Stability Analysis and Controller Synthesis for a Class of Piecewise Smooth Systems

Behzad Samadi

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ABSTRACT

Stability Analysis and Controller Synthesis for a Class of Piecewise Smooth Systems

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This thesis deals with the analysis and synthesis of piecewise smooth (PWS) systems. In general, PWS systems are nonsmooth systems, which means their vector fields are discontinuous functions of the state vector. Dynamic behavior of nonsmooth systems is richer than smooth systems. For example, there are phenomena such as sliding modes that occur only in nonsmooth systems. In this thesis, a Lyapunov stability theorem is proved to provide the theoretical framework for the stability analysis of PWS systems. Piecewise affine (PWA) and piecewise polynomial (PWP) systems are then introduced as important subclasses of PWS systems.

The objective of this thesis is to propose efficient *computational* controller synthesis methods for PWA and PWP systems. Three synthesis methods are presented in this thesis. The first method extends linear controllers for uncertain nonlinear systems to PWA controllers. The result is a PWA controller that maintains the performance of the linear controller while extending its region of convergence. However, the synthesis problem for the first method is formulated as a set of bilinear matrix inequalities (BMIs), which are not easy to solve. Two controller synthesis methods are then presented to formulate PWA and PWP controller synthesis as convex problems, which are numerically tractable. Finally, to address practical implementation issues, a time-delay approach to stability analysis of sampled-data PWA systems is presented. The proposed method calculates the maximum sampling time for a sampled-data PWA system consisting of a continuous-time plant and a discrete-time emulation of a continuous-time PWA state feedback controller.

Dedicated to Hermin,

my mother,

whose unconditional love is absolutely incredible

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Praise Lord of Life, God the Wise	به نام خداوند جان و خرد
A worthier notion shall not arise	کزین برتر اندیشه برنگذرد
The God of fame in whom powers reside	خداوند نام و خداوند جای
Provider, Sustainer, the Ultimate Guide.	خداوند روزی ده رهنمای
Creator of the world & the orderly universal run	خداوند کیوان و گردان سپهر
The light giver to the Moon, Mercury and Sun.	فروزنده ماه و ناهید و مهر
Transcends all name, label and notion	ز نام و نشان و گمان برترست
Is the author of form and motion	نگارنده‌ی برشده پیکرست
Capable is he who is wise	توانا بود هر که دانا بود
Happiness from wisdom will arise.	ز دانش دل پیر برنا بود
The Epic of the Kings, by Ferdowsi, Persian poet (935-1020), translated by Sh. Shahriari	شاهنامه فردوسی

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List of Symbols and Abbreviations

LMI	Linear Matrix Inequality
BMI	Bilinear Matrix Inequality
PWA	Piecewise Affine
PWL	Piecewise Linear
PWP	Piecewise Polynomial
PWQ	Piecewise Quadratic
PWS	Piecewise Smooth
SOS	Sum of Squares
$\mathcal{X}$	State space
$\mathcal{W}$	Disturbance space
$\mathcal{R}_i$	Polytopic region in $\mathcal{X}$
$\mathcal{R}_i^{\mathcal{X}\times\mathcal{W}}$	Polytopic region in $\mathcal{X} \times \mathcal{W}$
$\overline{\mathcal{R}}_i$	Closure of $\mathcal{R}_i$
$\mathcal{I}(x)$	$= \{i \mid x \in \overline{\mathcal{R}}_i\}$
$\mathcal{I}(x,w)$	$= \{i \mid (x,w) \in \overline{\mathcal{R}}_i^{\mathcal{X}\times\mathcal{W}}\}$

Chapter 1

Introduction

1.1 Motivation

Piecewise smooth (PWS) systems are multi-model systems that offer a good modeling framework for complex dynamical systems. For example, many engineering systems of practical interest have nonlinear components that can naturally be modeled by PWS characteristics. Some examples are:

- Saturation [56], [32]
- Dead-zone [32]
- Backlash [73]
- Electrical circuits with diodes [103], [22], [47]
- Mechanical oscillators with clearance [77]
- Moving parts with Coulomb friction [3]

To illustrate PWS systems, consider a nonlinear mechanical system consisting of a box and a beam (Fig. 1.1) described by the following model

$$
\begin{aligned}
\dot{x}_1 =& x_2 \\
\dot{x}_2 =& \begin{cases} -\mu g \cos(x_3) - g \sin(x_3) & \text{if } x_2 > 0 \\ \mu g \cos(x_3) - g \sin(x_3) & \text{if } x_2 < 0 \end{cases} \\
\dot{x}_3 =& \frac{1}{\tau}(-x_3 + u)
\end{aligned}
\tag{1.1}
$$

where $x_1 = x_b$, $x_2 = \dot{x}_b$, $x_3 = \theta$, $g = 9.8\text{m/s}^2$ and $\mu = 0.1$ is the static friction coefficient. u is the reference signal for θ and τ is the time-constant of the actuator. The set of state space equations (1.1) is a special case of PWS systems in the following form

$$
\dot{x}(t) = f_i(x(t)) + g_i(x(t))u(t), \ \ x(t) \in \mathcal{R}_i. \tag{1.2}
$$

where a subset of the state space $\mathcal{X} \subset \mathbb{R}^n$ is partitioned into regions $\mathcal{R}_i$ for i=1,...,M such that $\cup_{i=1}^{M} \overline{\mathcal{R}}_i = \mathcal{X}$, $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$, $i \neq j$, where $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. In the case of the box and beam system, the regions are defined as

$$
\begin{aligned}
\mathcal{R}_1 =& \{x \in \mathbb{R}^3 | x_2 > 0\}, \\
\mathcal{R}_2 =& \{x \in \mathbb{R}^3 | x_2 < 0\}
\end{aligned}
\tag{1.3}
$$

In general, PWS systems are nonsmooth systems, which means their vector fields are discontinuous functions of the state vector. The PWS system (1.1) is itself an example of a nonsmooth system. However, most of the classical methods for analysis and synthesis of dynamical systems concentrate on smooth systems. Dynamic behavior of nonsmooth systems is richer than smooth systems. For example, there

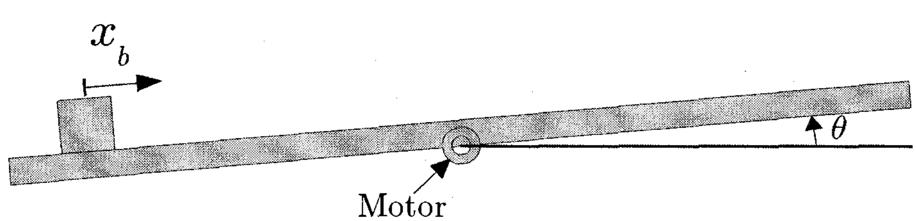

Figure 1.1: Box and beam model

are phenomena such as sliding modes that occur only in nonsmooth systems. In fact, even the definition of trajectories of a system must be generalized to describe sliding modes of nonsmooth systems. Existing literature on nonsmooth systems mainly concentrates on the analysis of the dynamics of nonsmooth systems, while controller synthesis has not received much attention.

1.2 Objective

The objective of this thesis is to propose efficient *computational* controller synthesis methods for the following subclasses of PWS systems:

- Piecewise polynomial (PWP) systems
- Piecewise affine (PWA) systems
- Piecewise linear (PWL) systems

The main focus will be on PWA and PWP systems. Figure 1.2 shows the relative hierarchy of these classes of PWS systems.

It is desired to address the controller synthesis problems with convex optimization techniques such as linear matrix inequalities (LMIs) and sum of squares (SOS) programs. The reason is that there exist numerically efficient tools to solve these convex optimization problems.

In the following section, the relevant literature is reviewed to show what has been done in the field and how the proposed methods relate to previous research.

1.3 Literature Review

A hybrid system is defined in [65] as a dynamical system with interacting continuous-time-driven and discrete-event-driven components. The continuous part of a hybrid

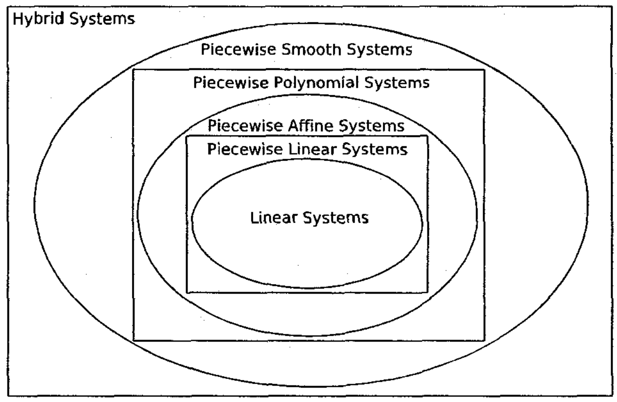

Figure 1.2: The class of PWS systems and its subclasses

system is usually described by a differential or difference equation and the discrete part is described by a finite-state machine or a set of logic-based rules. Therefore, a hybrid system has two distinct types of state variables: real-valued and discrete-valued state variables. Modern computer-based control systems that act on physical systems can be modeled by hybrid dynamical systems. As a result, the analysis and design of such systems has recently received great attention. In the following subsection, previous research on hybrid systems is briefly reviewed. The literature on PWP and PWA systems will be reviewed in separate subsections to describe in more detail the background of the proposed research.

1.3.1 Previous work on hybrid dynamical systems

Hybrid systems have attracted significant attention in recent years. Special Issues on Hybrid Control Systems of the *IEEE Transactions on Automatic Control* (April 1998), *Automatica* (March 1999), *Systems and Control Letters* (October 1999), *the Proceedings of the IEEE* (July 2000), *International Journal of Robust and Nonlinear Control* (April 2001) and a new journal, *Nonlinear Analysis: Hybrid Systems* (March 2007) illustrate the fast pace of advances in this field. The development of a unified and systematic hybrid systems theory is still a growing and vibrant research area. There have been some important research efforts toward an overall unified model

[13, 20, 58], a unified analysis methodology for a class of hybrid systems [20, 35], as well as a unifying view for the subclass of PWA systems [66]. Reference [105] is one of very few contributions toward a unified controller synthesis method that would provide a systematic control design tool for a large class of hybrid systems and enable designers to use the same methodology for a broad set of models and applications. A great deal of attention and efforts in hybrid systems have been focused on the modeling [11, 46, 111, 126] and stability [15, 19, 35, 61, 79, 91, 93, 128]. However, there are also many results on control design methods for hybrid systems. Most of the proposed controller synthesis methods fall into the following approaches:

- **Supervisory Control:** In this approach, continuous controllers are combined with discrete logic. A *supervisor* is used to effectively switch between several continuous control laws [60, 74, 127].

- **Hierarchical Control:** The controller is decomposed into hierarchical levels and it can guarantee a certain performance [21, 48, 86].

- **Optimal Control:** The optimal control problem is to find an input that drives the system to a desired state while minimizing a cost function that depends on the trajectory followed and the control input itself. Optimal control has recently been extended to discrete-event systems [112] and hybrid systems [20, 23, 28, 50, 113, 114].

- **Distributed Control:** The control task is divided among a collection of agents to increase reliability. These agents may communicate with each other to transfer information related to their sensing and decision making [2, 84].

- **Game theoretic approach:** Control problems in this category usually require that all trajectories of the system satisfy certain properties. Properties include safety properties (for example, requiring that the state of the system remains in a certain safety set) and liveness properties (requiring that the state eventually enters a certain target set or visits a set infinitely often) [10,125].

For a more detailed review on hybrid systems, the reader is referred to [5] and [33]. In the following, we review the special case of PWP and PWA systems and focus on the methods that have used the structure of these systems.

1.3.2 Previous work on PWP systems

PWP or spline approximation of curves and surfaces has been widely used in many different scientific contexts and engineering applications [1,34]. However, the lack of proper methods to check the sign of polynomials has prevented PWP systems to be commonly used in the field of control systems. Recently, Ebenbauer proposed analysis and design methods for polynomial systems using sum of squares techniques in [37]. For PWP systems, one of the first attempts to design controllers was made in [87]. Paul proposed in [87] to partition the state space of an affine in the input nonlinear system into cells and to approximate the dynamics of the system in each cell by a model that is polynomial in the state. A controller is then designed for each cell using feedback linearization. A global controller is then formed by joining the individual cell controllers. The proposed method was employed in [87] to design controllers for a few examples of nonlinear systems. However, there is no guarantee for the stability of the closed loop system because a switched system consisting of stable subsystems can be unstable in general.

Recently, the class of discrete-time PWP systems was defined in [45] and a new method based on Cylindrical Algebraic Decomposition (CAD) was proposed to address the constrained finite-time optimal control problem for this class of systems. This seems to be the first systematic approach to controller synthesis for discrete-time PWP systems. However, according to the authors of [45], the method suffers from excessive computational burden.

For continuous-time PWP systems, a stability analysis was proposed in [93] and [85] using PWP Lyapunov functions. The advantage of the proposed method is that the analysis problem is formulated as an SOS programming which is a convex optimization problem. There exist numerical tools such as SOSTOOLS [95] and Yalmip [76] to solve SOS programming problems efficiently. However, systems with infinitely fast switching or sliding modes are excluded from the discussion in [93] and [85]. This will be one of the main topics of the thesis.

1.3.3 Previous work on PWA systems

The roots of PWA systems date back to the pioneering work of Andronov (1901-1952). Andronov's first major piece of research into nonlinear dynamics concerned what is known in Russian as the *metod pripasovyvaniya*, the technique in which separate solutions for the various linear regimes of a PWL problem are joined to form a complete solution - they are "stitched together" as the graphic alternative Russian term *metod sshivaniya* puts it. More details of Andronov's research can be found in [14].

The theory of PWA systems was also used in the analysis and synthesis of nonlinear electrical circuits with most pioneering works done up until the 1970's [4, 29, 30, 120]. In the early 1980's, Sontag [117] developed a Piecewise Linear Algebra mainly for discrete-time PWA systems. For continuous-time dynamics, a technique based on vector field considerations was developed by Pettit [92] to provide a qualitative analysis of PWL systems. In the following, the analysis and synthesis methods for discrete-time and continuous-time PWA systems are briefly reviewed.

Discrete-time PWA systems

Algorithms for computing feedback controllers for constrained discrete-time PWA systems were presented for quadratic and linear objectives in [16] and [9], respectively. Instead of computing the feedback controllers that minimize a finite time cost objective, it is also possible to obtain the infinite time optimal solution for discrete-time PWA systems [8]. These problems are formulated as Mixed Integer Quadratic or Linear Programming. Even though these approaches rely on off-line computation of a feedback law, the computation can quickly become prohibitive for larger problems. This is not only due to the high complexity of the multi-parametric programs involved, but mainly because of the exponential number of transitions between regions which can occur when a controller is computed in a dynamic programming fashion [16, 69]. As a result, some methods were proposed to obtain controllers of low complexity for linear and PWA systems as presented in [52–54]. In general, synthesis methods for discrete-time PWA systems can be classified into the following groups:

- Infinite Time Optimal Control [8, 51]
- Finite Time Optimal Control [9, 12, 16, 78]
- Minimum Time Control [51, 53, 54]
- Bilinear matrix inequality (BMI) based methods for stabilization [116]
- PWA control with performance [40, 42, 43]

The controller designed using any of these methods is typically much more complex than the PWA system to be controlled. As an example, for a PWA system with 4 regions, the number of regions for the controller can range from 138 to 3904 [52]. In addition, one of the main drawbacks of the methods in [16] is the lack of an *a-priori* stability guarantee for the closed-loop system. This problem has recently been addressed in [52].

Continuous-time PWA systems

Sufficient conditions for analysis of continuous-time PWA systems by searching for a Lyapunov function to prove stability, can be formulated as convex optimization programs involving LMIs [17]. These mathematical programs can then be solved efficiently using polynomial-time algorithms [83]. The analysis methods are only approximate in the sense that there are no guarantees that a Lyapunov function can be found. However, if one is found, the result is unambiguous. This has been the trend of research in the linear parameter varying approach to gain scheduling (see [108] and references therein) and in the more recent work on the analysis of PWA systems based on Lyapunov functions and LMIs [18,35,49,56,67,90,91,103].

The work on switched linear systems initiated in [90] is one of the first attempts to apply Lyapunov-based methods to PWA systems. Following this work, and its extensions to nonlinear dynamics [18], a unified approach to the analysis of PWA systems and a class of hybrid systems was formulated in [35]. Several promising Lyapunov-based methods have recently been developed to analyze PWL and PWA systems [49, 56, 67, 91, 98]. Some synthesis methods [55, 56, 67, 101] have also been developed. A specific technique for state feedback control of PWA systems on simplices and rectangles was proposed in [55]. Synthesis methods using convex optimization programs based on the analysis methods in [56, 67, 91] were developed in [56, 96]. The resulting controllers designed by these methods are either patched LQRs [96] or cannot guarantee that sliding modes are avoided [56, 96] and, therefore, are not provably stabilizing. In [97, 103], a synthesis method based on BMIs has been proposed for state and output feedback stabilization of PWA systems. The method has the advantage of guaranteeing that sliding modes are not generated at the switching and the controllers are therefore provably stabilizing. Another important feature of this method for practical implementation of the controllers is that continuity of the control input can also be guaranteed at the switching. However, BMI problems are not generally convex problems and thus, are not easy to be solved efficiently.

A very important subclass of PWA systems is the class of PWA *slab systems* [101], for which the partition of the state space is a function of a scalar variable. Hassibi and Boyd [56] proposed methods for quadratic stabilizability and $\mathcal{L}_2$ gain synthesis for PWA systems using PWL controllers. Three different algorithms for PWA controller synthesis for slab PWA systems have also been proposed in [101]. It has been shown that by considering an affine term in the controller, the synthesis problem can be formulated as a set of Linear Matrix Inequalities (LMIs) parametrized by a vector. Furthermore, it has also been shown that by relaxing the problem to a finite set of LMIs, it can be solved efficiently to a point near the global optimum. In addition, the global solution can be exactly found under some conditions.

In general, design methods for continuous-time PWA systems can be classified into the following groups:

- Methods based on Hamilton Jacobi Bellman Inequality [66,96]
- PWL and PWA stabilization of slab PWA systems [56,66,101]
- BMI-based PWA controller design methods for stabilization of generic PWA systems [97,102,103]
- PWL control with L_2 gain performance [26,56,66]

Considering the existing approaches for PWA and PWP controller synthesis, the contributions of this thesis are stated in the next section.

1.4 Contributions

This thesis addresses the following questions

- How can we design a PWA controller that keeps the performance of a linear controller in a neighborhood of the equilibrium point and guarantees a larger region of attraction?
- Is it possible to formulate the PWA/PWP controller synthesis as a convex optimization problem?
- For a sampled-data implementation of a continuous-time PWA controller, how large can the sampling time be?

Therefore, the main contributions of this thesis are

1. To present a unified approach for stability analysis of PWA systems with continuous and discontinuous vector fields. The Filippov definition is considered for the solution of PWA systems and then a Lyapunov stability theorem is proved. The importance of this theorem is to show that sufficient conditions for the stability of a PWA system can be formed using a *differentiable Lyapunov function* without any need for a priori information about *attractive sliding modes* on switching surfaces. This is a great advantage over existing stability results for PWA systems in the literature because obtaining this a priori information is difficult in general. Sufficient conditions for quadratic and sum of squares (SOS) polynomial Lyapunov stability are then formulated as convex problems. The SOS conditions are less conservative than the quadratic conditions. It is shown in an example that the proposed SOS program can prove stability where quadratic and differentiable piecewise quadratic (PWQ) functions fail.

2. To propose a two-step controller synthesis method for a class of uncertain nonlinear systems described by PWA differential inclusions. In the first step, a robust linear controller is designed for the linear differential inclusion that describes the dynamics of the nonlinear system close to the equilibrium point. In the second step, a stabilizing PWA controller is designed that coincides with the linear controller in a region around the equilibrium point. The proposed method has two objectives: global stability and local performance. It thus enables to use well known techniques in linear control design for local stability and performance while delivering a global PWA controller that is guaranteed to stabilize the nonlinear system.

3. To introduce for the first time a duality-based interpretation of PWA systems. This enables controller synthesis for PWA slab systems to be formulated as a convex optimization problem. PWA L2-gain analysis and synthesis is also extended to PWA systems whose output is a PWA function of the state (as opposed to a PWL function). In addition, a convex optimization program is proposed to compute a PWA differential inclusion for nonlinear systems for which the nonlinearity is a function of one variable.

4. To propose a nonsmooth backstepping controller synthesis for PWP systems. The main contribution of the proposed method is to formulate controller design for a large class of PWP and PWA systems as a convex problem. The controller synthesis problem is divided in two cases. The first case consists of the construction of a sum of squares (SOS) Lyapunov function for PWP systems with discontinuous vector fields. The second case addresses the construction of a PWP Lyapunov function for PWP systems with continuous vector fields. After constructing a Lyapunov function, controller synthesis for a PWP system can be formulated as an SOS program, which is a convex optimization problem and can be efficiently solved.

5. To propose a time-delay approach to stability analysis of sampled data PWA systems consisting of a continuous-time plant and a discrete-time emulation of a continuous-time PWA state feedback controller. The sampled-data system is considered as a delayed system with a variable delay. Conditions under which the trajectories of the sampled data closed-loop system will converge to an attractive invariant set are then presented. It is also shown that when the sampling period converges to zero, the conditions of the proposed theorem coincide with sufficient conditions for the non-fragility of the stabilizing continuous-time PWA state feedback controller.

The results of the current research were submitted to and published in a few conferences and journals. The details of the publications is listed in the following subsection.

1.4.1 Publications

The following publications contain the main contributions of the thesis

1. B. Samadi and L. Rodrigues, "Controller synthesis for piecewise affine differential inclusions: a duality-based convex optimization approach," under second revision for publication in *Automatica.*

2. B. Samadi and L. Rodrigues, "Extension of local linear controllers to global piecewise affine controllers for uncertain nonlinear systems," accepted for publication in the *International Journal of Systems Science.*

3. B. Samadi and L. Rodrigues, "Sampled-Data Piecewise Affine Slab Systems: A Time-Delay Approach," in *Proc. of the American Control Conference,* Seattle, WA, Jun. 2008.

4. B. Samadi and L. Rodrigues, "Controller synthesis for piecewise affine slab

differential inclusions: a duality-based convex optimization approach," in *Proc. of the 46th Conference on Decision and Control*, New Orleans, LA, Dec. 2007.

5. B. Samadi and L. Rodrigues, "Backstepping Controller Synthesis for Piecewise Affine Systems: A Sum of Squares Approach," in *Proc. of the IEEE International Conference on Systems, Man, and Cybernetics (SMC 2007)*, Montreal, Oct. 2007.

6. B. Samadi and L. Rodrigues, "Extension of a local linear controller to a stabilizing semi-global piecewise-affine controller," *7th Portuguese Conference on Automatic Control*, Lisbon, Portugal, Sep. 2006.

1.5 Structure of the Thesis

The thesis is structured as shown in Figure 1.3. Chapter 2 defines PWS, PWP and PWA systems and presents a unified approach for stability analysis of PWS systems with continuous and discontinuous vector fields. In Chapter 3, continuous PWA differential inclusions are defined and sufficient conditions for monotonicity of PWQ Lyapunov functions for these inclusions are proved. A two-step controller synthesis method is then presented for a class of uncertain nonlinear systems described by PWA differential inclusions. Chapter 4 introduces the parameter set and the dual parameter set for PWA slab systems. It then provides stability and performance analysis and synthesis tools for PWA slab systems. In Chapter 5, a backstepping approach to controller synthesis for PWP systems in strict feedback form is presented. Chapter 6 addresses stability analysis of sampled-data PWA systems consisting of a continuous-time plant in feedback connection with a discrete-time emulation of a continuous-time PWA state feedback controller.

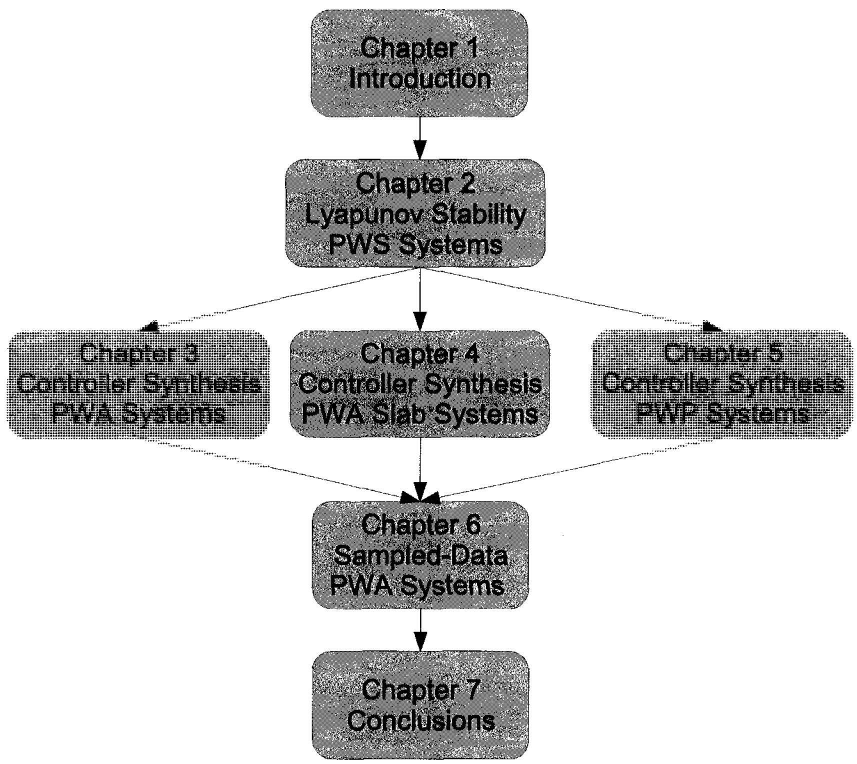

Figure 1.3: Structure of the thesis

Chapter 2

Lyapunov Stability for Piecewise Smooth Systems

The main objective of this chapter is to present a unified approach to stability analysis of PWS systems with continuous and discontinuous vector fields. The Filippov definition is considered for the solution of these systems and then a Lyapunov stability theorem is proved. The importance of this theorem is to show that sufficient conditions for the stability of a PWS system can be formed without any need for a-priori information about *attractive sliding modes* on switching surfaces. This is a significant advantage over existing stability results for switched systems in the literature because obtaining this a-priori information is difficult in general.

2.1 Introduction

There have been different approaches to construct a Lyapunov function to provide sufficient conditions for the stability of PWS systems. For a survey of existing approaches for stability analysis of hybrid systems and switched linear systems the readers are referred to [35] and [75]. A general framework for analyzing stability of nonlinear switched systems using multiple Lyapunov functions is presented in [25].

It is shown in [56] that by searching for quadratic Lyapunov functions, sufficient conditions for stability of PWA systems can be formulated as convex optimization problems with LMI constraints. Finding a common quadratic Lyapunov function for all linear modes is used to analyze the stability of switched linear systems under arbitrary switching in [74].

However, there are stable PWA systems for which a quadratic Lyapunov function does not exist. Examples of such systems are shown in [66, p. 47]. Conservativeness of a quadratic form is the motivation for studying nonquadratic Lyapunov functions. As an example, continuous PWQ Lyapunov functions were extensively investigated in recent years (see [19, 66, 91, 102]). However, it is a common misunderstanding in the literature to believe that if there is a continuous PWQ or PWP function that is positive definite and decreasing with time along each vector field of a switched affine system then the system is stable. A counter-example will be provided in section 2.6.2.

A recent result in [88] shows that the existence of a common quadratic Lyapunov function for the linear parts of a PWA system in every mode is sufficient for exponential convergence of the system if the vector field of the PWA system is continuous. Exponential convergence is defined in the same reference. The case of discontinuous vector fields is studied in [89] and it is shown that the existence of a common quadratic Lyapunov function for linear parts of the system is not a sufficient condition for convergence. Necessary and sufficient conditions for quadratic convergence of the special case of bimodal PWA systems are then derived.

SOS polynomials were also proposed as candidate Lyapunov functions. In fact, quadratic Lyapunov functions are a special class of SOS Lyapunov functions [94]. In addition, by using the SOS approach, it is possible to analyze the stability of systems with nonlinear polynomial vector fields. Stability analysis tools based on the SOS decomposition for classes of nonlinear systems, hybrid systems, switched systems, and time-delay systems are presented in [85]. In the same reference it is proposed to use PWP Lyapunov functions for hybrid systems, which is a generalization of PWQ Lyapunov functions. However, systems with infinitely fast switching or sliding modes are excluded from the discussion in [85].

Although there is a vast amount of work on stability of switched linear and PWA systems, sliding modes or infinitely fast switching are not usually considered in the literature. Important exceptions are the references [19, 66, 89, 103]. In [19], it is proposed to add the sliding modes and their associated sliding dynamics to the modes of the system before doing the stability analysis. However, this needs a-priori information about the sliding modes of the system, which is typically hard to get. In another approach, an extra condition is introduced in [66, p.64] to extend the analysis to systems with attractive sliding modes. However, one needs to identify potential sets in which sliding modes can occur and then the corresponding condition can be formed and added to the analysis problem. This might again be hard and make the problem complex if there is no previous information about sliding modes. In [103], a synthesis method based on BMIs was proposed for state and output feedback stabilization of PWA systems. The synthesis method includes linear constraints on controller gains to guarantee that sliding modes are not generated at the switching. Finally, reference [89] has addressed sliding modes but has concentrated on the specific case of common quadratic Lyapunov functions for bimodal PWA systems. A question that still remains to be answered is when the necessity to check the existence of unstable sliding modes of a general PWA system can be removed.

Based on the aforementioned limitations, this chapter will present a nonsmooth Lyapunov stability theorem. This theorem applies to PWS systems with continuous or discontinuous vector fields. The theorem states that a sufficient condition for the stability of a PWS system is the existence of a $\mathcal{C}^1$ positive definite function that decreases with time inside each region. The importance of the proposed theorem is to show that by using a $\mathcal{C}^1$ Lyapunov function, no additional condition is needed to address potential sliding modes of PWS systems and, therefore, no a-priori knowledge about sliding modes is necessary.

This chapter is structured as follows. PWS systems are defined in section 2.2. In section 2.3, a stability theorem is stated and proved. Sufficient condtions for monotonicity of nonsmooth functions and dissipativity of PWS systems are described in sections 2.4 and 2.5, respectively. PWA systems and PWA slab systems are then introduced in sections 2.6 and 2.7, respectively. For PWA systems, as the first choice for a $\mathcal{C}^1$ Lyapunov function, a quadratic form is considered in subsection 2.6.1. Then, PWQ Lyapunov functions are discussed in subsection 2.6.2. As a counter example, it is shown that it is possible to find a PWQ positive definite function which is decreasing with time in each region for an unstable PWA system. A proposition for SOS Lyapunov functions is then proved in subsection 2.6.3. It is also shown by an example that it is possible to find an SOS Lyapunov function for a PWA system which does not admit any quadratic or $\mathcal{C}^1$ PWQ Lyapunov functions. PWP systems are then introduced in section 2.8 and a stability proposition based on polynomial Lyapunov functions is presented.

2.2 PWS Systems

The dynamics of a PWS system can be written as

$$\dot{x} = f_i(x), \ x \in \mathcal{R}_i \tag{2.1}$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$ is the state vector and the initial state is $x(0) = x_0$. A subset of the state space $\mathcal{X}$ is partitioned into M regions, $\mathcal{R}_i$, $i = 1, \ldots, M$, such that

$$\cup_{i=1}^{M} \overline{\mathcal{R}}_i = \mathcal{X} \tag{2.2}$$

$$\mathcal{R}_i \cap \mathcal{R}_j = \emptyset, \ i \neq j \tag{2.3}$$

where $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. The function $f_i(x) : \overline{\mathcal{R}}_i \to \mathbb{R}^n$ is continuous and locally bounded. The Filippov definition of trajectories is considered for the solution of (2.1) (see [44] and [66]).

Definition 2.1. *(Filippov solution) A continuous function $x(t)$ is regarded to be a Filippov solution to (2.1) if it is a solution of the differential inclusion*

$$\dot{x}(t) \in \mathcal{F}(x) \tag{2.4}$$

where

$$\mathcal{F}(x) \triangleq \boldsymbol{co}\{f_i(x) | i \in \mathcal{I}(x)\} \tag{2.5}$$

co *stands for the convex hull of a set and*

$$\mathcal{I}(x) = \{i | x \in \overline{\mathcal{R}}_i\}. \tag{2.6}$$

Note that if $x \in \mathcal{R}_i$, then

$$\mathcal{F}(x) = \{f_i(x)\}. \tag{2.7}$$

Example 2.1. *Consider the following simple scalar differential equation*

$$\dot{x} = -\operatorname{sgn}(x), x(0) = 1 \tag{2.8}$$

It can easily be seen that the differential inclusion (2.4) becomes

$$\dot{x} \in -SGN(x) \tag{2.9}$$

where SGN is the set-valued sign function decribed below.

$$SGN(x) = \begin{cases} \{-1\} & , \textit{for } x < 0 \\ \{+1\} & , \textit{for } x > 0 \\ [-1, 1] & , \textit{for } x = 0 \end{cases} \tag{2.10}$$

2.3 Lyapunov stability

In this section, a Lyapunov stability theorem is proved for nonsmooth Lyapunov functions. This theorem forms the theoretical framework for using piecewise smooth Lyapunov functions in stability analysis of nonlinear systems. There are other nonsmooth versions of Lyapunov theorems in the literature e.g. [24, 31, 59, 106, 118]. However, certain conditions in these theorems (such as, for example, the conditions on the Dini derivative, the proximal subdifferential or the upper bound of the Lyapunov function) are difficult to check or not needed in the cases described in this thesis. The objective of Theorem 2.1 will thus be to extend the standard Lyapunov stability theorem in [70] to nonsmooth Lyapunov functions and to fit the framework needed in this thesis. To the best of the author's knowledge, this theorem in this exact form does not appear in the literature.

