On the Distribution of Discounted Compound Renewal Sums with PH Claims

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ABSTRACT

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The family of phase–type (PH) distributions has many good properties such as closure under convolution and mixtures and have rational Laplace transforms. PH distributions are widely used in applications of stochastic models such as in queueing systems, biostatistics and engineering. They are also applied to insurance risk.

In this thesis, we discuss the moment generating function (m.g.f.) of a compound present value risk process with phase–type (PH) deflated claim severities under a net interest $\delta \neq 0$. This represents a generalization of the classical risk model $\delta = 0$.

A closed form of the m.g.f. of a compound Poisson present value risk process with PH deflated claims is obtained. We also consider the discounted compound renewal process and get homogeneous differential equations for its m.g.f. in the case of PH deflated claims. Applications and some numerical examples are given to illustrate the results.

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Chapter 1

Introduction

As a generalization of the exponential distribution, phase–type distributions were first introduced by Neuts (1975). They have been used widely in applications of stochastic models such as in queueing systems, engineering, biostatistics and reliability. The random variable defined as the absorption time in a continuous-time Markov chain with n transient states $i = 1, 2, \ldots, n$ and one absorbing state 0 has a phase–type (PH) distribution. PH distribution and density functions are expressed in terms of a vector $\underline{\alpha}$ and a nonsingular matrix $\mathbf{A}$.

Neuts (1981) gives a detailed introduction of stochastic models in queueing theory with PH distributions. Asmussen (2003) presents more details about the properties of PH distributions and their applications to queuing theory. In the last decade PH distributions have been applied also to insurance risk. Asmussen (2000) studies the ruin probability in the compound Poisson model with PH claim severities. Asmussen and Rolski (1991) also introduced some numerical examples with PH claim severities to compute ruin probabilities. Asmussen, Avram and Pistorius (2004) show the potential of PH distributions in mathematical finance. Frostig (2004) obtains the distribution of the time to ruin and upper bounds with PH claim size distributions.

PH distributions form an interesting family. First it can be represented using matrices and vectors, which simplifies the computation with mathematical software such as Maple, Mathematica or Matlab. PH distributions generalize exponential, Erlang(n) and Cox distributions, which are already known. Moreover they are dense in the class of all distributions defined on the nonnegative real numbers, hence PH distributions enable algorithmically tractable solutions for stochastic models. Many papers and theses propose approximation methods using PH-distributions. For instance Fackwell (2003) presents estimation methods for PH distributions. Bladt, Gonzalez and Lauritzen (2003) consider the estimation of functionals depending on one or several PH distributions using Markov chain Monte Carlo methods. Asmussen, Avram and Usabel (2002) present a fast and simple algorithm for computing finite-horizon ruin probability using an Erlang (phase–type) approximation and an extrapolation scheme.

In this thesis, we study the problem of obtaining the moment generating function (m.g.f.) of a compound present value risk process. The m.g.f. is a classical technique to find the expectation and variance of a random variable, as well as its probability density function by inverting the moment generation function.

Léveillé and Garrido (2001a) derived the first two moments of a compound renewal present value risk (CRPVR) process using renewal theory arguments. In Léveillé and Garrido (2001b) they also obtained recursive formulas for all the moments of the CRPVR process. Léveillé (2002) discusses further the asymptotic and finite time distributions of the CRPVR process and gives an analytical expression for the m.g.f. of the CRPVR process. In this thesis we study the m.g.f. of the compound Poisson present value risk (CPPVR) process as well as the CRPVR with PH claim severities.

The thesis is organized as follows. Chapter 2 gives the formal definition of PH distributions and introduces some of its basic properties and examples. Chapter 3 introduces the model and gives a closed form of the m.g.f. of the CPPVR process with PH claim severities using matrix-exponential function arguments. Chapter 4 discusses the renewal model. An homogeneous differential equation is obtained for the m.g.f. of the CRPVR process when the inter–arrival times are Erlang(n) and claim severities are PH. The application of the results in Chapters 3 and 4 is discussed in Chapter 5. We compare the CPPVR and CRPVR models and see the impact of different claim frequency assumptions on the compound sum. We also consider the impact of net interest $\delta \neq 0$ on the CPPVR and CRPVR by computing expectations and variances. Some numerical examples are also given as illustrations.

Chapter 2

Phase–Type Distributions

In this chapter we briefly present the definition of phase–type (PH) distributions, some of their basic properties and examples. Since introduced by Neuts (1975) in queueing theory and reliability, PH distributions have been applied also to insurance risks. Now that matrix computations can be carried out with mathematical software such as Maple, Mathematica or Matlab, at least a numerical solution is possible for some of these applications. This has created an increased interest for PH distributions in many areas of applied probability.

2.1 The Definition of PH Distributions

A phase–type distribution is defined as a probability distribution that represents the time to absorption in a continuous-time Markov chain with n transient states $i = 1, 2, \ldots, n$ and one absorbing state 0. Here we consider the mathematical definition of PH distributions. For details on the probabilistic interpretation and the properties of PH distributions in Markov chains, please refer to Neuts (1981) and Asmussen (2003).

Definition 2.1.1. *Continuous phase–type Distribution*

Let $\boldsymbol{A}$ *be an arbitrary non-singular square matrix of order* n *such as* $\lim_{x\to\infty} e^{\boldsymbol{A}x} = \boldsymbol{0}$, $\underline{\alpha}$ *be a* n*-dimensional column vector such that* $\underline{\alpha}'\,\underline{1} = 1$, *where* $\underline{1}$ *is a* n*-dimensional column vector of* $1's$, *that is:*

$$\underline{\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}', \quad \sum_{i=1}^{n} \alpha_i = 1\,, \quad \alpha_i \geq 0 \quad and \quad \underline{1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}'. \tag{2.1}$$

If the distribution function F_X *of a random variable* X *can be written as:*

$$F_X(x) = 1 - \underline{\alpha}' e^{\boldsymbol{A}x}\underline{1}\,, \qquad x \geq 0, \tag{2.2}$$

then we say that F_X *is (or* X *has) a phase–type (PH) distribution with parameters* $(\underline{\alpha}\,, \boldsymbol{A})$.

Remark 2.1. Note that in the original definition of PH distributions, like that Neuts (1975) $\mathbf{A} = (a_{ij})$ was the rate matrix of a stationary Markov chain. Consequently it was assumed that $a_{ii} < 0$ and $\sum_i a_{ij} = 0$ for all j. From Neuts (1981) we can see these conditions imply those of Definition 2.1.1.

Hence taking the derivative of F_X (see Lemma A.2.4.), we obtain the density function of X:

$$f_X(x) = -\underline{\alpha}'\, e^{\mathbf{A}x}\,\mathbf{A}\,\underline{1}\,, \qquad x \geq 0. \tag{2.3}$$

The following are some examples of PH distributions (see Fackrell, 2003 and Neuts, 1981).

Example 2.1.1. If X has an exponential distribution with density function $f_X(x) = \lambda\, e^{-\lambda x}$, for $\lambda > 0$, then it is PH with

$$\underline{\alpha}' = (1), \qquad \mathbf{A} = (-\lambda)\,.$$

Example 2.1.2. If X has an hyper-exponential distribution (also called mixed exponential) with density function

$$f_X(x) = \sum_{i=1}^{n} \alpha_i \, \lambda_i \, e^{-\lambda_i x} \,, \quad x > 0 \,, \quad \lambda_i > 0 \,,$$

where $\alpha_i > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$, then it is also PH with $\underline{\alpha}$ and $\mathbf{A}$ given by:

$$\underline{\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}' \,,$$

and

$$\mathbf{A} = \begin{pmatrix} -\lambda_1 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\lambda_n \end{pmatrix} .$$

Example 2.1.3. If X has an Erlang(n) distribution with density

$$f_X(x) = \frac{\lambda^n \, x^{n-1} \, e^{-\lambda x}}{(n-1)!} \,, \quad x > 0 \,, \quad n \in \mathbb{N}^+ \,, \quad \lambda > 0,$$

then it is also PH with

$$\underline{\alpha} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}' \,,$$

a n-dimensional vector and the following matrix of order n

$$\mathbf{A} = \begin{pmatrix} -\lambda & \lambda & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & \lambda & \cdots & 0 & 0 \\ 0 & 0 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & \lambda \\ 0 & 0 & 0 & \cdots & 0 & -\lambda \end{pmatrix} . \tag{2.4}$$

Example 2.1.4. If we have n different values $\lambda_i > 0$ in the previous example, then it defines a generalized Erlang(n) distribution of order n with $\underline{\alpha}$ and $\mathbf{A}$ as follows:

$$\underline{\alpha} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}' \,,$$

and

$$\mathbf{A} = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_{n-1} & \lambda_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & -\lambda_n \end{pmatrix},$$

and the density function can be expressed as a mixture of exponentials $f_X(x) = \sum_{i=1}^{n} a_i e^{-\lambda_i x}$ for given polynomial coefficients a_i in terms of λ_i .

Example 2.1.5. If X has a n-phase *Coxian* distribution with the following parameters, then it is also PH distribution:

$$\underline{\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix}', \qquad \sum_{k=1}^{n} \alpha_k = 1, \qquad \alpha_i \geq 0,$$

and

$$\mathbf{A} = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_{n-1} & \lambda_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & -\lambda_n \end{pmatrix},$$

where $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

Example 2.1.6. If X has the following parameters $(\underline{\alpha}, \mathbf{A})$, it is called *unicycle* PH distribution:

$$\underline{\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{pmatrix}', \qquad \sum_{k=1}^{n} \alpha_k = 1, \qquad \alpha_i \geq 0,$$

and

$$\mathbf{A} = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_{n-1} & \lambda_{n-1} \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n-1} & -\lambda_n \end{pmatrix},$$

where $\mu_i \geq 0$, for $i = 1, 2, \cdots, n-1$ and $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

Example 2.1.7. If $\mathbf{A}$ is an upper triangular matrix, we call X *acylic* or triangle PH (TPH).

Note that in general, the parameters $(\underline{\alpha}, \mathbf{A})$ for PH distributions are not unique. Consider the following PH distribution with density

$$f_X(x) = (\frac{1}{4}x^2 + x - 1)e^{-x} + e^{-2x}, \qquad x > 0,$$

which can be parameterized either with $(\underline{\alpha}, \ \mathbf{A})$ or $(\underline{\beta}, \mathbf{B})$ given by:

$$\alpha = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}', \qquad \mathbf{A} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}', \qquad \mathbf{B} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

2.1.1 Expectation

From (2.1) the expectation of X is given by:

$$\mathbb{E}(X) = \int_0^\infty [1 - F_X(x)]\, dx = \underline{\alpha}' \int_0^\infty e^{\mathbf{A}x}\, dx\, \underline{1}\,. \tag{2.5}$$

From the definition of the matrix exponential function (see Appendix A, definition A.1.4), we have

$$\mathbf{A} \int_0^x e^{\mathbf{A}u}\, du = e^{\mathbf{A}x} - \mathbf{I} = \int_0^x e^{\mathbf{A}u}\, du\ \mathbf{A}\,, \tag{2.6}$$

where $\mathbf{I}$ is identity matrix of order n. Given that $\mathbf{A}^{-1}$ exists, and assuming that $\lim_{x\to\infty} e^{\mathbf{A}x} = \mathbf{0}$, we have the following:

$$\int_0^x e^{\mathbf{A}u}\, du = \mathbf{A}^{-1}\mathbf{A} \int_0^x e^{\mathbf{A}u}\, du = \mathbf{A}^{-1}(e^{\mathbf{A}x} - \mathbf{I})\,, \tag{2.7}$$

and hence

$$\int_0^\infty e^{\mathbf{A}u}\, du = \lim_{x\to\infty} \int_0^x e^{\mathbf{A}u}\, du = \lim_{x\to\infty} \mathbf{A}^{-1}(e^{\mathbf{A}x} - \mathbf{I}) = -\mathbf{A}^{-1}\,. \tag{2.8}$$

Substituting (2.8) into (2.5) gives:

$$\mathbb{E}(X) = -\underline{\alpha}'\mathbf{A}^{-1}\underline{1}\,. \tag{2.9}$$

Following a similar procedure, one can obtain the n-th moment of X as :

$$\mathbb{E}(X^n) = (-1)^n\, n!\, \underline{\alpha}'\mathbf{A}^{-n}\, \underline{1}\,, \qquad n \in \mathbb{N}^+\,. \tag{2.10}$$

2.1.2 Moment Generating Function

From (2.8) one can also obtain the moment generating function (m.g.f.) of X :

$$M_X(t) = \int_0^\infty e^{tx} f_X(x)\, dx = -\underline{\alpha}' \int_0^\infty e^{(t\mathbf{I}+\mathbf{A})x}\, dx \mathbf{A}\, \underline{1} = \underline{\alpha}'\, (t\mathbf{I} + \mathbf{A})^{-1}\mathbf{A}\, \underline{1}\,, \quad t \in \mathbb{C}\,. \tag{2.11}$$

The same procedure also gives the Laplace transform $\widehat{f}_X$ of X :

$$\widehat{f}_X(s) = -\underline{\alpha}'\, (\mathbf{sI} - \mathbf{A})^{-1}\mathbf{A}\, \underline{1}\,, \qquad s \in \mathbb{C}\,. \tag{2.12}$$

For additional properties of PH-distributions and detailed derivations see Asmussen (2003) and Neuts (1981).

Though in this thesis we consider continuous PH distributions. For completion, we also define discrete PH distributions. Traditionally a discrete PH random variable is defined as the absorption time of an evanescent discrete-time Markov chain $\{Y_k\}$, with $k = 0, 1, 2, \ldots$, on a finite phase space $S = \{0, 1, 2, \ldots, n\}$ where phase 0 is absorbing. Here we give an algebraïc definition.

Definition 2.1.2. *Discrete phase–type Distributions*
Let $\boldsymbol{A}$ be an arbitrary non-singular square matrix of order n, such that $\lim_{k\to\infty} \boldsymbol{A}^k = \boldsymbol{0}$ and $\boldsymbol{I} - \boldsymbol{A}$ is non-singular and $\underline{\alpha}$ be a n-dimensional column vector such that $\underline{\alpha}'\,\underline{1} = 1$, where $\underline{1}$ is a n-dimensional column vector of $1's$, that is:

$$\underline{\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}', \quad \sum_{i=1}^{n} \alpha_i = 1\,, \quad \alpha_i \geq 0 \quad \textit{and} \quad \underline{1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}'. \tag{2.13}$$

If the probability function $\{p_k\}$ of a random variable X is given by:

$$p_k = \underline{\alpha}'\boldsymbol{A}^{k-1}(\boldsymbol{I} - \boldsymbol{A})\underline{1}\,, \qquad k \geq 1\,. \tag{2.14}$$

Then X is called a discrete PH distribution with parameters $(\underline{\alpha}\,, \boldsymbol{A})$. The cumulative distribution function, defined for $k = 1, 2, \ldots,$ is given by:

$$F_k = 1 - \underline{\alpha}'\boldsymbol{A}^k\underline{1}\,.$$

From the definition of the probability function, the probability generating function is given:

$$\begin{aligned} G(z) &= \sum_{k=1}^{\infty} p_k z^k = \sum_{k=1}^{\infty} \underline{\alpha}'\mathbf{A}^{k-1}(\mathbf{I} - \mathbf{A})\underline{1}z^k\,, \qquad z \in \mathbb{C}\,, \\ &= z\,\underline{\alpha}'\sum_{k=1}^{\infty}(\mathbf{A}z)^{k-1}(\mathbf{I} - \mathbf{A})\underline{1} = z\,\underline{\alpha}'(\mathbf{I} - z\mathbf{A})^{-1}(\mathbf{I} - \mathbf{A})\underline{1}\,, \end{aligned} \tag{2.15}$$

where *sprad*$(z\mathbf{A}) < 1$ (*sprad*$(\mathbf{A})$ is defined in the Appendix A.1.3.). The expression (2.15) shows that the probability generating function is a rational function.

From Lemma A.2.7. in Appendix A, differentiating (2.15) with respect to z and letting $z = 1$ gives the first moment

$$\mathbb{E}(X) = \underline{\alpha}'(\mathbf{I} - \mathbf{A})^{-1}\underline{1}.$$

Similarly the n-th moment is given by:

$$\mathbb{E}(X^n) = n!\,\underline{\alpha}'(\mathbf{I} - \mathbf{A})^{-n}\mathbf{A}^{n-1}\underline{1}\,, \qquad n = 1, 2, \ldots .$$

Some discrete random variables have PH distributions. For example, the geometric distribution with probability function

$$p_k = (1-p)^{k-1}p\,, \qquad 0 \leq p < 1 \quad \text{and} \quad k \geq 1\,,$$

is a discrete PH distribution with parameters $(\underline{\alpha}\,, \mathbf{A})$ given by

$$\underline{\alpha} = 1\,, \qquad \mathbf{A} = 1 - P\,.$$

The properties of the following section show that mixtures of geometric and negative binomial distributions are also discrete PH distributions. In fact, one can verify that any distribution with finite support on the nonnegative integers is a discrete PH distribution. Thus, the binomial and hypergeometric distributions are discrete PH distributions also. However, the Poisson distribution is not a PH distribution since it does not have a rational probability generating function. For details on the properties of discrete PH distributions, see Neuts (1981) and Fackwell (2003).

2.2 Closure Properties

Apart from having analytical expressions for its moments and its m.g.f., the family of PH distributions is closed under convolution and mixtures.

Property 2.2.1. If the distributions F_X of X and F_Y of Y are both continuous PH-distributions with parameters $(\underline{\alpha}, \mathbf{A})$ of order n and $(\underline{\beta}, \mathbf{B})$ of order m respectively, then their convolution $F_X * F_Y$ is also a PH-distribution with parameter $(\underline{\gamma}, \mathbf{C})$. Here $\underline{\gamma}$ and $\mathbf{C}$ are given by

$$\underline{\gamma} = (\underline{\alpha}', \underline{0}'_m)' \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} \mathbf{A} & -\mathbf{A}\,\underline{1}_n\underline{\beta}' \\ \mathbf{0} & \mathbf{B} \end{pmatrix}, \tag{2.16}$$

where $\underline{1}_k = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}'$ is a vector of order $k \times 1$.

Proof. The Laplace transform of a PH $(\underline{\gamma}, \mathbf{C})$ random variable Z can be obtained from (2.12) as

$$\begin{aligned}
\hat{f}_Z(s) &= -\underline{\gamma}'(s\mathbf{I}_{n+m} - \mathbf{C})^{-1}\mathbf{C}\,\underline{1} \\
&= -(\underline{\alpha}', 0'_m)\begin{pmatrix} s\mathbf{I}_n - \mathbf{A} & \mathbf{A}\underline{1}_n\underline{\beta}' \\ 0 & s\mathbf{I}_m - \mathbf{B} \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{A} & -\mathbf{A}\,\underline{1}_n\underline{\beta}' \\ 0 & \mathbf{B} \end{pmatrix}\begin{pmatrix} \underline{1}_n \\ \underline{1}_m \end{pmatrix} \\
&= -(\underline{\alpha}', 0'_m)\begin{pmatrix} (s\mathbf{I}_n - \mathbf{A})^{-1} & -(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{A}\,\underline{1}_n\underline{\beta}'(s\mathbf{I}_m - \mathbf{B})^{-1} \\ \mathbf{0} & (s\mathbf{I}_m - \mathbf{B})^{-1} \end{pmatrix} \\
&\quad \times \begin{pmatrix} \mathbf{A} & -\mathbf{A}\underline{1}_n\underline{\beta}' \\ 0 & \mathbf{B} \end{pmatrix}\begin{pmatrix} \underline{1}_n \\ \underline{1}_m \end{pmatrix} \\
&= -\underline{\alpha}'(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{A}\underline{1}_n + \underline{\alpha}'(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{A}\underline{1}_n\underline{\beta}'\underline{1}_m \\
&\quad + \underline{\alpha}'(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{A}\underline{1}_n\underline{\beta}'(s\mathbf{I}_m - \mathbf{B})^{-1}\mathbf{B}\underline{1}_m.
\end{aligned}$$

Since $\underline{\beta}'\,\underline{1}_m = 1$, then

$$\hat{f}_Z(s) = (-\underline{\alpha}'(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{A}\underline{1}_n)(-\underline{\beta}'(s\mathbf{I}_m - \mathbf{B})^{-1}\mathbf{B}\underline{1}_m) = \hat{f}_X(s)\hat{f}_Y(s), \quad s \in \mathbb{C}, \tag{2.17}$$

where $\hat{f}_X$ and $\hat{f}_Y$ are the Laplace transforms of a PH$(\underline{\alpha}, \mathbf{A})$ random variable X of order n and a PH$(\underline{\beta}, \mathbf{B})$ random variable Y of order m, respectively.

