Introducing Angles in Grade Four:

a Realistic Approach Based on the van Hiele Model

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A Thesis

In

The Department

Of

Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements

for the Degree of Master in the Teaching of Mathematics

at Concordia University

Montreal, Québec, Canada

April 2008

Your file Votre référence
ISBN: 978-0-494-40862-9
Our file Notre référence
ISBN: 978-0-494-40862-9

ABSTRACT

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This thesis is a study of teaching and learning Geometry, and, in particular, angles. The theoretical framework used in the research is a combination of a teaching theory called Realistic Mathematics Education (RME) and a learning theory called the can Hiele Model of Geometric Thinking. These frameworks, backed up by a historical study of geometry, were used in the design, experimentation and evaluation of a lesson where fourth grade students were introduced, for the first time, to the idea of angle and relations between angles of different sizes. The in-class experiment was conducted in two different fourth grade classrooms. The lessons were taught by teachers of the respective classrooms, based on a detailed script and materials prepared by the researcher. At the end of the lesson the students answered, in writing, questions on what they had learned during the lesson. These answers, combined with the classroom observations, provided a basis for the evaluation of the experiment. The written answers were categorized into analytical and narrative statements. The analytical statements were divided into the different levels of the van Hiele Model. The narrative statements were divided into whether or not they mentioned angles. The research findings suggest the usefulness of using lesson plans based on the two theoretical frameworks in helping students develop an analytical conceptualization of mathematics.

ACKNOWLEDGEMENTS

I would like to acknowledge and offer my sincere thanks to Dr. Sierpinska for all of her help and guidance. This thesis would not be what it is without her. I would also like to thank Dr. Denis Connolly for introducing me to the wonderful world of Geometry and for always being there throughout this process to consult on any problems that arose. Lastly, I would like to acknowledge Dr. Hillel and Dr. Cummins and thank them for the time they spent reviewing my thesis and helping me improve it in every way.

DEDICATIONS

This thesis is dedicated –

- first, to my husband, Dylan, for knowing exactly what type of support to provide throughout my entire Masters degree. He has always known when I have needed to be pushed and when I have needed to be pulled.

- to my parents, Wally and Lorri Smart, for instilling in me the importance of education at the earliest age.

- and lastly, , to all the great Geometers throughout history for providing such enjoyable material to study.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES AND GRAPHS

INTRODUCTION

Geometry, Angles, Realistic Mathematics Education and the van Hiele Model

The Research Questions

Geometry is one of the oldest and best-documented subjects in mathematics. Geometry was so important in ancient times that Plato is even credited with writing

ΑΓΕΩΜΕΤΡΗΤΟΣ ΜΗΔΕΙΣ ΕΙΣΙΤΩ

or "Let no one unversed in geometry enter here" (Sibley, p. 2) over the door of his Academy of philosophy. This shows that one of the smartest men of ancient times believed in the value of knowing Geometry. This value is still recognized today, as evidenced in the ICMI initiative of mounting, in the 1990s, a study on perspectives and prospects of teaching and learning geometry for the 21st century and in the papers published in the related edited book (Mammana & Villani, 1998). However, this book also shows that geometry remains a difficult subject to teach and to learn, although a lot of research has already been done and many ingenious proposals – put forward and sometimes experimented. Even special "dynamic geometry" computer software has been developed for educational purposes. This software did not solve any of the problems of geometry teaching and learning but created new and very interesting ones for educational research and development.

This historical and modern day importance of geometry is two aspects that motivated me to developing a research project on the topic of teaching and learning

Geometry. The objectives of the project are not unrelated with theory of mathematics education, and therefore I have to talk about the theory before I can formulate my objectives.

Several theoretical frameworks for research and development in mathematics education hold particular relevance to the subject of Geometry. One of them is *Realistic Mathematics Education – RME* (Cooper and Harries, 2002; Fyhn, 2008; Gravemeijer and Doorman, 1999; Treffers, 1993; van den Heuvel-Panhuizen, 1994; 2001; 2003; Wubbels, Korthagen and Broekman, 1997). Another is the *van Hiele Model* of Geometric Thinking, extensively applied in research (Burger and Shaughnessy, 1986; Cannissaro and Menghini, 2006; Fyhn, 2008; Gutierrez, Jaime and Fortuny, 1991; Hoffer, 1983; Mayberry, 1983; NCTM, 1988; Senk, 1989). Using these frameworks I have developed a context for my research objectives:

1. To show how the history of geometry, and especially the history of the concept of angle, supports the principles of *RME* and *the van Hiele Model* of geometric thinking.

2. To use the teaching theory of *RME* in a teaching experiment aimed at initiating a successful conceptualization of Geometric concepts, particularly the concept of angle, as outlined by the *van Hiele Model* of Geometric Thinking.

For the second objective of this thesis a lesson plan about angles was developed. Two separate teachers then taught this lesson according to the plan to their own grade four classes in an Ontario elementary school. This was done during their regular class time. The reason for having the lesson plan taught by the students' regular teacher and during the regular class time was to evaluate whether this teaching method would be effective in real time classroom situations and not just in controlled research environments.

Outline of Thesis

The first chapter provides a survey of literature on, first, the concept of angle, and, then, the theories of RME and the *van Hiele Model.*

In the second chapter I address the first objective of the thesis, by trying to see how *RME* and the *van Hiele Model* are aligned with the historical development of Geometry.

The third chapter presents the theoretical framework of the thesis. To begin, a combination of the *van Hiele Model* and *Realistic Mathematics Education* is developed in order to provide a framework that includes both the teaching and learning aspects of Geometry. Then, the chapter outlines the concept of angle that will be used throughout the research.

Chapter four discusses the methodology of the empirical part of the research (objective 2). Here, a detailed summary of the lesson plan as well as all other materials used in the research study are presented.

Chapters five and six consist of the quantitative and qualitative results of the empirical study. A closer look is given here to the in-class observations, and the students' productions. A qualitative method of categorizing the written responses is given.

Chapter seven is devoted to a discussion of the results. Separate conclusions are given to the different data that were collected, namely the in-class observations, the students' transparencies and the students' written responses. As well, some implications of the research findings taken as a whole are proposed. This chapter also contains recommendations for future research on similar subjects.

CHAPTER ONE

Survey of Literature

The main themes of this thesis, which include angle conceptualization, the *van Hiele Model* of Geometric Thinking and teaching according to the *Realistic Mathematics Education* theory, have been popular topics for mathematics education researchers over the last few decades. Conducting research on the conceptualization of angles seems like a never-ending task. This is probably due to the fact that the notion of angles is quite complex. Just defining angles is a problem, not to mention the notion of how to teach angles effectively. Researchers have focused on a wide variety of ideas when it comes to teaching angles, such as teaching with *Logo* or using physical examples. As well, literature has focused on the more basic ideas of angles, for example, how it should be defined and in what context it should be presented to students. Likewise, the *van Hiele Model* of Geometric Thinking has also received a lot of attention among mathematics education researchers. Some researchers have tried to provide support for the *van Hiele Model, some* have tried to disprove it, and yet others have tried to adjust or improve the model. Mostly, the *van Hiele Model* has just been used as a theoretical framework in research that has studied a number of different topics in Geometry.

Realistic Mathematics Education has received much notice recently with the trend towards problem solving becoming an integral part of compulsory mathematics education. Most research on *Realistic Mathematics Education* is in support of the use of context problems as the initial teaching tool. Some research has focused more in depth on the whole theory of *Realistic Mathematics Education* and in particular the theories of model development. Overall, the core subject matter of this thesis has been much researched and written about by mathematicians and educators alike.

1.1. Angles

Research on Angles

As mentioned, the amount of research conducted on angles and teaching angles is overwhelming. Here, I will focus on a selection of papers only. The specific articles mentioned here were chosen because they represent a good overview of the variety of the different types of research available.

In the late 1980's and early 1990's there was a shift in angle research to incorporate the new accessibility of angle making computer software. Forefront of these software systems was *Logo* or *Turtle Geometry*. Extensive research was conducted on the educational potential of the *Logo* environment. For example, in (Clements & Battista, 1989) third grade students were given 26 weeks of *Logo* instructions and interaction. At the end of this time period, these students were interviewed and their responses were compared to a control group of students who were not instructed in *Logo*. One of the goals of the research was to see if *Logo* could be helpful in assisting students' conceptualization of angles, shapes and motion. The researchers found that although the *Logo* group of students preformed higher than the control group, neither group reflected a strong conceptualization of angle and angle measure. The *Logo* group, however, tended to conceptualize angles in terms of the action or procedure involved in making the angle. In other words, these students held the notion of angle as rotation. This could be attributed to their experience with *Logo* where angles are created based on the amount of rotation in which the user instructs the Turtle to turn.

In another article by the same authors (Clements and Battista, 1990), the research focused mainly on what each child's concepts of the notion of angle and polygons were after 40 sessions with *Logo*. The authors now found that students who were instructed with *Logo* tended to conceptualize geometric objects as the product of the pieces that make up the object as well as the processes involved to make the pieces and combine them to make the object. In other words, the students instructed with *Logo* had a perception that geometric objects had to be created the same way as when they are created with the *Logo* program. With regard to angles, the students developed the concept that angles were made up of an amount of turning and of a measure (of the ray after the turn). This is consistent with how angles are created using *Logo* as well as consistent with these authors' findings in their 1989 article.

Another research involving *Logo* (Hillel & Kieran, 1987) looked at how students chose their inputs when instructing the turtle to make an angle. Hillel and Kieran found that students input the commands on either a visual decision or an analytical decision and that the visual decisions were predominant. In other words, the students tended to use a guess and check method to see if the turtle would accomplish the given task. I tend to think that this has something to do with how young children approach computers. In my limited experience, children tend to be very comfortable with the idea of "guess and check", or in other words, just pressing buttons to see what the results are and learning by this method. Thus, I have interpreted the results from Hillel and Kieran's article as consistent with my observations.

As the 1990's came to a close and the areas of research investigation with *Logo* seemed to be exhausted, some researchers turned to the area of investigation with physical realistic models. For example, Mitchelmore (1998) examined how students in grades two, four and six conceptualize the concept of angle as rotation or turn. The experiment involved the students studying three-dimensional physical models that they could manipulate and move providing the students with a reality based foundation on which to examine angles. All of the models created some sort of angle through rotation or turning. Thus, in this research, the author used the notion of angle as a rotation. One of his conclusions was that introducing angles as rotations or turns seems premature in elementary school. In a follow up of this research, still using physical models, Mitchelmore & White (2000) asked whether students could move from a conceptualization of angles as turnings or rotations to a more abstract conceptualization of angles in general. This research highlighted the fact that there is more than one conception of angle.

Research that uses physical or realistic models to teach the notion of angle is still being conducted today. In one of the most recent articles, Fyhn (2008) takes an entire class of grade eight students to a climbing wall for a day in order to help them conceptualize the concept of angle. The students were responsible for investigating the different angles they created with the ropes and their bodies, while scaling the climbing wall. Through this investigation, the goal was to have the students reinvent for themselves some informal theories on angles and the best angles to use while climbing. This theoretical framework of using physical or realistic models in the hope of promoting student reinvention was based on the framework of *Realistic Mathematics*

Education. Fyhn also analyzed the students' results using the framework of the *van Hiele Model* of Geometric Thinking. As was already mentioned, both these frameworks were used as a foundation for my own thesis research. However, it is important to note that Fyhn's research article only became available after my research was already completed and any similarities are purely coincidental.

Literature and the Many Definitions of an Angle

If an examination of the above literature shows anything, it is that, in order to think about teaching angles, the meaning of this concept must first be clarified. A quick search through some of the available literature shows that there is no consensus on how to define an angle (Brown, Simon and Snader, 1970; Hartshorne, 1997; Sibley, 1998; Sobel, Maletsky, Golden, Lerner and Cohen, 1986; Webster Dictionary and Thesaurus, 2002). Instead, there are many different definitions covering a variety of concepts of what is an angle.

The Webster Dictionary gives two definitions of an angle, the first being; "The figure formed by two lines extending from the same point," and the second as; "A measure of an angle or the amount of turning necessary to bring one line or plane into coincidence with or parallel to another" (Webster Dictionary & Thesaurus, 2002, p. 37-38). The first definition defines a physical two-dimensional geometric object made up of three parts; two lines and a point. It refers only to the figure that is the angle and nothing else. The second definition does not talk about what an angle *is* but what an angle *is measuring*. Thus, these two definitions define the same object by completely different characteristics.

Throughout history, mathematicians have used their own definitions of angle in conjunction with geometric systems. Two of the most famous axiomatic systems of Geometry are the systems of Euclid and David Hilbert. In the *Elements*, Euclid defined an angle as

> "A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line" (Definition 8, Book I, *The Elements*; see also Sibley, 1998, p. 287).

In comparison, Hilbert's definition states that,

> "By an angle is meant a point (called the vertex of the angle) and two rays (called the sides of the angle) emanating from the point" (Sibley, 1998, p. 294).

Like the dictionary definitions, these two definitions also focus on different characteristics of angles. Hilbert's definition is very similar to the first definition in Webster's Dictionary (2002). It only differs in that it gives specific names to the objects that make up the angles, that is, the vertex and the sides. On the other hand, Euclid's definition defines an angle as a ratio of, or difference between, the inclinations of the lines. According to Euclid, an angle is the relationship between the positions in the plane of two intersecting lines, or their *relative* position. While, when thinking of angles as figures, one can see four angles in a pair of intersecting lines, there is only one angle according to *Definition 8*: knowing one of the figure-angles implies knowing them all; what matters, really, is the mutual position of two straight lines, and this is Euclid's notion of angle. Euclid's plane has no fixed frame of reference, no coordinates (so the "slope" of a straight line is not defined), no orientation, and therefore it does not matter which of the two lines is first and which is second. So, again, a pair of intersecting straight lines is inclined to each other in a unique way, and this is the Euclid's angle. The *measure* of this angle can be given in two ways, but each way determines the other one uniquely. Another interesting feature of Euclid's angles is that parallel lines do not determine an angle, since *Definition 8* requires that the two lines do "not to lie in a straight line". There would therefore be no angles of measure 0 in Euclid's system (see Hartshorne, 1997, p. 28), while, such angles are perfectly acceptable in Hilbert's system (Sibley, 1998). Thus, we see that there are important differences between these two definitions.

Examining a few secondary school mathematics textbooks also reveals a number of different definitions for angles. Two examples of these include the definition that "An angle is formed when two rays begin at the same point...The measure of an angle depends upon the amount of rotation and not upon the length of the sides..." (Brown, Simon and Snader, 1970, p. 205) and "A figure formed by two non-collinear rays with a common endpoint. The two rays are the sides of the angle. The common endpoint is the vertex of the angle" (Sobel, Maletsky, Golden, Lerner and Cohen, 1986, p. 578). The first definition is specifically interesting in that it defines the angle as an object and then discusses the notion of putting measurement to the angle based on the amount of rotation of one line-ray to the other. The second definition has a similar characteristic to Euclid's, in that it specifies that the rays are non-collinear. The first definition corresponds a lot to Hilbert's definition in how it gives names to the pieces or parts that makes up the angles. Overall, it is clear that textbooks, a very important teaching tool in most classrooms, do not even have a consensus as to how an angle should be defined.

As was already mentioned, angles are also the topic of many different research studies described in academic journal articles (Clements and Battista, 1989, 1990; Clements and Burns, 2000; Hillel and Kieran, 1987; Mitchelmore, 1998, Mitchelmore and White, 2000; Munier and Merle, 2007; Simmons and Cope, 1990). It would be impossible to mention every journal article that discusses a different definition for an angle. The articles that are mentioned here were chosen as examples because of the diverse way in which they define angles. It is interesting to note here that in some research articles no definition of what the researchers actually meant by angle is given. This is not the case of Mitchelmore (1998), where the concept is explicitly discussed. The author states how "…many curriculum documents now recommend treating angle in terms of turning…" (ibidem, p. 265) and goes on to define angle as "the amount of turning between two lines about a common point" (ibidem) According to Mitchelmore, "A focus on turning (a) emphasizes the relative inclination of the arms of an angle while showing that their length or orientation are irrelevant and (b) could make the arc marking the angle more meaningful" (ibidem).

In (Clements & Burns, 2000) two different definitions for an angle are described. The article states that "Angles have been defined as a part of the plane included between two rays meeting at their endpoint (static definition) and as the amount of rotation necessary to bring one of its rays to the other ray without moving out of the plane (dynamic definition)"(ibidem, p. 31). Here the researchers have defined an angle in two very different ways, similar to the many previous examples of angle definitions. The difference here is that researchers categorized the definitions depending on the context in which the angle is being examined. This special attention to context introduces the idea that the notion of angle will change depending on the situation.

Lastly, in a more recent article by Mitchelmore and White (2000), many definitions for an angle are given. At the beginning of the article it mentions that "Three particular classes of angle definitions occur repeatedly: an amount of turning about a point between two lines; a pair of rays with a common end-point; and the region formed by the intersection of two half planes"(Mitchelmore & White, 2000, p. 209). Later on Mitchelmore and White describe some findings of Davey and Pegg (1991) saying that these authors have "obtained a sequence of four definitions of angle: (a) a corner which is pointy or sharp; (b) a place where two lines meet; (c) the distance or area between two lines; and (d) the difference between the slope of the two lines" (Mitchelmore & White, 2000, p. 218). Thus, Mitchelmore and White also categorize different definitions. Overall, there seems to be a trend towards having more than one definition for an angle, depending on context.

From the above findings I was motivated to conduct a simple experiment. This experiment strictly involved asking a class of students enrolled in a college level "Vectors and matrices" course to write down what they understood to be an angle. The instructions that were given were "Please write down how you would define an angle." All of the students who participated were aspiring to major in either Engineering or Computer Science. This experiment was done only on a volunteer basis with nobody being forced to participate and with no penalties for those who chose not to take part. As well, all of the results were given anonymously. The students were made aware that the results could potentially be used in my thesis research.

This experiment resulted in an entire spectrum of different definitions for angles. For example, one student defined an angle as "Two unparallel rays that share a common point (vertex of the angle) and the value of this angle defines how far the second ray is from the first one, expressed in terms of degrees or radians" (see the list of all students' responses in Appendix A). This student understands an angle as an object and has concluded that this angle can have a value of measurement associated with it. Another student defined an angle to be the process where "I would take two lines that intersect and I would think about the angle as the rotation that it would take one of the lines to 'merge' with the other – basically so that they are one". This student does not define an angle as being an object but as rather something this object has the potential to do, the rotation of the lines. Another interesting definition given by a student was "An angle as the measure of the space between any two straight lines that meet up at one point". Thus, now we have an angle as a space or area between two lines that meet. But this last definition is confusing because it is seems impossible to define area in a plane with only two lines (at least in Euclidean Geometry). So this definition leaves us questioning what was actually meant and if in some way maybe the "space" referred to in the definition is a different concept of space or area when it refers to angles than to other geometric objects.

After examining the different definitions of angles in the literature and looking at the different results from my students, categories start to appear in which the angle definitions fit. One of these very noticeable categories includes definitions that discuss angles as a rotation or turning. Definitions in this category look at some physical attribute of the angle. That is, there is some type of movement, like a rotation, turning or revolution that is used to describe the angle. This movement is not necessarily involved in the construction of the angle, but could be potential movement (that is, how much it would have to move or could move). A review of the literature shows us that 'angle as rotation' is not a new mathematical concept. There have been many empirical studies conducted to gauge students understanding of the concept. For example, all three of the studies mentioned previously in this chapter cover the concept of angles as rotation or turning. As was previously mentioned, the article by Clements and Burns (2000) uses a classification by Kieren (1986) and distinguishes all definitions encompassing a movement aspect as "dynamic definitions." (Clements & Burns, 2000, p.31) In the postsecondary mathematics textbook by Henderson & Tainmina (2005), we have the explanation that "a dynamic notion of angle involves an action: a rotation, a turning point, or a change in direction between two lines" (ibidem, p. 38). Overall, dynamic definitions of an angle are probably the easiest definition category to recognize.

The obvious dual category to dynamic definitions is "static definitions" (Clements & Burns, 2000, p. 31). Like the name suggests, static definitions do not imply any movement at all. These definitions tend towards a more abstract concept. Clements and Burns state that static angles are "defined as part of the plane included between two rays meeting at their endpoint" (2000, p. 31). This explanation tends to beg the question as to the size of the part of the plane that is the angle. In Euclid's Geometry rays are infinitely long within a plane (although we do not tend to draw them as such). As well, there is no explanation as to which side of the two rays is the angle. Consider the picture below (Figure 1). Let's assume the rectangle is an infinite plane and the blue lines represent the two rays meeting at an endpoint. The red line arrows extending from the angle represent the infinite aspect of the rays that makes up the angle. Thus, from the picture we see that the Clements and Burns (2000) explanation of a static angle is very unclear as to what is actually meant to be the angle.

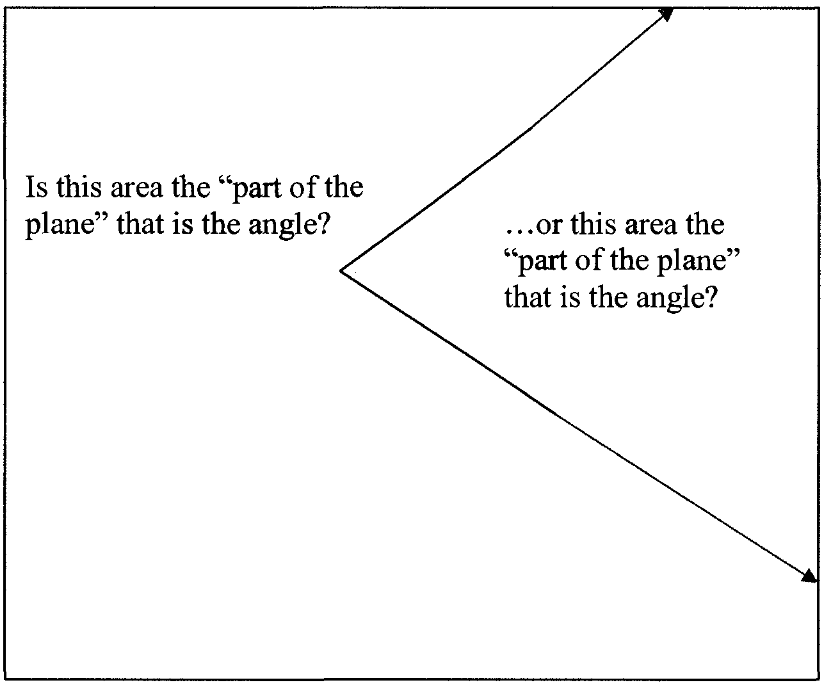

Figure 1. Illustration of Clements & Burns (2000) definition of angle

For another example of how the Clements and Burns (2000) explanation is confusing, consider the situation of a child being sent to stand in the corner. If we consider corners as a byproduct of angles, then a child being sent to stand in the corner is really being told to stand within the angle. Thus, according to the Clements and Burns (2000) definition of a static angle (p. 31), within a finite space (like a room) a child could potentially stand anywhere and still be standing in the corner.

Overall, categorizing angle definitions as "static," using Clements and Burns (2000) explanations does not give a very clear understanding of static angles. Other literature resources offer some alternative categorizations. For example, Mitchelmore and White (2000) take the categorizing of angles one step further and name not two but three particular classes of angle definitions. These include "an amount of turning about a point between two lines; a pair of rays with a common end-point; and the region formed by the intersection of two half-planes" (Michelmore & White, 2000, p.209). The first classification is clearly part of the dynamic category, as there is some movement involved and it is very similar to the Clements and Burns (2000) explanation of a dynamic angle definition. The next two classifications can both be considered as static definitions since there is no movement mentioned. The Clements and Burns (2000) definition of static angles is similar to the third definition used by Mitchelmore and White (2000). In contrast, Mitchelmore and White (2000) have actually divided the category of static angles into two sub-categories.

Another look at these two sub-categories of static angle definitions can be found in (Henderson & Tainmina, 2005). Here the authors have described static angle definitions as either concerning a measure or a geometric shape (Henderson & Tainmina, 2005, p. 38). "Angle as a measure may be thought of as the length of a circular arcs or the ratio between areas of circular sectors...[Angles] as a geometric shape...may be seen as the delineation of space by two intersecting lines" (ibidem, p. 39). In other words, angle definitions that concern measuring a distance or area between two intersecting lines or planes can be categorized as a static angle in the measure subcategory. Thus we can see that Mitchelmore and White's (2000) third definition can be categorized as a measure definition. Looking again at one of the definitions submitted by a university student we see that the definition, "An angle as the measure of the space between any two straight line that meet up at one point," (Appendix A) is an example of an angle as a measure type of definition. The other subcategory of angles as a geometric shape refers to definitions that only concern how the angle looks visually or physically. One of the best examples of this type of angle definitions is Hilbert's definition of "By an angle is meant a point (called the vertex of the angle) and two rays (called the sides of the angle) emanating from the point" (Sibley, 1998, p. 294). Overall, dividing angle definitions into three categories as in Henderson & Tainmina, 2005) seems like the best way to ensure that all different possible types of definitions of an angle are included.

One interesting feature of having three different categories of angle definitions is that each category has a unique way for determining whether two angles are congruent. With a dynamic angle it can be verified that two angles are congruent if the action required to create or recreate the angles is equal or if the potential movement available to the angles is equal. Two angles that are considered measurements are easy to examine for congruency by simply seeing if the measurement associated with each angle is equal (as long as the measuring is done with the same units). Lastly, if the angles are described as strictly geometric shapes then a process of superimposing one onto another will let you examine if they are equivalent. This last process is similar to Euclid's method of superposition used in proof I.4 (side-angle-side criterion for congruence of two triangles) (Hartshorne, 1997, p. 31). Thus, depending on which category the angle is being defined in will determine features like congruency as well as other geometric properties. Features like this are what promote some of the difficulties when trying to teach angles.

Overall, research involving the conceptualization of teaching of angles is a dynamic field. The research has been approached from numerous different directions, involving many different definitions for an angle or angle categorization. Researchers have tried to answer questions such as, how teachers should first introduce angles, whether angles should be presented as abstract or physical, what definition of angle is most appropriate or best suited to ensure students' understanding, what tools teachers should use and whether there is one method more suited than another. It seems that the notion of angles and hence the notion of teaching angles is such a dynamic concept in itself that the research possibilities might never be completely exhausted.

1.2. The *van Hiele Model* of Geometric Thinking

Description of the van Hiele Model

In the 1950's in Netherlands, a married couple of mathematics teachers started to question why it was that so many high school students struggled with Geometry. Pierre van Hiele and Dina van Hiele-Geldof published companion dissertations in 1957, describing their research and insightful findings. Their work came to be known as *The van Hiele Model of Geometric Thinking.* This forthcoming description of the *van Hiele Model* of Geometric Thinking was mostly taken from the literature provided by the NCTM Monograph Number 3, "The *Van Hiele Model* of Thinking in Geometry Among Adolescents" (1988).

Central to the *van Hiele Model* is the concept that there are five levels or stages (level 0-4) of Geometric Thinking that any one student could be at. These levels are assumed to be sequential and in order to ensure success at a higher level, a student should be successful at all of the levels below it. From their experience as teachers and researchers the van Hieles found that the majority of high school Geometry was being taught at a very high level of Geometric Thinking, while at the same time the students did not have the needed understanding of the lower levels. Overall, the van Hieles believed that if teachers were able to understand the five levels of Geometric Thinking, they would be able to assess at what level their students were at and adjust their teaching accordingly. As well, the *van Hiele Model* allows educators to develop lesson plans that move through the different levels sequentially.

The basic idea underlying the van Hiele levels is that each next level focuses on properties of the objects of attention in the previous level. Therefore, it is conceivable to generalize the van Hiele levels to topics other than Geometry. Understanding the levels in general is the first step to being able to identify the characteristics to look for when using the model to describe other topics in mathematics. It is this generalization that is targeted in the description of the levels below.

Level 0: Visualization Stage

Level 0 is the base stage. It is the first encounter with the objects of the mathematical domain. These objects will function as the foundation elements of everything that will be studied.

As the name of the level describes, comprehension at this stage involves visualizing these base objects. The visualization that is defined at this level can be described in terms of the students seeing or understanding these initial objects in their minds. For example, a base object in the domain of Real Numbers for students could be the number line. In the domain of Linear Algebra, the base objects would be vectors, perceived as, for example, directed segments, or n-tuples, and matrices, perceived as rectangular tables of numbers.

As any elementary teacher will know, it can take a few years of school for students to come to visualize real numbers in a number line format. Likewise, perception of an ordered pair of points, or an ordered list or array of numbers is not something that occurs to an untrained eye and mind. Thus, Level 0 objects are not assumed to be obvious or trivial for students. Their introduction requires serious teaching effort.

Geometry in elementary school begins by recognizing geometric characteristics in objects that can be physically seen. The Level 0 objects here are geometric shapes, such as circles, squares, triangles, straight lines, etc. At this level students are assumed to be able to categorize geometric shapes by visual recognition, and know their names. For example, if shown a picture of a square a student would be able to say that it is a square because it looks like one for him or her. A this stage, it is not required to think of a square, or any other geometric object, in terms of its properties, like saying a square has four sides. With visual recognition a student would be able to make a copy, by drawing or using some sort of physical manipulative, of a shape or configuration of shapes if they could be shown or told what it is they were supposed to be copying. The instructions would have to be based on the name the child has memorized for the object and not the object's properties. For example, the instruction would have to be, 'draw a square', not 'draw a polygon with four equal sides and angles'.

Level 1: Analysis Stage

At Level 1, students begin to analyze objects that were only visually perceived at Level 0, identifying their parts and relations among these parts. The focuses of attention are properties of these objects. Using the example of the Real Numbers again, it is at this stage that properties of order, as well as closure under operations could be noticed, leading to distinguishing subsets of Integers and Rational Numbers inside the set Real Numbers.