Consider the following autonomous nonlinear system

$$\dot{x}(t) = f(x(t)) \tag{2.11}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, the initial state, $x(0) = x_0$, is bounded and $f : \mathcal{X} \longrightarrow \mathbb{R}^n$ is piecewise continuous and bounded in $\mathcal{X} \subset \mathbb{R}^n$. The following theorem describes sufficient conditions for stability of system (2.11) in the sense of Lyapunov based on a continuous Lyapunov function that is not necessarily differentiable everywhere. Because of its importance, the theorem is proved here. The proof combines the proof of the standard Lyapunov theorem in [70] for stability and the proof of the nonsmooth Lyapunov theorem in [31] for asymptotic stability.

Theorem 2.1. *For nonlinear system (2.11), if there exists a continuous function $V(x)$ such that*

$$V(x^\star) = 0 \tag{2.12}$$

$$V(x) > 0 \textit{ for all } x \neq x^\star \textit{ in } \mathcal{X} \tag{2.13}$$

$$t_1 \leq t_2 \Rightarrow V(x(t_1)) \geq V(x(t_2)) \tag{2.14}$$

then $x = x^\star$ is a stable equilibrium point. Moreover if there exists a continuous function $W(x)$ such that

$$W(x^\star) = 0 \tag{2.15}$$

$$W(x) > 0 \text{ for all } x \neq x^\star \text{ in } \mathcal{X} \tag{2.16}$$

$$t_1 \leq t_2 \Rightarrow V(x(t_1)) \geq V(x(t_2)) + \int_{t_1}^{t_2} W(x(\tau))d\tau \tag{2.17}$$

and

$$\|x\| \to \infty \Rightarrow V(x) \to \infty \tag{2.18}$$

then all trajectories in $\mathcal{X}$ asymptotically converge to $x = x^\star$.

Proof. For stability, we want to prove

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ s.t. } \|x_0 - x^\star\| < \delta \Rightarrow \|x(t) - x^\star\| < \varepsilon, \forall t \geq 0 \tag{2.19}$$

Following [70], for a given $\varepsilon > 0$ we choose $r \in (0, \varepsilon]$ such that

$$\mathcal{B}_r = \{x | \|x - x^\star\| \leq r\} \subset \mathcal{X} \tag{2.20}$$

Let $\alpha = \min_{\|x-x^\star\|=r} V(x)$. Then $\alpha > 0$ by (2.13). Take $\beta \in (0, \alpha)$ and let $\Omega_\beta = \{x \in \mathcal{B}_r | V(x) \leq \beta\}$. Then Ω_β is in the interior of $\mathcal{B}_r$ (Figure 2.1). If $x_0 \in \Omega_\beta$ then (2.14) implies that $x(t) \in \Omega_\beta$ for all $t \geq 0$. As $V(x)$ is continuous and $V(x^\star) = 0$, there is a $\delta > 0$ such that $\|x - x^\star\| \leq \delta \Rightarrow V(x) < \beta$. Then

$$\mathcal{B}_\delta = \{x | \|x - x^\star\| \leq \delta\} \subset \Omega_\beta \subset \mathcal{B}_r \tag{2.21}$$

and $x_0 \in \mathcal{B}_\delta \Rightarrow x_0 \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in \mathcal{B}_r \Rightarrow x(t) \in \mathcal{B}_\varepsilon$. Therefore

$$\|x_0 - x^\star\| < \delta \Rightarrow \|x(t) - x^\star\| < r \leq \varepsilon, \forall t \geq 0 \tag{2.22}$$

This implies that $x = x^\star$ is a stable equilibrium point.

To prove asymptotic stability, following [31], we show that $x(t)$ converges to $x^\star$ as $t \to \infty$. It follows from $x(0) = x_0$ and (2.17) that

$$V(x(t)) + \int_0^t W(x(t))d\tau \leq V(x_0) \tag{2.23}$$

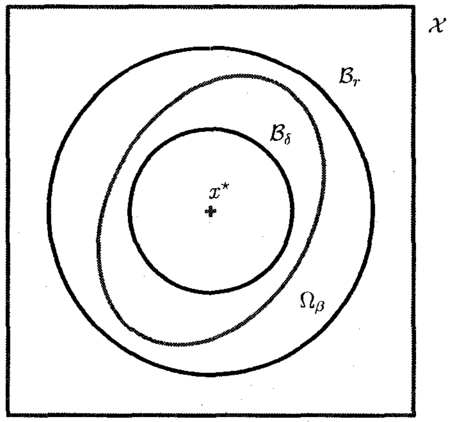

Figure 2.1: Geometric illustration of sets in the proof of Theorem 2.1.

Then (2.13), (2.16), (2.23) and the fact x_0 is bounded imply that $V(x(t))$ and $\int_0^t W(x(t))d\tau$ are bounded. Because $V(x(t))$ is bounded, it follows from (2.18) that $\|x(t)\|$ is bounded. Since $f(x)$ is bounded in $\mathcal{X}$, $\dot{x}(t)$ is bounded and $x(t)$ satisfies a global Lipschitz condition on $t \in [0, +\infty)$ with constant L.

Assume that $x(t)$ fails to converge to $x^\star$. Then for some $\varepsilon > 0$ there exists a sequence of points t_i tending to infinity such that

$$\|x(t_i) - x^\star\| \geq \varepsilon, \; i = 1, 2, \ldots \tag{2.24}$$

Without loss of generality, possibly by selecting a subsequence, the sequence t_i can always be chosen such that

$$|t_{i+1} - t_i| > \frac{\varepsilon}{2L} \tag{2.25}$$

Since $\|x(t)\|$ is bounded, there exists a $\lambda > 0$ such that $\|x(t) - x^\star\| \leq \lambda$. Consider

$$\mathcal{A}_{[\frac{\varepsilon}{2},\lambda]} = \{x | \; \frac{\varepsilon}{2} \leq \|x - x^\star\| \leq \lambda\} \tag{2.26}$$

$\mathcal{A}_{[\frac{\varepsilon}{2},\lambda]}$ is not empty since $\lambda \geq \varepsilon > \frac{\varepsilon}{2}$. Let $\eta > 0$ be such that

$$x \in \mathcal{A}_{[\frac{\varepsilon}{2},\lambda]} \Rightarrow W(x) \geq \eta \tag{2.27}$$

Such η exists because of (2.16) and the fact that $\mathcal{A}_{[\frac{\varepsilon}{2},\lambda]}$ is not empty. Consider t such that $|t - t_i| < \frac{\varepsilon}{2L}$. Since $x(t)$ is globally Lipschitz continuous with constant L, we have

$$\|x(t) - x(t_i)\| < \frac{\varepsilon}{2} \tag{2.28}$$

Inequalities (2.24), (2.28) and the following triangle inequality

$$\|x(t) - x^\star\| \geq \|x(t_i) - x^\star\| - \|x(t) - x(t_i)\| \tag{2.29}$$

imply $\|x(t) - x^\star\| > \frac{\varepsilon}{2}$ and consequently $x(t) \in \mathcal{A}_{[\frac{\varepsilon}{2},\lambda]}$. Therefore, from (2.27)

$$\int_{t_i-\frac{\varepsilon}{2L}}^{t_i+\frac{\varepsilon}{2L}} W(x(\tau))d\tau \geq \frac{\eta\varepsilon}{L} \tag{2.30}$$

and then using (2.25) and (2.16)

$$\int_{t_{i-1}}^{t_{i+1}} W(x(\tau))d\tau > \frac{\eta\varepsilon}{L} \tag{2.31}$$

This would imply that $\int_0^t W(x(t))d\tau$ diverges as $t \to \infty$, which is a contradiction with (2.23) and the conclusion that $\int_0^t W(x(t))d\tau$ is bounded. This proves that $x(t)$ converges to $x^\star$ as $t \to \infty$. □

Remark 2.1. *In this work, the equilibrium point $x = x^\star$ is said to be* globally stable *if all trajectories in $\mathcal{X}$, the domain of nonlinear system (2.11), asymptotically converge to $x = x^\star$.*

The next section presents conditions to verify the monotonicity of nonsmooth functions. Using these conditions, one can verify (2.14) and (2.17) for nonlinear system (2.11) without needing to know the trajectories of the system.

2.4 Monotonicity of nonsmooth functions

In this section, sufficient conditions for monotonicity of nonsmooth functions of state variables along the trajectories of a piecewise smooth system will be provided. In the following, the concept of generalized gradient is introduced. Note that the monotonicity conditions can also be described by the Dini derivative (such as it is done in [106]). However, in this chapter, the theorem of Rademacher [31, p. 93] is used to define the generalized gradient. This definition will be employed to formulate the monotonicity condition for continuous piecewise quadratic and piecewise polynomial functions.

Definition 2.2. *[31] For a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$, the generalized gradient is defined as*

$$\partial_C V(x) = \boldsymbol{conv}\{\lim_{i \to +\infty} \nabla V(x_i) | x_i \to x, x_i \notin N\} \tag{2.32}$$

where N is the set of measure zero where the gradient of V does not exist.

Proposition 2.1. *(Section 2.4 of [24]) Let $\mathcal{F} : \mathbb{R}^n \to 2^{\mathbb{R}^n} \setminus \emptyset$ be continuous and let $V : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous. V is nonincreasing along all solutions of*

$$\dot{x} \in \mathcal{F}(x) \tag{2.33}$$

if and only if

$$\forall x \in \mathbb{R}^n, \forall f \in \mathcal{F}(x), \max\{p^T f | p \in \partial_C V(x)\} \leq 0 \tag{2.34}$$

2.5 Dissipativity

Consider the following piecewise smooth system

$$\begin{aligned} \dot{x} &= f_i(x) + g_i(x)w, \;\; x \in \mathcal{R}_i \\ y &= h(x, w) \end{aligned} \tag{2.35}$$

where $x(t) \in \mathbb{R}^n$ denotes the state, $w(t) \in \mathbb{R}^{n_w}$ is the exogenous input and $y(t) \in \mathbb{R}^{n_y}$ is the output. The functions $f_i(x) : \overline{\mathcal{R}}_i \to \mathbb{R}^n$, $g_i(x) : \overline{\mathcal{R}}_i \to \mathbb{R}^{n \times n_w}$ and $h(x, w) : \mathcal{X} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_y}$ are continuous in x and locally bounded.

In this section, the notion of *dissipativity* is defined. Roughly speaking, a system is considered dissipative if the amount of energy that the system can provide to its environment is less than what it receives from external sources [110].

Definition 2.3. *[110] The system (2.35) is said to be* dissipative *with* supply rate $W(y, w)$ *and* storage function $V(x)$, *if* $V(x)$ *is a nonnegative function such that*

$$t_1 \leq t_2 \Rightarrow V(x(t_1)) + \int_{t_1}^{t_2} W(y(\tau), w(\tau))d\tau \geq V(x(t_2)) \tag{2.36}$$

Two propositions are provided in the following to describe the sufficient conditions for the system (2.35) to be dissipative in two cases of discontinuous and continuous vector fields. The importance of these propositions lies in the fact that to check the dissipativity of the system, it suffices to verify a condition on the storage function, the supply rate and the vector field of the subsystem in each region separately. There is therefore no need to examine the storage function in one region with the vector field of another region, which would make the problem much more complicated.

2.5.1 Piecewise smooth systems with discontinuous vector fields

Proposition 2.2. *(Smooth storage functions) The piecewise smooth system (2.35) is dissipative with a storage function $V(x)$ and a supply rate $W(y, w)$ if $V(x)$ is a nonnegative C^1 function, $W(y, w)$ is a continuous function in y and for all $x \in \overline{\mathcal{R}}_i$, $i = 1, \ldots, M$ and any $w \in \mathbb{R}^{n_w}$*

$$\nabla V(x)^T (f_i(x) + g_i(x)w) \leq W(y, w) \tag{2.37}$$

Proof. The inequality (2.37) can be rewritten as

$$\begin{bmatrix} \nabla V(x) \\ -W(y, w) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} f_i(x) + g_i(x)w \\ 1 \end{bmatrix} \leq 0 \tag{2.38}$$

By appending time (t) to the state vector of the system (2.35), we have the following differential equation

$$\begin{bmatrix} \dot{x} \\ \dot{t} \end{bmatrix} = \begin{bmatrix} f_i(x) + g_i(x)w \\ 1 \end{bmatrix}, \quad x \in \mathcal{R}_i \tag{2.39}$$

In the following, using Proposition 2.1 the following function

$$S(x,t) = V(x) - \int_0^t W(y,w)d\tau \tag{2.40}$$

is shown to be non-increasing along the trajectories of (2.39). The fact that $V(x)$ is a C^1 function implies that

$$\partial_c S(x,t) = \mathbf{conv}\left\{ \begin{bmatrix} \nabla V(x(t)) \\ -W(y(t), w(\tau)) \end{bmatrix} \middle| \tau \to t \right\} \tag{2.41}$$

Let $x(t)$ be a Filippov solution of (2.35). Therefore, $x(t)$ is a solution of the following differential inclusion

$$\begin{bmatrix} \dot{x} \\ \dot{t} \end{bmatrix} \in \mathbf{conv}\left\{ \begin{bmatrix} f_i(x(t)) + g_i(x(t))w(\tau) \\ 1 \end{bmatrix} \middle| i \in \mathcal{I}(x),\ \tau \to t \right\} \tag{2.42}$$

Note that if $x(t) \in \mathcal{R}_i$ and $w(t)$ is continuous at t, the vector field of (2.35) is continuous and we have

$$\begin{bmatrix} \dot{x} \\ \dot{t} \end{bmatrix} = \begin{bmatrix} f_i(x(t)) + g_i(x(t))w(\tau) \\ 1 \end{bmatrix} \tag{2.43}$$

Since (2.38) is satisfied for any w, it follows from (2.41) that (2.34) is satisfied for the differential inclusion (2.42). Therefore by Proposition 2.1, $S(x,t)$ is nonincreasing along the trajectories of (2.35) in $\mathcal{X}$ i.e.

$$t_1 \le t_2 \Rightarrow V(x(t_1)) - \int_0^{t_1} W(y,w)d\tau \ge V(x(t_2)) - \int_0^{t_2} W(y,w)d\tau \tag{2.44}$$

Therefore (2.36) is satisfied and the system (2.35) is dissipative with storage function $V(x)$ and supply rate $W(y,w)$. □

2.5.2 Piecewise smooth systems with continuous vector fields

Proposition 2.3. *(Piecewise smooth storage functions) The piecewise smooth system (2.35) is dissipative with a storage function $V(x)$ and a supply rate $W(y,w)$ if*

- *$V(x)$ is a nonnegative continuous function where*

$$V(x) = V_i(x), x \in \overline{\mathcal{R}}_i \tag{2.45}$$

where $V_i : \overline{\mathcal{R}}_i \to \mathbb{R}$ is a $\mathcal{C}^1$ function,

- *$W(y, w)$ is a continuous function in y,*

- *the vector field of the system (2.35) is continuous in x, i.e. for any $i, j \in \{1, \ldots, M\}$ such that $\overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset$,*

$$\begin{cases} f_i(x) = f_j(x) \\ g_i(x) = g_j(x) \end{cases}, \ \forall x \in \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \tag{2.46}$$

- *for all $x \in \overline{\mathcal{R}}_i$, $i = 1, \ldots, M$ and any $w \in \mathbb{R}^{n_w}$*

$$\nabla V_i(x)^T (f_i(x) + g_i(x)w) \leq W(y, w) \tag{2.47}$$

Proof. The inequality (2.47) can be rewritten as

$$\begin{bmatrix} \nabla V_i(x) \\ -W(y, w) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} f_i(x) + g_i(x)w \\ 1 \end{bmatrix} \leq 0 \tag{2.48}$$

By appending time (t) to the state vector of the system (2.35), we have the following differential equation

$$\begin{bmatrix} \dot{x} \\ \dot{t} \end{bmatrix} = \begin{bmatrix} f_i(x) + g_i(x)w \\ 1 \end{bmatrix}, \ x \in \mathcal{R}_i \tag{2.49}$$

In the following, using Proposition 2.1 the following function

$$S(x, t) = V_i(x) - \int_0^t W(y, w) d\tau, x \in \overline{\mathcal{R}}_i \tag{2.50}$$

is shown to be non-increasing along the trajectories of (2.49). The fact that $V(x)$ is a piecewise $\mathcal{C}^1$ function implies that

$$\partial_c S(x, t) = \mathbf{conv} \left\{ \begin{bmatrix} \nabla V_i(x(t)) \\ -W(y(t), w(\tau)) \end{bmatrix} \middle| i \in \mathcal{I}(x), \tau \to t \right\} \tag{2.51}$$

Let $x(t)$ be a Filippov solution of (2.35). Therefore, $x(t)$ is a solution of the following differential inclusion

$$\begin{bmatrix} \dot{x} \\ \dot{t} \end{bmatrix} \in \mathbf{conv} \left\{ \begin{bmatrix} f_i(x(t)) + g_i(x(t))w(\tau) \\ 1 \end{bmatrix} \middle| \, i \in \mathcal{I}(x), \; \tau \to t \right\} \tag{2.52}$$

Consider the following two cases

- If $x(t) \in \mathcal{R}_i$, we have

$$\partial_c S(x,t) = \mathbf{conv} \left\{ \begin{bmatrix} \nabla V_i(x(t)) \\ -W(y(t), w(\tau)) \end{bmatrix} \middle| \, \tau \to t \right\} \tag{2.53}$$

  and

$$\begin{bmatrix} \dot{x} \\ \dot{t} \end{bmatrix} \in \mathbf{conv} \left\{ \begin{bmatrix} f_i(x(t)) + g_i(x(t))w(\tau) \\ 1 \end{bmatrix} \middle| \, \tau \to t \right\} \tag{2.54}$$

  Since (2.47) is satisfied for any w, it follows from (2.53) that (2.34) is satisfied for the differential inclusion (2.54).

- If $x(t)$ is on the boundary of two or more regions i.e. $x(t) \in \bigcap_{i \in \mathcal{I}(x)} \overline{\mathcal{R}}_i$, it follows from (2.47) that for any j, k in $\mathcal{I}(x)$,

$$\begin{bmatrix} \nabla V_j(x) \\ -W(y,w) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} f_j(x) + g_j(x)w \\ 1 \end{bmatrix} \leq 0 \tag{2.55}$$

$$\begin{bmatrix} \nabla V_k(x) \\ -W(y,w) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} f_k(x) + g_k(x)w \\ 1 \end{bmatrix} \leq 0 \tag{2.56}$$

  In addition, the continuity condition (2.46) implies that

$$\begin{bmatrix} \nabla V_j(x) \\ -W(y,w) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} f_k(x) + g_k(x)w \\ 1 \end{bmatrix} = \begin{bmatrix} \nabla V_j(x) \\ -W(y,w) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} f_j(x) + g_j(x)w \\ 1 \end{bmatrix} \leq 0 \tag{2.57}$$

  and

$$\begin{bmatrix} \nabla V_k(x) \\ -W(y,w) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} f_j(x) + g_j(x)w \\ 1 \end{bmatrix} = \begin{bmatrix} \nabla V_k(x) \\ -W(y,w) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} f_k(x) + g_k(x)w \\ 1 \end{bmatrix} \leq 0 \tag{2.58}$$

From (2.55-2.58), it follows that (2.34) is satisfied for the differential inclusion (2.52).

In conclusion, by Proposition 2.1, $S(x,t)$ is nonincreasing along the trajectories of (2.35) in $\mathcal{X}$ i.e.

$$t_1 \leq t_2 \Rightarrow V(x(t_1)) - \int_0^{t_1} W(y,w)d\tau \geq V(x(t_2)) - \int_0^{t_2} W(y,w)d\tau \tag{2.59}$$

Therefore (2.36) is satisfied and the system (2.35) is dissipative with storage function $V(x)$ and supply rate $W(y,w)$. □

2.6 PWA Systems

A PWA system is described by

$$\dot{x} = A_i x + a_i, \quad \text{for } x \in \mathcal{R}_i \tag{2.60}$$

where $A_i \in \mathbb{R}^{n \times n}$, $a_i \in \mathbb{R}^n$ for $i = 1, \ldots, M$. It is assumed that $a_i = 0$ for $i \in \mathcal{I}(0)$. Therefore, the origin is an equilibrium point of the system. A subset of the state space $\mathcal{X} \subset \mathbb{R}^n$ is partitioned into M polytopic regions $\mathcal{R}_i$. Each region is constructed as the intersection of a finite number of half spaces

$$\mathcal{R}_i = \{x | E_i x + e_i \succ 0\}, \tag{2.61}$$

where $E_i \in \mathbb{R}^{p_i \times n}$, $e_i \in \mathbb{R}^{p_i}$ and $\succ$ represents an elementwise inequality. Each polytopic region $\mathcal{R}_i$ can be outer approximated by a (possibly degenerate) quadratic curve ε_i

$$\mathcal{R}_i \subseteq \varepsilon_i = \{x | \bar{x}^T \bar{E}_i^T \bar{\Lambda}_i \bar{E}_i \bar{x} > 0\} \tag{2.62}$$

where $\bar{\Lambda}_i \in \mathbb{R}^{(p_i+1)\times(p_i+1)}$ is a matrix with nonnegative entries and

$$\bar{E}_i = \begin{bmatrix} E_i & e_i \\ 0 & 1 \end{bmatrix} \tag{2.63}$$

A parametric description of the boundaries between two regions $\mathcal{R}_i$ and $\mathcal{R}_j$ where $\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}_j} \neq \emptyset$ can also be obtained as (see [56] and [103] for more details)

$$\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}_j} \subseteq \{x | x = F_{ij}s + f_{ij},\ s \in \mathbb{R}^{n-1}\} \tag{2.64}$$

In this chapter, following [66], the following notation is considered

$$\forall x \in \mathbb{R}^n, \bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix} \tag{2.65}$$

Equation (2.60) can then be written as

$$\dot{\bar{x}}(t) = \bar{A}_i \bar{x}(t),\ \ x(t) \in \mathcal{R}_i \tag{2.66}$$

where

$$\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix} \tag{2.67}$$

In the following, three types of candidate Lyapunov functions are examined for stability analysis of PWA systems.

2.6.1 Quadratic Lyapunov Function

Perhaps the simplest candidate for a $\mathcal{C}^1$ Lyapunov function is the quadratic form

$$V(x) = x^{\mathrm{T}} P x \tag{2.68}$$

where $P = P^{\mathrm{T}} > 0$. The following proposition describes the sufficient conditions for the stability of the PWA system (2.60) using a quadratic Lyapunov function.

Proposition 2.4. *(Sufficient conditions for quadratic Lyapunov stability) If for a given decay rate $\alpha > 0$, there exists $P = P^T > 0$ satisfying*

$$\begin{cases} PA_i + A_i^T P \leq -\alpha P, & \text{if } a_i = 0 \text{ and } e_i \neq 0 \\ \\ PA_i + A_i^T P + E_i^T \Lambda_i E_i \leq -\alpha P, & \text{if } a_i = 0 \text{ and } e_i = 0 \\ \\ \bar{P}\bar{A}_i + \bar{A}_i^T \bar{P} + \bar{E}_i^T \Lambda_i \bar{E}_i \leq -\alpha \bar{P}, & \text{otherwise} \end{cases} \tag{2.69}$$

for $i = 1, \ldots, M$ *where*

$$\bar{P} = \begin{bmatrix} P & 0_{n\times 1} \\ 0_{1\times n} & 0 \end{bmatrix}, \tag{2.70}$$

$\Lambda_i \in \mathbb{R}^{p_i \times p_i}$ *and* $\bar{\Lambda}_i \in \mathbb{R}^{(p_i+1)\times(p_i+1)}$ *have nonnegative entries,* $x = 0$ *is asymptotically stable for the PWA system (2.60).*

Proof. Consider $V(x) = x^{\mathrm{T}}Px$ as the candidate Lyapunov function. For this function, $\nabla V(x) = 2Px$. In the following, the regions $\mathcal{R}_i$ will be divided into three groups:

1. If $a_i = 0$ and $e_i \neq 0$, we conclude from (2.69) that for all $x \in \mathbb{R}^n$

$$\begin{aligned} \nabla V(x)^{\mathrm{T}} A_i x &= 2x^{\mathrm{T}} P A_i x \\ &= x^{\mathrm{T}}(PA_i + A_i^{\mathrm{T}} P)x \\ &\leq -\alpha x^{\mathrm{T}} P x = -\alpha V(x) \end{aligned} \tag{2.71}$$

2. If $a_i = 0$ and $e_i = 0$, $\overline{\mathcal{R}}_i = \{x | E_i x \geq 0\}$ and for any $\Lambda_i \in \mathbb{R}^{p_i \times p_i}$ with nonnegative entries and for all $x \in \overline{\mathcal{R}}_i$

$$x^{\mathrm{T}} E_i^{\mathrm{T}} \Lambda_i E_i x \geq 0 \tag{2.72}$$

In this case, (2.69) leads to the following inequality for all $x \in \overline{\mathcal{R}}_i$.

$$\begin{aligned} \nabla V(x)^{\mathrm{T}} A_i x &= 2x^{\mathrm{T}} P A_i x \\ &= x^{\mathrm{T}}(PA_i + A_i^{\mathrm{T}} P)x \\ &\leq -\alpha x^{\mathrm{T}} P x - x^{\mathrm{T}} E_i^{\mathrm{T}} \Lambda_i E_i x \\ &\leq -\alpha x^{\mathrm{T}} P x = -\alpha V(x) \end{aligned} \tag{2.73}$$

3. If $a_i \neq 0$, we have $\overline{\mathcal{R}}_i = \{x | \bar{E}_i \bar{x} \geq 0\}$ and similarly to the previous case, condition (2.69) implies that for all $x \in \overline{\mathcal{R}}_i$

$$
\begin{aligned}
\nabla V(x)^{\mathrm{T}}(A_i x + a_i) &= 2\bar{x}^{\mathrm{T}}\bar{P}\bar{A}_i\bar{x} \\
&= \bar{x}^{\mathrm{T}}(\bar{P}\bar{A}_i + \bar{A}_i^{\mathrm{T}}\bar{P})\bar{x} \\
&\leq -\alpha\bar{x}^{\mathrm{T}}\bar{P}\bar{x} - \bar{x}^{\mathrm{T}}\bar{E}_i^{\mathrm{T}}\Lambda_i\bar{E}_i\bar{x} \\
&\leq -\alpha\bar{x}^{\mathrm{T}}\bar{P}\bar{x} = -\alpha x^{\mathrm{T}}Px = -\alpha V(x) \qquad (2.74)
\end{aligned}
$$

In summary, for all $x \in \overline{\mathcal{R}}_i, i = 1, \ldots, M$,

$$
\nabla V(x)^{\mathrm{T}}(A_i x + a_i) \leq -\alpha V(x) \qquad (2.75)
$$

Therefore using Proposition 2.2, the system (2.60) is dissipative with the storage functinon $V(x)$ and the supply rate $-\alpha V(x)$. Therefore, considering (2.40), $S(x,t) = V(x) + \int_0^t \alpha V(x(\tau))d\tau$ is nonincreasing along the trajectories of (2.60). This implies that the conditions of Theorem 2.1 are satisfied and all trajectories of the PWA system (2.60) in $\mathcal{X}$ asymptotically converge to $x = 0$. □

Remark 2.2. *In Proposition 2.4, the origin is not required to be the equilibrium point of all the subsystems the PWA system (2.60). This makes Proposition 2.4 different from the common Lyapunov function approach in [75] which requires the origin to be the equilibrium point for all vector fields of the system.*

Proposition 2.4 provides sufficient conditions for quadratic stability of PWA systems as a set of linear matrix inequalities (LMIs). LMIs can be solved efficiently using interior point algorithms implemented in software packages such as Yalmip [76] and SeDuMi [121]. The following examples illustrate Proposition 2.4.

Example 2.2. *Consider the following Piecewise Linear (PWL) system:*

$$
\dot{x} = \begin{cases} A_1 x, & x_2 > 0 \\ \\ A_2 x, & x_2 < 0 \end{cases} \qquad (2.76)
$$

where

$$A_1 = \begin{bmatrix} -1 & -2 \\ 2 & -2 \end{bmatrix}, \; A_2 = \begin{bmatrix} -1 & 2 \\ -2 & -2 \end{bmatrix} \tag{2.77}$$

For this system, we have:

$$E_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & -1 \end{bmatrix} \tag{2.78}$$

Solving the following LMIs based on Proposition 2.4

$$\begin{cases} A_1^T P + PA_1 + \lambda_1 E_1^T E_1 < -\alpha P \\ \\ A_2^T P + PA_2 + \lambda_2 E_2^T E_2 < -\alpha P \\ \\ P > 0, \; \lambda_1 > 0, \; \lambda_2 > 0 \end{cases} \tag{2.79}$$

where $\alpha = 0.1$, *yields*

$$P = \begin{bmatrix} 0.6002 & 0 \\ 0 & 0.5817 \end{bmatrix}, \; \lambda_1 = 1.1329, \; \lambda_2 = 1.1329. \tag{2.80}$$

Therefore $x = 0$ *is asymptotically stable. It is interesting to note that this system has an attractive sliding mode on the negative side of the* x_1 *axis (Fig. 2.2). However, no separate condition was considered to check the existence or stability of the sliding mode.*

Example 2.3. *Consider the following system:*

$$\dot{x} = \begin{cases} A_1 x, & x_2 > 0 \\ \\ A_2 x, & x_2 < 0 \end{cases} \tag{2.81}$$

where

$$A_1 = \begin{bmatrix} 1 & -2 \\ 2 & -2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 2 \\ -2 & -2 \end{bmatrix} \tag{2.82}$$

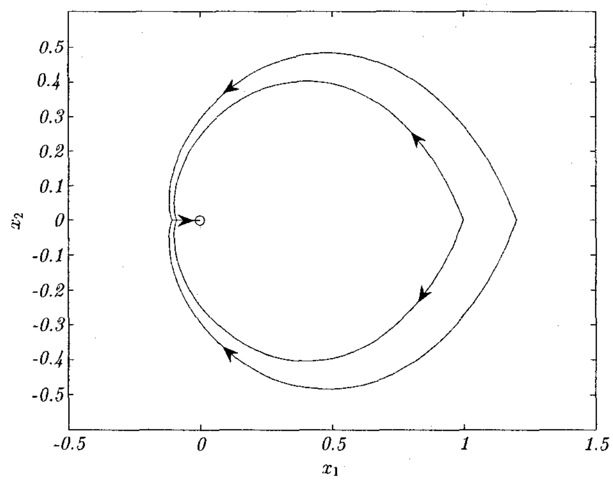

Figure 2.2: Trajectories of a stable PWA system with a sliding mode

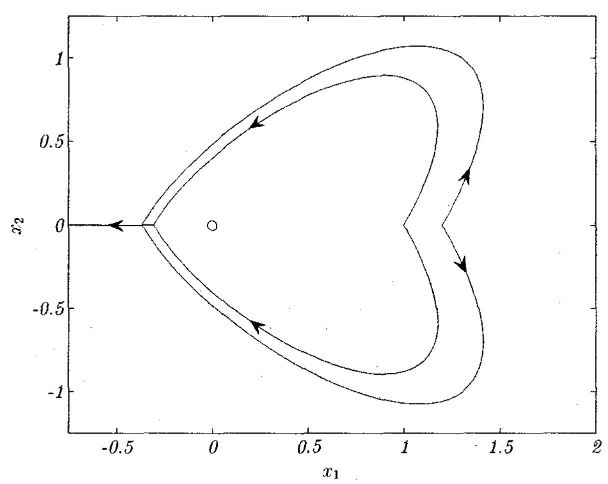

Figure 2.3: Trajectories of an unstable PWA system with a sliding mode

For this system, E_1 and E_2 are defined as in Example 2.2. The LMI set (2.79) is infeasible in this case. In fact, although A_1 and A_2 are Hurwitz, there exists an unstable sliding mode and system (2.81) is unstable (Fig. 2.3).

2.6.2 Piecewise Quadratic Lyapunov Function

For stability analysis of PWA systems, PWQ functions are less conservative than quadratic Lyapunov functions [66]. However, PWA systems with sliding modes are not usually considered in the literature of multiple Lyapunov functions. The reason is that the existence of a continuous positive definite PWQ function that decreases with time inside the regions is not a sufficient condition for stability of a PWA system. This is shown by the following counter-example.

Example 2.4. *Consider the PWA system (2.81) and the following PWQ Lyapunov function candidate.*

$$V(x) = \begin{cases} x^T P_1 x, & x_2 \geq 0 \\ \\ x^T P_2 x, & x_2 \leq 0 \end{cases} \tag{2.83}$$

The following set of constraints is a sufficient condition for (2.83) to be continuous, positive definite and decreasing with time inside the regions.

- *Continuous at $x_2 = 0$:*

$$\begin{bmatrix} 1 & 0 \end{bmatrix} (P_1 - P_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \tag{2.84}$$

- *Positive definite:*

$$P_1 > 0, \;\; P_2 > 0 \tag{2.85}$$

- *Decreasing with time inside the regions:*

$$\begin{cases} A_1^T P_1 + P_1 A_1 + \lambda_1 E_1^T E_1 < -\alpha P_1 \\ \\ A_2^T P_2 + P_2 A_2 + \lambda_2 E_2^T E_2 < -\alpha P_2 \\ \\ \lambda_1 > 0, \;\; \lambda_2 > 0, \;\; \alpha = 0.1 \end{cases} \tag{2.86}$$

One solution of the above problem is

$$\begin{aligned} P_1 &= \begin{bmatrix} 1.8073 & -1.0745 \\ -1.0745 & 1.4261 \end{bmatrix}, \\ P_2 &= \begin{bmatrix} 1.8073 & 1.0745 \\ 1.0745 & 1.4261 \end{bmatrix}, \\ \lambda_1 &= 0.5755, \lambda_2 = 0.5755. \end{aligned} \tag{2.87}$$

$V(x)$ in (2.83) is a continuous PWQ positive definite function that decreases with time inside the regions of system (2.81). However, system (2.81) is unstable. Therefore, the existence of such a function is not a sufficient condition for stability.

In [66, p.64], an extra condition is introduced for PWA systems which have sliding modes. However, the limitation of this method is that it requires previous knowledge of geometrical properties of the sliding modes. One way to solve this problem is to use $\mathcal{C}^1$ PWQ Lyapunov functions, which is also proposed in [66, p. 84].

