{-(s-1)k} = 2^{(1-\beta-s)k}.$$

Case 3, $s = 1$: then $r = 1$, and

$$\phi^s(T^k) = (\eta_1^k)^{s-r+1} = 2^{-\beta k}.$$

Now, calculate $d(T_1, T_2)$ in (4.9):

Since for all k-term sequences $\{i_1, \cdots, i_k\}$ with $i_j = 1$ or 2 ($j = 1, \cdots, k$), we have $T_{i_j} = T$ and $T_{i_1} \circ \cdots \circ T_{i_k} = T^k$, so

$$\sum_{S_k} \phi^s(T_{i_1} \circ \cdots \circ T_{i_k}) = 2^k \phi^s(T^k).$$

Case 1, $s = 2$:

$$\sum_{k=1}^{\infty} 2^k \phi^s(T^k) = \sum_{k=1}^{\infty} 2^k 2^{-(1+\beta)k} = \sum_{k=1}^{\infty} 2^{-\beta k} = \frac{2^{-\beta}}{1 - 2^{-\beta}} < \infty.$$

Case 2, $1 < s < 2$:

$$\sum_{k=1}^{\infty} 2^k \phi^s(T^k) = \sum_{k=1}^{\infty} 2^k 2^{(1-\beta-s)k} = \sum_{k=1}^{\infty} 2^{k(2-\beta-s)} < \infty \quad \Leftrightarrow \quad s > 2 - \beta.$$

Case 3, $s = 1$:

$$\sum_{k=1}^{\infty} 2^k \phi^s(T^k) = \sum_{k=1}^{\infty} 2^k 2^{-\beta k} = \sum_{k=1}^{\infty} 2^{(1-\beta)k} = \infty.$$

Consequently,

$$d(T_1, T_2) = \inf\{s : s > 2 - \beta\} = 2 - \beta.$$

Second, for $\beta \geq 1$, the singular values of T are $\eta_1 = 2^{-1} \geq \eta_2 = 2^{-\beta}$, and the singular values of T^k are $\eta_1^k = 2^{-k} \geq \eta_2^k = 2^{-\beta k}$.

We only need to look at $1 < s < 2$. Thus, for $r = 2$, the singular function

$$\phi^s(T^k) = \eta_1^k (\eta_2^k)^{s-r+1} = 2^{-k} 2^{-(s-1)\beta k} = 2^{-k-(s-1)\beta k},$$

and

$$\sum_{k=1}^{\infty} 2^k \phi^s(T^k) = \sum_{k=1}^{\infty} 2^k 2^{-k-(s-1)\beta k} = \sum_{k=1}^{\infty} 2^{-(s-1)\beta k} < \infty.$$

Therefore,

$$d(T_1, T_2) = \inf\{s : s > 1\} = 1.$$

We now generalize Proposition 3 to $\mathbb{R}^n$ for $n > 1$.

Let $x = (x_1, ..., x_n) \in I_0 = [0,1]^n$, and consider the graph $G_f = \{(x, f(x)). x \in I_0\} \subset \mathbb{R}^n \times \mathbb{R}^1$. suppose the contracting mappings $\tau_i : \mathbb{R}^n \times \mathbb{R}^1 \longrightarrow \mathbb{R}^n \times \mathbb{R}^1$ are affine transformations given by $\tau_i(x,y) = T_i(x,y) + b_i$, where T_i are $n+1$ dimension diagonal matrices, with $T_i = T = \mathrm{diag}\{2^{-1}, \cdots, 2^{-1}, 2^{-\beta}\}$ $\forall i$, $b_i = (b_{i,1}, ..., b_{i,n}, b_{i,n+1})$, with $b_{i,j} = 0$ or $\frac{1}{2}$ for $j = 1, ...n$ and $b_{i,n+1} = (-1)^{P(b_i)}$, where $P(b_i)$ denotes the number of times $\frac{1}{2}$ occurs in $b_{i,1}, ..., b_{i,n}$. There are in total 2^n such distinct τ_i.