In elementary geometry, the analysis stage is where students begin seeing the properties associated with the different shapes or configurations. A square will now become a shape with four equal sides and four right angles or a parallelogram will became a shape with four sides where opposite sides are parallel, and having opposite sides and opposite angles equal, as well as having the diagonals intersect in their middle. However, at this stage, it is not assumed that students will be seeking logical relationships between properties such as knowing that it is enough to define a parallelogram as a quadrilateral with parallel opposite sides and all the other properties follow. Neither is it assumed that students will think about a square as a special type of parallelogram. Therefore, students will identify shapes and configurations based on the entirety of their properties. In other words, relationships between shapes and configurations remain solely on the list of properties they have. At this stage if a student were asked to describe a shape or configuration, the description would be based on the objects properties. At the same time, if a student were asked to reproduce a shape or configuration based on the list of properties, they would be able to do so. Students would also be able to use the properties of a shape or configuration to solve some simple geometric problems. For example, knowing the property that the angles of a triangle add up to 180°, the student would be able to deduce that the angles of a quadrilateral add up to 360° since a quadrilateral can be made by putting two triangles together.

Level 2: Informal Deduction Stage

A full understanding of Level 1 is concluded when the objects of study have become the properties of the base elements, introduced at Level 0. At Level 2 the focus of attention are "properties of sets of properties", that is, relations among properties. Students functioning at this level will try to group properties into subsets based on relationships between them. They will aim to recognize properties that are equivalent in certain situations and also be able to recognize the minimum amount of properties needed to describe one of the initial base elements. Overall, the main focus here is on the many different mathematical relationships between the properties. Finding and understanding these relationships is a type of informal deduction.

For the Real Numbers, it would be at this level that students would start to develop the idea that some properties of operations and order in real numbers follow from a small set of basic properties, thus making a step towards understanding the axioms of the Real Numbers as an ordered commutative field. But it only at the next stage that they would be able to produce proofs of such informal observations. That's where using the tools and techniques of algebra would start to play an important role.

Understanding Geometry at the informal deduction stage is a big leap for most students. Students would now be able to place properties into sets and identify the minimum amount of properties needed. A square, which might have had at Level 1 the properties of four equal sides, four equal angles, equal diagonals and parallel sides would now be described with a smaller set of properties like four equal sides and four equal angles. From this, students would now start formulating definitions for classes of figures. For example, a triangle would be defined as an enclosed shape with three rectilinear sides and a right triangle would be defined as a triangle where one of the angles is a right angle (or two sides are perpendicular). Students would also be able to recognize subsets of geometric objects or figures. Rectangles and parallelograms would no longer be independent shapes. Instead, rectangles would now be a special type of parallelogram. Students would also start to recognize which properties were subsets of each other. For example, having four parallel sides and four equal angles infers that the diagonals are also equal.

One of the most important aspects of this level of Geometric thinking is that students start to think deductively about Geometry. At this level a student would be able to give informal arguments to prove geometric results. These arguments might follow the simple logic of something similar to stating that if angle A = angle B and angle B = angle C then angle A = angle C. Or given a triangle ABC, if the midpoint of AB was T and the midpoint of AC was S and if TS was parallel to BC then BC = 2TS. Students would also be able to justify arguments they are presented with informal logical relationships. Thus, at this level a student can use and give informal deductive arguments about previously known properties. As well, a student could use deductive arguments to discover new properties. Overall, students now start to recognize the importance of logic and deduction in Geometry.

Level 3: Deduction

The objective of Level 3 is the organization of the statements about relationships from Level 2 into deductive proofs. At Level 2, relationships among properties of the base elements were discovered. At Level 3 these relationships are used to deduce theorems about the base elements according to the laws of deductive logic.

Referring to the Real Numbers example, it would be at this level that students would be expected to prove, for example, that for all real numbers a and b, $(-a)(-b) = ab$.

At this stage students are ready to accept a system of definitions, axioms (or postulates) and theorems. What was previously informally proved at Level 2 using diagrams and informal arguments can now be formally proved using definitions and axioms. Students can now create the proofs from the axioms and only use diagrams or models as a support for the argument. Students also begin to recognize the need for undefined terms in Geometry, which can be a very hard concept to understand in a purely logical system. At this level students also start to identify and understand the difference between a theorem and its converse, and contrapositive. They would also be able to prove/disprove any of these relationships. Students would also be able to see connections and relationships between theorems and group these accordingly. Thus, we have now reached the level at which traditionally high school Geometry has been taught in North America.

Level 4: Rigor

Level 4 looks at relationships among the organizations identified at Level 3. In other words, the deductive proofs from Level 3 are now hyper-analyzed. This analysis looks for associations between the proofs. For example, at this level the questions of "are the proofs consistent with each other", "how strong of a relationship is described in the proof" and "how do they compare with other proofs" would be asked. The level of Rigor involves a deep questioning of all of the assumptions that have come before. This type of questioning also involves a comparison to other mathematical systems of similar qualities. For example, in Level 4 if we considered Real Numbers we would begin to compare them as a field to other fields in general. It is fair to say that this level is usually only undertaken by professional mathematicians.

Level 4 of geometric understanding is very rarely met by high school students and is usually only attributed to further postsecondary education at college or university, if even then. Here, students have the ability to work in geometric systems that are non-Euclidean and thus, the system usually is not able to produce a lot of visual models for recognition or rather that the models produced are not very useful and thus, focuses mainly on the abstract. Most of the Geometry done at this level is strictly theoretical, done on an abstract, proof-oriented basis. At this level students are able to compare axiomatic systems, like Euclidean and non-Euclidean. Students who have also reached this level are able to carefully develop theorems in different axiomatic geometric systems. Therefore, as was mentioned initially, this is usually the work of professional mathematicians and their students who conduct research in different areas of geometry.

As previously stated, the van Hieles began their research after they observed that the majority of high school students struggled with Geometry, even if other mathematical topics were easily understood. From their research, they concluded that most high school Geometry is taught at Level 3. The van Hieles were then able to deduce that most students did not have a good enough grasp of Geometry at Level 2 to be able to move onto comprehending Level 3. Thus, from the van Hieles research it can be concluded that more focus needs to be placed on the pre-deduction, informal argument stage of geometric thinking, with more emphasis on informal everyday reasoning in order to expect students to be able to succeed at the deduction level.

Overall, the *van Hiele Model* provides us with a unique learning theory that can be related to Geometry and other areas of mathematics as well. The hope is that teachers who understand the different levels of the *van Hiele Model* will be able to recognize the level their students are currently functioning at and adjust their teaching accordingly. At the same time, the van Hieles always professed that success is very much based on adequate teaching.

Research Involving the van Hiele Model of Geometric Thinking

The *van Hiele Model* of Geometric Thinking has shown up in a variety of different ways in research. Some researchers have used the *van Hiele Model* as the theoretical framework on which to base some of their research. An example of this was already mentioned above with Fyhn (2008), where the author categorized students' responses according to the van Hiele Levels. Mayberry (1983) designed tasks to see at what levels of the *van Hiele Model* pre-service teachers were functioning. In this research, like in Fyhn's (ibidem), the model was neither proved nor disproved but just accepted as an analytic framework.

Research that used the *van Hiele Model* as an accepted framework covers an assortment of different topics. For example, Senk (1989) discusses how secondary school students were tested at the beginning of the school year and at the end of the school year to judge at what level of the *van Hiele Model* they were functioning. This was then related to whether the students had the ability to compose appropriate geometric proofs. In another article, Burger and Shaughnessy (1986) discuss how students from grade one all the way to first year university were tested to ascertain at what level the students were functioning with regard to triangles and quadrilaterals. As the topics and combination of topics in Geometry are pretty much endless, the research possibilities in this area could potentially be never ending, although the usefulness of this is debatable.

A massive three-year research project that strictly focused on the *van Hiele Model*, was undertaken by professors at Brooklyn College, City University of New York. Their goal was to develop a working foundation of the *van Hiele Model* so curricula could be developed using the Model. This project was published as a Journal for Research in Mathematics Education Monograph under the title "The *van Hiele Model* of Thinking in Geometry Among Adolescents" (1988). As was already mentioned, the previous in depth description of the *van Hiele Model* was adapted from this research projects publication. Like the previously mentioned research, this research took the *van Hiele Model* as being the correct way in which to categorize Geometric Thinking.

Alternatively, some research has not taken the *van Hiele Model* as being correct and instead tried to prove whether it is or is not an appropriate model of Geometric

Thinking. An example of this approach can be found in (Gutierrez, Jaime & Fortuny, 1991). Here the researchers propose an adjustment to the *van Hiele Model* that they believe will improve its use as an analytic tool. Thus, the different levels of the *van Hiele Model* are subdivided into even more categories with the goal of reflecting more accurately exactly where specific students are functioning. This is just one example of how the *van Hiele Model* has been scrutinized or adjusted in different research settings.

In general, the *van Hiele Model* has been used in research as an analytic tool; it has been used to prove the possible correctness of research results; it has been applied as a foundation for developing curricula and it has been scrutinized and adjusted to fit different researchers projects. Overall, it is a topic that has been very thoroughly studied in mathematics education research and covered in a lot of available literature.

1.3. Realistic Mathematics Education

This section will first describe the RME approach to teaching and then survey the literature on this approach.

Description of Realistic Mathematics Education as a theory of teaching

Mathematics education researchers have long felt that the traditional methods of teaching mathematics are not best suited for a student's education. This is at least the case from a Western Culture point of view. Traditional mathematics education is sometimes referred to as mechanistic mathematics education (van den Heuvel-Panhuizen, 2001, p. 1). Students tend to be introduced to mathematics as though it were a ready made system of symbols and solutions, where they mechanically drill problems by mimicking the procedure the teacher has demonstrated for them. Thus, mathematics becomes a very mechanical process in which a true understanding is never really achieved.

Realistic Mathematics Education is the Dutch response to such mentioned traditional mathematics education techniques. Founded from the works of Hans Freudenthal, Realistic Mathematics Education (hereafter denoted as RME) places an emphasis on making mathematics education a relatable, usable experience for the students. According to RME, this is achieved by emphasizing the connections between the real world and the mathematical world. By staying close to reality, educators are able to give mathematics a sense of value in which students can find real life associations. If students are able to see how mathematics applies in real life situations they might better understand the importance of the topic. It is important to mention here that these realistic situations that RME proposes do not refer to the so-called "everyday life" contexts. It refers to situations that are "experientially real" to students, and these include mathematical situations they are familiar with, and which they see as making sense and important. For a graduate student in mathematics, groups, rings and fields may be "experientially real" objects; they are usually not part of the reality for high school students. One researcher described the RME teaching approach as the "emphasis on making something real in [the students] mind" (van den Heuvel-Panhuizen, 2001, p. 3).

Freudenthal believed that a true understanding in mathematics would come through the process of 'mathematizing' knowledge. Mathematizing or mathematization can be defined as the process of "organizing from a mathematical perspective" (Gravemeijer & Doorman, 1999, p. 116). In other words, as a student takes the knowledge of mathematics and organizes it to be able to think about it in a mathematical perspective the student will develop a true understanding. Having a true understanding of concepts in mathematics will almost guarantee the student the ability to apply the concepts successfully. Since Freudenthal initially developed the idea of mathematization, it has been formulated into two distinct types, namely horizontal mathematization and vertical mathematization. Marja van den Heuvel-Panhuizen describes these two types in the following way: "In horizontal mathematization, the students come up with mathematical tools, which can help to organize and solve a problem...Vertical mathematization is the process of reorganization within the mathematical system itself, like, ... finding shortcuts and discovering connections between concepts and the strategies and then applying these discoveries" (van den Heuvel-Panhuizen, 2001, p. 3). Overall, the key features of RME can be summarized as mathematizing or thinking mathematically, about reality-based mathematical problems.

The teaching theory of RME can be shown to have five main characteristics, all of which include some form of the key features mentioned above (van den Heuvel-Panhuizen, 2001). First among these is the dominating use of context problems in the teaching/learning process. In other words, the mathematical concepts are taught using applications from real life situations, or experientially real situations. This differs from traditional methods of mathematical education in that context problems or applications were usually only found as end-of-the-chapter problems, where it was implicitly assumed that students are supposed to apply the concepts and theorems presented in the chapter, but were not expected to develop any new ones. Also, in the practice of teaching, such context problems tended not to be focused on and were often just considered for *extra credit*. Thus, the shift to using context problems as the initial tool to convey the mathematical concepts is quite a radical methodological change from the traditional way of teaching. The major reason for this focus on context problems goes along with Hans Freudenthal's belief in instilling human value or importance in mathematics education. If the students are able to relate to the context problems, if they recognize that the problems could potentially appear in real life situations or are experientially real, then they are better able to accept the importance of learning how to solve such problems. This use of context problems is thus used to apply an element of realism to the mathematics process.

Another role of context problems is to help students in the reinvention process of mathematics. This emphasis on the reinvention of mathematics is a characteristic that will be discussed later. For now, it is sufficient to say that context problems facilitate the processes of reinvention. As the students come into contact with a context problem they examine the knowledge that they already have, apply this knowledge where they can and generally, reinvent a method for solving this new mathematical situation. This very much mimics the methods mathematicians have followed throughout history when presented with new mathematical problems to solve.

Some examples of successful use of context problems in elementary school can be found in van den Heuvel-Panhuizen (2001) paper. The examples include the use of context problems in introducing long division to students who have no previous experience with division. One problem, which the author names the "Stickers Problem" is as follows: "342 match stickers are fairly distributed among five children.

How many does each of them get?" (van den Heuvel-Panhuizen, 2001, p. 6). This is a context problem to which elementary school children can definitely relate. It involves a familiar item (match stickers) and a familiar process (fair or equal sharing of objects among several people). Thus, there is little question as to whether young school aged children can find human value to this type of question. The problem requires students to mathematize this familiar situation.

On a different note, if we consider Geometry there is an endless amount of context problems available. Geometry initially developed from physical (realistic) situations. An example of how to introduce students to the idea of thinking about Geometry in a non-traditional sense is "Rush Hour Geometry" (I have also heard it called "School Bus Geometry" or "Taxi Cab Geometry", which are all essentially identical). Rush Hour Geometry takes place on a grid of city streets, thus placing the constraint that you must stay on the road (no driving through backyards or buildings) to get from one point to another. As students start to investigate geometric concepts like circles, midpoints and distances between points relative to this new notion of distance, they come to recognize that these geometric concepts look and behave differently in the world of Rush Hour Geometry than in a Geometry that takes place on a surface or a plane. This situation has strong connections to both the everyday life experience and the mathematical experience of students, yet, at the same time, it gives students a chance to go beyond this experience and understand it better by becoming aware of the existence of different notions of distance.

The next key characteristic of RME is the development of models as part of the mathematics education. Using models might not seem like a new characteristic of mathematics, but RME does not just use models, it develops the models in a very specific way to help facilitate the learning processes. When a student is first presented with a context problem, they are encouraged to use the knowledge they have to develop a method for solving the problem. This is where the first type of model is introduced. Here students work out an answer using a particular model of the situation in the given, concrete problem. Here it is interesting to note that when students use the knowledge they already have to obtain a model of a problem, they are undertaking horizontal mathematization. Once this initial model of the context problem is developed students are encouraged to refine the process over a number of steps until they obtain a model that will solve other similar problems. This has been termed as going from the "model of" to the "model for" stage of mathematical learning (van den Heuvel-Panhuizen, 2001, p. 4). Also, being able to go from a model of a particular question to a model for a general concept undertakes the process of vertical mathematization.

We can use the previously mentioned examples to demonstrate this idea more fully. In the situation of the Stickers Problem, van den Heuvel-Panhuizen (2001) illustrates how students begin solving this problem by developing a model of the particular problem. Students begin by repeatedly allocating stickers to five positions on a table until they would have exhausted the pile of 342 stickers and were left with 2 stickers. They would count the number of stickers in one of the five equal groups they thus formed, or they would count the number of times they had subtracted 5 from 342, where division becomes understood as repeated subtraction. Both are time consuming procedures. So some students start keeping tract of the process in a symbolic graphical form (van den Heuvel-Panhuizen, 2001, p. 6, Figure 3), and subtracting multiples of 5, not just 5 each time. For example, they subtract 10 times 5 or 50, from 342, obtaining 292; then subtracting 50 times 5 from 292, obtaining 42; and then subtracting 8 times 5 from 42, obtaining 2. Adding the number of times 5 was subtracted, 10+50+8 gives 68 and the remainder is 2. Students here use only previous knowledge they have learned, namely grouping and repeated subtraction, as well as their understanding of the concept of fairly distributing, to develop a model of this context problem. These are their "models of" the given problem.

The next step involves teacher intervention aimed at engaging students in the process of "progressive schematization" of their strategies to arrive at a "model for" solving this type of division problems, namely the strategy of long division.

The process is generally the same if we consider Rush Hour Geometry as a context problem that can introduce the concepts of alternative metric spaces in Geometry. Initially, students begin with the grid world of Rush Hour Geometry and investigate common geometrical definitions. Once these geometrical definitions have been shown to produce different results or characteristics in Rush Hour Geometry than in ordinary plane Geometry, a model with the attributes of Rush Hour Geometry can start to be developed. Following this process and investigating other metric spaces or by placing different types of constraints on Rush Hour Geometry (roads that are closed, time constraint when traveling a certain distance, etc.) the student can begin to develop a model for understanding metric spaces in general. Thus, the student has moved from a model of one type of geometrical space to a model for defining and working with many different types of geometrical spaces. Unlike the long division example that seems very systematic and straightforward, this example of Rush Hour Geometry might not seem to have the same simplicity. It must be remembered that Geometry is a very unique type of mathematics that constantly requires an understanding of the space in which the problems are taking place. Thus, Geometry examples might tend towards the more abstract.

As was mentioned previously, another of the five characteristic of RME is the trend of student reinvention. The fundamental idea of student reinvention is that students undertake the same process of investigation that mathematicians initially took when certain concepts were first discovered. By following along the same path as mathematicians, it is believed that students will be able to come to a better understanding of the concepts. This is in contrast to the usual approach where what mathematicians discovered is the starting point for students' learning and understanding of mathematics. The student contribution comes from students using their previous knowledge as well as the techniques they know for solving specific problems to help reinvent a new method for solving the new problem.

It is thus rather clear why context problems play such a central role in the process of reinvention. If we think about the historical processes of discovery we can see that in many situations, the desire to find an answer to a specific problem is because the problem arose in a real life situation. For example, the Egyptians became masters of finding areas of land plots when it became necessary to know the area of a certain amount of land for tax purposes (Coolidge, 1963, p. 9). These methods for figuring out the amount of taxes someone should pay were refined into general formulas for finding the areas of specific geometric shapes. Thus, we have the invention of area formulas. By using context problems students are able to retrace the steps of former mathematicians and reinvent the mathematical concepts for themselves. The process of reinvention also helps students travel through many steps of horizontal and vertical mathematization.

Considering once again the example of long division from van den Heuvel-Panhuizen (2001) we can see that once the students have developed the model for solving long division problems they would have actually reinvented the long division algorithm. Traditional mathematics education methods would usually dictate that the long division algorithm be the starting point for teaching long division. A teacher would demonstrate the method with some problems involving only numbers and would then expect the students to be able to mimic the method. By the method of reinvention the students are helped to develop the algorithm on their own and thus have a very solid understanding of how it works and where it can be applied. Another really good example is the use of triangle tessellations to help students rediscover that the angles of a triangle add up to a straight angle. This is accomplished by instructing the students cover an entire area with identical triangle tiles (leaving no gaps). Then, usually with some teacher direction, students are asked to find some straight lines throughout the tiled area. It will be discovered (through a process of discussion and investigation) that the straight lines are produced whenever the three different angles of the triangles meet. Thus, through reinvention, the students can now physically see that the angles of a triangle add up to a straight angle.

All of the previous characteristics discussed would not amount to much if RME did not also emphasize the characteristics of strong interactive learning processes. In a traditional mathematics education setting students are usually left to work alone and silently in their individual workspaces without any interaction whatsoever (unless it involved asking the teacher a question). In a RME classroom, student interaction is a necessary and encouraged part to learning mathematics (although usually strongly supervised and sometimes redirected by the teacher). It is easy to recognize that most mathematical problems can be solved with a variety of different techniques, some more efficient than others. To be able to move through the processes of developing a model of a problem to refining a model for some general problems, students will pass through a number of different techniques for solving specific problems. It is too much to hope that each individual student will be able to come up with every step of the solution by him or herself. Instead, some sort of external stimuli is usually necessary to help students make these cognitive leaps. By utilizing student interaction, students can have the opportunity to learn from their peers. This can help students who are solving a problem in a less efficient manner recognize the process for solving the same problem in a more efficient manner. It is thus very evident that the interactive learning process is one of the main characteristics in the progression of vertical mathematization.

Referring one more time to the long division Sticker Problem example of van den Heuvel-Panhuizen (2001), it is easy to believe that in a classroom of students the problem could initially be undertaken in many different ways. While most students will begin by repeated subtraction of small numbers of fives, some students might be able to initially comprehend that using bigger numbers will limit the number of steps the process will take. When each method is then discussed in the classroom setting, the students who did not use bigger numbers at first will now have the opportunity to experiment with this technique and draw their own conclusions.

The last defining characteristic of RME is its unique tendency to focus not only on the micro-didactic perspective of the learning process but on the macro-didactic perspective as well. This characteristic stems from the concern that teachers focus too much on meeting a list of educational objectives they are required to teach to their specific grade level while giving no thought as to what has come before or what might come after. With respect to mathematics, this means that a teacher would focus on the specific topics in mathematics that need to be taught to the grade level they are teaching as a separate piece of knowledge with no foundation whatsoever. Thus, with this method, as the school years progress, students tend to see mathematics as individual chunks of knowledge, or random facts, which are only vaguely related. In contrast, RME emphasizes the need to demonstrate to students the connections within mathematics. This might not seem like a specifically unique characteristic as the assumption can be made that teachers should have this goal in mind already. But with the constraints placed on educational systems it is very common to find that teachers only focus on the present material without much thought given to the future mathematics the students will study in the coming years. RME curricula are written to incorporate what has been taught previously and to leave space available to add to the knowledge in later years. This type of long term curriculum planning hopes to allow students to recognize recurring patterns of previous knowledge. The goal of providing a long-term perspective on the learning/teaching process of students is put in place with hope to help students move with ease from one subject to the next in mathematics.

Overall, RME is a teaching theory that combines the aspects of the short term teaching goals with the long term teaching goals. It focuses on how students can play a role in their own education and encourages interaction and questions. This interaction also provides the teachers with many opportunities to judge whether or not the students actually understand the concept that is being taught. Thus, alone the many characteristics of RME do not have much to offer the education setting, but combined in the proper way they provide a very "mathematized" approach to teaching mathematics.

Research involving Realistic Mathematics Education

Literature related with *Realistic Mathematics Education* suggests that most research has been conducted to determine whether using context realistic problems is, in fact, helpful. For example, the research outlined in (Cooper & Harries, 2002), proposes that children are suited for context or realistic problems as long as they are written in such a way to persuade appropriate realistic responses. Thus, Cooper and Harries support *Realistic Mathematics Education* as long as the initial context problems are suitable for the situation. Similar to Cooper and Harries, Marja van den Heuvel-Panhuizen also conducted research on the approach to take when choosing context problems. In (van den Heuvel-Panhuizen, 1994) she outlines how improving the context problems that students are asked to solve will improve a teacher's ability to assess at what level the student is working. Treffers (1993) also studied finding appropriate or improved context problems. In this article Treffers makes obvious how a simple newspaper article can be used as a very good context problem. Thus, Treffers attempts to demonstrate a successful way in which to develop an appropriate context problem from a realistic situation, just like the ones searched for by Cooper and Harries (2002) and van den Heuvel-Panhuizen (1994).

Realistic Mathematics Education research has also looked at the aspect of developing models of context problems. This is a vital part of the teaching theory of *Realistic Mathematics Education.* Another article by van den Heuvel-Panhuizen (2003), discusses how the use of context problems will come to nothing if students do not develop appropriate models with which to expound the initial problems. According to this research, models of the initial problem must be developed in such a way that they can be adjusted and formed to fit the general situation represented by the context problem.

Lastly, some literature has been devoted to the implications of implementing the teaching theory of *Realistic Mathematics Education.* Wubbles, Korthagen and Broekman (1997) investigate the changes taken to prepare prospective teachers to be able to teach according to the principles of *Realistic Mathematics Education.* Since *Realistic Mathematics Education* has become the predominate theory for teaching mathematics in the Netherlands, preservice teaching programs had to be adjusted to accommodate these changes. These researchers found that while most prospective teachers quickly warmed up to the concept of teaching using realistic or context problems, they struggled at being able to help students move through the stages of model development, which is deemed a necessary part of *Realistic Mathematics Education.* Overall, the literature suggests that there are still many more topics that could still be studied with regard to *Realistic Mathematics Education.*

CHAPTER TWO

How the history of geometry, and especially the history of the concept of angle, supports the principles of RME and the van Hiele Model of geometric thinking

There are many external influences that affect the development of any type of knowledge. This can be shown and will be shown in this chapter to be the case even with the historical development of Geometry. The following interpretation of the history of Geometry was composed from facts taken from the resources which include "A History of Geometrical Methods" by Julian Lowell Coolidge (1963), "Space Through the Ages: The Evolution of Geometrical Ideas from Pythagoras to Hilbert and Einstein" by Cornelius Lanczos (1970), "The Geometric Viewpoint: A Survey of Geometries" by Thomas Q. Sibley (1998), "Euclid's Window: The Story of Geometry from Parallel Lines to Hyperspace" by Leonard Mlodinow (2001), "Geometry: Euclid and Beyond" by Robin Hartshorne (1997) "Mathematical Visions: The Pursuit of Geometry in Victorian England" by Joan L. Richards (1988) and "The Changing Shape of Geometry: Celebrating a Century of Geometry and Geometry Teaching," Edited by Chris Pritchard (2003).

The development of Geometry was immensely influenced by the different world cultures during its time, like most knowledge has been throughout history. With this comes the fact that Geometry knowledge has developed in stages. In this chapter I intend to demonstrate, among other things, that the historical development of Geometry very much follows a pattern similar to the five stages of the *van Hiele Model* of Geometric Thinking. At the same time, it is possible to take ideas from the historical development and apply them to the teaching of Geometry in modern day classrooms in order following the stages of van Hiele. These examples can also help facilitate the development of context problems needed when teaching according to the methods of RME, as RME is strictly based around using context problems. Throughout history, Geometry has made developmental leaps when certain context problems needed to be solved. Thus, the history of Geometry can provide ample sources of context problems that were successful in developing the understanding of Geometry at one time and might have the potential of still being successful now. Overall, the history of Geometry and the historical development of Geometry can provide an abundant amount of learning and teaching material to any educator as long as the educator with these histories.

2.1 Ancient Geometry (Pre-Euclid)

Ancient Geometry was a science of observation and experiment. There is a limited amount known about ancient, pre-Euclid Geometry, because of the lack of records that would have survived to our day. Those records that survived can give anthropologists and mathematicians a lot of insight into what a certain society actually knew in relation to Geometry. Deductive proofs were not a characteristic of Geometry found in early civilizations' records. Instead, many records pertaining to Geometry discuss ways of solving specific problems and then comment on how similar problems can be solved in the same fashion. This is similar to the RME teaching method of using models to solve a specific problem and then developing the model to solve similar problems. Approximation was a very common approach among ancient Geometers; with every culture having its own approximation for π. Some records, not just from the same society either, have very significant mistakes in them as well. This demonstrates to us that Geometry was still in a development stage (or at the very least the authors of the surviving documents were). There has also been speculation that certain cultures may have influenced each other's knowledge of Geometry. Almost all cultures in one way or another knew the Pythagorean Theorem, but the question is whether each culture developed it from their own experimentation or observation or whether it was a result transferred over from a sharing of knowledge with another culture. A few specific cultures are mentioned below in more detail but it should be kept in mind that many other cultures probably had similar knowledge of Geometry even if records have not survived to this day. From simply examining how African tribes built huts with perfect right angle corners or how Inuit built igloos from blocks laid in a spiral it is easy to see that Ancient Geometry knowledge is probably not limited to only what we have on record.

Babylonian Geometry

The earliest records in the field of Geometry come from the Babylonians. These are clay tablets from approximately 3000 B.C. The dates of these records are estimates with a margin of error of $\pm$ 500 years (Coolidge, 1963, p.5). These records show significant knowledge in the field of Geometry. They also show that the Babylonians were generally interested in studying and recording their findings in Geometry. These Babylonian records deal with a lot of plane Geometry, showing measurements such as finding areas of regular polygons, subdividing irregular plots of land and measurements of quadrilaterals (Coolidge, 1963, p.5-8). The Babylonians had formulas with two variables, for rectangles, and three variables, for parallelograms. There are also some records pertaining to volumes, but not as many. One example of this is finding the volume of a basket (Coolidge, 1963, p.7). Some surveying and area problems even lead to systems of linear equations (Coolidge, 1963, p.5). The Babylonian had an idea of the Pythagorean Theorem and understood the essential principle behind it (Coolidge, 1963, p.8). They also took π to equal 3 in calculations, but were aware that this was an estimate (Coolidge, 1963, p.6). Probably the most remarkable feature is the fact that the Babylonians solved all of these problems with a base 60 numerical system, showing their extraordinary arithmetic abilities (Lanczos, 1970, p.8).

The clay tablet records that have survived represent probably only a fraction of what the Babylonians actually knew. But what they do show can be pieced together to demonstrate what the Babylonians felt was important. All of the knowledge of Geometry that we can attribute to the Babylonians falls into the first two levels of the *van Hiele Model*, namely Level 0 and Level 1. Babylonian Geometry deals with shapes or objects in a flat (Euclidean) universe. Thus there must have been an understanding of the visual aspects of all of these shapes or objects or a Level 0 comprehension. As well, the activity of finding areas, volumes and doing general measurement can be attributed to Level 1 of the *van Hiele Model.* Measurement is an activity usually attributed to a deeper analysis of the Geometric figure or objects that are of concern. It can also be assumed that the Geometry that the Babylonians undertook was done so for practical, real life application purposes. There was probably no reason to explore further (if Geometry was explored further no records have survived). Measuring areas of land can help when knowing how much land each person has and so forth. Thus, we are provided with an idea for a context problem.