□

Remark 2.2. The convolution $F_X * F_Y$ does not have a unique PH representation. For instance, it can also be written as a PH($\underline{\gamma}$, **C**) distribution with

$$\underline{\gamma} = (\underline{\beta}', \underline{0}'_n)', \qquad \mathbf{C} = \begin{pmatrix} \mathbf{B} & -\mathbf{B}\underline{1}_m\underline{\alpha}' \\ \mathbf{0} & \mathbf{A} \end{pmatrix}. \tag{2.18}$$

Property 2.2.2. The mixture $\theta F_X + (1-\theta)F_Y$, where $0 \le \theta \le 1$, is also a PH distribution with parameters $(\underline{\gamma}, C)$, where $\underline{\gamma}$ and **C** are given by:

$$\underline{\gamma} = \left(\theta\underline{\alpha}', \quad (1-\theta)\underline{\beta}'\right)', \qquad \mathbf{C} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}. \tag{2.19}$$

Proof. Let $\hat{f}_Z$ be the Laplace transform of $\theta F_X + (1-\theta)F_Y$, then:

$$\begin{aligned} \hat{f}_Z(s) &= \theta \hat{f}_X(s) + (1-\theta)\hat{f}_Y(s) \\ &= -\theta\underline{\alpha}'(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{A}\underline{1}_n - (1-\theta)\underline{\beta}'(s\mathbf{I}_m - \mathbf{B})^{-1}\mathbf{B}\underline{1}_m \\ &= -(\theta\underline{\alpha}', (1-\theta)\underline{\beta}')' \begin{pmatrix} s\mathbf{I}_n - \mathbf{A} & \mathbf{0} \\ \mathbf{0} & s\mathbf{I}_m - \mathbf{B} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \end{pmatrix} \begin{pmatrix} \underline{1}_n \\ \underline{1}_m \end{pmatrix} \\ &= -\underline{\gamma}'(s\mathbf{I}_{n+m} - \mathbf{C})^{-1}\mathbf{C}\,\underline{1}, \end{aligned}$$

where $\underline{\gamma}$ and **C** are given in (2.19).

□

Property 2.2.3. If the random variable X is a PH $(\underline{\alpha}, \mathbf{A})$, then θX also is a PH $(\underline{\gamma}, \mathbf{C})$, where

$$\underline{\gamma} = \underline{\alpha} \quad \text{and} \quad \mathbf{C} = \tfrac{1}{\theta}\mathbf{A}, \qquad \text{for } \theta > 0.$$

Proof. Let $Y = \theta X$, then

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(\theta X \le y) = \mathbb{P}(X \le \frac{1}{\theta}y) = 1 - \underline{\alpha}' e^{\frac{1}{\theta}\mathbf{A}y}\underline{1}, \qquad y > 0.$$

□

From Properties 2.2.1 and 2.2.3 we obtain the following results.

Corollary 2.2.1. *If X and Y are both PH-distributions with parameters $(\underline{\alpha}, \boldsymbol{A})$ and $(\underline{\beta}, \boldsymbol{B})$ respectively, then $aX + bY$ also is PH $(\underline{\gamma}, \boldsymbol{C})$, where*

$$\underline{\gamma} = (\underline{\alpha}' \quad 0)' , \quad \boldsymbol{C} = \begin{pmatrix} \frac{1}{a}\boldsymbol{A} & -\frac{1}{a}\boldsymbol{A}\underline{1}_n\underline{\beta}' \\ 0 & \frac{1}{b}\boldsymbol{B} \end{pmatrix} , \quad a > 0 , \quad b > 0 .$$

Note that the parameters $(\underline{\gamma}, \mathbf{C})$ are not unique. Another representation could be:

$$\underline{\gamma} = (\underline{\beta}' \quad 0)' , \quad \mathbf{C} = \begin{pmatrix} \frac{1}{b}\mathbf{B} & -\frac{1}{b}\mathbf{B}\underline{1}_m\underline{\alpha}' \\ 0 & \frac{1}{a}\mathbf{A} \end{pmatrix} , \quad a > 0 , \quad b > 0 .$$

Property 2.2.4. Let p_k be the probability function of a discrete PH with parameters $(\underline{\beta}, \mathbf{S})$ and F_X be a continuous PH distribution with parameters $(\underline{\alpha}, \mathbf{A})$. Then the mixture $\sum_{k=0}^{\infty} p_k\, F_X^{*k}$ is also a PH with parameter $(\underline{\gamma}, \mathbf{C})$ given by:

$$\begin{aligned} \gamma &= \alpha' \otimes \beta , \\ \mathbf{C} &= \mathbf{A} \otimes \mathbf{I} - \mathbf{A}\underline{1}\alpha' \otimes \mathbf{S} , \end{aligned}$$

where F_X^{*k} denotes the $k-$fold convolution of F_X (where $F_X^{*0} = 1[x \geq 0]$), $\mathbf{I}$ is identity matrix of order n and $\otimes$ is the Kronecker product. For a proof see Neuts (1981).

From the definition of the Laplace transform of a PH distribution, we see that it is a rational polynomial in s. If the maximal degree of the denominator is p then degree of the numerator is $q < p$, since the limit of the Laplace transform goes to zero as s tends to ∞. The question is whether a rational polynomial corresponds to a PH distribution. The answer is given in O'Cinneide (1990) and is reproduced here with the following result.

Property 2.2.5. A distribution defined on $(0, \infty)$ is a PH distribution if and only if it satisfies the following conditions:

1. it has the point mass at zero, or

2. it has

   - a strictly positive density function on $(0, \infty)$, and

   - a rational Laplace transform (LST) such that there exists a pole of maximal real part $-\gamma$ that is real, negative and such that $-\gamma > Re(-\xi)$, where $-\xi$ is any other pole.

Chapter 3

Moments of Compound Poisson Sums with Discounted PH Claims

The influence of economical and other unstable factors on risk processes makes the study of inflation and interest on the present value of the surplus process become very important. In classical risk models, the inflation experienced on claim severities is assumed to cancel the interest earned on the investment of the surplus. In this case the analysis is simple and produces very elegant results. However, in the long run, this assumption may not be true because of unforeseeable factors in the economy. Then the study of discounted claims becomes necessary, see Garrido and Léveillé (2004) for a detailed motivation of discounted models.

Léveillé (2002) presents an analytic form of the m.g.f. of a compound Poisson present value (CPPVR) process. He also gives the asymptotic form of the CPPVR m.g.f. as time t tends to infinity. In this chapter we consider the m.g.f. of the CPPVR process with PH claim severities. A nice form of the m.g.f. is obtained in terms of matrices and vectors using matrix–exponential theory.

The definition of the Poisson process and the model are introduced in the first section. In the second section, the m.g.f. of the CPPVR process is produced and extensions, limit and asymptotic results are also given. Lastly we give some illustrative examples.

3.1 Definitions and Model Assumptions

In this section we introduce the compound Poisson model with discounted PH claims. As one of the most important stochastic counting processes, the Poisson process has been studied for many years. Some beautiful results have been obtained. It has been used in classical risk models, queueing systems, genetic studies and many other fields of applied probability. Traditionally there are three equivalent definitions of the Poisson process. Here we present one that is commonly used in many probability books, such as Ross (2003).

Definition 3.1.1. *Poisson process*
The counting process $N = \{N(t), t \geq 0\}$ *is said to be a Poisson process with intensity* $\lambda > 0$, *iff*

1. $N(0) = 0$,
2. *The process* N *has independent increments,*
3. *The number of events in any interval of length* t *is Poisson distributed with mean* λt. *That is for all* $s, t \geq 0$

$$P\{N(t+s) - N(s) = n\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 0, 1, 2, \ldots .$$

In the classical risk model the aggregate claims can be written as

$$S(t) = \sum_{i=1}^{N(t)} X_i, \qquad t \geq 0, \tag{3.1}$$

where N is a Poisson process, $S(0) = 0$ and X_i represents the severity of the i-th claim $i = 1, 2, \ldots, n$.

Now consider the effect of inflation on claims and interest earned on the investment of the surplus. Léveillé and Garrido (2001a) impose the following model assumptions:

- Assume that there is an inflationary impact on the risk business and the inflation rate acting on claim severities at time t is known and denoted α_t. The claim severities, $\{Y_k\}_{k\geq 1}$, are then inflated. Claim occurrence times are represented by $\{T_k\}_{k\geq 1}$.
- Let $N(t) = \sup\{k \in \mathbb{N}; T_k \leq t\}$ for each $t > 0$, where $\sup\emptyset = 0$ and $N(0) = 0$, count the number of claims recorded over the time interval $[0, t]$.

Also, if β_s is the known force of interest earned at time $s \in (0, t]$, then

$$Z(t) = \sum_{k=1}^{N(t)} e^{-B(T_k)} Y_k \,, \qquad t \geq 0 \,, \tag{3.2}$$

where $B(s) = \int_0^s \beta_u du$ for $s \in (0, t]$ and $Z(0) = 0$ if $N(0) = 0$, defines the aggregate discounted value at time 0 of all claims recorded over $[0, t]$.

The definition of model:

1. the claim number process $N = \{N(t)\,, t \geq 0\}$ forms a renewal process. The inter–arrival times, denoted by $\tau_k = T_k - T_{k-1}\,,\ k \geq 2$ and $\tau_1 = T_1$ have a common distribution say F_τ. Here the T_k are the claim occurrence times.
2. The claim severities $\{Y_k\}_{k\geq 1}$ are defined as random variables. Let the deflated claim severities

$$X_k = e^{-A(T_k)} Y_k \,, \qquad k \geq 1 \,,$$

where $A(t) = \int_0^t \alpha_s \, ds$ for any $t \geq 0$, satisfy the following assumptions:

- $\{X_k\}_{k\geq 1}$ are independent and identically distributed (i.i.d.),
- $\{X_k\,,\tau_k\}_{k\geq 1}$ are mutually independent.

From the definition of model, the aggregate discounted sum in (3.2) is

$$Z(t)=\sum_{k=1}^{N(t)} e^{-B(T_k)}Y_k=\sum_{k=1}^{N(t)} e^{-D(T_k)}X_k\,,\qquad t>0\,, \tag{3.3}$$

where $D(T_t)=B(t)-A(t)=\int_0^t(\beta_s-\alpha_s)\,ds=\int_0^t\delta_s\,ds$.
If the net interest rates are constant but not zero, that is $\delta_t=\beta_t-\alpha_t=\delta>0$, then the aggregate discounted value at time 0 of the total claims recorded over the period $[0\,,t]$ is then given by

$$Z(t)=\sum_{k=1}^{N(t)} e^{-\delta T_k}X_k\,,\qquad t\geq 0\,, \tag{3.4}$$

with $Z(t)=0$ if $N(t)=0$. If $\delta=0$ then $Z(t)$ simply yields the Sparre Andersen model [see Andersen (1957)].

The following theorem is a key result for the rest of the thesis.

Theorem 3.1.1. *For any $t>0\,,\delta>0$ and any s such the integrals converge*

$$\begin{aligned} M_{Z(t)}(s) &= 1+\sum_{k=0}^{\infty}\int_0^t\int_0^{t-x_1}\cdots\int_0^{t-\sum_{i=1}^k x_i}\prod_{i=1}^{k+1}\left[M_X\left(se^{-\delta\sum_{j=1}^i x_j}\right)-1\right] \\ &\quad \times dm(x_{k+1})\ldots dm(x_2)\,dm(x_1)\,, \end{aligned} \tag{3.5}$$

where $m(x)$ is the renewal function defined in (4.2) and $\sum_{i=1}^k x_i=0$ for $k=0$.

Proof. See Léveillé (2002). □

In the following section we study the moments of the discounted process $Z=\{Z(t)\,,t\geq 0\}$ in (3.4), that is for $\delta>0$, but in the special case where the renewal process N is a Poisson process and the distribution of the claim severities F_X is PH.

3.2 Moments of Compound Poisson Sums with PH Claims

When we consider the net interest $\delta = 0$ our model becomes the classical risk model. In that case the moments of the compound Poisson process are well known and can be found in any risk theory textbook such as Klugman, Panjer and Willmot (2004). Léveillé (2002) gives an analytic expression for the m.g.f. of the compound Poisson present value risk (CPPVR) process with net interest rate $\delta > 0$. In this section we study the m.g.f. of the CPPVR process when claim severities have a PH distribution.

Corollary 3.2.1. *Let $N = \{N(t), t \geq 0\}$ be a Poisson process with parameter $\lambda > 0$. Then for any fixed $t > 0$ and $\delta > 0$, the moment generating function (m.g.f.) of $Z(t)$ is given by*

$$M_{Z(t)}(s) = e^{\lambda \int_0^t [M_X(se^{-\delta u})-1]du}, \qquad s \in \mathbb{R}, \tag{3.6}$$

where M_X is the m.g.f. of the claim severity X.

Proof. This result is given in Léveillé (2002), but we give a different proof here. For any $t > 0, \delta > 0$ and $s \in \mathbb{C}$. Since here the inter–arrival times are exponential distributed with parameter λ, the renewal function is

$$m(x) = \lambda x, \qquad x \geq 0.$$

Hence (3.5) reduces to:

$$M_{Z(t)}(s) = 1 + \sum_{k=0}^{\infty} \lambda^{k+1} \int_0^t \int_0^{t-x_1} \cdots \int_0^{t-\sum_{i=1}^k x_i} \prod_{i=1}^{k+1} \left[M_X\left(se^{-\delta \sum_{j=1}^i x_j}\right) - 1\right] dx_{k+1} \ldots dx_1.$$

Let $y_i = x_1 + x_2 + \cdots + x_i$ and $i = 1, 2, \cdots, k+1$, then

$$\begin{aligned} M_{Z(t)}(s) &= 1 + \sum_{k=0}^{\infty} \lambda^{k+1} \int_0^t \int_{y_1}^t \cdots \int_{y_k}^t \prod_{i=1}^{k+1} \left[M_X(se^{-\delta y_i}) - 1 \right] dy_{k+1} \dots dy_2 \, dy_1 \\ &= 1 + \sum_{k=0}^{\infty} \lambda^{k+1} \int_0^t \int_0^{y_{k+1}} \cdots \int_0^{y_2} \prod_{i=1}^{k+1} \left[M_X(se^{-\delta y_i}) - 1 \right] dy_1 \dots dy_k \, dy_{k+1} \, . \end{aligned} \tag{3.7}$$

Differentiating both sides of (3.7) with respect to t yields

$$\begin{aligned} \frac{\partial}{\partial t} M_{Z(t)}(s) &= \lambda \left[M_X(se^{-\delta t}) - 1 \right] \Big(1 + \big(\sum_{k=1}^{\infty} \lambda^k \int_0^t \int_0^{y_k} \cdots \int_0^{y_2} \\ &\qquad \left[M_X(se^{-\delta y_1}) - 1 \right] \cdots \left[M_X(se^{-\delta y_k}) - 1 \right] dy_1 \, dy_2 \cdots dy_k \big) \Big) \\ &= \lambda \left[M_X(se^{-\delta t}) - 1 \right] M_{Z(t)}(s) \, , \end{aligned}$$

which is a first order homogeneous ordinary differential equation with initial value $M_{Z(0)}(s) = 1$. The solution is

$$M_{Z(t)}(s) = e^{\lambda \int_0^t [M_X(se^{-\delta u}) - 1] du} \, , \qquad s \in \mathbb{R} \, . \tag{3.8}$$

□

When the claim severity distribution F_X belongs to the PH family, then the m.g.f., above can be written more explicitly in matrix form as follows.

Corollary 3.2.2. *If the deflated claims* $\{X_k\}_{k \ge 1}$ *have a PH*$(\underline{\alpha}, \boldsymbol{A})$ *distribution such that sprad*$\{s\boldsymbol{A}^{-1}\} < 1$ *(see Appendix A, Definition A.1.3.) and* $N = \{N(t), t \ge 0\}$ *forms a Poisson process, then for* $\delta > 0$

$$M_{Z(t)}(s) = \exp \left\{ \tfrac{\lambda}{\delta} \underline{\alpha}' \ln \left[(\boldsymbol{I} + se^{-\delta t} \boldsymbol{A}^{-1})(\boldsymbol{I} + s\boldsymbol{A}^{-1})^{-1} \right] \underline{1} \right\} , \qquad s \in \mathbb{R} \, . \tag{3.9}$$

Proof. The moment generating function of a PH$(\underline{\alpha}, \mathbf{A})$ is given in (2.11), hence from Proposition 3.2.1, the moment generating function of $Z(t)$ is given by

$$\begin{aligned} M_{Z(t)}(s) &= \exp \Big(\lambda \int_0^t \left[\underline{\alpha}' (se^{-\delta u} \mathbf{I} + \mathbf{A})^{-1} \mathbf{A} \underline{1} - 1 \right] du \Big) \\ &= \exp \Big(\lambda \Big[\underline{\alpha}' \int_0^t (se^{-\delta u} \mathbf{I} + \mathbf{A})^{-1} du \, \mathbf{A} \underline{1} \Big] - \lambda t \Big) \, , \qquad s \in \mathbb{R} \, . \end{aligned} \tag{3.10}$$

First we need to calculate $\int_0^t (se^{-\delta u}\mathbf{I}+\mathbf{A})^{-1}du$. If $\mathbf{A}^{-1}$ exists, then

$$\begin{aligned}\int_0^t (se^{-\delta u}\mathbf{I}+\mathbf{A})^{-1}du &= \int_0^t (se^{-\delta u}\mathbf{A}\mathbf{A}^{-1}+\mathbf{A})^{-1}du \\ &= \int_0^t (se^{-\delta u}\mathbf{A}^{-1}+\mathbf{I})^{-1}\,du\,\mathbf{A}^{-1}.\end{aligned}$$