Proposition 4. *$\bar{G}_f$ is the affine invariant set on $I_0 \times [-C_\beta, C_\beta]$, which satisfies*

$$G = \bigcup_{i=1}^{2^n} \tau_i(G), \tag{4.17}$$

where τ_i $(i = 1, \cdots, 2^n)$ are 2^n distinct affine contractions as described above. Moreover,

$$\dim_F \bar{G}_f = d(T_1, \cdots, T_{2^n}) = \begin{cases} n+1-\beta & \text{if } \ 0 < \beta < 1, \\ n & \text{if } \ \beta \geq 1. \end{cases} \tag{4.18}$$

Proof: The proof of Proposition 4 is analogous to that of Proposition 3. Here we just show the details of calculation of $s = d(T_1, \cdots, T_{2^n})$ for $0 < \beta < 1$.

Since $T^k = \mathrm{diag}\{2^{-k}, \cdots, 2^{-k}, 2^{-\beta k}\}$ $(k \geq 1)$, its singular values are

$$\eta_1^k = 2^{-\beta k} > \eta_2^k = \cdots = \eta_{n+1}^k = 2^{-k}.$$

Recall the singular function of T^k, $\phi^s(T^k) = \eta_1^k \eta_2^k \cdots \eta_r^{k(s-r+1)}$. Naturally, we just consider $n \le s \le n+1$:

Case 1, $s = n+1$: then $r = n+1$ and

$$\phi^s(T^k) = \eta_1^k \eta_2^k \cdots \eta_{n+1}^k = 2^{-\beta k} 2^{-nk} = 2^{-k(n+\beta)};$$

Case 2, $n < s < n+1$: also $r = n+1$,

$$\phi^s(T^k) = \eta_1^k \eta_2^k \cdots \eta_n^k \eta_{n+1}^{k(s-n)} = 2^{-\beta k} 2^{-(n-1)k} 2^{-k(s-n)} = 2^{k(1-\beta-s)};$$

Case 3, $s = n$: then $r = n$,

$$\phi^s(T^k) = \eta_1^k \eta_2^k \cdots \eta_n^k = 2^{-\beta k} 2^{-(n-1)k} = 2^{-k(n-1+\beta)}.$$

Since for any $1 \le i_j \le 2^n$, $T_{i_j} = T$ and $T_{i_1} \circ \cdots \circ T_{i_k} = T^k \ \forall (i_1, \cdots, i_k)$, we have

$$\sum_{S_k} \phi^s(T_{i_1} \circ \cdots \circ T_{i_k}) = 2^{nk} \phi^s(T^k).$$

Case 1, $s = n+1$:

$$\sum_{k=1}^{\infty} 2^{nk} \phi^s(T^k) = \sum_{k=1}^{\infty} 2^{nk} 2^{-\beta k} 2^{-nk} = \sum_{k=1}^{\infty} 2^{-\beta k} = \frac{2^{-\beta}}{1 - 2^{-\beta}} < \infty.$$

Case 2, $n < s < n+1$:

$$\sum_{k=1}^{\infty} 2^{nk} \phi^s(T^k) = \sum_{k=1}^{\infty} 2^{nk} 2^{k(1-\beta-s)} = \sum_{k=1}^{\infty} 2^{k(n+1-\beta-s)} < \infty \Leftrightarrow s > n+1-\beta.$$

Case 3, $s = n$:

$$\sum_{k=1}^{\infty} 2^{nk} \phi^s(T^k) = \sum_{k=1}^{\infty} 2^{nk} 2^{-k(n-1+\beta)} \sum_{k=1}^{\infty} 2^{(1-\beta)k} = \infty.$$

Consequently,

$$\dim_F \bar{G}_f = d(T_1, \cdots, T_m) = \inf\{s : s > n+1-\beta\} = n+1-\beta.$$

Now we come to the proof of

Theorem 5. *Let $G_{f_\beta} = \{(x, f_\beta(x)), x \in I_0\}$, the graph of f_β over I_0. Then $\bar{G}_{f_\beta}$ is an affine invariant set on $I_0 \times [-C_\beta, C_\beta]$. Moreover,*

$$\dim_B G_{f_\beta} = \dim_F \bar{G}_{f_\beta} = \begin{cases} n+1-\beta & \text{if } \ 0 < \beta < 1, \\ n & \text{if } \ \beta \geq 1, \end{cases} \tag{4.19}$$

Proof: This could be a corollary of Proposition 3 or 4 by noting Theorem 8. However, Theorem 8 gives no clue for which $b_1, \cdots, b_m$ the Hausdorff dimension and the Box dimension agree with the Falconer dimension (4.9). So, in what follows we calculate the Box dimension of the graph G_f directly.