Egyptian Geometry

The records of Egyptian Geometry have survived to our day on a few different pieces of papyrus. These records are dated between 1900 and 1800 B.C. (Coolidge, 1963, p.9). Like the clay tablets of the Babylonians, a lot of Egyptian records concern plane Geometry and problems of area and volume for shapes or objects, again all in a Euclidean universe. For example, one papyrus discusses the area of a rectangle as well as a formula for finding the volume of a truncated pyramid with $V=1/3h(a^2+ab+b^2)$ (Coolidge, 1963, p.9). There is also mention of right triangles with 3, 4, 5 side measurements and many more such "Pythagorean triples", showing at least a beginning understanding the Pythagorean Theorem. One of the best examples of the Egyptians using Geometry for practical purposes occurred in their surveying of land. Every farmer was allocated an equal amount of land along the Nile. But the Nile tended to flood a lot and recede to different levels or flow along different paths. Thus the Egyptians would resurvey the land and reallocate the appropriate amount to each farmer (Coolidge, 1963, p.8-9). This alone shows an extraordinary understanding of measurement and plane Geometry. The Egyptians also had a much more accurate estimation of π, taking it to be $(16/9)^2$ which is approximately 3.16. Despite all of this, even if no records had survived at all we would still be able to conclude that the Egyptians were wonderful architectural Geometers by simply examining the complex, yet painfully accurate, constructions of the great pyramids.

It would be easy to understand that Egyptian Geometry grew out of an observation from the world around them by their desire to create geometric structures. A very simple context problem could be developed using the idea of a river changing course and needing to re-divide land. Students would be able to relate to a problem such as this and history has shown that it was once successful in helping people discover the mathematics needed to solve these problems. From the viewpoint of the *van Hiele Model*, the Egyptians were very much working at the Level 1 stage of Analysis. Beginning deductive thinking was not necessarily needed for the activities for which the Egyptians used Geometry. Understanding the properties of the objects and shapes was enough for them. Therefore, the Egyptians seemed to have no desire to move beyond the practical uses of Geometry as it was not culturally necessary.

Indian Geometry

Indian records about Geometry date back to approximately 800 B.C. (Coolidge, 1963 p.13). Like the Babylonians and Egyptians, the Hindus had an understanding and interest in areas and volumes of shapes and objects, again in Euclidean space. This understanding was well developed and the Hindus used it to help solve other problems of a similar nature. In India, the practical side of Geometry was used mostly in the construction of sacrificial altars and other religious artifacts, which could be likened with the Egyptians' use of Geometry to build pyramids as tombs for their pharaohs. This was done by the technique of rope stretching or "rules of the cord" (Coolidge, 1963, p. 13). For example, finding a right triangle or right angle was done by stretching ropes of particular lengths. This also provided records of the Pythagorean Theorem. Unlike the previously mentioned cultures, the Hindus have recorded more than one approximation for π. The oldest records give $\pi = 3.088$. This value was calculated from the problem of constructing a circle with area equivalent to the area of a given square. According to Coolidge, "When it comes to constructing a circle equivalent to a given square, we are told to increase half the length of one side by one-third of the difference between itself and half the length of the diagonal, an easy, but inaccurate construction with gives $\pi = 3.088$" (Coolidge, 1963, p.15). Then there are records dating from approximately the 6th century A.D. which give π as equal to 62,832/20,000 = 3.1416. There is however, a lot of speculation as to whether this result is original or if it was influenced by Greek mathematics. Nevertheless, it is the best approximation of π by any non-Greek society.

The example of the Hindus using Pythagorean Theorem to solve similar problems is a step towards the Level 2 stage of the *van Hiele Model.* At the Level 2 stage the deep understanding of the properties would allow the geometer to relate the situation to a similar situation. This characteristic of pre-deductions shows that the Geometry knowledge is coming together into a comprehensive system and is no longer just a collection of unrelated facts. The other example of the Hindus changing their calculation of π shows that they were not satisfied with an approximation and wanted more accuracy. Such work would also reflect the pre-deduction stage of the *van Hiele Model*, demonstrating a move towards accuracy and consistency and a move away from the purely practical sphere.

Chinese Geometry

In 213 B.C. the Emperor of China decreed that all books in the Empire were to be burned (Coolidge, 1963, p.19). Therefore, although we can assume that it would be hard for the Emperor to enforce this completely, it is really difficult to be sure if something from China is dated correctly if it is older than 213 B.C. Despite this misfortune to record keeping, we can still confidently point out some strong points in Chinese Geometry. For one thing the Chinese were and still are excellent Astronomers. Instead of using the Earth, the Chinese used the stars to trace out shapes. They studied right triangles, quadrilaterals and many other objects from patterns in the stars. The Chinese are also the only culture, other than the Greeks, to give an attempt at a proof for the Pythagorean Theorem, although a different type of proof than the Greek undertook. The picture in Figure 2 shows the main idea of the proof.

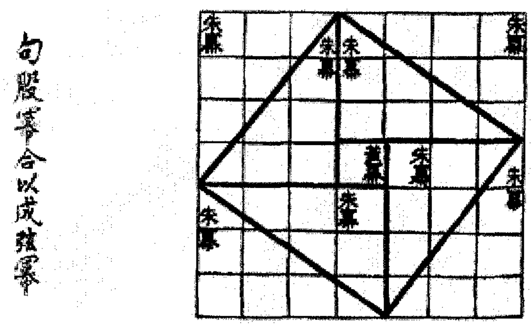

Figure 2. Hsuan thu – the diagram used to demonstrate the Pythagorean relation (cf. Swetz, 1994, p. 323)[1]

This proof is a simplified informal deductive proof using diagrams (like the Hsuan thu diagram). It uses a special case of the right triangle of dimensions 3, 4, 5 (Coolidge, 1963, p. 20-21), but the diagram schematizes the reasoning so that it can be generalized to any right angled triangle. If we look at the above diagram we see that the oblique square is made of four congruent right triangles and a little square in the middle, whose side is the difference between the sides of the triangles. Let us use

[1] This picture was downloaded from http://en.wikipedia.org/woki/Pythagorean_theorem. This picture is similar although not identical with Figure 6 in the cited chapter by Swetz.

modern letter notation and algebra, to explain the main idea of the proof. Suppose the sides of the triangles are a, b, ($b \geq a$) and the hypotenuse is c. Then the side of the little square is $b - a$, because the oblique square is inscribed in a large square of side $a + b$. So $c^2 = 4(1/2)ab + (b - a)^2 = a^2 + b^2$.

The records also indicate that the Chinese mathematicians were satisfied using 3 for π, which is not surprising given the practical context of land surveying and elementary engineering work in which it appeared (Coolidge, 1963, p. 21). One interesting, unique feature of Chinese Geometry, however, was their interest in similar triangles and the many complex results they could obtain from this.

This brief account of Chinese Geometry shows that these mathematicians were able to progress through a few levels of the *van Hiele Model.* From discovering or visualizing shapes to making informal proofs and relating similar triangles, the Chinese moved from Level 0 through Level 1 to Level 2, at least when it relates to these few topics. Why the Chinese were able to advance to Level 2, while other cultures were satisfied staying at Level 1 could probably be attributed to their culture. The Chinese were philosophers in their own way and sought after knowledge not just for its practical applications. One idea we can take from the Chinese and use in the classroom as a context problem is the idea of connecting points to make shapes. This is similar, although not exactly the same, as connecting stars. Already many early elementary school classrooms do this by using pegboards and rubber bands. With a bit of teacher instruction and expansion on the activity, the geometric concept of area could be developed out of the use of pegboards.

2.2 Greek Geometry and Euclid

According to the surviving records, the Greek approach to Geometry was the first of its kind. Using the knowledge from the Egyptians and Babylonians that was for practical purposes as well as some knowledge they had discovered on their own, the Greeks developed a deductive system of mathematics based on axioms, theorems and most importantly, proofs. Geometry was no longer strictly an applied science of the physical world. For the Greeks, Geometry represented fundamental truths of the universe. Many elements of the Greek culture influenced their need for mathematical proofs. The Greeks tended to be "thinkers" or philosophers who would debate ideas using arguments, counter-arguments, and *reductio ad absurdum* forms of reasoning This different way of approaching thought allowed the Greeks to abstract to the appropriate level needed to look at Geometry in a deductive perspective. For example, it was no longer necessary to represent a rectangle by the base of a house; there could just be an abstract rectangle. The Greek philosophers organized themselves into "schools" dedicated to the study of philosophy of nature and Geometry. These schools, including those of Pythagoras (approximately 500 B.C.), Plato (429-348 B.C.) and Aristotle (384-322 B.C.), helped guide Greek Geometry to its climax, around 300 B.C., of Euclid's *Elements* (Sibley, 1998, p.2-4).

Pythagoras has become a household name in western culture to anyone with at least a middle school education. One part of the history of Pythagoras' Theorem that is usually left out of a child's education is that the Theorem is attributed to a cult-like secret society, the Pythagoreans. The Pythagoreans were made up of the followers of the teachings of the philosopher and mathematician, Pythagoras. As it has already been shown, many cultures held an understanding of Pythagoras' Theorem, at least in its most basic applicable context. The reason this knowledge is attributed to Pythagoras is because of the simple fact that he was the very first person to prove it. That is to say, he was the first to provide a deductive proof based on logic, not diagrams. This reflects on our current mathematical cultural idea of mathematics needing proofs to be considered true and complete. Previous cultures, like the ones mentioned above, were content knowing that $a^2 + b^2 = c^2$, at least for specific number combinations, like 3, 4, 5 or 5, 12, 13, and used these facts successfully in architectural pursuits. Pythagoras offered a simple geometric proof based on the Greek system of deduction. Thus, it has been named the Pythagorean Theorem.

This theorem offered much more to the advancement of Geometry than simply its statement. Once a theorem is proved using deduction it is easy to conclude the truth of the statement in every situation. The Greek philosophers now used the Pythagorean Theorem to calculate the hypotenuse of a right triangle with sides both equal to 1. This lead them to the irrational number we now call $\sqrt{2}$, but which the Greeks did not understand (Sibley, 1998, p. 4). This brought with it all sorts of problems, mathematically and philosophically. All numbers previously known in mathematics were either whole numbers or fractions of whole numbers. The Greeks looked at numbers as ratios of magnitudes. Since $\sqrt{2}$ is not a number that can be written as a ratio of integers, the Greeks had never come across it before. Irrational numbers jeopardized the Greek thesis that all mathematics could be built from whole numbers. They needed a new foundation. This would eventually become Geometry combined with careful proofs (Mlodinow, 2001, p. 25-26).

The Pythagoreans would now adopt Geometry as their new basis for mathematical philosophy. They continued their studies and proved many more Geometry results. As time progressed, philosophers like Plato and his student Aristotle also embraced Geometry as essential to mathematics as well as appropriate for philosophical thought. Plato was known for teaching Geometry and encouraged Geometric proofs as vital for any philosopher (Sibley, 1998, p.5). Aristotle thought that all mathematics held its certainty in careful proof (Sibley, 1998, p.5). Thus, the Greeks took Geometric knowledge that had been around for over 2000 years and proved the accuracy of results around which many pre-Greek cultures had built a great deal of their applied science.

The new philosophy of proving Geometry continued in Greece and eventually led one Greek man to write a book, which would become one of the most famous mathematics books every written. Euclid lived in about 300 B.C. and very little is known about him other than the fact that he was a philosopher who loved Geometry and that he wrote down what is assumed to be everything the Greeks knew about Geometry at that time (Mlodinow, 2001, p.29). What was revolutionary about Euclid's book was not so much its content, but its format. Euclid's method defined the way in which we still approach mathematics to this day. Euclid never credited himself with any of the proofs in the *Elements*, but simply acted as the organizer as he systematized all of the Greek knowledge of Geometry. Thus, we do not know who truly discovered the majority of the proofs within the *Elements,* only who compiled them.

Despite its revolutionary impact, Euclid's arrangement of the proofs of the geometric facts is very simple. He begins by listing twenty-three definitions

(Hartshorne, 1997, p.27). These definitions covered everything from a point and line to the different types of quadrilateral figures. Some people have argued that a few of the definitions are not needed as they do not say much and the knowledge is just intuitively known. But Euclid's ultimate goal was to design a system free of guesswork, inexactness and preconceived ideas. Thus, we have the very precise definitions of terms never before thought to need a definition. This was Euclid's attempt to do away with any misunderstandings.

After the definitions come the five postulates and the five common notions. Another name used for postulates or common notions in mathematics is axioms. Axioms describe how to use terms and how they relate to each other. They do not need to be proved as they usually stem from logical common sense. Euclid probably separated his axioms into postulates and common notions because postulates are specific to geometry and common notions refer to general relations between magnitudes of any kind, whether geometric or arithmetic. There is also the fact that Aristotle had used this distinction before. Euclid's common notions are thus common to all of mathematics. They may also be seen as representing the most general common sense notions. For example, common notion #5 states, "The whole is greater than the part." (Sibley, 1998, p.289) It is hard to logically argue that this statement is false. The postulates, on the other hand, formed the foundation of the geometric content of the *Elements*.

As was mentioned above, axioms tend to make some sort of logical common sense. Euclid's first four axioms appear, at least on the surface, to be obviously true. This is not the case for the fifth postulate. This postulate is long-winded and wordy and has been nicknamed the "parallel postulate" over the years. For years mathematicians were convinced that it should not be a postulate at all but instead it should be proved from the previous definitions and axioms. Many mathematicians spent years trying to prove just that, with no success. It is now known that Euclid's fifth postulate cannot be proven (which will be discussed in more detail later). Mathematicians now assert that it is necessary to the nature of the Geometry Euclid describes and shows just how much of a genius Euclid actually was for including it as a postulate in the first place.

What we get as an end result of the definitions, common notions and postulates are 465 Geometric theorems all proved using only diagrams and logical reasoning. Armed with only these tools, as well as a straight edge and collapsible compass, Euclid changed the way the world would forever view mathematics. Mathematics had now crossed into the domain of the abstract. No longer did a line have to represent the edge of some physical object, a line could just be a line in a space. Mathematicians took Euclid's system of using logic and developed future mathematics in such a way that applications and examples were no longer considered creditable until they could be associated with a proof. This is the system that is still used today. As well, a lot of the essentials of Euclid's *Elements* are still seen as an accentual key part of a complete Geometry education.

The Greek development of deductive proofs is a textbook example of moving from the lower levels of the *van Hiele Model* to Level 3, the stage of Deduction. As was already mentioned, the Greeks used knowledge that had previously been discovered, as well as knowledge they discovered on their own to develop proofs. Thus, they followed the characteristic of the *van Hiele Model* of using prior understanding to move from one level to the next in sequential steps. As well, Euclid developed his deductive system exactly how the van Hieles describe a full comprehension of Level 3 (Deduction) should be developed. Of course, it can be speculated here that the van Hieles took these ideas of development from the success of the Greek system. At the same time, the van Hieles observed systems successfully being developed in such a way throughout their research experiments. Thus, according to the *van Hiele Model*, following the example that the history of Geometry has demonstrated up to this point could be a successful way to help students progress through the levels of Geometric thinking. The Greek history of Geometry also provides us with many ideas of context problems to initiate the development of models in the classroom. For example, following along with the Pythagoreans it can be demonstrated that the length of the hypotenuse found from a right triangle with two sides equal to 1 is a number previously unknown. This is an excellent way to introduce irrational numbers to students.

2.3 Post-Euclid to Analytical Geometry

Despite the phenomenal revolutionary advances the Greeks and especially Euclid made for the subject of Geometry, after the fall of the Greek Empire it took close to two millennia for Geometry to progress much further. There are many reasons for this delay in academic progress. The initial source can be thought to be the Roman Empire. The Roman Empire looked on mathematics as something that only held practical purposes (Mlodinow, 2001, p. 45-46). The Romans felt that mathematics was not a necessary component of education or worth their philosophers' time unless it could be used to help defeat an enemy Rome was trying to concur or improve exchange rates on international trade. Thus, any previous advances in abstract mathematics were disregarded. Euclid's *Elements* was translated into Latin but excluded any proofs (Mlodinow, 2001, p. 45-46). Only the theorems and their practical applications were felt to be of any worth. Where the Greeks had produced some of the best mathematicians ever known to mankind, the Romans produced none at all.

Even after the Roman Empire fell, abstract mathematical knowledge had been such an ignored and abused subject that it would take many centuries to heal those wounds of neglect. The Roman Empire ended (approximately 476 A.D.) at about the same time that the Catholic Church started to gain a true hold on the political and social dealings of European life (Mlodinow, 2001, p. 49). Anyone who felt brave enough to propose new theories in science, mathematics or astronomy was quickly silenced by the Catholic Church. Ironically, the Catholic Church did have one attribute that became a stepping-stone to the progression of academic freedom, their tendency to insist on at least a basic education for all of their clergymen. Charles the Great, a European Emperor who reigned during the late eighth and early ninth century, was a conquering monarch and imposed Roman Catholicism wherever he went (Mlodinow, 2001, p.60). With the new ideals of the Church also came the education and academic pursuits that the clergymen felt necessary in order to interpret the Bible appropriately. Thus, Church schools were established all across Europe, many of which would become the Universities we still know today (Mlodinow, 2001, p.61-62). Charles the Great had started a small boulder rolling, which, although it would take another 800 years, would eventually bring down a mountain.

Over the next 800 years academic importance in Europe would increase while the Church's stronghold over knowledge would slowly lessen as more and more people, not just clergymen, came to the Church schools to be educated. This would eventually help some great thinkers come up with innovative new themes in mathematics. Unfortunately, anyone proposing ideas seen as too radical, opposing the Bible or just heresy would bring the wrath of the Catholic Church upon their heads. Many people were skeptical and passive in their academic pursuits beyond the basics. On a positive note, Euclid's *Elements* was restored to include proofs and was considered necessary for a young man's well-rounded education (Mlodinow, 2001, p. 62).

One of these young men who got a "well rounded education" was René Descartes. Born in 1596, at the age of eight Descartes was sent to La Flèche, a Jesuit school (Mlodinow, 2001, p.79). According to the history related in "Euclid's Window" by Leonard Mlodinow (2001), Descartes was a good student and keen about mathematics, but always seemed bored with what he was learning. He was also skeptical, even at such a young age, that a lot of what he had been taught was either useless or mistaken. Upon finishing La Flèche and strictly to comply with his father's wishes, Descartes spent two years receiving a law degree. This, again, did not really interest him. Therefore he joined the army to try to find some adventure and give himself time to think about mathematics, his favorite subject.

Descartes' military career took him to many places in Europe and eventually led to Holland in 1618 (Mlodinow, 2001, p.80). On a notice board in a small town he was stationed in, there was a poster advertising a mathematical challenge to the public. Descartes mentioned out loud that it looked easy and an old man standing near by challenged him to solve it. To the old man's astonishment Descartes did solve the problem and thus began Descartes' relationship with the greatest Dutch mathematician of this time, Isaac Beekman; the old man (Mlodinow, 2001, p. 80-81). Descartes now had someone to correspond with and discuss the mathematical ideas that came to his mind. He credits Beekman for inspiring a lot of his studies and the letters between them are filled with references to Descartes' realization of a relationship between numbers and space (Mlodinow, 2001, p.81).

Another factor that helped motivate Descartes towards developing his system of Analytical (or Coordinate) Geometry was the fact that Descartes always liked to do as little work as possible. He was known to constantly complain about the Greeks and Euclid's proofs, finding them tiresome and too complicated. He felt that for such simple ideas there should be an easy way to prove they were true without having to do so much intellectual work. Descartes began his process of simplification by introducing a coordinate system into the plane. With a horizontal axis and a vertical axis, any point in the plane could be represented by a horizontal distance and a vertical distance or an ordered pair of numbers. Essentially, Descartes took the model of deductive Geometric proofs and changed and adapted it to fit into a new system of mathematics. It is interesting to note that this adapting of the model is basically a progression along vertical mathematization, as mentioned in the theory of RME.

This method might seem a long way away from Euclid's proofs but as Descartes developed his new system further he was able to find algebraic formulas to represent lines and curves in a space. With these algebraic formulas representing geometric properties Descartes was able to prove a lot of Greek Geometry using his new coordinate system and algebra. Thus, Descartes bridged the divide between Geometry and Algebra that would open the door towards the invention of Calculus and the modern age of Physics. It is a fact that can easily be proven that without Analytical Geometry, our modern engineering accomplishments would be impossible. Analytical Geometry helped bring forth the development of Calculus, which is an integral part of modern engineering. Descartes is also credited with creating a system so simple to use that students find it easy to understand and apply. When it comes to the *van Hiele Model* of Geometric Thinking, it can be shown that Descartes began by having a full comprehension of Level 3. He then developed his coordinate system and undertook a progression through all of the Levels of van Hiele again up to Level 3, this time looking at Euclidean Geometry in terms of a coordinate system. Or, as was already mentioned above, Descartes refined the model and moved vertically through mathematization.

It is known that another mathematician, Pierre de Fermat, invented a system very similar to Descartes' Analytical Geometry at about the same time (Sibley, 1998, p.60). Although most of the credit is given to Descartes, as Fermat never published any of his work, probably because he was worried about the consequences he would face from the Catholic Church (who was very public about silencing Galileo at this time in history). For this very same reason Descartes delayed publishing his discoveries until he was 40 years old. At that time he was still very cautious of the assertions he made and probably never published his findings to their fullest (Mlodinow, 2001, p.87). Despite receiving a massive amount of negative feedback from the Church after publishing, as well as criticism from other mathematicians, Descartes was not punished as some of his predecessors had been before him. Descartes continued his work until he passed away at the age of 53 (Sibley, 1998, p. 61).

This example of Descartes and the development of Analytical Geometry is a perfect way to demonstrate the forever-intertwining branches that represent mathematical knowledge. Quite simply, Descartes did not change the knowledge or disagree with the knowledge; he just found a different way to represent what was already proved in Euclid's *Elements*. Ideally, it would seem best to teach both types of Geometry at the same time to help demonstrate the ties between them. Students would still need to progress through the stages of the *van Hiele Model* but instead of being introduced to Analytical Geometry in high school and never recognizing the connections with Euclidean Geometry, it would seem to make more sense to introduce coordinate systems earlier in the education system. Descartes' way of approaching Euclid's Geometry can also provide a good base for developing context problems and models to hopefully ensure vertical mathematization through the topic.

2.4 The Introduction of Non-Euclidean Geometry

Geometry can be thought of as the science and mathematics of objects in a space. Geometry overcame a major hurdle when a few mathematicians in the 19th century were able to realize that space did not necessarily have to be the space of Euclid's Geometry or Descartes' Analytical Geometry. Euclid's space can be described in many ways with one of its defining characteristics being the behavior of parallel lines. Euclid recognized this and included his fifth postulate that specified how parallel lines behave in flat Euclidean space. It was mathematicians' never ending lack of satisfaction with Euclid's fifth postulate, more commonly called the Parallel Postulate, that led to the discovery of alternative Geometries. Throughout the centuries mathematicians never considered Euclid was wrong. Euclid's *Elements* was revered in the world of mathematics as being the pure language of Geometry. The problem was, however, that axioms are supposed to be obvious enough so that they do not need proof. The Parallel Postulate, on the contrary, is not really an intuitively obvious fact. Thus, mathematicians tried for centuries, without success, to prove the Parallel Postulate from Euclid's other four postulates and common notions. A major leap was made the day one mathematician decided that Euclid's fifth postulate could be changed and that there could exist a space in which parallel lines behaved differently. This introduced a step towards the last van Hiele Level of Rigor.

This mathematician was Carl Friedrich Gauss (1777-1855). According to Mlodinow (2001), Gauss began showing genius mathematical characteristics at the young age of two (ibidem, p. 108). By the age of three his father was getting him to check his payroll arithmetic. Gauss had a teacher at the age of seven who recognized his genius for what it was. This teacher was not a genius but was able to introduce Gauss to tutors and companions with whom Gauss could pursue higher mathematics and discuss his ideas with. Because of the age in which Gauss lived, Euclid's *Elements* was definitely an area of which he, and all mathematicians, were expected to have a very comprehensive knowledge.

At the age of 18 Gauss entered the University of Göttingen. It was here that he received a Ph.D. degree and eventually spent his career teaching mathematics (Mlodinow, 2001, p.113). Gauss made many outstanding discoveries and advances to a number of fields of mathematics. Ironically, he never published his findings in the area of Geometry (they were discovered after this death). Gauss did have plenty of people he corresponded with and kept in frequent touch with them. He discussed his findings on what would later be known as Hyperbolic Geometry with many different people. One of the reasons Gauss might never have published is because he lived at a time when, although he no longer needed to fear the church, science and philosophy were not completely separated. This caused issues because philosophers were known to be strict critiques of new ideas that might contradict their foundations of logic or were not logically obvious. Some philosophers even believed that only intuition needed to be embraced and that proofs should be done away with altogether. This, combined with the fact that Gauss was already recognized for many other accomplishments in mathematics, probably stayed his hand in publishing his discoveries in Geometry.

Two mathematicians did publish on Hyperbolic Geometry during Gauss' time. They were János Bolyai and Nikolay Ivanovich Lobachevsky (Mlodinow, 2001, p.100). Both men were in contact with some of the same mathematicians Gauss corresponded with and it is believed that the spirit of the idea entered their thoughts that way. Neither works these men published received much attention, mostly attributed to the fact that they were not published in very well known journals and that both men were not very well known mathematicians themselves. But after Gauss' death and the discovery of his notebooks, these previous articles were also dug back up and received great notice at that time.

The Geometry that Gauss, Bolyai and Lobachevsky discovered takes place in what we call Hyperbolic Space. What makes this space Non-Euclidean is a straightforward changing of the axiom that describes the behavior of parallel lines in the space, namely, the Parallel Postulate. Quite simply, these men thought about what would happen if there were many lines parallel to a given one through any given point and not just one as postulated in Euclid's *Elements*. The result was a system of Geometry that was just as consistent as Euclidean Geometry but described a different type of space. This change in postulates produced some interesting geometric results. For example, triangles no longer had angles that added up to 180°, but instead every triangle's angles added up to less than 180°. However, it would take many years before there was an uncomplicated way to visualize hyperbolic space. In the 1880's Henri Poincaré began the same way as Euclid had by defining *lines, points* and *planes* but with different definitions. He then constructed a model of the new theory, using what we now call a Poincaré circle, which is an infinite space with a finite boundary (Sibley, 1998, p.104). This model is now commonly used today when studying Hyperbolic Geometry.

Just how some mathematicians questioned what would happen if parallel lines behaved such that each line had many parallel lines through one point; some other mathematicians questioned what would happen if we never had parallel lines at all. The result was another type of Non-Euclidean Space and Non-Euclidean Geometry that we call Spherical Geometry. The Greeks and other civilizations had known about spherical spaces, but because of the simplicity that Euclid's *Elements* offered, Spherical Geometry was hardly given any attention at all (Mlodinow, 2001, p. 135). It was Georg Friedrich Bernhard Riemann (1826-1866), who developed the theory into a workable comprehensive type of Geometry (Mlodinow, 2001, p.135). Ironically, Riemann first presented these ideas at a job interview at the University of Göttingen in 1851, for which Gauss was actually evaluating him (Mlodinow, 2001, p.139). Gauss, who was known for hardly being impressed with anything, sincerely enjoyed the theories

Riemann put forward. Unlike Hyperbolic Geometry (the Poincaré disc was not invented until the 1880's), Riemann did have a working visual model. Like Euclid, Riemann began by defining *points, lines* and *planes* along with a unique rule that parallel lines did not exist. To visualize Spherical Space, we simply need to visualize the surface of a sphere. Like Hyperbolic Space, Spherical Space also has some interesting geometric results. One example is that angles of triangles add up to more than 180°. Another is that in Euclidean Geometry we can only ever have one right angle in a triangle whereas in Spherical Geometry we can make triangles where all three angles are right angles.

Non-Euclidean Geometry led the way for many other advances in mathematics and Geometry. For the first time, people started to question whether Euclidean Geometry was consistent. This was a question that nobody had felt the urge to ask for more than 2000 years. There were quite a few logical errors and taken for granted implicit assumptions throughout Euclid's proofs that Non-Euclidean Geometry helped to point out and clarify. Some of these errors were not spotted beforehand because nobody had questioned the nature of the space in which this Geometry was defined. With this new reexamination of Euclid's proofs, mathematicians were able to mend these logical mistakes and improve the existing Geometry. Hence, this groundbreaking new way to think about space also meant an improvement to Euclid's Geometry.

The discovery of non-Euclidean Geometry only came about because certain mathematicians were now able to advance to Level 4 of Geometric Thinking. Level 4 of the *van Hiele Model* is the stage of Rigor. It is at this stage that the knowledge shown to be true at Level 3 is questioned. The consistency of the axioms are examined and changed. Overall, it is the stage of complete scrutiny where everything is questioned. These types of investigations are exactly what happened to Euclidean Geometry in order to bring about the discovery of non-Euclidean Geometry. This level is not usually the goal of most high school curricula. Non-Euclidean Geometry is a subject that tends to get left for university. At the same time though, the ideas of non-Euclidean spaces can be introduced at the high school level by using context problems. One of these context problems could be the activity of drawing triangles and other shapes on spherical surfaces and analyzing how the different surface makes the shape change.

2.5 The Continuing Evolution of Geometry (Beyond Non-Euclidean)

Advances in Geometry did not end with the distinction between Euclidean and Non-Euclidean Space. There have been developments in other areas of Geometry for about the past 500 years. It is true however, that a lot of the most recent advances were encouraged by the revolutionary ideas of Non-Euclidean Geometry. Just like the Greeks were revolutionary in developing a new way of looking at Geometry when they started insisting on proofs, non-Euclidean Geometry was revolutionary in helping mathematicians look at Geometry from a mainly abstract perspective. The following types of Geometry were all either further developed or newly discovered when mathematicians were able to reach Level 4 of the *van Hiele Model.*

Projective Geometry

The biggest distinction between Renaissance art and Pre-Renaissance is the use of perspective employed by the artists to paint the way people saw the world. Pre-Renaissance art tends to have a flat, two-dimensional perspective, whereas Renaissance artists were able to convey a three dimensional effect on canvas. Artists like Albrecht Dürer (1471-1528) and Leonardo da Vinci (1452-1519) developed the geometric rules of painting with perspective (Sibley, 1998, p. 226). These geometric rules did tend to be a bit contradictory to Euclid's Geometry in that all parallel lines meet at a point on the horizon. But Euclid understood this effect of perspective and had written briefly about it in his work *The Optics*.