Since $sprad\{s\mathbf{A}^{-1}\}<1$, then we have $sprad\{se^{-\delta t}\mathbf{A}^{-1}\}<1$. From Lemma A.2.2 in the Appendix we obtain

$$\begin{aligned}&\int_0^t \left(se^{-\delta u}\mathbf{A}^{-1}+\mathbf{I}\right)^{-1}du\,\mathbf{A}^{-1}\\ =& \int_0^t \Big(\mathbf{I}+(-1)^1(se^{-\delta u}\mathbf{A}^{-1})+(-1)^2(se^{-\delta u}\mathbf{A}^{-1})^2+\cdots\\ &\quad +(-1)^k(se^{-\delta u}\mathbf{A}^{-1})^k+\cdots\Big)du\,\mathbf{A}^{-1}\\ =& \Big\{\mathbf{I}t+(-1)^1\big[\frac{s}{\delta}(1-e^{-\delta t})\big]\mathbf{A}^{-1}+(-1)^2\big[\frac{s^2}{2\delta}(1-e^{-2\delta t})\big](\mathbf{A}^{-1})^2+\cdots\\ &\quad +(-1)^k\big[\frac{s^k}{k\delta}(1-e^{-k\delta t})\big](\mathbf{A}^{-1})^k+\cdots\Big\}\mathbf{A}^{-1}\\ =& \Big\{\mathbf{I}t+\big[(-1)^1\frac{s}{\delta}\mathbf{A}^{-1}+(-1)^2\frac{s^2}{2\delta}(\mathbf{A}^{-1})^2+\cdots+(-1)^k\frac{s^k}{k\delta}(\mathbf{A}^{-1})^k+\cdots\big]-\\ &\quad \big[(-1)^1\frac{s}{\delta}e^{-\delta t}\mathbf{A}^{-1}+(-1)^2\frac{s^2}{2\delta}e^{-2\delta t}(\mathbf{A}^{-1})^2+\cdots\\ &\quad +(-1)^k\frac{s^k}{k\delta}e^{-k\delta t}(\mathbf{A}^{-1})^k+\cdots+\big]\Big\}\mathbf{A}^{-1}\\ =& \Big\{\mathbf{I}t-\frac{1}{\delta}\Sigma_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}(s\mathbf{A}^{-1})^k+\frac{1}{\delta}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}(se^{-\delta t}\mathbf{A}^{-1})^k\Big\}\mathbf{A}^{-1}\\ =& \Big\{\mathbf{I}t-\frac{1}{\delta}\ln(\mathbf{I}+s\mathbf{A}^{-1})+\frac{1}{\delta}\ln(\mathbf{I}+se^{-\delta t}\mathbf{A}^{-1})\Big\}\mathbf{A}^{-1}, \qquad (3.11)\end{aligned}$$

the last equality follows from definition A.1.5 of the Appendix. Substituting (3.11) into (3.10), then the m.g.f. of $Z(t)$ becomes

$$\begin{aligned}M_{Z(t)}(s) &= \exp\Big\{\lambda\underline{\alpha}'\big[\mathbf{I}t-\frac{1}{\delta}\ln(\mathbf{I}+s\mathbf{A}^{-1})+\frac{1}{\delta}\ln(\mathbf{I}+se^{-\delta t}\mathbf{A}^{-1})\big]\mathbf{A}^{-1}\mathbf{A}\underline{1}-\lambda t\Big\}\\ &= \exp\Big\{\frac{\lambda}{\delta}\underline{\alpha}'\big[\ln(\mathbf{I}+se^{-\delta t}\mathbf{A}^{-1})-\ln(\mathbf{I}+s\mathbf{A}^{-1})\big]\underline{1}\Big\}, \qquad s\in\mathbb{R}. \quad (3.12)\end{aligned}$$

From Lemma A.2.6 then (3.9) follows. □

Corollary 3.2.3. *For $\delta>0$ we have:*

$$\mathbb{E}[Z(t)] = -\frac{\lambda}{\delta}(1-e^{-\delta t})\,\underline{\alpha}'\boldsymbol{A}^{-1}\underline{1}, \qquad t>0, \qquad (3.13)$$

and

$$\mathbb{V}[Z(t)] = \frac{\lambda}{\delta}\,(1-e^{-2\delta t})\,\underline{\alpha}'\mathbf{A}^{-2}\underline{1}\,, \qquad t>0\,. \tag{3.14}$$

Proof. From (3.10) we have

$$\begin{aligned} M_{Z(t)}(s) &= \exp\Big(\lambda\underline{\alpha}'\{\mathbf{I}t + (-1)^1[\frac{s}{\delta}(1-e^{-\delta t})]\mathbf{A}^{-1} + \cdots \\ &\quad +(-1)^k[\frac{s^k}{k\delta}(1-e^{-k\delta t})]\mathbf{A}^{-k} + \cdots\}\mathbf{A}^{-1}\mathbf{A}\underline{1} - \lambda t\Big)\,, \end{aligned} \tag{3.15}$$

and hence

$$\mathbb{E}[Z(t)] = \frac{\partial}{\partial s}M_{Z(t)}(s)|_{s=0} = -\frac{\lambda}{\delta}\,(1-e^{-\delta t})\,\underline{\alpha}'\mathbf{A}^{-1}\underline{1}\,,$$

and

$$\mathbb{E}[Z(t)^2] = \frac{\partial^2}{\partial s^2}M_{Z(t)}(s)|_{s=0} = \frac{\lambda}{\delta}\,\underline{\alpha}'(1-e^{-2\delta t})\mathbf{A}^{-2}\underline{1} + \{-\frac{\lambda}{\delta}\,(1-e^{-\delta t})\,\underline{\alpha}'\mathbf{A}^{-1}\underline{1}\}^2\,.$$

$$\Rightarrow\ \mathbb{V}[Z(t)] = \mathbb{E}[Z(t)^2] - \{\mathbb{E}[Z(t)]\}^2 = \frac{\lambda}{\delta}\,(1-e^{-2\delta t})\,\underline{\alpha}'\mathbf{A}^{-2}\underline{1}\,.$$

These moments are consistent with those from Léveillé and Garrido (2001a), which were obtained using renewal arguments. □

Remark 3.1. Following the same procedure as in Corollary 3.2.1, higher order moments $\mathbb{E}[Z(t)^k]$ can be obtained for $k = 0, 1, 2, \ldots$. The regularity shown by the PH structure leads itself well to a systematic treatment by a symbolic computational software.

Let $p(s)$ be a polynomial in s:

$$p(s) = a_1 s + \frac{a_2}{2}s^2 + \frac{a_3}{3}s^3 + \cdots + \frac{a_k}{k}s^k + \cdots\,,$$

where

$$a_k = (-1)^k\frac{\lambda}{\delta}\,(1-e^{-k\delta t})\,\underline{\alpha}'\mathbf{A}^{-k}\underline{1}\,, \qquad k\in\mathbb{N}\,.$$

Then from (3.15) we have that

$$M_{Z(t)}(s) = e^{p(s)}\,, \qquad s\in\mathbb{R}\,.$$

Using Maple one easily verifies the following results:

$$\begin{aligned}
\mathbb{E}[Z(t)^3] &= 2a_3 + 3a_2a_1 + a_1^3\,,\\
\mathbb{E}[Z(t)^4] &= 6a_4 + 8a_3a_1 + 3a_2^2 + 6a_2a_1^2 + a_1^4\,,\\
\mathbb{E}[Z(t)^5] &= 24a_5 + 30a_4a_1 + 20a_3a_2 + 20a_3a_1^2 + 15a_2^2a_1 + 10a_2a_1^3 + a_1^5\,,\\
\mathbb{E}[Z(t)^6] &= 120a_6 + 144a_5a_1 + 90a_4a_2 + 90a_4a_1^2 + 40a_3^2 + 120a_3a_2a_1\\
&\qquad +40a_3a_1^3 + 15a_2^3 + 45a_2^2a_1^2 + 15a_2a_1^4 + a_1^6\,,
\end{aligned}$$

Following the same method we can find $\mathbb{E}[Z(t)^k]$, for any $k \geq 7$.

Corollary 3.2.4. *If $\delta \to 0$, then the m.g.f. in (3.9) reduces to that of the classical risk model with PH claims:*

$$M_{Z(t)}(s) = e^{\lambda t[\underline{\alpha}'(\boldsymbol{I}+s\boldsymbol{A}^{-1})^{-1}\underline{1}-1]}\,, \tag{3.16}$$

where $Z(t) = S(t)$ in (3.1) and the X_i are i.i.d. with a common PH($\underline{\alpha}$, $\boldsymbol{A}$) distribution.

Proof. Using (3.12) leads to

$$\lim_{\delta\to 0} M_{Z(t)}(s) = \lim_{\delta\to 0} \exp\left\{\frac{\lambda}{\delta}\underline{\alpha}'\left[\ln(\mathbf{I} + se^{-\delta t}\mathbf{A}^{-1}) - \ln(\mathbf{I}+s\mathbf{A}^{-1})\right]\underline{1}\right\}.$$

From Definition A.1.5 of Appendix A: $\ln[\mathbf{I}+se^{-\delta t}\mathbf{A}^{-1}] = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}(se^{-\delta t}\mathbf{A}^{-1})^k$ and l'Hospital's rule implies that

$$\begin{aligned}
\lim_{\delta\to 0} M_{Z(t)}(s) &= \lim_{\delta\to 0}\exp\left\{\lambda\underline{\alpha}'\left[t\sum_{k=1}^{\infty}(-1)^k(se^{-\delta t}\mathbf{A}^{-1})^k\right]\underline{1}\right\}\\
&= \lim_{\delta\to 0}\exp\left\{\lambda\underline{\alpha}'\left[t\sum_{k=1}^{\infty}(-1)^k(s\mathbf{A}^{-1})^k\right]\underline{1}\right\}\\
&= \exp\left\{\lambda\underline{\alpha}'\left[t(\mathbf{I}+s\mathbf{A})^{-1} - t\mathbf{I}\right]\underline{1}\right\}\\
&= \exp\left\{\lambda t\left[\underline{\alpha}'(\mathbf{I}+s\mathbf{A})^{-1}\underline{1} - 1\right]\right\},
\end{aligned}$$

which is the moment generating function of the classical risk model for $S(t) = \sum_{i=1}^{N(t)} X_i$, with X_i having a PH($\underline{\alpha}$, $\mathbf{A}$) distribution.

□

Remark 3.2.

$$\lim_{\delta\to 0}\mathbb{E}[Z(t)] = \lim_{\delta\to 0} -\frac{\lambda\underline{\alpha}'}{\delta}(1-e^{-\delta t})\mathbf{A}^{-1}\underline{1} = \lim_{\delta\to 0} -\frac{1-e^{-\delta t}}{\delta}\lambda\underline{\alpha}'\mathbf{A}^{-1}\underline{1}.$$

By the l'Hospital's rule we have

$$\lim_{\delta\to 0}\mathbb{E}[Z(t)] = \lim_{\delta\to 0} -\frac{(1-e^{-\delta t})}{\delta}\lambda\underline{\alpha}'\mathbf{A}^{-1}\underline{1} = -\lambda t\underline{\alpha}'\mathbf{A}^{-1}\underline{1} = \mathbb{E}[N(t)]\mathbb{E}[X]. \quad (3.17)$$

By (2.9) we see that this is consistent with the result for the classical risk model $\mathbb{E}[S(t)] = \mathbb{E}[N(t)]\,\mathbb{E}[X]$, when X is a PH distributed claim severity. Similarly

$$\begin{aligned}\lim_{\delta\to 0}\mathbb{E}[Z(t)^2] &= \lim_{\delta\to 0}\frac{(1-e^{-2\delta t})}{\delta}\lambda\underline{\alpha}'\mathbf{A}^{-2}\underline{1} - \lim_{\delta\to 0}\Big[-\frac{(1-e^{-\delta t})}{\delta}\lambda\underline{\alpha}'\mathbf{A}^{-1}\underline{1}\Big]^2\\ &= 2\lambda t\underline{\alpha}'\mathbf{A}^{-2}\underline{1} + (\lambda t\underline{\alpha}'\mathbf{A}^{-1}\underline{1})^2.\end{aligned}$$

Then

$$\lim_{\delta\to 0}\mathbb{V}\big[Z(t)\big] = \lim_{\delta\to 0}\mathbb{E}\big[Z(t)^2\big] - \lim_{\delta\to 0}\Big\{\mathbb{E}\big[Z(t)\big]\Big\}^2 = \lambda t(2\underline{\alpha}'\mathbf{A}^{-2}\underline{1}), \quad (3.18)$$

again, by (2.10) this is consistent with the classical result $\mathbb{V}[S(t)] = \mathbb{E}[N(t)]\mathbb{E}[X^2]$, with PH distributed claim severities. A similar procedure gives $\lim_{\delta\to 0}\mathbb{E}[Z(t)^k]$, for $k > 2$ using Remarks 3.1 and 3.2.

Consider now the asymptotic behavior of $M_{Z(t)}$, as $t \to \infty$.

Corollary 3.2.5. *For $\delta > 0$ we have:*

$$M_{Z_{(\infty)}}(s) = \lim_{t\to\infty} M_{Z_{(t)}}(s) = \exp\Big(\frac{\lambda}{\delta}\underline{\alpha}'\ln(\boldsymbol{I} + s\boldsymbol{A}^{-1})^{-1}\underline{1}\Big), \qquad s\in\mathbb{R}. \quad (3.19)$$

Proof. The result easily follows from (3.9). □

Remark 3.3. From Equations (3.13) and (3.14) we have

$$\lim_{t\to\infty}\mathbb{E}[Z(t)] = \lim_{t\to\infty} -\frac{\lambda}{\delta}\underline{\alpha}'(1-e^{-\delta t})\mathbf{A}^{-1}\underline{1} = -\frac{\lambda}{\delta}\underline{\alpha}'\mathbf{A}^{-1}\underline{1}, \quad (3.20)$$

and

$$\lim_{t\to\infty}\mathbb{V}[Z(t)] = \lim_{t\to\infty}\frac{\lambda}{\delta}\underline{\alpha}'(1-e^{-2\delta t})\mathbf{A}^{-2}\underline{1} = \frac{\lambda}{\delta}\underline{\alpha}'\mathbf{A}^{-2}\underline{1}. \quad (3.21)$$

By a similar procedure we can obtain the asymptotic expression $\lim_{t\to\infty}\mathbb{E}[Z(t)^k]$, for any $k \geq 1$, using Maple.

3.3 Examples

In this section, we illustrate the above results with some examples and discuss the asymptotic behavior as t goes to ∞. First consider the following generalized Erlang(2) example.

Example 3.3.1. (Generalized Erlang(2) distribution) Let $N = \{N(t), t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$ and generalized Erlang(2) be deflated claims with density function:

$$f_X(x) = \frac{\lambda_1 \lambda_2 (e^{-\lambda_1 x} - e^{-\lambda_2 x})}{\lambda_2 - \lambda_1}, \qquad x > 0,$$

where $\lambda_1 < \lambda_2$, $\underline{\alpha} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ 0 & -\lambda_2 \end{pmatrix}$.

By Theorem 3.2.1, the moment generating function of $Z(t)$ here takes the following form:

$$M_{Z(t)}(s) = \left(\frac{se^{-\delta t} - \lambda_1}{s - \lambda_1}\right)^{\frac{\lambda\lambda_2}{\delta(\lambda_2 - \lambda_1)}} \left(\frac{se^{-\delta t} - \lambda_2}{s - \lambda_2}\right)^{\frac{\lambda\lambda_1}{\delta(\lambda_2 - \lambda_1)}}, \qquad s < \lambda_1 .$$

According to Corollary 3.2,1 the expectation and variance of $Z(t)$ are given by

$$\mathbb{E}[Z(t)] = \frac{\lambda(1 - e^{-\delta t})}{\delta\lambda_1} + \frac{\lambda(1 - e^{-\delta t})}{\delta\lambda_2}, \qquad t > 0,$$

and

$$\mathbb{V}[Z(t)] = \frac{\lambda(1 - e^{-2\delta t})}{\delta\lambda_1^2} + \frac{\lambda(1 - e^{-2\delta t})}{\delta}\Big(\frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2^2}\Big), \qquad t > 0.$$

Consider the classical risk model, by Corollary 3.2.2 we can compute the limit value as $\delta \to 0$:

$$\lim_{\delta \to 0} M_{Z(t)}(s) = \exp\Big[\lambda t\,\Big(\frac{\lambda_1}{\lambda_1 - s} + \frac{s\lambda_1}{(\lambda_1 - s)(\lambda_2 - s)} - 1\Big)\Big], \qquad s < \lambda_1,$$

which implies that

$$\lim_{\delta \to 0} \mathbb{E}[Z(t)] = \lambda t\,\Big(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\Big), \qquad t > 0,$$

and

$$\lim_{\delta \to 0} \mathbb{V}[Z(t)] = 2\lambda t\,\Big(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2^2}\Big), \qquad t > 0.$$

Similarly we consider the asymptotic value of $Z(t)$ as t increases to infinity.

$$\lim_{t\to\infty} M_{Z(t)}(s) = \left(\frac{\lambda_1}{\lambda_1 - s}\right)^{\frac{\lambda\lambda_2}{\delta(\lambda_1-\lambda_2)}} \left(\frac{\lambda_2}{\lambda_2 - s}\right)^{\frac{\lambda\lambda_1}{\delta(\lambda_1-\lambda_2)}},$$

which implies that

$$\lim_{t\to\infty} \mathbb{E}[Z(t)] = \frac{\lambda}{\delta}\Big(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\Big),$$

and

$$\lim_{t\to\infty} \mathbb{V}[Z(t)] = \frac{\lambda}{\delta}\Big(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2^2}\Big).$$

Remark 3.4. A similar approach gives $\lim_{\delta\to 0} \mathbb{E}[Z(t)^k]$ and $\lim_{t\to\infty} \mathbb{E}[Z(t)^k]$ using Maple, as in Remark 3.1.

Example 3.3.2. (Generalized Erlang(3)) Let N be a Poisson process with intensity $\lambda > 0$ and generalized Erlang(3) be deflated claims with the following density function, for $x > 0$:

$$f(x) = \frac{\lambda_1\lambda_2\lambda_3\Big((\lambda_3 - \lambda_2)e^{-\lambda_1 x} + (\lambda_1 - \lambda_3)e^{-\lambda_2 x} + (\lambda_2 - \lambda_1)e^{-\lambda_3 x}\Big)}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)},$$

where $\lambda_1 < \lambda_2 < \lambda_3$.

By Theorem 3.2.1 we have that the m.g.f. of $Z(t)$ is given by

$$\begin{aligned} M_{Z(t)}(s) &= \left(\frac{\lambda_1 - se^{-\delta t}}{\lambda_1 - s}\right)^{\frac{\lambda\lambda_2\lambda_3}{\delta(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}} \left(\frac{\lambda_2 - se^{-\delta t}}{\lambda_2 - s}\right)^{\frac{\lambda\lambda_1\lambda_2}{\delta(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)}} \\ &\quad \times \left(\frac{\lambda_3 - se^{-\delta t}}{\lambda_3 - s}\right)^{\frac{\lambda\lambda_1\lambda_2}{\delta(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}}, \qquad s < \lambda_1. \end{aligned}$$

Then the asymptotic values of $\mathbb{E}[Z(t)]$ and $\mathbb{V}[Z(t)]$ are given by Corollary 3.2.1 to be:

$$\mathbb{E}[Z(t)] = \frac{\lambda(1 - s^{-\delta t})}{\delta}\Big(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\Big), \qquad t > 0,$$

and

$$\mathbb{V}[Z(t)] = \frac{\lambda(1 - se^{-2\delta t})}{\delta}\Big(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_1\lambda_3} + \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2}\Big), \qquad t > 0.$$

By Corollary 3.2.2 we reproduce the following classical results:

$$\lim_{\delta\to 0} M_{Z(t)}(s) = \exp\left(\frac{\lambda_1\lambda t}{\lambda_1 - s} + \frac{s\lambda_1\lambda t}{(\lambda_1 - s)(\lambda_2 - s)} + \frac{s\lambda_1\lambda_2\lambda t}{(\lambda_1 - s)(\lambda_2 - s)(\lambda_3 - s)} - 1\right),$$

for $s < \lambda_1$ and

$$\lim_{\delta\to 0} \mathbb{E}[Z(t)] = \lambda t\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right), \qquad t > 0,$$

while

$$\lim_{\delta\to 0} \mathbb{V}[Z(t)] = 2\lambda t\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_1\lambda_3} + \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2}\right), \qquad t > 0.$$

Finally we show some asymptotic results by the Corollary 3.2.3 and Remark 3.3.

$$\begin{aligned}\lim_{t\to\infty} M_{Z(t)}(s) &= \left(\frac{\lambda_1}{\lambda_1 - s}\right)^{\frac{\lambda\lambda_2\lambda_3}{\delta(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}} \left(\frac{\lambda_2}{\lambda_2 - s}\right)^{-\frac{\lambda\lambda_1\lambda_3}{\delta(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)}} \\ &\quad \times \left(\frac{\lambda_3}{\lambda_3 - s}\right)^{\frac{\lambda\lambda_1\lambda_2}{(\delta\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}}, \qquad s < \lambda_1\end{aligned}$$

and

$$\lim_{t\to\infty} \mathbb{E}[Z(t)] = \frac{\lambda}{\delta}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right),$$

while

$$\lim_{t\to\infty} \mathbb{V}[Z(t)] = \frac{\lambda}{\delta}\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_1\lambda_3} + \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2}\right).$$

Remark 3.5. Here again $\lim_{\delta\to 0} \mathbb{E}[Z(t)^k]$ and $\lim_{t\to\infty} \mathbb{E}[Z(t)^k]$ can be obtained by the same procedure as in Remark 3.1 and Maple.