Consider the piecewise quadratic candidate Lyapunov function continuous at the boundaries and defined in $\mathcal{X}$ by the expression

$$V(x) = \bar{x}^T \bar{P}_i \bar{x}, \text{ for } x \in \overline{\mathcal{R}}_i \tag{2.88}$$

where $\bar{P}_i = \bar{P}_i^T \in \mathbb{R}^{(n+1)\times(n+1)}$ and

$$\bar{P}_i = \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix} \tag{2.89}$$

with $P_i \in \mathbb{R}^{n \times n}$, $q_i \in \mathbb{R}^n$ and $r_i \in \mathbb{R}$. To simplify the notation, define

$$\bar{F}_{ij} = \begin{bmatrix} F_{ij} & f_{ij} \\ 0 & 1 \end{bmatrix} \tag{2.90}$$

The following proposition describes the sufficient conditions for the stability of the PWA system (2.60) based on a PWQ Lyapunov function.

Proposition 2.5. *(Sufficient conditions for PWQ Lyapunov stability) Let there exist matrices $\bar{P}_i = \bar{P}_i^T$ defined in (2.89), Z_i, $\bar{Z}_i$, Λ_i and $\bar{\Lambda}_i$ that verify the following conditions for all $i = 1, \ldots, M$ and a given decay rate $\alpha > 0$*

- *Conditions on the vector field:*

$$a_i = 0, \text{ if } 0 \in \overline{\mathcal{R}}_i \tag{2.91}$$

$$(\bar{A}_i - \bar{A}_j)\bar{F}_{ij} = 0, \text{ if } \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset \tag{2.92}$$

- *Continuity of the Lyapunov function:*

$$\bar{F}_{ij}^T(\bar{P}_i - \bar{P}_j)\bar{F}_{ij} = 0, \ \ if \ \ \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset \tag{2.93}$$

- *Positive definiteness of the Lyapunov function:*

$$q_i = 0, \ \ r_i = 0, \ \ if\, 0 \in \overline{\mathcal{R}}_i \tag{2.94}$$

$$P_i \geq \epsilon I, \ \ if\, 0 \in \overline{\mathcal{R}}_i \ and \ e_i \neq 0 \tag{2.95}$$

$$\begin{cases} Z_i \succeq 0 \\ P_i - E_i^T Z_i E_i \geq \epsilon I \end{cases}, \ \ if\, 0 \in \overline{\mathcal{R}}_i \ and \ e_i = 0 \tag{2.96}$$

$$\begin{cases} \bar{Z}_i \succeq 0 \\ \bar{P}_i - \bar{E}_i^T Z_i \bar{E}_i \geq \epsilon \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \end{cases}, \ \ if\, 0 \notin \overline{\mathcal{R}}_i \tag{2.97}$$

- *Monotonicity of the Lyapunov function:*

$$P_i A_i + A_i^T P_i \leq -\alpha P_i, \ \ if\, 0 \in \overline{\mathcal{R}}_i \ and \ e_i \neq 0 \tag{2.98}$$

$$\begin{cases} \Lambda_i \succeq 0 \\ P_i A_i + A_i^T P_i + E_i^T \Lambda_i E_i \leq -\alpha P_i \end{cases}, \ \ if\, 0 \in \overline{\mathcal{R}}_i \ and \ e_i = 0 \tag{2.99}$$

$$\begin{cases} \bar{\Lambda}_i \succeq 0 \\ \bar{P}_i \bar{A}_i + \bar{A}_i^T \bar{P}_i + \bar{E}_i^T \bar{\Lambda}_i \bar{E}_i \leq -\alpha \bar{P}_i \end{cases}, \ \ if\, 0 \notin \overline{\mathcal{R}}_i \tag{2.100}$$

Then all the trajectories of (2.60) in $\mathcal{X}$ asymptotically converge to $x = 0$.

Proof. Consider $V(x) = \bar{x}^{\mathrm{T}} \bar{P}_i \bar{x}$ for $x \in \overline{\mathcal{R}}_i$ as the candidate Lyapunov function. It follows from (2.64) and (2.93) that for any $x \in \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j$, $V_i(x) = V_j(x)$. Therefore $V(x)$ is continuous over $\mathcal{X}$. In addition, constraint (2.94) implies that $V(0) = 0$. The rest of the proof is divided into three parts:

1. If $a_i = 0$ and $e_i \neq 0$, we conclude from (2.95) that for all $x \neq 0$ in $\mathcal{R}_i$

$$V(x) \ = \ x^{\mathrm{T}} P_i x \geq \epsilon \|x\|^2 > 0 \tag{2.101}$$

and from (2.98) that for all $x \neq 0$

$$\begin{aligned}
\nabla V_i(x)^{\mathrm{T}} A_i x &= 2x^{\mathrm{T}} P_i A_i x \\
&= x^{\mathrm{T}}(P_i A_i + A_i^{\mathrm{T}} P_i)x \\
&\leq -\alpha x^{\mathrm{T}} P_i x = -\alpha V(x)
\end{aligned} \tag{2.102}$$

2. If $a_i = 0$ and $e_i = 0$, we have $\overline{\mathcal{R}}_i = \{x | E_i x \geq 0\}$ and for any $Z_i \in \mathbb{R}^{p_i \times p_i}$ and $\Lambda_i \in \mathbb{R}^{p_i \times p_i}$ with nonnegative entries and for all $x \in \overline{\mathcal{R}}_i$

$$x^{\mathrm{T}} E_i^{\mathrm{T}} Z_i E_i x \geq 0 \tag{2.103}$$

$$x^{\mathrm{T}} E_i^{\mathrm{T}} \Lambda_i E_i x \geq 0 \tag{2.104}$$

In this case, (2.96) leads to the following inequality for all $x \neq 0$ in $\overline{\mathcal{R}}_i$

$$\begin{aligned}
V_i(x) &= x^{\mathrm{T}} P_i x \\
&> x^{\mathrm{T}} E_i^{\mathrm{T}} Z_i E_i x + \epsilon \|x\|^2 \\
&\geq \epsilon \|x\|^2 > 0
\end{aligned} \tag{2.105}$$

and (2.99) leads to the following inequality for all $x \neq 0$ in $\overline{\mathcal{R}}_i$

$$\begin{aligned}
\nabla V_i(x)^{\mathrm{T}} A_i x &= 2x^{\mathrm{T}} P_i A_i x \\
&= x^{\mathrm{T}}(P_i A_i + A_i^{\mathrm{T}} P_i)x \\
&\leq -\alpha x^{\mathrm{T}} P_i x - x^{\mathrm{T}} E_i^{\mathrm{T}} \Lambda_i E_i x \\
&\leq -\alpha x^{\mathrm{T}} P_i x = -\alpha V(x)
\end{aligned} \tag{2.106}$$

3. If $a_i \neq 0$, we have $\overline{\mathcal{R}}_i = \{x | \bar{E}_i \bar{x} \geq 0\}$ and similarly to the previous case, condition (2.97) implies that for all $x \neq 0$ in $\overline{\mathcal{R}}_i$

$$\begin{aligned}
V_i(x) &= \bar{x}^{\mathrm{T}} \bar{P}_i \bar{x} \\
&> \bar{x}^{\mathrm{T}} \bar{E}_i^{\mathrm{T}} \bar{Z}_i \bar{E}_i \bar{x} + \epsilon \|x\|^2 \\
&\geq \epsilon \|x\|^2 > 0
\end{aligned} \tag{2.107}$$

and condition (2.100) implies that for all $x \neq 0$ in $\overline{\mathcal{R}}_i$

$$\begin{aligned}
\nabla V_i(x)^{\mathrm{T}}(A_i x + a_i) &= 2\bar{x}^{\mathrm{T}}\bar{P}_i\bar{A}_i\bar{x} \\
&= \bar{x}^{\mathrm{T}}(\bar{P}_i\bar{A}_i + \bar{A}_i^{\mathrm{T}}\bar{P}_i)\bar{x} \\
&\leq -\alpha\bar{x}^{\mathrm{T}}\bar{P}_i\bar{x} - \bar{x}^{\mathrm{T}}\bar{E}_i^{\mathrm{T}}\Lambda_i\bar{E}_i\bar{x} \\
&\leq -\alpha\bar{x}^{\mathrm{T}}\bar{P}_i\bar{x} = -\alpha V(x) \qquad (2.108)
\end{aligned}$$

In summary, for all $x \in \overline{\mathcal{R}}_i$ and for $i = 1, \ldots, M$,

$$V_i(x) \geq \epsilon\|x\|^2 \qquad (2.109)$$

$$\nabla V_i(x)^{\mathrm{T}}(A_i x + a_i) \leq -\alpha V_i(x) \qquad (2.110)$$

Therefore using Proposition 2.3, the system (2.60) is dissipative with the storage functinon $V(x)$ and the supply rate $-\alpha V(x)$. Therefore $S(x,t) = V(x) + \int_0^t \alpha V(x(\tau))d\tau$ is nonincreasing along the trajectories of (2.60). This implies that the conditions of Theorem 2.1 are satisfied and all trajectories of the PWA system (2.60) in $\mathcal{X}$ asymptotically converge to $x = 0$. □

PWQ Lyapunov functions are less conservative than quadratic Lyapunov functions. However, it is stated in [66] that for the following PWA system, it is not possible to find a $\mathcal{C}^1$ PWQ Lyapunov function although the system is stable.

$$\dot{x} = \begin{cases} A_1 x = \begin{bmatrix} -2 & -2 \\ 4 & 1 \end{bmatrix} x, & x_2 > 0 \\ \\ A_2 x = \begin{bmatrix} -2 & 2 \\ -4 & 1 \end{bmatrix} x, & x_2 < 0 \end{cases} \qquad (2.111)$$

In the next section, it is shown that a polynomial Lyapunov function exists for this system and can be found using the proposed method in section 2.6.3.

2.6.3 Sum of Squares Lyapunov Function

In this section, it is proposed to consider a sum of squares polynomial as a Lyapunov function candidate. For a tutorial about recent system analysis techniques based on the sum of squares decomposition see [85]. A sum of squares polynomial is defined in the following.

Definition 2.4. *[94] A multivariate polynomial $p(x_1, \ldots, x_n) \triangleq p(x)$ is a sum of squares, if there exist polynomials $p_1(x), \ldots, p_m(x)$ such that*

$$p(x) = \sum_{i=1}^{m} p_i^2(x). \tag{2.112}$$

SOS polynomials $p(x)$ are globally nonnegative. Although verifying nonnegativity of a polynomial is an NP-hard problem [81], the SOS condition can be formulated as a convex problem in polynomial coefficients [94]. However, note that not all nonnegative polynomials are SOS. In the following proposition, nonnegativity of SOS polynomials is used to construct a Lyapunov function for PWA systems.

Proposition 2.6. *If for the PWA system (2.60), there exists a polynomial $V(x)$ satisfying*

$$V(x) - \lambda(\|x\|^2) \text{ is SOS.} \tag{2.113}$$

$$-\nabla V(x)^T(A_i x + a_i) - \Gamma_i(x)^T(E_i x + e_i) - \alpha V(x) \text{ is SOS for all } i. \tag{2.114}$$

where $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing polynomial function, $\lambda(0) = 0$, $\alpha > 0$ and $\Gamma_i : \mathbb{R}^n \to \mathbb{R}^{p_i \times 1}$ is a vector of SOS polynomials, $x = 0$ is asymptotically stable.

Proof. Conditions (2.113) imply

$$V(x) \geq \lambda(\|x\|^2) \tag{2.115}$$

Condition (2.114) leads to the following inequality

$$\nabla V(x)^{\mathrm{T}}(A_i x + a_i) \leq -\Gamma_i(x)^{\mathrm{T}}(E_i x + e_i) - \alpha V(x) \tag{2.116}$$

Since $\Gamma_i(x)$ is a vector of SOS polynomials and $E_i x + e_i \geq 0$ for all x in $\overline{\mathcal{R}}_i$, we have

$$\Gamma_i(x)^{\mathrm{T}}(E_i x + e_i) \geq 0, \ \forall x \in \overline{\mathcal{R}}_i \tag{2.117}$$

Therefore (2.116) and (2.117) imply

$$\nabla V(x)^{\mathrm{T}}(A_i x + a_i) \leq -\alpha V(x) < 0, \ \forall x \in \overline{\mathcal{R}}_i, x \neq 0 \tag{2.118}$$

Thus, using Proposition 2.2, the system (2.60) is dissipative with the storage functinon $V(x)$ and the supply rate $-\alpha V(x)$. Therefore, considering (2.40), $S(x,t) = V(x) + \int_0^t \alpha V(x(\tau))d\tau$ is nonincreasing along the trajectories of (2.60). This implies that the conditions of Theorem 2.1 are satisfied and all trajectories of the PWA system (2.60) in $\mathcal{X}$ asymptotically converge to $x = 0$. $\square$

Remark 2.3. *The parameters of the Lyapunov function can be computed by solving the SOS program in Proposition 2.6 using Yalmip [76] and SeDuMi [121].*

Example 2.5. *Consider the PWA system (2.111). There is no quadratic or C^1 PWQ Lyapunov function for this system [66, p.84]. However, by solving the following SOS program we can find a sixth order polynomial Lyapunov function.*

$$V(x) - 0.001\|x\|^2 \textit{ is SOS.}$$

$$-\nabla V.(A_1 x) - \Gamma_1(x)(x_2) - 0.01V(x) \textit{ is SOS.}$$

$$-\nabla V.(A_2 x) - \Gamma_2(x)(-x_2) - 0.01V(x) \textit{ is SOS.}$$

where $\Gamma_1(.)$ and $\Gamma_2(.)$ are fourth order SOS polynomials. This is a convex problem and can be solved by SOSTOOLS [95] or Yalmip [76]. One feasible solution to the problem is

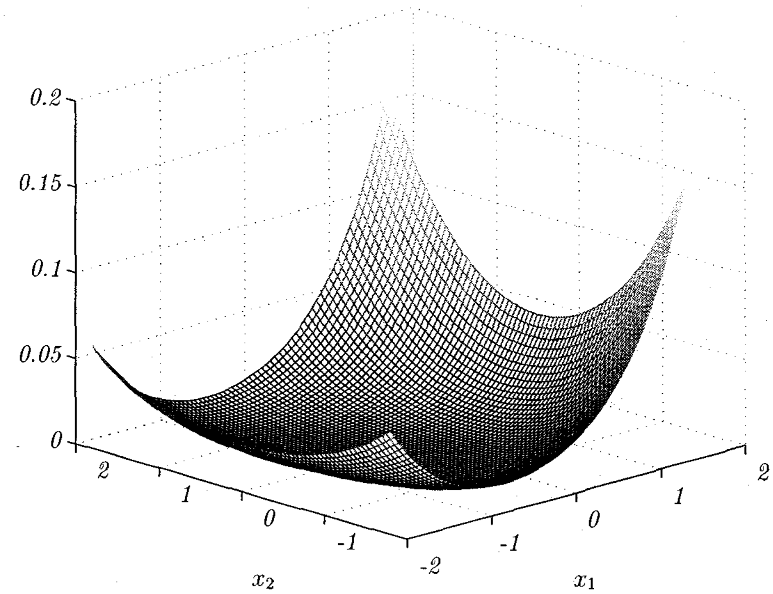

Figure 2.4: The SOS Lyapunov function

$$\begin{aligned}
V(x) &= (-0.0589x_1 - 0.0567x_1^2 - 0.00143x_1x_2 - 0.02x_2^2 \\
&\quad -0.00269x_1^3 - 0.00147x_1x_2^2)^2 + (0.00102x_1 - 0.0323x_2 \\
&\quad -0.0651x_1x_2 - 0.00265x_1^2x_2)^2 + (-0.0156x_1 + 0.0253x_1^2 - 0.0257x_2^2 \\
&\quad +0.00182x_1^3 - 0.00207x_1x_2^2)^2 + (+0.00717x_1 - 0.00439x_1^2 - 0.00863x_2^2)^2 \\
&\quad +(-0.00957x_2 + 0.00474x_1x_2)^2 + (-0.0036x_1^2x_2 + 0.00118x_2^3)^2 \\
&\quad +(-0.00166x_1^3 + 0.00232x_1x_2^2)^2 + (0.00124x_1^3)^2 + (0.00104x_2^3)^2 \qquad (2.119)
\end{aligned}$$

This SOS Lyapunov function is shown in Fig. 2.4. Trajectories of the system (2.111) and contours of the SOS Lyapunov function are shown in Fig. 2.5. Notice that there is a stable sliding mode in this system.

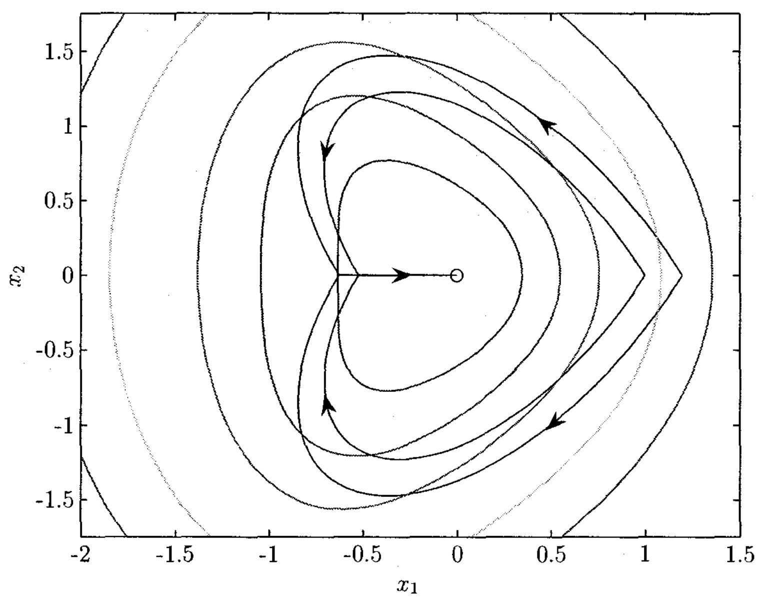

Figure 2.5: Trajectories of a stable PWA system (black) and contours of the Lyapunov function (gray)

2.7 PWA Slab Systems

A PWA slab system can be described by

$$\dot{x} = A_i x + a_i, \quad \text{for } x \in \mathcal{R}_i \tag{2.120}$$

where $A_i \in \mathbb{R}^{n \times n}$, $a_i \in \mathbb{R}^n$. It is assumed that $a_i = 0$ for $i \in \mathcal{I}(0)$. Therefore, the origin is an equilibrium point of the system. The slab regions $\mathcal{R}_i$, $i = 1, \ldots, M$ partitioning a slab subset of the state space $\mathcal{X} \subset \mathbb{R}^n$ are defined as

$$\mathcal{R}_i = \{x \mid \sigma_i < C_{\mathcal{R}} x < \sigma_{i+1}\}, \tag{2.121}$$

where $C_{\mathcal{R}} \in \mathbb{R}^{1 \times n}$ and σ_i for $i = 1, \ldots, M+1$ are scalars such that

$$\sigma_1 < \sigma_2 < \ldots < \sigma_{M+1} \tag{2.122}$$

Each slab region can alternatively be described by the following degenerate ellipsoid

$$\mathcal{R}_i = \{x \mid \|L_i x + l_i\| < 1\} \tag{2.123}$$

where $L_i = 2C_{\mathcal{R}}/(\sigma_{i+1} - \sigma_i)$ and $l_i = -(\sigma_{i+1} + \sigma_i)/(\sigma_{i+1} - \sigma_i)$.

We are interested to know if all possible trajectories in $\mathcal{X}$ asymptotically converge to the origin. Note that the right-hand-side of (2.120) is not necessarily continuous and therefore there might exist attractive sliding modes. The following proposition provides sufficient conditions for the stability of system (2.120) based on Theorem 2.1.

Proposition 2.7. *All trajectories of the PWA slab system (2.120) in $\mathcal{X}$ asymptotically converge to $x = 0$ if for a given decay rate $\alpha > 0$, there exist $P \in \mathbb{R}^{n\times n}$ and $\lambda_i \in \mathbb{R}$ for $i = 1, \ldots, M$ such that*

$$P > 0, \tag{2.124}$$

$$A_i^T P + PA_i + \alpha P < 0, \ \forall i \in \mathcal{I}(0), \tag{2.125}$$

$$\left\{ \begin{array}{l} \lambda_i < 0, \\ \begin{bmatrix} A_i^T P + PA_i + \alpha P + \lambda_i L_i^T L_i & Pa_i + \lambda_i l_i L_i^T \\ a_i^T P + \lambda_i l_i L_i & \lambda_i (l_i^2 - 1) \end{bmatrix} < 0, \end{array} \right. \quad \textit{for } i \notin \mathcal{I}(0) \tag{2.126}$$

Proof. Consider the candidate Lyapunov function $V(x) = x^{\mathrm{T}} Px$ for the PWA slab system (2.120) where $P > 0$. Consider the following function

$$S(x, t) = V(x) + \int_0^t \alpha V(x(\tau))d\tau, \tag{2.127}$$

where $\alpha > 0$. In the following, we will show that $S(x, t)$ is nonincreasing along the trajectories of (2.120):

1. For $x \in \overline{\mathcal{R}}_i$ where $i \in \mathcal{I}(0)$, multiplying the inequality (2.125) by x^{T} and x from left and right, respectively, implies

$$\nabla V(x)^{\mathrm{T}} A_i x + \alpha V(x) < 0, \text{ for } x \in \overline{\mathcal{R}}_i, i \in \mathcal{I}(0) \tag{2.128}$$

2. For $x \in \overline{\mathcal{R}}_i$ where $i \notin \mathcal{I}(0)$, it follows from the constraint (2.126) that

$$\nabla V(x)^{\mathrm{T}}(A_i x + a_i) + \alpha V(x) + \lambda_i(\|L_i x - l_i\|^2 - 1) < 0, \tag{2.129}$$

Since $\lambda_i < 0$, conditions (2.129) and (2.123) imply

$$\nabla V(x)^{\mathrm{T}}(A_i x + a_i) + \alpha V(x) < 0, \text{ for } x \in \overline{\mathcal{R}}_i, i \notin \mathcal{I}(0) \tag{2.130}$$

Now, it follows from (2.128), (2.130) and Proposition 2.2 that the system (2.120) is dissipative with the storage functinon $V(x)$ and the supply rate $-\alpha V(x)$. Therefore $S(x,t)$ is nonincreasing along the trajectories of (2.120). This implies that the conditions of Theorem 2.1 are satisfied and all trajectories of the PWA slab system (2.120) in $\mathcal{X}$ asymptotically converge to $x = 0$. □

2.8 PWP Systems

The dynamics of a PWP system can be written as follows.

$$\dot{x}(t) = f_i(x(t)), \text{ if } x(t) \in \mathcal{P}_i \tag{2.131}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector and $f_i(x) \in \mathbb{R}^n$ are polynomial functions of x. The regions $\mathcal{P}_i$, $i \in \mathcal{I} = \{1, \ldots, M\}$, partition a subset of the state space $\mathcal{X} \subset \mathbb{R}^n$ such that $\cup_{i=1}^{M} \overline{\mathcal{P}}_i = \mathcal{X}$, $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$, $i \neq j$, where $\overline{\mathcal{P}}_i$ denotes the closure of $\mathcal{P}_i$. Each region is described by

$$\mathcal{P}_i = \{x | E_i(x) \succ 0\} \tag{2.132}$$

where $E_i(x) \in \mathbb{R}^{p_i}$ is a vector polynomial function of x and $\succ$ represents an elementwise inequality.

Proposition 2.8. *If for the PWP system (2.131), there exists a polynomial $V(x)$ satisfying*

$$V(x) - \lambda(\|x\|^2) \text{ is SOS.} \tag{2.133}$$

$$-\nabla V(x)^T f_i(x) - \Gamma_i^T(x) E_i(x) - \alpha V(x) \text{ is SOS for all } i. \tag{2.134}$$

where $\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a strictly increasing polynomial function, $\lambda(0) = 0$, $\alpha > 0$ and $\Gamma_i : \mathbb{R}^n \rightarrow \mathbb{R}^{p_i \times 1}$ is a vector of SOS polynomials, $x = 0$ is asymptotically stable.

Proof. Condition (2.133) implies

$$V(x) \geq \lambda(\|x\|^2) \tag{2.135}$$

Condition (2.134) leads to the following inequality

$$\nabla V(x)^{\mathrm{T}} f_i(x) \leq -\Gamma_i^{\mathrm{T}}(x) E_i(x) - \alpha V(x) \tag{2.136}$$

Since $\Gamma_i(x)$ is a vector of SOS polynomials and $E_i(x) \geq 0$ for all x in $\overline{\mathcal{P}}_i$, we have

$$\Gamma_i^{\mathrm{T}}(x) E_i(x) \geq 0, \ \forall x \in \overline{\mathcal{P}}_i \tag{2.137}$$

Therefore (2.136) and (2.137) imply

$$\nabla V(x)^{\mathrm{T}} f_i(x) \leq -\alpha V(x) < 0, \ \forall x \in \overline{\mathcal{R}}_i, x \neq 0 \tag{2.138}$$

Thus, using Proposition 2.2, the system (2.131) is dissipative with the storage function $V(x)$ and the supply rate $-\alpha V(x)$. Therefore, considerong (2.40), $S(x,t) = V(x) + \int_0^t \alpha V(x(\tau))d\tau$ is nonincreasing along the trajectories of (2.131). This implies that the conditions of Theorem 2.1 are satisfied and all trajectories of the PWP system (2.131) in $\mathcal{X}$ asymptotically converge to $x = 0$. □

2.9 Conclusions

In this chapter a general nonsmooth theorem for stability of nonlinear systems was stated and proved. Then, sufficient conditions for stability of PWA slab systems, PWA systems and PWP systems were formulated as convex problems subject to LMIs. The importance of the results of this chapter is to show that sufficient conditions for the stability of PWA and PWP systems can be formed without any need for a-priori information about attractive sliding modes on switching surfaces. This is very important given that such information is very difficult to obtain for complex systems and, even when this information is available, it might lead to unnecessary conservativeness.

Chapter 3

Extension of local linear controllers to global piecewise affine controllers for uncertain nonlinear systems

A two-step controller synthesis method is proposed in this chapter for a class of uncertain nonlinear systems described by piecewise affine differential inclusions. In the first step, a robust linear controller is designed for the linear differential inclusion that describes the dynamics of the nonlinear system close to the equilibrium point. In the second step, a stabilizing piecewise affine controller is designed that coincides with the linear controller in a region around the equilibrium point. The proposed method has two objectives: global stability and local performance. It thus enables to use well known techniques in linear control design for local stability and performance while delivering a global piecewise affine controller that is guaranteed to stabilize the nonlinear system. The new method will be applied to numerical examples.

3.1 Introduction

Linear control theory provides a variety of well established tools to guarantee robust stability and performance [36]. The controller is, however, valid only locally if the controller is designed for the linearization of a nonlinear system. In fact, the linear controller may not even stabilize the nonlinear system if the initial condition is far from the linearization point. On the other hand, most of the methods in nonlinear control theory address global asymptotic stability but not necessarily performance. Designing a controller that has both a large region of attraction and a good local performance is therefore one of the most interesting research problems in nonlinear control theory [80]. Having this problem in mind, a two-step method is proposed in this chapter to design a piecewise affine (PWA) controller for uncertain nonlinear systems described by piecewise affine differential inclusions (PWADI). The objective of the proposed method is to design a controller to satisfy a local performance requirement and to globally stabilize the nonlinear system. The method extends a linear controller designed for performance to a globally stabilizing PWA controller. One of the main advantages of this method is that it can be employed in many practical problems for which linear controllers currently exist without changing the local performance of the system.

The structure of the proposed method is shown in Figure 3.1. In the first step, a robust linear controller is designed for the linear differential inclusion (LDI) that approximates the local behaviour of the nonlinear system in a neighbourhood of the desired operating point. Then, a PWA controller that coincides with the linear controller in a region around the equilibrium point and globally stabilizes the nonlinear system is designed in the second step. Since the design approach is based on finding a piecewise quadratic Lyapunov function, it is only approximate in the sense that there is no guarantee that a Lyapunov function can be found. If one is found, global stability is guaranteed. Otherwise, the method is inconclusive.

In spite of their approximate nature, Lyapunov-based methods for PWA controller design appear to work well in practice and have been widely used in the literature [40, 56, 66, 96, 103, 104].

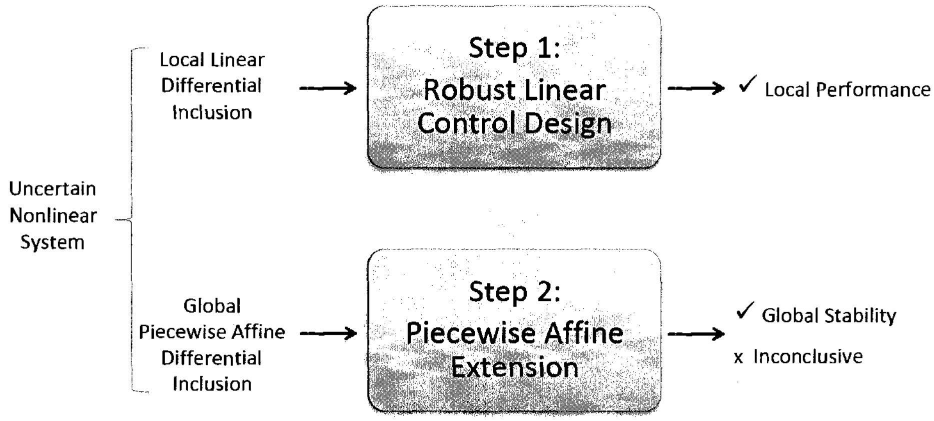

Figure 3.1: Structure of the proposed PWA controller design method

The main result of this chapter is proved in Theorem 3.2. The contribution of this result is to provide the theoretical framework for extending a local linear controller to a global PWA controller based on piecewise quadratic Lyapunov functions. In previous research, reference [103] has also used piecewise quadratic Lyapunov functions to synthesize PWA controllers. However, the method of [103] does not enable one to extend a local linear controller to a global PWA controller. Futhermore, it is assumed in [103] that there is one equilibrium point for the dynamic equations of each region. The equilibrium points of all regions are then selected a-priori by solving an optimization problem. It is also required that each of the equilibrium points be the extrema of the corresponding sector of any candidate Lyapunov function. By contrast, Theorem 3.2 now shows that it is in fact not necessary to compute those equilibrium points. This has the important advantage of relieving the designer from this tedious and non-intuitive task.

Note that a PWQ function is not necessarily differentiable everywhere and therefore it is a nonsmooth function. Despite this fact, none of the previously existing approaches to PWA controller design have developed a nonsmooth theory nor have they considered using well-developed nonsmooth analysis theory in the literature e.g. [31]. By contrast, in this chapter, we depart from previous approaches to PWA controller design by using Theorem 2.1, a Lyapunov theorem for nonsmooth Lyapunov functions. The theorem has the advantage of including the standard Lyapunov stability theorem in [70] as its special case for $\mathcal{C}^1$ Lyapunov functions. The proposed PWA controller in this chapter has the additional advantage of coinciding locally with a robust linear controller designed using linear control methods. It therefore combines local performance with global stability. One important application of the proposed method can thus be to extend the region of convergence of existing linear controllers for nonlinear systems.

The rest of this chapter is organized as follows. An illustrative example is employed in section 3.2 to clarify the need for the proposed method. Section 3.3 then explains the proposed method which consists of a robust linear controller design and its PWA extension. Finally, PWA controllers are designed for numerical examples in section 3.4 and conclusions are drawn in section 3.5.

3.2 Illustrative Example

In this section, the following nonlinear system is used to illustrate the design procedure:

$$\dot{x} = 0.5(1 - x^2) + u \tag{3.1}$$

The open loop system has two equilibrium points (figure 3.2), one at $x = -1$ (unstable) and the other one at $x = 1$ (stable). The goal is to design a controller so that for any $x(0) \in \mathcal{X} = [-4, 4]$, the trajectory of the system asymptotically converges to $x^\star = 1$. It is also required that for any $x(0) \in (0, 2)$, the following cost function

$$J = \int_0^\infty [Q(x-1)^2 + Ru^2]dt \tag{3.2}$$

be minimized where $Q = 2$ and $R = 1$.

To achieve this goal, continuous PWA functions $\sigma_1(x)$ and $\sigma_2(x)$ (Figure 3.3) are first defined so that

$$\dot{x} \in \mathbf{conv}\{\sigma_1(x) + u, \sigma_2(x) + u\} \tag{3.3}$$

where **conv** stands for the closed convex hull [107] of a set and $\sigma_1(x)$ and $\sigma_2(x)$ are affine in x inside each of the following regions:

$$\mathcal{R}_1 = (-4, -2), \mathcal{R}_2 = (-2, 0), \mathcal{R}_3 = (0, 2), \mathcal{R}_4 = (2, 4) \tag{3.4}$$

In $\mathcal{R}_3$ (where $x^\star$ is located), the dynamics of the system are described by the following LDI,

$$\dot{x} \in \mathbf{conv}\{-1.6(x-1) + u, -0.4(x-1) + u\} \tag{3.5}$$

Defining $z = x - x^\star = x - 1$, we have

$$\dot{z} \in \mathbf{conv}\{-1.6z + u, -0.4z + u\} \tag{3.6}$$

An LQR controller can be designed for (3.6) using the design method for robust linear controllers described in subsection 3.3.1. The resulting controller for $\mathcal{R}_3$ is then described by

$$u = -1.07x + 1.07 \tag{3.7}$$

Figure 3.4 shows the phase plane of the nonlinear system in feedback connection with the linear controller. It can be clearly seen that the system still has two equilibrium points. Therefore, although the closed-loop system locally satisfies the required performance measure, it is not globally stable. In the following sections, a method for extending the designed LQR controller to a PWA controller will be presented. It will be shown in section 3.4 that the resulting PWA controller has the same local performance and, in addition, it is globally stablizing.