The first person to prove Geometric properties about perspective that were not needed by artists was Girard Desargues (1593-1662) (Sibley, 1998, p.226). Few mathematicians paid his work any attention or built on it any way. This might have been because perspective Geometric facts were seen as the tricks artists used and not serious mathematics. At the same time, there were such immense advances in Analytical Geometry and Calculus that a lot of good mathematics was overshadowed. Another mentionable person, who understood the mathematics of perspective, at least as far as it refers to conics, was Johannes Kepler (1571-1630) (Sibley, 1998, p. 226). Kepler saw conics as a perspective transformation from one shape to another. To illustrate this, imagine shining a flashlight on a wall. When the flashlight is held perpendicular to the wall the light shines in a circle. As we change the angle in which we shine the light at we move from a circle to an ellipse to a parabola and finally to a hyperbola. This idea was a focal point of many of Kepler's discoveries. Using a flashlight in the way described could easily be done in a classroom to introduce the subject of conics to students as a context problem[2].

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[2] Ideas and materials for such context-based approach can be gleaned from many websites devoted to the teaching of conics. A particularly interesting and rich one is http://britton.disted.camosun.bc.ca/jbconics.htm

As was already mentioned, these early breakthroughs were largely ignored by the mathematics world. A few hundred years later, in the 1800's, Gaspard Monge and one of his students, Jean Victor Poncelet, began a revitalization of the study of perspective, which would become Projective Geometry (Sibley, 1998, p.226). This time the work was not overlooked. Poncelet developed the subject comprehensively and published a book on his research which would help fuel others doing research in this area. One of the most important ideas from Poncelet is the idea of duality; the concept that lines and points are completely interchangeable and have the same function (Sibley, 1998, p.226). Some mathematicians were critical of Projective Geometry because it did not work within Analytical Geometry methods. Augustus Möbius and Julius Plücker, solved this problem by developing a coordinate system for Projective Geometry on its own (Sibley, 1998, p.226-227). Mathematicians eventually came to realize that both Projective and Analytical Geometries were valid, consistent models that actually complement each other. Mathematicians would also eventually show that Euclidean, Hyperbolic, Spherical and even the Geometry of the special theory of relativity are all contained within Projective Geometry.

Differential Geometry

Once mathematicians were able to recognize that consistent models of Geometry could exist in Non-Euclidean Spaces, it opened the imagination to the possibility of a number of different types of curvature. If we consider that Euclidean Geometry is flat, then Spherical Geometry is curved positively and Hyperbolic Geometry has a uniform negative curvature. Riemann started to investigate Geometries in a number of different dimensions and with non-constant curvature (Sibley, 1998, p.

102). To simplify, Differential Geometry combines Geometry and Calculus in a way that we are able to study the geometric properties of curved space. When investigating a model of Geometry it is important to investigate measurement within that model. Einstein would later use these ideas of measurement, initiated by Riemann, to develop his General Theory of Relativity that integrated measurement of space with gravity (Sibley, 1998, p.102-103).

Finite Geometries

The same way Non-Euclidean Geometry influenced the investigation of our notion of space, it also influenced the way we look at axioms. Axioms no longer had to be self-evident truths. Mathematicians began looking at what could be proved given a limited number of axioms. Geometers found this subject particularly interesting. They discovered that when only a specific few axioms are to be satisfied, the results could lead to a model with only a finite amount of points, hence Finite Geometry. Gino Fano developed the first Finite Geometry in 1892 (Sibley, 1998, p.264). This system was three dimensional with only 15 points. Since then many Finite Geometries have been discovered and have been applied to many different areas in mathematics. For example, Finite Geometry in combination with Algebra, Combinatorics or Group Theory provides insights into Geometry that traditionally could never be understood. As well, many famous mathematical problems have been solved because of the combination of Combinatorics and Finite Geometry. One example is Euler's problem known as the Seven Bridges of Königsberg.

These alternative areas of Geometry, the old and the new, are now progressing through levels of "super-analysis". At Level 4 when the rigorous analysis of the

Geometric system produces a new idea to be investigated, mathematicians will then begin their progression again at the beginning of the *van Hiele Model*, this time with these new ideas. Thus, these are at the level of hyper-analysis since they are developed from previous complete analysis of other Geometric ideas. Therefore, the levels of van Hiele progression of Geometric Thought do not end with a high school education. They are ongoing even in modern research in Geometry. As well, some of the more modern topics in Geometry are probably not the best place from which to draw educational material. Since these modern topics only came about after thousands of years of development of previous Geometric knowledge, a thorough comprehension of original, older Geometry is probably necessary before studying the more modern topics.

Conclusion to Chapter 2

Overall, it seems evident that the history of Geometry should not be ignored when analyzing and developing Geometry curricula. History has demonstrated that some of the biggest advances in Geometry followed a developmental pattern similar to the *van Hiele Model*. Thus, it could be concluded that the history of Geometry could have a significant influence when studying Geometry topics in a school setting, especially if the goal is to progress through the levels of the *van Hiele Model*. Likewise, history also demonstrates that context problems play a vital role in stimulating the investigation of certain topics in Geometry. Overall, both the *van Hiele Model* as well as context problems in the nature of RME can be supported by historical facts.

CHAPTER THREE

Theoretical Frameworks

Research in mathematics education, and in particular Geometry education, has been conducted with many different theoretical frameworks including *Realistic Mathematics Education* and the *van Hiele Model* of Geometric Thinking. *Realistic Mathematics Education* is a *teaching* theory that provides a good framework in which to study different teaching approaches. The *van Hiele Model* of Geometric Thinking is a theory of *learning* that can illustrate the steps students go through in their learning of Geometry, or any domain of mathematics. Alone, these two frameworks do not help much with the goal of figuring out a way to teach mathematics which will likely ensure that the students will learn it. On the other hand, taken together these two theories have the potential to do just that. In this chapter I will propose a combination of these two theories, *Realistic Mathematics Education* and the *van Hiele Model*, which should hopefully provide a working theory for teaching mathematics that takes into account the laws of learning mathematics.

This chapter will also describe the notion of angle and angle definition that I intend to use throughout my research study. Choosing an appropriate definition depends on many factors including the context in which the students will be examining angles and the van Hiele Level at which the teaching will be taking place. As well, other challenges with regard to teaching angles should also be addressed when determining an appropriate angle definition. Overall, using my research goals and the above-mentioned determining factors, I have been able to choose an angle definition that I am confident was best suited to the planned research experiment.

3.1 Teaching with RME and Learning with The *van Hiele Model*

RME and the *van Hiele Model* are very unique theories in mathematics education. RME is a teaching theory that can be applied to probably any mathematical teaching situation. The *van Hiele Model* of Geometric Thinking is a learning theory that was initially designed to describe a student's learning of Geometry but with some adaptation it can also be applied to other areas of mathematics as well. Separately these theories do not provide us a lot of concrete substance to go about teaching Geometry, so that we can hopefully ensure that the students will be able to learn it. On the other hand, it might prove successful to take a combination of the two theories and develop a theoretical framework for which the teaching and learning of Geometry are both considered. Recognizing where the learning progression within the *van Hiele Model* takes place with respect to the teaching steps of RME and situating them appropriately can carry out the combination of the theories.

As was stated previously, to begin teaching a topic or area of mathematics according to RME there must be a stated problem. According to RME, this problem should be presented in a contextual situation. A context problem does not necessarily have to be a real life application. It is considered a context problem as long as it has the potential to be experientially real to the student. In other words, when thinking about geometric shapes, we do not need to provide objects that have certain shapes. For example, we do not need to produce the top of a book or a window frame if we want to speak about a rectangle. Shapes in their abstract form can constitute a sufficiently real context for the students. For the sake of this theoretical framework combination, let us consider the example of studying the different types of quadrilaterals with students. Therefore, the problem is to identify quadrilaterals, to eventually group them, be able to recognize the differences between them and develop some theoretical understanding of them.

Once we are presented with a problem we can identify the objects of it. These objects represent the base elements of the problem to be studied. For quadrilaterals, these base elements would be any four-sided two-dimensional geometric figure. The RME teaching method now expects an initial model of this problem to be developed. It is here that the student would be developing a comprehension at Level 0, (visualization), of the *van Hiele Model*. It is also at this stage that the students undergo horizontal mathematization of the problem at hand. This initial model of the problem to be solved will be developed as the student begins to visualize the base elements of the problem. The initial model will eventually be made of a visual understanding of the base elements in the problem. For the problem of studying quadrilaterals, at the visualization stage students would be able to pick out from pictures which shapes are quadrilaterals. This would be done on the basis of memorization sense. That is, students would include squares, rectangles, parallelograms, and so on because they have been told that those are quadrilaterals and they can visually recognize them. According to RME, this model is considered a "model of" the problem.

The next step in RME is to develop or refine the initial model. This is where the majority of the learning Levels of the *van Hiele Model* are situated. This is also where the majority of the mathematization that goes on is vertical mathematization.

RME teaches that to refine the initial model the class should engage in a comparison of ideas and an open discussion. If we consider Level 1, the level of analysis, the class would begin to analyze, through open discussion, the different properties they recognized from the base elements in the initial model. For example, it could be here that students start to recognize that all quadrilaterals have four sides. At the same time, the analysis might lead to them seeing that a square has four equal sides and four equal angles. A full analysis would be complete when the students are able to recognize all of the properties of each different type of quadrilateral (multiple classroom discussions would probably be necessary). This new understanding of recognizing quadrilaterals by their properties and not just memorizing how they look can now be developed into a new model.

The next step, according to RME, would be to refine the model once again. In this new model students should be able to identify any relationships between the properties of the base elements. This is similar to Level 2 of the *van Hiele Model*. Level 2 is considered the Pre-deduction or informal deduction stage in which students find connections and links between the properties. Using these connections, relationships can be developed that describe a particular aspect of the base elements. Students really begin to move towards a "model for" the initial problem at this stage. It is here that we would find relationships such as; the diagonals of a rectangle are equal, all squares are rhombuses but not all rhombuses are squares, opposite angles of a parallelogram are equal, and any other types of relationships that a quadrilateral might hold.

A completely refined finished "model for" the initial problem will depend at what Level of the *van Hiele Model* the students need to be able to function. If the end goal is to have students be able to use deduction and prove results about the relationships found at Level 2, a completely developed model would come once a student is able to perform and operate at Level 3 or at the stage of Deduction. At Level 3, the relationships from Level 2 that were deduced are now proved, leaving no doubt as to their truth. Thus, the model would represent a system of deductive proofs with necessary definitions and axioms included. In the case of our example, these proofs would have to do with quadrilaterals. Also contained inside the model would be any definitions, axioms and postulates that are needed to prove the theorems regarding quadrilaterals. There is however, one more level of the *van Hiele Model* that can be reached. This highest level, Level 4 or Rigor, is usually not sought in high school curricula. It would, however, still be possible to situate this Level within the teachings of RME. Basically, once a student has reached Level 3 the teacher would introduce the next task that would involve the rigorous analysis of the final model obtained in Level 3. Nevertheless, once this final model is reached, be it the model at the Deduction Level or the model at the Rigor Level, the end result is a complete model for the initial problem. When this final model is developed the students have also undertaken one other characteristic of RME, namely the rediscovery or reinvention of the knowledge.

It is important to understand that the progression through the Levels of the *van Hiele Model* as well as this refining and developing of new models can take many years of education to accomplish. For example, students in the early elementary grades might spend their time memorizing the names of different quadrilaterals and recognizing them from pictures. In the later elementary grades the focus might shift to looking at the properties of these different quadrilaterals. Early secondary school is probably the time when the relationships between the properties are first examined. Finally, in the later years of secondary school, near graduation, proofs and proving relationships about quadrilaterals could be introduced. The problem of understanding quadrilaterals is a very long-term example. Other mathematical problems might not require the dedication of years of education. It would all depend on the subject being taught. Thus, just like the last characteristic of RME focus on the long-term, macro-dynamic aspects of the mathematics education, it is essential to keep the long-term goals of the education in mind.

3.2 Notion of Angle and Angle Definition

Angles are a fundamental concept of Geometry. From classic Euclidean shapes to modern day Fractals, angles play a key role in almost all geometric objects. Angles also play a major role in the modern day mathematics curricula. This includes the elementary school curricula, where angles are first introduced, all the way to the curricula of university mathematics courses where the concept of angles is still used. Angles are also a foundational topic in Geometry. They are not a stand-alone topic, but a topic that arises repeatedly as students progress through mathematics. Thus, a good understanding of angles can be deemed necessary for successful conceptualization of Geometry.

Angle Context Situations

To determine what type of angle definition is appropriate to use in a particular teaching setting will depend on the context of the situation in which the angles are represented. Beginning with the dynamic notion of angle, as it is probably the most simple, it can easily be simplified to the fact that all dynamic angle context situations must require some sort of action. For example, what if a class of students were asked to examine the angle created by the hands of a clock? The arms of the clock would represent two line segments and the angles created would be determined by how much one arm (or line segment) rotates around the central point with respect to the other arm. Then again, sometimes angles are just required to represent a context situation in a visual or physical form. When this is the case, the notion of angles as geometric shapes is most appropriate. An angle as a geometric shape could represent a context situation like how steep a hill is in comparison to the flat ground. In a more abstract domain, an angle as a geometric shape can represent two lines or planes intersecting at a point. In both of the previous examples, a measurement for the angles could then be determined. Angles are thought of as measures in context situations where a measurement is required. A measurement might be required to know how far one arm has rotated from another or the difference between the slopes of two lines. Once a measurement label is given to the rotation then the rotation is no longer being examined by its movement but by a quantitative (measured) amount. Likewise, once a measurement label is given to a geometric shape, the angle is then represented by that measurement, or the measurement is a measurement of the geometric shape (which is the angle).

Another interesting context that sometimes arises with regard to angles is that of directed angles. Directed angles can be categorized along with angles as measurement. However, in the case of directed angles, the angle's measurement is also determined by the direction of a previous or potential rotation. Thus, two angles can have the same magnitude measurement but different direction measurements. Directed angles are similar to the three previous categories of angles the same way line segments are similar to vectors. But just like line segments and vectors, one of the most useful features of angles and directed angles is the magnitude and thus directed angles usually only arise in situations where the direction is absolutely necessary to the context of the problem at hand. Interestingly, no university student in my small survey defined angle as a directed angle (Appendix A) and I only found one mention of them in all the resources (Henderson & Taimina, 2005, p.39). Overall, it can be concluded that the definition of an angle that should be used in research studies and classroom teaching should depend on the context of the situations the angles are presented.

The Challenges of Teaching Angles

As was demonstrated in the literature review, there is no single way to define angles. Hence, this provides one of the biggest challenges when it comes to teaching angles. The common way of categorizing the different angle definitions is through their context. This leaves students needing to develop an understanding of the different angle context situations in order to successfully understand what an angle is. Likewise, the angle definitions are ambiguous. An angle that is defined in one context can also be defined in another context. Therefore, students would need to be able to move between the different angle contexts as well. Overall, angles are naturally an abstract concept in terms of Euclidean Geometry, which just compounds their difficulty. These reasons, to name a few, contribute to the difficulty in teaching angles.

To look into why the different definitions would cause issues in teaching we need to look into the definition context categories in a bit more detail. The context categories are divided into dynamic, measure and geometric shape (Henderson & Taimina, 2005, p.38). In order to understand these different context situations a student will need to be able to identify which angles are represented in a dynamic sense, which angles are measures and which angles are just geometric shapes. In the dynamic situations difficulties can arise when the angle is a product of a single-armed (or no armed) object being moved. Let's consider the wheel of a ship. If the wheel is turned a quarter turn this movement has actually produced a 90° (or right) angle. But this angle is with respect to what? We could consider the point at the very top of the wheel as a starting place and see how far it has turned after the rotation is complete. Likewise, any point on the wheel would produce the same result. By labeling this quarter turn as 90° we are jumping ahead of ourselves and assigning a measure to the angle. First, students need to be able to recognize that this situation and others like it produce angles. Therefore, in dynamic context situations involving angles, students will have to learn to recognize which situations have dynamic angles.

When it comes to angles as measurements it is not as simple as pulling out a ruler and lining it up properly. We are not measuring a line or an area. Measuring an angle involves measuring the arc or amount of rotation it would take to get from one line ray to another. But there is a problem with this as well. We need to know whether we are to measure the smaller arc contained between the two rays (like the one marked by the blue arrow in Figure 3) or the larger arc on the outside of the two rays (like the one marked by the red arrow Figure 3).

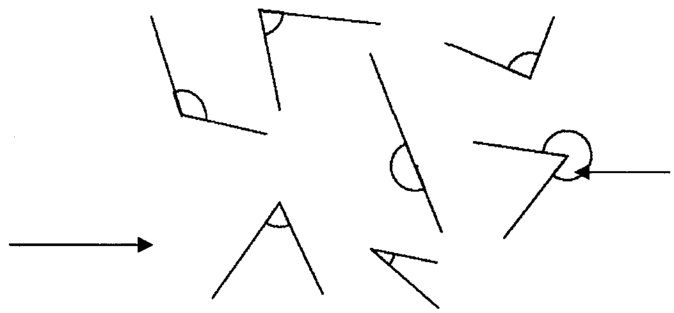


Figure 3. Marked angles

If we were to consider the idea of teaching angles following through the levels of van Hiele, then measuring angles (a property of the angles) should not even be considered until the student has a sense of angles at the Level 0, that is, where to locate them and how to identify them (the visual aspects of angles).

Angles as a geometric shape are probably the most commonly used angle context. Using the word shape to describe an angle might be misleading. Angles are better considered as geometric objects. It can be challenging to teach angles as geometric objects since they are objects that represent an abstract concept. Thus, students should not become too familiar with the visual aspects of angles or this could cause a hindrance to being able to conceptualize the abstract notions that angles can represent.

One of the most important features about angles and another teaching challenge is the fact that any angle can been thought of in any context. If we are presented with an angle that was developed from a rotation of a wheel then we can physically draw that angle as a geometric object and eventually take a measure of that angle. The measure of the angle can then be interpreted as the amount of the rotation. This is a wonderful characteristic of angles but could be challenging to students. In order to be able to successfully use the characteristic of ambiguity the students would need to have a thorough comprehension of each individual angle context. But as was mentioned above, understanding the angle context situations does not happen at the same stage within the *van Hiele Model*. As well, if the teacher did not understand the ambiguity between angle definitions it would be very difficult for the teacher to be able to identify where a student's knowledge was lacking.

Lastly, teaching angles is difficult because of their abstract nature. Angles are not like geometric shapes in that they have physical properties that must remain constant (that is, a triangle must always have three sides, etc.) The properties of an angle depend on the context they are in. In the situations of dynamic angles and angles as measurement they are nonfigurative. Angle as a geometric object is a little less abstract in that it does maintain the physical property of two rays crossing. But at the same time, teaching young students the difference between a line segment and a line ray is also an abstract concept. By and large, the challenges presented by teaching angles can provide researchers with an ample amount of experiment ideas.

Chosen Angle Definition

I have chosen to focus my research project on angles because of the challenges and difficulties outlined above. Since angles are such a challenging topic in Geometry to understand and to teach, I believe focus should be placed in this area to help find improvements for the current educational methods. I have chosen to use Euclid's definition of an angle for my research purposes. That is, "A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line." (Sibley, 1998, p.287) I have chosen this definition because I believe it will best convey the physical and visual properties of an angle in a context manner. My research does not cover angle measurement so this is not a necessary part of the definition which is used. Likewise, the different contexts in which the angles are examined in the research are all visual or physical. Thus, a static definition of an angle is appropriate. The research does, however, introduce the notion of a flat angle. Euclid's definition particularly mentions that the rays of an angle do not lie in a straight line. Thus, in my research, I have introduced a flat angle as the result of adding angles together that produce a straight line. Overall, angles are introduced by the focus being placed on recognizing the different inclinations between two lines, which is exactly how Euclid defined an angle.

CHAPTER FOUR

Methodology and Materials of the Research Study

The aim of the study was to determine whether using the theoretical foundation of RME to develop a lesson plan would help promote conceptualization of angles. This conceptualization was to be judged according to the *van Hiele Model* of Geometric Thinking. The lesson plan, which was the primary tool used for the study, was meant to get students to be able to understand angles in an analytical sense. The particular classrooms were chosen because according to curriculum the students should not have had any prior formal instruction with regard to angles. Of course, some students might have had some informal instruction on angles in previous school years or outside of the school system. This could not be helped. Lastly, it was also essential to the aim of the study that the lesson plans take place outside of a controlled (experimental) environment. This was in order to help judge the success of the lesson plan in ordinary classes.

4.1 Materials and Methods

The study consisted of a classroom experiment in which two grade-four classes participated one week apart from each other. The students in these classrooms were all fluent in English. The students in both classrooms were from many different racial backgrounds and may have spoken a language other than English in their homes.

The experiment consisted in implementing a lesson plan to be taught by the respective teachers in their own classrooms. The lesson was designed to take close to an hour to complete and be taught to the students during their normal mathematics period. It was imperative to the goals of this thesis that the lesson plan was written in such a manner that it may be used in the ordinary classroom setting and not a controlled research setting.

It was not required that I receive written consent from the parents of the individual students in the classrooms in order to conduct this experiment. This is because the lesson, which was the foundation of the experiment, was taught by the students' actual teacher. The teachers also had the choice to agree or not agree to implement the lesson plan in their classrooms, and they could agree as long as they felt it was appropriate and covered the material they needed to teach. My lesson plan did both these things. As well, the lesson plan took place during the regular class time. I received permission from the school to attend the classes as a volunteer. I checked in and out of the office every time I was at the school.

Lesson plan

The lesson plan, which is reproduced in Appendix B, consisted of five activities to be done sequentially in the classroom. The materials needed for the lesson included a handout booklet for each student (Appendix D), a set of blank transparencies for each student, a pen for each student to write on the transparencies and the teacher's set of transparencies that were used for the different activities in the lesson (Appendix C).

The intention of Activity 1 in the lesson plan is to introduce the concept of angles and right angles to the students using transparencies on the overhead. Following the theory of RME, angles are introduced in a realistic context situation, namely different roads. The activity includes showing roads that have a turn and so form an angle and those that do not. As well, the activity also shows roads intersecting at different angles and specified how to identify right angles. This notion of angles and right angles is taken from Euclid's method of defining angles and right angles. Activity 1 also consists of an activity for the students. Using the roadmap on the first page of the student's booklet and a blank transparency students are supposed to copy different angles they recognize onto the transparency and identify the right angles. The process of identifying, recognizing, finding and reproducing angles is consistent with teaching the topic of angles beginning with the first level (Level 0) of the *van Hiele Model* of Geometric Thinking.

The next activity, Activity 2, is intended to help solidify student's visual recognition of angles. Thus, this activity is designed to ensure students are functioning at or within the visual level of the *van Hiele Model.* As well, this activity continues to use context situations as the model from which to teach, consistent with RME. In this activity students continue to use transparencies to find angles in different context pictures, namely in mountains and a pair of scissors. These context pictures are also found in the students' booklet.

Activity 3 is designed to introduce another notion of angles; the flat angle. Using the overhead, the teacher uses two right angles and slides them together to make a flat angle. After using the overhead to demonstrate this concept the teacher also gets the students to stand up and, with a partner, use their arms to make two right angles and bring them together in order to make a flat angle (Figure 4). Students also use their arms to make different angles in groups of two and four. A diagram is provided for the teacher at the end of lesson plan to clarify the body actions for this activity.

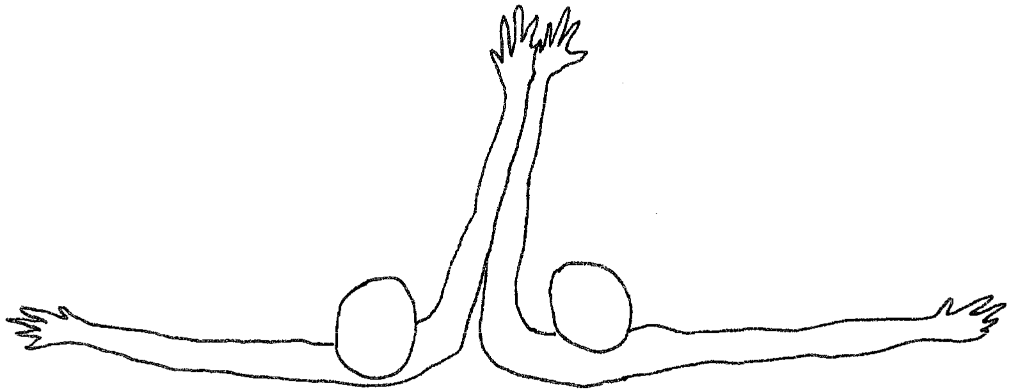


Figure 4. Embodied supplementary angles

This activity is partially in both the first and second levels of the *van Hiele Model.* The activity is in the first level or the visual level in that a new notion of angles is introduced, a flat angle, and the students are taught to visually recognize and reproduce this type of angle. Likewise, the activity covers some of the second level or the properties level in that it uses the property of adding up two angles to make a different angle. This activity also stays within the theory of RME in that the realistic context used to teach flat angles are the student's bodies themselves. It is hard to find a more realistic situation than those produced by your own body. This type of activity is a way to embody the notion of angles and give a tangible or visible form to something abstract.

The fourth activity, which is the last activity to teach anything new, is designed to see if the students can use the visual knowledge they have received in the previous activities to discover specific properties about angles. Students are asked to draw an arbitrary triangle and quadrilateral in their booklets in the space provided and copy the angles each onto separate pieces of transparency. They are then told to "add up" the angles and see what they produce. This activity assumes that the students have a good visual understanding of angles (van Hiele Level 0). It then asks the students to perform an action (adding up the angles) and interpret the results. This is also consistent with the teaching theory of RME in that it uses a hands-on activity to promote discovery of mathematical properties.

The last activity that the students are asked to complete is to answer two questions at the back of the student's booklet (Appendix D). These questions are "What was math class about today?" and "What did you learn?" The students should be asked to answer both questions in full sentences. This activity is mainly designed so that data can be collected at the end of the lesson plan to analyze what the students learned.

As was mentioned above, two teachers, in the same school, implemented this lesson plan in two separate grade four classes. The lessons were taught a week apart from each other. Initially the teachers were presented with a folder, which contained the lesson plan (Appendix B), a set of all of the transparencies to be used by the teacher (Appendix C), a copy of the student's booklet (Appendix D) and a set of the transparencies to be used by the students. Each teacher was given instructions to read the lesson plan and attempt the activities themselves. This instruction to attempt the activities themselves was given so that the teachers would have a very clear idea of activities and be able to answer student's questions from their own experience. The teachers were given an opportunity to talk with me a few days before the experiment was to take place in case they had any questions or concerns. On the day of the experiment I brought all of the other materials required for the students and had them waiting on the student's desk for them when they began their regular math period.

CHAPTER FIVE

Classroom Observations

The observations were gathered from two in-class experiments, prepared as described in Chapter 5. On the days of the respective experiments I collected written data from my observations as the lesson plan was taking place, recorded data in the form of a digital voice recorder to have a record of what was said and the students' transparencies when the lesson plan was complete. The booklets in which the students recorded their written responses were collected immediately after class in the first class and a few days after the lesson was completed from the second class.

A comparison of the two classrooms is important because it outlines just how differently the two teachers taught the same lesson plan. To ensure an appropriate analysis of the collected results it is important to understand how these results were affected by the presentation of the material. Thus detailed descriptions of the in-class observations are given below.

5.1 Observations in Classroom 1

The teacher from classroom 1 (hereafter denoted Teacher1) spoke to me briefly a few days before I was to come to her classroom. At this time she wanted to clarify how the experiment would take place. That is, we clarified that she would be teaching the lesson plan provided and that I would be attending the class and silently observing. When I spoke to her at this time she had still not yet completely read through the lesson plan or attempted any of the activities. Teacher1 also informed me at the time that I should not expect anything out of the ordinary from her students and in her opinion they were not very smart. She did not get into details about why she felt this way about her class. She felt that she was simply warning me that I might be "wasting my time" in her classroom.

On the day of the experiment I arrived early to the classroom in order to place the booklets and transparencies on each student's desk. The students were at lunch during this time. Teacher1 had the overhead projector ready in front of the board with the lesson plan and transparencies on a table next to it. Teacher1 had read through the lesson plan but had still not tried the activities herself.

To begin the lesson Teacher1 asked all of the students to sit on the floor in front of the board and all look at the overhead projector. Thus, for the initial stages of the lesson, the students were not at their desks. Teacher1 made this decision because she felt that the students would all be able to see the board better if they were all on the floor in front of it. At the beginning of the lesson Teacher1 followed along with the instructions given in the lesson plan for the first two slides. She then started talking about roads crossing perpendicular to each other before she had actually put up slide #2a (Appendix C). She noticed her mistake and quickly changed the slides and continued the description of roads that cross at right angles and those that do not.

At this point Teacher1 instructed the class to look at the roads that do not cross at right angles (on slide #2a, Appendix C) and instructed them to call the smaller angle an acute angle and the bigger angle an obtuse angle. This is a complete divergence from the lesson plan provided because there is no mention of acute or obtuse angles in the plan at all. My observations are that Teacher1 must have felt that this was an appropriate time for her to provide her students with these categorizations of angles based on size. Being a veteran teacher, Teacher 1 probably felt comfortable diverging from the lesson plan at this time.

Teacher1 then skipped slides #2b and #3 completely and jumped right to slide #4, the roadmap (Appendix C). Interestingly, when Teacher1 placed the roadmap slide on the overhead she did so with the words written at the bottom of the slide. The students started to say that Teacher1 had it sideways. At this point, Teacher1 asked the class whether it really mattered which way the road map was facing if we are just examining the angles. The class decided that it did not matter and that the angles did not change depending on the way they were facing. In the end, however, Teacher1 did rotate the road map to place the words the right way up. Teacher1 then demonstrated to the students how to find and copy a few angles from the road map. When she placed her blank transparency over the road map she did not move it or shift it in any way as she was demonstrating copying angles. Also, Teacher1 verbally instructed her students to draw a square in the corner of all angles that were right angles and to draw an arc in all of the angles that were not. In my opinion she explained this in a way that would make the students perceive that they had not drawn their angles correctly unless they had either the square or the arc in the corner.

The students then all returned to their desks to complete the copying phase of Activity 1. While the students were copying the angles from the roadmap, I observed that they did not move their transparencies at all. In fact, it was almost as though they placed the transparency on top of the picture and were tracing the road map exactly. To confirm this, when one student realized that she was going to have to shift the transparency in order to copy all of the angles, this student promptly asked if she could start again. This student explained that she had made a mistake because the picture was not going to "line up" properly because she would have to move it "off track". This is interesting because as a class they had previously decided that it did not matter which way the angles were facing because it did not change them.