Let $\delta_k = 2^{-k}$. By (4.7), the number of δ_k-mesh cubes in $\mathbb{R}^{n+1}$ in the column over each $J \in \mathcal{D}_k(I_0)$ which intersect G_f is at most $2C_\beta 2^{-\beta k}/2^{-k} + 2$. Since there are in total 2^{nk} many such J, $N_{\delta_k} \leq 2^{nk} \cdot (2C_\beta 2^{-\beta k}/2^{-k} + 2)$. So

$$\overline{\dim}_B G_f \leq \lim_{k\to\infty} \frac{\log[2^{nk} \cdot (2C_\beta 2^{(1-\beta)k} + 2)]}{-\log 2^{-k}} = \begin{cases} n+1-\beta & \text{if } \ 0 < \beta < 1; \\ n & \text{if } \ \beta \geq 1. \end{cases} .$$

When $\beta \geq 1$, $\dim_B G_f = n$ since $\underline{\dim}_B G_f \geq \underline{\dim}_B(\mathrm{Proj}_{\mathbb{R}^n} G_f) = \dim_B I_0 = n$ (see [16], Chapter 6).

When $0 < \beta < 1$, since the part of G_f over $J \in \mathcal{D}_k$ is affine to G_f, f restricted to J is onto from J to $[\inf_{x\in J} f(x),\ \sup_{x\in J} f(x)]$ by Lemma 3. It follows that the number of δ_k-mesh cubes in $\mathbb{R}^{n+1}$ in the column over each $J \in \mathcal{D}_k(I_0)$ intersecting G_f

is at least $2C_\beta 2^{-\beta k}/2^{-k}$. Therefore, $N_{\delta_k} \geq 2^{nk} \cdot 2C_\beta 2^{-\beta k}/2^{-k}$, and

$$\underline{\dim}_B G_f \geq \lim_{k\to\infty} \frac{\log[2C_\beta \cdot 2^{(n+1-\beta)k}]}{-\log 2^{-k}} = n+1-\beta,$$

Hence, $\dim_B G_f = n + 1 - \beta$.

Remark: As a counterexample, the function f_β is interesting only for $0 < \beta < \min\{1, \frac{n}{2}\}$ due to the properties of Q_α spaces. However, f_β is well defined for all $\beta > 0$. Moreover, the related mappings $\{\tau_1, \cdots, \tau_{2^n}\}$ is a system of affine contractions for all $\beta > 0$ (a system of similar contractions when $\beta = 1$.) Theorems 5 tells us that f_β is a fractal function if $0 < \beta < 1$, since for β in this range, $\dim_B G_f = n + 1 - \beta > n$. As a corollary of Theorem 4 and 5, the fractal dimension of f_β is related to the space $Q_\alpha(\mathbb{R}^n)$ by the fact that $f_\beta \in Q_\alpha$ $(0 < \alpha < 1)$ if and only if $n \leq \dim_B G_{f_\beta} < n+1-\alpha$. This raises an interesting question: is this a characterization of functions in the space $Q_\alpha(\mathbb{R}^n)$?

Chapter 5

The tent spaces T_q^p for $q \neq 2$

When Coifman, Meyer and Stein [3] studied the BMO-Hardy duality in the setting of tent spaces T_q^p, they gave a projection mapping from T_2^p to L^p, H^p, or BMO for $1 < p < \infty$, $p \leq 1$, or $p = \infty$, respectively. Conversely, by any standard square function, such as the Lusin area integral, one can map functions in L^p, H^p, or BMO to T_2^p for p in the corresponding range (see also [17]). Hence T_2^p are bijective to L^p, H^p, or BMO. In this chapter, we explore what happens to the relationship between tent spaces and L^p, H^p, and BMO when one modifies their theorem by change $q = 2$ to $q \neq 2$.