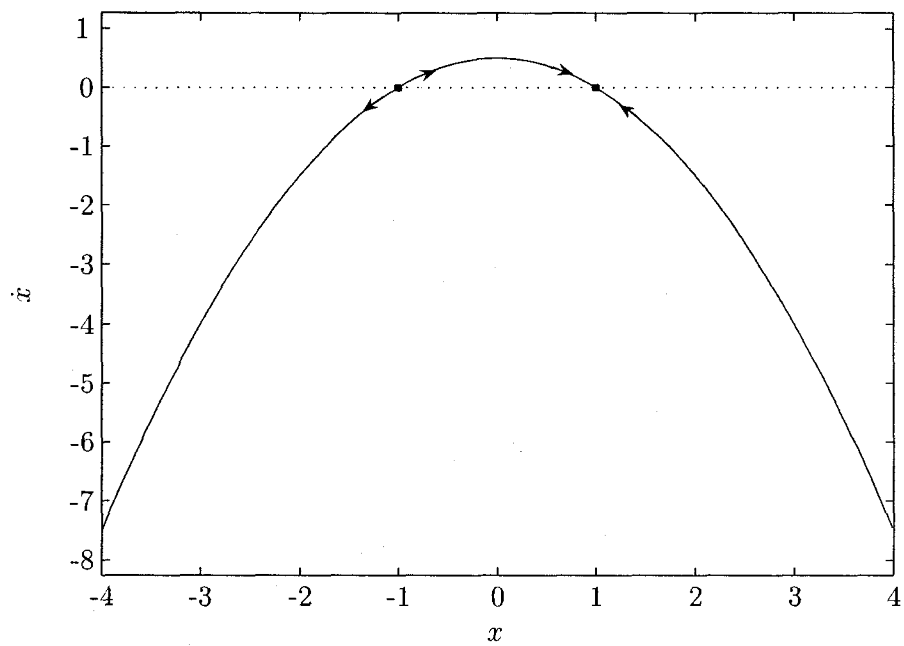

Figure 3.2: The phase plane of the open loop system.

3.3 Extension of a Linear Controller to a PWA Controller

This section proposes a method to extend a local linear controller to a global PWA controller. The method consists of two steps. In the first step, a robust linear controller will be designed for a nonlinear system that is affine in the input. In this step, the designer can benefit from well established methods for designing robust linear controllers to make the nonlinear system locally stable and to satisfy a performance requirement in a neighbourhood of the desired equilibrium point. In the second step, the objective is to design a PWA controller that coincides with the linear controller in the neighbourhood of the equilibrium point and guarantees global stability of the nonlinear closed-loop system.

Consider the following nonlinear system

$$\dot{x} = f(x) + g(x)u \tag{3.8}$$

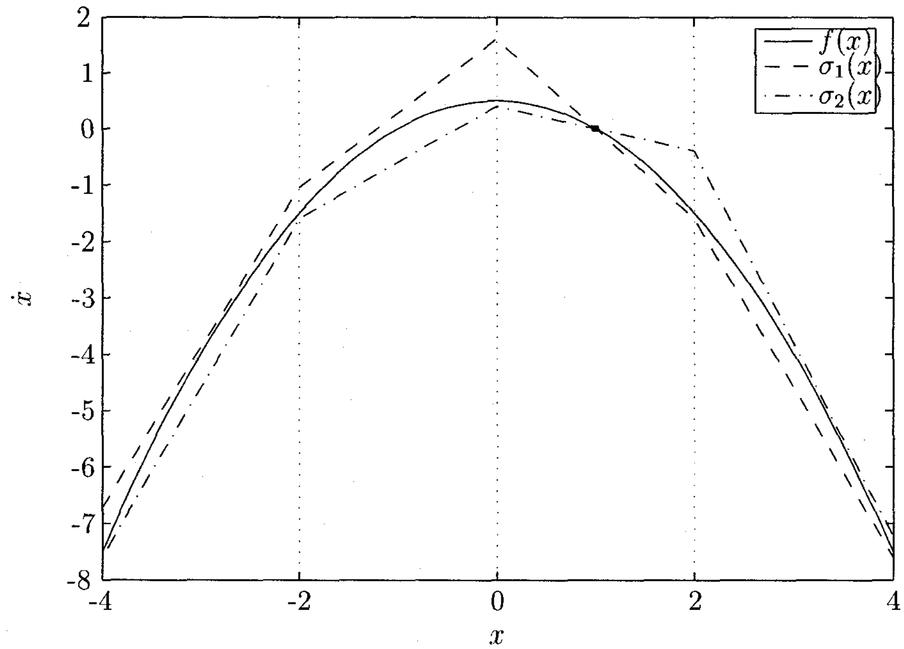

Figure 3.3: PWA differential inclusion

where $x \in \mathcal{X} \subset \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Let

$$\dot{x} \in \mathbf{conv}\{\sigma_1(x,u), \ldots, \sigma_{\mathcal{K}}(x,u)\} \tag{3.9}$$

where $\sigma_\kappa(x,u)$ is defined as

$$\sigma_\kappa(x,u) = A_{i\kappa}x + a_{i\kappa} + B_{i\kappa}u, x \in \overline{\mathcal{R}}_i, \tag{3.10}$$

with $A_{i\kappa} \in \mathbb{R}^{n \times n}$, $a_{i\kappa} \in \mathbb{R}^n$, $B_{i\kappa} \in \mathbb{R}^{n \times m}$ for $i = 1, \ldots, M$ and $\kappa = 1, \ldots, \mathcal{K}$. The polytopic regions $\mathcal{R}_i$ are constructed as the intersection of a finite number of half spaces

$$\mathcal{R}_i = \{x | E_i x + e_i \succ 0\}, \text{ for } i = 1, \ldots, M \tag{3.11}$$

where $E_i \in \mathbb{R}^{p_i \times n}$, $e_i \in \mathbb{R}^{p_i}$ and $\succ$ represents an elementwise inequality.

The objective is to stabilize system (3.8) to $x = x^\star$ while satisfying a performance requirement for x close to $x^\star$. The two steps of the proposed method will be presented in the following subsections.

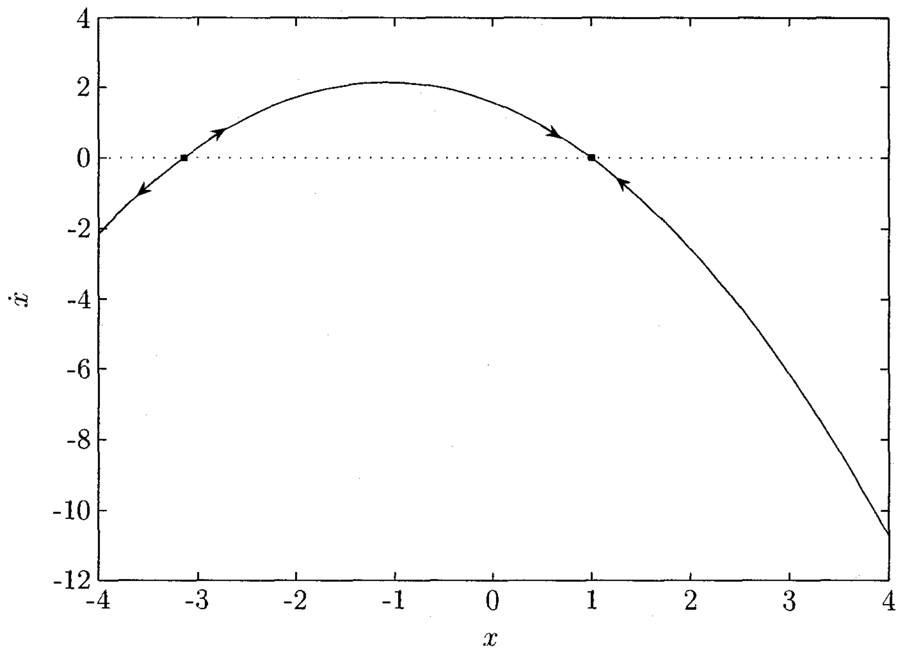

Figure 3.4: The phase plane of the system with the linear controller.

3.3.1 Step 1: Robust linear controller design

The first step is to design a robust linear controller for the LDI describing local behaviour of the nonlinear system. Consider a region $\mathcal{R}_{i^\star}$ such that

$$x^\star \in \overline{\mathcal{R}}_{i^\star}$$

The dynamics of system (3.8) in this region can be described by the following LDI.

$$\dot{x} \in \mathbf{conv}\{A_{i^\star\kappa}x + a_{i^\star\kappa} + B_{i^\star\kappa}u|\ \kappa = 1, \ldots, \mathcal{K}\} \tag{3.12}$$

Changing variables to $z = x - x^\star$ and assuming a state feedback control input $u = K_{i^\star}x + k_{i^\star}$ yields

$$\dot{z} \in \mathbf{conv}\{(A_{i^\star\kappa} + B_{i^\star\kappa}K_{i^\star})(z + x^\star) + a_{i^\star\kappa} + B_{i^\star\kappa}k_{i^\star}|\ \kappa = 1, \ldots, \mathcal{K}\} \tag{3.13}$$

To make $z = 0$ an equilibrium point of the system, the following condition must be satisfied.

$$(A_{i^\star\kappa} + B_{i^\star\kappa}K_{i^\star\kappa})x^\star + a_{i^\star\kappa} + B_{i\kappa}k_{i^\star\kappa} = 0,\ \ \kappa = 1, \ldots, \mathcal{K} \tag{3.14}$$

The closed-loop dynamics of the system can then be written as

$$\dot{z} \in \mathbf{conv}\{(A_{i^*\kappa} + B_{i^*\kappa}K_{i^*\kappa})z | \ \kappa = 1, \ldots, \mathcal{K}\} \tag{3.15}$$

The matrix gain K_{i^*} can be designed using robust linear control methodologies to satisfy desired design objectives for the differential inclusion (3.15). The affine term of the controller k_{i^*} can then be computed if the linear equation (3.14) has a solution. The choice of the required performance measure depends on the application. In this work, a robust LQR is designed for the LDI (3.12) using the following result taken from [64].

Theorem 3.1. *[64] Consider the cost function*

$$J = \int_0^\infty (z^T Q z + u^T R u) dt \tag{3.16}$$

where $Q \geq 0$ *and* $R > 0$ *for the following LDI*

$$\dot{z} \in \boldsymbol{conv}\{A_\kappa z + B_\kappa u | \kappa = 1, \ldots, \mathcal{K}\} \tag{3.17}$$

where $z \in \mathbb{R}^n$ *and* $u \in \mathbb{R}^m$. *If there exist* S *and* Y *so that*

$$S > 0 \tag{3.18}$$

$$\begin{bmatrix} SA_\kappa^T + A_\kappa S + Y^T B_\kappa^T + B_\kappa Y & SQ^{1/2} & Y^T R^{1/2} \\ Q^{1/2} S & -I_n & 0 \\ R^{1/2} Y & 0 & -I_m \end{bmatrix} < 0 \tag{3.19}$$

for $\kappa = 1, \ldots, \mathcal{K}$, *then for* $u = Kz$ *where* $K = YS^{-1}$, *we have*

$$J < z(0)^T S^{-1} z(0) \tag{3.20}$$

□

To avoid the dependency of the upper bound of the cost function on initial conditions of the system, it is proposed in [64] to assume that the initial condition is a random vector with zero mean and identity covariance, i.e.,

$$\begin{aligned}\mathbb{E}\{z(0)\} &= 0 \\ \mathbb{E}\{z(0)z(0)^T\} &= I\end{aligned} \tag{3.21}$$

It is shown in [64] that **Trace** (S^{-1}) is an upper bound on $\mathbb{E}\{J\}$. Therefore it is proposed in the same reference to solve the following optimization problem to minimize the upper bound on the cost function.

$$\begin{aligned}&\max \textbf{Trace } (S) \\ &\qquad \text{subject to} \quad (3.18) \text{ and } (3.19)\end{aligned} \tag{3.22}$$

This optimization problem can be solved using SeDuMi [121] and Yalmip [76] to compute the controller gain $K_{i^\star}$. The affine term $k_{i^\star}$ can then be computed by solving (3.14). The controller in region $\mathcal{R}_{i^\star}$ can then be written as

$$u = \bar{K}_{i^\star}\bar{x}, \text{ where } \bar{K}_{i^\star} = \begin{bmatrix} K_{i^\star} & k_{i^\star} \end{bmatrix} \text{ and } \bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix} \tag{3.23}$$

The next step is to find a PWA controller that coincides with the linear controller (3.23) in $\mathcal{R}_{i^\star}$ and guarantees the stability of the closed-loop system in $\mathcal{X}$.

3.3.2 Step 2: PWA state feedback design

The second step is to extend the robust linear controller to a PWA state feedback controller that stabilizes the nonlinear system (3.8) at the equilibrium point $x^\star$. A PWA control input of the following form is considered for this purpose

$$u = K_i x + k_i = \bar{K}_i \bar{x}, \text{ for } x \in \mathcal{R}_i \tag{3.24}$$

where

$$\bar{K}_i = \begin{bmatrix} K_i & k_i \end{bmatrix} \tag{3.25}$$

The closed loop system is therefore described by

$$\dot{x} = f(x) + g(x)(\bar{K}_i\bar{x}) \text{ for } x \in \mathcal{R}_i, \tag{3.26}$$

Consider the piecewise quadratic candidate Lyapunov function continuous at the boundaries and defined in $\mathcal{X}$ by the expression

$$V(x) = \bar{x}^T\bar{P}_i\bar{x}, \text{ for } x \in \overline{\mathcal{R}}_i \tag{3.27}$$

where $\bar{P}_i = \bar{P}_i^T \in \mathbb{R}^{(n+1)\times(n+1)}$ and

$$\bar{P}_i = \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix} \tag{3.28}$$

with $P_i \in \mathbb{R}^{n\times n}$, $q_i \in \mathbb{R}^n$ and $r_i \in \mathbb{R}$. To simplify the notation, define

$$\bar{A}_{i\kappa} = \begin{bmatrix} A_{i\kappa} & a_{i\kappa} \\ 0 & 1 \end{bmatrix}, \bar{B}_{i\kappa} = \begin{bmatrix} B_{i\kappa} \\ 0 \end{bmatrix}, \bar{F}_{ij} = \begin{bmatrix} F_{ij} & f_{ij} \\ 0 & 1 \end{bmatrix}, \bar{x}^\star = \begin{bmatrix} x^\star \\ 1 \end{bmatrix} \tag{3.29}$$

$$\bar{I} = \begin{bmatrix} I & 0 \\ -x^{\star\mathrm{T}} & 1 \end{bmatrix}\begin{bmatrix} I & -x^\star \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & -x^\star \\ -x^{\star\mathrm{T}} & x^{\star\mathrm{T}}x^\star \end{bmatrix} \tag{3.30}$$

The following theorem describes sufficient conditions for the existence of a continuous piecewise quadratic Lyapunov function of the form (3.27) and a PWA controller of the form (3.24) that coincides with the robust linear controller in the region where $x^\star$ lies and guarantees global stability.

Theorem 3.2. *Let there exist matrices $\bar{P}_i = \bar{P}_i^T$ defined in (3.28), $\bar{K}_i$ defined in (3.25), Z_i, $\bar{Z}_i$, $\Lambda_{i\kappa}$ and $\bar{\Lambda}_{i\kappa}$ that verify the following conditions for all $i = 1, \ldots, M$, $\kappa = 1, \ldots, \mathcal{K}$ and for a given decay rate $\alpha > 0$, desired equilibrium point $x^\star$, linear controller gain $\bar{K}_{i^\star}$ defined in (3.23) and $\epsilon > 0$*

- *Conditions on the PWA controller:*

$$\bar{K}_i = \bar{K}_{i^\star}, \text{ if } x^\star \in \overline{\mathcal{R}}_i \tag{3.31}$$

$$(\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i)\bar{x}^\star = 0, \text{ if } x^\star \in \overline{\mathcal{R}}_i \tag{3.32}$$

$$(\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i)\bar{F}_{ij} = (\bar{A}_{j\kappa} + \bar{B}_{j\kappa}\bar{K}_j)\bar{F}_{ij}, \text{ if } \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset \tag{3.33}$$

- *Continuity of the Lyapunov function:*

$$\bar{F}_{ij}^T(\bar{P}_i - \bar{P}_j)\bar{F}_{ij} = 0, \;\; \textit{if} \;\; \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset \tag{3.34}$$

- *Positive definiteness of the Lyapunov function:*

$$\bar{P}_i \bar{x}^\star = 0, \;\; \textit{if}\; x^\star \in \overline{\mathcal{R}}_i \tag{3.35}$$

$$P_i > \epsilon I, \textit{if}\; x^\star \in \overline{\mathcal{R}}_i, E_i x^\star + e_i \neq 0 \tag{3.36}$$

$$\begin{cases} Z_i \in \mathbb{R}^{n \times n}, Z_i \succeq 0 \\ P_i - E_i^T Z_i E_i > \epsilon I \end{cases}, \textit{if}\; x^\star \in \overline{\mathcal{R}}_i, E_i x^\star + e_i = 0 \tag{3.37}$$

$$\begin{cases} \bar{Z}_i \in \mathbb{R}^{(n+1)\times(n+1)}, \bar{Z}_i \succeq 0 \\ \bar{P}_i - \bar{E}_i^T \bar{Z}_i \bar{E}_i > \epsilon \bar{I} \end{cases}, \textit{if}\; x^\star \notin \overline{\mathcal{R}}_i \tag{3.38}$$

- *Monotonicity of the Lyapunov function:*

for i such that $x^\star \in \overline{\mathcal{R}}_i, E_i x^\star + e_i \neq 0$,

$$P_i(A_{i\kappa} + B_{i\kappa}K_i) + (A_{i\kappa} + B_{i\kappa}K_i)^T P_i < -\alpha P_i \tag{3.39}$$

for i such that $x^\star \in \overline{\mathcal{R}}_i, E_i x^\star + e_i = 0$,

$$\begin{cases} \Lambda_{i\kappa} \in \mathbb{R}^{n\times n}, \;\; \Lambda_{i\kappa} \succeq 0 \\ P_i(A_{i\kappa} + B_{i\kappa}K_i) + (A_{i\kappa} + B_{i\kappa}K_i)^T P_i + E_i^T \Lambda_{i\kappa} E_i < -\alpha P_i \end{cases} \tag{3.40}$$

for i such that $x^\star \notin \overline{\mathcal{R}}_i$,

$$\begin{cases} \bar{\Lambda}_{i\kappa} \in \mathbb{R}^{(n+1)\times(n+1)}, \;\; \bar{\Lambda}_{i\kappa} \succeq 0 \\ \bar{P}_i(\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i) + (\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i)^T \bar{P}_i + \bar{E}_i^T \bar{\Lambda}_{i\kappa} \bar{E}_i < -\alpha \bar{P}_i \end{cases} \tag{3.41}$$

Then for the nonlinear system (3.26), all trajectories in $\mathcal{X}$ asymptotically converge to $x = x^\star$.

Proof. Consider the change of coordinates $z = x - x^\star$ or equivalently

$$\begin{cases} \bar{z} = \bar{T}_{zx}\bar{x} \\ \bar{x} = \bar{T}_{zx}^{-1}\bar{z} \end{cases} \tag{3.42}$$

where

$$\bar{T}_{zx} = \begin{bmatrix} I & -x^\star \\ 0 & 1 \end{bmatrix}, \ \bar{T}_{zx}^{-1} = \begin{bmatrix} I & x^\star \\ 0 & 1 \end{bmatrix} \tag{3.43}$$

With this change of variables, the differential inclusions (3.9) with the control input $u = K_i x + k_i$ for $x \in \mathcal{R}_i$ is transformed into

$$\dot{z} \in \mathbf{conv}\{\sigma_1^z(z), \ldots, \sigma_{\mathcal{K}}^z(z)\} \tag{3.44}$$

where $\sigma_\kappa^z(z)$ is defined as

$$\sigma_\kappa^z(z) = A_{i\kappa}^z z +, a_{i\kappa}^z, \ z \in \overline{\mathcal{R}}_i^z, \tag{3.45}$$

for $i = 1, \ldots, M$ and $\kappa = 1, \ldots, \mathcal{K}$ where

$$\bar{A}_{i\kappa}^z = \begin{bmatrix} A_{i\kappa}^z & a_{i\kappa}^z \\ 0 & 0 \end{bmatrix} = (\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i)\bar{T}_{zx}^{-1} \tag{3.46}$$

and therefore

$$A_{i\kappa}^z = A_{i\kappa} + B_{i\kappa}K_i \tag{3.47}$$

$$a_{i\kappa}^z = (A_{i\kappa} + B_{i\kappa}K_i)x^\star + a_{i\kappa} + B_{i\kappa}k_i \tag{3.48}$$

Polytopic regions $\mathcal{R}_i^z$ can be written as

$$\mathcal{R}_i^z = \{z | E_i^z z + e_i^z \succ 0\}, \text{ for } i = 1, \ldots, M \tag{3.49}$$

where $E_i^z = E_i$ and $e_i^z = E_i x^\star + e_i$. If $\overline{\mathcal{R}}_i^z \bigcap \overline{\mathcal{R}}_j^z \neq \emptyset$ (i.e. $\overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset$) it follows from (2.64) that

$$\forall x \in \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j, \ \exists s \in \mathbb{R}^{n-1}, \ x = F_{ij}s + f_{ij} \tag{3.50}$$

Thus

$$\forall z \in \overline{\mathcal{R}}_i^z \bigcap \overline{\mathcal{R}}_j^z, \ \exists s \in \mathbb{R}^{n-1}, \ z = F_{ij}s + f_{ij} - x^\star \tag{3.51}$$

and one can define $F_{ij}^z = F_{ij}$, $f_{ij}^z = f_{ij} - x^\star$ and

$$\bar{F}_{ij}^z = \begin{bmatrix} F_{ij}^z & f_{ij}^z \\ 0 & 1 \end{bmatrix} = \bar{T}_{zx}\bar{F}_{ij} \tag{3.52}$$

For the candidate Lyapunov function $V^z(z) = V(x)$, for $z \in \mathcal{R}_i^z$ (i.e. $x \in \mathcal{R}_i$) we have

$$V^z(z) = V_i^z(z), \tag{3.53}$$

where

$$V_i^z(z) = \bar{x}^{\mathrm{T}} \bar{P}_i \bar{x} = \bar{z}^{\mathrm{T}} \bar{T}_{zx}^{-\mathrm{T}} \bar{P}_i \bar{T}_{zx}^{-1} \bar{z} = \bar{z}^{\mathrm{T}} \bar{P}_i^z \bar{z} \tag{3.54}$$

and thus

$$\bar{P}_i^z = \begin{bmatrix} P_i^z & q_i^z \\ q_i^{z\mathrm{T}} & r_i^z \end{bmatrix} = \left[\begin{array}{c|c} P_i & P_i x^\star + q_i \\ \hline x^{\star\mathrm{T}} P_i + q_i^{\mathrm{T}} & x^{\star\mathrm{T}} P_i x^\star + q_i^{\mathrm{T}} x^\star + x^{\star\mathrm{T}} q_i + r_i \end{array}\right] \tag{3.55}$$

In the following, it is shown that for any $\kappa \in \{1, \ldots, \mathcal{K}\}$, the PWA system $\dot{z} = \sigma_\kappa^z(z)$ satisfies the conditions of Proposition 2.5.

- Conditions on the vector field:
  - It follows from (3.32) that if $x^\star \in \mathcal{R}_i$ i.e. $0 \in \mathcal{R}_i^z$

    $$\left(\begin{bmatrix} A_{i\kappa} & a_{i\kappa} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_{i\kappa} \\ 0 \end{bmatrix} \begin{bmatrix} K_i & k_i \end{bmatrix} \right) \begin{bmatrix} x^\star \\ 1 \end{bmatrix} = 0 \tag{3.56}$$

    This is equivalent to $a_{i\kappa}^z = 0$ and therefore (2.91) is satisfied.
  - Using (3.33), for i and j such that $\overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset$ we have

    $$\left[(\bar{A}_{i\kappa} + \bar{B}_{i\kappa} \bar{K}_i) - (\bar{A}_{j\kappa} + \bar{B}_{j\kappa} \bar{K}_j) \right] \bar{T}_{zx}^{-1} \bar{T}_{zx} \bar{F}_{ij} = 0 \tag{3.57}$$

    and therefore from (3.46) and (3.52)

    $$(\bar{A}_{i\kappa}^z - \bar{A}_{j\kappa}^z) \bar{F}_{ij}^z = 0, \quad \text{if } \overline{\mathcal{R}}_i^z \bigcap \overline{\mathcal{R}}_j^z \neq \emptyset \tag{3.58}$$

    Thus (2.92) is satisfied.

- Continuity of the Lyapunov function: It follows from (3.34) that

$$\bar{F}_{ij}^{\mathrm{T}} \bar{T}_{zx}^{\mathrm{T}} \bar{T}_{zx}^{-\mathrm{T}} (\bar{P}_i - \bar{P}_j) \bar{T}_{zx}^{-1} \bar{T}_{zx} \bar{F}_{ij} = 0, \text{ if } \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset \tag{3.59}$$

and from (3.52)

$$\bar{F}_{ij}^{z^{\mathrm{T}}}(\bar{P}_i^z - \bar{P}_j^z)\bar{F}_{ij}^z = 0, \text{ if } \overline{\mathcal{R}}_i^z \bigcap \overline{\mathcal{R}}_j^z \neq \emptyset \tag{3.60}$$

Therefore (2.93) is satisfied.

- Positive definiteness of the Lyapunov function:

  - The condition (3.35) can be rewritten as

$$\begin{cases} P_i x^\star + q_i = 0 \\ q_i^{\mathrm{T}} x^\star + r_i = 0 \end{cases}, \quad \text{if } x^\star \in \mathcal{R}_i \tag{3.61}$$

  and therefore from (3.55),

$$\begin{cases} q_i^z = 0 \\ r_i^z = 0 \end{cases}, \quad \text{if } 0 \in \mathcal{R}_i^z \tag{3.62}$$

  Thus (2.94) is satisfied.

  - It follows from (3.36), (3.55), (3.62) and (3.49) that

$$P_i^z > \epsilon I, \text{ if } 0 \in \overline{\mathcal{R}}_i^z \text{ and } e_i^z \neq 0 \tag{3.63}$$

  Therefore (2.95) is satisfied.

  - It follows from (3.37), (3.55), (3.62) and (3.49) that

$$P_i^z - E_i^{\mathrm{T}} Z_i E_i > \epsilon I, \text{ if } 0 \in \overline{\mathcal{R}}_i^z \text{ and } e_i^z = 0 \tag{3.64}$$

  Therefore (2.96) is satisfied.

  - The condition (3.37) can be rewritten as

$$\bar{T}_{zx}^{-\mathrm{T}} \bar{P}_i \bar{T}_{zx}^{-1} - \bar{T}_{zx}^{-\mathrm{T}} \bar{E}_i^T \bar{Z}_i \bar{E}_i \bar{T}_{zx}^{-1} > \epsilon \bar{T}_{zx}^{-\mathrm{T}} \bar{I} \bar{T}_{zx}^{-1} \tag{3.65}$$

  and then from (3.54), (3.49) and (3.30) that

$$\bar{P}_i^z - \bar{E}_i^{z^{\mathrm{T}}} Z_i \bar{E}_i^z > \epsilon \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \text{ if } 0 \notin \overline{\mathcal{R}}_i^z \tag{3.66}$$

  Therefore (2.97) is satisfied.

- Monotonicity of the Lyapunov function:

  - The condition (3.39) can be rewritten using (3.49), (3.55) and (3.56) as

$$P_i^z A_{i\kappa}^z + A_{i\kappa}^{z\,\mathrm{T}} P_i^z < -\alpha P_i^z, \quad \text{if } 0 \in \mathcal{R}_i^z,\ e_i^z \neq 0 \tag{3.67}$$

  This is equivalent to (2.98).

  - It follows from (3.40), (3.49) and (3.55) that

$$P_i^z A_{i\kappa} + A_{i\kappa}^{\mathrm{T}} P_i^z + E_i^{z\,\mathrm{T}} \Lambda_i E_i^z < -\alpha P_i^z, \text{ if } 0 \in \overline{\mathcal{R}}_i^z \text{ and } e_i^z = 0 \tag{3.68}$$

  Therefore (2.99) is satisfied.

  - Multiplying (3.41),by $\bar{T}_{zx}^{-\mathrm{T}}$ and $\bar{T}_{zx}^{-1}$ from left and right yields

$$\begin{aligned} &\bar{T}_{zx}^{-\mathrm{T}} \bar{P}_i (\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i)\bar{T}_{zx}^{-1} + \bar{T}_{zx}^{-\mathrm{T}} (\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i)^T \bar{P}_i \bar{T}_{zx}^{-1} \\ &+ \bar{T}_{zx}^{-\mathrm{T}} \bar{E}_i^T \bar{\Lambda}_{i\kappa} \bar{E}_i \bar{T}_{zx}^{-1} < -\alpha \bar{T}_{zx}^{-\mathrm{T}} \bar{P}_i \bar{T}_{zx}^{-1} \end{aligned} \tag{3.69}$$

  Since $T_{zx}^{-1} T_{zx} = I$, the condition (3.69) can be rewritten as

$$\begin{aligned} &\bar{T}_{zx}^{-\mathrm{T}} \bar{P}_i T_{zx}^{-1} T_{zx} (\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i)\bar{T}_{zx}^{-1} + \bar{T}_{zx}^{-\mathrm{T}} (\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i)^T T_{zx}^{\mathrm{T}} T_{zx}^{-\mathrm{T}} \bar{P}_i \bar{T}_{zx}^{-1} \\ &+ \bar{T}_{zx}^{-\mathrm{T}} \bar{E}_i^T \bar{\Lambda}_{i\kappa} \bar{E}_i \bar{T}_{zx}^{-1} < -\alpha \bar{T}_{zx}^{-\mathrm{T}} \bar{P}_i \bar{T}_{zx}^{-1} \end{aligned} \tag{3.70}$$

  Note that from (3.43), it follows

$$\begin{aligned} \bar{T}_{zx}(\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i) &= \begin{bmatrix} I & -x^\star \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{i\kappa} + B_{i\kappa}K_i & a_{i\kappa} + B_{i\kappa}k_i \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A_{i\kappa} + B_{i\kappa}K_i & a_{i\kappa} + B_{i\kappa}k_i \\ 0 & 0 \end{bmatrix} \\ &= (\bar{A}_{i\kappa} + \bar{B}_{i\kappa}\bar{K}_i) \end{aligned} \tag{3.71}$$

  Now using (3.71), (3.70), (3.46), (3.49) and (3.54), we have

$$\bar{P}_i^z \bar{A}_i^z + \bar{A}_i^{z\,\mathrm{T}} \bar{P}_i^z + \bar{E}_i^{z\,\mathrm{T}} \bar{\Lambda}_i \bar{E}_i^z < -\alpha \bar{P}_i^z, \text{ if } 0 \notin \overline{\mathcal{R}}_i^z \tag{3.72}$$

  Therefore (2.100) is satisfied.

Since all the conditions of Proposition 2.5 are satisfied, it can be concluded that for any $\kappa \in \{1, \ldots, \mathcal{K}\}$, the PWA system $\dot{z} = \sigma_{\kappa}^{z}(z)$ is stable at the origin ($z = 0$) and

$$V_i^z(x) > \epsilon \|z\|^2, z \in \overline{\mathcal{R}}_i^z \tag{3.73}$$

$$\nabla V_i^z(x)^{\mathrm{T}}(A_{i\kappa}^z z + a_{i\kappa}^z) < -\alpha V_i^z(x), z \in \overline{\mathcal{R}}_i^z \tag{3.74}$$

for $i = 1, \ldots, M$.

The differential inclusion (3.44) can be described as

$$\dot{z} = f_i^z(z), \;\; z \in \mathcal{R}_i^z \tag{3.75}$$

where

$$f_i(z)^z = \sum_{\kappa=1}^{\mathcal{K}} \omega_\kappa(t)(A_{i\kappa}^z z + a_{i\kappa}^z) \tag{3.76}$$

and $\omega_\kappa(t) \geq 0$ for $\kappa = 1, \ldots, \mathcal{K}$ and all $t \geq 0$ are piecewise smooth functions such that

$$\sum_{\kappa=1}^{\mathcal{K}} \omega_\kappa(t) = 1 \tag{3.77}$$

From (3.76) and (3.74), it follows

$$\nabla V_i^z(x)^{\mathrm{T}} f_i(z) < -\alpha V_i^z(x), z \in \overline{\mathcal{R}}_i^z \tag{3.78}$$

Now using Proposition 2.3, the system (3.75) is dissipative with the storage functinon $V^z(z)$ and the supply rate $-\alpha V^z(x)$. Therefore $S(z,t) = V^z(z) + \int_0^t \alpha V^z(z(\tau))d\tau$ is nonincreasing along the trajectories of (3.75). This implies that the conditions of Theorem 2.1 are satisfied and all trajectories of (3.75) (the nonlinear closed loop system (3.26)) asymptotically converge to $z = 0$ ($x = x^\star$). $\square$

Remark 3.1. *Theorem 3.2 shows that there is no need to assume that there is one equilibrium point for the dynamic equations of each region and to select them a-priori by solving an optimization problem (such as it was done in [103]).*

Remark 3.2. *The conditions in Theorem 3.2 include bilinear matrix inequalities (BMI) which make the problem nonconvex. Reference [124] showed that the problem of checking the solvability of a BMI is $\mathcal{NP}$-hard. The complexity of the synthesis problem increases with the order of the system, the dimension of the partitioned space and the number of regions. However, PENBMI [71], a recent software package providing algorithms with local optimality guarantees, can be used in practice to search for a local solution to the problem as it will be shown in the next section.*

3.4 Numerical Examples

Example 3.1. *For the illustrative example in section 3.2, a PWA controller is designed to extend the region of convergence of the robust LQR controller. A feasible solution to the synthesis problem described in Theorem 3.2, was calculated using PENBMI [71] and Yalmip [76]. Figure 3.5 depicts the resulting piecewise quadratic Lyapunov function. The designed PWA controller (Figure 3.6) is described by the following gains.*

$$
\begin{aligned}
\bar{K}_1 &= [-4.08 \ -0.437], \ \bar{K}_2 = [-3.32 \ 1.07] \\
\bar{K}_3 &= [-1.07 \ 1.07], \ \bar{K}_4 = [2.45 \ -5.97]
\end{aligned} \tag{3.79}
$$

Note that the PWA controller coincides with the linear LQR controller in $(0,2)$. *Figure 3.7 shows the phase plane of the closed-loop system consisting of the nonlinear system in feedback connection with the PWA controller. Notice that the closed-loop system has now only one equilibrium point in* $\mathcal{X} = [-4,4]$ *and it is stable for all initial conditions in* $\mathcal{X}$.