One interesting occurrence happened when a student asked if angles could be straight and if so, were straight angles included. The teacher responded by saying that anything he thought was an angle should be copied. Thus, before the class even spoke about straight angles, one student had recognized that they could exist. After this communication passed between the student and the teacher, the other students in the class began including straight lines as angles to copy.

Teacher1 kept all of the students in their desks and moved them onto Activity 2 immediately after completion of Activity 1. When she was describing the activity to the students, one student said that he did not think there were any angles in the picture of the mountains. Many other students in the class quickly corrected him and said that they could see angles in the mountains. This activity caused no serious problems for any of the students.

Once completed, Teacher1 moved her class back to the floor in front of the overhead projector. The students seemed to do this quickly and quietly with no serious difficulty. When Teacher1 placed transparencies #5a and #5b (Appendix C) on the overhead (the identical right angles) she asked the students if they could predict what they would make if she slide them together. No student was able to predict that it would make a straight line. Teacher1 slid the transparencies together to show this to the class. She then skipped slides #6a and #6b (Appendix C) and moved on to the students creating angles with their arms. Teacher1 had two girls demonstrate to the class how to make a straight angle with two people before she instructed the rest of the class to join in and try.

After making angles with their arms the students were sent back to their desks and told to turn to the page in their booklets that asked them to draw a triangle and a quadrilateral. Teacher1 instructed each student to draw a triangle and then to copy the angles onto little transparencies and put them together and add them up. Teacher1 initially gave no other explanation as to how to add up angles. Once the students began with the activity and Teacher1 realized that they did not know what she meant by adding up angles she rephrased and told the students roughly to put all of the points together and line up the outside lines. Many students still struggled but as a few students realized how to do this they helped the students sitting around them. Teacher1 asked the class to tell her out loud what the angles of a triangle made and then confirmed with them that it was a straight line by demonstrating it on the overhead with one student's transparencies. At this point Teacher1 felt that she was running out of class time and rushed the students through the quadrilateral. She did not take the time to walk around and talk to students about it and she did not discuss the results out loud.

On the final activity, the one where the students were asked to write what they did in math class and what they learned Teacher1 instructed her students that they would not be finished unless they had written four lines. More than one student raised their hand and asked Teacher1 to come and check whether they had written the correct answer. Many students found it difficult to understand that the correct answer was what they felt was correct and that they were just supposed to answer the questions from their own perspective. I was under the impression that the students are not asked to answer questions of this nature very often.

I am not able to judge whether the students reacted differently to this math lesson than to their regular math lessons. The class, however, was attentive and well behaved. They raised their hands when they asked questions or had comments. They seemed interested in the hands-on activities and were excited to have their own transparencies and pens to individually work with. The students all worked pretty silently and individually on the different activities with the transparencies.

5.2 Observation of Classroom 2

I will refer to Teacher2 when I comment on the teacher in the second classroom I attended. From the very first time I spoke to Teacher2 about the possibility of using her classroom for an experiment she expressed great interest and excitement about the idea. When I spoke to Teacher2 prior to the lesson plan she had thoroughly read the plan and tried all of the activities herself. Her only question was to clarify how the students were to make the angles with their arms. She said it was a little hard trying this part of the lesson plan by herself. As with Teacher1's classroom I came early and placed all of the materials on each student's desk while they were at lunch. I also asked Teacher2 how she perceived the intelligence of her students so that I could compare with Teacher1's volunteered information. Teacher2 referred to her students as average. She said they were neither over intelligent or under intelligent but just an average group of forth grade students.

Unlike the experiment conducted in Teacher1's classroom, Teacher2 did not deviate from the lesson plan very much. Teacher2 did not miss any slides, rush any parts or skip through any explanations. There were only a few major differences. Teacher2 used sticky notes to cover up the words in slides #1a and #2a (Appendix C). She then asked the class if they could tell her the difference between the different roads pictured, instead of her just stating the differences. Teacher2 used this method of asking the students what they thought throughout most of the lesson plan. When I enquired about this afterwards, she said that is just how she teaches and she did not even really notice that she was doing it, other than when she purposely covered up the words. Teacher2's students also stayed in their desks the entire time, except for when they made angles with their bodies. Teacher2's classroom was better suited to have the students stay at their desks and watch the overhead projector at the same time.

I observed that when Teacher2 drew her angles she automatically drew an arc in with them or a square, depending on whether the angle was right or not. Unlike Teacher1, Teacher2 did not instruct the students to do so or even mention why she was drawing the arcs or squares. It was my impression that Teacher2 did not even realize what she was doing. Teacher2 never actually instructed the students how to draw an angle and realized once the class had started copying the angles from the road map that some of the students did not know how she wanted them to be copied. When she was instructing the students on the overhead projector she used the word "find" and told the students to find the angles. She never actually told them to copy them once they were found. Also, when Teacher2 was demonstrating on the overhead projector she put a little dot next to the angles she found that were right angles. To rectify the fact that she never told the students how to copy or draw an angle, Teacher2 walked around and privately demonstrated how to copy the angles to a few groups of the students who needed help. This seemed to be successful and on Activity 2 there were no problems with copying the angles.

Teacher2 spent more time than Teacher1 on the Activity 3 where the students make angles with their arms. She really spent a lot of time making sure that the students understood that they were making a straight angle and that by moving their arms that were touching, the straight angle was not changing. She walked through the groups of students and helped wherever it was necessary.

With Activity 4, Teacher2 also found that the students had some initial trouble understanding how they were supposed to add up the angles. Teacher2 did demonstrate adding up angles on the overhead projector but still found that she needed to walk around and help a lot of students with the task. Teacher2 had left more time for this activity and the students were able to spend their time on both the triangle and the quadrilateral.

At the end of the lesson plan it was time for the students to go to recess. Teacher2 told me that she would give the students' time to fill in the last page of the booklet after recess and that I could pick them up in a few days. Thus, where Teacher1's class answered the questions immediately following the lesson plan, but were rushed for time, Teacher2's class took a recess break and answered the questions afterwards and were given as much time as they needed. I collected the students' transparencies while I was in the classroom and collected the booklets from Teacher2 a few days later. All of the booklets were accounted for.

The students in Teacher2's class also reacted well to the lesson. Like Teacher1's class, they were excited to each have their own materials to work with. I supposed that the alternative is watching the teacher work with materials or having to share materials in a group. Teacher2's class was a little less disciplined than Teacher1's classroom. Students would shout out comments or questions and were much more apt to talk to each other while performing the activities. This behavior did not seem to bother Teacher2 in any way. She mostly ignored it and sometimes would mildly remind a student to get back on task. I was under the impression that this is just the way Teacher2 allows her classroom to be and has no problem with that type of behavior.

CHAPTER SIX

Analysis of the Classroom Observations

The data used for this analysis came from the students written responses to the two questions at the end of the work booklets, the transparencies that the students used throughout the lesson's activities, my personal observations of the students performing these activities on the days I was in the classrooms and the audio transcripts from the recording made during the classroom experiments.

6.1 Written Responses

Each student was asked to answer two questions at the end of the work booklet that they were using throughout the lesson (Appendix D). The questions were "What was math class about today (answer in full sentences)?" and "What did you learn (answer in full sentences)?" The responses to these questions were categorized as either Analytical or Narrative.

Narrative Responses

A student's response was categorized as Narrative if it did not contain any mathematical descriptions in response to the question asked. Instead, a Narrative response describes what took place or tells a story. This category also contains responses that express a personal opinion of the student. Narrative responses are divided into two types, namely N1 or N2. An N1 response is a completely narrative response that includes some mention of angles. An N2 response is a completely narrative response that has no mention of angles in it at all.

Analytical Responses

A student's response was categorized as Analytical if it contained words that describe angles in an appropriate mathematical context. Analytical responses were divided into three types, A1, A2 and A3, based on the corresponding level of the *van Hiele Model*. A response was considered A1 if the response contained at least one comment of an analytical nature that describes angles at the stage of van Hiele Level 0 (visual). That is, the description included mention of angles in a visual sense. Some guidelines for assigning a response as A1 included whether the student mentioned anything about finding angles in pictures, because that can be taken to imply that the student can visually recognize angles. As well, a response was considered A1 if the student mentioned different types of angles because that can be taken to imply that the student has an understanding that the angles have different name categories, which is also a visual (recognition) trait.

A response was considered to be A2 if the response contained at least one comment of an analytical nature that described angles at the stage of van Hiele Level 1 (descriptive, describes properties). That is, the description included mention of some sort of property pertaining to angles. The best guideline for assigning a response as A2 is if the student mentioned anything about adding angles up to get different angles or putting angles together. This implies that the student understood the property about angles that two angles added together make a different angle.

The category of A3 was very rare but included, for those situations that the response mentioned, at least one comment at the stage of van Hiele Level 2 (informal deduction). The only responses considered in this experiment as A3 were responses that mentioned the results from Activity 4. That is, if the student mentioned that the angles of a triangle make a straight line or that the angles of a quadrilateral make a circle.

Other Categories

There are two other categories used to classify the student's responses. One is the category of N/A or not applicable. This was saved for responses that were either left blank, incomplete, or illegible. The other is the category of AE or "Analytical with Error." A response was considered AE if it made mathematically analytical statements about angles but the statements were false. For example, one student stated, "Math class was about right angles and left angles…" (Appendix F).

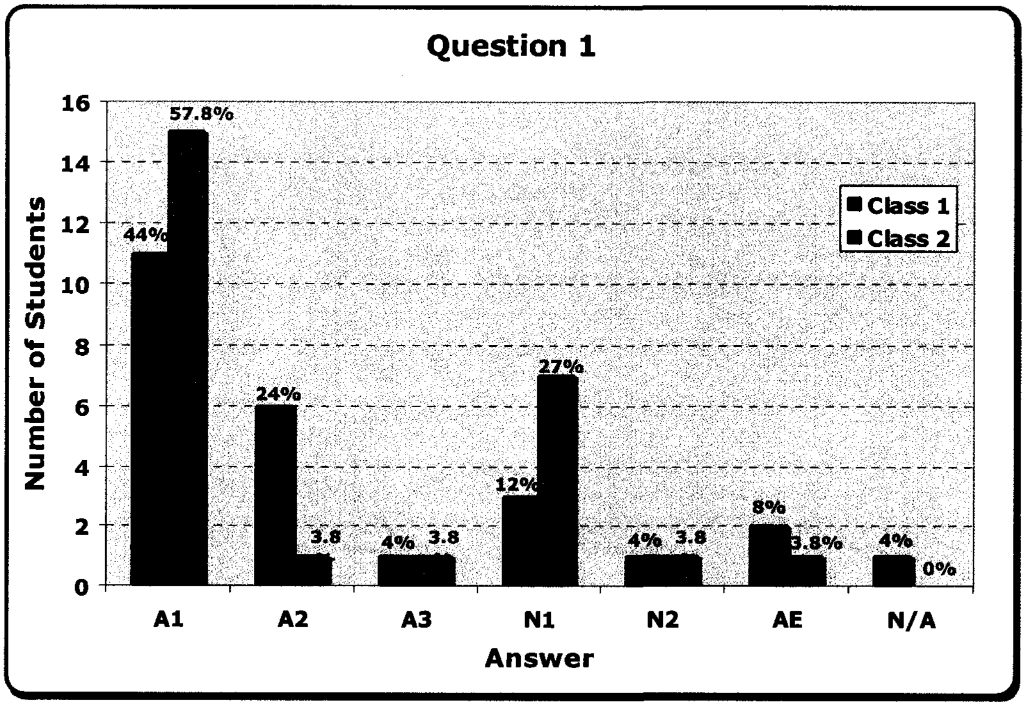
A complete list of all of the students written responses along with the categorization they were given can be found in Appendix F. The student's responses are represented below in Tables 1a and 1b and Graphs 1 and 2.

Class 1 (25 Total)		
	Q1	Q2
A1	11 (44%)	3 (12%)
A2	6 (24%)	11 (44%)
A3	1 (4%)	0
N1	3 (12%)	1 (4%)
N2	1 (4%)	5 (20%)
AE	2 (8%)	3 (12%)
N/A	1 (4%)	2 (8%)

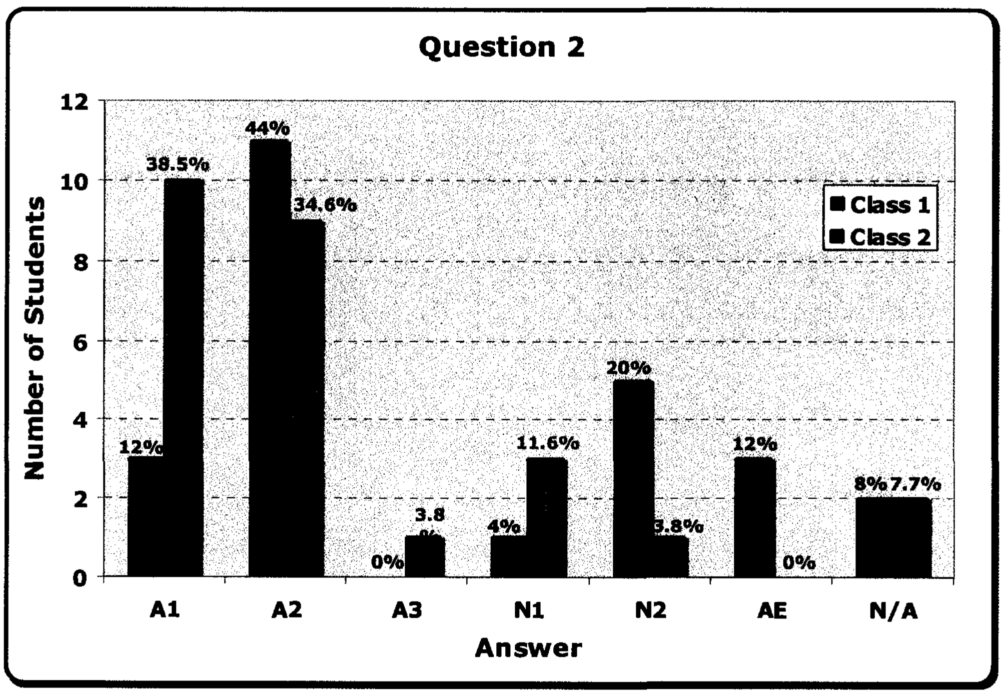
Table 1a. Classroom 1 written responses

Class 2 (26 Total)		
	Q1	Q2
A1	15 (57.8%)	10 (38.5%)
A2	1 (3.8%)	9 (34.6%
A3	1 (3.8%)	1 (3.8%)
N1	7 (27%)	3 (11.6%)
N2	1 (3.8%)	1 (3.8%)
AE	1 (3.8%)	0
N/A	0	2 (7.7%)

Table 1b. Classroom 2 written responses



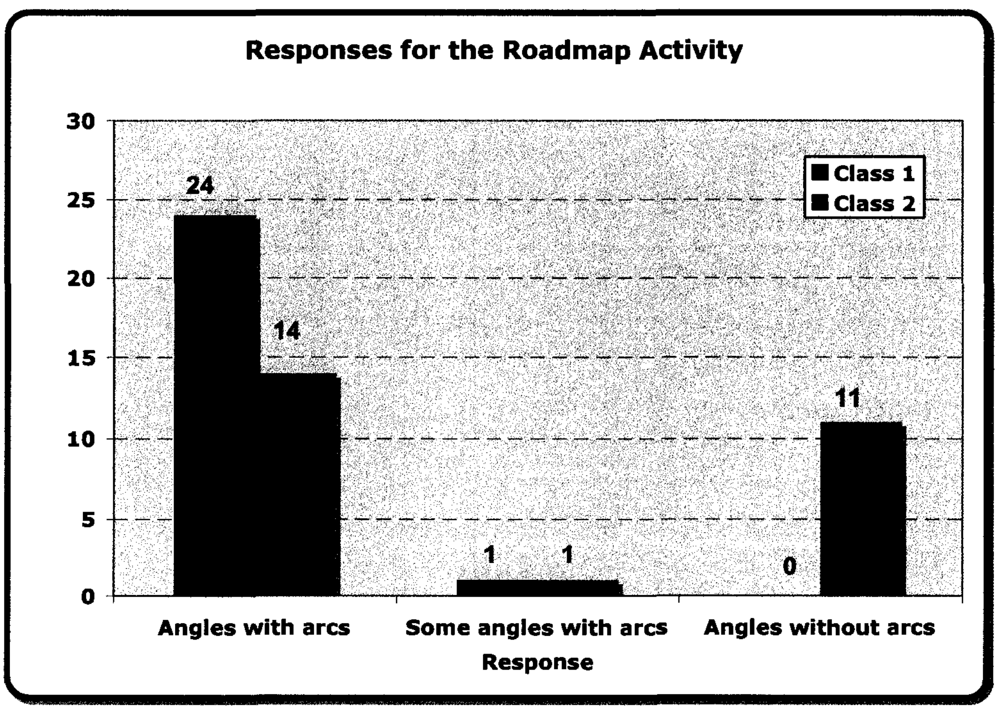
Graph 1. Distribution results for Question 1



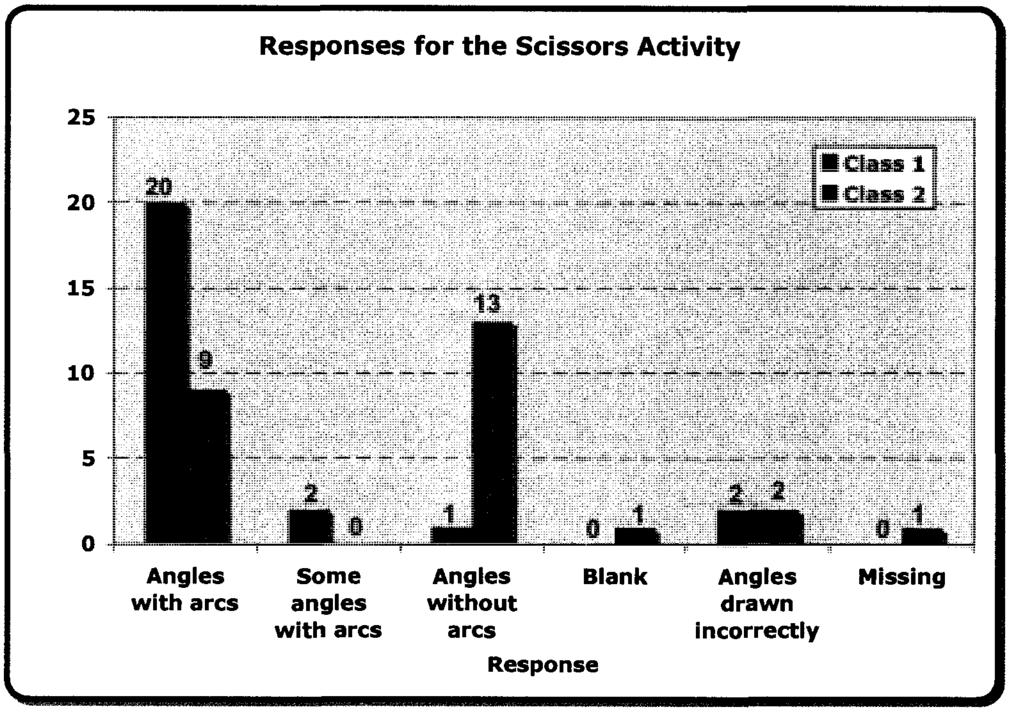
Graph 2. Distribution results for Question 2

6.2 Analysis of students' transparencies

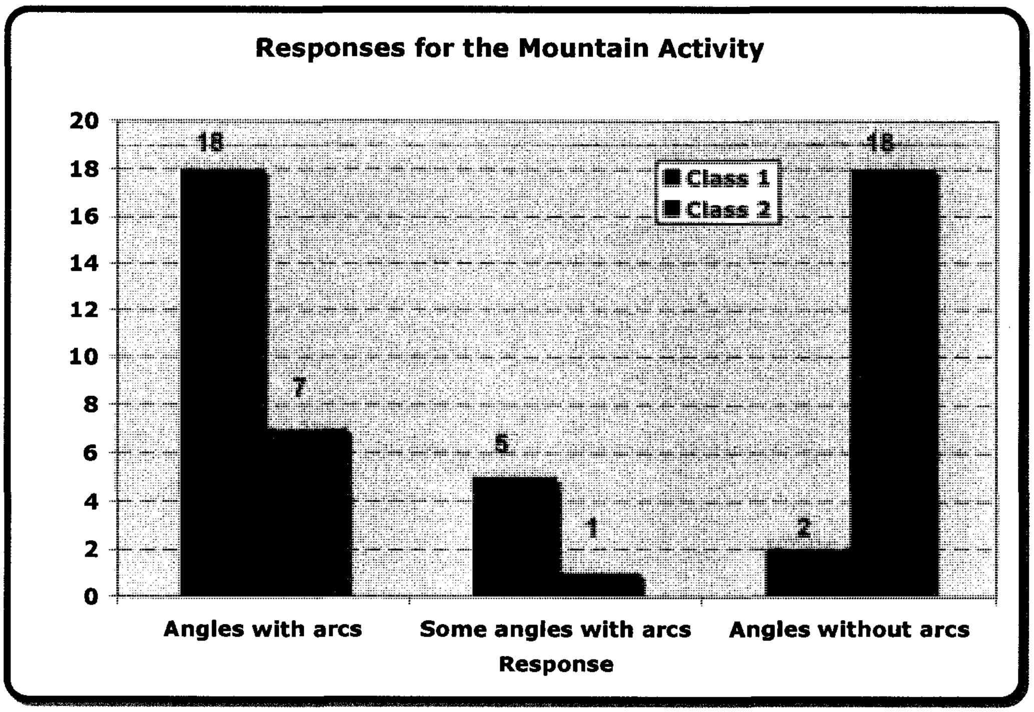
The transparencies that the students drew on were collected after the activities. They were analyzed based on whether or not the angles were drawn correctly and whether the angles included arcs or squares (for right angles) in the corners. Arcs or squares in the corners were included because it was interesting to see whether the students thought this was necessary to complete a drawing of an angle. Drawing an arc is not something I preserved to be a significant part to learning of the notion of angle. In general, all students drew relatively the same number of angles on each of the transparencies used to copy angles from pictures. As well, the students each drew their angles with rays of relatively the same length. Graphs 3 through 6 represent the results collected from the students' transparencies.



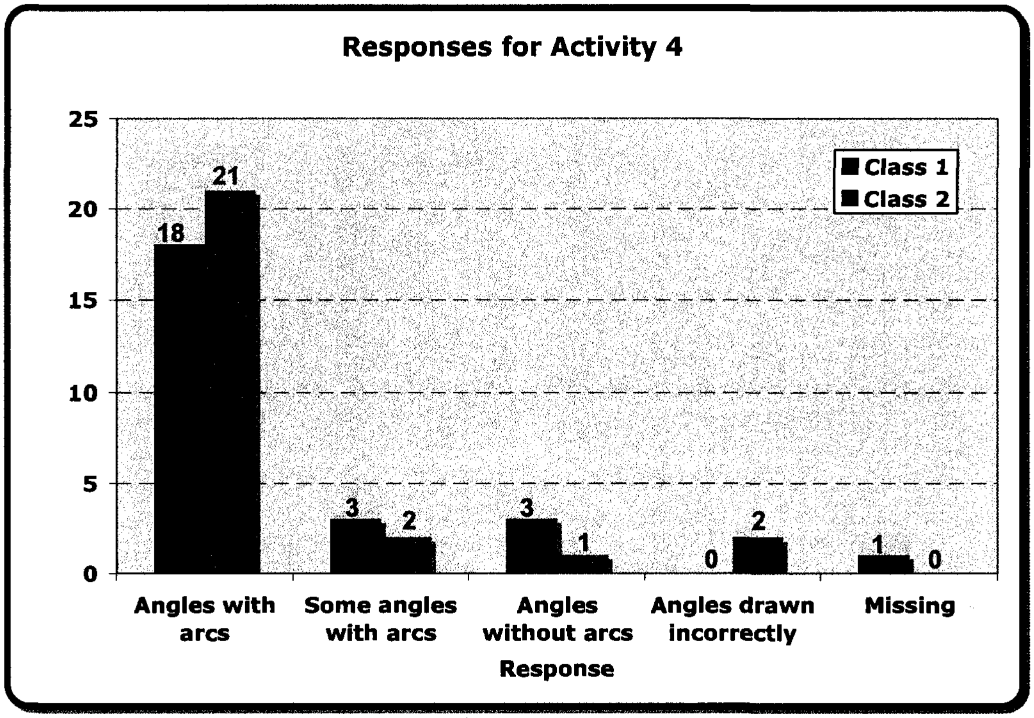
Graph 3. Transparency Results for Roadmap Activity



Graph 4. Transparency Results for Scissors Activity



Graph 5. Transparency Results for Mountain Activity



Graph 6. Transparency Results for Activity 4

There were some interesting results with the transparencies that are not represented in the graphs. With regard to the Roadmap, only one student in Classroom 1 drew less than 20 angles (this student drew only seven). In Classroom 2, the students who drew their angles with no arcs had a variety of ways in which they labeled their right angles. Two students marked their right angles with a dot (like the teacher initially did in front of the entire class). Eight students marked their right angles with a big 'R'. One student coloured in a complete triangle for the each of their angles. With the Scissors, the tables do show that two students in each class drew some of their angles incorrectly, but it does not describe how these were drawn. In Classroom 1, one student drew lines on the sides of the scissors but did not connect the lines at the tip of the scissors, thus excluding what is actually the angle. The other student copied the round shape from the scissors handles and identified those as angles. In Classroom 2, both students drew the angles incorrectly with more than two rays. The results from the Mountains transparencies are represented accurately by Graph 5, with no other variations to mention. In Activity 4, the only thing worth mentioning is that of the two transparencies that had angles drawn incorrectly from Classroom 2, one set had the edges of the shapes copied but the edges did not meet up with the corner and the other set had coloured in triangles for the angles.

6.3: In-Class notes and Audio Transcripts

The observations I made from the in-class visits were analyzed in a suggestive manner with no categorization assigned. I made note of observations which I felt were important to mention. The complete audio transcripts of the experiments are reproduced in Appendix E. There were quite a few differences between the two class experiments, as would be expected in an uncontrolled classroom setting. Some of these differences include the fact that Teacher1 gave me the impression of not being very impressed with my lesson plan and was only doing it as a favor to me. Teacher2 was excited about the lesson plan and wanted to know more about my research and whether I had developed other lesson plans she could try in her classroom. Teacher1 was an older lady who had actually returned from retirement to teach one more year of school. Teacher2 was relatively young and near the beginning of her career as a teacher. Teacher1 missed a number of transparencies during the lesson, namely #2b, #3, #6a, and #6b (Appendix C). Teacher2, on the other hand, did not miss any transparencies at all while implementing the lesson plan and did everything in the order that I had provided it in. Teacher1 instructed her students to find as many angles as they could in the roadmap, mountains and scissor pictures. Teacher2 limited the amount of angles the students were supposed to find and gave them a shorter amount of time to do the activity. Teacher1 acted as though she found the activity where the students form angles with their arms was a waste of time. Teacher2 was very enthusiastic about that part of the lesson plan and spent quite a bit of time on it to ensure that all the students could form the angles correctly with their arms and that all the students understood the purpose of the activity. Teacher1 maintained a very high level of discipline in her classroom and barely allowed any extra talking from the students while they were completing the activities at their desks. Teacher2 was very lenient about discipline. She ignored the extra chatter that was going on when the students were working at their desks, even though it largely had nothing to do with the activity and mainly just reflected that students were off task. Overall, the two classrooms were probably as different as you expect two random classrooms to be.

There was also a big difference in the length of the audio transcripts – containing mainly whole class teacher-student interchanges – and the amount of student responses on the transcripts. The audio transcript from Classroom 1 typed up to 11 pages, while the audio transcript from Classroom 2 typed up to just over 7 pages. Both of the transcripts were typed up in the exact same format. Thus, this reflects the amount of verbal instructions that were actually given to the students. As well, there were 50 recorded responses made by students in Classroom 1 (some students responded more than once) and in only 23 recorded responses made by students in Classroom 2 (again, some students responded more than once). Thus, in Classroom 1 there was more interaction between individual students and the teacher while the entire class was listening. On the other hand, I observed that Teacher2 spent a lot more time helping students individually during the time that the students were to be completing the activity tasks. Unfortunately, the audio recording did not catch the conversations between the teacher and individual students because of the background talking that was prevalent in Classroom 2.

CHAPTER SEVEN

Discussion of Research Study Results and Conclusions

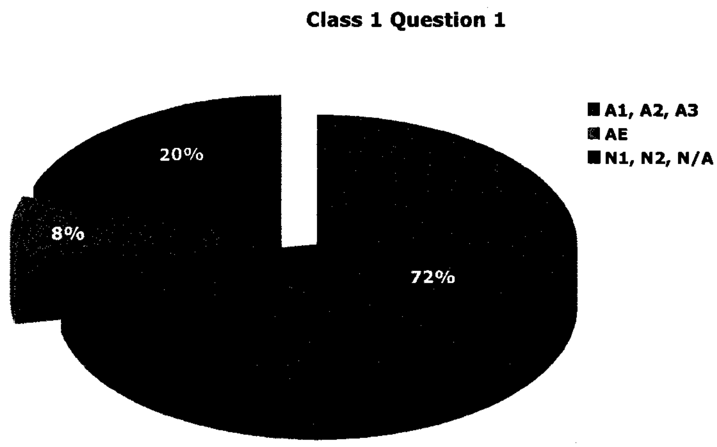
This chapter discusses the major findings from the research study and the implications of these findings. As well, some suggestions for future experimentation are suggested. Suggestions are made in regard to both the lesson plan used for this thesis as well as for other research involving a similar theoretical framework.