5.1 Connection of T_q^p with L^p, H^1 and BMO

We adopt the following notations from [3]:

Let $\Gamma(x)$ be the cone in $\mathbb{R}_+^{n+1}$ with aperture 1 whose vertex is at x, i.e.

$$\Gamma(x) = \{(y,t), |x-y| < t\}.$$

Denote by A_q and C_q two functionals mapping functions on $\mathbb{R}_+^{n+1}$ to functions on

$\mathbb{R}^n$, defined as:

$$A_q(f)(x) = \left(\int_{\Gamma(x)} |f(y,t)|^q \frac{dydt}{t^{n+1}} \right)^{1/q}, \quad 0 < q < \infty, \tag{5.1}$$

and

$$C_q(f) = \sup_{x \in B} \left(\frac{1}{B} \int_{\hat{B}} |f(y,t)|^q \frac{dydt}{t} \right)^{1/q}, \quad 0 < q < \infty, \tag{5.2}$$

where $\hat{B}$ is the tent over ball B which is defined in (1.8), and the sup is taken over all balls B containing x.

Definition 5. Tent space:

Let $0 < q < \infty$. The tent space T_q^p is defined as the collection of all function f in $\mathbb{R}_+^{n+1}$ such that

$$\|f\|_{T_q^p} := \|A_q(f)\|_{L^p} < \infty$$

when $0 < p < \infty$, and

$$\|f\|_{T_q^\infty} = \|C_q(f)\|_{L^\infty} < \infty.$$

when $p = \infty$.

Definition 6. T_q^p atom:

Let $1 < q \leq \infty$ and $0 < p \leq 1$. A T_q^p atom is a function $a(x,t)$ which is supported in the tent $\hat{B}$ for some ball $B \subset \mathbb{R}^n$, and satisfies

$$\left(\int_{\hat{B}} |a(x,t)|^q \frac{dxdt}{t} \right)^{\frac{1}{q}} \leq |B|^{1/q-1/p}. \tag{5.3}$$

With p and q in this range, functions f in T_q^p can be decomposed as

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \tag{5.4}$$

where a_j are T^p_q atoms, $\lambda_j \in \mathbb{C}$ and $\sum |\lambda_j|^p \leq c\|f\|^p_{T^p_q}$ (see [3], Proposition 5).

Let ψ be a function defined in $\mathbb{R}^n$ such that

1. ψ has compact support in the unit ball $B(0,1)$);
2. $|\psi(x)| \leq M$ and $\exists \epsilon > 0$, such that, $|\psi(x+h) - \psi(x)| \leq M(|h|/|x|)^\epsilon$;
3. $\int \psi(x)dx = 0$;
4. $\int x^r \psi(x)dx = 0$, for all $|r| \leq N$, where $N \geq \left\lfloor n(\frac{1}{p} - 1)\right\rfloor$, the greatest integer in $n(\frac{1}{p} - 1)$.

Put $\psi_t(x) = t^{-n}\psi(x/t)$, $t > 0$. Define an operator π_ψ on T^p_q by

$$\pi_\psi(f)(\cdot) = \int_0^\infty (f(\cdot,t) * \psi_t)(\cdot)\frac{dt}{t} = \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_\varepsilon^N (f(\cdot,t) * \psi_t)(\cdot)\frac{dt}{t}, \quad f \in T^p_q. \tag{5.5}$$

where the limit is taken in the sense of distribution. Namely, $\forall f \in T^p_q$ and $G \in \mathcal{S}$, the Schwartz space, the action of $\pi_\psi(f)$ on G is defined as

$$< \pi_\psi(f), G >= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{\mathbb{R}^n} \left(\int_\varepsilon^N (f(\cdot,t) * \psi_t)(x)\frac{dt}{t}\right) G(x)dx.$$

Note that when f has compact support in $\mathbb{R}^{n+1}_+$, the limit in (5.5) can be taken in the pointwise sense, so, as in [3], the operator π_ψ can initially be defined on this dense subset of T^p_q.

In the case $q = 2$, Coifman, Meyer and Stein proved (see [3], Theorem 6),

Theorem 9 (CMS). *Under conditions* 1, 2, and 3 *on* ψ, $\pi_\psi(f)$ *extends to a bounded linear operator from:*

(i). T^p_2 *to* $L_p(\mathbb{R}^n)$, *if* $1 < p < \infty$*;*

(ii). T^1_2 *to* $H^1(\mathbb{R}^n)$*;*

(iii). T^∞_2 *to BMO;*

(iv) T^p_2 *to* $H^p(\mathbb{R}^n)$, *for* $p < 1$, *if condition* 4 *is included.*

As an analogue of their result to $q \neq 2$, we have theorem (6) stated in the introduction with the same projection π_ψ under the same condition for ψ.