Chapter 4

Renewal Processes

In this chapter we consider the compound renewal present value risk (CRPVR) process, for the case where the inter–arrival times are PH distributed and the deflated claims are also PH. We obtain an homogeneous differential equation for the moment generation function (m.g.f) of the CRPVR sum at time t and discuss the asymptotic behavior of this m.g.f. as t goes to infinity. We also give examples to illustrate the results.

The first section gives a definition of the PH-renewal process and discusses its first properties. In the second section the special case of the distribution Erlang(n) is considered.

4.1 The PH-Renewal Process

In this section we define the PH-renewal process and some of its properties, especially the renewal function and the renewal density.

4.1.1 Definitions

A Poisson process is a special case of a renewal process with exponential inter–arrival times. It is discussed in detail in Chapter 3. Here we consider a more general family of inter–arrival time distributions.

As in the model of Andersen (1957), let $\{T_k\}_{k\geq 1}$ be the claim occurrence times and $\tau_k = T_k - T_{k-1}$, for $k \geq 2$ and $\tau_1 = T_1$ be the claim inter–arrival times. If $\{\tau_k\}_{k\geq 1}$ are mutually independent and have the same distribution F_τ, then $\{T_k\}_{k\geq 1}$ are called renewal times.

Now let $N(t) = \max\{k \in \mathbb{N}\,; T_k \leq t\}$ for each t, where $N(0) = 0$, represent the number of renewals up to time t, then $N = \{N(t)\,; t \geq 0\}$ is a renewal process. If in addition the inter–arrival times have a PH distribution, this defines a PH-renewal process.

Definition 4.1.1. *The counting process $N = \{N(t)\,; t \geq 0\}$ is said to be a PH-renewal process if the inter–arrival times $\tau_k = T_k - T_{k-1}$, $k \geq 2$ and $\tau_1 = T_1$, have a common PH distribution say F_τ, and are independent. Here the $\{T_k\}_{k\geq 1}$ are the arrival times.*

In this thesis we are particularly concerned with the mean of $N(t)$. The function

$$m(t) = \mathbb{E}\big[N(t)\big]\,, \qquad t \geq 0\,, \tag{4.1}$$

is called *the renewal function. The renewal density* is then defined as

$$m'(t) = \lim_{\Delta t \to 0} \frac{\mathbb{E}\big[N(t + \Delta\, t)\big] - \mathbb{E}\big[N(t)\big]}{\Delta\, t}\,, \qquad t \geq 0\,.$$

For instance, the renewal function of a Poisson process is easily obtained to be

$$m(t) = \lambda t\,, \qquad t \geq 0\,,$$

where λ is the parameter of the exponential inter–arrival times.

Now consider inter–arrival times that are PH-distributed with parameters $(\underline{\alpha}\,, \mathbf{A})$, then we have the following result.

Proposition 4.1.1. *Consider a renewal process with inter–arrival times which are PH distributed with parameters* $(\underline{\alpha}\,, \boldsymbol{A})$. *Then the renewal density exists and is given by*

$$m'(x) = \underline{\alpha}' e^{\boldsymbol{A}(\boldsymbol{I}-\underline{1}\,\underline{\alpha}')x}(-\boldsymbol{A})\underline{1}\,.$$

Here we use probabilistic methods to prove the result. For the Markov chain method see Asmussen (2003).

Proof. From Cox (1970) we have:

$$m(x) = \mathbb{E}\big[N(x)\big] = \sum_{k=1}^{\infty} F_{\tau}^{*k}(x)\,, \qquad t \geq 0\,, \tag{4.2}$$

where F_{τ}^{*k} denotes the k–fold convolution of F_{τ}, which is the distribution of inter–arrival times. When the inter–arrival times have PH $(\underline{\alpha}, \mathbf{A})$ distributions, then

$$F_{\tau}(x) = 1 - \underline{\alpha}' e^{\mathbf{A}x}\underline{1}\,, \qquad x \geq 0. \tag{4.3}$$

By the closure property, the convolution of F_{τ} is also PH distribution. Let the F_{τ}^{*k} for $k \geq 2$ be the following PH $(\underline{\alpha_k}, \mathbf{C}_k)$ forms:

$$\underline{\alpha}_1 = \underline{\alpha}\,, \qquad \underline{\alpha}_k = (\underline{\alpha}'\,, \underline{0}'_{(k-1)n})'\,, \qquad \mathbf{C}_1 = \mathbf{A}\,, \qquad \mathbf{C}_k = \begin{pmatrix} \mathbf{A} & -\mathbf{A}\underline{1}\,\underline{\alpha}_{k-1} \\ \mathbf{0} & \mathbf{C}_{k-1} \end{pmatrix}, \tag{4.4}$$

then from (4.2) and Appendix Definition A.1.4. we have

$$\begin{aligned} m(x) &= 1 - \underline{\alpha}' e^{\mathbf{A}x}\underline{1} + 1 - \underline{\alpha}_2' e^{\mathbf{C}_2 x}\underline{1}_{2n} + \cdots + 1 - \underline{\alpha}_k' e^{\mathbf{C}_k x}\underline{1}_{kn} + \cdots \\ &= 1 - \underline{\alpha}' \sum_{r=0}^{\infty} \frac{(\mathbf{A}x)^r}{r!}\underline{1} + 1 - \underline{\alpha}_2' \sum_{r=0}^{\infty} \frac{(\mathbf{C}_2 x)^r}{r!}\underline{1}_{2n} + \cdots \\ &\quad + 1 - \underline{\alpha}_k' \sum_{r=0}^{\infty} \frac{(\mathbf{C}_k x)^r}{r!}\underline{1}_{kn} + \cdots \\ &= -\Big[\big(\underline{\alpha}'\mathbf{A}\underline{1} + \underline{\alpha}_2'\mathbf{C}_2\underline{1}_{2n}\cdots\big)x + \frac{1}{2!}\big(\underline{\alpha}'\mathbf{A}^2\underline{1} + \underline{\alpha}_2'\mathbf{C}_2^2\underline{1}_{2n} + \cdots\big)x^2 + \cdots\Big]. \end{aligned} \tag{4.5}$$

First let us prove the following the result using induction on j:

$$\underline{\alpha}'\mathbf{C}_k^j\,\underline{1}_{kn} = 0\,, \qquad \text{for} \quad k \geq j+1 = 0\,. \tag{4.6}$$

When $j = 1$ and $k \geq j+1$, we have

$$\begin{aligned} \underline{\alpha}_k'\,\mathbf{C}_k\,\underline{1}_{kn} &= \begin{pmatrix} \underline{\alpha}' & \underline{0}'_{(k-1)n} \end{pmatrix} \begin{pmatrix} \mathbf{A} & -\mathbf{A}\underline{1}\,\underline{\alpha}_{k-1} \\ \mathbf{0} & \mathbf{C}_{k-1} \end{pmatrix} \begin{pmatrix} \underline{1} \\ \underline{1}_{(k-1)n} \end{pmatrix} \\ &= \underline{\alpha}'\mathbf{A}\underline{1} - \underline{\alpha}'\mathbf{A}\underline{1}\underline{\alpha}_{k-1}'\,\underline{1}_{k-1} = 0\,, \end{aligned}$$

from $\underline{\alpha}_{k-1}'\underline{1}_{k-1} = 1$. Suppose $j \leq n-1$

$$\underline{\alpha}'\mathbf{C}_k^j\,\underline{1}_{kn} = 0\,, \qquad \text{for} \quad k \geq j+1\,. \tag{4.7}$$

Now prove (4.7) is true, when $j = n$:

$$\begin{aligned} \underline{\alpha}_k'\,\mathbf{C}_k^n\,\underline{1}_{kn} &= \begin{pmatrix} \underline{\alpha}' & \underline{0}'_{(k-1)n} \end{pmatrix} \begin{pmatrix} \mathbf{A}^n & -\sum_{i=0}^{n-1}\mathbf{A}^{n-i}\underline{1}\,\underline{\alpha}_{k-1}'\mathbf{C}_{k-1}^i \\ \mathbf{0} & \mathbf{C}_{k-1}^n \end{pmatrix} \begin{pmatrix} \underline{1} \\ \underline{1}_{(k-1)n} \end{pmatrix} \\ &= \underline{\alpha}'\mathbf{A}^n\underline{1} - \sum_{i=0}^{n-1} \underline{\alpha}'\mathbf{A}^{n-i}\underline{1}\,\underline{\alpha}_{k-1}'\mathbf{C}_{k-1}^i\underline{1}_{(k-1)n} \\ &= -\sum_{i=1}^{n-1} \underline{\alpha}'\mathbf{A}^{n-i}\underline{1}\,\underline{\alpha}_{k-1}'\mathbf{C}_{k-1}^i\underline{1}_{(k-1)n}\,. \end{aligned} \tag{4.8}$$

Assumption (4.7) leads to $\underline{\alpha}'_{k-1}\mathbf{C}^i_{k-1}\underline{1}_{(k-1)n} = 0$, for $k \geq n \geq i+1$, then the result $\underline{\alpha}'_k\,\mathbf{C}^n_k\,\underline{1}_{kn} = 0$ holds. So we have

$$\begin{aligned} m(x) &= -\Big\{\underline{\alpha}'\mathbf{A}\underline{1}x + \frac{1}{2!}\left[\underline{\alpha}'\mathbf{A}^2\underline{1} + \underline{\alpha}'_2\mathbf{C}^2_2\underline{1}_{2n}\right]x^2 + \frac{1}{3!}\left[\underline{\alpha}'\mathbf{A}^3\underline{1} + \underline{\alpha}'_2\mathbf{C}^3_2\underline{1}_{2n} + \underline{\alpha}'_3\mathbf{C}^3_3\underline{1}_{3n}\right] \\ &\quad \times x^3 + +\frac{1}{n!}\left[\underline{\alpha}'\mathbf{A}^n\underline{1} + \underline{\alpha}'_2\mathbf{C}^n_2\underline{1}_{2n} + \cdots + \underline{\alpha}'_n\mathbf{C}^n_n\underline{1}_{nn}\right]x^n + \cdots\Big\}. \end{aligned} \tag{4.9}$$

Now we show the following result by induction:

$$\underline{\alpha}'\mathbf{A}^n\underline{1} + \underline{\alpha}'_2\mathbf{C}^n_2\underline{1}_{2n} + \cdots + \underline{\alpha}'_n\mathbf{C}^n_n\underline{1}_{nn} = \alpha'\left[\mathbf{A}(\mathbf{I} - \underline{1}\,\alpha')\right]^{n-1}\mathbf{A}\underline{1}\,, \tag{4.10}$$

When $n = 1$, obviously (4.10) holds. Suppose then that when $j \leq n-1$ the following result is true.

$$\underline{\alpha}'\mathbf{A}^j\underline{1} + \underline{\alpha}'_2\mathbf{C}^j_2\underline{1}_{2n} + \cdots + \underline{\alpha}'_j\mathbf{C}^j_j\underline{1}_{jn} = \alpha'\left[\mathbf{A}(\mathbf{I} - \underline{1}\,\alpha')\right]^{j-1}\mathbf{A}\underline{1}\,. \tag{4.11}$$

Consider $j = n$, from (4.8) we have:

$$\begin{aligned} &\underline{\alpha}'\mathbf{A}^n\underline{1} + \underline{\alpha}'_2\mathbf{C}^n_2\underline{1}_{2n} + \cdots + \underline{\alpha}'_n\mathbf{C}^n_n\underline{1}_{nn} \\ &= \underline{\alpha}'\mathbf{A}^n\underline{1} - \sum_{i=1}^{n-1}\underline{\alpha}'\mathbf{A}^{n-i}\underline{1}\,\underline{\alpha}'_{2-1}\mathbf{C}^i_{2-1}\underline{1}_{(2-1)n} - \cdots - \sum_{i=1}^{n-1}\underline{\alpha}'\mathbf{A}^{n-i}\underline{1}\,\underline{\alpha}'_{n-1}\mathbf{C}^i_{n-1}\underline{1}_{(n-1)n} \\ &= \underline{\alpha}'\mathbf{A}^n\underline{1} - \sum_{i=1}^{n-1}\underline{\alpha}'\mathbf{A}^{n-i}\underline{1}\left[\underline{\alpha}'\mathbf{A}^i\underline{1} + \underline{\alpha}'_2\mathbf{C}^i_2\underline{1}_{2n} + \cdots + \underline{\alpha}'_{n-1}\mathbf{C}^i_{n-1}\underline{1}_{(n-1)n}\right]. \end{aligned} \tag{4.12}$$

Assumptions (4.11) and (4.6) lead to following form of (4.12):

$$= \underline{\alpha}'\mathbf{A}^n\underline{1} - \sum_{i=1}^{n-1}\underline{\alpha}'\mathbf{A}^{n-i}\underline{1}\,\underline{\alpha}'\left[\mathbf{A}(\mathbf{I} - \underline{1}\,\alpha')\right]^{i-1}\underline{1} = \underline{\alpha}'\left[\mathbf{A}(\mathbf{I} - \underline{1}\,\alpha')\right]^{n-1}\underline{1}\,. \tag{4.13}$$

Hence (4.9) can be simplified as:

$$\begin{aligned} m(x) &= -\Big\{\underline{\alpha}'\mathbf{A}\underline{1}\,x + \frac{1}{2!}\,\underline{\alpha}'\mathbf{A}(\mathbf{I} - \underline{1}\,\underline{\alpha}')\mathbf{A}\underline{1}\,x^2 + \frac{1}{3!}\,\underline{\alpha}'\left[\mathbf{A}(\mathbf{I} - \underline{1}\,\underline{\alpha}')\right]^2\mathbf{A}\underline{1}x^3 + \cdots \\ &\quad + \frac{1}{(k+1)!}\,\underline{\alpha}'\left[\mathbf{A}(\mathbf{I} - \underline{1}\,\underline{\alpha}')\right]^k\mathbf{A}\underline{1}\,x^{k+1} + \cdots\Big\}. \end{aligned} \tag{4.14}$$

Differentiating (4.14) with respect to x yields

$$\begin{aligned} m'(x) &= -\Big\{\underline{\alpha}'\mathbf{A}\underline{1} + \underline{\alpha}'\mathbf{A}(\mathbf{I} - \underline{1}\,\underline{\alpha}')\mathbf{A}\underline{1}\,x + \frac{1}{2!}\,\underline{\alpha}'\left[\mathbf{A}(\mathbf{I} - \underline{1}\,\underline{\alpha}')\right]^2\mathbf{A}\underline{1}x^2 + \cdots \\ &\quad + \frac{1}{k!}\,\underline{\alpha}'\left[\mathbf{A}(\mathbf{I} - \underline{1}\,\underline{\alpha}')\right]^k\mathbf{A}\underline{1}\,x^k + \cdots\Big\}. \end{aligned} \tag{4.15}$$

Using Definition A.1.4. from Appendix A leads to $m'(x) = \underline{\alpha}' e^{\mathbf{A}(\mathbf{I}-\underline{1}\,\underline{\alpha}')x}(-\mathbf{A})\underline{1}$ which completes the proof. □

Additional details on the PH-renewal process and its applications to ruin probabilities and queueing systems can be found in Neuts (1981), Chakravarthy and Alfa (1997) and Asmussen (2003).

4.1.2 The Moment Generating Function of $Z(t)$

Consider now the m.g.f. of $Z(t)$, for fixed t, when N is a renewal process, we have an analytic expression in (3.5). In particular, when inter–arrival times are PH $(\underline{\alpha}, \mathbf{A})$ distributed, the renewal density can be written as

$$dm(x) = \underline{\alpha}'\, e^{\mathbf{B}x}\, (-\mathbf{A})\underline{1}\, dx\,, \qquad \text{for } \mathbf{B} = \mathbf{A}(\mathbf{I} - \underline{1}\,\underline{\alpha}').$$

Hence from Theorem 3.1.1.

$$\begin{aligned} M_{Z(t)}(s) &= 1 + \sum_{k=0}^{\infty} \int_0^t \int_0^{t-x_1} \cdots \int_0^{t-\sum_{i=1}^{k} x_i} \prod_{i=1}^{k+1} \left(\left[M_X\left(se^{-\delta \sum_{j=1}^{i} x_j}\right) - 1 \right] \right. \\ &\qquad \left. \times\, \underline{\alpha}'\, e^{\mathbf{B}x_i}(-\mathbf{A})\underline{1} \right) dx_{k+1} \ldots dx_2\, dx_1\,, \qquad t > 0\,, s \in \mathbb{R}\,. \end{aligned}$$

Let $y_i = x_1 + x_2 + \cdots + x_i$ for $i = 1, 2, \cdots, k+1$, then

$$\begin{aligned} & M_{Z(t)}(s) \\ = {} & 1 + \sum_{k=0}^{\infty} \int_0^t \int_{y_1}^t \cdots \int_{y_k}^t \left(\prod_{i=1}^{k+1} \left[M_X\left(se^{-\delta y_i}\right) - 1 \right] \underline{\alpha}'\, e^{\mathbf{B}(y_{k+1}-y_k)}(-\mathbf{A})\underline{1} \right. \\ & \left. \cdots\, \underline{\alpha}'\, e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\, \underline{\alpha}'\, e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1} \right) dy_{k+1} \ldots dy_2\, dy_1 \\ = {} & 1 + \sum_{k=0}^{\infty} \int_0^t \int_0^{y_{k+1}} \cdots \int_0^{y_2} \left(\prod_{i=1}^{k+1} \left[M_X\left(se^{-\delta y_i}\right) - 1 \right] \underline{\alpha}'\, e^{\mathbf{B}(y_{k+1}-y_k)}(-\mathbf{A})\underline{1} \right. \\ & \left. \cdots\, \underline{\alpha}'\, e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\, \underline{\alpha}'\, e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1} \right) dy_1 \ldots dy_k\, dy_{k+1}\,, \qquad t > 0\,, s \in \mathbb{R}\,. \end{aligned} \tag{4.16}$$

Differentiating both sides of (4.16) with respect to t yields

$$\begin{aligned}\frac{\partial}{\partial t}M_{Z(t)}(s) &= \left[M_X(se^{-\delta t})-1\right]\underline{\alpha}'\,e^{\mathbf{B}t}(-\mathbf{A})\underline{1}+\left[M_X(se^{-\delta t})-1\right] \\ &\quad\times\left(\sum_{k=1}^{\infty}\int_0^t\int_0^{y_k}\cdots\int_0^{y_2}\prod_{i=1}^{k}\left[M_X\left(se^{-\delta y_i}\right)-1\right]\right. \\ &\quad\times\underline{\alpha}'\,e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}(y_k-y_{k-1})}(-\mathbf{A})\underline{1}\cdots\underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1} \\ &\quad\left.\times\,\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1}\,dy_1\cdots dy_{k-1}\,dy_k\right), \qquad t>0\,,s\in\mathbb{R}\,.\end{aligned}$$

Now let

$$\begin{aligned}f(t,s) &= \sum_{k=1}^{\infty}\int_0^t\int_0^{y_k}\cdots\int_0^{y_2}\prod_{i=1}^{k}\left[M_X\left(se^{-\delta y_i}\right)-1\right]\underline{\alpha}'\,e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1} \\ &\quad\times\underline{\alpha}'\,e^{\mathbf{B}(y_k-y_{k-1})}(-\mathbf{A})\underline{1}\cdots\underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1} \\ &\quad\times dy_1\cdots dy_{k-1}\,dy_k\,, \qquad t>0\,,s\in\mathbb{R}\,, \end{aligned} \tag{4.17}$$

therefore we can rewrite (4.17) as

$$\frac{\partial}{\partial t}M_{Z(t)}(s)=\left[M_X(se^{-\delta t})-1\right]\underline{\alpha}'\,e^{\mathbf{B}t}(-\mathbf{A})\underline{1}+\left[M_X(se^{-\delta t})-1\right]f(t,s)\,. \tag{4.18}$$

This differential equation for $M_{Z(t)}(s)$ can be solved for certain PH distribution, as we can see in the following section.

4.2 Erlang(n) inter-arrival times

Consider a special case of the PH family of distributions, when inter-arrival times are Erlang(n) (see example 2.1.3) then we obtain an homogeneous differential equation for the moment generating function of $Z(t)$.