7.1 Discussion and Findings

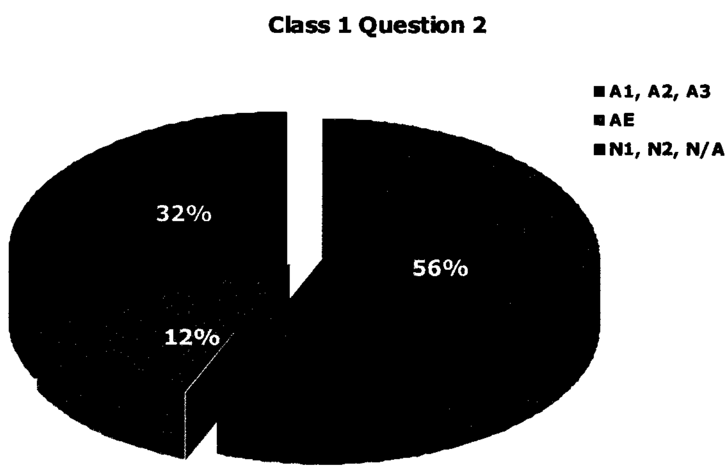
The written responses suggest that the majority of students finished the lesson with an analytical understanding of angles. In Classroom 1 a total of 20 students (out of 25), or 80%, answered analytically to at least one of the questions and twelve students answered analytically to both. Only one student made an analytical response with an error on both questions. As Table 1a in the Chapter 6 shows, only 1 student gave a response that could be classified as A3. I am inclined to think that this was because of the way in which Teacher1 rushed through activity 4 in the lesson plan. Another interesting feature is that a lot more students give A2 responses for Question 2 than A1 responses. That makes me conclude that the students might have had some previous experience with angles and thus the knowledge that they considered they learned was the new information they have been shown, like adding up angles.

In Classroom 2 a total of 21 students (out of 26), or 81%, answered analytically to at least one of the questions and sixteen students answered analytically to both. I am inclined to believe that more students answered analytically to both question in Class 2 than in Classroom 1 because of the comment Teacher 1 made when she instructed the students to be "very detailed" when filling in their answers (Appendix E). Only one analytical response with an error was made overall. One student answered both questions at an A3 level (Appendix F). This is not very different from Classroom 1, even though more time was spent on the activity. However, Teacher1 did particularly point out to the entire class the results of adding up the angles of a triangle. In Classroom 2, Teacher2 specifically left out the conclusions. Thus, it could be concluded that the student in Classroom 1 was repeating the information Teacher1 told the class, while in Classroom 2 the student had to discover the information on his or her own. Another remarkable feature is that Classroom 2 had less N2, AE and N/A responses than Classroom 1. From my personal observations and a review of the audio transcripts, I can see no evident reason for this difference in responses.

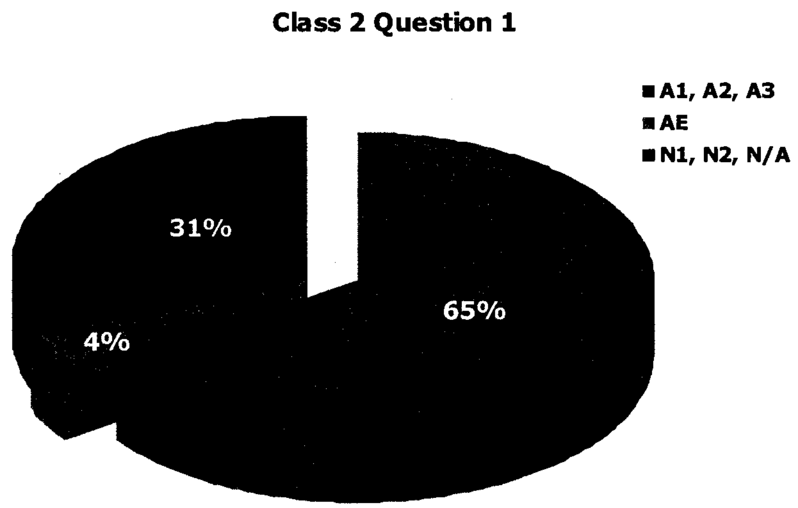
From these results I have concluded that the majority of students in both classes gained an analytical understanding of the first van Hiele Level, at the very least. Despite the differences in how the lesson plans were implemented and the different classroom settings, it seems that the activities provided did help with the initial conceptualization of the notion of angles. I also consider the lesson plan to be a success in introducing the concept of angles to students because it was tested in two real classroom experiments with different external variables and the overall results were



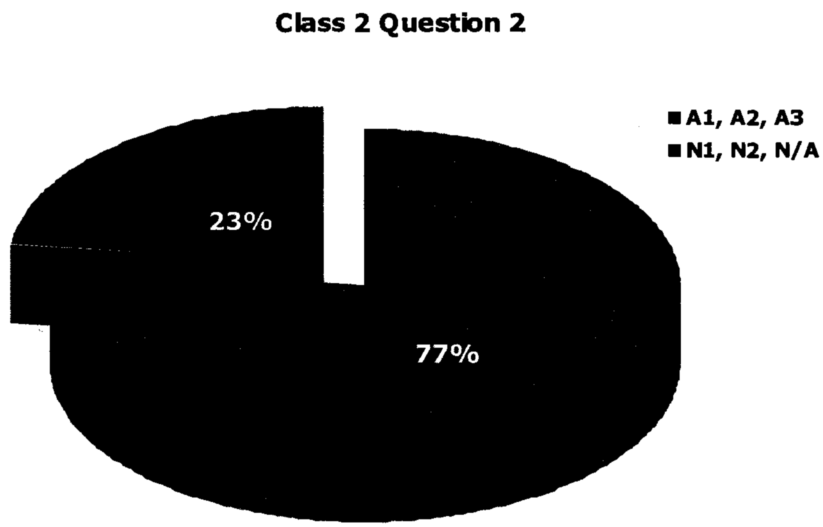
Graph 7. Written results distribution on Question 1 for Classroom 1



Graph 8. Written results distribution on Question 2 for Classroom 1



Graph 9. Written results distribution on Question 1 for Classroom 2



Graph 10. Written results distribution on Question 2 for Classroom 2

Transparencies

The transparencies can be analyzed from the perspective of whether or not the students drew their angles correctly and whether they included arcs or not. Drawing (or replicating) a geometric figure properly is a skill designated at the van Hiele Level 0 stage (visual). After reviewing the transparencies it is apparent that the majority of students can draw an angle correctly, which correspond with the written responses that suggest the majority of students understand angles at least at the visual stage.

Whether or not arcs were included is a different subject. As is shown in Tables 1a and 1b in Chapter 6, a lot more students drew angles without arcs in Classroom 2, where the teacher never mentioned it. It is interesting though that some students still drew their angles with arcs. I am under the impression that these students have probably seen pictures of angles elsewhere, e.g. in math textbooks or workbooks, and picked up the idea of drawing arcs from these sources. Although I do not know why, the majority of the class (Classroom 2) resorted to drawing arcs in activity 4. I do not recall whether the teacher drew arcs with her angles when she was demonstrating this activity. On the other hand, maybe some students picked this up from their peers as the activity was taking place.

Classroom 1 had a very interesting outcome in this regard on the majority of their transparencies. If you recall from the observations as well as the audio transcript, during Activity 1 while the students were copying the angles from the roadmap, one student asked if angles could be straight (Appendix E). The teacher responded with, "I said do whatever angle you can find" (Appendix E). I believe that since the teacher did not answer this question with a direct "no" that the students got the impression that straight lines contain angles. Thus, the majority of the transparencies from Classroom 1 have straight lines drawn with arcs in the middle of them as though they are straight angles.

The interesting results on the transparencies from Classroom 2 were mostly on the roadmap transparencies. As was mentioned in the observations, the teacher did not instruct the students on how to draw an angle and marked the right angles with a dot. A few of the transparencies reflected this in that the students drew dots to point out which angles they thought were right angles. Another variation was that some students marked their right angles with an R. This was the instruction that the teacher gave to the students as she was walking around helping them with the activity after the general instructions to the entire class. It is easy to conclude that the students identified their right angles this way because of how they were instructed by the teacher. In Classroom 1, the majority of students identified their right angles with a square in the corner. Nonetheless, the angles marked with dots and R's in Classroom 2 are for the most part, right angles. Thus, despite the difference in identifying the right angles between the two classes, it seems the students were successfully able to find right angles.

I got the impression from the sets of student transparencies from both classes that the transparencies that had some angles with arcs and some angles without arcs were a result of the student not recognizing that they had actually drawn an angle. For example, in the picture of the mountains, some students drew a straight line along the bottom of the mountain with a half circle arc in the middle (to indicate a straight angle) as well as the sides of the mountains with the arc at the mountain peak. The students however did not draw an arc where the mountainside meets the ground and I am under the impression that this is because they did not recognize that there was actually an angle there.

On the subject of the mountain picture it is remarkable that not one student in either class actually drew an angle from the mountainside to the horizon intentionally (as I mentioned above, in a few cases it appeared it seemed unintentional). Maybe they considered that the horizon is fixed and does not produce angles with what is on top of it. At the same time, it may be unreasonable to expect grade 4 students to think of slope in this context. Overall, I get the impression from the transparencies used to find angles in the pictures that the students have a very good understanding at the visual stage of the *van Hiele Model.* Some students still need some work on finding angles and drawing angles, but that is consistent with the findings from the written responses.

The angles drawn on transparencies for Activity 4 also show a very good understanding at the visual stage with only two errors. The only remarkable thing worth mentioning in this case is that the majority of the students (18 out of 25 from Classroom 1 and 21 out of 26 from Classroom 2) felt that they needed to include arcs with their angles with this activity. Thus I get the impression that teachers need to be careful when demonstrating to students how to draw angles and need to make sure to clarify that the arc is not a necessary part of the angle and is actually used to mostly demonstrate where the angle should be measured. Because this activity had nothing to do with angle measuring and since angle measuring is not part of the first level of the *van Hiele Model*, it was not necessary for the teachers to insist on draw the angles this way.

In-class Observations and Audio Transcripts

Despite the differences in how the lesson plans were implemented, the results from the two classes were strikingly similar. This leads me to conclude that the activities were at least initially successful in helping the students conceptualize analytically the notion of angle. I cannot even conclude that there were deficiencies in the students' knowledge in Classroom 1 where the teacher skipped instructions or rushed through activities. This also means that there are certain instructional transparencies that could be taken out of the lesson and still have a high level of success. Likewise, the sometimes un-attentive behavior of the students in Classroom 2 does not seem to have made a big difference either.

7.2 Implications and Conclusions

The results of this research show that the RME approach to teaching using realistic context problems is a promising method when introducing angles. The research in this thesis supports the hypothesis that using the RME approach can help students gain an appropriate understanding of the angles as determined by the *van Hiele Model* of Geometric Thinking. I have concluded, at least for the classrooms where the lesson plan was tried, that this research has been a success. I also have no reason to believe that the results would be much different in other classrooms of the same type.

There are some improvements that could be appropriate to make to the lesson plan if it were to be used in the future. I feel that it would be appropriate to insert instructions on how to draw or copy an angle within the first activity of the lesson plan. This can be done while the teacher is demonstrating how to find the angles within the Roadmap. On the same note, students tended to struggle with adding up the angles in activity four. It might be necessary to leave this as a separate lesson altogether. If students were given more instruction on angle congruency before adding up angles, I believe it would help facilitate this activity. Thus, if activity four were removed from the lesson plan, an activity on angle congruency could be added. I can also assume that using transparencies would be a very successful way to teach angle congruency, as students would again be able to compare different angles by moving them to coincide on one of their arms.

More research could be conducted on the topic of angle conceptualization. A teacher could develop more lesson plans using the methods of RME to help students discover more properties pertaining to angles. The concept of measure of angles is also important and a natural next step after the initial conceptualization of what an angle is. Likewise, it would be interesting to conduct future research pertaining to other subjects in Geometry but using the same theoretical frameworks as this thesis.

Overall, using realistic context problems and the teaching theory of *Realistic Mathematics Education*, seems to be able to help shape a student's understanding of angles in the particular ways outlined by the *van Hiele Model* of Geometric Thinking.

REFERENCES

Bishop, A. J. (1983). Space and geometry. In R. Lesh & M. Landau (Eds.) *Acquisition of Mathematics Concepts and Processes* (pp. 175-203). New York: Academic Press.

Brahier, D. J. (2000). *Teaching secondary and middle school mathematics.* United States of America: Allyn & Bacon.

Brown, K. E., Simon, L., & Snader, D. W. (1970). *Applying high school mathematics.* Toronto: Laidlaw Brothers.

Burger, W. F., & Shaughnessy, J. M. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education,* 17(1), 31-48.

Cannissaro, L., & Menghini, M. (2006). From geometrical figures to definitional rigour: teachers' analysis of teaching units mediated through van Hiele's theory. *Canadian Journal of Science, Mathematics and Technology Education,* 6(4), 369-386.

Clements, D. H., & Battista, M. T. (1989). Learning of geometric concepts in a *Logo* environment. *Journal for Research in Mathematics Education,* 20(5), 450-467.

Clements, D. H., & Battista, M. T. (1990). The effects of *Logo* on children's conceptualization of angles and polygons. *Journal for Research in Mathematics Education,* 21(5), 356-371.

Clements, D. H., & Burns, B. A. (2000). Students' development of strategies for turn and angle measure. *Educational Studies in Mathematics,* 41(1), 31-45.

Coolidge, J. L. (1963). *A history of geometrical methods.* New York: Dover Publications.

Cooper, B., & Harries, T. (2002). Children's responses to contrasting 'realistic' mathematics problems: just how realistic are children ready to be?. *Educational Studies in Mathematics,* 49(1), 1-23.

Coxeter, H. S. M. (1964). *Projective geometry.* United States of America: Blaisdell Publishing Company.

Freudenthal, H. (1967). *Mathematics observed.* Verona: Officine Grafiche Arnoldo Mondadori.

Freudenthal, H. (1973). *Mathematics as an educational task.* Holland: D. Reidel Publishing Company.

Fyhn, A. B. (2008). A climbing class' reinvention of angles. *Educational Studies in Mathematics,* 67, 19-35.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: a calculus course as an example. *Educational Studies in Mathematic,* 39, 111-129.

Gutierrez, A., Jaime, A., & Fortuny, J. M. (1991). An alternative paradigm to evaluate the acquisition of the van Hiele levels. *Journal for Research in Mathematics Education,* 22(3), 237-251.

Hartshorne, R. (1997). *Geometry: Euclid and beyond.* New York: Springer.

Henderson, D. W., & Tainmina, D. (2005). *Experiencing geometry: euclidean and non-euclidean with history* (3rd ed.). New Jersey: Pearson Education.

Hillel, J., & Kieran, C. (1987). Schemas used by 12-year-olds in solving selected turtle geometry task. *Recherches en Didactique des Mathematiques,* 8(1.2), 61-102.

Hoffer, A. (1983). Van Hiele- based research. In R. Lesh & M. Landau (Eds.) *Acquisition of Mathematics Concepts and Processes* (pp. 205-227). New York: Academic Press.

Knorr, W. R. (1986). *The ancient tradition of geometric problems.* Boston: Birkhauser.

Lanczos, C. (1970). *Space through the ages: the evolution of geometrical ideas from Pythagoras to Hilbert and Einstein.* London: Academic Press.

Magnani, L. (2001). *Philosophy and geometry: theoretical and historical issues.* Netherlands: Kluwer Academic Publishers.

Malkevitch, J. (1998). Geometry in our world. In C. Mammana & V. Villani (Eds.), *Perspective on teaching of geometry for the 21st century: an ICMI study* (pp. 85-105). Netherlands: Kluwer Academic Publishers.

Mammana, C., & Villani, V. (Eds.). (1998). *Perspectives on teaching of geometry for the 21st century: an ICMI study.* Netherlands: Kluwer Academic Publishers.

Mayberry, J. (1983). The van Hiele levels of geometric thoughts in undergraduate preservice teachers. *Journal for Research in Mathematics Education,* 14(1), 58-69.

Mitchelmore, M. C. (1998). Young students' concepts of turning and angle. *Cognition and Instruction,* 16(3), 265-284.

Mitchelmore, M. C., & White, P. (2000). Development of angle concepts by progressive abstraction and generalization. *Educational Studies in Mathematics,* 41, 209-238.

Mlodinow, L. (2001). *Euclid's window: a story of geometry from parallel lines to hyperspace.* New York: Simon & Schuster.

Munier, V., & Merle, H. (2007). Une approche interdisciplinaire mathematiques: physique du concept d'angle a l'ecole elementaire. *Recherches en Didactiques des Mathematiques,* 27(3), 349-388.

National Council of Teachers of Mathematics. (1988). *The van Hiele Model of thinking in geometry among adolescents: Journal for Research in Mathematics Education, monograph number 3.* United States of America: Author.

Ness, D., & Farenga, S. J. (2007). *Knowledge under construction: the importance of play in developing children's spatial and geometric thinking.* United States of America: Rowman & Littlefield Publishers.

Pedoe, D. (1976). *Geometry and the visual arts.* Toronto: General Publishing Company.

Pritchard, C. (2003). Introductory essay: a concise and selective history of geometry from Ur to Erlangen. In Pritchard, C. (Ed.) *The changing shape of geometry: celebrating a century of geometry and geometry teaching.* Cambridge: Cambridge University Press.

Richards, J. L. (1988). *Mathematical visions: the pursuit of geometry in Victorian England.* Boston: Academic Press.

Senk, S. L. (1989). van Hiele levels and achievement in writing geometry proofs. *Journal for Research in Mathematics Education,* 20(3), 309-321.

Sibley, T. Q. (1998). *The Geometric Viewpoint: A Survey of Geometries.* Massachusetts: Addison Wesley Longman.

Simmons, M., & Cope, P. (1990). Fragile knowledge of angle in turtle geometry. *Educational Studies in Mathematics,* 21(4), 375-382.

Sobel, M. A., Maletsky, E. M., Golden, B. N., Lerner, N., & Cohen, L. S. (1986). *Geometry.* New York: Scribner Educational Publishers.

Swetz, F. J. (1994). 'The evolution of mathematics in Ancient China'. In F. J. Swetz (Ed.), *From Five Fingers to Infinity* (pp. 319-329). Chicago and La Salle, Illinois: Open Court.

Treffers, A. (1993). Wiskobas and Freudenthal *Realistic Mathematics Education. Educational Studies in Mathematics,* 25(1/2), 89-108.

van den Heuvel-Panhuizen, Marja. (1994). Improvement of (didactical) assessment by improvement of problems: an attempt with respect to percentage. *Educational Studies in Mathematics,* 27(4), 341-372.

van den Heuvel-Panhuizen, M. (2001, November). *Realistic Mathematics Education as work in progress.* Paper presented at the meeting of The Netherlands and Taiwan Conference on Mathematics Education, Taipei, Taiwan.

van den Heuvel-Panhuizen, Marja. (2003). The didactical use of models in realistic mathematics education: an example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics,* 54(1), 9-35.

Webster Dictionary & Thesaurus: of the English Language. (2002). *Angle, (*pp. 37-38). Canada: Trident Press.

Wortzman, R., Harcourt, L., Kelly, B., Morrow, P., Charles, R., Brummett, D. C., et. al. (1996). *Quest 2000: exploring mathematics (teacher's guide & journal).* Canada: Addison Wesley Publishers.

Wubbels, T., Korthagen, F., & Broekman, H. (1997). Preparing teachers for realistic mathematics education. *Educational Studies in Mathematics,* 32(1), 1-28.

Appendix A

University Students' Angle Definitions

This appendix contains the responses to the question of "How do you define an angle?" All of the responses were collected from university students in the same first year Linear Algebra course. I asked the class to write their responses on pieces of paper that I collected. All of the students who responded were either majoring in Engineering or Computer Science. The students were told that it was completely voluntary to respond. The entire class was informed that the responses would be used informally in my thesis research.

Student Responses (in no particular order):

1. The amount of rotation that is formed from the point where 2 lines intersect.

2. An angle is the degree at which two intersecting lines rotate off each other.

3. An angle is any straight lines intersecting together forming a corner between them it could be 2 or more lines intersecting together.

4. The opening created at the point of intersection between two lines, planes, surfaces, etc.

5. Its one aspect of an event or problem. An angular projection (projecting corner). Maybe to turn in a different angle?

6. Two unparallel rays that share a common point (vertex of the angle) and the value of this angle defines how far the second ray is from the first one, expressed in terms of degrees or radians.

7. Later it became clear to me that value of an angle is defined by dependence between length of the arc formed by the rays mentioned above and distance to the arc from the common point these rays share (radius of the circle).

8. The figure formed when 2 "vectors" or "lines" share a common endpoint.

9. An angle is any spaces formed by 2 lines that have 1 point in common.

10. An angle is the measurement of rotation between two lines that intersect.

11. An angle is the magnitude between two rays that are connected at a vertex. The magnitude can be described as the amount of rotation of one ray in relation to the other

12. Any 2 lines that coincide at one point and then each one goes in a different direction

will form an angle.

13. I would take two lines that intersect and I would think about the angle as the rotation that it would take one of the lines to "merge" with the other - basically so that they are one. Easier to visualize rather than describe.

14. Angle is the part formed by the joint point of two lines.

15. An angle is the inclination between two lines (or planes) measured in degrees.

16. An angle is the union of two rays having the same end point which is called a vertex of the angle, these rays are called the sides of the angle.

17. An angle is the distance in degrees, between two lines that intersect at one end.

18. Angle is a length of arc cut from unit circle by two intersecting lines of some length.

19. An angle is a pair of lines that intersect at one point to form a corner (of sorts) that can be measured by the arc the two lines create from their center point.

20. An angle as the measure of the space between any two straight lines that meet up at one point.

21. I would define an angle as 2 line segments with one common vertex, the angle is the degree of separation between the two lines.

22. The assumption of things that between some lines starting from one point are angles.

23. The term angle means, two lines that cross at a point in space, which can be perpendicular, acute, or obtuse. Two lines cannot be parallel (usually written in degrees).

24. An Angle: Measurement of a circle from some origin. By going clockwise or counter clockwise from some origin.

25. An angle is the difference of inclination of two segments.

26. An angle is the distance between 2 lines when the 2 lines are connected at a point.

27. Its an intersection of two lines at a point, or two lines that start at one point and goes in different directions and by that they form an angle.

28. Measures the rate of how sharp or unsharp two lines intersect.

29. Angle is the measurement of space between two intersecting lines.

30. A angle is the relationship between two intersecting lines and their direction of path.

31. An angle is the opening between two lines that have the same initial point.

32. An angle is the steepness of a slope.

Appendix B
Lesson Plan

Directions: Follow the activities in order listed.

Materials needed:
-Handout containing context pictures and worksheets for each student
-Set of blank transparencies for each student
-Non-permanent pen for each student
-Transparencies referred to throughout the lesson for teacher

Activity 1: Introducing angles and right angles as well as identifying them.

Place transparency #1a on the overhead
Refer to the first picture and tell the class "This road has a turn, or an angle."
Refer to the second picture and tell the students "This road does not."
Place transparency #1b onto #1a on the overhead
Refer to the first picture and tell the class "The red line and the blue line cross."
Refer to the second picture and tell the class "The red line and the blue line do not cross."

Place transparency #2a on the overhead
Refer to the first picture and tell the students "These roads cross at an angle."
Refer to the second picture and tell the students "These roads cross at a right angle; they are perpendicular."
Place transparency #2b onto #2a on the overhead
Refer to the first picture and tell the class "The lines cross at an angle."
Refer to the second picture and tell the class "The line cross at a right angle."

Place transparency #3 on the overhead
Refer to the first picture and tell the students "Where the lines cross the angles are not equal on both sides."

Refer to the second picture and tell the students "Where the lines cross the angles are equal on both sides. When the angles on equal on both sides, they are right angles."

Place transparency #4 (Road map) on the overhead
Ask the students to turn to his or her copy of the road map that is in the handout.
Tell the class "This road map is full of angles, some of them are right angles and some of them are not."
Using a clean transparency, ruler and non-permanent pen, demonstrate to the class how to copy an angle from the road map.
Ask the class to follow your example and copy a number of angles (maybe 10) from the road map onto the clean transparency labeled "Road Map".
Ask the students to label the right angles in the angles they have copied.

Activity 2: Finding angles in context pictures.

Ask the students to look at the pictures on page 2 of the handout.
Tell the student "Each of these pictures have angles in them."
Ask the students to find the angles in the pictures and copy the angles onto the clean transparencies labeled "Scissors" and "Mountains".

Activity 3: Introduce the flat angle

Place transparencies #5a and #5b (the identical right angles) on the overhead. Make sure the right angles open in opposition directions and are on the same axis. (See diagram 1 for an example)
Tell the students "Two right angles make a flat angle." While you tell them this, slide the two transparencies so that the two transparencies line up and form a t-intersection.
Refer to the new figure and tell the students "The arms of a flat angle lie in a straight line."
Place transparencies #6a and #6b on the overhead
Like with the previous transparencies, slide these two together to line up.
Tell the students "If two angles are added together and make a straight line, their sum is a flat angle."

Ask two students to stand up. Have both students make a right angle with their arms in opposition directions. Have the two students slide shoulder to shoulder to demonstrate that the two right angles make a flat angle. Then have the students hook pinky fingers on the hands that are held out in front and touching. Then demonstrate how the students can move those arms back and forth to form different sizes and angles that will still sum up to a flat angle. Get the rest of the class to stand up and try the activity with a partner. (See diagram 2 for an example)

While students are still standing, have two groups of two students each stand back to back and have the students with the outside arms hook pinky fingers. This demonstrates to the students that the flat angle is definitely a line.

Activity 4: Adding up other angles

Adding up a triangle's angles: Have the students draw any type of triangle in the handout on the page labeled "triangle". Make sure that each student makes a triangle that is different than his or her neighbor's. Tell them to use three little pieces of transparency to copy the three different corner angles. Then have students add up the angles and see if they can see something special about the resulting added up angle. Ask the students to look at his neighbors work and see if there is something special about their added up angles. If they do not come to the conclusion that the angles add up to make a straight line (or some variation of that conclusion), then have the students repeat the exercise with a different triangle.

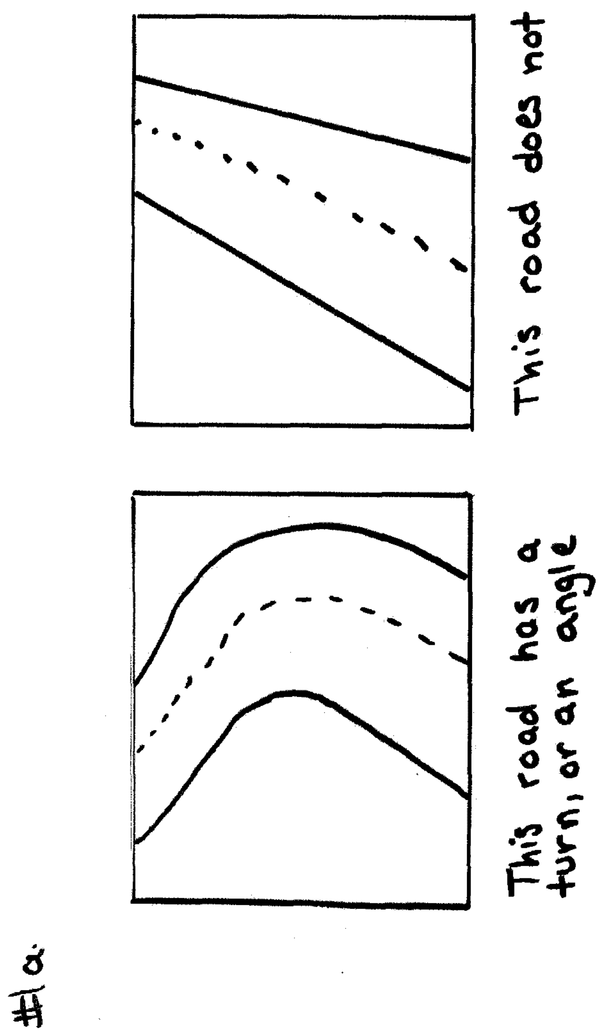
Adding up quadrilateral angles: Have students draw any four-sided shape (does not have to be a square or even have right angles) in the handout on the page labeled "quadrilateral". Make sure that each student makes a quadrilateral that is different than his or her neighbor's. Tell them to use four little pieces of transparency to copy the four corner angles. Then have the students add up the angles and see if they can see something special about the resulting added up angles. Ask the students to look at his neighbor's work and see if there is something special about their added up angles. If they do not come to the conclusion that the angles add up to make a complete circle (or some variation of that conclusion), then have the student repeat the exercise with a different four-sided shape.

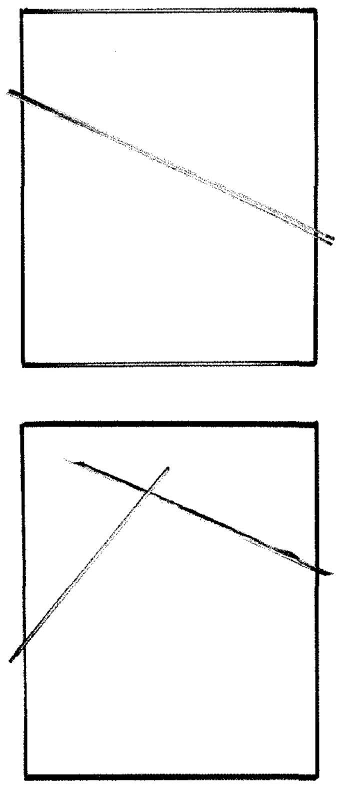
Activity 5: Journal entry

On the last page of the handout ask students to answer the two questions in complete sentences.