Example 3.2. *Consider the following simple PWA system (adopted from [96] with slight modification)*

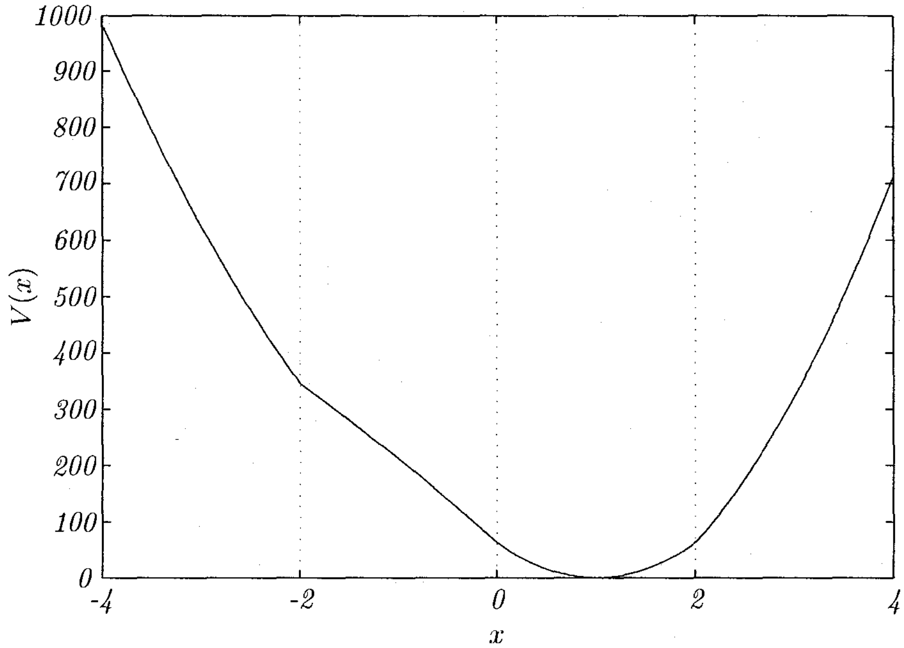

Figure 3.5: The computed Lyapunov function - Example 3.1

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -0.1x_2 + g(x_1) + u \end{aligned} \tag{3.80}$$

where $g(x_1)$ is the PWA function depicted in Figure 3.8. It is desired to stabilize the origin ($x_1 = x_2 = 0$) for this system. The local performance criterion is

$$J(x,u) = \int_0^\infty 4x_1^2(t) + 4x_2(t)^2 + u(t)^2 dt \tag{3.81}$$

At first, a PWA controller was designed by applying the synthesis method proposed by [96] using PWLTOOL [57]. Figure 3.9 shows the trajectories of the closed loop system. It can be seen that, in this case, the PWA controller designed by PWLTOOL does not stabilize the origin even locally.

We then employ Theorem 3.2 to stabilize the origin and to extend the following LQR controller with the cost function (3.81) to a PWA controller

$$u = -3.2361x_1 - 3.1376x_2 \tag{3.82}$$

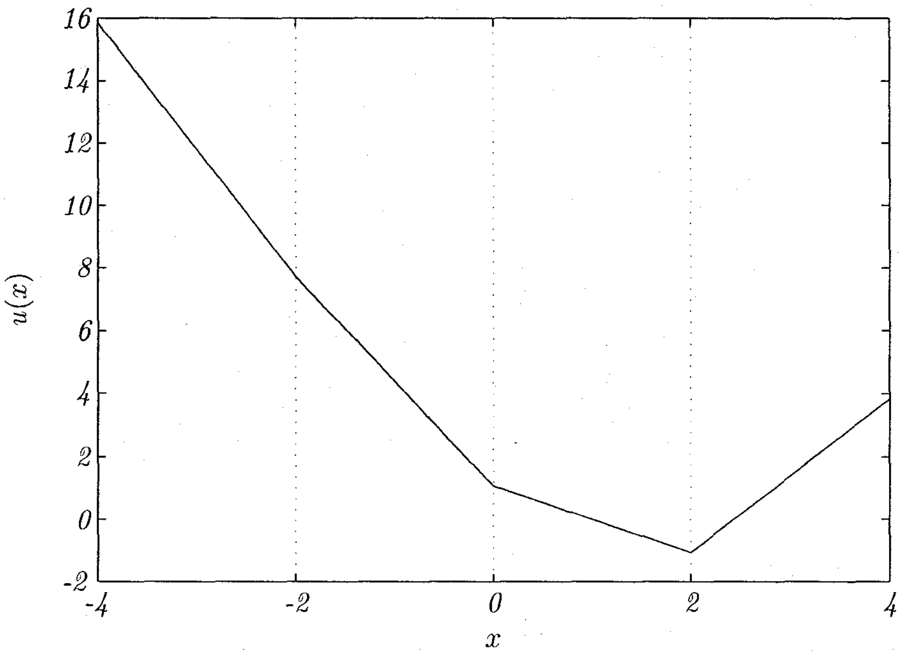

Figure 3.6: The designed PWA controller - Example 3.1

Figure 3.10 depicts the trajectories of the closed loop system. The PWA controller stabilizes the origin while it coincides with the LQR controller (3.82) for the center region ($-1 < x_1 < 1$).

Example 3.3. *Consider the following second order system*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + 0.5x_2 - 0.5x_1^2 x_2 + u \end{aligned} \tag{3.83}$$

Figure 3.11 shows the trajectories of the open loop system. A linear controller $u = -198x_1 - 101x_2$ can extend the region of convergence to the origin as depicted in Figure 3.12. However, there still exist initial conditions in

$$\mathcal{X} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| -30 < x_1 < 30,\ -60 < x_2 < 60 \right\} \tag{3.84}$$

for which the trajectories of the system do not converge to the origin.

To design a PWA controller, the nonlinear system (3.83) should first be included by a PWADI. This is done by computing upper and lower PWA bounds on

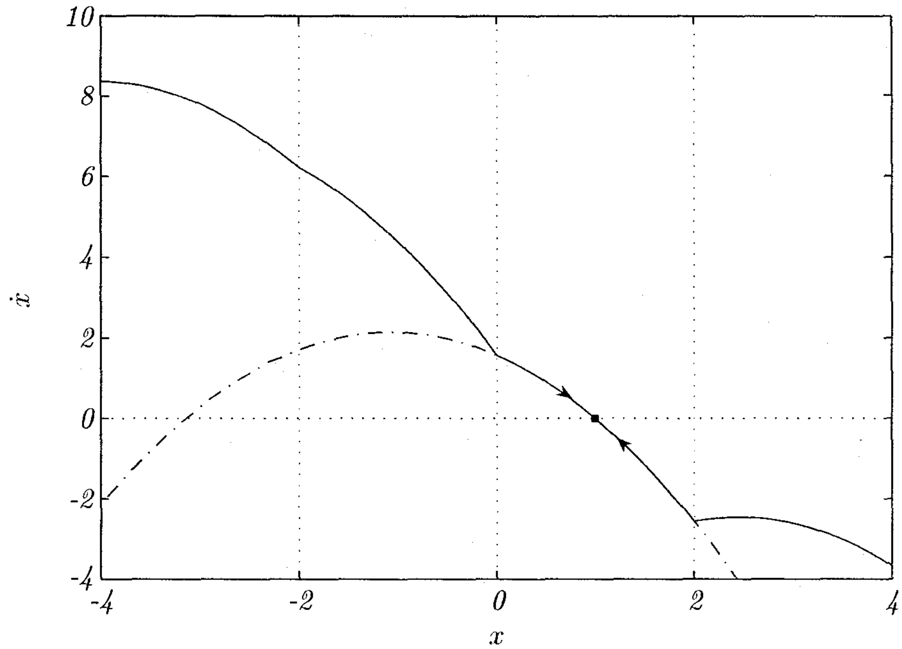

Figure 3.7: The phase plane of the closed-loop system with the PWA controller (solid) and the linear controller (dashed) - Example 3.1

the nonlinear function $h(x) = 0.5x_1^2x_2$ *and then substituting this nonlinear function in (3.83) by its PWA bounds. Figure 3.13 shows the regions (triangles) for which the PWA bounds were computed.*

A PWA controller was then designed that satisfies all the conditions of Theorem 3.2. The corresponding piecewise quadratic Lyapunov function is depicted in Figure 3.14. The trajectories in Figure 3.13 clearly show that the PWA controller enlarges the region of convergence.

3.5 Conclusions

This chapter proposed a two-step synthesis method to achieve both local performance and global stability for a class of uncertain nonlinear systems. In this method, a local robust linear controller is first designed for a neighborhood of the desired equilibrium point to satisfy a local performance requirement. The local linear controller is then extended to a PWA controller to globally stabilize the nonlinear system.

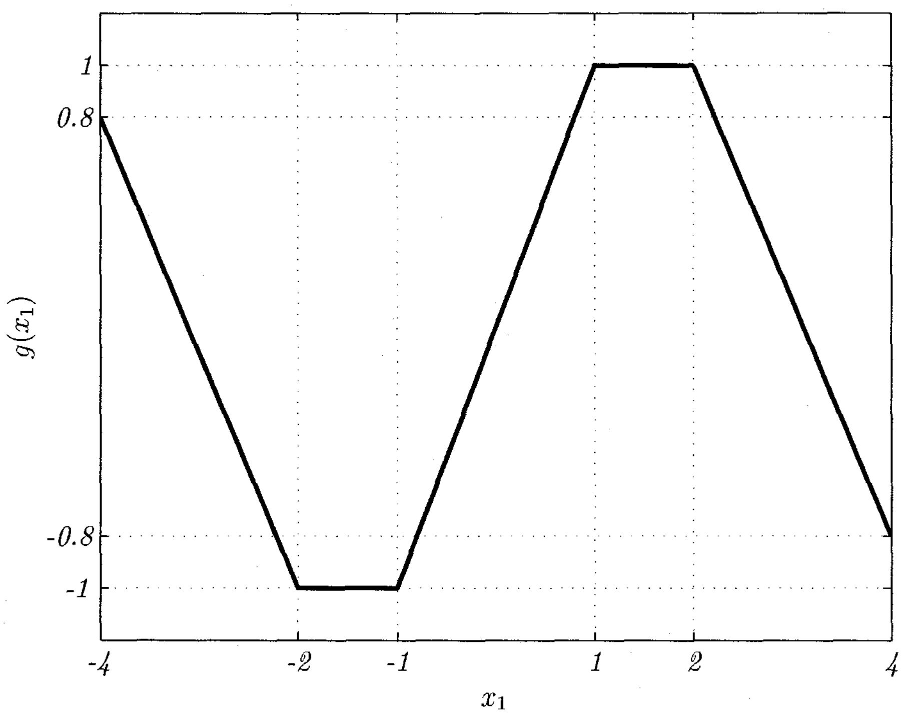

Figure 3.8: PWA function - Example 3.2

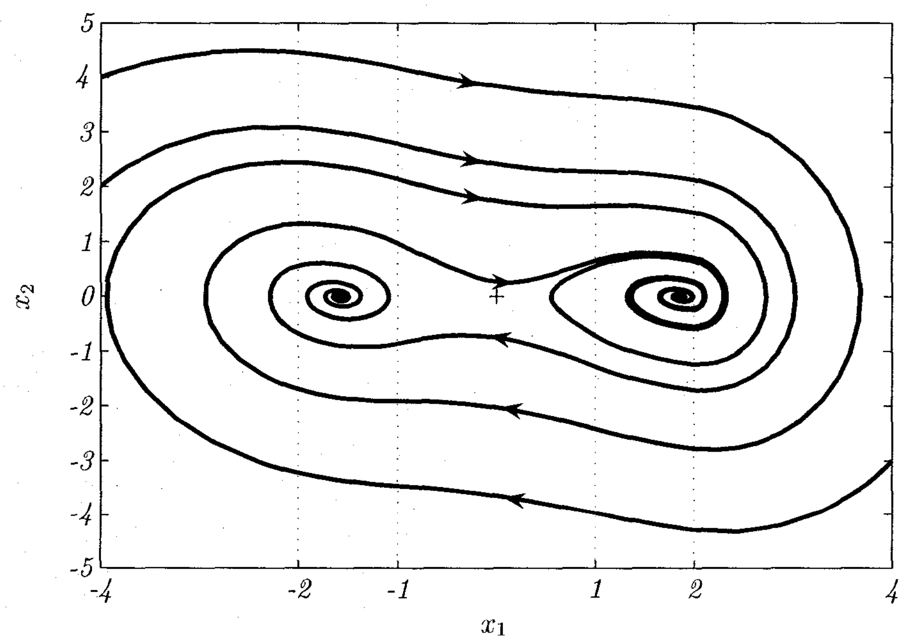

Figure 3.9: Trajectories of the closed-loop system for the PWA controller proposed in [96] - Example 3.2

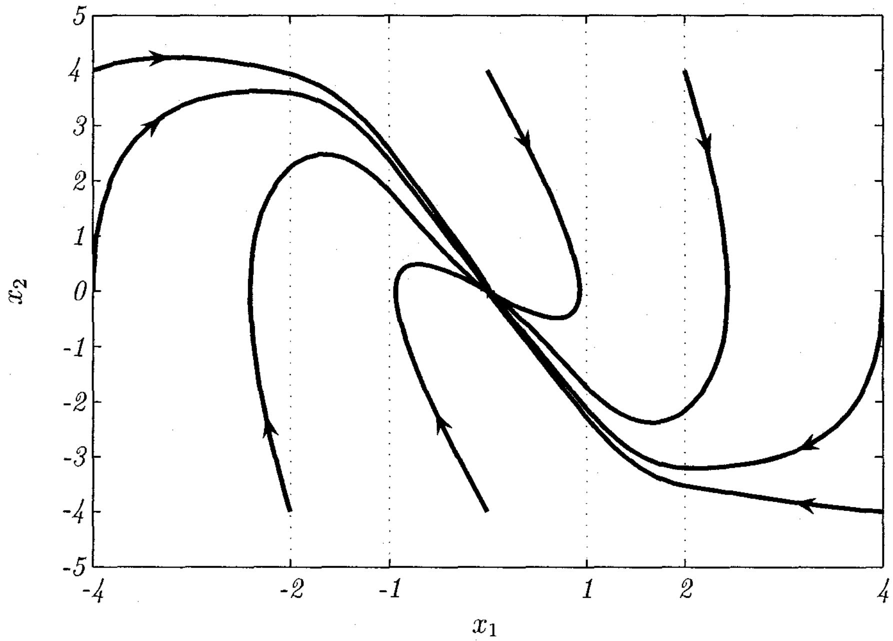

Figure 3.10: Trajectories of the closed-loop system for the proposed PWA controller - Example 3.2

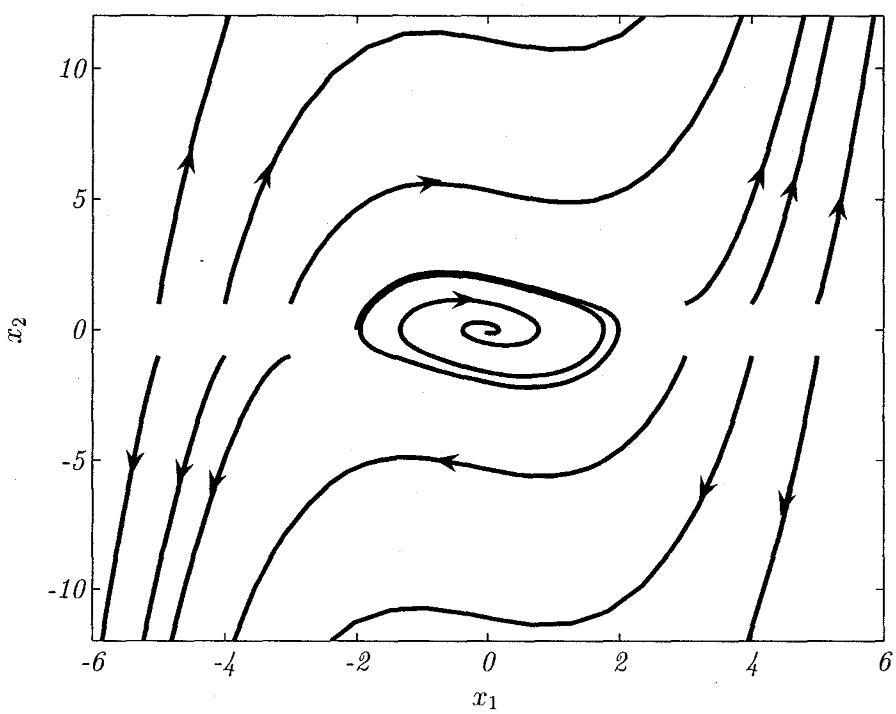

Figure 3.11: Trajectories of the open loop system - Example 3.3

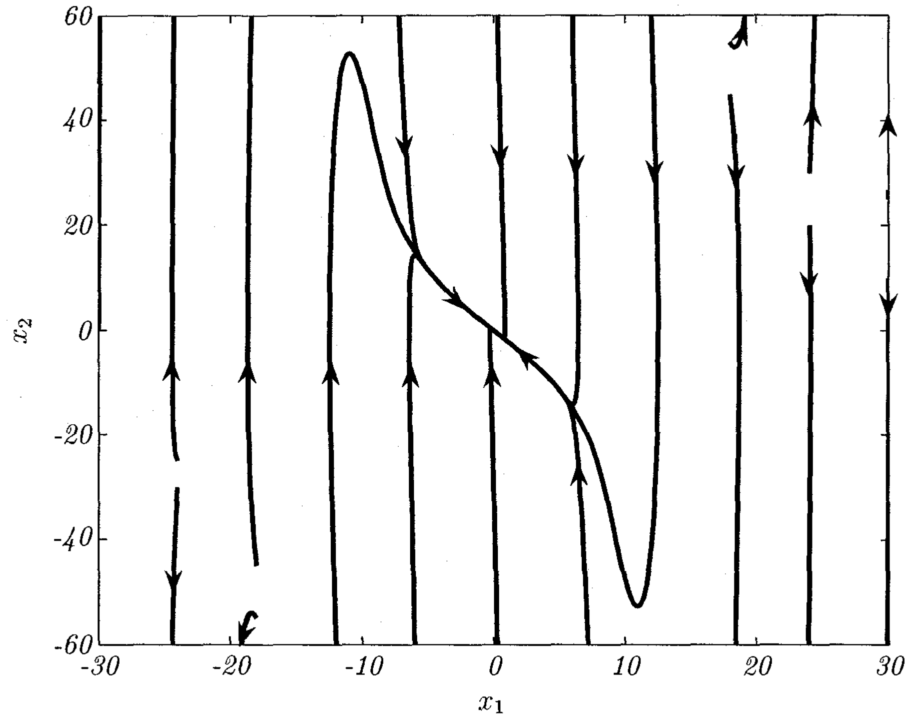

Figure 3.12: Trajectories of the closed-loop system for the linear controller - Example 3.3

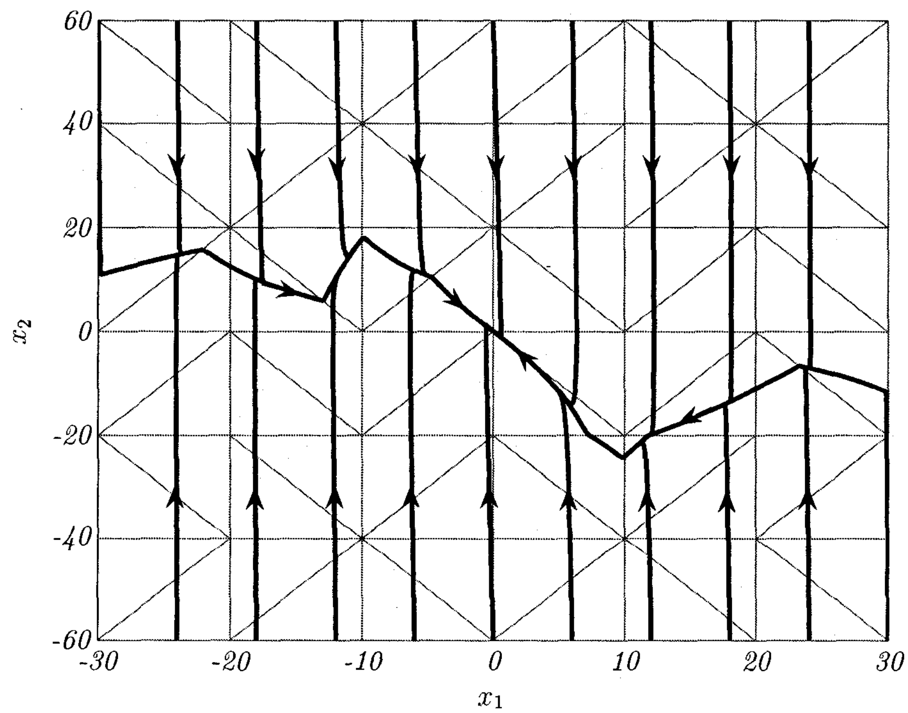

Figure 3.13: Trajectories of the closed-loop system for the PWA controller - Example 3.3

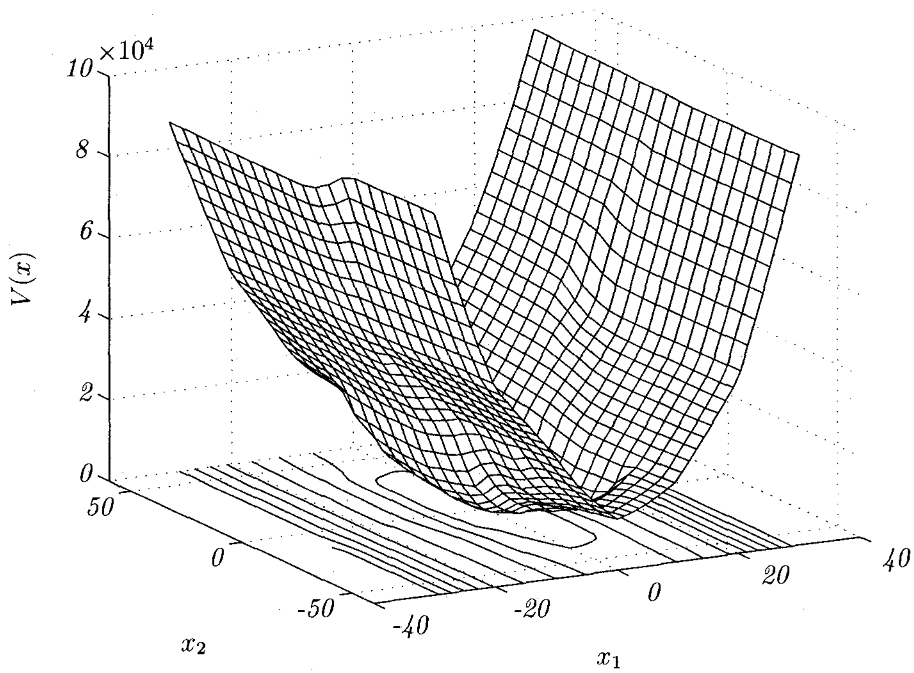

Figure 3.14: Piecewise quadratic Lyapunov function - Example 3.3

The PWA controller locally coincides with the linear controller and therefore has the same local performance.

Chapter 4

Controller synthesis for piecewise affine slab differential inclusions: a duality-based convex optimization approach

The main contribution of this chapter is to introduce for the first time a duality-based interpretation of piecewise affine (PWA) systems. This is a key concept to enable a convex formulation of PWA controller synthesis for PWA slab differential inclusions using a new convex relaxation. A convex optimization program is also proposed to compute a PWA differential inclusion that *includes* a nonlinear system for which the nonlinearity is a function of one variable. Therefore, by formulating the synthesis problem for PWA differential inclusions, the proposed method can also guarantee stability and performance for the original nonlinear system. Another important contribution of the chapter is to present stability and performance analysis and synthesis results that extend PWA L_2-gain analysis and synthesis to PWA systems whose output is also a PWA function of the state (as opposed to a piecewise-linear function). To this end, the definition of the regions of a PWA system is generalized in this chapter. These results work even when the PWA systems include sliding modes. Numerical examples illustrate the new approach.

4.1 Introduction

Controller synthesis for L_2-gain performance of PWA systems has attracted growing attention in recent years [40–42]. Reference [40] formulates the L_2-gain controller synthesis problem for uncertain PWA systems as a set of LMIs based on a piecewise quadratic (PWQ) Lyapunov function provided that the structure of the PWA controller is constrained. Reference [42] proposed a method to design PWL controllers for PWL systems based on a PWQ Lyapunov function to limit the L_2-gain of the system. The method was later extended to uncertain PWL systems in [41]. However, the approaches in [40, 42] and [41] do not use any S-procedure in the design process, which means that each closed-loop subsystem of the PWL system has to be stable and this makes the proposed methods conservative. Attractive sliding modes are also ignored in [40, 42] and [41]. There is therefore no guarantee for the closed-loop system to be stable in general. In addition, no method is proposed to obtain bounds on the uncertainty for nonlinear systems that are approximated by PWA systems.

A very important subclass of PWA systems is the class of PWA *slab systems* [101], for which the partition of the state space is a function of a scalar variable. The synthesis of PWL controllers for stability and performance of PWA slab systems is formulated in [56] as a set of LMIs. Reference [100] applied PWL L_2-gain controller synthesis to the problem of inventory control of production systems. Additional constraints were introduced in this chapter to limit the control input. However, for PWA controllers, it is said in [56] that "*It doesn't seem that* *the condition for stabilizability using this type of input command can be cast as an LMI*". Reference [101] showed that by considering an affine term in the controller, the synthesis problem for PWA slab systems can be formulated as a set of LMIs parametrized by a vector. Three different algorithms for controller synthesis have been proposed in [101] and the bisection method has been used to find the controller that maximizes the decay rate of the trajectories.

However, no convex optimization problem has been proposed for PWA controller design for stability and performance without limiting the structure of the controller. To fill this gap in the literature, this chapter formulates PWA controller synthesis for stabilization as a set of LMIs. Then, this formulation is extended to L_2-gain controller design. These results are based on a new key concept - the *dual parameter set* - that is introduced in this chapter for the first time. In addition, PWA slab differential inclusions (as opposed to equations) are considered here. This enables the design for stability and performance of nonlinear systems that can be *included* by a PWA envelope. Furthermore, for nonlinear systems for which the nonlinearity is a function of one variable, a convex optimization method is proposed in this chapter to compute the PWA envelope that includes the nonlinear system.

The structure of the chapter is as follows. The definition of L_2 gain of a nonlinear system and PWA differential inclusions are introduced in section 4.2 and section 4.3, respectively. A convex optimization method is then described in section 4.3.1 to compute a PWA envelope for a class of nonlinear systems. Stability and performance analysis are presented in section 4.4. Section 4.5 addresses stabilization and L_2-gain control design. Finally, numerical examples are shown in section 4.6 and conclusions are drawn in section 4.7.

4.2 Mathematical Preliminaries

Consider the nonlinear system

$$\begin{cases} \dot{x} = f(x) + g(x)w \\ y = h(x) \end{cases} \tag{4.1}$$

where $f(x)$ and $g(x)$ are defined *almost everywhere* and bounded for bounded $\|x\|$.

The L_2 gain from w to y is defined as

$$\sup_{0<\|w\|_2<\infty} \frac{\|y\|_2}{\|w\|_2} \tag{4.2}$$

where the L_2 norm of a signal z is defined as

$$\|z\|_2 = \left[\int_0^\infty z^{\mathrm{T}}(\tau)z(\tau)d\tau\right]^{\frac{1}{2}} \tag{4.3}$$

and the supremum is taken over all nonzero trajectories assuming $x(0) = 0$.

Lemma 4.1. *[7] The nonlinear system (4.1) has finite L_2-gain less than $\gamma > 0$ if there exists a locally bounded storage function $V : \mathbb{R}^n \to \mathbb{R}$, such that $V(x) \geq 0$ for all $x \in \mathbb{R}^n$, $V(0) = 0$ and*

$$\forall t \geq 0, V(x(t)) \leq V(x_0) + \int_0^t W(\tau)d\tau \tag{4.4}$$

for the supply rate $W(\tau) = -\|y(\tau)\|_2^2 + \gamma^2\|w(\tau)\|_2^2$.

4.3 Polytopic PWA Slab Differential Inclusions

Polytopic PWA slab differential inclusions are a generalization of polytopic linear differential inclusions in [17]. A polytopic PWA slab differential inclusion is described by

$$\begin{aligned} \dot{x} &= A(x,t)x + a(x,t) + B_u(x,t)u + B_w(x,t)w \\ y &= C(x,t)x + c(x,t) + D_u(x,t)u + D_w(x,t)w \end{aligned} \tag{4.5}$$

where $x(t) \in \mathbb{R}^n$ denotes the state, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $w(t) \in \mathbb{R}^{n_w}$ is the exogenous input and $y(t) \in \mathbb{R}^{n_y}$ is the output. The initial state is $x(0) = x_0$. It is assumed that system (4.5) satisfies

$$\begin{aligned} \dot{x} &\in \mathbf{conv}\{A_{i\kappa}x + a_{i\kappa} + B_{u_{i\kappa}}u + B_{w_{i\kappa}}w, \kappa = 1,2\} \\ y &\in \mathbf{conv}\{C_{i\kappa}x + c_{i\kappa} + D_{u_{i\kappa}}u + D_{w_{i\kappa}}w, \kappa = 1,2\} \end{aligned} \quad \text{for } (x,w) \in \mathcal{R}_i^{\mathcal{X}\times\mathcal{W}} \tag{4.6}$$

where **conv** stands for the convex hull of a set and $\mathcal{R}_i^{\mathcal{X}\times\mathcal{W}}$, $i = 1, \ldots, M$ are M slab regions partitioning the cross product of a slab subset of the state space $\mathcal{X} \subset \mathbb{R}^n$ and the space of the exogenous input $\mathcal{W}$ defined as

$$\mathcal{R}_i^{\mathcal{X}\times\mathcal{W}} = \{(x,w) \mid \sigma_i < C_{\mathcal{R}}x + D_{\mathcal{R}}w < \sigma_{i+1}\}, \tag{4.7}$$

where $C_{\mathcal{R}} \in \mathbb{R}^{1\times n}$, $D_{\mathcal{R}} \in \mathbb{R}^{1\times n_w}$ and σ_i for $i = 1, \ldots, M+1$ are scalars such that

$$\sigma_1 < \sigma_2 < \ldots < \sigma_{M+1} \tag{4.8}$$

Each slab region can be described by the following degenerate ellipsoid

$$\mathcal{R}_i^{\mathcal{X}\times\mathcal{W}} = \{(x,w) \mid \|L_i x + l_i + M_i w\| < 1\} \tag{4.9}$$

where $L_i = 2C_{\mathcal{R}}/(\sigma_{i+1} - \sigma_i)$, $l_i = -(\sigma_{i+1} + \sigma_i)/(\sigma_{i+1} - \sigma_i)$ and $M_i = 2D_{\mathcal{R}}/(\sigma_{i+1} - \sigma_i)$.

It is assumed that $a_{i\kappa} = 0$ and $c_{i\kappa} = 0$ for $i \in \mathcal{I}(0,0)$ and $\kappa = 1,2$ where

$$\mathcal{I}(x,w) = \{i \mid (x,w) \in \overline{\mathcal{R}}_i^{\mathcal{X}\times\mathcal{W}}\} \tag{4.10}$$

The following subsection will formulate the computation of PWA envelopes for a class of nonlinear systems as convex optimization programs. These envelopes will be used to design PWA controllers for nonlinear systems.

4.3.1 PWA envelope for nonlinear systems

In this subsection, a numerical method is proposed to compute a PWA envelope (bounding differential inclusion) for the following nonlinear system

$$
\begin{aligned}
\dot{x} &= Ax + B_p p + B_u u + B_w w, \ x(0) = x_0, \\
q &= C_{\mathcal{R}} x + D_{\mathcal{R}} w \\
y &= Cx + D_p p + D_u u + D_w w
\end{aligned}
\tag{4.11}
$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_u}$, $w \in \mathbb{R}^{n_w}$, $y \in \mathbb{R}^{n_y}$ and the vector $p \in \mathbb{R}^{n_p}$ is a nonlinear function of the scalar $q \in \mathbb{R}$

$$p = f(q), \tag{4.12}$$

with $f(0) = 0$. It is assumed that $f(q)$ is measured at a finite number of sampling points q_k for $k = 1, \ldots, N_s$.

The objective is to find PWA vector functions $\delta_1(q)$ and $\delta_2(q)$ defined as

$$\delta_\kappa(q) = A_{q_{i\kappa}} q + a_{q_{i\kappa}}, \ \text{for } \sigma_i < q < \sigma_{i+1}, \ \kappa = 1, 2 \tag{4.13}$$

for $i = 1, \ldots, M$ such that

$$f(q) \in \mathbf{conv}\{\delta_1(q), \delta_2(q)\} \tag{4.14}$$

and $\delta_\kappa(0) = 0, \ \kappa = 1, 2$. It is also required to make the bounding envelope as *tight* as possible. Given σ_i for $i = 1, \ldots, M+1$, the computation of the bounding envelope consisting of $\delta_1(q)$ and $\delta_2(q)$ can be formulated as the following optimization problems.