5.2 Proof of Theorem 6

Theorem 9 is based on the relation of the square function and Littlewood-Paley g_ψ-function, and is proved by vector-valued singular integrals applied to g_ψ (see [26] Chapter 1, Section 8.23 and Chapter 3, Section 4.4). However, in order to generalize this theory to the case $q \neq 2$, we need some knowledge about Triebel-Lizorkin spaces (see [27], [18]):

Definition 7. Triebel-Lizorkin spaces:

Let φ be a function in Schwartz space $\mathcal{S}$, and let $\hat{\varphi}$ be the Fourier transformation of φ. Assume that $\hat{\varphi}$ is supported in $\{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$ and $|\hat{\varphi}| \geq c > 0$ if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$. Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$, and $f \in \mathcal{S}'$. Define

$$\|f\|_{\dot{F}_p^{s,q}} = \left\| \left\{ \sum_{\nu \in \mathbb{Z}} (2^{\nu s} |\varphi_{2^{-\nu}} * f|)^q \right\}^{1/q} \right\|_{L^p}. \tag{5.6}$$

The homogeneous Triebel-Lizorkin space $\dot{F}_p^{s,q}$ consists of all distribution f such that the norm $\|f\|_{\dot{F}_p^{s,q}} < \infty$.

Triebel connected these spaces to the spaces L^p, H^p and BMO by the following theorems: (see [27] Section 1.7, [18] Section 5)

Theorem 10 (Triebel). *The homogeneous $\dot{F}_p^{s,q}$ Triebel-Lizorkin spaces can be identified with the spaces L^p, H^p and BMO as follows:*

- $L^p \cong \dot{F}_p^{0,2}$ *when* $1 < p < \infty$*;*
- $H^p \cong \dot{F}_p^{0,2}$ *when* $0 < p \leq 1$*;*

- $BMO \cong \dot{F}^{0,2}_{\infty}$.

Moreover, we have, (see [28] Section 2.2, [24])

Theorem 11 (Triebel). *For $0 < p < \infty, 0 < q \leq \infty$, $G \in \dot{F}^{0,q}_{p}(\mathbb{R}^n)$, ψ as in Section 5.1, and $g(x,t) = G * \psi_t(x)$,*

$$\|G\|_{\dot{F}^{0,q}_{p}(\mathbb{R}^n)} \cong \|A_q(g)\|_{L^p(\mathbb{R}^n)}. \tag{5.7}$$

Define the truncated cone with aperture 1, whose vertex is at x, as

$$\Gamma^h(x) = \{(y,t), |x-y| < t < h\},$$

and put

$$A_q(f|h)(x) = \left(\int_{\Gamma^h(x)} |f(y,t)|^q \frac{dydt}{t^{n+1}}\right)^{1/q}. \tag{5.8}$$

As in [3] (in the case $q = 2$), we define a stopping time for the function g (relative to some large constant M, chosen to depend only on n) as

$$h(x) = \sup_h \{A_q(g|h)(x) \leq MC_q(g)(x)\}. \tag{5.9}$$

From this we can derive, in the same way as for $q = 2$, that for a ball $B \subset \mathbb{R}^n$ of radius r,

$$|\{x \in B : h(x) \geq r\}| \geq cr^n,$$

Thus we have from Fubini's Theorem (see [3], inequality (4.3)),

Lemma 4 (CMS). *Assume $\Phi(y,t)$ is a non-negative function, then,*

$$\int_{\mathbb{R}^{n+1}_+} \Phi(y,t)t^n dydt \leq c\int_{\mathbb{R}^n} \left\{\int_{\Gamma^{h(x)}(x)} \Phi(y,t)dydt\right\} dx. \tag{5.10}$$

Replacing $\Phi(y,t)$ by $|f(y,t)g(y,t)|t^{-(n+1)}$ in (5.10) and applying Hölder's inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} |f(y,t)g(y,t)|\frac{dydt}{t} &\leq c\int_{\mathbb{R}^n}\left\{\int_{|x-y|<t}|f(y,t)g(y,t)|\frac{dydt}{t^{n+1}}\right\}dx \quad (5.11)\\ &\leq c\int_{\mathbb{R}^n} A_q(f|h(x))(x)A_{q'}(g|h(x))(x)dx. \quad (5.12)\end{aligned}$$