Differentiating both sides of (4.17) with respect to t yields

$$
\begin{aligned}
&\frac{\partial}{\partial t} f(t,s) \\
= \ & \underline{\alpha}'\mathbf{B}e^{\mathbf{B}t}\Big\{ \sum_{k=1}^{\infty} \int_0^t \int_0^{y_k} \cdots \int_0^{y_2} \prod_{i=1}^{k} \Big[M_X\big(se^{-\delta y_i}\big) - 1\Big] e^{-\mathbf{B}y_k}(-\mathbf{A})\underline{1} \\
& \times \underline{\alpha}'\, e^{\mathbf{B}(y_k - y_{k-1})}(-\mathbf{A})\underline{1} \cdots \underline{\alpha}'\, e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\, e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1} \\
& dy_1 \cdots dy_{k-1}\, dy_k \Big\} + \underline{\alpha}'\, e^{\mathbf{B}t}\Big\{ \big[M_X(se^{-\delta t}) - 1\big] e^{-\mathbf{B}t}\,(-\mathbf{A})\,\underline{\mathbf{1}}\,\underline{\alpha}'\, e^{\mathbf{B}t}(-\mathbf{A})\underline{1} \\
& + \big[M_X(se^{\delta t}) - 1\big] \sum_{k=2}^{\infty} \int_0^t \int_0^{y_{k-1}} \cdots \int_0^{y_2} \prod_{i=1}^{k-1} \Big[M_X\big(se^{-\delta y_i}\big) - 1\Big] \\
& \times e^{-\mathbf{B}t}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\, e^{\mathbf{B}(t-y_{k-1})}(-\mathbf{A})\underline{1} \cdots \underline{\alpha}'\, e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\, e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1} \\
& \times dy_1 \cdots dy_{k-2}\, dy_{k-1} \Big\}, \qquad t > 0\,, s \in \mathbb{R}\,.
\end{aligned}
$$

Since here matrix $\mathbf{A}$ simplifies to the form (2.4) given in Example 2.1.3, we get that in the Erlang(n) case the sum of the first row of $\mathbf{A}$ equal to 0. Hence $\underline{\alpha}'\, e^{-\mathbf{B}t}e^{\mathbf{B}t}\,(-\mathbf{A})\underline{1} = \underline{\alpha}'\,(-\mathbf{A})\underline{1} = 0$. Then

$$
\begin{aligned}
\frac{\partial}{\partial t} f(t,s) \ = \ & \sum_{k=1}^{\infty} \int_0^t \int_0^{y_k} \cdots \int_0^{y_2} \prod_{i=1}^{k} \Big[M_X\big(se^{-\delta y_i}\big) - 1\Big] \underline{\alpha}'\,\mathbf{B}e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1} \\
& \times\ \underline{\alpha}'\, e^{\mathbf{B}(y_k-y_{k-1})}(-\mathbf{A})\underline{1} \cdots \underline{\alpha}'\, e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1} \\
& \times \underline{\alpha}'\, e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1}\, dy_1 \cdots dy_{k-1}\, dy_k\,, \qquad t > 0\,, s \in \mathbb{R}\,. \qquad (4.19)
\end{aligned}
$$

4.2.1 The Erlang(2) Case

To simplify the first derivations, consider the special case of Erlang(2) inter–arrival times, that means here $(\underline{\alpha}\,, \mathbf{A})$ are given by:

$$
\underline{\alpha} = \begin{pmatrix} 1 & 0 \end{pmatrix}', \qquad \mathbf{A} = \begin{pmatrix} -\lambda & \lambda \\ 0 & -\lambda \end{pmatrix}, \qquad (4.20)
$$

where $\lambda > 0$. Then we have the following result.

Theorem 4.2.1. *If the inter–arrival times are Erlang(2), then the m.g.f. of $Z(t)$ satisfies:*

$$
\frac{\partial^2}{\partial t^2} M_{Z(t)}(s) = a_1(t) \frac{\partial}{\partial t} M_{Z(t)}(s) + a_0(t) M_{Z(t)}(s)\,, \qquad t \geq 0\,,\ s \in \mathbb{R}\,, \qquad (4.21)
$$

with initial values $M_{Z(0)}(s) = 1$ *and* $\frac{\partial}{\partial t} M_{Z(t)}(s)|_{t=0} = 0$, *where* $a_1(t) = \frac{\partial}{\partial t}\big[M_X(se^{-\delta t}) - 1\big]/\big[M_X(se^{-\delta t}) - 1\big] - 2\lambda$, $a_0(t) = \lambda^2\big[M_X(se^{-\delta t}) - 1\big]$ *and* M_X *is the m.g.f. of the deflated claim severity* X.

Before we prove Theorem 4.2.1, we need to show the following lemma.

Lemma 4.2.1. *If inter–arrival times are Erlang(2), then*

$$\frac{\partial}{\partial t} f(t,s) = -2\lambda f(t,s) + \lambda^2 M_{Z(t)}(s) - \lambda^2\,, \qquad t > 0\,,\ s \in \mathbb{R}\,. \tag{4.22}$$

Proof. Since the inter–arrival times are Erlang(2), then by (4.20), $\mathbf{B} = \mathbf{A}(\mathbf{I} - \underline{1}\underline{\alpha}')$, where using Maple we obtain the following results:

$$\underline{\alpha}'\,\mathbf{B}e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1} = \lambda^2 e^{-2\lambda(t-y_k)}\,, \qquad t \geq 0, \tag{4.23}$$

and

$$\underline{\alpha}'\,e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1} = -\frac{1}{2}\lambda e^{-2\lambda(t-y_k)} + \frac{1}{2}\lambda\,, \qquad t \geq 0\,,\ s \in \mathbb{R}\,. \tag{4.24}$$

combining (4.23) and (4.24) we have:

$$\underline{\alpha}'\,\mathbf{B}e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1} = -2\lambda\underline{\alpha}'\,e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1} + \lambda^2\,. \tag{4.25}$$

Substituting (4.25) into (4.19) yields

$$\begin{aligned} \frac{\partial}{\partial t} f(t,s) \;=\;& -2\lambda \sum_{k=1}^{\infty} \int_0^t \int_0^{y_k} \cdots \int_0^{y_2} \prod_{i=1}^{k} \Big[M_X\big(se^{-\delta y_i}\big) - 1\Big]\underline{\alpha}'\,e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1} \\ & \times \underline{\alpha}'\,e^{\mathbf{B}(y_k - y_{k-1})}(-\mathbf{A})\underline{1} \cdots \underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1} \\ & \times dy_1 \cdots dy_{k-1}\,dy_k + \lambda^2 \sum_{k=1}^{\infty} \int_0^t \int_0^{y_k} \cdots \int_0^{y_2} \prod_{i=1}^{k} \Big[M_X\big(se^{-\delta y_i}\big) - 1\Big] \\ & \times \underline{\alpha}'\,e^{\mathbf{B}(y_k - y_{k-1})}(-\mathbf{A})\underline{1} \cdots \underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1} \\ & dy_1 \cdots dy_{k-1}\,dy_k\,. \end{aligned} \tag{4.26}$$

Combining (4.16), (4.17) and (4.26) gives (4.22) □

Returning to the proof of Theorem 4.2.1, from expression (4.18) we have

$$\begin{aligned} \frac{\partial^2}{\partial t^2} M_{Z(t)}(s) \;=\;& \frac{\partial}{\partial t}\big[M_X(se^{-\delta t}) - 1\big]\underline{\alpha}'\,e^{\mathbf{B}t}(-\mathbf{A})\underline{1} + \big[M_X(se^{-\delta t}) - 1\big] \\ & \times \underline{\alpha}'\,\mathbf{B}\,e^{\mathbf{B}t}(-\mathbf{A})\underline{1} + \frac{\partial}{\partial t}\big[M_X(se^{-\delta t}) - 1\big] f(t,s) \\ & + \big[M_X(se^{-\delta t}) - 1\big]\frac{\partial}{\partial t} f(t,s)\,. \end{aligned} \tag{4.27}$$

Substituting (4.22) into (4.27) and combining (4.16) and (4.17) yields Theorem 4.2.1.

Remark 4.1. If $\delta = 0$, then the homogeneous differential equation in (4.21) is given by:

$$\frac{\partial^2}{\partial t^2} M_{Z(t)}(s) = a_1 \frac{\partial}{\partial t} M_{Z(t)}(s) + a_0 M_{Z(t)}(s) , \qquad t > 0 , \; s \in \mathbb{R} , \tag{4.28}$$

with coefficients $a_1 = -2\lambda , a_0 = \lambda^2 \left[M_X(s) - 1 \right]$ that are constant with respect to t. Solving the differential equation yields the m.g.f. of the Sparre Andersen sum with Erlang(2) inter–arrival times $S(t) = \sum_{i=1}^{N(t)} X_i$:

$$M_{Z(t)}(s) = e^{-\lambda t} \left[M_X^{-\frac{1}{2}}(s) \sinh \left(\lambda t \, M_X^{\frac{1}{2}}(s) \right) + \cosh \left(\lambda t \, M_X^{\frac{1}{2}}(s) \right) \right] .$$

This result is consistent with an example given earlier in Léveillé (2002) that he obtained using techniques other than differential equations.

Example 4.2.1. Let the inter–arrival times be Erlang(2) that is for $\underline{\alpha} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} -\lambda & \lambda \\ 0 & -\lambda \end{pmatrix}$. Furthermore, let the deflated claim X have an exponential(θ) distribution and $\delta = 0.01 , \lambda = 0.01 , \theta = 1$. Then from Theorem 4.2.1, we have an homogeneous differential equation

$$\frac{\partial^2}{\partial t^2} M_{Z(t)}(s) = a_1(t) \frac{\partial}{\partial t} M_{Z(t)}(s) + a_0(t) M_{Z(t)}(s) , \tag{4.29}$$

where

$$\begin{aligned} a_1(t) &= \frac{\frac{\partial}{\partial t}\left[M(t,s)\right]}{M(t,s)} - 2\lambda = \frac{0.01(2se^{-0.01t} - 3)}{1 - se^{-0.01t}} , \\ a_0(t) &= \lambda^2 M(t,s) = \frac{0.0001se^{-0.01t}}{1 - se^{-0.01t}} , \quad M(t,s) = \frac{\theta}{\theta - se^{-\delta t}} - 1 . \end{aligned}$$

Solving the differential equation with Maple yields

$$\begin{aligned} & M_{Z(t)}(s) \\ = \; & \frac{1}{s^2} \left\{ (s-1) \left[se^{-0.01t} - 2 \right] \ln \left[\frac{1-s}{1 - se^{-0.01t}} \right] + se^{-0.01t}(s-2) + 2s \right\} , \qquad s < 1 . \end{aligned} \tag{4.30}$$

Remark 4.2. From (4.30) the asymptotic behavior of $M_{Z(t)}(s)$ as $t \to \infty$ is given as:

$$M_{Z(\infty)}(s) = \frac{2}{s} + \frac{2(1-s)\ln(1-s)}{s^2}, \qquad s < 1,$$

which is consistent with Léveillé (2002) who obtained this by solving a second hypergeometric differential equation in $M_{Z(\infty)}(s)$.

Remark 4.3. Taking first and second derivatives of $M_{Z(t)}(s)$ with respect to s in (4.30) and letting $s \to 0$, we get the first and second moment of $Z(t)$:

$$\begin{aligned}
\mathbb{E}\big[Z(t)\big] &= \frac{1}{6}\Big(e^{-0.03t} - 3e^{-0.01t} + 2\Big), \qquad t \geq 0, \\
\mathbb{E}\big[Z(t)^2\big] &= \frac{1}{3}\Big(e^{-0.04t} - e^{-0.03t} - e^{-0.01t} + 1\Big), \qquad t \geq 0,
\end{aligned}$$

which is consistent with an example Léveillé and Garrido (2001a).

4.2.2 The Erlang(3) Case

Now consider the moment generating function of Z(t) in the Erlang(3) case. The function f and its derivative with respect to t, $\frac{\partial}{\partial t}f(t,s)$, are defined in (4.17) and (4.19) for any PH distributed inter–arrival times. Now differentiating $\frac{\partial}{\partial t}f(t,s)$ again with respect to t yields:

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}f(t,s) &= \underline{\alpha}'\mathbf{B}^2 e^{\mathbf{B}t}\Bigg\{\sum_{k=1}^{\infty}\int_0^t\int_0^{y_k}\cdots\int_0^{y_2}\prod_{i=1}^{k}\Big[M_X\big(se^{-\delta y_i}\big)-1\Big]e^{-\mathbf{B}y_k}(-\mathbf{A})\underline{1} \\
&\quad \times e^{\mathbf{B}(y_k-y_{k-1})}(-\mathbf{A})\underline{1}\cdots\underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1}\,dy_1\cdots \\
&\quad dy_{k-1}\,dy_k\Bigg\} + \underline{\alpha}'\,\mathbf{B}\,e^{\mathbf{B}t}\Bigg\{\big[M_X(se^{-\delta t})-1\big]e^{-\mathbf{B}t}\,(-\mathbf{A})\,\underline{\mathbf{1}}\,\underline{\alpha}'\,e^{\mathbf{B}t}(-\mathbf{A})\underline{1} \\
&\quad +\Big[M_X\big(se^{-\delta t}\big)-1\Big]\sum_{k=2}^{\infty}\int_0^t\int_0^{y_{k-1}}\cdots\int_0^{y_2}\prod_{i=1}^{k-1}\Big[M_X\big(se^{-\delta y_i}\big)-1\Big] \\
&\quad \times e^{-\mathbf{B}t}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}(t-y_{k-1})}(-\mathbf{A})\underline{1}\cdots\underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\,\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1} \\
&\quad \times dy_1\cdots dy_{k-2}\,dy_{k-1}\Bigg\}.
\end{aligned}$$

As for (4.19) again here $\underline{\alpha}'\,\mathbf{B}(-\mathbf{A})\underline{1}=0$, then

$$\begin{aligned}\frac{\partial^2}{\partial t^2}f(t,s) &= \sum_{k=1}^{\infty}\int_0^t\int_0^{y_k}\cdots\int_0^{y_2}\prod_{i=1}^{k}\Big[M_X\big(se^{-\delta y_i}\big)-1\Big]\underline{\alpha}'\,\mathbf{B}^2e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1}\\ &\quad\times\underline{\alpha}'\,e^{\mathbf{B}(y_k-y_{k-1})}(-\mathbf{A})\underline{1}\cdots\underline{\alpha}'\,e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1}\\ &\quad\times\underline{\alpha}'\,e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1}\,dy_1\cdots dy_{k-1}\,dy_k\,. \qquad (4.31)\end{aligned}$$

Similarly to equations (4.23) and (4.24), with Maple we have:

$$\begin{aligned}\underline{\alpha}'\,\mathbf{B}^2e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1} &= \lambda^3e^{-\frac{3\lambda(t-y_k)}{2}}\Big[\cos\big(\frac{\lambda(t-y_k)\sqrt{3}}{2}\big)-\sqrt{3}\sin\big(\frac{\lambda(t-y_k)\sqrt{3}}{2}\big)\Big],\\ \underline{\alpha}'\,\mathbf{B}\,e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1} &= \frac{2\sqrt{3}}{3}\lambda^2e^{-\frac{3\lambda(t-y_k)}{2}}\sin\big(\frac{\lambda(t-y_k)\sqrt{3}}{2}\big),\\ \underline{\alpha}'\,e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1} &= \frac{\lambda}{3}e^{-\frac{3\lambda(t-y_k)}{2}}\Big[1-\cos\big(\frac{\lambda(t-y_k)\sqrt{3}}{2}\big)-\\ &\quad\sqrt{3}\sin\big(\frac{\lambda(t-y_k)\sqrt{3}}{2}\big)\Big],\end{aligned}$$

then the following result holds.

$$\underline{\alpha}'\,\mathbf{B}^2e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1}=-3\lambda\,\underline{\alpha}'\,\mathbf{B}\,e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1}-3\lambda^2\underline{\alpha}'\,e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1}+\lambda^3\,. \qquad (4.32)$$

Substituting (4.32) into (4.31) and combining (4.19) and (4.17) shows the following result.

Lemma 4.2.2. *If the inter-arrival times are Erlang(3), then*

$$\frac{\partial^2}{\partial t^2}f(t,s)=-3\lambda\,\frac{\partial}{\partial t}f(t,s)-3\lambda^2\,f(t,s)+\lambda^3\,M_{Z(t)}(s)-\lambda^3\,,\qquad t\geq 0\,,\ s\in\mathbb{R}\,. \qquad (4.33)$$

A method similar to that used in the previous section and differentiating (4.18) respect to t yields

$$\begin{aligned}&\frac{\partial^3}{\partial t^3}M_{Z(t)}(s)\\ =\;&\frac{\partial^2}{\partial t^2}\big[M_X(se^{-\delta t})-1\big]\underline{\alpha}'\,e^{\mathbf{B}t}(-\mathbf{A})\underline{1}+2\frac{\partial}{\partial t}\big[M_X(se^{-\delta t})-1\big]\underline{\alpha}'\,\mathbf{B}\;e^{\mathbf{B}t}(-\mathbf{A})\underline{1}\\ &+\big[M_X(se^{-\delta t})-1\big]\,\underline{\alpha}'\,\mathbf{B}^2\,e^{\mathbf{B}t}(-\mathbf{A})\underline{1}+\frac{\partial^2}{\partial t^2}\big[M_X(se^{-\delta t})-1\big]f(t,s)\\ &+2\frac{\partial}{\partial t}\big[M_X(se^{-\delta t})-1\big]\frac{\partial}{\partial t}f(t,s)+\big[M_X(se^{-\delta t})-1\big]\frac{\partial^2}{\partial t^2}f(t,s)\,. \qquad (4.34)\end{aligned}$$

Substituting (4.32) into (4.34) and combining (4.19) and (4.17) yields the following result.

Theorem 4.2.2. *If the inter-arrival times are Erlang(3), then:*

$$\frac{\partial^3}{\partial t^3}M_{Z(t)}(s) = a_2(t)\frac{\partial^2}{\partial t^2}M_{Z(t)}(s)+a_1(t)\frac{\partial}{\partial t}M_{Z(t)}(s)+a_0(t)M_{Z(t)}(s)\,, \qquad t\geq 0\,,\ s\in\mathbb{R}\,, \tag{4.35}$$

with initial values $M_{Z(0)}(s) = 1\,, \frac{\partial}{\partial t}M_{Z(t)}(s)|_{t=0} = 0\,, \frac{\partial^2}{\partial t^2}M_{Z(t)}(s)|_{t=0} = 0$, *and where*

$$\begin{aligned}
a_2(t) &= \frac{2\frac{\partial}{\partial t}\left[M_X(se^{-\delta t})-1\right]}{\left[M_X(se^{-\delta t})-1\right]} - 3\lambda\,,\\
a_1(t) &= \frac{\frac{\partial^2}{\partial t^2}\left[M_X(se^{-\delta t})-1\right] - 3\lambda^2\left[M_X(se^{-\delta t})-1\right] - a_2(t)\,\frac{\partial}{\partial t}\left[M_X(se^{-\delta t})-1\right]}{\left[M_X(se^{-\delta t})-1\right]}\,,\\
a_0(t) &= \lambda^3\left[M_X(se^{-\delta t})-1\right].
\end{aligned}$$

Remark 4.4. If $\delta = 0$, then we obtain an homogeneous differential equation:

$$\frac{\partial^3}{\partial t^3}M_{Z(t)}(s) = a_2\frac{\partial^2}{\partial t^2}M_{Z(t)}(s) + a_1\frac{\partial}{\partial t}M_{Z(t)}(s) + a_0M_{Z(t)}(s)\,, \tag{4.36}$$

with coefficients $a_2 = -3\lambda\,, a_1 = -3\lambda^2$ and $a_0 = \lambda^3[M_X(s)-1]$ that are constant with respect to t. Solving this differential equation yields the moment generating function of the Sparre Andersen sum $S(t) = \sum_{i=1}^{N(t)} X_i$ for Erlang(3) inter-arrival times:

$$\begin{aligned}
&M_{Z(t)}(s)\\
&= \frac{1}{3M_X(s)^{\frac{2}{3}}}\Bigg\{\Bigg(\left[2M_X(s)^{\frac{2}{3}} - M_X(s)^{\frac{1}{3}} - 1\right]\cos\left[\frac{\sqrt{3}\lambda t}{2}M_X(s)^{\frac{1}{3}}\right] - \sqrt{3}\left[1-M_X(s)^{\frac{1}{3}}\right]\\
&\qquad\times\sin\left[\frac{\sqrt{3}\lambda t}{2}M_X(s)^{\frac{1}{3}}\right]e^{[-\frac{1}{2}M_X(s)^{\frac{1}{3}}-1]\lambda t}\Bigg)\\
&\qquad+\left[M_X(s)^{\frac{2}{3}} + 1 + M_X(s)^{\frac{1}{3}}\right]e^{[M_X(s)^{\frac{1}{3}}-1]\lambda t}\Bigg\}.
\end{aligned}$$

4.2.3 The Erlang(n) Case

From the results of Erlang(2) and Erlang(3) we can give a conjecture when the inter–arrival times are Erlang(n), there is an n-th order homogeneous differential equation $M_{Z(t)}(s)$. In order to prove our conjecture we need to prove the following Lemmas.