1. Optimization problem to compute $\delta_1(q)$:

$$
\begin{aligned}
\min_{A_{q_{i1}}, a_{q_{i1}}} \quad & \Sigma_{k=1,\ldots,N_s} \| f(q_k) - \delta_1(q_k)) \|^2 \\
\text{s.t.} \quad & \delta_1(q) = A_{q_{i1}} q + a_{q_{i1}}, \ \text{for } \sigma_i < q < \sigma_{i+1}, \\
& \delta_1(q_k) < f(q_k), \ \text{for } q_k > 0 \\
& \delta_1(q_k) > f(q_k), \ \text{for } q_k < 0
\end{aligned}
\tag{4.15}
$$

2. Optimization problem to compute $\delta_2(q)$:

$$\begin{aligned}
\min_{A_{q_{i2}}, a_{q_{i2}}} \quad & \Sigma_{k=1,\ldots,N_s} \|f(q_k) - \delta_2(q_k))\|^2 \\
\text{s.t.} \quad & \delta_2(q) = A_{q_{i2}} q + a_{q_{i2}}, \text{ for } \sigma_i < q < \sigma_{i+1}, \\
& \delta_2(q_k) > f(q_k), \text{ for } q_k > 0 \\
& \delta_2(q_k) < f(q_k), \text{ for } q_k < 0
\end{aligned} \tag{4.16}$$

Continuity of $\delta_1(q)$ and $\delta_2(q)$ can be written as a set of additional linear constraints on $A_{q_{i\kappa}}$ and $a_{q_{i\kappa}}$ as

$$A_{q_{i\kappa}} \sigma_{i+1} + a_{q_{i\kappa}} = A_{q_{(i+1)\kappa}} \sigma_{i+1} + a_{q_{(i+1)\kappa}}, \tag{4.17}$$

for $i = 1, \ldots, M-1$ and $\kappa = 1, 2$. The quadratic optimization problems (4.15) and (4.16) (including constraints (4.17) if needed) can be solved efficiently because they are convex. The nonlinear system (4.11) can then be embedded in the PWA differential inclusion (4.6) replacing $p = f(q)$ by the inclusion (4.14) where

$$\begin{aligned}
& A_{i\kappa} = A + B_p A_{q_{i\kappa}}, \ a_{i\kappa} = B_p a_{q_{i\kappa}}, \ B_{u_{i\kappa}} = B_u, \\
& B_{w_{i\kappa}} = B_w, \ C_{i\kappa} = C + D_p A_{q_{i\kappa}}, \ c_{i\kappa} = D_p a_{q_{i\kappa}}, \\
& D_{u_{i\kappa}} = D_u, \ D_{w_{i\kappa}} = D_w
\end{aligned}$$

The next section addresses stability and performance analysis of PWA differential inclusions.

4.4 Analysis

In this section, stability and performance analysis of PWA differential inclusions are considered. The concept of the *parameter set* of a PWA differential inclusion is also introduced. This concept will be used to derive equivalent sets of conditions for the analysis.

4.4.1 Stability analysis

In this section, the following PWA slab differential inclusion is considered

$$\dot{x} \in \mathbf{conv}\{A_{i\kappa}x + a_{i\kappa} | \kappa = 1, 2\}, \; x \in \mathcal{R}_i \tag{4.18}$$

$$\mathcal{R}_i = \{x| \; \|L_i x + l_i\| < 1\} \tag{4.19}$$

where $L_i \in \mathbb{R}^{1\times n}$, $l_i \in \mathbb{R}$, $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{M} \overline{\mathcal{R}}_i = \mathcal{X}$. It is assumed that $a_{i\kappa} = 0$ for $i \in \mathcal{I}(0)$ and $\kappa = 1, 2$.

Definition 4.1. *The* parameter set *of the differential inclusion (4.18) is defined as*

$$\Omega = \left\{ \begin{bmatrix} A_{i\kappa} & a_{i\kappa} \\ L_i & l_i \end{bmatrix} \middle| i = 1, \ldots, M, \; \kappa = 1, 2 \right\} \tag{4.20}$$

We are interested to know if all possible trajectories in $\mathcal{X}$ asymptotically converge to the origin. Note that the right-hand-side of (4.18) is not necessarily continuous and therefore there might exist attractive sliding modes. This prevents us from using standard Lyapunov theorems. The following proposition provides sufficient conditions for the stability of system (4.18) based on Theorem 2.1.

Proposition 4.1. *All trajectories of the PWA slab differential inclusion (4.18) in $\mathcal{X}$ asymptotically converge to $x = 0$ if for a given decay rate $\alpha > 0$, there exist $P = P^T \in \mathbb{R}^{n\times n}$ and $\lambda_{i\kappa} \in \mathbb{R}$ for $i = 1, \ldots, M$ and $\kappa = 1, 2$ such that*

$$P > 0, \tag{4.21}$$

$$A_{i\kappa}^T P + P A_{i\kappa} + \alpha P < 0, \; \forall i \in \mathcal{I}(0), \tag{4.22}$$

$$\begin{cases} \lambda_{i\kappa} < 0, \\ \begin{bmatrix} A_{i\kappa}^T P + P A_{i\kappa} + \alpha P + \lambda_{i\kappa} L_i^T L_i & P a_{i\kappa} + \lambda_{i\kappa} l_i L_i^T \\ a_{i\kappa}^T P + \lambda_{i\kappa} l_i L_i & \lambda_{i\kappa}(l_i^2 - 1) \end{bmatrix} < 0, \end{cases} \tag{4.23}$$

for $i \notin \mathcal{I}(0)$.

Proof. Consider the candidate Lyapunov function $V(x) = x^{\mathrm{T}}Px$ for the differential inclusion (4.18) where $P > 0$. It follows from Proposition 2.7 and the inequalities (4.19), (4.21),(4.22) and (4.23) that

$$\nabla V(x)^{\mathrm{T}}(A_{i1}x + a_{i1}) + \alpha V(x) < 0, \text{ for } x \in \overline{\mathcal{R}}_i, \tag{4.24}$$

and

$$\nabla V(x)^{\mathrm{T}}(A_{i2}x + a_{i2}) + \alpha V(x) < 0, \text{ for } x \in \overline{\mathcal{R}}_i \tag{4.25}$$

Therefore by performing a convex combination of (4.24) and (4.25)

$$\nabla V(x)^{\mathrm{T}}f + \alpha V(x) < 0, \text{ for } x \in \mathcal{R}_i, \forall f \in \mathbf{conv}\{A_{i\kappa}x + a_{i\kappa} | i \in \mathcal{I}(x), \kappa = 1, 2\} \tag{4.26}$$

Now, it follows from (4.26) and Proposition 2.2 that the differential inclusion (4.18) is dissipative with the storage functinon $V(x)$ and the supply rate $-\alpha V(x)$. Therefore the following function

$$S(x,t) = V(x) + \int_0^t \alpha V(x(\tau))d\tau, \tag{4.27}$$

is nonincreasing along the trajectories of (4.18). This implies that the conditions of Theorem 2.1 are satisfied and all trajectories of the differential inclusion (4.18) in $\mathcal{X}$ asymptotically converge to $x = 0$. □

In the following, the new concept of the *dual parameter set* of the differential inclusion (4.18) is introduced.

Definition 4.2. *The* dual parameter set *of (4.18) is defined as*

$$\Omega^T = \left\{ \begin{bmatrix} A_{i\kappa}^T & L_i^T \\ a_{i\kappa}^T & l_i \end{bmatrix} \middle| \; i = 1, \ldots, M, \;\kappa = 1, 2 \right\} \tag{4.28}$$

The importance of the dual parameter set is that if we write the LMIs in Proposition 4.1 for Ω^{T}, the resulting LMIs are stability conditions equivalent to those of Proposition 4.1. This is shown in the following.

Proposition 4.2. *All trajectories of the PWA slab differential inclusion (4.18) in $\mathcal{X}$ asymptotically converge to $x = 0$ if for a given decay rate $\alpha > 0$, there exist $Q = Q^T \in \mathbb{R}^{n \times n}$ and $\mu_{i\kappa} \in \mathbb{R}$ for $i = 1, \ldots, M$ and $\kappa = 1, 2$ such that*

$$Q > 0, \tag{4.29}$$

$$A_{i\kappa}Q + QA_{i\kappa}^T + \alpha Q < 0, \ \forall i \in \mathcal{I}(0), \ \kappa = 1, 2 \tag{4.30}$$

$$\left\{ \begin{array}{c} \mu_{i\kappa} < 0 \\ \begin{bmatrix} A_{i\kappa}Q + QA_{i\kappa}^T + \alpha Q + \mu_{i\kappa} a_{i\kappa} a_{i\kappa}^T & QL_i^T + \mu_{i\kappa} l_i a_{i\kappa} \\ L_i Q + \mu_{i\kappa} l_i a_{i\kappa}^T & \mu_{i\kappa}(l_i^2 - 1) \end{bmatrix} < 0, \end{array} \right. \tag{4.31}$$

for $i \notin \mathcal{I}(0)$ and $\kappa = 1, 2$.

Proof. The conditions of Proposition 4.1 and Proposition 4.2 will be shown to be equivalent with the change of variables $Q = P^{-1}$, $\mu_{i\kappa} = \frac{1}{\lambda_{i\kappa}}$. Multiplying (4.21) and (4.22) by Q from the left and right leads to (4.29) and (4.30) respectively. To show that (4.31) is equivalent to (4.23), we multiply the matrix inequality in (4.23) by the following matrix from both sides

$$\begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix} \tag{4.32}$$

to get the following inequality

$$\begin{bmatrix} \begin{pmatrix} QA_{i\kappa}^{\mathrm{T}} + A_{i\kappa}Q + \alpha Q \\ +\frac{1}{\mu_{i\kappa}} QL_i^{\mathrm{T}} L_i Q \end{pmatrix} & a_{i\kappa} + \frac{1}{\mu_{i\kappa}} l_i Q L_i^{\mathrm{T}} \\ a_{i\kappa}^{\mathrm{T}} + \frac{1}{\mu_{i\kappa}} l_i L_i Q & \frac{1}{\mu_{i\kappa}}(l_i^2 - 1) \end{bmatrix} < 0 \tag{4.33}$$

Using the Schur complement, inequality (4.33) is equivalent to

$$\frac{1}{\mu_{i\kappa}}(l_i^2 - 1) < 0 \tag{4.34}$$

$$QA_{i\kappa}^{\mathrm{T}} + A_{i\kappa}Q + \alpha Q + \frac{1}{\mu_{i\kappa}} QL_i^{\mathrm{T}} L_i Q - (a_{i\kappa} + \frac{1}{\mu_{i\kappa}} l_i Q L_i^{\mathrm{T}})$$
$$\mu_{i\kappa}(l_i^2 - 1)^{-1}(a_{i\kappa}^{\mathrm{T}} + \frac{1}{\mu_{i\kappa}} l_i L_i Q) < 0 \tag{4.35}$$

Replacing $(l_i^2-1)^{-1}$ by the following expression

$$(l_i^2-1)^{-1} = -1 + l_i^2(l_i^2-1)^{-1} \tag{4.36}$$

leads to the following inequality after some manipulations.

$$\begin{aligned}&A_{i\kappa}Q + QA_{i\kappa}^{\mathrm{T}} + \alpha Q + \mu_{i\kappa}a_{i\kappa}a_{i\kappa}^{\mathrm{T}} - (QL_i^{\mathrm{T}} + \mu_{i\kappa}l_i a_{i\kappa})\\ &\frac{1}{\mu_{i\kappa}}(l_i^2-1)^{-1}(L_iQ + \mu_{i\kappa}l_i a_{i\kappa}^{\mathrm{T}}) < 0\end{aligned} \tag{4.37}$$

From (4.34) and the fact that $\mu_{i\kappa} < 0$, it follows that

$$\mu_{i\kappa}(l_i^2-1) < 0 \tag{4.38}$$

Using the Schur complement once again, inequalities (4.37) and (4.38) are equivalent to (4.31). Therefore, all the constraints of Proposition 4.2 and Proposition 4.1 are equivalent and all trajectories of the PWA slab differential inclusion (4.18) in $\mathcal{X}$ asymptotically converge to $x = 0$. □

Proposition 4.2 will be used in section 4.5 to formulate the PWA controller synthesis problem as a convex problem. In the next subsection, the L_2-gain analysis of PWA differential inclusions is discussed.

4.4.2 L_2-gain analysis

In this subsection, the L_2-gain analysis of the following system is considered

$$\begin{aligned}&\dot{x} \in \mathbf{conv}\{A_{i\kappa}x + a_{i\kappa} + B_{w_{i\kappa}}w, \kappa = 1,2\},\ (x,w) \in \mathcal{R}_i^{\mathcal{X}\times\mathcal{W}}\\ &y \in \mathbf{conv}\{C_{i\kappa}x + c_{i\kappa} + D_{w_{i\kappa}}w, \kappa = 1,2\}\\ &\mathcal{R}_i^{\mathcal{X}\times\mathcal{W}} = \{(x,w)|\ \|L_ix + l_i + M_iw\| < 1\}\end{aligned} \tag{4.39}$$

It is assumed that $a_{i\kappa} = 0,\ c_{i\kappa} = 0$ for $i \in \mathcal{I}(0,0)$ and $\kappa = 1,2$. It is also assumed that the PWA function $C_{i\kappa}x + c_{i\kappa} + D_{w_{i\kappa}}w$ is continuous in x for $\kappa = 1$ and 2. Similar to the case of stability analysis, the *parameter set* of (4.39) is defined in the following.

Definition 4.3. *The* parameter set *of the differential inclusion (4.39) is defined as*

$$\Phi = \left\{ \begin{bmatrix} A_{i\kappa_1} & a_{i\kappa_1} & B_{w_{i\kappa_1}} \\ L_i & l_i & M_i \\ C_{i\kappa_2} & c_{i\kappa_2} & D_{w_{i\kappa_2}} \end{bmatrix} \middle| \; i = 1, \ldots, M, \;\; \kappa_1 = 1, 2, \;\; \kappa_2 = 1, 2 \right\} \tag{4.40}$$

The following proposition describes sufficient conditions for the differential inclusion (4.39) to have a finite L_2-gain less than γ from w to y.

Proposition 4.3. *For a given $\gamma > 0$, the PWA slab differential inclusion (4.39) has a finite L_2-gain less than $\sqrt{2}\gamma$ from w to y if there exists $P = P^T \in \mathbb{R}^{n \times n}$ such that*

1. $P > 0$

2. *for $i \in \mathcal{I}(0,0)$, $\kappa_1 = 1, 2$ and $\kappa_2 = 1, 2$,*

$$\begin{bmatrix} A_{i\kappa_1}^T P + P A_{i\kappa_1} + C_{i\kappa_2}^T C_{i\kappa_2} & * \\ B_{w_{i\kappa_1}}^T P + D_{w_{i\kappa_2}}^T C_{i\kappa_2} & -\gamma^2 I + D_{w_{i\kappa_2}}^T D_{w_{i\kappa_2}} \end{bmatrix} < 0 \tag{4.41}$$

3. *there exists $\lambda_{i\kappa_1\kappa_2} < 0$ for $i \notin \mathcal{I}(0,0)$, $\kappa_1 = 1, 2$ and $\kappa_2 = 1, 2$ such that*

$$\begin{bmatrix} \begin{pmatrix} A_{i\kappa_1}^T P + P A_{i\kappa_1} \\ + C_{i\kappa_2}^T C_{i\kappa_2} \\ + \lambda_{i\kappa_1\kappa_2} L_i^T L_i \end{pmatrix} & * & * \\ a_{i\kappa_1}^T P + c_{i\kappa_2}^T C_{i\kappa_2} + \lambda_{i\kappa_1\kappa_2} l_i L_i & \lambda_{i\kappa_1\kappa_2}(l_i^2 - 1) + c_{i\kappa_2}^T c_{i\kappa_2} & * \\ \begin{pmatrix} B_{w_{i\kappa_1}}^T P \\ + D_{w_{i\kappa_2}}^T C_{i\kappa_2} \\ + \lambda_{i\kappa_1\kappa_2} M_i^T L_i \end{pmatrix} & D_{w_{i\kappa_2}}^T c_{i\kappa_2} + \lambda_{i\kappa_1\kappa_2} l_i M_i^T & \begin{pmatrix} -\gamma^2 I \\ + D_{w_{i\kappa_2}}^T D_{w_{i\kappa_2}} \\ + \lambda_{i\kappa_1\kappa_2} M_i^T M_i \end{pmatrix} \end{bmatrix} < 0 \tag{4.42}$$

Proof. Consider the storage function $V(x) = x^{\mathrm{T}} P x$ for the differential inclusion (4.39) where $P > 0$.

1. For $i \in \mathcal{I}(0,0)$, inequality (4.41) implies

$$\nabla V(x)^{\mathrm{T}} f_1 + \|y_1\|^2 - \gamma^2 \|w\|^2 < 0 \tag{4.43}$$

$$\nabla V(x)^{\mathrm{T}} f_2 + \|y_1\|^2 - \gamma^2 \|w\|^2 < 0 \tag{4.44}$$

$$\nabla V(x)^{\mathrm{T}} f_1 + \|y_2\|^2 - \gamma^2 \|w\|^2 < 0 \tag{4.45}$$

$$\nabla V(x)^{\mathrm{T}} f_2 + \|y_2\|^2 - \gamma^2 \|w\|^2 < 0 \tag{4.46}$$

where

$$f_1 = A_{i1}x + a_{i1} + B_{w_{i1}}w \tag{4.47}$$

$$f_2 = A_{i2}x + a_{i2} + B_{w_{i2}}w \tag{4.48}$$

$$y_1 = C_{i1}x + c_{i1} + D_{w_{i1}}w \tag{4.49}$$

$$y_2 = C_{i2}x + c_{i2} + D_{w_{i2}}w \tag{4.50}$$

From (4.43) and (4.44), it follows

$$\nabla V(x)^{\mathrm{T}} f + \|y_1\|^2 - \gamma^2 \|w\|^2 < 0 \tag{4.51}$$

for any $f \in \mathbf{conv}\{A_{i\kappa}x + B_{w_{i\kappa}}w | i \in \mathcal{I}(0,0), \kappa = 1,2\}$. It also follows from (4.45) and (4.46),

$$\nabla V(x)^{\mathrm{T}} f + \|y_2\|^2 - \gamma^2 \|w\|^2 < 0 \tag{4.52}$$

Consider $y \in \mathbf{conv}\{y_1, y_2\}$ so that

$$y = \alpha y_1 + (1-\alpha) y_2 \tag{4.53}$$

where $0 \leq \alpha \leq 1$. One can write

$$\|y\|^2 \leq 2(\alpha^2 \|y_1\|^2 + (1-\alpha)^2 \|y_2\|^2) \tag{4.54}$$

From (4.51) and (4.52), one has

$$\|y\|^2 \leq 2[\alpha^2 + (1-\alpha)^2](-\nabla V(x)^{\mathrm{T}} f + \gamma^2 \|w\|^2) \tag{4.55}$$

It is easy to verify that for $0 \leq \alpha \leq 1$,

$$\alpha^2 + (1-\alpha)^2 \leq 1 \tag{4.56}$$

Thus,

$$\|y\|^2 \leq 2(-\nabla V(x)^{\mathrm{T}} f + \gamma^2 \|w\|^2) \tag{4.57}$$

or

$$2\nabla V(x)^{\mathrm{T}} f + \|y\|^2 - 2\gamma^2 \|w\|^2 \leq 0 \tag{4.58}$$

2. For $i \notin \mathcal{I}(0,0)$, it follows from the inequality (4.42) that

$$\nabla V(x)^{\mathrm{T}} f_1 + \|y_1\|^2 - \gamma^2 \|w\|^2 + \lambda_{i11}(\|L_i x + l_i + M_i w\|^2 - 1) < 0 \tag{4.59}$$

$$\nabla V(x)^{\mathrm{T}} f_2 + \|y_1\|^2 - \gamma^2 \|w\|^2 + \lambda_{i21}(\|L_i x + l_i + M_i w\|^2 - 1) < 0 \tag{4.60}$$

$$\nabla V(x)^{\mathrm{T}} f_1 + \|y_2\|^2 - \gamma^2 \|w\|^2 + \lambda_{i12}(\|L_i x + l_i + M_i w\|^2 - 1) < 0 \tag{4.61}$$

$$\nabla V(x)^{\mathrm{T}} f_2 + \|y_2\|^2 - \gamma^2 \|w\|^2 + \lambda_{i22}(\|L_i x + l_i + M_i w\|^2 - 1) < 0 \tag{4.62}$$

Since $\lambda_{i\kappa_1\kappa_2} < 0$ for $\kappa_1 = 1, 2$ and $\kappa_2 = 1, 2$, the expression (4.9) and the inequalities (4.59)-(4.62) imply that inequalities (4.43)-(4.46) are satisfied for $x \in \overline{\mathcal{R}}_i$ and $i \notin \mathcal{I}(0,0)$. Using the same arguments as above, one can conclude that the inequality (4.58) is satisfied $x \in \overline{\mathcal{R}}_i$ and $i \notin \mathcal{I}(0,0)$.

Therefore the PWA differential inclusion (4.39) is dissipative with the storage function $2V(x)$ and the supply rate $W(x,w) = 2\gamma^2\|w\|^2 - \|y\|^2$ and the L_2-gain of (4.39) from w to y is less than $\sqrt{2}\gamma$. □

The *dual parameter set* is defined in the following. Note that, in this chapter, we consider polytopic PWA slab differential inclusions. Therefore $l_i = l_i^{\mathrm{T}}$ is a scalar.

Definition 4.4. *The* dual parameter set *of (4.39) is defined as*

$$\Phi^T = \left\{ \begin{bmatrix} A_{i\kappa_1}^T & L_i^T & C_{i\kappa_2}^T \\ a_{i\kappa_1}^T & l_i & c_{i\kappa_2}^T \\ B_{w_{i\kappa_1}}^T & M_i^T & D_{w_{i\kappa_2}}^T \end{bmatrix} \middle|\, i = 1, \ldots, M,\ \kappa_1 = 1, 2,\ \kappa_2 = 1, 2 \right\} \tag{4.63}$$

The following proposition shows that the L_2-gain LMIs for Φ in Proposition 4.3 are equivalent to the L_2-gain LMIs for Φ^{T}.

Proposition 4.4. *For a given $\gamma > 0$, the PWA slab differential inclusion (4.39) has a finite L_2-gain less than $\sqrt{2}\gamma$ from w to y if there exists $Q = Q^T \in \mathbb{R}^{n\times n}$ such that*

1. $Q > 0$,
2. *for* $i \in \mathcal{I}(0,0)$, $\kappa_1 = 1, 2$ *and* $\kappa_2 = 1, 2$,

$$\begin{bmatrix} A_{i\kappa_1}Q + QA_{i\kappa_1}^T + B_{w_{i\kappa_1}}B_{w_{i\kappa_1}}^T & * \\ C_{i\kappa_2}Q + D_{w_{i\kappa_2}}B_{w_{i\kappa_1}}^T & -\gamma^2 I + D_{w_{i\kappa_2}}D_{w_{i\kappa_2}}^T \end{bmatrix} < 0 \tag{4.64}$$

3. *there exists* $\mu_{i\kappa_1\kappa_2} < 0$ *for* $i \notin \mathcal{I}(0,0)$, $\kappa_1 = 1, 2$ *and* $\kappa_2 = 1, 2$ *such that*

$$\begin{bmatrix} \begin{pmatrix} A_{i\kappa_1}Q + QA_{i\kappa_1}^T \\ +B_{w_{i\kappa_1}}B_{w_{i\kappa_1}}^T \\ +\mu_{i\kappa_1\kappa_2}a_{i\kappa_1}a_{i\kappa_1}^T \end{pmatrix} & * & * \\ L_iQ + M_iB_{w_{i\kappa_1}}^T + \mu_{i\kappa_1\kappa_2}l_i a_{i\kappa_1}^T & \mu_{i\kappa_1\kappa_2}(l_i^2 - 1) + M_iM_i^T & * \\ \begin{pmatrix} C_{i\kappa_2}Q + D_{w_{i\kappa_2}}B_{w_{i\kappa_1}}^T \\ +\mu_{i\kappa_1\kappa_2}c_{i\kappa_2}a_{i\kappa_1}^T \end{pmatrix} & D_{w_{i\kappa_2}}M_i^T + \mu_{i\kappa_1\kappa_2}l_i c_{i\kappa_2} & \begin{pmatrix} -\gamma^2 I \\ +D_{w_{i\kappa_2}}D_{w_{i\kappa_2}}^T \\ +\mu_{i\kappa_1\kappa_2}c_{i\kappa_2}c_{i\kappa_2}^T \end{pmatrix} \end{bmatrix} < 0 \tag{4.65}$$

Proof. In the following, it will be shown that the conditions of Proposition 4.4 and that of Proposition 4.3 are equivalent with the change of variables $Q = \gamma^2 P^{-1}$ and $\mu_{i\kappa_1\kappa_2} = \frac{1}{\lambda_{i\kappa_1\kappa_2}}$. Multiplying (4.41) from both sides by the following matrix

$$\begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \tag{4.66}$$

leads to the following inequality.

$$\begin{bmatrix} \gamma^2 QA_{i\kappa_1}^{\mathrm{T}} + \gamma^2 A_{i\kappa_1}Q + QC_{i\kappa_2}^T C_{i\kappa_2}Q & * \\ \gamma^2 B_{w_{i\kappa_1}}^{\mathrm{T}} + D_{w_{i\kappa_2}}^{\mathrm{T}} C_{i\kappa_2}Q & -\gamma^2 I + D_{w_{i\kappa_2}}^{\mathrm{T}} D_{w_{i\kappa_2}} \end{bmatrix} < 0 \tag{4.67}$$

It follows from applying the Schur complement that

$$-\gamma^2 I + D_{w_{i\kappa_2}}^{\mathrm{T}} D_{w_{i\kappa_2}} < 0 \tag{4.68}$$

$$\gamma^2 Q A_{i\kappa_1}^{\mathrm{T}} + \gamma^2 A_{i\kappa_1} Q + Q C_{i\kappa_2}^{T} C_{i\kappa_2} Q - (\gamma^2 B_{w_{i\kappa_1}} + Q C_{i\kappa_2}^{\mathrm{T}} D_{w_{i\kappa_2}})$$
$$(-\gamma^2 I + D_{w_{i\kappa_2}}^{\mathrm{T}} D_{w_{i\kappa_2}})^{-1} (\gamma^2 B_{w_{i\kappa_1}}^{\mathrm{T}} + D_{w_{i\kappa_2}}^{\mathrm{T}} C_{i\kappa_2} Q) < 0 \tag{4.69}$$

Using the matrix inversion lemma [68], we have

$$(-\gamma^2 I + D_{w_{i\kappa_2}}^{\mathrm{T}} D_{w_{i\kappa_2}})^{-1} = -\frac{1}{\gamma^2}(I - \frac{1}{\gamma^2} D_{w_{i\kappa_2}}^{\mathrm{T}} D_{w_{i\kappa_2}})^{-1}$$
$$= -\frac{1}{\gamma^2}\left(I + D_{w_{i\kappa_2}}^{\mathrm{T}} (\gamma^2 I - D_{w_{i\kappa_2}} D_{w_{i\kappa_2}}^{\mathrm{T}})^{-1} D_{w_{i\kappa_2}}\right) \tag{4.70}$$

Replacing (4.70) into (4.69), after some manipulations we get

$$A_{i\kappa_1} Q + Q A_{i\kappa_1}^{\mathrm{T}} + B_{w_{i\kappa_1}} B_{w_{i\kappa_1}}^{\mathrm{T}} - (Q C_{i\kappa_2}^{\mathrm{T}} + B_{w_{i\kappa_1}} D_{w_{i\kappa_2}}^{\mathrm{T}})$$
$$(-\gamma^2 I + D_{w_{i\kappa_2}} D_{w_{i\kappa_2}}^{\mathrm{T}})^{-1} (C_{i\kappa_2} Q + D_{w_{i\kappa_2}} B_{w_{i\kappa_2}}^{\mathrm{T}}) < 0 \tag{4.71}$$

From (4.68) and the fact that nonzero eigenvalues of $D_{w_{i\kappa_2}}^{\mathrm{T}} D_{w_{i\kappa_2}}$ and $D_{w_{i\kappa_2}} D_{w_{i\kappa_2}}^{\mathrm{T}}$ are equal, one can conclude

$$-\gamma^2 I + D_{w_{i\kappa_2}} D_{w_{i\kappa_2}}^{\mathrm{T}} < 0 \tag{4.72}$$

Applying the Schur complement to (4.71) and (4.72) yields inequality (4.64).

The steps of the proof for inequality (4.42) are similar. If one writes (4.42) as

$$\begin{bmatrix} \psi_{11} & * \\ \psi_{21} & \psi_{22} \end{bmatrix} < 0 \tag{4.73}$$

where

$$\psi_{11} = \begin{bmatrix} A_{i\kappa_1}^{\mathrm{T}} & 0 \\ a_{i\kappa_1}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & -\frac{\lambda_{i\kappa_1\kappa_2}}{2} \end{bmatrix} + \begin{bmatrix} P & 0 \\ 0 & -\frac{\lambda_{i\kappa_1\kappa_2}}{2} \end{bmatrix} \begin{bmatrix} A_{i\kappa_1} & a_{i\kappa_1} \\ 0 & 1 \end{bmatrix}$$
$$+ \begin{bmatrix} C_{i\kappa_2}^{\mathrm{T}} & L_i^{\mathrm{T}} \\ c_{i\kappa_2}^{\mathrm{T}} & l_i \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \lambda_{i\kappa_1\kappa_2} I \end{bmatrix} \begin{bmatrix} C_{i\kappa_2} & c_{i\kappa_2} \\ L_i & l_i \end{bmatrix} \tag{4.74}$$

$$\psi_{21} = \begin{bmatrix} B_{w_{i\kappa_1}}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & -\frac{\lambda_{i\kappa_1\kappa_2}}{2} \end{bmatrix} + \begin{bmatrix} D_{w_{i\kappa_2}}^{\mathrm{T}} & M_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \lambda_{i\kappa_1\kappa_2} I \end{bmatrix} \tag{4.75}$$

$$\psi_{22} = -\gamma^2 I + \begin{bmatrix} D_{w_{i\kappa_2}}^{\mathrm{T}} & M_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \lambda_{i\kappa_1\kappa_2} I \end{bmatrix} \begin{bmatrix} D_{w_{i\kappa_2}} \\ M_i \end{bmatrix} \tag{4.76}$$

and multiply it from both sides by the following matrix

$$\begin{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & \mu_{i\kappa_1\kappa_2} \end{bmatrix} & 0 \\ 0 & I \end{bmatrix} \tag{4.77}$$

Then, using the following matrix inversion

$$\begin{aligned} &\left(-\gamma^2 I + \begin{bmatrix} D_{w_{i\kappa_2}}^{\mathrm{T}} & M_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \lambda_{i\kappa_1\kappa_2} I \end{bmatrix} \begin{bmatrix} D_{w_{i\kappa_2}} \\ M_i \end{bmatrix}\right)^{-1} = \\ &\qquad -\frac{1}{\gamma^2}\left(I + \begin{bmatrix} D_{w_{i\kappa_2}}^{\mathrm{T}} & M_i^{\mathrm{T}} \end{bmatrix} (\gamma^2 \begin{bmatrix} I & 0 \\ 0 & \mu_{i\kappa_1\kappa_2} I \end{bmatrix}\right. \\ &\qquad\qquad \left. - \begin{bmatrix} D_{w_{i\kappa_2}} \\ M_i \end{bmatrix} \begin{bmatrix} D_{w_{i\kappa_2}}^{\mathrm{T}} & M_i^{\mathrm{T}} \end{bmatrix})^{-1} \begin{bmatrix} D_{w_{i\kappa_2}} \\ M_i \end{bmatrix}\right) \end{aligned} \tag{4.78}$$

and the Schur complement, after some manipulation, we get (4.65). Therefore all the constraints of Propositions 4.4 and 4.3 are equivalent and the PWA slab differential inclusion (4.39) has a finite L_2-gain less than γ from w to y. $\square$

Proposition 4.4 is an important result, which enables us to formulate the L_2-gain synthesis as a convex optimization problem in the next section.

4.5 Controller Synthesis

In this section, PWA controller synthesis for PWA slab differential inclusions will be formulated as a convex optimization program.

4.5.1 Stabilizability

Consider the following PWA slab differential inclusion.

$$\begin{aligned} \dot{x} &\in \mathbf{conv}\{A_{i\kappa}x + a_{i\kappa} + B_{u_{i\kappa}}u \mid \kappa = 1,2\},\ x \in \mathcal{R}_i \\ \mathcal{R}_i &= \{x \mid \|L_i x + l_i\| < 1\} \end{aligned} \tag{4.79}$$

We seek a PWA control signal of the form $u = K_i x + k_i$ for $x \in \mathcal{R}_i$ to stabilize (4.79) to the origin. It follows from Proposition 4.2 that there exists a control signal

$$u = K_i x + k_i,\ x \in \mathcal{R}_i \tag{4.80}$$

such that all trajectories of the PWA slab differential inclusion (4.79) in $\mathcal{X}$ asymptotically converge to $x = 0$ if for a given decay rate $\alpha > 0$, there exist $Q \in \mathbb{R}^{n\times n}$ and $\mu_i \in \mathbb{R}$ such that

$$Q > 0, \tag{4.81}$$

$$A_{i\kappa}Q + QA_{i\kappa}^{\mathrm{T}} + B_{u_{i\kappa}}Y_i + Y_i^{\mathrm{T}}B_{u_{i\kappa}}^{\mathrm{T}} + \alpha Q < 0, \tag{4.82}$$

for $i \in \mathcal{I}(0)$, $\kappa = 1, 2$, and

$$\mu_i < 0 \tag{4.83}$$

$$\begin{bmatrix} \begin{pmatrix} A_{i\kappa}Q + QA_{i\kappa}^{\mathrm{T}} \\ +B_{u_{i\kappa}}Y_i + Y_i^{\mathrm{T}}B_{u_{i\kappa}}^{\mathrm{T}} \\ +\alpha Q + \mu_i a_{i\kappa}a_{i\kappa}^{\mathrm{T}} \\ +a_{i\kappa}Z_i^{\mathrm{T}}B_{u_{i\kappa}}^{\mathrm{T}} + B_{u_{i\kappa}}Z_i a_{i\kappa}^{\mathrm{T}} \\ +B_{u_{i\kappa}}W_i B_{u_{i\kappa}}^{\mathrm{T}} \end{pmatrix} & * \\ \begin{pmatrix} L_i Q + \mu_i l_i a_{i\kappa}^{\mathrm{T}} \\ +l_i Z_i^{\mathrm{T}} B_{u_{i\kappa}}^{\mathrm{T}} \end{pmatrix} & \mu_{i\kappa}(l_i^2 - 1) \end{bmatrix} \leq 0, \tag{4.84}$$

for $i \notin \mathcal{I}(0)$ and $\kappa = 1, 2$ where

$$Y_i = K_i Q \tag{4.85}$$

$$Z_i = \mu_i k_i \tag{4.86}$$

$$W_i = \mu_i k_i k_i^{\mathrm{T}} \tag{4.87}$$

The main problem in designing a PWA controller using constraints (4.81)–(4.87) is the equality constraint (4.87) because it prevents the problem to be formulated as a convex optimization. We propose two methods to overcome this difficulty.