Hence, by (5.9),

$$\int_{\mathbb{R}^{n+1}_+} |f(y,t)g(y,t)|\frac{dydt}{t} \leq c\int_{\mathbb{R}^n} C_q(g)(x)A_{q'}(f)(x)dx. \quad (5.13)$$

If we also apply Hölder's inequality to the integral (5.12) and note that $A(f|h)(x)$ is increasing in h, we get

$$\int_{\mathbb{R}^{n+1}_+} |f(y,t)g(y,t)|\frac{dydt}{t} \leq c\|A_q(f)\|_{L^p}\|A_{q'}(g)\|_{L^{p'}}. \quad (5.14)$$

Now we prove Theorem 6. More precisely, let $1 < q \leq 2$, we show that if ψ satisfies conditions 1, 2, 3, $\pi_\psi(f)$ extends to a bounded linear operator from

(a). *T^p_q to $L_p(\mathbb{R}^n)$, for $1 < p < \infty$;*

(b). *T^1_q to $H^1(\mathbb{R})$;*

(c). *T^∞_q to BMO;*

(d). *T^p_q to $H^p(\mathbb{R}^n)$, for $p < 1$, provided condition 4 is also included.*

Proof of (a):

Let $1 \leq p < \infty$, $1 < q < \infty$, and let p' and q' be the conjugate exponents of p and q, respectively, that is, $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

For $f \in T^p_q$ with compact support and $G \in \mathcal{S}$, we have, by Fubini's Theorem,

$$\begin{aligned} |< \pi_\psi(f), G >| &= \left| \int_{\mathbb{R}^n} \left[\int_0^\infty \int_{\mathbb{R}^n} f(y,t) \frac{1}{t^n} \psi\left(\frac{x-y}{t}\right) \frac{dydt}{t} \right] G(x) dx \right| \\ &= \left| \int_{\mathbb{R}^n} \int_0^\infty f(y,t) \left[\int_{\mathbb{R}^n} G(x) \frac{1}{t^n} \psi\left(\frac{x-y}{t}\right) dx \right] \frac{dydt}{t} \right| \\ &= \left| \int_{\mathbb{R}^n} \int_0^\infty f(y,t)(G * \tilde{\phi}_t) \frac{dydt}{t} \right| \\ &= \left| \int_{\mathbb{R}^{n+1}} f(y,t) g(y,t) \frac{dydt}{t} \right| \\ &\leq c \|A_q(f)\|_{L^p} \|A_{q'}(g)\|_{L^{p'}}, \quad \text{by (5.14)}, \end{aligned}$$

where $\tilde{\psi}(x) = \psi(-x)$, and $g(x,t) = G * \tilde{\psi}_t$. Since such f is dense in T^p_q, (5.14) holds for all $f \in T^p_q$.

We have that $\|A_{q'}(g)\|_{L^{p'}} \leq c\|G\|_{L^{p'}}$. To see this, first for $0 < p \leq \infty$, we use the fact that $\dot{F}^{0,2}_p \subset \dot{F}^{0,q}_p$ is a continuous embedding for $q \geq 2$ (see [18]). If $G \in L^{p'}$, then for $q' \geq 2$ (or $q \leq 2$), we have, by Theorem 10 and (11),

$$\|A_{q'}(g)\|_{L^{p'}} \leq c\|G\|_{\dot{F}^{0,q}_p} \leq c\|G\|_{\dot{F}^{0,2}_p} \leq c\|G\|_{L^{p'}}.$$

It follows that

$$|< \pi_\psi(f), G >| \leq c\|f\|_{T^p_q}\|G\|_{L^{p'}}, \quad \forall G \in \mathcal{S}.$$

Therefore, for $f \in T^p_q$, $1 \leq p < \infty$, and $1 < q \leq 2$,

$$\|\pi_\psi(f)\|_{L^p} \leq c\|f\|_{T^p_q}. \tag{5.15}$$

Proof of (b): Based on the atomic decomposition for the space T^1_q, we just need to show that π_ψ maps a T^1_q atom to a multiple of H^1 atom. We use the following definition of an H^p-atom (see [26], Chapter 3, Section 2.2) with L^q size condition, $1 < q \leq \infty$:

Definition 8. <u>H^p atom</u>: *Let $p \leq 1$. an H^p atom is a function a such that*

1. *a is supported in a ball B,*

2. *a satisfies the size condition $\|a\|_{L^q} \leq |B|^{1/q-1/p}$,*

3. *a satisfies the moment conditions $\int x^r a(x)dx = 0$, for all $|r| \leq n(1/p-1)$.*

First, note that $\pi_\psi(a)$ is supported in the closed ball $\bar{B} \subset \mathbb{R}^n$ if a is a T_q^1 atom supported in the tent $\hat{B}$. To see this, note if that $x \notin B$, then $\Gamma(x) \cap \hat{B} = \emptyset$, and since ψ is supported in the unit ball,

$$\pi_\psi(a)(x) = \int_{\Gamma(x)\cap\hat{B}} a(y,t)\psi(\frac{x-y}{t})\frac{dydt}{t^{n+1}} = 0.$$

Second, letting $1 < q \leq 2$, and putting $p = q$ in (5.15), we get, by Fubini's theorem and (5.3),

$$\begin{aligned}
\|\pi_\psi(a)\|_{L^q} &\leq c\|a\|_{T_q^q} = c\left[\int_{\mathbb{R}^n}\int_{\Gamma(x)} |a(y,t)|^q \frac{dydt}{t^{n+1}}dx\right]^{1/q} \\
&= c\left[\int_{\mathbb{R}^{n+1}_+} |a(y,t)|^q \frac{dydt}{t}\right]^{1/q} \\
&= c\left[\int_{\hat{B}} |a(y,t)|^q \frac{dydt}{t}\right]^{1/q} \leq c\cdot|B|^{1/q-1},
\end{aligned}$$

This shows that $\pi_\psi(a)$ satisfies the L^q size condition for an H^1 atom.

Third, since $\int_{\mathbb{R}^n} \psi(x)dx = 0$, $\pi_\psi(a)$ satisfies the moment condition

$$\int_{\mathbb{R}^n} \pi_\psi(a)(x)dx = 0.$$

<u>Proof of (d):</u>

If we also assume condition (4) on ψ, we will get the H^p moment condition on

$\pi_\psi(a)$. Moreover the size condition (5.3) will imply the condition

$$\|\pi_\psi(a)\|_{L^q} \leq c \cdot |B|^{1/q-1/p}.$$

Proof of (c): To prove (c), we show that for $f \in T_q^\infty$, $\pi_\psi(f)$ can be paired with H^1. Similarly to the proof of (a), but now using (5.13), we have

$$\begin{aligned} |<\pi_\psi(f), G>| &= \int_{\mathbb{R}^{n+1}} f(y,t) g(y,t) \frac{dydt}{t} \\ &\leq c \int_{\mathbb{R}^n} C_q(f)(x) A_{q'}(g)(x) dx \\ &\leq c\|C_q(f)\|_{L^\infty} \|A_{q'}(g)\|_{L^1}, \end{aligned}$$

again where $\tilde{\psi(x)} = \psi(-x)$, $g(x,t) = G * \tilde{\psi_t}(x)$.

Since $\|C_q(f)\|_{L^\infty} = \|f\|_{T_q^\infty}$, and as in the proof of (a), $\|A_q(g)\|_{L^1} \leq c\|G\|_{H^1}$ by putting $p = 1$ in Theorem 10 and (5.7), we have

$$|<\pi_\psi(f), G>| \leq c\|f\|_{T_q^\infty} \|G\|_{H^1}, \quad 1 < q \leq 2,$$

so by the duality of BMO with H^1,

$$\|\pi_\psi(f)\|_{BMO} \leq c\|f\|_{T_q^\infty}, \quad 1 < q \leq 2. \tag{5.16}$$

This completes the proof of Theorem 6.

Remark The discussion in this chapter raises the interesting question of what will happen if the tent spaces defined by Dafni and Xiao [8] are modified by changing 2 to $q \neq 2$, and whether there is a relation between these tent spaces and the generalized Q (Morrey) spaces $Q^\alpha_{p,q}$ defined by Cui and Yang [5].

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