Appendix C

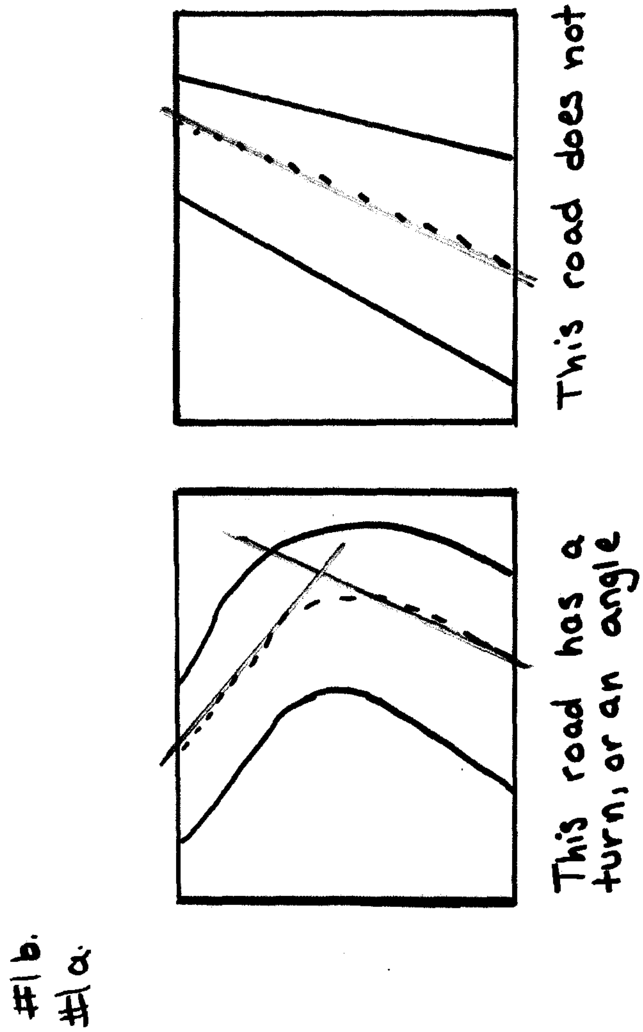
Teachers' Transparencies



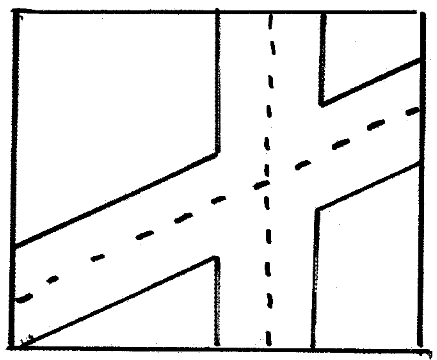


#16

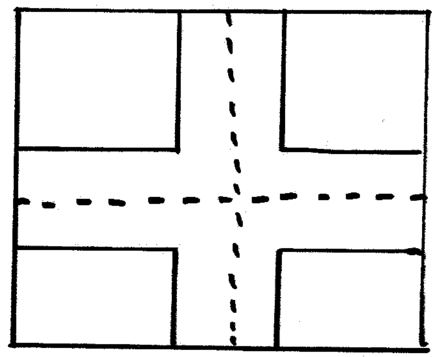
Transparencies #1a and #1b were meant to be shown together as in the figure below.



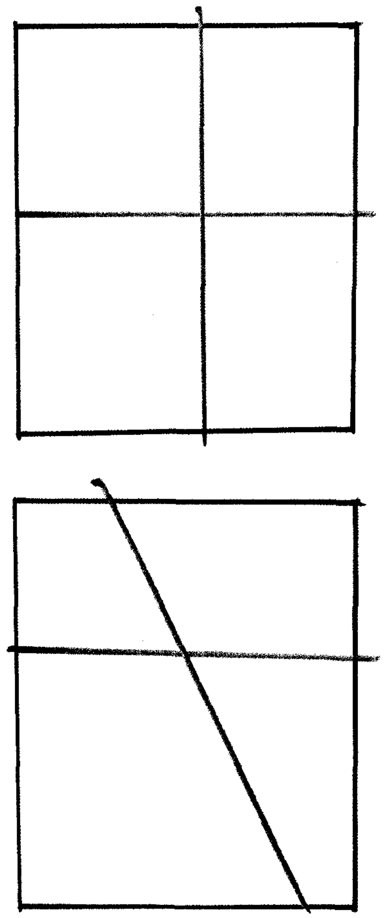
#2a.



These roads cross at an angle.

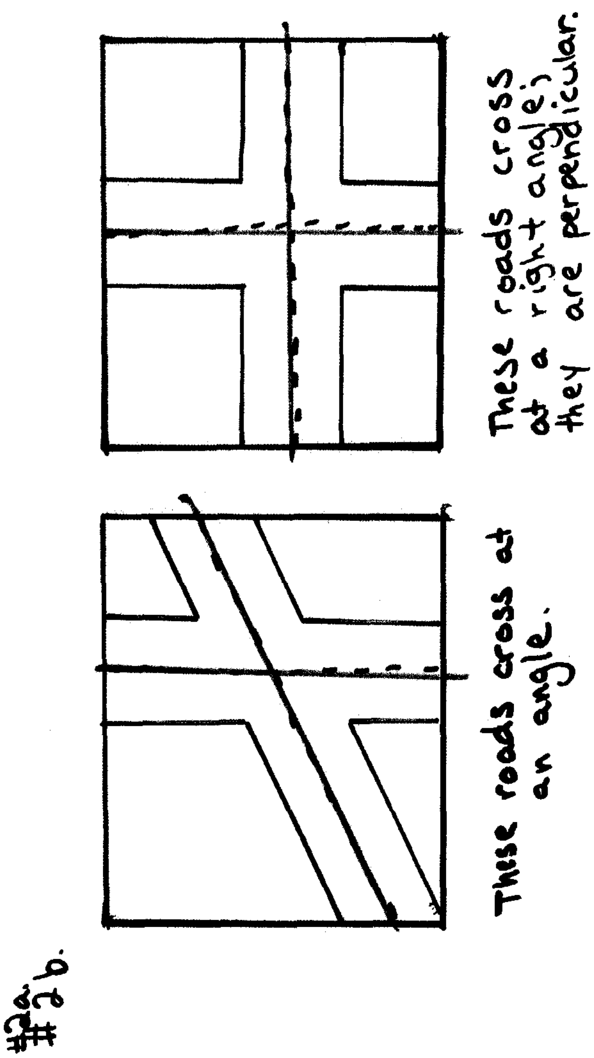


These roads cross at a right angle; they are perpendicular.

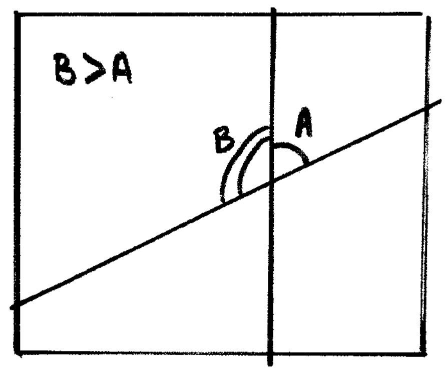


#3b.

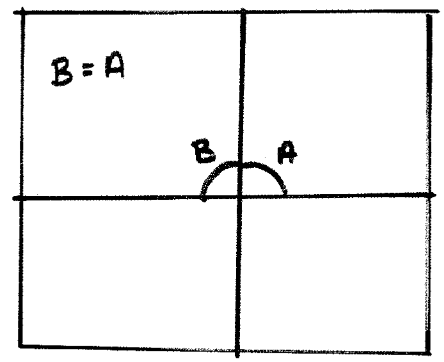
Transparencies #2a and #2b were meant to be shown together as in the figure below.



#3



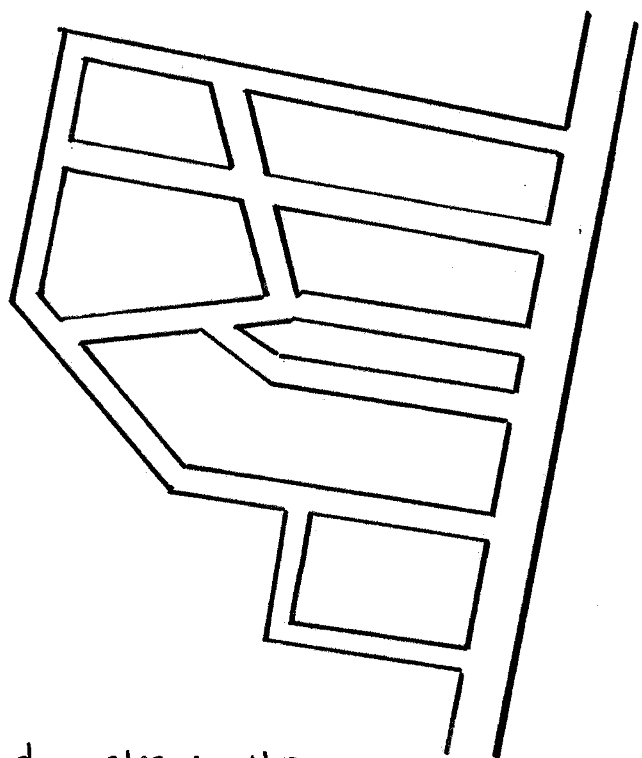
Angle B is larger than angle A.



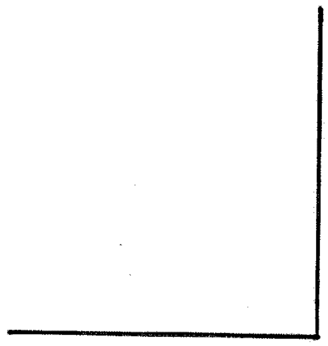
Angle B is equal to angle A.

#4

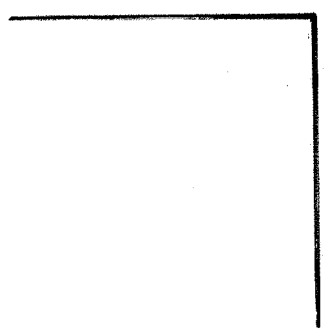
Road Map



- Find angles in the figure
- Label the angles that are right angles

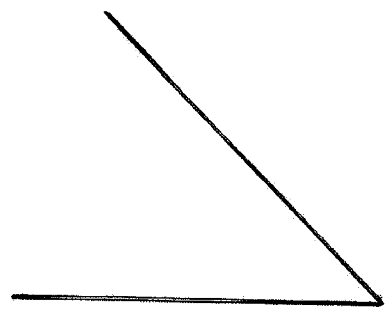


5a.



5b.

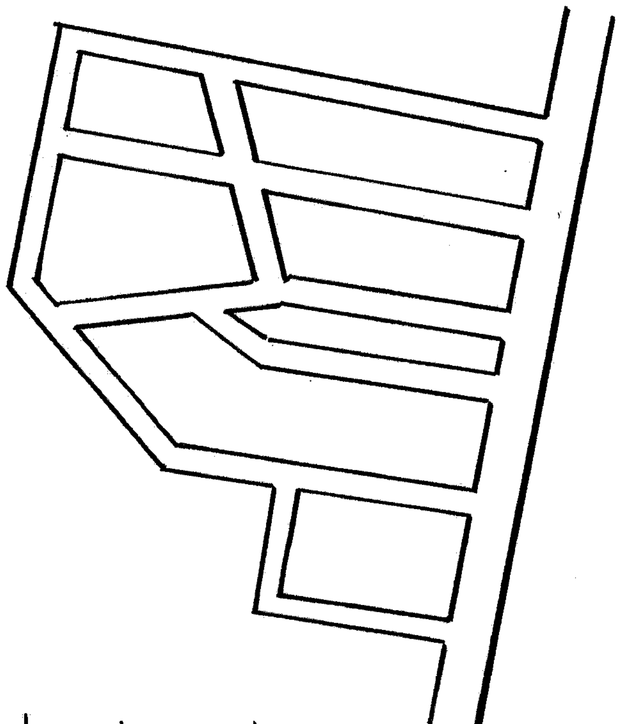
#6a.



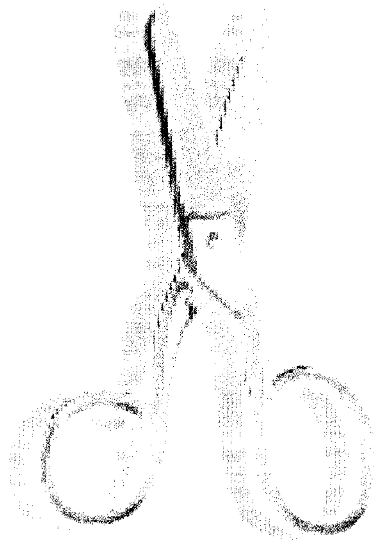
#6b.

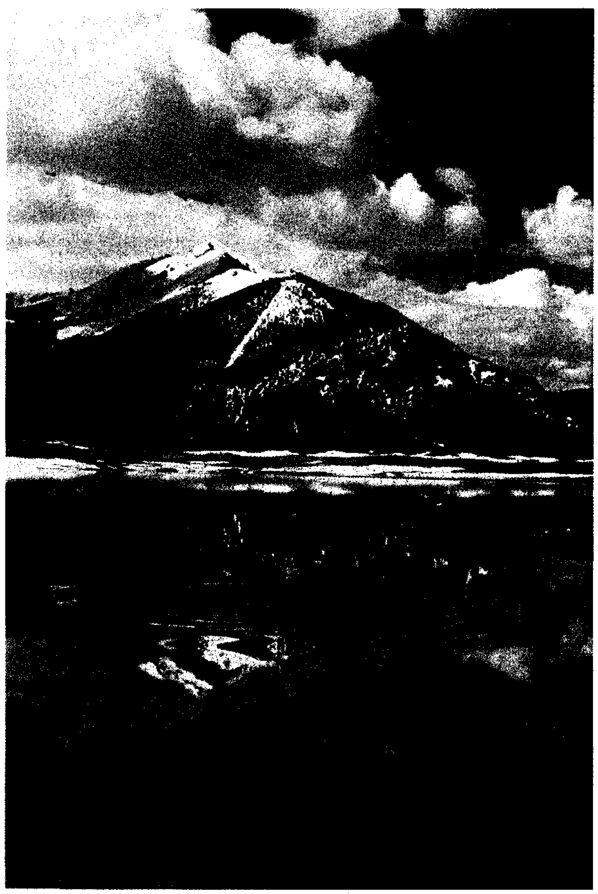
#4

Road Map



- Find angles in the figure
- Label the angles that are right angles





Triangle:

Quadrilateral (four-sided shape):

What was math class about today (answer in full sentences)?

What did you learn (answer in full sentences)?

Appendix E

Audio Transcripts from In-Class Studies

Part I: Classroom 1

Classroom 1 audio transcript

Note: The audio recording was done as the lesson took place in the classroom during school time. The students were unaware that they were being recorded and were note distracted by the recording device. The "T" stands for audio responses that were made by the teacher and "S" followed by a number refers to a response given by a student in the class. The number does not refer to any particular student, only to the chronological order in which the response was made. As well, if it was very obvious that the same student made two comments in a row, the number was repeated to show this.

T: Boys and girls come up to the front and be quiet please. Face the board and make sure you can see.

Hear background noise of people moving around.

T: Okay Boys and girls. Now we have a picture of some roads up here. The road on this side has a turn or it could also be called an angle *(teacher points to the road on the left of transparency #1a)*, if we put the lines on properly. *(Can hear one student make an awing sound like they are seeing something new for the first time. Cannot really tell if it is sincere.)* The road on the other side does not *(teachers points to the road on the right of transparency #1a)*. Okay. Now as we put lines on here *(teacher puts transparency #1b on top of #1a)* and you look at this one *(teacher points to the road on the right)* we can see that is does not have any turns in it because the two lines go side by side. What is it called when lines goes sides by side? Hands up!

S1: Parallel

T: Parallel, perfect. But if you look at the second picture over here the red line and the blue line cross *(teacher is referring to the road on the left.)* And what do they form when the two lines cross?

S2: It forms a triangle so it is able to measure the angle.

T: It is not forming a triangle honey. Just over here *(teacher points specifically to the areas where the lines intersect)*. The red and the blues lines cross and what do they make.

S3: A "T".

T: It does make a "T" and what does it form with the "T". What is it making?

S4: It is making a little "X"

T: *(teacher doesn't respond to that student and calls on another student to answer).*

S5: An angle

T: Nice and loud dear.

S5: An angle

T: It is making an angle. Okay. These roads cross at an angle. The roads cross at a right angle. If the roads cross at a right angle, if they cross like thing. Does anyone know what the words are that tell what these line are like?

S6: Parallel

T: Um, we talked about parallel lines, we really didn't talk about these. They are perpendicular. That is a great big huge math term. And they are perpendicular when they cross like this *(teacher is still referring to slides #1a and #1b)*

Silence and pause while the teacher gets organized

T: Okay, here we have some more roads *(teacher is referring to slide #2a).* These roads cross at an angle. These roads cross at a right angle. And what kind of lines are they when they cross at a right angle.

S7: Um they are... *(student trails off and doesn't answer immediately so the teacher calls on someone else.)*

S8: I think they are perpendic... *(trails off as though he does not remember how to say the word.)*

T: They are perpendicular. Now if you look at the angles here what can you tell me about them *(teacher is referring to the roads on the right of slide #2a).* Hands up if you can tell me. The lines are perpendicular. What kind of angles are made? Only one hand. I don't think so. I'm going to wait for some more. *(Pause while teacher waits for students to think and then calls on one student).*

S9: They form right angles.

T: They form right angles. They are equal on both sides. One is not larger than the other, which is what is happening over here *(referring to he roads on the left of slide #2a).* We have what type of angle right here? The little small angle.

S10: Acute angle

T: Acute angle. What type of angle do we make over here?

S11: Obtuse

T: Obtuse, good job! But over here if you look where the lines are perpendicular all of the angles are equal and they are all right angles.

Silence as teacher moves onto the slide

T: Okay, we have a roadmap here *(referring to slide #4).*

S12: It is upside down!! *(other students join in and agree).*

T: Let's put it that way *(puts transparency #4 so that the words are on their side).*

S13: No! Look at the words.

T: But let's put it that way.

Many students start calling out that it is upside down and yelling for it to be turned.

T: Do you think it makes a difference?

Some students call out yes and some students call out no.

T: Explain why? Hands up. Other than the words. Forget the words. Does it make a difference?

S14: Umm, I'm trying to guess.

T: Okay, look at the angles. If they are this way, this way, this way or this way *(teacher rotates the transparency while saying this)*? Do the angles change?

S14: No

T: No, so it really does not matter. But we will put it this way if that is what you would like *(puts the transparencies so that the words are facing the right way).*

T: Alright. Now, this roadmap is full of angles. Some of them are right angles and some of them are not. Opps, I think I missed something *(teacher realizes right now that she missed slides #2b and #3 previous but does not go back to them and keeps referring to slide #4)*. Anyway, some of them are not. Now if I put this transparency over this *(places a blank transparency over slide #4)*, which is what you are going to be doing in a few moments. What you will need to do is go through like that and outline the angles *(teacher copies an angle from the Roadmap)*. What type of angle did I just make?

S15: 90 degree *(student shouts out without being called on and the teacher ignores him)*

S16: *(Teacher actually calls on this student)* Right angle

T: A right angle. This angle *(teacher refers to a different angle she has copied onto the transparency)*, hands up. Yes?

S17: A right angle

T: No, this one right here.

S18: Acute angle

T: Acute angle. Umm, this one *(teacher refers to a different angle she has copied onto the transparency)*.

S19: Do we have to do all of them *(student interrupts and is ignored by the teacher)*

S20: A right angle

T: A right angle. This one *(teacher refers to a different angle she has copied onto the transparency)*.

S21: Obtuse

T: Obtuse angle. Now in a moment I am going to have you go back. In the little booklet you have on your desk there is this very same roadmap on paper. And in it is a blank transparency. I want you with the marker and the transparency to find at least ten angles. Now, any one that is a right angle I want you to mark it. Now who can remember how we mark right angles. Other than just saying it is a right angle. What little thing did we do with them?

S22: We made a bump.

T: No *(teach calls on someone else)*.

S23: We do a square.

T: We put a square in there *(referring to the corner of the angle)*. So that is what I would like you to do, if it is a right angle. If it is not just do the hump.

S24: Do we need to write the number?

T: No, we are not measuring. Now are there questions.

S25: Do we have to do like those lines?

T: *(Teacher did not really understand what the student was asking and neither do I)* I want you to go over and trace over top of every angle you can find and you should have at least 10. If it is a right angle you put the square in there *(referring to the corner)*. If it is not you put the curve. Now are there any questions. Okay *(teacher dismisses students by name to go back to their desks.)*

Noise of students moving around and going back to their seats

S26: What do we do with these things on our desk? *(referring to the materials on the desk)*

T: You use them *(teacher responds in a very irritated voice)*

T: Boys and girls, find the overhead that says roadmap and use it *(when the says overhead she is referring to a transparency)*. You don't need anything else except that. Now I did put a ruler on your desk if you need it to help you make the lines a little straighter. Use it please.

Noise of activity taking place

T: *(teacher stops at one desk)* You have to mark the right angles. I don't want the words, I want a square.

S27: Umm, are there really right angles on these.

T: Well, that's you job to find out.

Noise of activity taking place

S28: Are we allowed to do more than 10?

T: You do as many as you can.

Noise of activity continuing

S29: Do we do straight angles too? *(this was asked about half way through the activity)*

T: I said do whatever angle you can find.

Noise of activity continuing (classroom is relatively very quiet and teacher can be heard shushing students as well)

T: *(after about five minutes)* Boys and girls put your hand up if you think you need a little bit more time. Okay, how about one more minute. If you are finished just wait nicely.

Noise of students finishing up and getting a little restless

T: Boys and girls, put the lid of your market back on, put it on the table and just watch from in your seat. Now, just turn so you can see me. Marker away, let go. Now, I want you to turn to the second page and then look back at me. *(Noise of pages turning).* The picture of the top is what?

S30: Scissors

T: Bottom picture is what?

S31: Mumble

T: Can't hear you!

S31: Mountains

T: Mountains, right. So if you are looking at those, hands up if you can tell me if they have any angles or not. If you look at the scissors, hands up if you think there might be some angles in there.

S32: Do you want me to show you?

T: No, just do you think that you can find angles in a pair of scissors or in every day life things. How about the picture at the bottom, the mountains, do you think we will find angles in the mountains?

S33: Do you mean in the picture.

T: Yes, in the picture

S33: Then I would say probably not.

T: What do you think *(teacher asks a different student)?*

S34: I think we are going to.

T: Can you give me an example of what you think might end up being an angle in that picture?

S34: The clouds

T: Possibly some portion or some point of the clouds. How about you *(referring to a different student)?*

S35: The tip of the mountain.

T: Yes, the tip of the mountains. Anything else you think you might find in there. Well actually, I'm going to let you do that and you might see. On your desk you have a little overhead that says scissors and one that says mountains *(again, by overhead the teacher actually means transparency).* Use your marker and let us see how many angles you can create from those pictures. Make sure you put them on the right overheads or the right little transparencies.

Noise of activity taking place (students working pretty quietly).

T: Boys and girls if you are finished with your roadmap would you just hold it up in the air please and we will collect them. *(Noise of teacher collecting roadmap to give more room on the desks)*

Noise of activity taking place (about five minutes)

T: Hands up if you need a little bit more time. Okay, take your time.

Noise of activity continuing

T: Boys and girls, hold the mountain one up *(noise of teacher going around and collecting them).* Just your mountain one right now.

Noise of students continuing to work.

T: Now hold your scissors one up. And put the lid on your marker. *(Teacher moves around and collects scissors transparency).* Boys and girls, for a couple of minutes I need you to come and sit again on the carpet. Leave your marker where it is.

Noise of students moving

T: Now boys and girls, what types of angles are these *(referring to slides #5a and #5b, the two right angles).*

S36: Right angles

T: They are right angles. What do you think might happen if I slide them over together so that they line up? What do you think might happen?

S37: They will make a triangle.

T: What do you think *(referring to a different student)?*

S38: They will make a square if you put that one on top.

T: Yes, if I turn them around. But what if I just slide them over easily like this *(demonstrates sliding the slides a bit).* What do you think is going to happen? What is it going to make?

S39: An upside down "T"

T: No, what if I slide them like this *(slides the angles again).* What if I slide them all of the way together *(slides the angles all of the way together)*?

S40: A straight angle.

T: We have a straight angle. So if we put two right angles together the line is flat, it is nice and straight. When you put them together it makes a straight angle.

S41: And it also makes two right angles.

T: Yes, you are right but we are talking about putting the angles together. It is just a total straight line, isn't it?

Teacher pauses before moving on to next activity

T: I need two volunteers. *(calls on two students)*

T: Now stand up, right up there in front. I will turn off the overhead for a moment so you are not in the spotlight. Now if you used your arms, do you think you could make a right angle. Let's see, how could you do that.

The students try but don't really succeed.

T: No, make them with two arms. Okay, there is one way. No, you need to put both arms. Can you do it in a different way? Does anyone have a different way?

Student stands up and does a right angle in different ways.

T: But I said you needed to use two arms hon.

Teacher gets some other students to try it. Nobody comes up with the configuration that is in the lesson plan.

T: What if I put my arm out like that *(teacher puts arm straight in front)*? Where would I put my other arm to make an angle. Can you show me now? If you have one arm out straight where would the other one go. Okay, but where else could it go. *(Teacher let's some students try).* Okay, how about we do it like this *(teacher just shows the students the configuration that is in the lesson plan).*

T: Okay, *(turning back to the two students up front)*, let's see if you both do that. Now girls slide together. What are they making? What type of angle are they making when they put the two together?

S42: A straight angle.

T: A straight angle. See, from the tip of this hand, straight across to the tip of that hand, we got that to work. Now let's see if you can with whoever you are closest to stand and up see if you can also make two right angles and bring them together to make a straight angle. Whoever you are beside.

Noise of students trying the activity and the teacher helping out where necessary. A lot of background noise.

T: Alright *(teacher pauses as though reviewing what to do next and then makes a sigh of irritation)*, I need you four boys to stand up. The rest of you sit down. You two put your backs to them and do the exact same thing with two right angles and a straight angle. Now, lock your hand together and there you might even be able to see a little bit better how the line goes from here straight across to here. Okay. Now you need to sit down. Actually you can return to your desk and turn to the page that says triangle and just wait there for me.

Noise of students going back to their desks

T: Now, rulers down, pencils down, markers down, everything down.

Noise of students putting odd items back on their desks

T: Nothing in your hands and make sure you are listening. Now, what I need you to do on this page at the top, not at the bottom, is I need you to draw a triangle with the ruler. If you don't have a ruler on your desk you have a protractor which is something you can make a straight line with. When you are finished that, in the pocket, in the envelope, you need to take out three of the pieces of transparency and trace your angles *(does not demonstrate this at this point).*

Teacher pauses

T: With those three pieces of paper *(teacher holds up three pieces of transparencies)* I want you to put them together, move your angles together here and see them here and see what happens. Make sure you use three little transparencies to trace the angles from the triangle you have drawn.

Noise of students starting to work

T: We are not measuring the angles, we are tracing them and putting them together to see what happens. Draw the triangle with the pencil and then trace each of your angles.

Noise of students working

T: Any triangle, does not matter what triangle.

A lot more background noise of the teacher having to help individual students with the initial tracing of the angles. A lot of students did not understand what to trace.

T: Now see if you can fit the angles together in some way. Put your hand up when you put your angles together to make some *(teacher does not explain how to put the angles together).*

Background noise of work going on

T: Okay, boys and girls look up at the board and see how these angles are touching *(referring to the two right angles on the board from transparencies #5a and #5b).* All of your angles should touch.

Noise of teacher moving around and helping out

T: We have someone who has discovered what it makes. How about anyone else?

Noise of some students making responses, but nothing that is understandable.

T: Okay, Billy *(name has been changed)* put his angles together. Can I use yours on the overhead Billy. Billy put his angles together and it ended up like this.

S43: That's what mine did too *(sounded excited).*

T: What did they end up making when they were all together.

S44: A straight angle

T: It ended up making a straight angle. They all fit. When they are all side-by-side like that they make a straight angle.
T: Now on the bottom of that page I need you do draw a quadrilateral, which is what?

S45: A four-sided figure

T: Perfect. Remember it doesn't matter what the shape is as long as it has four sides. Do the same thing as the triangle and see what shape you can make or what will happen when you put the angles all together.

Noise of activity taking place

T: Boys and girls, when you are finished with little transparencies they go back in your envelopes.

Noise of activity taking place

T: Once you have your quadrilateral trace your angles and fit them together and see what happens.

Noise of activity taking place

T: Okay, we have a few people who got it. It works out to be a complete circle around. Now boys and girls please stop and look over here. I need you to put the little pieces back in the envelope. On the last page it says, "What was math class about today (answer in full sentences)?" and "What did you learn (answer in full sentences)?" Write on every other line. You finish this before you go to recess. I expect at least four lines.

Noise of students completing activity.

S46: Um, I want to know if this is right *(asking the teacher to come and look at what he had written).*

T: You know what, it is not my answer but what you think is right.

Noise of students completing activity. Can hear the teacher encouraging students to keep working when they were getting distracted and off task.

T: Boys and girls, be very detailed. We just spent an hour doing this activity so I should be able to see more complete answers then what I am seeing.

Noise of students completing activity. No more instructions given to the class in general. Recorded a few minutes of the students finishing up and handing in their booklets.

Part II: Classroom 2

<u>*Classroom 2 audio transcript*</u>

Note: The audio recording was done as the lesson took place in the classroom during school time. The students were unaware that they were being recorded and were note distracted by the recording device. The "T" stands for audio responses that were made by the teacher and "S" followed by a number refers to a response given by a student in the class. The number does not refer to any particular student, only to the chronological order in which the response was made. As well, if it was very obvious that the same student made two comments in a row, the number was repeated to show this.

T: So before we begin does anybody have any idea what we are going to talk about today? We are going to talk about Geometry. So first of all, you all seem very bright and awake, which is important. Can you guys tell me what is different or what major difference there is between these two roads *(teacher refers to slide #1a)*. Yes *(teacher calls on student).*

S1: One is straight and one curved

T: Yes, this first one is curved *(refers to road on the left)* and the second one is straight *(refers to road on the right)*. So this road has a turn or an angle *(refers to the road on the left of transparency #1a)*. Who has heard the word angle before? Raise your hands if you have. *(A majority of the students raised their hands)*. Good, okay. Now, this one of course, does not have an angle *(Refers to the road on the right of transparency #1a)*.

T: Now if I put another transparency on top *(teacher puts up transparency #1b on top of #1a)* and see where it has drawn two lines in the middle of the road you will see that over here *(refers to the road on the left of transparency #1a)*, as you already have seen, these two lines form an angle and over here these two lines do not cross so they do not form and angle.

T: Okay, moving on to the next slide *(teacher puts up transparency #2a but actually places it on opposite of how the roads are drawn. That is, the transparency is ink side down. This does not make a big difference by is opposite to how it is discussed in the lesson plan)*. Who can tell me what the major difference is between these two roads? Or if you can even explain to me how it looks to you. *(Teacher calls on a student)*

S2: Um, one is a complete straight four-way and the other one is a diagonal four-way.

T: Okay, which one, the first one is the *(teacher points to the roads on the left)*...

S2: The first one is the straight four-way.

T: What do you mean by four-way.

S2: It is like a four-way stop.