In the special case when a PH $(\underline{\alpha}, \mathbf{A})$ distribution is an Erlang(n) distribution we have

Lemma 4.2.3. *If the inter-arrival times have an Erlang(n) distribution with parameters* $(\underline{\alpha}, \boldsymbol{A})$*, then*

$$\underline{\alpha}'\,\boldsymbol{B}^k\,(-\boldsymbol{A})\underline{1} = 0\,, \qquad \textit{for } k \leq n-2\,. \tag{4.37}$$

where $\boldsymbol{B} = \boldsymbol{A}(\boldsymbol{I} - \underline{1}\,\underline{\alpha}')$.

Proof. First prove the following result

$$\underline{\alpha}'\mathbf{A}^k\underline{1} = 0\,, \qquad \text{for } k \leq n-1\,. \tag{4.38}$$

Let

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} -\lambda & \lambda & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & \lambda & \cdots & 0 & 0 \\ 0 & 0 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda & \lambda \\ 0 & 0 & 0 & \cdots & 0 & -\lambda \end{pmatrix} = \lambda \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \\ &= \lambda\big(-\mathbf{I} + \mathbf{D}\big)\,,\end{aligned}$$

where $\mathbf{I}$ is an identity matrix and

$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ then } \mathbf{D}^k = \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ \underbrace{0 & \cdots & 0}_{k} & 0 & \cdots & 0 \end{pmatrix}. \tag{4.39}$$

Hence

$$\mathbf{A}^k = \lambda^k\big(-\mathbf{I}+\mathbf{D}\big)^k = \lambda^k \sum_{r=0}^{k}(-1)^r \binom{k}{r}\mathbf{D}^{k-r}\,, \qquad \text{for } \mathbf{D}^0 = \mathbf{I}\,. \tag{4.40}$$

From the fact that $\underline{\alpha}'\mathbf{A}^k\underline{1}$ is sum of the first row of $\mathbf{A}^k$ and combining (4.38) and (4.39) we have

$$\underline{\alpha}'\mathbf{A}^k\underline{1} = \lambda^k \sum_{r=0}^{k}(-1)^r\binom{k}{r} = \lambda^k(1-1)^k = 0\,. \tag{4.41}$$

Now we prove the Lemma. From $\mathbf{B} = \mathbf{A}(\mathbf{I} - \underline{1}\,\underline{\alpha}')$, we have

$$\underline{\alpha}'\,\mathbf{B}^k\,(-\mathbf{A})\underline{1} = \underline{\alpha}'\,\mathbf{B}^{k-1}\,\mathbf{A}\big(\mathbf{I}-\underline{1}\,\underline{\alpha}'\big)(-\mathbf{A})\underline{1} = -\underline{\alpha}'\,\mathbf{B}^{k-1}\,\mathbf{A}^2\underline{1} + \underline{\alpha}'\,\mathbf{B}^{k-1}\mathbf{A}\underline{1}\,\underline{\alpha}'\mathbf{A}\underline{1}\,. \tag{4.42}$$

Consider (4.38) then

$$-\underline{\alpha}'\,\mathbf{B}^{k-1}\,\mathbf{A}^2\underline{1} + \underline{\alpha}'\,\mathbf{B}^{k-1}\mathbf{A}\underline{1}\,\underline{\alpha}'\mathbf{A}\underline{1} = -\underline{\alpha}'\,\mathbf{B}^{k-1}\,\mathbf{A}^2\underline{1}\,. \tag{4.43}$$

Recursively applying $\mathbf{B} = \mathbf{A}(\mathbf{I} - \underline{1}\,\underline{\alpha}')$ and (4.38) to (4.43) we have

$$\begin{aligned} -\underline{\alpha}'\,\mathbf{B}^{k-1}\,\mathbf{A}^2\underline{1} &= -\underline{\alpha}'\,\mathbf{B}^{k-2}\,\mathbf{A}\big(\mathbf{I}-\underline{1}\,\underline{\alpha}'\big)\mathbf{A}^2\underline{1} = -\underline{\alpha}'\,\mathbf{B}^{k-2}\,\mathbf{A}^3\underline{1} + \underline{\alpha}'\,\mathbf{B}^{k-2}\mathbf{A}^2\underline{1}\,\underline{\alpha}'\mathbf{A}^2\underline{1} \\ &= -\underline{\alpha}'\,\mathbf{B}^{k-2}\,\mathbf{A}^3\underline{1} \\ &= \cdots \\ &= -\underline{\alpha}'\,\mathbf{A}^{k+1}\underline{1} = 0\,, \end{aligned} \tag{4.44}$$

where $k+1 \leq n-1$. Hence $\underline{\alpha}'\,\mathbf{B}^k\,(-\mathbf{A})\underline{1} = 0$ for $k \leq n-2$. □

In order to prove the Theorem we also need the following Lemmas.

Lemma 4.2.4. *If $(\underline{\alpha}, \boldsymbol{A})$ are the parameters for the Erlang(n) distribution and $\boldsymbol{B} = \boldsymbol{A}(\boldsymbol{I} - \underline{1}\,\underline{\alpha}')$ then*

$$\underline{\alpha}' e^{\boldsymbol{B}x}\boldsymbol{B}^{n-1}\,(-\boldsymbol{A})\underline{1} = \lambda^n - \sum_{k=1}^{n-1}\lambda^k\binom{n}{k}\,\underline{\alpha}' e^{\boldsymbol{B}x}\boldsymbol{B}^{n-1-k}\,(-\boldsymbol{A})\underline{1}\,. \tag{4.45}$$

Proof. By Definition A.1.4. in Appendix A

$$\begin{aligned}
\underline{\alpha}' e^{\mathbf{B}x} \mathbf{B}^{n-1} (-\mathbf{A}) \underline{1} &= \sum_{k=0}^{\infty} \frac{1}{k!} \underline{\alpha}' \mathbf{B}^{k+n-1} (-\mathbf{A}) \underline{1}\, x^k \\
\underline{\alpha}' e^{\mathbf{B}x} \mathbf{B}^{n-2} (-\mathbf{A}) \underline{1} &= \sum_{k=0}^{\infty} \frac{1}{k!} \underline{\alpha}' \mathbf{B}^{k+n-2} (-\mathbf{A}) \underline{1}\, x^k \\
&\cdots \\
\underline{\alpha}' e^{\mathbf{B}x} (-\mathbf{A}) \underline{1} &= \sum_{k=0}^{\infty} \frac{1}{k!} \underline{\alpha}' \mathbf{B}^{k} (-\mathbf{A}) \underline{1}\, x^k .
\end{aligned}$$

The coefficient of x^k , $k \geq 1$ for the right hand side in (4.45) is

$$\begin{aligned}
& -\frac{1}{k!}\binom{n}{1}\lambda\, \underline{\alpha}' \mathbf{B}^{k+n-1} (-\mathbf{A}) \underline{1} - \frac{1}{k!}\binom{n}{2}\lambda^2 \underline{\alpha}' \mathbf{B}^{k+n-2} (-\mathbf{A}) \underline{1} - \cdots \\
& \quad -\frac{1}{k!}\binom{n}{n-1}\lambda^{n-1} \underline{\alpha}' \mathbf{B}^{k} (-\mathbf{A}) \underline{1} \\
= \; & -\frac{1}{k!}\underline{\alpha}' \mathbf{B}^{k-1} \left(\binom{n}{1}\lambda\, \mathbf{B}^{n-1} + \binom{n}{2}\lambda^2\, \mathbf{B}^{2-1} + \cdots + \binom{n}{n-1}\lambda^{n-1}\, \mathbf{B} \right) (-\mathbf{A}) \underline{1} \\
= \; & -\frac{1}{k!}\underline{\alpha}' \mathbf{B}^{k-1} \Big[\big(\lambda \mathbf{I} + \mathbf{B}\big)^n - \lambda^n \mathbf{I} - \mathbf{B}^n \Big] (-\mathbf{A}) \underline{1} \\
= \; & \frac{1}{k!}\underline{\alpha}' \mathbf{B}^{k+n-1} (-\mathbf{A}) \underline{1} + \frac{1}{k!}\lambda^n\, \underline{\alpha}'\, \mathbf{B}^{k-1} (-\mathbf{A}) \underline{1} - \frac{1}{k!}\underline{\alpha}'\, \mathbf{B}^{k-1} (\lambda \mathbf{I} + \mathbf{B})^n (-\mathbf{A}) \underline{1} .
\end{aligned} \tag{4.46}$$

Next we want to show that $\big(\lambda \mathbf{I} + \mathbf{B}\big)^n = \lambda^n\, \mathbf{I}$, let

$$\big(\lambda \mathbf{I} + \mathbf{B}\big)^n = \lambda^n\, \mathbf{C}^n \tag{4.47}$$

where

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}^n .$$

Since

$$\mathbf{C}^2 = \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}, \cdots, \mathbf{C}^{n-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \mathbf{C}^n = \mathbf{I}. \tag{4.48}$$

So (4.46) can be simplified as $\frac{1}{k!}\underline{\alpha}'\mathbf{B}^{k+n-1}(-\mathbf{A})\underline{1}$.

Comparing the x^k term for $k \geq 1$ in the series expansions for both sides of (4.45), they have the same coefficients. Now for $k = 0$, using the same procedure in (4.44) and applying (4.40), the constant term for the left hand side is

$$\underline{\alpha}'\mathbf{B}^{n-1}(-\mathbf{A})\underline{1} = -\underline{\alpha}'\mathbf{A}^n\underline{1} = -\lambda^n\underline{\alpha}'\sum_{r=0}^{n}(-1)^r\binom{n}{r}\mathbf{D}^{n-r}\underline{1}. \tag{4.49}$$

Since (4.49) is the sum of the first row in a matrix $-\lambda^n\sum_{r=0}^{n}(-1)^r\binom{n}{r}\mathbf{D}^{n-r}$ and $\mathbf{D}^n = \mathbf{0}$, then

$$-\lambda^n\underline{\alpha}'\sum_{r=0}^{n}(-1)^r\binom{n}{r}\mathbf{D}^{n-r}\underline{1} = -\lambda^n\big[(1-1)^n - 1\big] = \lambda^n, \tag{4.50}$$

hence the constant term for the left hand side in (4.45) is λ^n. From Lemma 4.2.3. the constant term for the right hand side is also λ^n. Hence we conclude that the coefficients of the right and left hand sides of term x^k for are equal for $k \geq 0$, which leads to Lemma 4.2.4. □

Similar results hold for Lemmas 4.2.1 and 4.2.2.

Lemma 4.2.5. *If the inter–arrival times are Erlang(n), then the $(n-1)$-th derivative of $f(t,s)$ in t is given by*

$$\frac{\partial^{n-1}}{\partial t^{n-1}}f(t,s) = -\sum_{k=1}^{n-1}\lambda^k\binom{n}{k}\frac{\partial^{n-1-k}}{\partial t^{n-1-k}}f(t,s) + \lambda^n M_{Z(t)}(s) - \lambda^n. \tag{4.51}$$

Proof. By Lemma 4.2.3. we have

$$\begin{aligned}\frac{\partial^{n-1}}{\partial t^{n-1}} f(t,s) &= \sum_{k=1}^{\infty} \int_0^t \int_0^{y_k} \cdots \int_0^{y_2} \prod_{i=1}^{k} \left[M_X\left(se^{-\delta y_i}\right) - 1\right] \underline{\alpha}'\, \mathbf{B}^{n-1} e^{\mathbf{B}(t-y_k)}(-\mathbf{A})\underline{1} \\ &\quad \times \underline{\alpha}'\, e^{\mathbf{B}(y_k - y_{k-1})}(-\mathbf{A})\underline{1} \cdots \underline{\alpha}'\, e^{\mathbf{B}(y_2-y_1)}(-\mathbf{A})\underline{1} \\ &\quad \times \underline{\alpha}'\, e^{\mathbf{B}y_1}(-\mathbf{A})\underline{1}\, dy_1 \cdots dy_{k-1}\, dy_k\,. \end{aligned} \tag{4.52}$$

Substituting (4.45) in Lemma 4.2.4. into (4.51) and combining with (4.16) yields the result. □

Now we prove the conjecture that there is an homogeneous differential equation for the m.g.f. of $Z(t)$.

Theorem 4.2.3. *If the inter-arrival times are Erlang(n), then the n-th derivative of the m.g.f. $M_{Z(t)}(s)$ with respect to t is given by:*

$$\begin{aligned}\frac{\partial^n}{\partial t^n} M_{Z(t)}(s) &= a_{n-1}(t) \frac{\partial^{n-1}}{\partial t^{n-1}} M_{Z(t)}(s) + a_{n-2}(t) \frac{\partial^{n-2}}{\partial t^{n-2}} M_{Z(t)}(s) + \cdots \\ &\quad + a_1(t) \frac{\partial}{\partial t} M_{Z(t)}(s) + a_0(t) M_{Z(t)}(s)\,, \end{aligned} \tag{4.53}$$

with initial values

$$M_{Z(0)}(s) = 1\,,\ \frac{\partial}{\partial t} M_{Z(t)}(s)|_{t=0} = 0\,,\ \frac{\partial^2}{\partial t^2} M_{Z(t)}(s)|_{t=0} = 0\,, \cdots,\ \frac{\partial^{n-1}}{\partial t^{n-1}} M_{Z(t)}(s)|_{t=0} = 0\,,$$

where

$$a_k(t) = \frac{\binom{n-1}{n-k} \frac{\partial^{n-k}}{\partial t^{n-k}} M(t,s) - \sum_{i=1}^{(n-2)-(k-1)} a_{k+i}(t) \binom{k+i-1}{i} \frac{\partial^i}{\partial t^i} M(t,s)}{M(t,s)} - \binom{n}{n-k} \lambda^{n-k}\,,$$

with $M(t,s) = M_X(se^{-\delta t}) - 1$, for $k = 1, 2, \cdots, n-1$ and $a_0(t) = \lambda^n M(t,s)$.

Proof. Differentiating (4.18) yields

$$\begin{aligned}\frac{\partial^n}{\partial t^n}M_{Z(t)}(s) &= \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\partial^k}{\partial t^k}M(t,s)\underline{\alpha}'\, e^{\mathbf{B}t}\mathbf{B}^{n-1-k}(-\mathbf{A})\underline{1} \\ &\quad +\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\partial^k}{\partial t^k}M(t,s)\frac{\partial^{n-1-k}}{\partial t^{n-1-k}}f(t,s) \\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\partial^k}{\partial t^k}M(t,s)\underline{\alpha}'\, e^{\mathbf{B}t}\mathbf{B}^{n-1-k}(-\mathbf{A})\underline{1} \\ &\quad +\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{\partial^k}{\partial t^k}M(t,s)\frac{\partial^{n-1-k}}{\partial t^{n-1-k}}f(t,s) \\ &\quad +M(t,s)\frac{\partial^{n-1}}{\partial t^{n-1}}f(t,s)\,, \end{aligned} \tag{4.54}$$

where $M(t,s) = M_X(se^{-\delta t}) - 1$.

Substituting (4.51) into (4.54) leads to:

$$\begin{aligned}\frac{\partial^n}{\partial t^n}M_{Z(t)}(s) &= \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\partial^k}{\partial t^k}M(t,s)\underline{\alpha}'\, e^{\mathbf{B}t}\mathbf{B}^{n-1-k}(-\mathbf{A})\underline{1} \\ &\quad +\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{\partial^k}{\partial t^k}M(t,s)\frac{\partial^{n-1-k}}{\partial t^{n-1-k}}f(t,s) \\ &\quad +M(t,s)\Big[-\sum_{k=1}^{n-1}\lambda^k\binom{n}{k}\frac{\partial^{n-1-k}}{\partial t^{n-1-k}}f(t,s)+\lambda^n M_{Z(t)}(s)-\lambda^n\Big] \\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\partial^k}{\partial t^k}M(t,s)\underline{\alpha}'\, e^{\mathbf{B}t}\mathbf{B}^{n-1-k}(-\mathbf{A})\underline{1} \\ &\quad +\sum_{k=2}^{n-1}\Big[\binom{n-1}{k}\frac{\partial^k}{\partial t^k}M(t,s)-\binom{n}{k}\lambda^k M(t,s)\Big]\frac{\partial^{n-1-k}}{\partial t^{n-1-k}}f(t,s) \\ &\quad +\Big[\binom{n-1}{1}\frac{\partial}{\partial t}M(t,s)-\binom{n}{1}\lambda M(t,s)\Big]\frac{\partial^{n-2}}{\partial t^{n-2}}f(t,s) \\ &\quad +M(t,s)\Big[\lambda^n M_{Zt(s)]}-\lambda^n\Big] \end{aligned} \tag{4.55}$$

Since

$$\frac{\partial^{n-1}}{\partial t^{n-1}} M_{Z(t)}(s) = \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\partial^k}{\partial t^k} M(t,s) \underline{\alpha}' e^{\mathbf{B}t} \mathbf{B}^{n-2-k} (-\mathbf{A}) \underline{1} + \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\partial^k}{\partial t^k} M(t,s) \frac{\partial^{n-2-k}}{\partial t^{n-2-k}} f(t,s). \quad (4.56)$$

Substituting (4.56) into (4.55) we have:

$$\begin{aligned} &\frac{\partial^n}{\partial t^n} M_{Z(t)}(s) \\ = \ & a_{n-1}(t) \frac{\partial^{n-1}}{\partial t^{n-1}} M_{Z(t)}(s) + \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\partial^k}{\partial t^k} M(t,s) \underline{\alpha}' e^{\mathbf{B}t} \mathbf{B}^{n-1-k} (-\mathbf{A}) \underline{1} \\ & + \sum_{k=2}^{n-1} \left[\binom{n-1}{k} \frac{\partial^k}{\partial t^k} M(t,s) - \binom{n}{k} \lambda^k M(t,s) - a_{n-1}(t) \binom{n-2}{k-1} \frac{\partial^{k-1}}{\partial t^{k-1}} \right. \\ & \left. \times M(t,s) \right] \frac{\partial^{n-1-k}}{\partial t^{n-1-k}} f(t,s) - a_{n-1}(t) \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\partial^k}{\partial t^k} M(t,s) \\ & \times \underline{\alpha}' e^{\mathbf{B}t} \mathbf{B}^{n-2-k} (-\mathbf{A}) \underline{1} + M(t,s) \Big[\lambda^n M_{Zt(s)]} - \lambda^n \Big], \end{aligned} \quad (4.57)$$

where $a_{n-1}(t) = \frac{\binom{n-1}{1} \frac{\partial}{\partial t} M(t,s) - \binom{n}{1} \lambda M(t,s)}{M(t,s)}$.