1. *Convex relaxation:* The first approach to formulate PWA controller synthesis as a convex program comes from the observation that considering $\mu_i < 0$ and (4.87), we have

$$W_i \leq 0 \tag{4.88}$$

Therefore the term $B_{u_{i\kappa}} W_i B_{u_{i\kappa}}^{\mathrm{T}}$ in (4.84) is always negative semi-definite and can be omitted to make the problem convex. This idea leads to the following proposition.

Proposition 4.5. *There exists a PWA controller of the from (4.80) such that all trajectories of the PWA slab differential inclusion (4.79) in $\mathcal{X}$ asymptotically converge to $x = 0$ if for a given decay rate $\alpha > 0$, there exist $Q = Q^T \in \mathbb{R}^{n \times n}$ and $\mu_i \in \mathbb{R}$ such that for all $i \in \mathcal{I}(0)$ and $\kappa = 1, 2$*

$$Q > 0, \tag{4.89}$$

$$A_{i\kappa} Q + Q A_{i\kappa}^T + B_{u_{i\kappa}} Y_i + Y_i^T B_{u_{i\kappa}}^T + \alpha Q < 0, \tag{4.90}$$

and for all $i \notin \mathcal{I}(0)$ and $\kappa = 1, 2$

$$\mu_i < 0 \tag{4.91}$$

$$\begin{bmatrix} \begin{pmatrix} A_{i\kappa} Q + Q A_{i\kappa}^T \\ B_{u_{i\kappa}} Y_i + Y_i^T B_{u_{i\kappa}}^T \\ + \alpha Q + \mu_i a_{i\kappa} a_{i\kappa}^T \\ + a_{i\kappa} Z_i^T B_{u_{i\kappa}}^T + B_{u_{i\kappa}} Z_i a_{i\kappa}^T \end{pmatrix} & * \\ \begin{pmatrix} L_i Q + \mu_i l_i a_{i\kappa}^T \\ + l_i Z_i^T B_{u_{i\kappa}}^T \end{pmatrix} & \mu_{i\kappa}(l_i^2 - 1) \end{bmatrix} \leq 0, \tag{4.92}$$

2. *Trace heuristic:* Equality constraint (4.87) can be written as the following rank minimization.

$$\begin{aligned} &\min \textbf{Rank } X_i \\ &\text{s.t. } X_i = \begin{bmatrix} W_i & Z_i \\ Z_i^{\mathrm{T}} & \mu_i \end{bmatrix} \leq 0 \end{aligned} \tag{4.93}$$

A well-known heuristic to solve the rank minimization problem (4.93) is to maximize the trace instead of minimizing the rank (for a negative semi-definite matrix) [39]. Using this approach, the following proposition formulates sufficient conditions for the existence of a PWA controller for differential inclusion (4.79) as a convex optimization problem.

Proposition 4.6. *There exists a PWA controller of the form (4.80) such that all trajectories of the PWA slab differential inclusion (4.79) in $\mathcal{X}$ asymptotically converge to $x = 0$ if for a given decay rate $\alpha > 0$, there exist $Q = Q^T \in \mathbb{R}^{n \times n}$ and $\mu_i \in \mathbb{R}$ such that the following optimization problem*

$$\max \sum_{i=1}^{M} \textbf{\textit{Trace}}\ X_i \tag{4.94}$$

subject to

$$X_i = \begin{bmatrix} W_i & Z_i \\ Z_i^T & \mu_i \end{bmatrix} \leq 0, \forall i \notin \mathcal{I}(0,0), \tag{4.95}$$

and (4.81)-(4.84) has a feasible solution such that

$$\mu_i W_i = Z_i Z_i^T \tag{4.96}$$

For both Propositions 4.5 and 4.6 , the PWA controller gains can be computed as

$$K_i = Y_i Q^{-1} \tag{4.97}$$

$$k_i = \begin{cases} 0 & \text{if } i \in \mathcal{I}(0) \\ \frac{1}{\mu_i} Z_i & \text{otherwise} \end{cases} \tag{4.98}$$

In the next subsection, L_2-gain PWA controller synthesis is formulated as a convex optimization problem.

4.5.2 L_2-gain synthesis

In this subsection, the objective is to design a PWA control signal of the form (4.80) to limit the L_2-gain from w to y for the following differential inclusion

$$\begin{aligned}
&\dot{x} \in \mathbf{conv}\{A_{i\kappa}x + a_{i\kappa} + B_{u_{i\kappa}}u + B_{w_{i\kappa}}w \mid \kappa = 1,2\},\\
&y \in \mathbf{conv}\{C_{i\kappa}x + c_{i\kappa} + D_{u_{i\kappa}}u + D_{w_{i\kappa}}w\},\\
&\text{for } (x,w) \in \mathcal{R}_i^{\mathcal{X}\times\mathcal{W}} = \{(x,w) \mid \ \|L_i x + l_i + M_i w\| < 1\}
\end{aligned} \tag{4.99}$$

Similar to the case of stabilizability, the same convex relaxations can be used to yield the following propositions.

Proposition 4.7. *For a given $\gamma > 0$, there exists a PWA controller of the form (4.80) such that the PWA slab differential inclusion (4.99) has a finite L_2-gain less than $\sqrt{2}\gamma$ from w to y if there exist $Q = Q^T \in \mathbb{R}^{n\times n}$ and $\mu_i \in \mathbb{R}$, such that for all $i \in \mathcal{I}(0,0)$ and $\kappa_1 = 1,2$ and $\kappa_2 = 1,2$*

$$Q > 0, \tag{4.100}$$

$$\begin{bmatrix}
\begin{pmatrix} A_{i\kappa_1}Q + B_{u_{i\kappa_1}}Y_i \\ +QA_{i\kappa_1}^T + Y_i^T B_{u_{i\kappa_1}}^T \\ +B_{w_{i\kappa_1}}B_{w_{i\kappa_1}}^T \end{pmatrix} & * \\
\begin{pmatrix} C_{i\kappa_2}Q + D_{u_{i\kappa_2}}Y_i \\ +D_{w_{i\kappa_2}}B_{w_{i\kappa_1}}^T \end{pmatrix} & -\gamma^2 I + D_{w_{i\kappa_2}}D_{w_{i\kappa_2}}^T
\end{bmatrix} < 0 \tag{4.101}$$

and for all $i \notin \mathcal{I}(0,0)$, $\mu_i < 0$, $\kappa_1 = 1,2$ and $\kappa_2 = 1,2$

$$\begin{bmatrix} \begin{pmatrix} A_{i\kappa_1}Q + B_{u_{i\kappa_1}}Y_i \\ +QA_{i\kappa_1}^T + Y_i^T B_{u_{i\kappa_1}}^T \\ +B_{w_{i\kappa_1}}B_{w_{i\kappa_1}}^T + \mu_i a_{i\kappa_1} a_{i\kappa_1}^T \\ a_{i\kappa_1} Z_i^T B_{u_{i\kappa_1}}^T + B_{u_{i\kappa_1}} Z_i a_{i\kappa_1}^T \\ +B_{u_{i\kappa_1}} W_i B_{u_{i\kappa_1}}^T \end{pmatrix} & * & * \\ \begin{pmatrix} L_i Q + M_i B_{w_{i\kappa_1}}^T \\ +\mu_i l_i a_{i\kappa}^T + l_i Z_i^T B_{u_{i\kappa}}^T \end{pmatrix} & \mu_i(l_i^2 - 1) + M_i M_i^T & * \\ \begin{pmatrix} C_{i\kappa_2}Q + D_{u_{i\kappa_2}}Y_i \\ +D_{w_{i\kappa_2}} B_{w_{i\kappa_1}}^T + \mu_i c_{i\kappa_2} a_{i\kappa_1}^T \\ c_{i\kappa_2} Z_i^T B_{u_{i\kappa_1}}^T + D_{u_{i\kappa_2}} Z_i a_{i\kappa_1}^T \end{pmatrix} & \begin{pmatrix} D_{w_{i\kappa_2}} M_i^T \\ +\mu_i l_i c_{i\kappa_2} \\ +l_i D_{u_{i\kappa_2}} Z_i \end{pmatrix} & \begin{pmatrix} -\gamma^2 I + D_{w_{i\kappa_2}} D_{w_{i\kappa_2}}^T \\ +\mu_i c_{i\kappa_2} c_{i\kappa_2}^T + c_{i\kappa_2} Z_i^T D_{u_{i\kappa_2}}^T \\ +D_{u_{i\kappa_2}} Z_i c_{i\kappa_2}^T \end{pmatrix} \end{bmatrix} < 0 \tag{4.102}$$

Proposition 4.8. *For given $\gamma > 0$, there exists a PWA controller of the form (4.80) such that the PWA slab differential inclusion (4.99) has finite L_2-gain less than $\sqrt{2}\gamma$ from w to y if there exist $Q = Q^T \in \mathbb{R}^{n \times n}$ and $\mu_i \in \mathbb{R}$ such that the following optimization problem*

$$\max \sum_{i=1}^{M} \textbf{\textit{Trace}}\ X_i \tag{4.103}$$

subject to (4.100)–(4.102) and

$$\mu_i < 0,\ X_i = \begin{bmatrix} W_i & Z_i \\ Z_i^T & \mu_i \end{bmatrix} \leq 0, \tag{4.104}$$

has a feasible solution that satisfies

$$\mu_i W_i = Z_i Z_i^T \tag{4.105}$$

Note that the PWA controller gains can be computed using (4.97) and (4.98).

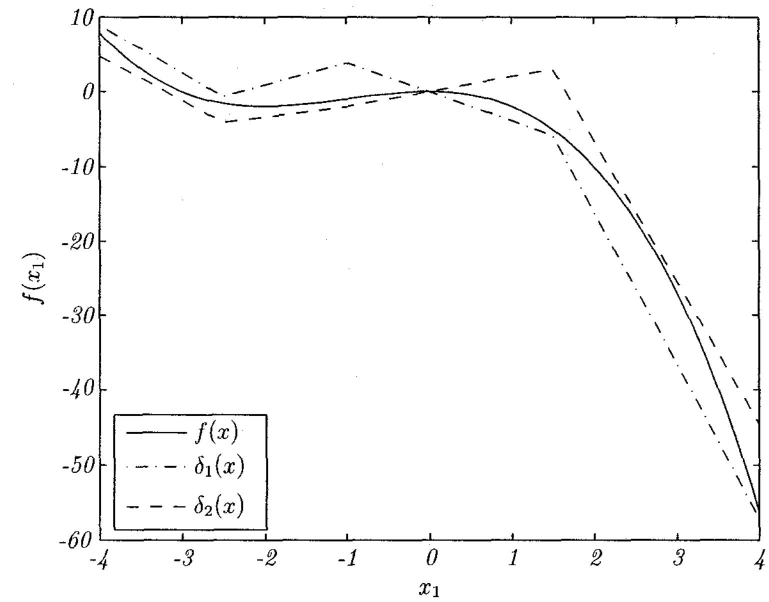

Figure 4.1: The computed bounding envelope for a scalar function

4.6 Numerical Examples

Example 4.1. *In this example, we consider the surge model of a jet engine taken from [72]. The model is described by the following state equations.*

$$\begin{cases} \dot{x}_1 = -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 \\ \dot{x}_2 = u \end{cases} \tag{4.106}$$

Using the proposed method in subsection 4.3.1, a bounding envelope is computed for the nonlinear function $f(x_1) = -\frac{3}{2}x_1^2 - \frac{1}{2}x_1^3$ *which is shown in Fig. 4.1.*

By substituting the PWA bounds in (4.106), we get a differential inclusion with

$$\mathcal{R}_1 = \begin{pmatrix} -4 & -2.5 \end{pmatrix}, \qquad \mathcal{R}_2 = \begin{pmatrix} -2.5 & -1 \end{pmatrix},$$

$$\mathcal{R}_3 = \begin{pmatrix} -1 & 1.5 \end{pmatrix}, \qquad \mathcal{R}_4 = \begin{pmatrix} 1.5 & 4.0 \end{pmatrix},$$

$$A_{11} = \begin{bmatrix} -19 & -1 \\ 0 & 0 \end{bmatrix}, \qquad A_{21} = \begin{bmatrix} 2.013 & -1 \\ 0 & 0 \end{bmatrix},$$

$$A_{31} = \begin{bmatrix} -5.938 & -1 \\ 0 & 0 \end{bmatrix}, \qquad A_{41} = \begin{bmatrix} 1.413 & -1 \\ 0 & 0 \end{bmatrix},$$

$$a_{11} = \begin{bmatrix} 31.512 \\ 0 \end{bmatrix}, \qquad a_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$a_{31} = \begin{bmatrix} -18.978 \\ 0 \end{bmatrix}, \qquad a_{41} = \begin{bmatrix} -0.601 \\ 0 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} -20.465 & -1 \\ 0 & 0 \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} -3.8912 & -1 \\ 0 & 0 \end{bmatrix},$$

$$A_{32} = \begin{bmatrix} -6.44 & -1 \\ 0 & 0 \end{bmatrix}, \qquad A_{42} = \begin{bmatrix} 3.034 & -1 \\ 0 & 0 \end{bmatrix},$$

$$a_{12} = \begin{bmatrix} 24.861 \\ 0 \end{bmatrix}, \qquad a_{22} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$a_{32} = \begin{bmatrix} -16.759 \\ 0 \end{bmatrix}, \qquad a_{42} = \begin{bmatrix} 6.925 \\ 0 \end{bmatrix}$$

The approximation error of the nonlinear function is considered as a disturbance input (w) and the objective is to limit the L_2-gain from w to x_1. Using the computed PWA approximation, the nonlinear system (4.106) can be described by the following differential inclusion

$$\begin{aligned} \dot{x} &\in \boldsymbol{conv}\{A_{i\kappa}x + a_{i\kappa} + B_u u + B_w w | \ \kappa = 1,2\}, \ x \in \mathcal{R}_i \\ y &= Cx + D_w w + D_u u \end{aligned} \tag{4.107}$$

where $i = 1, \ldots, 4$, $\kappa = 1, 2$ and

$$B_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ D_u = 0, \ D_w = 0 \tag{4.108}$$

Using Proposition 4.7, the following PWA controller was obtained for $\gamma = 0.2$ using

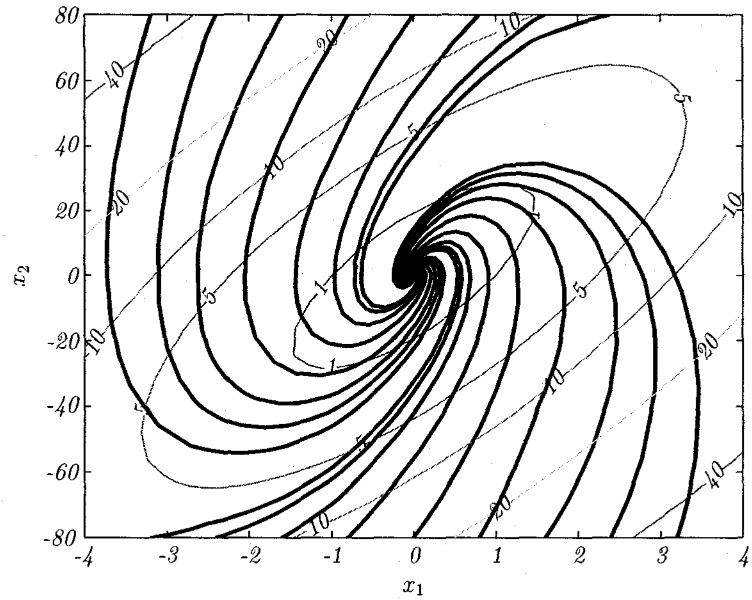

Figure 4.2: Trajectories of the closed loop nonlinear system

SeDuMi [121] and Yalmip [76].

$$\begin{aligned} K_1 &= \begin{bmatrix} 1036.3 & -40.37 \end{bmatrix}, \; k_1 = 280.40 \\ K_2 &= \begin{bmatrix} 769.20 & -30.27 \end{bmatrix}, \; k_2 = 0.03 \\ K_3 &= \begin{bmatrix} 899.15 & -34.97 \end{bmatrix}, \; k_3 = 0 \\ K_4 &= \begin{bmatrix} 597.57 & -23.57 \end{bmatrix}, \; k_4 = -17.39 \end{aligned} \tag{4.109}$$

Figure 4.2 shows the trajectories of the nonlinear system (4.106) in closed loop connection with the PWA controller. The contours of the storage function are also shown in Figure 4.2. Note that the PWA controller designed for the differential inclusion is guaranteed to stabilize the nonlinear system.

Example 4.2. *In this example, we consider a nonlinear system with a discontinuous vector field. The model is described by the following state equations.*

$$\begin{aligned} \dot{x}_1 &= f(x_1) - 2\,\mathrm{sgn}(x_1)x_2 + w \\ \dot{x}_2 &= 2|x_1| - 2x_2 + u \\ y &= x_1 + x_2 - w \end{aligned} \tag{4.110}$$

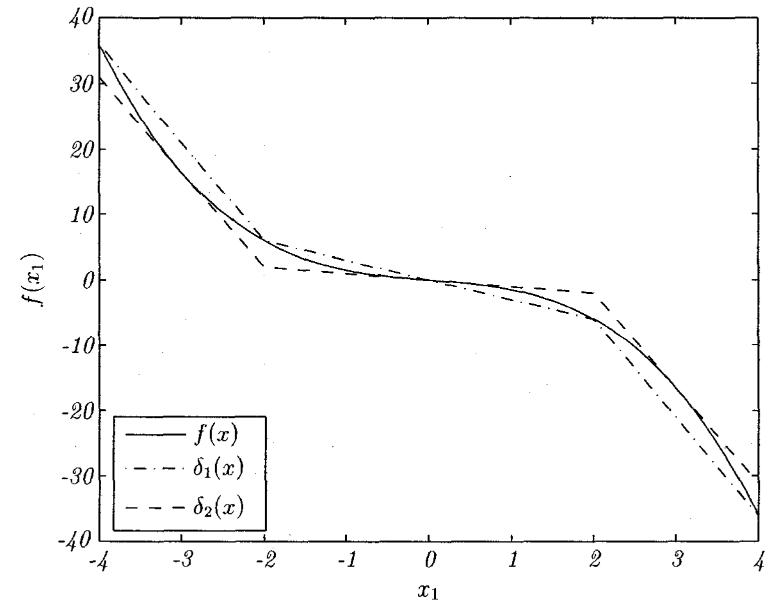

Figure 4.3: The computed bounding envelope for a scalar function

where $f(x_1) = -x_1 - 0.5x_1^3$. *Using the proposed method in [109], a bounding envelope is computed for the nonlinear function* $f(x_1)$ *which is shown in Fig. 4.3. Trajectories of the open loop system are shown in Fig. 4.4. Notice that a sliding mode exists on* $x_1 = 0$.

The objective is to design a PWA controller u *to limit the* L_2*-gain from the disturbance* w *to the output* y. *Substituting* $f(x_1)$ *by its PWA bounds in (4.110), one gets a PWA differential inclusion with*

$$\mathcal{R}_1 = \begin{pmatrix} -4 & -2 \end{pmatrix}, \qquad \mathcal{R}_2 = \begin{pmatrix} -2 & 0 \end{pmatrix},$$
$$\mathcal{R}_3 = \begin{pmatrix} 0 & 2 \end{pmatrix}, \qquad \mathcal{R}_4 = \begin{pmatrix} 2 & 4 \end{pmatrix}$$

Using the PWA approximation proposed in subsection 4.3.1, the nonlinear system (4.110) can be described by the following differential inclusion

$$\begin{aligned} \dot{x} &\in \textbf{conv}\{A_{i\kappa}x + a_{i\kappa} + B_u u + B_w w\}, \ x \in \mathcal{R}_i \\ y &= Cx + D_w w + D_u u \end{aligned} \tag{4.111}$$

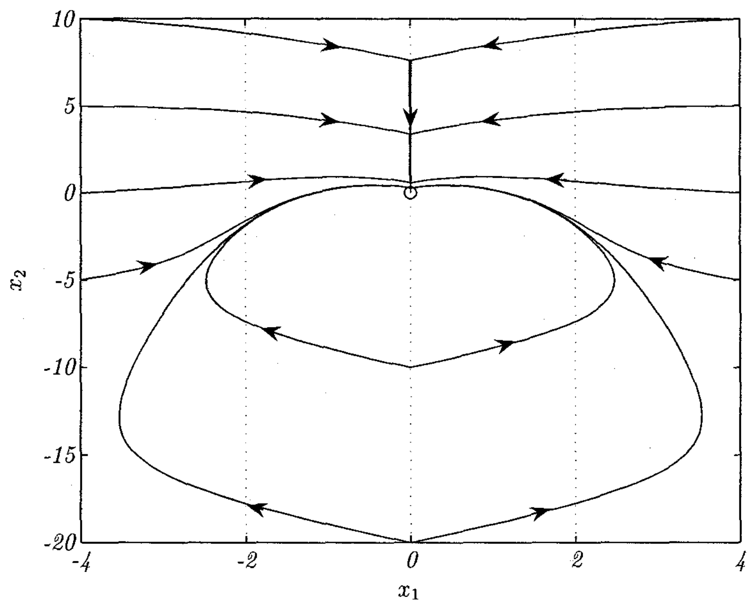

Figure 4.4: Trajectories of the closed loop nonlinear system

where $i = 1, \ldots, 4$, $\kappa = 1, 2$ *and*

$$B_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ D_u = 0, \ D_w = 0 \tag{4.112}$$

Using Proposition 4.4, one can solve a set of LMIs using SeDuMi [121] and Yalmip [76] to compute $\gamma = 1.2$ *for the open loop system. The following PWA controller can then be calculated using Proposition 4.7 to achieve* $\gamma = 1$ *for the closed loop system.*

$$\begin{aligned} K_1 &= \begin{bmatrix} -22.4985 & -83.6140 \end{bmatrix}, \ k_1 = 0.7609 \\ K_2 &= \begin{bmatrix} -31.6286 & -117.2289 \end{bmatrix}, \ k_2 = 0 \\ K_3 &= \begin{bmatrix} -25.6282 & -96.8814 \end{bmatrix}, \ k_3 = 0 \\ K_4 &= \begin{bmatrix} -35.2165 & -114.3215 \end{bmatrix}, \ k_4 = 0.0489 \end{aligned} \tag{4.113}$$

Figure 4.5 shows the trajectories of the nonlinear system (4.110) in closed loop connection with the PWA controller. Note that the PWA controller designed for the differential inclusion is guaranteed to limit the L_2 *gain of the nonlinear system.*

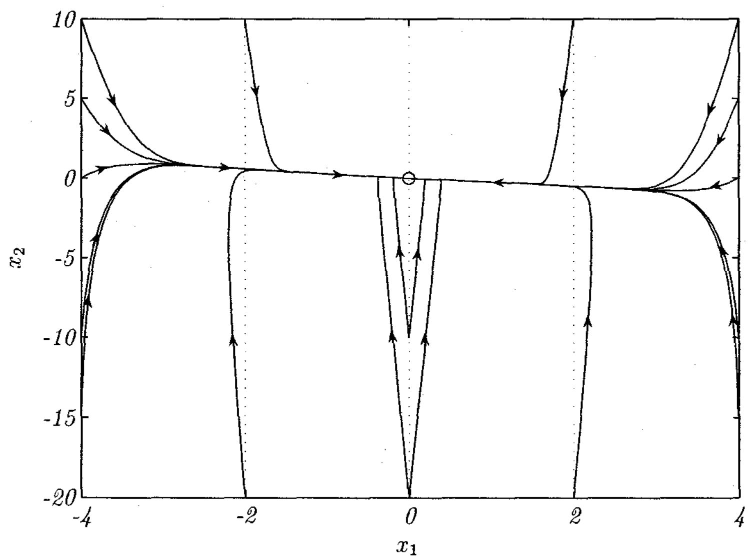

Figure 4.5: Trajectories of the closed loop nonlinear system

4.7 Conclusions

In this chapter, an interesting duality relation was revealed in the LMIs describing sufficient conditions for the stability of PWA slab differential inclusions. This concept was then employed to find the duality relation for the L_2-gain design. As a result, the definition of the regions of a PWA slab system was extended, the L_2-gain controller design was formulated as a set of LMIs and this design method was extended to PWA slab systems with an output that is also a PWA function of the state. The new method presented in the chapter enables stability and performance analysis, as well as controller synthesis, for a large class of nonlinear systems as a solution of convex optimization problems.

Chapter 5

Backstepping Controller Synthesis for Piecewise Polynomial Systems: A Sum of Squares Approach

This chapter addresses backstepping controller synthesis for piecewise polynomial (PWP) systems. The main contribution of the chapter is to formulate controller design for a large class of PWP systems as a convex problem. Integrator backstepping is proposed as the principal design step in constructing Lyapunov functions for PWP systems in *strict feedback form.* The controller synthesis problem is divided into two cases. The first case consists of the construction of a sum of squares (SOS) Lyapunov function for PWP systems with discontinuous vector fields. The second case addresses the construction of a piecewise polynomial Lyapunov function for PWP systems with continuous vector fields. After constructing a (piecewise) polynomial Lyapunov function, controller synthesis for a PWP system can be formulated as an SOS program, which is a convex optimization problem and can be solved efficiently using available software [76]. The new synthesis method is applied to two numerical examples to illustrate its effectiveness.

5.1 Introduction

PWP or spline approximation of curves and surfaces has been widely used in many different scientific contexts and engineering applications [1,34]. However, the lack of practical methods to check the sign of polynomials has prevented PWP systems to be commonly used in the field of control systems. One of the first attempts to design controllers for PWP systems was made in [87]. Paul proposed in [87] to partition the state space of a nonlinear system that is affine in the input into cells and to approximate the dynamics of the system in each cell by a model that is polynomial in the state. A controller is then designed for each cell using feedback linearization. A global controller is then formed by joining the individual cell controllers. The proposed method was employed in [87] to design controllers for a few examples of nonlinear systems. However, there is no guarantee for the stability of the closed loop system because a switched system consisting of stable subsystems can be unstable in general.

Recently, the class of discrete-time PWP systems was defined in [45] and a new method was proposed to address the constrained finite-time optimal control problem for this class of systems. This seems to be the first systematic approach to controller synthesis for discrete-time PWP systems. However, according to the authors of [45], the method suffers from excessive computational burden.

For continuous time PWP systems, a stability analysis was proposed in [93] and [85] using piecewise polynomial Lyapunov functions. The advantage of the proposed method is that the analysis problem is formulated as a sum of squares (SOS) programming which is a convex optimization problem. There exist numerical tools such as SOSTOOLS [95] to solve SOS programming problems efficiently. However, systems with infinitely fast switching or sliding modes are excluded from the discussion in [93] and [85].

The main contribution of this chapter is to propose a backstepping technique to construct control Lyapunov functions for a class of PWP systems. The proposed method formulates the control synthesis problem for PWP systems in *strict feedback form* as an SOS feasibility problem. The synthesis of PWP controllers is formulated for two cases. The first case addresses the construction of (SOS) Lyapunov functions [85] for PWP systems with discontinuous vector fields. The second case deals with the construction of piecewise polynomial Lyapunov functions for PWP systems with continuous vector fields. After constructing a (piecewise) polynomial Lyapunov function, controller synthesis for a PWP system can be formulated as an SOS program, which is a convex optimization problem and therefore can be solved efficiently.

The chapter is organized as follows. Integrator backstepping is addressed in section 5.2. Controller design for PWP systems in strict feedback form is then described in section 5.3. Finally, a numerical example is demonstrated in section 5.4.

5.2 Integrator Backstepping

Before introducing the recursive backstepping controller design, integrator backstepping is presented in this section for its simplicity. Consider the following PWP system

$$\dot{x} = f_i(x) + g_i(x)z, \;\; x \in \mathcal{P}_i \tag{5.1}$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^{n_z}$ and $\mathcal{P}_i$ for $i = 1, \ldots, M$ is defined in section 2.8. Assume that there exist a stabilizing polynomial controller $z = \gamma(x)$ and a Lyapunov function $V(x)$ which proves the stability of the closed loop system. Consider adding an integrator to this system, which yields the following PWP system

$$\begin{aligned} \dot{x} &= f_i(x) + g_i(x)z, \;\; x \in \mathcal{P}_i \\ \dot{z} &= u \end{aligned} \tag{5.2}$$

The objective is to design a controller to stabilize the augmented system (5.2). In the following, two approaches to this problem are discussed. The first approach is to construct an SOS Lyapunov function for the case of a PWP system with discontinuous vector fields. The second approach builds a piecewise polynomial Lyapunov function for the case where the vector field of the PWP system (5.2) is continuous.

5.2.1 PWP systems with discontinuous vector fields

Consider the PWP system (5.1). Assume that there exists a polynomial control $z = \gamma(x)$ where $\gamma(x) \in \mathbb{R}^{n_z}$ is a vector of polynomials so that $\gamma(0) = 0$ and $V(x)$ is an SOS Lyapunov function for the closed loop system verifying

$$\begin{cases} V(x) - \lambda(x) \text{ is SOS} \\ -\nabla V(x)^{\mathrm{T}}(f_i(x) + g_i(x)\gamma(x)) - \Gamma_i(x)^{\mathrm{T}} E_i(x) - \alpha V(x) \text{ is SOS} \end{cases} \tag{5.3}$$

for $i = 1, \ldots, M$ and any $\alpha > 0$, where $\lambda(x)$ is a positive definite SOS polynomial, $\Gamma_i(x)$ is an SOS vector function and $E_i(x)$ is defined in section 2.8. Consider now the following candidate Lyapunov function for system (5.2),

$$V_\gamma(x, z) = V(x) + \frac{1}{2}(z - \gamma(x))^{\mathrm{T}}(z - \gamma(x)) \tag{5.4}$$

Note that $V_\gamma(x, z)$ is a positive definite function. To compute a PWP controller, one can write

$$\begin{aligned} \nabla V_\gamma(x, z)^{\mathrm{T}} \begin{bmatrix} f_i(x) + g_i(x)z \\ u \end{bmatrix} =& \nabla V(x)^{\mathrm{T}}(f_i(x) + g_i(x)z) \\ &+ (z - \gamma(x))^{\mathrm{T}}[u - \frac{d\gamma(x)}{dx}(f_i(x) + g_i(x)z)] \end{aligned}$$

$$
\begin{aligned}
=&\nabla V(x)^{\mathrm{T}}(f_i(x)+g_i(x)\gamma(x))+\nabla V(x)^{\mathrm{T}}g_i(x)(z-\gamma(x))\\
&+(z-\gamma(x))^{\mathrm{T}}[u-\frac{d\gamma(x)}{dx}(f_i(x)+g_i(x)z)]\\
=&\nabla V(x)^{\mathrm{T}}(f_i(x)+g_i(x)\gamma(x))+(z-\gamma(x))^{\mathrm{T}}[u+g_i^{\mathrm{T}}(x)\nabla V(x)\\
&-\frac{d\gamma(x)}{dx}(f_i(x)+g_i(x)z)]
\end{aligned}
\tag{5.5}
$$

Using the following expression

$$
u(x,z)=-g_i^{\mathrm{T}}(x)\nabla V(x)+\frac{d\gamma(x)}{dx}(f_i(x)+g_i(x)z)-\frac{\alpha}{2}(z-\gamma(x)), \tag{5.6}
$$

and the SOS constraints (5.3) leads to the following inequality

$$
\nabla V_\gamma(x,z)^{\mathrm{T}}\begin{bmatrix} f_i(x)+g_i(x)z \\ u(x,z)\end{bmatrix}\le -\alpha V_\gamma(x,z)-\Gamma_i(x)^{\mathrm{T}}E_i(x) \tag{5.7}
$$

Therefore it follows from (2.137) that for $x\in\overline{\mathcal{P}}_i$ and $i=1,\dots,M$

$$
\nabla V_\gamma(x,z)^{\mathrm{T}}\begin{bmatrix} f_i(x)+g_i(x)z \\ u(x,z)\end{bmatrix}\le -\alpha V_\gamma(x,z) \tag{5.8}
$$

From (5.8) and based on Proposition 2.2, it follows that the PWP system (5.2) with the following controller

$$
u(x,z)=-g_i^{\mathrm{T}}(x)\nabla V(x)+\frac{d\gamma(x)}{dx}(f_i(x)+g_i(x)z)-\frac{\alpha}{2}(z-\gamma(x)),\ x\in\mathcal{P}_i \tag{5.9}
$$

is dissipative with the storage function $V_\gamma(x,z)$ and supply rate $-\alpha V_\gamma(x,z)$. Therefore it follows from Theorem 2.1 that the PWP system (5.2) with controller (5.9) is asymptotically stable.

In summary, integrator backstepping consists of two steps:

- *Lyapunov function construction*: The candidate Lyapunov function (5.4) was constructed using a known Lyapunov function $V(x)$ and polynomial controller $\gamma(x)$ for the PWP system (5.1).
- *Controller synthesis*: The control law (5.9) was designed to make the candidate Lyapunov function (5.4) decreasing with time.

5.2.2 PWP systems with continuous vector fields

Assume that the vector field of the PWP system (5.1) is continuous for $x \in \mathcal{X}$ and there exists a continuous PWP control $z = \gamma(x)$ where

$$\gamma(x) = \gamma_i(x), \ x \in \mathcal{P}_i \tag{5.10}$$

where $\gamma_i(x) \in \mathbb{R}^{n_z}$ is a vector polynomial so that the continuous piecewise polynomial

$$V(x) = V_i(x), \ x \in \mathcal{P}_i \tag{5.11}$$

where $V_i(x)$ is a polynomial function verifying the following constraints

$$\begin{cases} V_i(x) - \Lambda_i^{\mathrm{T}}(x)E_i(x) - \lambda(x) \text{ is SOS} \\ -\nabla V_i(x)^{\mathrm{T}}(f_i(x) + g_i(x)\gamma_i(x)) - \Gamma_i^{\mathrm{T}}(x)E_i(x) - \alpha V_i(x) \text{ is SOS} \end{cases} \tag{5.12}$$

for $i = 1, \ldots, M$ where α is a positive scalar, $\Lambda_i(x)$ and $\Gamma_i(x)$ are SOS vector functions and $E_i(x)$ is defined in section 2.8. In the following, the objective is to construct a PWP Lyapunov function and a PWP controller for the PWP system (5.2).