T: Okay, and these roads *(referring to the diagonal intersection or the roads on the right)* are four-way as well.

S2: But they are across.

T: But these ones *(referring to the right angle intersection)* are across as well.

S2: Yes, but that one is straight across and the other one is diagonal at an angle.

T: Okay, can you tell me which one has an angle *(referring to the roads that cross diagonal)?*

S2: The one going that way *(referring to the line that is not horizontal in the diagonal intersection roads).*

T: Okay, so we all see that these roads cross at an angle. Can everyone see that? If not, I'll show it on another transparency in a minute. These roads cross at a right angle. Do they remind you of a letter perhaps?

S3: It looks like a "T"

T: Looks like a "T", okay. And when we have these roads crossing straight like this they are perpendicular. So we can see that if I put this other transparency over it we see that this one is like a "T" and a right angle *(teacher places transparency #2b on top of #2a).* Who can point to the right angles for me? *(calls a student up to the board to find a right angle)*

S4: This one

T: That's right. Okay, I'm going to show you something else with another transparency *(teacher puts up transparency #3 and takes away transparency #2b).* So you see where I've drawn those little curves. Now if you look at over here at the first picture, the angle over here and the angle over here are. Who can tell me if A is a bigger angle or if B is a bigger angle? Who says B, raise your angles if you say B *(most students raise their hands).* Oh boy, I guess I can't trick you guys. Okay, so angle B is larger than angle A. Now how about this one over here? We see that B is equal to A. So these angles are exactly the ...

Class: Same

T: Same, very good. So now that I have seen how much you guys know about angle, which is a lot, I would like you to look on your desk. You will find a package of sheets. Now on the first page you will find a picture of a roadmap. The page should look like this *(teacher puts up transparency #4)*. Now what you need to do with the little transparency you have that is called roadmap, it is blank and says roadmap at the top. Did everyone find it? Okay, hold it up once you have it. Now I want you to lay it on the picture. And we should probably just find two angles together up on the board first. So I need somebody to come up and here and tell me where on this roadmap you see angles. *(teacher calls on a student)*

S5: Here *(student follows a straight line)*

T: Are you sure here? How about over here? *(teacher points to an angle and the student agrees)* So there is one over here. *(Teacher copies the angle onto her transparency and automatically puts an arc in)*. And now let's try to find another one. Now is there another angle somewhere in this roadmap? *(teacher calls on a student to come up and point)*. Ahh, over there, there is another one. So let's copy it and if you need to use a ruler to copy it you should. So we have another angle over there. So once you have drawn those two angles you can find eight more angles and I'm giving you three to four minutes. And if the angles are right angles I want you to label them *(teacher puts a little dot up by the right angle she had previously copied).*

Noise of children working

S6 (question): Do you mean that if it goes to the right hand side I label it right?

T: No, remember the roads we looked at and what a right angle looks like. Look for the letter "T" to find the right angles. And then label all of those you think are right angles.

Noise of children working

T: You could also write a big R for a right angle to label those that are right angles.

Noise of children working

T: Okay great. Now, can everyone look up here. My next question is, do we only find angles in textbooks and math workbooks. Raise you hands.

S7: No

T: Okay, so can anybody give me some other examples of where we can find angles?

S8: Your leg.

T: Okay, your leg. Next?

S9: Umm, with your hand.

T: Can you show me? *(student points to angles between their fingers)* Good. Anyone else?

S10: With your arm.

T: Can you show me how? *(student bends their elbow and makes an angle)* Pretty good. Okay I would now like you all to turn to this page of your handout *(shows the students the page with the pictures on it).* We have scissors and we have mountains. Now before everyone just mentioned body parts, I want to know from your guys, can you find angle in these two things, the scissors and the mountains. Raise your hand if you think you can only find angles in the scissors *(some students raise their hands approx. 5-6).* Okay raise your hand if you think you can only find angles in the mountains *(again some students raised their hands, about the same amount).* Raise your hand if you think you can find angles in both of them *(many more students raised their hands making me think some people changed their answers once this became an option).* Okay, the task for you is for you guys to find all of the angles. You have two transparencies, one labeled mountains and one labeled scissors. Okay, you have two of them. Lay them on your picture and you do exactly the same thing you did with the roadmap and you are going to trace the angles that you find. Okay, whatever size they may be, whether they are right angles or not. So you have two minutes for that.

S11: How much do we do

T: As many as you can find.

Noise of students working

T: Someone just had a good question. Does the reflection of the mountain count.

Class: Yes

Noise of students working

T: Okay, 60 more seconds.

Students working

T: Okay, time is up please. Eyes up front. Over here now, I have two angles which are called what again? *(Teacher is referring to slides #5a and #5b)* There lines are perpendicular to each other so what are they.

S12: "can't hear what is being said"

T: That's true that it starts with the letter R. Can you tell it to me *(teacher asks the class in general)*?

Class: Right angle.

T: Now what would happen if I push these transparencies closer and closer together? What is going to happen? I have haven't pushed them all of the way together yet. What is going to happen? *(Teacher slides the transparencies slowly together but not all of the way.)*

S13: They are going to have a car accident *(students tone suggests that he knows he is making a silly comment).*

T: These are not roads, they are just lines *(teacher responds as though the comment was not silly at all).*

S13: Okay, so they will bump into each other *(still has a silly tone of voice).*

T: Okay, so they will bump into each other. Anything else?

S14: The lines with meet up with each other.

T: Okay, that's right. So they will meet up with each other *(Teacher slides transparencies #5a and #5b together).* And if I stop them right there *(teacher stops them when they are just touching)*, now who can tell me *(nobody responds)* … okay I'm going to do it again *(teacher pulls the transparencies apart).* I had two right angles and I pushed them together and then what happened *(demonstrates pushing the right angles together again). (No students respond).* So you see these two lines here, I put them together and how can tell me about this line on the bottom.

S15: It has turned into one line that is straight.

T: So together when they fit together if I had to draw an angle here and here *(referring to the straight angle)* would they be bigger or smaller.

S16: Bigger

T: That's right. So when you have two right angles together they make a bigger angle called a straight angle. Does that make sense? A straight angle. It is easy to remember. It is straight because it is flat. Okay now, I need two volunteers. Come up to the front please.

Noise of two students coming up the front

T: Okay, so can everybody see. If you can't see you can stand up at your desks. I want you girls *(teacher is referring to the volunteers)* to make a right angle with your arms. Just by yourselves. Try to make a right angle with your arms. Okay, now put your arms like this *(teacher demonstrates making a right angle with her arms for the students).*

Noises of helping the students get their arms right

T: Now, just like we did on the overhead, get closer, closer, closer and you touch. Can anyone see what they made? Can you all see this straight angle? *(Teacher refers to the angle made from the tip of one student's hand to the tip of the other student's hand)* Good, we are going to try it in just one moment. Actually I want everybody to try it right now. Everybody stand up.

A lot of noise as the students try this. The teacher can be heard walking around helping some students and asking other students to get back on task. Can't hear the actually directions

T: *(Once the teacher regains control)* Okay I want you to stay where you are and I look at these two different angles here *(teacher has transparencies #6a and #6b on the overhead).* What would happen when I move the smaller angle and the bigger angle toward one another? What is going to happen?

S17: *a student responds but it is inaudible*

T: Okay, so the same thing basically. Now what happened here, these are two different angles but I still have a straight angle. Even though we have two different size angles we have a straight angle. Now what I want you to do is with your partner and with your pinkies attached I want you to move your arms so you can see the different angles that will make a straight angle *(teacher demonstrates this with a group of students that are near the front of the room).*

Noise of the students doing this. Again, very loud!!! Can't clearly make anything out. Sounds like a lot of students are off task

T: Okay back in your seats please *(teacher is shouting over the noise).*

Still a lot of noise as the students settle down

T: For this next activity I would like you… Okay, I'm just going to demonstrate first. I'm going to draw a triangle with the ruler. You are going to draw a triangle on your paper on the page that say triangle. So I want you all to flip over to the page that says triangle. Don't draw one yet because I'm not done explaining. Then once you are going to do after you have drawn your triangle, you are going to … Um, you should all have little pieces of transparencies in an envelope. Can everyone take out only three? I know you have more than three, just take out three transparencies.

Noise of students finding their transparencies

T: Okay, now you are going to…no what I'm going to do first, so everyone watch me now, I'm going to draw the angle that I find in the triangle *(the teachers cell phone rang in her purse at this moment and all of the students were completely distracted).*

Noise of students as the teacher breaks to turn off her cell phone

T: Okay moving on. You are going to draw each of the three angles and then I want you to try to put these angles together so that … well we are trying to put them together to see what will happen if you put them together. However, do not draw your triangle too small and make sure when you copy the angles you draw them bigger. You can start. Use a pencil on the paper and use the pen on the transparencies.

Noise of student doing activity. Sounds of teacher helping out individual students. Noise level slowly raises

T: Okay now. Eyes up here please. Now you are going to do the same thing except with a quadrilateral. Now who can tell me what a quadrilateral is?

S18: A four-sided shape

T: A four-sided shape. Yes. We can draw a rectangle or a square or whatever you may wish. And you do the exact same thing. You line the points together and the outside lines of the angles. You should have four more transparencies in the envelope.

Noise of students working. Sounds of teacher helping out individual students but can't make out what is said. Noise level slowly raises. Can even hear some students singing at one point.

T: Okay, look up here. Settle down. Now that you are having done those little experiments with the angles with two different shapes I haven't really told you what the answer is or what I want you to discover because it was up to you to discover. So now I need you to flip over to the last page.

Can hear students moving paper. Hear general groans when the students realize there is some writing involved.

T: Exactly… "Can't hear what the teacher is saying because someone is making loud groans and boos right near the recorder."

Noise of class, very loud.

T: Okay, now that there are no more activities and only writing everyone needs to quite, mouths closed, eyes on your paper and pencils writing.

After about two minutes the recess bell went.

T: Okay, leave your booklets on your desk please.

Sounds of the class leaving for recess. Since nobody was finished with their write ups the teacher asked if she could have the students finish them later in the day and then I could get the booklets from her later in the week. I agreed and only took the students transparencies with me at that this time. I received that the booklets three days later and all were accounted for.

Appendix F

Student's Written Responses

What the students wrote is written first in normal writing. The italicized writing in brackets is how I interpreted what the student wrote. The words and/or phrases that are underlined are what determined the categorization of the responses. If nothing was underlined then the entire responses contributed to the categorization. Note that the numbers correspond to students. That is, Question 1, #1 and Question 2, #1 is the same student and so on.

Part I: Classroom 1 student responses

Question 1: What was math class about today (answer in full sentences)?

1. Math class was about agles. And droing angles. Like a pare of sizers. And a road map and a maotin one. Obtuse angle, acute angle and a right angle. And a straight. *(Math class was about angles. And drawing angles. Like a pair of scissors. And a road map and a mountain one. Obtuse angle, acute angle and a right angle. And a straight.)* – A1

2. In math class I lernd aboute side and someone came in to are class. We talke about everything it was preetey fun I love it a little bit it was fun. *(In math class I learned about side and some came into our class. We talked about everything it was pretty fun. I love it a little bit, it was fun.)* – N2

3. Math was about angels and that we can find angels in pictures. We traced angels in fake roads it was one of the best math classes no make it the best. I loved it. I hope we can do it again. I liked that we used the overhear pejuctor because it helps you understand. *(replace angels with angles and pejector with projector)* – A1

4. Math is about angels and right angels and obtuse angels and straight angels and acute angel. We saw that there are a lot of angels in pictures. Everything you see sometimes has a angels. We have worked with a lot of angels and sometimes you see know angels but there could be a lot in one picture. *(replace angels with angles)* – A1

5. Today in math class we did some activities on angles. We had to trace some angles on a see through-piece of paper. We traced the angles off of a picture of a pair of scissors, some mountains and a road map. We had to try and find as many as we could. Next we traced some angles on some shapes that we made. There was also a lady here today who collected our results. – A1

6. Math calass was about angles and linig up lines a cugelteralls and math ageling and we put stuff together and we merde it was kiof hard and... *(Math class was about*

angles and lining up lines and quadrilaterals and math angling and we put stuff together and we made it was kind of hard and ...) – N1

7. Math class was about angals and we had a book of it and we lirend how to measer angals like a strate, right and acoute angle and it was rilly fun. *(Math class was angles and we had a book of it and we learned how to measure angles like a straight, right and acute angle and it was really fun.)* - AE

8. Math class was adout angels, right angels and strat angels. It was like what we did last time. And we put them together and they mad strat angels and angels, and right angels. It was also adout montens, scissors and rodes. *(Math class was about angles, right angles and straight angles...And <u>we put them together and they made straight angles</u> and angles and right angles. It was also about mountains, scissors and roads.)* – A2

9. Measuring angles and using plastic sheets. We measured angles. If the lines are cruvid *(curved)* you can use the rulers. You measure pictures like a mountain, sissors *(scissors)* and we measure angles with the pictures. - AE

10. Math class was about angles. We had to <u>copy the angles</u> on to plastick pices *(plastic piece)* of paper and then we had to draw a shape and out line the angles. After that we had to wright *(write)* what we did in class today after that we wroght *(wrote)* what we learned. – A1

11. Math class today was about angles and making your own and then putting them together to make a shape with pen and we answered on the overhead. – N1

12. Today math class today was about looking at shapes and pictures and being able to find all sorts of angles. We also made our own shapes and finding the angles that we made ourselves. We also did some stuff on the overhear projector. – A1

13. In math class today we wored whith angols. We allso did some fan ateifates. The ativaties were to fiended lots of other angols in to pethers. We did some thing weth awer armes it was to make a strat angol and a right angol. *(In math class today we worked with angles. We also did some fan activities. The activities were to find a lot of other angles in to pictures. We <u>did some thing with our arms it was to make a straight angle and a right angle.</u>)* – A2

14. Today math class was about angles and roads and if you <u>attach some angles together you make different angles</u>. On one of them it made this circle shape and at first we looked a picture of two different roads one of them had a angle of a turn one didn't and if you put a line thru *(through)* it makes a angle. – A2

15. Math class was about angles and how to put things tighter to see another shape. Drawing shapes and <u>finding angles</u> and <u>drawing the angles in the shapes</u>. – A1

16. Math class was about quadrilateral and angles. We used those see thow *(through)* paper for a projucteure *(projector)*. We found the angles in mountains and scissors. There was all these lines on the first page and we used a see throw *(through)* paper and saw how many angles there are and I counted 34 lines in the see throw paper that we had. – A1

17. Math class was about teching *(teaching)* us… - N/A

18. It was about angles and what they are, how to find one, finding angles in picshurs and objaiets and combinding angles to maek new angles. *(It was about angles and what they are, how to find one, finding angles in pictures and objects and combining angles to make new angles.)* – A2

19. Math class was about angle. We used roads and diffrint *(different)* stuff to find angles. I fond that the roads were very esey *(easy)*. We also lerand *(learned)* that if you put angels together you could make a straigt *(straight)* angle by putting to two angles together. – A2

20. Today math class was about looking at angles. When we stated math we looked at a road map. We looked for angles in the road map. After that we did sisers *(scissors)* and montenos *(mountains)*. We made a triangle and trased *(traced)* all of the angles, then tried to make a angle. We did a 4 sided shape and did the asact *(exact)* same thing as the triangle. – A2

21. Today in math class we were doing angels. We had to traces and find. First we had to do the car map then we had to do sissors *(scissors)* and montams *(mountains)* and last of all we made triangles. – A1

22. Math class was about angles. We also did shining on mekering all defrent cands of things like montens sesers and roeds. *(We also did something on marking all different kinds of things like mountains, scissors and roads.)* – N1

23. Math class was about angles and what they make when you put them together. We learned that a four sided shapes angle can make different angles. We had to use a picture to trase *(trace)* the angles and make a shape. – A3

24. Today's math class was about finding strat, right, obtos and acut angls. *(Today's math class was about finding straight, right, obtuse and acute angles.)* – A1

25. It's about angles. Today we looked for angles on a road map, scissors and a mountain. I found a lot. – A1

Question 2: What did you learn (answer in full sentences)?

1. What we learn is the learning all the angles agien *(again)*. – N2

2. We lernd adoute it side shaps and we had lerd adoute angles I had a preety hard time. It was not bad I like it. *(We learned about it side shapes and we had learned about angles. I had a pretty hard time...)* – N1

3. I learned that if we look close anoufe *(enough)* we can <u>find</u> angels anywhere. I learned that we can <u>find</u> angels with our body and in shapes. I knew most of this already and I can't waite *(wait)* till we do something like this any time again. *(replace angels with angles)* – A1

4. I learned that there are many angels everywhere you see and on drive way there are many angels. We learned that <u>when you pout *(put)* it together you can get other angels</u>. I learned all ready a lot about angels but I learned more then before. *(replace angels with angles)* – A2

5. I learned that anything can have angles if you look hard enough. I also improved my tracing skills with the excersises *(exersice)*. I also learned that if you <u>take all of the angles from a shape and connect the sides, it makes a straight angle</u> at first I didn't understand but then they explained what everything really ment *(meant)*. I learned a lot today. – A2

6. I lerd how to put liners on the rite sopet and how to mersh other stat that wert shaps and we shaped and linde up line to put them str and make them rite. *(I learned how to put lines on the right spot and how to make? other stuff? that weren't shapes and we shaped and lined up line to put them straight and make them right.)* – N2

7. We learnd how to measer angles. We used prowtraters to make the angles like the cute, strate, and the strate angles. *(We learned how to measure angles. We used protractors to make the angles like the acute, straight and straight angles.)* - AE

8. I learned that if you put stuff together it maches biffrent angels like strate angels and right angels and the ordinary angels. We learned that theres all cinds of angels and there's not guste one. *(I learned that if you <u>put stuff together it makes different angles like straight angles and right angles and the ordinary angles</u>. We learned that there's all kind of angles and there's not just one.)* – A2

9. I learned that you use plastic to measure you could use a inky pen and measure. I allso *(also)* learned to measure better and to help me learned I think it is good to do this type of math. – N2

10. In class today I learned that <u>if you put two right angles together you can make a straite *(straight)* angle</u>. – A2

11. I learned in this math class that <u>you can two right angles make a straight angle</u>. – A2

12. Today we learned how to make angles with ordinary things like our hands. We also learned how to find angles in sissors *(scissors)* and mountains. – A1

13. I learnt that if you put to right angols together it makes a strat angol. If you crose to liese together it will make a pendikler angol. *(I learned that if you put two right angles together it makes a straight angle. If you cross to lines together it will make a perpendicular angle.)* – A2

14. I learned if you use some angles and attach them together it makes anothere *(another)* angle and if you draw a line thru *(through)* a road that has a angle it makes a angle. – A2

15. I leaned *(learned)* that two strait *(straight)* angles can attach and make a strait *(straight)* angle. (I mosiety *(mostly)* knew all this.) AE

16. I learned that 2 right angles make a straight angle. *Two girls* went up front and put 2 right angles to make a straight angle. *(Names of girls were removed for confidentiality reasons.)* – A2

17. I learned about angles in pictures and solving the right angles, acute angles, straight angles and obtuse angles. I also learned what every body learned is that there is a lot of angles everywhere you go and maby *(maybe)* its on you or even close to you. – A1

18. I learned that there is angles all aroad us, combing angles togastr can maek one single angle and how to nontis a angle. *(I learned that there is angles all around us, combining angles together can make one single angle and how to notice an angle.)* – A2

19. We learned that you don't have to use angels. You can use different stuff like sissers *(scissors)* or montins *(mountains)* or even roads. Those were the one that we use today. We also learned that if you put two angels together you can make a big one. *(replace angels with angles)* – A2

20. Today we learned how to make angles. We made angles with a road map. We did sisers *(scissors)* and monteons *(mountains)*. We learned that angles can make other angles. – A2

21. Illegible. – N/A

22. I leaned adout angles veters lining paring. It was so much fun but it could have been a little bite esiyer. *(I learned about angles veters?? lining pairing. It was so much fun but it could have been a little bit easier.)* – N/A

23. I didn't learn much. – N2

24. I learnt that combinding *(combining)* two thing together can make diffent *(different)* things like 4 thing make a corkol *(circle)*. - AE

25. I didn't learn much. Actually, I learned nothing. – N2

Part II: Classroom 2 student responses

Question 1: What was math class about today (answer in full sentences)?

1. It was about difrent kins of egels like right egels and dieganel egels with tips of shapes. And we fond egels with pecturs of motins and sisers. (*It was about <u>different kinds of angles</u> like right angles and diagonal angles with types of shapes. And <u>we found angles</u> with pictures of mountains and scissors.)* – A1

2. It was abut tow diffit kind of angels. It was very fun! The angles was the right angel. *(It was about two <u>different kinds of angles</u>. It was very fun! The angles were the right angle.)* – A1

3. In math class today we learned about angles and a little bit of geometry. The angles we learned about were <u>straight angles and right angles</u>. In geometry, we <u>found angles</u> in a triangle and a quadrilateral. It was fun to do the example. – A1

4. I drew a triangle and a rectangle. I fond angles in a road in a picture and in sisors. I drew. *(I drew a triangle and a rectangle. I <u>found angles</u> in a road in a picture and in scissors. I drew.)* – A1

5. Math class was about angles. – N1

6. Today in maths class we did setaf with sapas and it was fun. It was with agles. And with a road map. *(Today in math class we did stuff? with shapes and it was fun. It was with angles. And with a road map.)* – N1

7. Today math class was about <u>straight angles, right angles, how to draw angles and where to find angles.</u> – A1

8. Math class was about right angles and left angles. And what pesishion ther in. And how they cross. And where they go. *(Math class was about right angles and left angles. And what position they're in. And how they cross. And where they go.)* - AE

9. It was about angales and we hade to finde the angales in the drin. We hade to make one stra line wath tow peoples. We allso hade to finde right angales. *(It was about angles and we had to <u>find the angles</u> in the drawing. We had to make one straight line with two people. We also had to find right angles.)* – A1

10. Math class whas about right angles and non right angles. Math class whas how to know what is a right angle and a non right angle. Math class whas also about if you mix right angles then what do you get? In math class we made right angles with our hands. *(Math class was about right angles and non right angles. Math class was how to know what is a right angle and a non angles. Math class was also about if you mix right angles then what do you get? In math class we made right angles with our hands.)* – A1

11. Math class was about angles. We talked about right angles. We also talked about strate angles. We talked about evrything you can do with angles. *(Math class was about angles. We talked about right angles. We also talked about straight angles. We talked about everything you can do with angles.)* – A1

12. The math class was about angels. The left angle and the right angle. It was fun. We had to find angeles in roads and cloud and sissors. Then we had to wight about it. *(The math class was about angles. The left angle and the right angle. It was fun. We had to find angles in roads and cloud and scissors. Then we had to right about it.)* – A1

13. Today I learned about angles. We did lots of experiments that were fun. We used lines and roads. I dident quite get what a angle was but I had fun. *(...I didn't quite get what an angle was but I had fun.)* – N1

14. Math class was about different angles. And also different shapes. It was partly about writing sentences. It was partly about a booklet. – N1

15. This math class has been about all kinds of lines. I loved doing the first activity because I love doing math. I like it because it think that youring lines is really fun. *(...I like it because I think that drawing? lines is really fun.)* – N2

16. The math was about angels and right angles. We learned that right angles form a difrent way. Math class was also about finding angles anywhere. The most intresting part was that angles can be anywhere. *(The math was about angles and right angles. We learned that right angles form a different way. Math class was also about finding angles anywhere. The most interesting part was that angles can be anywhere.)* – A1

17. There is many names to sai that well math was about angles, roads, shapes, pictures and writing. Well I like that we can talk in English. And I guess that I sort of liket the writing. And I felt comfribel beacus I know people in that classe. *(There is many names to say that well was about angles, roads, shapes, pictures and writing. Well I like that we can talk in English. And I guess that I sort of liked the writing. And I felt comfortable because I know people in the class.)* – N1

18. Angles was today's math class. Also to discover of angles. We found angles in the mountains, siccors and roads. We also found angles in shapes. *(...We found angles in the mountains, scissors and roads...)* – A1

19. Today's math class was about discovering angles. We found angles in charts, pictures and different shapes. We looked for angles in triangles and quadrilaterals. A quadrilateral is a four-sided shape. – A1

20. It was about geometry and angles. Plus combinding angles to make new angles. Plus we did experiments. Plus it was about shape angles. *(...Plus combining angles to make new angles...)* – A2

21. In math class today it was about geometry. Today is was about …angles. Today it was also about if you trace a four sided shape and put all the angles together what you got. Today was also about what you get if you put a three angled shape together what shape you get. – A3

22. Math class was about geometry. It was about angles, right angles. We had partners to make angles with our body. – A1

23. Math class was about learning how to know about right angles, strait angles and an angle. *(Math class was about learning how to know about right angles, straight angles and an angle.)* – A1

24. It was about angles. It was about dricton. It was about lings. It was about turns. *(...It was about direction. It was about lines...)* – N1

25. The class was about angle. I liked it. We learned about angles. We learned about shapes. – N1

26. It was about left and right angles and a few shapes. Now I know that you can't only find angles in: Maths text books and sheets of paper and pictures and other stuff. I must be realy hard to make a city without angles. *(...Math textbooks and sheets of paper and pictures and other stuff. It must be really hard to make a city without angles.)* – A1

Question 2: What did you learn (answer in full sentences)?

1. I learnd how to find egels. And I learned tips of egels. And I leard how to ad up egels. And where to fined egels. *(I learned how to find angles. And I learned types of angles. And I learned how to add up angles. And where to find angles.)* – A2

2. That I lernd to draw angels and I lernd how to ade up angels. It was fun! *(That I learned to draw angles an I learned how to add up angles. It was fun!)* – A2

3. I learned that when two right angles crash you get a bigger forme. I also learned that you can find angles anywere, even in your body. *(I learned that* *<u>when two right angles crash you get a bigger form</u>. I also learned that you can find angles anywhere, even in your body.)* – A2

4. "Student left section blank" – N/A

5. I learnt that angles… - N/A

6. I learn about the agles. *(I learned about the angles.)* – N1

7. I learned about <u>straight angles</u>, how to <u>draw angles</u>. – A1

8. I learned that when two angles cross tgether the botom lines macke a streigt line. And that angles and be enywhere. *(I learned that when* *<u>two angles cross together the bottom lines make a straight line</u>. And that angles can be anywhere.)* – A2

9. I learned how too make one right angale. *(I learned how to* *<u>make one right angles</u>.)* – A1

10. I learnt how to tell the diffrinsis between the angles. I learnt how to find angles. I learnt how to find angles out of…*(I learned how to tell the* *<u>difference between the angles</u>. I learned how to* *<u>find angles</u>. I learned how to find angles out of…)* – A1

11. I learned that you can <u>add angles up</u>. I learned that you can make angles out of parts of your body. I learned that you can see angles in mountins. I learned that there are angles in a pair of siscors. *(…I learned that you can see angles in mountains. I learned that there are angles in a pair of scissors.)* – A2

12. I learned nothing. I new you could find angeles in mountins and sirssors. I also knew that right angles and left angles agsistied. *(I learned nothing. I knew you could* *<u>find angles</u> in mountains and scissors. I also knew that right angles and left angles existed.)* – A1

13. Today I learned about angles. She *(the teacher)* showed us what angles were with lots of examples. It was fun learning about angles. I would love to learn more. – N1

14. I learned how to <u>drawn an angle</u>. I learned that there was this booklet. I learned to hard this was. I learned that <u>scissors have angles</u>. – A1

15. I learned lots of kinds of line lake right lines. Ther was one of the best activities that was the one with the mountains and the scissors. You would put one of these on one of the angels and then we had to go and do a triangle and then do one of these and then you would do 3 of them and then thy to put them together. *(I learned* *lots of kinds of lines like right lines. There was one of the best activities that was the one with the mountains and the scissors. You would put on of these on one of the angles and then we had to go and do a triangle and then do one of these and you would do of them and then they to put them together.)* – N1

16. I learned that angles can be anywhere. I learned that when you put them together it will form a difrent angle. I learned that there are many difrent typs of angles. I also learned that can come out of difrent shapes. *(I learned that angles can be anywhere. I learned that when you put them together it will form a different angle. I learned that there are many different types of angles. I also learned that can come out of different shapes.)* – A2

17. I learned about angles, shapes, math and lignes. Wene two whrite angles crash it makes one big lingn at the bottom and tow lingnes that will make one ligne. Well I don't know what els I lerned. At my classe learned about most of that. And it was entertaining. *(I learned about angles, shapes, math and lines. When two right angles crash it makes one big line at the bottom and two lines that will make one line. Well I don't know what else I learned. At my class learned about most of that. And it was entertaining.)* – A2

18. I learned that when you push an angle together it becomes this *(student drew two right angles together, making a straight angle)*. Also you can find angles in your body. You can find angles in reflections you can find angles in almost anything. – A1

19. I learned that angles are everywhere. Angles can be found in things such as on your body, in maps and in pictures. When you put two right angles together, it makes a bigger angle. I also learned you can find angles in shapes. – A2

20. I learnd that you can find angles on body parts. Plus you can find angles anywere. Plus you can find angles in shapes. Plus on roads. Plus angles are in angles. *(I learned that you can find angles on body parts. Plus you can find angles anywhere…)* – A1

21. Today I learnt that if you put a four angled shape and traced all the angles and put them together you get a circle. Also I learned that you have angles that you can make angles out of your arm body. I also learned that if you put two right angles together you can make a strait angle. Also I learned that almost everything has an angle in it. *(…I also learned that if you put two right angles together you can make a straight angle…)* – A3

22. I learned a lot about angles. It was so fun that I wish I can do it everyday. I learned that almost everything has angles include our body's too. Shapes have angles as well! *(…I learned that almost everything has angles including our bodies too…)* – A1

23. I learned that when two road that cross strait is a strait angle and a cruved road with a strait road is a right angle. *(I learned that* *when two roads that cross straight is a straight angle* *and a curved road with a straight road is a right angle.)* – A1

24. I learned some weird roads. I learned some drictons. I learned some shapes. I learned some turns. *(…I learned some directions…)* – N2

25. I learned about angles. I also learned about shapes. I also learned about when two right angles cross they make a straight angle. Angles are everywhere. Two objects can make angles. – A1

26. We learned about angles that could easyly colide into each other and that its almost impossible to make stuff without angles. *(We learned about angles that could easily collide into each other and that it is almost impossible to make stuff without angles.)* – A1