Substituting (4.45) into (4.57) yields:

$$\begin{aligned} &\frac{\partial^n}{\partial t^n} M_{Z(t)}(s) \\ = \ & a_{n-1}(t) \frac{\partial^{n-1}}{\partial t^{n-1}} M_{Z(t)}(s) + \sum_{k=1}^{n-1} \binom{n-1}{k} \frac{\partial^k}{\partial t^k} M(t,s) \underline{\alpha}' e^{\mathbf{B}t} \mathbf{B}^{n-1-k} (-\mathbf{A}) \underline{1} + M(t,s) \\ & \times \left[\lambda^n - \sum_{k=1}^{n-1} \lambda^k \binom{n}{k} \underline{\alpha}' e^{\mathbf{B}x} \mathbf{B}^{n-1-k} (-\mathbf{A}) \underline{1} \right] + \sum_{k=2}^{n-1} \left[\binom{n-1}{k} \frac{\partial^k}{\partial t^k} M(t,s) \right. \\ & \left. - \binom{n}{k} \lambda^k M(t,s) - a_{n-1}(t) \binom{n-2}{k-1} \frac{\partial^{k-1}}{\partial t^{k-1}} M(t,s) \right] \frac{\partial^{n-1-k}}{\partial t^{n-1-k}} f(t,s) \\ & - a_{n-1}(t) \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\partial^k}{\partial t^k} M(t,s) \underline{\alpha}' e^{\mathbf{B}t} \mathbf{B}^{n-2-k} (-\mathbf{A}) \underline{1} \\ & + M(t,s) \Big[\lambda^n M_{Z(t)}(s) - \lambda^n \Big]. \end{aligned} \quad (4.58)$$

Simplifying (4.58) gives:

$$\begin{aligned}
&\frac{\partial^n}{\partial t^n} M_{Z(t)}(s) \\
= \;& a_{n-1}(t) \frac{\partial^{n-1}}{\partial t^{n-1}} M_{Z(t)}(s) + \sum_{k=2}^{n-1} \left[\binom{n-1}{k} \frac{\partial^k}{\partial t^k} M(t,s) - \binom{n}{k} \lambda^k M(t,s) \right. \\
& \left. - a_{n-1}(t) \binom{n-2}{k-1} \frac{\partial^{k-1}}{\partial t^{k-1}} M(t,s) \right] \frac{\partial^{n-1-k}}{\partial t^{n-1-k}} f(t,s) + \sum_{k=2}^{n-1} \left[\binom{n-1}{k} \frac{\partial^k}{\partial t^k} M(t,s) \right. \\
& \left. - \binom{n}{k} \lambda^k M(t,s) - a_{n-1}(t) \binom{n-2}{k-1} \frac{\partial^{k-1}}{\partial^{k-1}} M(t,s) \right] \underline{\alpha}' \, e^{\mathbf{B}t} \mathbf{B}^{n-1-k} (-\mathbf{A}) \underline{1} \\
& + \lambda^n M_{Z(t)}(s) M(t,s) \,. \qquad (4.59)
\end{aligned}$$

Now rewrite (4.59) as:

$$\begin{aligned}
&\frac{\partial^n}{\partial t^n} M_{Z(t)}(s) \\
= \;& a_{n-1}(t) \frac{\partial^{n-1}}{\partial t^{n-1}} M_{Z(t)}(s) + \sum_{k=3}^{n-1} \left[\binom{n-1}{k} \frac{\partial^k}{\partial t^k} M(t,s) - \binom{n}{k} \lambda^k M(t,s) \right. \\
& \left. - a_{n-1}(t) \binom{n-2}{k-1} \frac{\partial^{k-1}}{\partial t^{k-1}} M(t,s) \right] \frac{\partial^{n-1-k}}{\partial t^{n-1-k}} f(t,s) + \left[\binom{n-1}{2} \frac{\partial^2}{\partial t^2} M(t,s) \right. \\
& \left. - \binom{n}{2} \lambda^2 M(t,s) - a_{n-1}(t) \binom{n-2}{1} \frac{\partial^2}{\partial t^2} M(t,s) \right] \frac{\partial^3}{\partial t^3} f(t,s) \\
& + \sum_{k=2}^{n-1} \left[\binom{n-1}{k} \frac{\partial^k}{\partial t^k} M(t,s) - \binom{n}{k} \lambda^k M(t,s) - a_{n-1}(t) \binom{n-2}{k-1} \frac{\partial^{k-1}}{\partial^{k-1}} \right. \\
& \left. \times M(t,s) \right] \underline{\alpha}' \, e^{\mathbf{B}t} \mathbf{B}^{n-1-k} (-\mathbf{A}) \underline{1} + \lambda^n M_{Z(t)}(s) M(t,s) \,. \qquad (4.60)
\end{aligned}$$

Finally

$$\begin{aligned}
\frac{\partial^{n-2}}{\partial t^{n-1}} M_{Z(t)}(s) \;=\; & \sum_{k=0}^{n-3} \binom{n-3}{k} \frac{\partial^k}{\partial t^k} M(t,s) \underline{\alpha}' \, e^{\mathbf{B}t} \mathbf{B}^{n-3-k} (-\mathbf{A}) \underline{1} \\
& + \sum_{k=0}^{n-3} \binom{n-3}{k} \frac{\partial^k}{\partial t^k} M(t,s) \frac{\partial^{n-3-k}}{\partial t^{n-2-k}} f(t,s) \,. \qquad (4.61)
\end{aligned}$$

Substituting (4.61) into (4.60) and simplifying yields:

$$\begin{aligned}
&\frac{\partial^n}{\partial t^n} M_{Z(t)}(s) \\
= \; & a_{n-1}(t)\frac{\partial^{n-1}}{\partial t^{n-1}} M_{Z(t)}(s) + a_{n-2}(t)\frac{\partial^{n-2}}{\partial t^{n-2}} M_{Z(t)}(s) + \lambda^n M_{Z(t)}(s) M(t,s) \\
& + \sum_{k=3}^{n-1} \left[\binom{n-1}{k} \frac{\partial^k}{\partial t^k} M(t,s) - \binom{n}{k} \lambda^k M(t,s) - a_{n-1}(t) \binom{n-2}{k-1} \right. \\
& \left. \times \frac{\partial^{k-1}}{\partial t^{k-1}} M(t,s) - a_{n-2}(t) \binom{n-3}{k-2} \frac{\partial^{k-2}}{\partial t^{k-2}} M(t,s) \right] \frac{\partial^{n-1-k}}{\partial t^{n-1-k}} f(t,s) \\
& + \sum_{k=3}^{n-1} \left[\binom{n-1}{k} \frac{\partial^k}{\partial t^k} M(t,s) - a_{n-1}(t) \binom{n-2}{k-1} \frac{\partial^{k-1}}{\partial t^{k-1}} M(t,s) \right. \\
& \left. - a_{n-2}(t) \binom{n-3}{k-2} \frac{\partial^{k-2}}{\partial t^{k-2}} M(t,s) \right] \underline{\alpha}' e^{\mathbf{B}t} \mathbf{B}^{n-1-k} (-\mathbf{A}) \underline{1}, \qquad (4.62)
\end{aligned}$$

where $a_{n-2}(t) = \frac{\binom{n-1}{2} \frac{\partial^2}{\partial t^2} M(t,s) - \binom{n}{2} \lambda^n M(t,s) - a_{n-1}(t) \binom{n-2}{1} \frac{\partial}{\partial t}}{M(t,s)}$.

Recursively applying $\frac{\partial^{n-3}}{\partial t^{n-3}} M_{Z(t)}(s), \cdots, \frac{\partial^2}{\partial t^2} M_{Z(t)}(s)$ into (4.62) and repeating the procedures as (4.60), (4.61) and (4.62), then we have the following result:

$$\begin{aligned}
&\frac{\partial^n}{\partial t^n} M_{Z(t)}(s) \\
= \; & a_{n-1}(t)\frac{\partial^{n-1}}{\partial t^{n-1}} M_{Z(t)}(s) + \cdots + a_2(t)\frac{\partial^2}{\partial t^2} M_{Z(t)}(s) + \lambda^n M_{Z(t)}(s) M(t,s) \\
& + \sum_{k=n-1}^{n-1} \left[\binom{n-1}{k} \frac{\partial^k}{\partial t^k} M(t,s) - \binom{n}{k} \lambda^k M(t,s) - a_{n-1}(t) \binom{n-2}{k-1} \right. \\
& \left. \times \frac{\partial^{k-1}}{\partial t^{k-1}} M(t,s) - \cdots - a_2(t) \binom{1}{n-k} \frac{\partial}{\partial t} M(t,s) \right] \frac{\partial^{n-1-k}}{\partial t^{n-1-k}} f(t,s) \\
& + \sum_{k=n-1}^{n-1} \left[\binom{n-1}{k} \frac{\partial^k}{\partial t^k} M(t,s) - \binom{n}{k} \lambda^k M(t,s) - a_{n-1}(t) \binom{n-2}{k-1} - \cdots \right. \\
& \left. \times \frac{\partial^{k-1}}{\partial t^{k-1}} M(t,s) - a_2(t) \binom{1}{n-k} \frac{\partial}{\partial t} M(t,s) \right] \underline{\alpha}' e^{\mathbf{B}t} \mathbf{B}^{n-1-k} (-\mathbf{A}) \underline{1}, \qquad (4.63)
\end{aligned}$$

where $a_j(t) = \frac{\binom{n-1}{j} \frac{\partial^{n-j}}{\partial t^{n-j}} M(t,s) - \binom{n}{j} \lambda^n M(t,s) - a_{n-1}(t) \binom{n-2}{n-j-1} \frac{\partial^{n-j-1}}{\partial t^{n-j-1}} M(t,s) - \cdots - a_2(t) \binom{j}{1} \frac{\partial}{\partial t} M(t,s)}{M(t,s)}$, for $j \leq 2$. Substituting (4.18) into (4.63) and writing in a compact form for the coefficients gives (4.53). □

Chapter 5

Applications

In this chapter the actuarial applications of the results obtained in Chapters 3 and 4 are discussed. The risk theory literature is rich in comparisons between compound Poisson sum models for different choices of the claim severity distribution (heavy tailed, etc..). Here we use the results in Chapter 4 to compare discounted compound Poisson and renewal sums for the same claim severity distribution.

In the first section we compare the discounted compound Poisson to the renewal processes. By comparing the expectations and variances of discounted compound sums $Z(t)$ under a fixed net interest δ we highlight the differences between the Poisson and renewal assumptions. By contrast, the second section considers the impact of varying δ values on the compound Poisson and renewal sum processes.

5.1 Comparison of Poisson and Renewal Processes

In this section we assume that the two processes have the same expectation of inter–arrival times and the same deflated claim size distribution. We illustrate the behavior of the expectation and variance of the discounted compound sums under a fixed interest with the following numerical example.

Example 5.1.1. In the Poisson case, let the inter–arrival times be exponentially distributed with mean $1/0.005$ and the deflated claims be also exponentially distributed with mean 1. For the renewal process, the inter–arrival times are Erlang(2) with parameters $\underline{\alpha} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} -\lambda & \lambda \\ 0 & -\lambda \end{pmatrix}$, where $\lambda = 0.01$. Hence they have the same expectation as the Poisson process inter–arrival times. In addition, as above, deflated claims are exponentially distributed with mean 1. By the results of Chapters 3 and 4 we can compute the expectation and variance of $Z(t)$ under a fixed interest, say $\delta = 0.01$. Let $\mathbb{E}_P\big[Z(t)\big]$ and $\mathbb{E}_R\big[Z(t)\big]$ denote the expectation of the Poisson and renewal models, respectively, while $\mathbb{V}_P\big[Z(t)\big]$ and $\mathbb{V}_R\big[Z(t)\big]$ are the variances. Then from (3.13) and Remark 4.3 we have:

$$\begin{aligned} \mathbb{E}_P\big[Z(t)\big] &= 0.5 - 0.5e^{-0.01t}, \qquad t \geq 0, \\ \mathbb{E}_R\big[Z(t)\big] &= \frac{1}{6}\left(e^{-0.03t} - 3e^{-0.01t} + 2\right), \qquad t \geq 0. \end{aligned}$$

Similarly, from (3.14) and Remark 4.3 it follows that

$$\begin{aligned} \mathbb{V}_P\big[Z(t)\big] &= 0.5 - 0.5e^{-0.02t}, \qquad t \geq 0, \\ \mathbb{V}_R\big[Z(t)\big] &= \frac{2}{9} + \frac{1}{2}e^{-0.04t} - \frac{4}{9}e^{-0.03t} - \frac{1}{36}e^{-0.06t} - \frac{1}{4}e^{-0.02t}, \qquad t \geq 0. \end{aligned}$$

The above results are consistent with the results of Léveillé and Garrido (2001a).

Figures 5.1 and 5.2 shows the behavior of the mean and variance of the compound sums as functions of time t. We observe that:

- Both expectations and variances in the Poisson and renewal cases converge to their asymptotic "perpetuity" values as time t goes to infinity. The moments of such perpetuities are studied in Dufresne (1990).
- Expectations are larger in the Poisson case than in the renewal case.
- Variances are also larger in the Poisson case than in the renewal case.

To explain these differences recall that the inter–arrival times have the same mean in both models, say $\mathbb{E}[\tau_i^P] = 1/0.005 = 2/0.01 = \mathbb{E}[\tau_i^R]$, but have different variances.

In the Poisson case the exponential τ_i^P's have a variance of $\mathbb{V}[\tau_i^P] = (1/0.05)^2 = 40,000$, while in the renewal case the τ_i^R's are Erlang(2) with a much smaller variance of $\mathbb{V}[\tau_i^R] = 2/(0.01)^2 = 20,000 < 40,000 = \mathbb{V}[\tau_i^P]$.

Under the conditions above, this means that in same sense the discounted compound renewal sum is less risky than the discounted compound Poisson sum. Its claims are recorded essentially at every average inter–arrival times of $2/0.01 = 200$ with a small standard deviation around it of $\sqrt{2}/(0.01) = 141.42$. This prevents the shorter inter–arrival times that occur with a large probability in the Poisson case and that lead to larger present values.

If we were to extend the comparison to an equal–on–average, Erlang(n) inter–arrival times, this would even further decrease the variance of the τ_i^R's, for a fixed expectation. We can conclude from this analysis that the discounted compound Poisson process is more risky, in mean and variance, than the corresponding discounted compound renewal process.

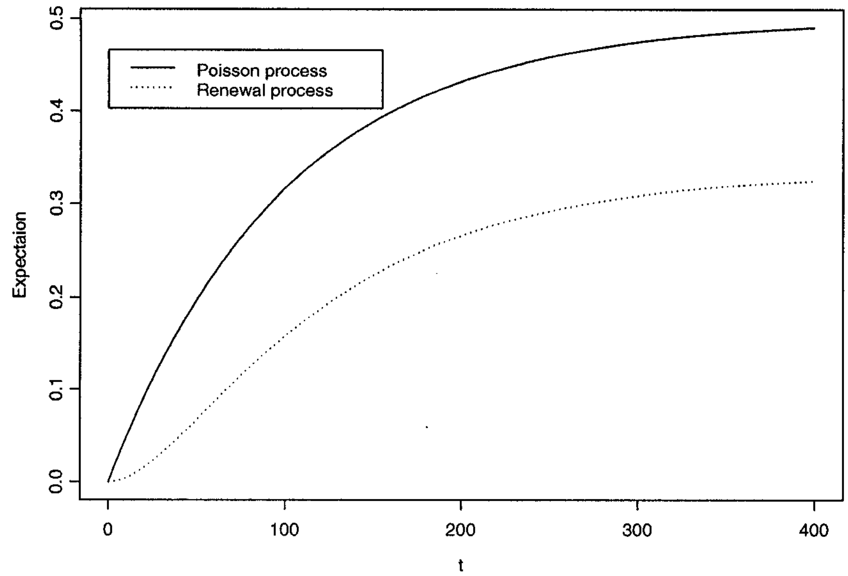

Figure 5.1: The expectations of the discounted compound Poisson and renewal sums.

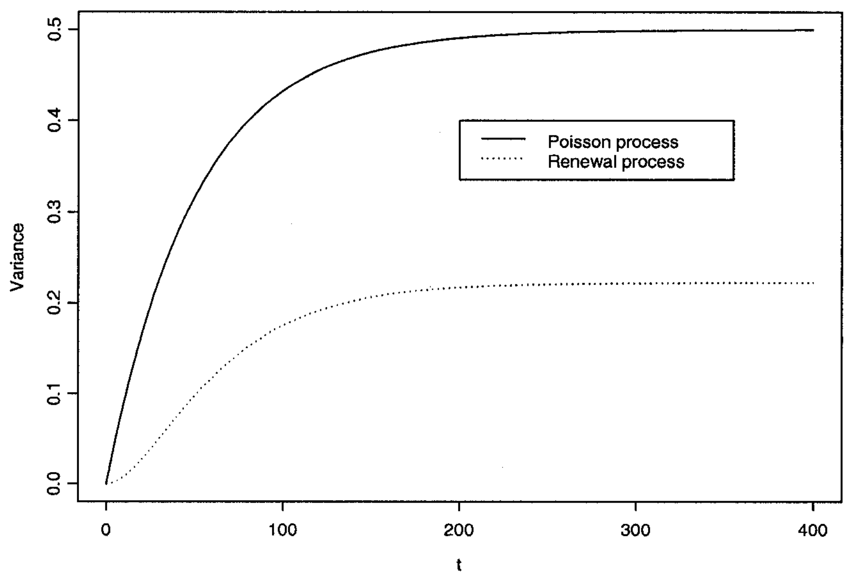

Figure 5.2: The variances of the discounted compound Poisson and renewal sums.

5.2 Impact of the Net Interest Rate δ

This section studies the impact of the net interest rate value δ on the expectations and variances of the discounted compound Poisson and renewal processes. First, we consider the Poisson case.

The same set of parameters as for the numerical examples in the previous section is used. The inter–arrival times are exponentially distributed with mean 1/0.005 and the deflated claims are exponential distributions with mean 1. Let $\mathbb{E}_{P0.01}\big[Z(t)\big]$, $\mathbb{E}_{P0.005}\big[Z(t)\big]$ and $\mathbb{E}_{P0}\big[Z(t)\big]$ denote the expectations under the net interest rate $\delta = 0.01$, 0.005 and 0. $\mathbb{V}_{P0.01}\big[Z(t)\big]$, while $\mathbb{V}_{P0.005}\big[Z(t)\big]$ and $\mathbb{V}_{P0}\big[Z(t)\big]$ are the corresponding variances. These can be calculated as:

$$\begin{aligned} \mathbb{E}_{P0.01}\big[Z(t)\big] &= 0.5 - 0.5e^{-0.01t}, \quad \mathbb{E}_{P0.005}\big[Z(t)\big] = 1 - 1e^{-0.005t}, \qquad t \geq 0, \\ \mathbb{E}_{P0}\big[Z(t)\big] &= 0.005t, \qquad \mathbb{V}_{P0.01}\big[Z(t)\big] = 0.5 - 0.5e^{-0.02t}, \qquad t \geq 0, \\ \mathbb{V}_{P0.005}\big[Z(t)\big] &= 1 - 1e^{-0.01t}, \qquad \mathbb{V}_{P0}\big[Z(t)\big] = 0.01t, \qquad t \geq 0. \end{aligned}$$

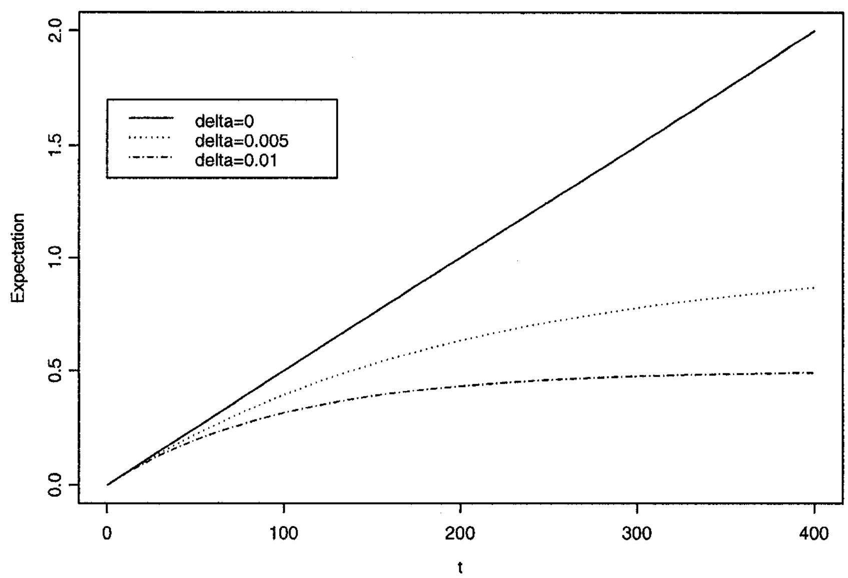

Figure 5.3: The expectation of the discounted compound Poisson sums.

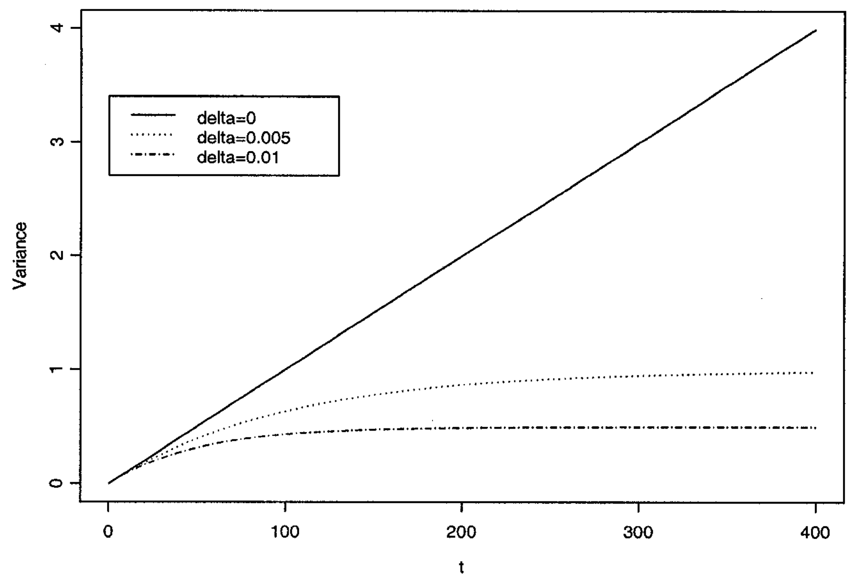

Figure 5.4: The variance of the discounted compound Poisson sums.

Figures 5.3 and 5.4 present the behavior of the expectations and variances for the discounted compound Poisson sum. As functions of time t we observe that:

- The expectations and variances under $\delta = 0.01$ and $\delta = 0.005$ converge as time t goes to infinity, except in the classical risk model $\delta = 0$. The latter go to infinity as $t \longrightarrow \infty$.

- The larger the net interest rate δ, the smaller the expectations and variances.

From (3.13) and (3.14) we can see that expectations and variances are the decreasing function in δ, so larger interest rates correspond to smaller expectations and variances. For $\delta = 0$, the expectation and variance are linear functions in t with positive slopes, going to infinity with t.

For the renewal case, expectations and variances are calculated under the same net interest rates $\delta = 0.01$, 0.005 and 0. Again we use Erlang(2) inter–arrival times with parameters $\underline{\alpha} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} -0.01 & 0.01 \\ 0 & -0.01 \end{pmatrix}$ and the deflated claims are exponentially distributed with mean 1. By similar notations to those in the Poisson case, we have:

$$\begin{aligned} \mathbb{E}_{R0.01}\big[Z(t)\big] &= \frac{1}{6}\left(e^{-0.03t} - 3e^{-0.01t} + 2\right), \\ \mathbb{E}_{R0.005}\big[Z(t)\big] &= \frac{4}{5} + \frac{1}{5}e^{-0.025t} - e^{-0.005t}, \\ \mathbb{E}_{R0}\big[Z(t)\big] &= \frac{1}{4}e^{-0.02t} + \frac{1}{200}t - \frac{1}{4}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{V}_{R0.01}\big[Z(t)\big] &= \frac{2}{9} + \frac{1}{2}e^{-0.04t} - \frac{4}{9}e^{-0.03t} - \frac{1}{36}e^{-0.06t} - \frac{1}{4}e^{-0.02t}, \\ \mathbb{V}_{R0.005}\big[Z(t)\big] &= \frac{14}{25} + e^{-0.03t} - \frac{1}{25}e^{-0.05t} - \frac{2}{3}e^{-0.01t} - \frac{64}{75}e^{-0.025t}, \\ \mathbb{V}_{R0}\big[Z(t)\big] &= \frac{1}{4}e^{-0.02t} + \frac{3}{400}t - \frac{1}{200}te^{-0.02t} - \frac{1}{16}e^{-0.04t} - \frac{3}{16}. \end{aligned}$$

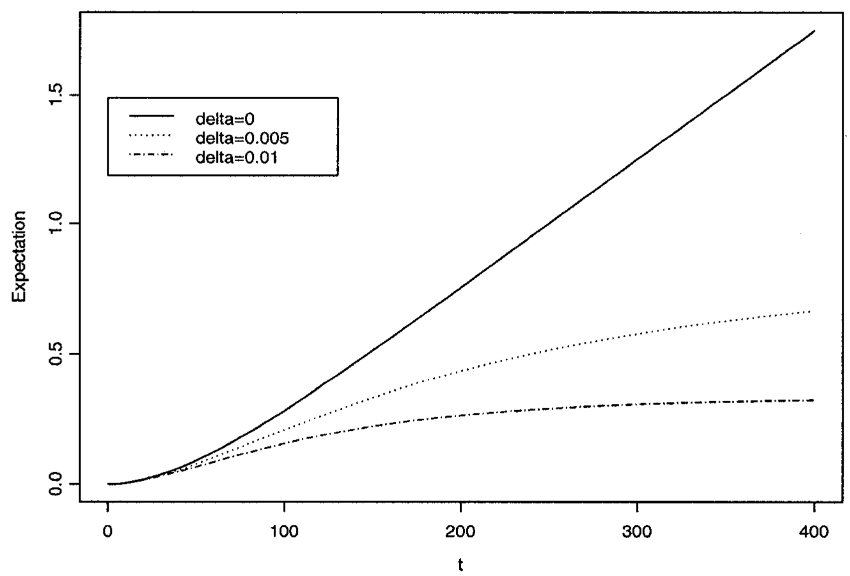

Figure 5.5: The expectation of the renewal process under different δ.

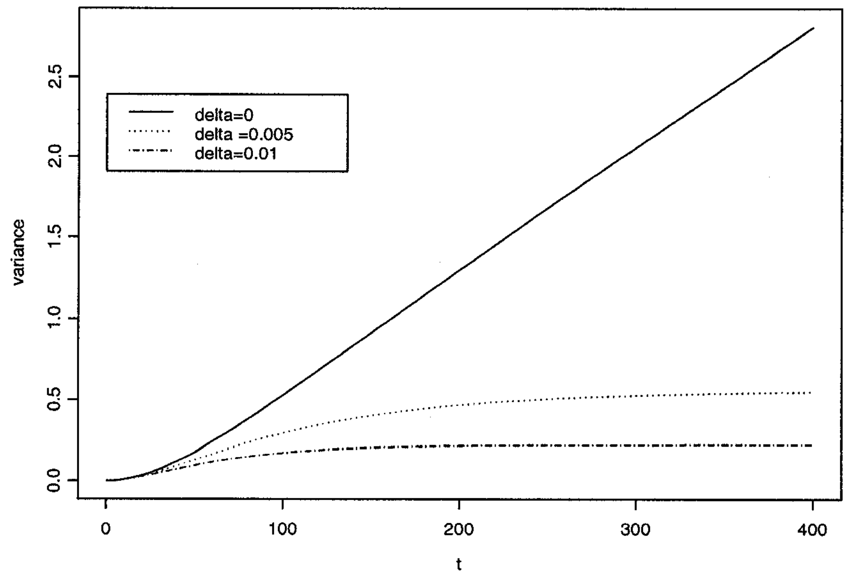

Figure 5.6: The variance of the renewal process under different interest rates.

Figures 5.5 and 5.6 show similar trend as the expectations and variances in Figures 5.3 and 5.4.

- The variances and expectations converge as t goes to infinity, except in the classical risk model.
- Larger δ corresponds to smaller expectations and variances.

The analysis is the same for the expectations and variance in the Poisson case. The expectations and variances are decreasing functions in δ. Now for $\delta = 0$, for the small t, the expectation and variance are decreasing functions in δ. When t gets larger the linear term with positive slope dominates the expectation and variance, hence it goes to infinity with t.

Conclusion

This thesis studies the m.g.f. of the compound Poisson present value risk (CP-PVR) process and the compound renewal present value risk (CRPVR) process. We obtain an explicit form for the m.g.f. the discounted compound Poisson sum with deflated PH claim severities using matrix–exponential arguments. We also study the asymptotic behavior of the m.g.f. of $Z(t)$ as the time $t \to \infty$ and of the moments of the CPPVR process.

The CRPVR process is considered for the case when the inter–arrival times have PH distributions and so do the deflated claim severities. We get an homogeneous differential equation for the m.g.f. when the inter–arrival times have an Erlang(n) distribution. The asymptotic behavior is also discussed. Some numerical examples are given to illustrate the results.

The last chapter presents some actuarial applications of the results obtained in the Chapters 3 and 4. A comparison of the discounted compound Poisson and renewal sums is presented. We can conclude that the discounted compound Poisson process is more risky in the mean-variance sense, than an equivalent discounted compound renewal process.

Some future work on the same topic may be considered. Since we now have the m.g.f. of $Z(t)$ we may try to find the distribution of $Z(t)$. The following suggestions lead in that direction:

- generalize the results for discounted compound renewal sums to any PH inter–arrival distribution and find the close form of the m.g.f. in terms of vectors and matrices.
- obtain the distribution of the CPPVR and CRPVR sums by inversion of the Laplace transform.
- give approximations for the distribution of the CPPVR and CRPVR processes.

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Appendix A

Matrix Exponential

From the definition, we can see that matrices play a crucial role in PH distributions. The characteristics of the distribution depend on matrix **A**. This is illustrated by the examples of Chapter 1. Hence we give here a brief introduction to matrices. We discuss some basic properties and definitions that are used in this thesis.

A.1 Definition

In this section first we introduce some definitions, especially that of the matrix exponential function and some of its properties. We also prove theorems that are used in the thesis. For the more details on matrix theory readers can refer to the book Bernstein (2005), that gives formulas and applications to the theory of linear systems. An alternative choice is Ortega (1987).

Definition A.1.1. *Nonsingular and singular matrices*

If the determinant $|\boldsymbol{A}| \neq 0$*, we call* $\boldsymbol{A}$ *nonsingular, otherwise it is called singular.*

Note that throughout this thesis, we assume that $\mathbf{A}$ is a nonsingular matrix, hence its inverse $\mathbf{A}^{-1}$ exists.

Definition A.1.2. *Eigenvalues and Eigenvectors*

An eigenvalue of a square matrix $\boldsymbol{A}$ *of order* n *is a real or complex scalar* λ *satisfying the equation:*

$$\boldsymbol{A}\underline{x} = \lambda \underline{x},$$

for some nonzero vector $\underline{x}$*, we call* λ *an eigenvalue and* $\underline{x}$ *an eigenvector of* $\boldsymbol{A}$.

Definition A.1.3. *Spectral radius of a matrix* $\boldsymbol{A}$ *of order* n

Let $\lambda_1, \lambda_1, \dots, \lambda_n$ *be eigenvalues of* $\boldsymbol{A}$*, we define*

$$sprad(\boldsymbol{A}) = \max\{|\lambda_i|, 1 \le i \le n\}.$$

Definition A.1.4. *Matrix exponential*

Let $\boldsymbol{A}$ *be square matrix of order* n*, then we call matrix exponential, denoted* $e^{\boldsymbol{A}}$ *or* $\exp(\boldsymbol{A})$*, the matrix:*

$$e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{1}{k!} \boldsymbol{A}^k, \tag{A.1}$$

with $e^{\boldsymbol{0}} = \boldsymbol{I}_n$*, where* $\boldsymbol{0}$ *is a zero matrix of order* n.

Definition A.1.5. *Logarithm of* $\boldsymbol{A}$

Let $\boldsymbol{A}$ *be square matrix with order* n *, then we call* $\boldsymbol{B}$ *a logarithm of* $\boldsymbol{A}$ *if* $\boldsymbol{B}$ *satisfies:*

$$e^{\boldsymbol{B}} = \boldsymbol{A}. \tag{A.2}$$

Then if $sprad(\boldsymbol{A} - \boldsymbol{I}) < 1$*, we can define*

$$\boldsymbol{B} = \ln \boldsymbol{A} = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} (\boldsymbol{A} - \boldsymbol{I})^i. \tag{A.3}$$

This leads to the following notion of $\ln(\mathbf{I} - \mathbf{A})$ and $\ln(\mathbf{I} + \mathbf{A})$

$$\ln(\mathbf{I} - \mathbf{A}) = -\sum_{i=1}^{\infty} \frac{\mathbf{A}^i}{i}, \tag{A.4}$$

and

$$\ln(\mathbf{I}+\mathbf{A}) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}\mathbf{A}^{i}, \tag{A.5}$$

if $sprad(\mathbf{A}) < 1$.

A.2 Lemmas

Lemma A.2.1. *If λ is an eigenvalue of $\boldsymbol{A}$, then λ^{-1} is eigenvalue of $\boldsymbol{A}^{-1}$.*

Proof. Since $\mathbf{A}$ is nonsingular, then λ^{-1} exits, then we have:

$$|\lambda\mathbf{I}-\mathbf{A}| = |\lambda\mathbf{A}\mathbf{A}^{-1}-\mathbf{A}| = |\mathbf{A}||\lambda\mathbf{A}^{-1}-\mathbf{I}| = -|\mathbf{A}||\lambda||\lambda^{-1}\mathbf{I}-\mathbf{A}^{-1}|.$$

Hence λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$. □

Lemma A.2.2. *If $\boldsymbol{A}$ is a matrix of order n with $sprad(\boldsymbol{A}) < 1$, then $(\boldsymbol{I}-\boldsymbol{A})^{-1}$ exists and*

$$(\boldsymbol{I}-\boldsymbol{A})^{-1} = \lim_{k\to\infty}\sum_{i=0}^{k}\boldsymbol{A}^{k} = \sum_{k=0}^{\infty}\boldsymbol{A}^{k}.$$

Proof. See Ortega (1987). □

Lemma A.2.3. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of order n. Then*

$$e^{t\boldsymbol{A}}e^{t\boldsymbol{B}} = e^{t(\boldsymbol{A}+\boldsymbol{B})}, \qquad t \in R, \tag{A.6}$$

if $\boldsymbol{AB}=\boldsymbol{BA}$.

Proof. See Bernstein (2005). □

Lemma A.2.4. *The derivative of a matrix exponential function is given by:*

$$\frac{d}{dt}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}, \qquad t \in \mathbb{R}. \tag{A.7}$$

Proof. Take a derivative term by term in the series expansion, as in Definition A.1.4. □

Lemma A.2.5. *Let $\boldsymbol{A}$ be square nonsingular matrices of order n and sprad$(\boldsymbol{I} - \boldsymbol{A}) < 1$. Then we have the following results:*

$$-\ln \boldsymbol{A} = \ln \boldsymbol{A}^{-1}. \tag{A.8}$$

Proof.

$$\begin{aligned}
\ln \mathbf{A} &= (\mathbf{A}-\mathbf{I}) - \frac{1}{2}(\mathbf{A}-\mathbf{I})^2 + \frac{1}{3}(\mathbf{A}-\mathbf{I})^3 + \cdots + (-1)^{k+1}\frac{1}{k}(\mathbf{A}-\mathbf{I})^k + \cdots \\
-\ln \mathbf{A} &= -[(\mathbf{A}-\mathbf{I}) - \frac{1}{2}(\mathbf{A}-\mathbf{I})^2 + \frac{1}{3}(\mathbf{A}-\mathbf{I})^3 + \cdots \\
&\quad + (-1)^{k+1}\frac{1}{k}(\mathbf{A}-\mathbf{I})^k + \cdots],
\end{aligned} \tag{A.9}$$

then

$$\ln \mathbf{A}^{-1} = (\mathbf{A}^{-1}-\mathbf{I}) - \frac{1}{2}(\mathbf{A}^{-1}-\mathbf{I})^2 + \frac{1}{3}(\mathbf{A}^{-1}-\mathbf{I})^3 + \cdots + (-1)^{k+1}\frac{1}{k}(\mathbf{A}^{-1}-\mathbf{I})^k + \cdots.$$

Since $\mathbf{A}^{-1} = \left[\mathbf{I} + (\mathbf{A}-\mathbf{I})\right]^{-1}$, from Lemma A.2.2 we have:

$$\begin{aligned}
A^{-1} &= [\mathbf{I} + (\mathbf{A}-\mathbf{I})]^{-1} \\
&= \mathbf{I} - (\mathbf{A}-\mathbf{I}) + (\mathbf{A}-\mathbf{I})^2 - (\mathbf{A}-\mathbf{I})^3 + \cdots + (-1)^k(\mathbf{A}-\mathbf{I})^k + \cdots,
\end{aligned}$$

and hence

$$\begin{aligned}
\ln\ \mathbf{A}^{-1} &= \left[-(\mathbf{A}-\mathbf{I}) + (\mathbf{A}-\mathbf{I})^2 - (\mathbf{A}-\mathbf{I})^3 + \cdots + (-1)^k(\mathbf{A}-\mathbf{I})^k + \cdots\right] \\
&\quad - \frac{1}{2}\left[-(\mathbf{A}-\mathbf{I}) + (\mathbf{A}-\mathbf{I})^2 - (\mathbf{A}-\mathbf{I})^3 + \cdots + (-1)^k(\mathbf{A}-\mathbf{I})^k + \cdots\right]^2 \\
&\quad + \frac{1}{3}\left[-(\mathbf{A}-\mathbf{I}) + (\mathbf{A}-\mathbf{I})^2 - (\mathbf{A}-\mathbf{I})^3 + \cdots + (-1)^k(\mathbf{A}-\mathbf{I})^k + \cdots\right]^3 \\
&\quad + \cdots + (-1)^{k+1}\frac{1}{k}\left[-(\mathbf{A}-\mathbf{I}) + (\mathbf{A}-\mathbf{I})^2 - (\mathbf{A}-\mathbf{I})^3 + \cdots \right. \\
&\quad \left. + (-1)^k(\mathbf{A}-\mathbf{I})^k + \cdots\right]^k + \cdots.
\end{aligned} \tag{A.10}$$

By comparing the polynomials in (A.9) and (A.10) for $(\mathbf{A}-\mathbf{I})$, we see that they are exactly the same, hence

$$-\ln \mathbf{A} = \ln \mathbf{A}^{-1}.$$

□

Lemma A.2.6. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be square nonsingular matrices. If* $\boldsymbol{AB} = \boldsymbol{BA}$ *, then we have*

$$\ln \boldsymbol{A} - \ln \boldsymbol{B} = \ln \boldsymbol{AB}^{-1} .$$

Proof.

$$e^{\ln \mathbf{A} - \ln \mathbf{B}} = e^{\ln \mathbf{A}} e^{-\ln \mathbf{B}} = A e^{\ln \mathbf{B}^{-1}} = \mathbf{AB}^{-1} ,$$

from Lemma A.2.4 and Lemma A.2.5 and using that $e^{\ln \mathbf{AB}^{-1}} = \mathbf{AB}^{-1}$, then $\ln \mathbf{A} - \ln \mathbf{B} = \ln \mathbf{AB}^{-1}$. □

Lemma A.2.7. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be the same order of square matrices, if the inverse of* $\boldsymbol{A} + s\boldsymbol{B}$ *exists then*

$$\frac{d}{ds}(\boldsymbol{A} + s\boldsymbol{B})^{-1} = -(\boldsymbol{A} + s\boldsymbol{B})^{-1}\boldsymbol{B}(\boldsymbol{A} + s\boldsymbol{B})^{-1} . \tag{A.11}$$

For the proof see Bernstein (2005).