- *Lyapunov function construction*: Consider now the following candidate Lyapunov function for system (5.2)

$$V_\gamma(x, z) = V_{\gamma_i}(x, z), \ x \in \mathcal{P}_i \tag{5.13}$$

where

$$V_{\gamma_i}(x, z) = V_i(x) + \frac{1}{2}(z - \gamma_i(x))^{\mathrm{T}}(z - \gamma_i(x)) \tag{5.14}$$

Note that $V_\gamma(x, z)$ is a continuous piecewise polynomial function because $V(x)$ and $\gamma(x)$ are continuous piecewise polynomial functions. To compute a PWP controller, we write

$$
\begin{aligned}
\nabla V_{\gamma_i}(x,z)^{\mathrm{T}} \begin{bmatrix} f_i(x)+g_i(x)z \\ u \end{bmatrix} &= \nabla V_i(x)^{\mathrm{T}}(f_i(x)+g_i(x)z) \\
&+ (z-\gamma_i(x))^{\mathrm{T}}[u - \frac{d\gamma_i(x)}{dx}(f_i(x)+g_i(x)z)] \\
&= \nabla V_i(x)^{\mathrm{T}}(f_i(x)+g_i(x)\gamma_i(x)) \\
&+ (z-\gamma_i(x))^{\mathrm{T}}[u + g_i(x)^{\mathrm{T}}\nabla V_i(x) - \frac{d\gamma_i(x)}{dx}(f_i(x)+g_i(x)z)] \quad (5.15)
\end{aligned}
$$

- *Controller synthesis*: Using the following expression

$$
u(x,z) = -g_i(x)^{\mathrm{T}}\nabla V_i(x) + \frac{d\gamma_i(x)}{dx}(f_i(x)+g_i(x)z) - \frac{\alpha}{2}(z-\gamma_i(x)), \quad (5.16)
$$

  the SOS constraint (5.12) and also (2.137) leads to the following inequality

$$
\nabla V_{\gamma_i}(x,z)^{\mathrm{T}} \begin{bmatrix} f_i(x)+g_i(x)z \\ u \end{bmatrix} \le -\alpha V_{\gamma_i}(x,z) \quad (5.17)
$$

  for $x \in \overline{\mathcal{P}}_i$ and $i = 1, \ldots, M$. Therefore if the following controller

$$
u(x,z) = -g_i(x)^{\mathrm{T}}\nabla V_i(x) + \frac{d\gamma_i(x)}{dx}(f_i(x)+g_i(x)z) - \frac{\alpha}{2}(z-\gamma_i(x)), \ x \in \mathcal{P}_i \quad (5.18)
$$

  is a continuous function for $x \in \mathcal{X}$, based on Proposition 2.3, the PWP system (5.2) with controller (5.18) is dissipative with the storage function $V_\gamma(x,z)$ and supply rate $-\alpha V_\gamma(x,z)$. Therefore it follows from Theorem 2.1 that the PWP system (5.2) with controller (5.18) is asymptotically stable. However, there is no guarantee that the control input in (5.18) is continuous.

The more general case of recursive backstepping controller design for PWP systems is formulated as a set of SOS programs in the next section.

5.3 Recursive Backstepping Controller Design

In this section, a recursive PWP controller synthesis method is proposed for *strict feedback* PWP systems of the following form

$$\begin{cases} \dot{x}_1 = f_{1i_1}(x_1) + g_{1i_1}(x_1)x_2, & \text{for } x_1 \in \mathcal{P}_{1i_1} \\ \dot{x}_2 = f_{2i_2}(x_1,x_2) + g_{2i_2}(x_1,x_2)x_3, & \text{for } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{P}_{2i_2} \\ \vdots & \\ \dot{x}_k = f_{ki_k}(x_1,x_2,\ldots,x_k) + g_{ki_k}(x_1,x_2,\ldots,x_k)u, & \text{for } \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} \in \mathcal{P}_{ki_k} \end{cases} \tag{5.19}$$

where the state vector of the system (5.19) is divided into k subvectors:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} \in \mathbb{R}^n, \;\; x_j \in \mathbb{R}^{n_j}, \tag{5.20}$$

with $\sum_{j=1}^{k} n_j = n$. For each $j \in \{1,2,\ldots,k\}$, the regions $\mathcal{P}_{ji_j}$ for $i_j = 1,\ldots,M_j$ are disjoint sets defined as

$$\mathcal{P}_{ji_j} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_j \end{bmatrix} \middle| \, E_{ji_j}(x_1,\ldots,x_j) \succ 0 \right\} \tag{5.21}$$

where $E_{ji_j}(x_1,\ldots,x_j) \in \mathbb{R}^{p_j}$ is a vector polynomial function and $\succ$ denotes an elementwise inequality. For a given j, the regions $\mathcal{P}_{ji_j}$ for $i_j = 1,\ldots,M_j$ partition the projection of the state space $\mathcal{X} \subset \mathbb{R}^n$ onto the $(x_1,\ldots,x_j)$ space.

Assumption 5.1. *It is assumed that for $1 \leq j_1 < j_2$, the projection of each region $\mathcal{P}_{j_2 i_{j_2}}$ for $i_{j_2} = 1,\ldots,M_{j_2}$ on the $(x_1,\ldots,x_{j_1})$ space is a subset of only* one *of the* *regions $\mathcal{P}_{j_1 i_{j_1}}$ for $i_{j_1} = 1, \ldots, M_{j_1}$. In other words, for each $1 < j_2 \leq k$, $j_1 < j_2$ and $i_{j_2} \in \{1, \ldots, M_{j_2}\}$ there exists a unique number $i(j_1, j_2, i_{j_2})$ in $\{1, \ldots, M_{j_1}\}$ such that*

$$\begin{bmatrix} x_1 \\ \vdots \\ x_{j_2} \end{bmatrix} \in \mathcal{P}_{j_2 i_{j_2}} \Rightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_{j_1} \end{bmatrix} \in \mathcal{P}_{j_1 i(j_1, j_2, i_{j_2})} \tag{5.22}$$

Assumption 5.2. *It is also assumed that*

$$f_{j i_j^\star}(0, \ldots, 0) = 0, \ \forall i_j^\star \in \mathcal{I}_j(0, \ldots, 0) \tag{5.23}$$

where

$$\mathcal{I}_j(x_1, \ldots, x_j) := \left\{ i_j \left| \begin{bmatrix} x_1 \\ \vdots \\ x_j \end{bmatrix} \in \mathcal{P}_{j i_j} \right. \right\} \tag{5.24}$$

In what follows the stabilization problem for PWP systems in strict feedback form is discussed for two cases of PWP systems: discontinuous and continuous vector fields.

5.3.1 PWP systems with discontinuous vector fields

To design a PWP controller for (5.19), we start from the following subsystem

$$\dot{x}_1 = f_{1i_1}(x_1) + g_{1i_1}(x_1)x_2, \ \text{for } x_1 \in \mathcal{P}_{1i_1}, \tag{5.25}$$

with $i_1 = 1, \ldots, M_1$. It is assumed that there exist a polynomial Lyapunov function $V_1(x_1)$ and a polynomial controller $x_2 = \gamma_1(x_1)$ such that for $i_1 = 1, \ldots, M_1$

$$\begin{cases} \gamma_1(0) = 0 \\ V_1(0) = 0 \\ V_1(x_1) - \lambda(x_1) \text{ is SOS} \\ -\nabla V_1(x_1)^{\mathrm{T}}(f_{1i_1}(x_1) + g_{1i_1}(x_1)\gamma_1(x_1)) - \Gamma_{1i_1}(x_1)^{\mathrm{T}} E_{1i_1}(x_1) - \alpha V_1(x_1) \text{ is SOS} \\ \Gamma_{1i_1}(x_1) \in \mathbb{R}^{p_1} \text{ is SOS} \end{cases} \tag{5.26}$$

where $\alpha > 0$ and $\lambda(x_1)$ is a positive definite polynomial.

Next, a polynomial controller should be designed for the following subsystem

$$\begin{cases} \dot{x}_1 = f_{1i_1}(x_1) + g_{1i_1}(x_1)x_2, \text{ for } x_1 \in \mathcal{P}_{1i_1} \\ \dot{x}_2 = f_{2i_2}(x_1, x_2) + g_{2i_2}(x_1, x_2)x_3, \text{ for } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{P}_{2i_2} \end{cases} \tag{5.27}$$

Note that if $f_{2i_2}(x_1, x_2) = 0$ and $g_{2i_2}(x_1, x_2) = 1$, this would be an integrator back-stepping problem. The design process consists of two steps.

1. *Lyapunov function construction*: Consider the following Lyapunov function

$$V_2(x_1, x_2) = V_1(x_1) + \frac{1}{2}(x_2 - \gamma_1(x_1))^{\mathrm{T}}(x_2 - \gamma_1(x_1)) \tag{5.28}$$

2. *Controller synthesis*: The synthesis problem can be formulated as the following SOS program.

$$\begin{aligned} \text{Find} \quad & x_3 = \gamma_2(x_1, x_2), \ \Gamma_{2i_2}(x_1, x_2) \\ \text{s.t.} \quad & -\nabla_{x_1} V_2(x_1, x_2)^{\mathrm{T}}(f_{1i(1,2,i_2)}(x_1) + g_{1i(1,2,i_2)}(x_1)x_2) \\ & -\nabla_{x_2} V_2(x_1, x_2)^{\mathrm{T}}(f_{2i_2}(x_1, x_2) + g_{2i_2}(x_1, x_2)x_3) \\ & -\Gamma_{2i_2}(x_1, x_2)^{\mathrm{T}} E_{2i_2}(x_1, x_2) - \alpha V_2(x_1, x_2) \text{ is SOS,} \\ & \Gamma_{2i_2}(x_1, x_2) \text{ is SOS} \\ & \gamma_2(0, 0) = 0 \end{aligned} \tag{5.29}$$

where $i_2 = 1, \ldots, M_2$ and $\gamma_2(x_1, x_2)$ is a polynomial function of x_1 and x_2.

If this SOS program is feasible then the procedure can be repeated for the next steps by adding the dynamics of x_3 and so on.

Assume that all the SOS programs in the backstepping procedure are feasible and we reach the last step with the following candidate Lyapunov function.

$$V_k(x_1, \ldots, x_k) = V_{k-1}(x_1, \ldots, x_{k-1}) + \frac{1}{2}(x_k - \gamma_{k-1}(x_1, \ldots, x_{k-1}))^{\mathrm{T}}(x_k - \gamma_{k-1}(x_1, \ldots, x_{k-1})) \tag{5.30}$$

where $\gamma_{k-1}(x_1, \ldots, x_{k-1})$ is a polynomial function. Note that $V_k(x_1, \ldots, x_k)$ is also a polynomial function. The final controller $u = \gamma_k(x_1, \ldots, x_k)$ will not be used to construct another SOS Lyapunov function. Therefore it does not have to be continuously differentiable. One can hence search for a PWP control of the form

$$u = \gamma_{ki_k}(x_1, \ldots, x_k), \text{ for } \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \in \mathcal{P}_{ki_k} \tag{5.31}$$

for $i_k = 1, \ldots, M_k$. In case that all g_{ki_k} for $i_k = 1, \ldots, M_k$ are invertible, this step of the controller synthesis can be converted to integrator backstepping. In general, this step can be formulated as the following SOS program

$$\begin{aligned}
\text{Find} \quad & u = \gamma_{ki_k}(x_1, \ldots, x_k), \ \Gamma_{ki_k}(x_1, \ldots, x_k) \\
\text{s.t.} \quad & -\nabla_{x_1} V_k(x_1, \ldots, x_k)^{\mathrm{T}}(f_{1i(1,k,i_k)}(x_1) + g_{1i(1,k,i_k)}(x_1)x_2) \\
& -\nabla_{x_2} V_k(x_1, \ldots, x_k)^{\mathrm{T}}(f_{2i(2,k,i_k)}(x_1, x_2) + g_{2i(2,k,i_k)}(x_1, x_2)x_3) \\
& - \ldots - \nabla_{x_k} V_k(x_1, \ldots, x_k)^{\mathrm{T}}(f_{ki_k}(x_1, \ldots, x_k) + g_{ki_k}(x_1, \ldots, x_k)u) \\
& -\Gamma_{ki_k}(x_1, \ldots, x_k)^{\mathrm{T}} E_{ki_k}(x_1, \ldots, x_k) - \alpha V_k(x_1, \ldots, x_k) \text{ is SOS}, \\
& \Gamma_{ki_k}(x_1, \ldots, x_k) \text{ is SOS}
\end{aligned} \tag{5.32}$$

for $i_k = 1, \ldots, M_k$. The following theorem shows that if the SOS program (5.32) is feasible then the PWP controller (5.31) stabilizes the PWP system (5.19).

Theorem 5.1. *Let there exist polynomial functions $V_1(x_1)$ and $\gamma_1(x_1)$ satisfying (5.26) and let $V_j(x_1, \ldots, x_j)$ for $j = 2, \ldots, k$ be defined as*

$$\begin{aligned}
V_j(x_1, \ldots, x_j) \quad = \quad & V_{j-1}(x_1, \ldots, x_{j-1}) \\
& +\frac{1}{2}(x_j - \gamma_{j-1}(x_1, \ldots, x_{j-1}))^T(x_j - \gamma_{j-1}(x_1, \ldots, x_{j-1}))
\end{aligned} \tag{5.33}$$

where

$$\gamma_j(\overbrace{0, \ldots, 0}^{j \text{ arguments}}) = 0, \ j = 1, \ldots, k-1 \tag{5.34}$$

Also assume that the PWP control (5.31) satisfies the conditions of the SOS program (5.32). Then the PWP control (5.31) makes the trajectories of the PWP system (5.19) in $\mathcal{X}$ asymptotically converge to the origin.

Proof. It follows from (5.26) that $V_1(x_1) \geq \lambda(x_1)$ and since $\lambda(x_1)$ is positive definite,

$$V_1(x_1) > 0, \text{ if } x_1 \neq 0 \tag{5.35}$$

From (5.33) we have

$$\begin{aligned} V_k(x_1, \ldots, x_k) &= V_1(x_1) \\ &+ \sum_{j=2}^{k} \frac{1}{2}(x_j - \gamma_{j-1}(x_1, \ldots, x_{j-1}))^{\mathrm{T}}(x_j - \gamma_{j-1}(x_1, \ldots, x_{j-1})) \end{aligned} \tag{5.36}$$

Therefore $V_k(x_1, \ldots, x_k)$ is nonnegative. Now assume for some $x_1, x_2, \ldots$ and x_k, we have $V_k(x_1, \ldots, x_k) = 0$. It follows from (5.36) that

$$V_1(x_1) = 0 \tag{5.37}$$

and

$$x_j = \gamma_{j-1}(x_1, \ldots, x_{j-1}), \; j = 2, \ldots, k \tag{5.38}$$

Now, from (5.34) and positive definiteness of $V_1(x_1)$ it follows that $x_1 = 0, x_2 = 0, \ldots$ and $x_k = 0$. Therefore $V_k(x_1, \ldots, x_k)$ is a positive definite function.

From (5.21) and (5.32), it follows that

$$\nabla_x V_k(x_1, \ldots, x_k)^{\mathrm{T}} \dot{x} \leq -\alpha V_k(x_1, \ldots, x_k), \text{ for } x \in \mathcal{P}_{ki_k}, i_k = 1, \ldots, M_k \tag{5.39}$$

Now, from Proposition 2.2 it follows that the PWP system (5.19) is dissipative with the storage function $V_k(x_1, \ldots, x_k)$ and supply rate $-\alpha V_k(x_1, \ldots, x_k)$. From the fact that $V_k(x_1, \ldots, x_k)$ is a positive definite function and Theorem 2.1 it then follows that the trajectories of the PWP system (5.19) in $\mathcal{X}$ asymptotically converge to the origin. $\square$

5.3.2 PWP systems with continuous vector fields

In this section, it is assumed that the vector field of PWP system (5.19) is continuous for $x \in \mathcal{X}$. It is also assumed that for the following subsystem

$$\dot{x}_1 = f_{1i_1}(x_1) + g_{1i_1}(x_1)x_2, \text{ for } x_1 \in \mathcal{P}_{1i_1}, \tag{5.40}$$

with $i_1 = 1, \ldots, M_1$, there exist a continuous piecewise polynomial Lyapunov function $V_1(x_1)$ and a continuous PWP controller $x_2 = \gamma_1(x_1)$ with

$$\begin{cases} V_1(x_1) = V_{1i_1}(x_1) \\ \\ \gamma_1(x_1) = \gamma_{1i_1}(x_1) \end{cases}, \text{ for } x_1 \in \mathcal{P}_{i_1}, \tag{5.41}$$

such that $\gamma_{1i_1}(x_1)$ and $V_{1i_1}(x_1)$ are polynomials and for $i_1 = 1, \ldots, M_1$

$$\begin{cases} V_1(0) = 0 \\ \gamma_1(0) = 0 \\ V_{1i_1}(x_1) - \Lambda_{1i_1}(x_1)^{\mathrm{T}} E_{1i_1}(x_1) - \lambda(x_1) \text{ is SOS} \\ -\nabla V_{1i_1}(x_1)^{\mathrm{T}}(f_{1i_1}(x_1) + g_{1i_1}(x_1)\gamma_{1i_1}(x_1)) - \Gamma_{1i_1}(x_1)^{\mathrm{T}} E_{1i_1}(x_1) - \alpha V_{1i_1} \text{ is SOS} \\ \Lambda_{1i_1}(x_1) \text{ and } \Gamma_{1i_1}(x_1) \text{ are SOS} \end{cases} \tag{5.42}$$

where $\alpha > 0$ and $\lambda(x_1)$ is a positive definite polynomial.

Then, a PWP controller should be designed for the following subsystem

$$\begin{cases} \dot{x}_1 = f_{1i_1}(x_1) + g_{1i_1}(x_1)x_2, \text{ for } x_1 \in \mathcal{P}_{1i_1} \\ \dot{x}_2 = f_{2i_2}(x_1, x_2) + g_{2i_2}(x_1, x_2)x_3, \text{ for } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{P}_{2i_2} \end{cases} \tag{5.43}$$

Considering the following PWP Lyapunov function

$$V_2(x_1, x_2) = V_{2i_2}(x_1, x_2), \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{P}_{2i_2} \tag{5.44}$$

where

$$V_{2i_2}(x_1, x_2) = V_{1i(1,2,i_2)}(x_1) + \frac{1}{2}(x_2 - \gamma_{1i(1,2,i_2)}(x_1))^{\mathrm{T}}(x_2 - \gamma_{1i(1,2,i_2)}(x_1)) \tag{5.45}$$

the synthesis problem can be formulated as the following SOS program

$$\begin{aligned}
\text{Find} \quad & x_3 = \gamma_{2i_2}(x_1, x_2), \Gamma_{2i_2}(x_1, x_2), c_{i_{21}i_{22}}(x_1, x_2) \\
\text{s.t.} \quad & -\nabla_{x_1} V_{2i_2}(x_1, x_2)^{\mathrm{T}}(f_{1i(1,2,i_2)}(x_1) + g_{1i(1,2,i_2)}(x_1)x_2) \\
& -\nabla_{x_2} V_{2i_2}(x_1, x_2)^{\mathrm{T}}(f_{2i_2}(x_1, x_2) + g_{2i_2}(x_1, x_2)x_3) \\
& -\Gamma_{2i_2}(x_1, x_2)^{\mathrm{T}} E_{2i_2}(x_1, x_2) - \alpha V_{2i_2}(x_1, x_2) \text{ is SOS,} \\
& \Gamma_{2i_2}(x_1, x_2) \text{ is SOS} \\
& \gamma_{2i_{21}}(x_1, x_2) - \gamma_{2i_{22}}(x_1, x_2) = c_{i_{21}i_{22}}(x_1, x_2) E_{2i_{21}i_{22}}(x_1, x_2)
\end{aligned} \tag{5.46}$$

for $i_2 = 1, \ldots, M_2$ and all i_{21} and i_{22} in $\{1, \ldots, M_2\}$ such that $\mathcal{P}_{2i_{21}}$ and $\mathcal{P}_{2i_{22}}$ are neighboring cells and $E_{2i_{21}i_{22}}(x_1, x_2) = 0$ contains their boundary, i.e.

$$\overline{\mathcal{P}}_{2i_{21}} \cap \overline{\mathcal{P}}_{2i_{22}} \subset \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| E_{2i_{21}i_{22}}(x_1, x_2) = 0 \right\} \tag{5.47}$$

In addition, $\gamma_{2i_2}(x_1, x_2)$ and $c_{i_{21}i_{22}}(x_1, x_2)$ are polynomial functions.

If this SOS program is feasible then the procedure can be repeated for the next steps by adding the dynamics of x_3 and so on until x_k. If all SOS programs in the backstepping procedure are feasible, a continuous PWP controller

$$u = \gamma_{ki_k}(x_1, \ldots, x_k), \text{ for } \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \in \mathcal{P}_{ki_k} \tag{5.48}$$

can be designed using the final SOS program

$$
\begin{aligned}
\text{Find} \quad & u = \gamma_{ki_k}(x_1,\dots,x_k),\ \Gamma_{ki_k}(x_1,\dots,x_k), c_{i_{k1}i_{k2}}(x_1,\dots,x_k) \\
\text{s.t.} \quad & -\nabla_{x_1} V_{ki_k}(x_1,\dots,x_k)^{\mathrm{T}}(f_{1i(1,k,i_k)}(x_1) + g_{1i(1,k,i_k)}(x_1)x_2) \\
& -\nabla_{x_2} V_{ki_k}(x_1,\dots,x_k)^{\mathrm{T}}(f_{2i(2,k,i_k)}(x_1,x_2) + g_{2i(2,k,i_k)}(x_1,x_2)x_3) \\
& -\dots - \nabla_{x_k} V_{ki_k}(x_1,\dots,x_k)^{\mathrm{T}}(f_{ki_k}(x_1,\dots,x_k) + g_{ki_k}(x_1,\dots,x_k)u) \\
& -\Gamma_{ki_k}(x_1,\dots,x_k)^{\mathrm{T}} E_{ki_k}(x_1,\dots,x_k) - \alpha V_{ki_k}(x_1,\dots,x_k) \text{ is SOS,} \\
& \Gamma_{ki_k}(x_1,\dots,x_k) \text{ is SOS} \\
& f_{ki_{k1}}(x_1,x_2,\dots,x_k) + g_{ki_{k1}}(x_1,x_2,\dots,x_k)\gamma_{ki_{k1}}(x_1,\dots,x_k) \\
& -f_{ki_{k2}}(x_1,x_2,\dots,x_k) + g_{ki_{k2}}(x_1,x_2,\dots,x_k)\gamma_{ki_{k2}}(x_1,\dots,x_k) \\
& = c_{i_{k1}i_{k2}}(x_1,\dots,x_k) E_{ki_{k1}i_{k2}}(x_1,\dots,x_k)
\end{aligned} \tag{5.49}
$$

for $i_k = 1,\dots,M_k$ and all i_{k1} and i_{k2} in $\{1,\dots,M_k\}$ such that $\mathcal{P}_{ki_{k1}}$ and $\mathcal{P}_{ki_{k2}}$ are neighboring cells where $\gamma_{ki_k}(x_1,\dots,x_k)$ and $c_{i_{k1}i_{k2}}(x_1,\dots,x_k)$ are polynomial functions.

Remark 5.1. *The equality constraint in (5.49) is equivalent to the continuity of the vector fields of the closed loop system. Therefore, if the vector field of the open loop system is discontinuous, the controller should be discontinuous to make the resulting vector field continuous. Note that the final controller will not be used to construct a Lyapunov function.*

Theorem 5.2. *Let there exist a PWP function $V_1(x_1)$ satisfying (5.42) and $V_j(x_1,\dots,x_j)$ for $j = 2,\dots,k$ is defined as*

$$
\begin{aligned}
V_j(x_1,\dots,x_j) &= V_{j-1}(x_1,\dots,x_{j-1}) \\
&\quad + \frac{1}{2}(x_j - \gamma_{j-1}(x_1,\dots,x_{j-1}))^T(x_j - \gamma_{j-1}(x_1,\dots,x_{j-1}))
\end{aligned} \tag{5.50}
$$

where

$$\gamma_j(\overbrace{0,\ldots,0}^{j\ arguments}) = 0,\ j = 1,\ldots,k-1 \tag{5.51}$$

Also assume that the PWP control (5.48) satisfies the conditions of the SOS program (5.49). Then the PWP control (5.48) makes the trajectories of the PWP system (5.19) in $\mathcal{X}$ asymptotically converge to the origin.

Proof. It follows from (5.42) and (5.21) that $V_1(x_1) \geq \lambda(x_1)$ and since $\lambda(x_1)$ is positive definite,

$$V_1(x_1) > 0,\ \text{if } x_1 \neq 0 \tag{5.52}$$

From (5.50) we have

$$\begin{aligned} V_k(x_1,\ldots,x_k) &= V_1(x_1) \\ &+ \sum_{j=2}^{k} \frac{1}{2}(x_j - \gamma_{j-1}(x_1,\ldots,x_{j-1}))^{\mathrm{T}}(x_j - \gamma_{j-1}(x_1,\ldots,x_{j-1})) \end{aligned} \tag{5.53}$$

Therefore $V_k(x_1,\ldots,x_k) \geq 0$. Now assume for some $x_1, x_2, \ldots$ and x_k, we have $V_k(x_1,\ldots,x_k) = 0$. It follows from (5.53) that

$$V_1(x_1) = 0 \tag{5.54}$$

and

$$x_j = \gamma_{j-1}(x_1,\ldots,x_{j-1}),\ j = 2,\ldots,k \tag{5.55}$$

Now, from (5.51) and positive definiteness of $V_1(x_1)$ it follows that $x_1 = 0, x_2 = 0, \ldots$ and $x_k = 0$. Therefore $V_k(x_1,\ldots,x_k)$ is a positive definite function.

From (5.21) and (5.49), it follows that

$$\nabla_x V_k(x_1,\ldots,x_k)^{\mathrm{T}}\dot{x} \leq -\alpha V_k(x_1,\ldots,x_k),\ \text{for } x \in \mathcal{P}_{ki_k}, i_k = 1,\ldots,M_k \tag{5.56}$$

Now, from Proposition 2.3 it follows that the PWP system (5.19) is dissipative with the storage function $V_k(x_1,\ldots,x_k)$ and supply rate $-\alpha V_k(x_1,\ldots,x_k)$. From the fact that $V_k(x_1, \ldots, x_k)$ is a positive definite function and Theorem 2.1 it then follows that the trajectories of the PWP system (5.19) in $\mathcal{X}$ asymptotically converge to the origin. $\square$

5.4 Numerical Examples

Example 5.1. *Consider the following PWA system:*

$$\begin{aligned} \dot{x}_1 &= \begin{cases} -0.25x_1 + 0.05x_2 & \text{if } x_1 < 0.2 \\ 0.1x_1 + 0.05x_2 - 0.07 & \text{if } 0.2 < x_1 < 0.6 \\ -0.2x_1 + 0.05x_2 + 0.11 & \text{if } x_1 > 0.6 \end{cases} \\ \dot{x}_2 &= -20x_1 - 30x_2 + 24 + 20u \end{aligned} \tag{5.57}$$

The objective is to stabilize the system to $x_{cl} = \begin{bmatrix} 0.6429 & 0.3714 \end{bmatrix}^T$. *Consider* $V_1(x_1) = \frac{1}{2}(x_1 - 0.6429)^2$ *and the following system*

$$\dot{x}_1 = \begin{cases} -0.25x_1 + 0.05x_2 & \text{if } x_1 < 0.2 \\ 0.1x_1 + 0.05x_2 - 0.07 & \text{if } 0.2 < x_1 < 0.6 \\ -0.2x_1 + 0.05x_2 + 0.11 & \text{if } x_1 > 0.6 \end{cases} \tag{5.58}$$

The following expression for x_2 *can stabilize this system to* $x_1 = 0.6429$

$$x_2 = \gamma(x_1) = 0.3714 - 4.8344(x_1 - 0.6429) \tag{5.59}$$

Considering the following candidate Lyapunov function

$$V_2(x_1, x_2) = \frac{1}{2}(x_1 - 0.6429)^2 + \frac{1}{2}(x_2 - 0.3714 + 4.8344(x_1 - 0.6429))^2, \tag{5.60}$$

the following PWA control input can be computed for the whole PWA system using the method presented in subsection 5.3.2.

$$u = \begin{cases} -0.35009 - 0.1216x_1 + 1.2572x_2, & x_1 < 0.2 \\ -0.34175 - 0.20165x_1 + 1.2603x_2, & 0.2 < x_1 < 0.6 \\ -0.3784 - 0.13739x_1 + 1.2567x_2, & x_1 > 0.6 \end{cases} \tag{5.61}$$

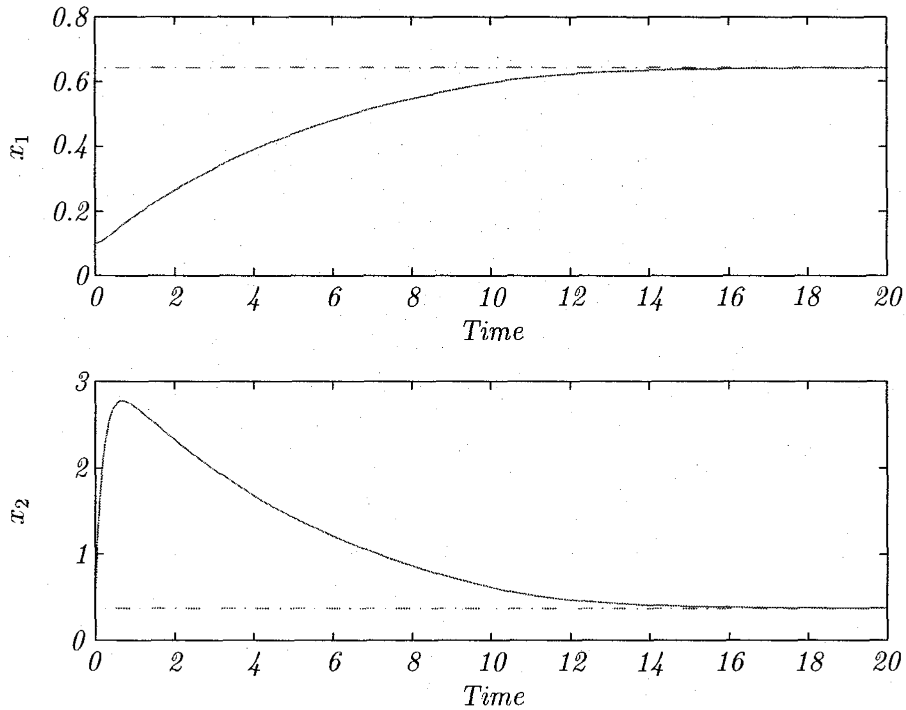

Figure 5.1: The trajectory of the closed-loop PWA system for $x(0) = [0.1\ 0.5]^T$

The trajectory of the closed-loop PWA system for $x(0) = [0.1\ 0.5]^T$ is shown in Fig. 5.1.

Example 5.2. *Consider the single-link flexible-joint robot in Fig. 5.2. The dynamic equations of the robot are given by [115]*

$$\dot{x}_1 = x_2 \tag{5.62}$$

$$\dot{x}_2 = -\frac{MgL}{I}\sin(x_1) - \frac{K}{I}(x_1 - x_3) \tag{5.63}$$

$$\dot{x}_3 = x_4 \tag{5.64}$$

$$\dot{x}_4 = -\frac{T_f}{J} + \frac{K}{J}(x_1 - x_3) + \frac{1}{J}u \tag{5.65}$$

where $x_1 = \theta_1$, $x_2 = \dot{\theta}_1$, $x_3 = \theta_2$ and $x_4 = \dot{\theta}_2$. u is the motor torque and $T_f = f_2(x_4)$ denotes the motor friction which is described by [119]

$$T_f = b_m x_4 + \operatorname{sgn}(x_4)\left(F_{cm} + (F_{sm} - F_{cm})\exp(-\frac{x_4^2}{c_m^2})\right) \tag{5.66}$$

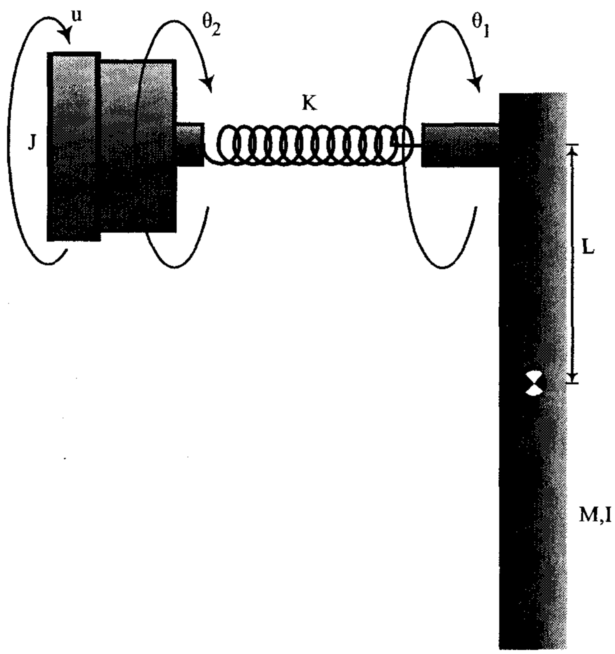

Figure 5.2: Single-link flexible-joint robot

The numerical values of the parameters are given as follows

$$M = 0.25kg \qquad L = 1m \qquad I = \frac{ML^2}{3}$$

$$K = 7.47Nm/rad \qquad J = 0.216kgm^2 \qquad g = 9.8m/s^2$$

$$c_m = 1.2rad/sec \qquad F_{cm} = 1.2Nm \qquad F_{sm} = 1.75Nm$$

$$b_m = 0.17Nm/(rad/sec)$$

Fig. 5.3 depicts the state response of the open loop nonlinear model of the robot with the initial condition $x_0 = [\pi \ 0 \ 0.8\pi \ 0]^T$. *It can be seen that the system converges to a limit cycle. The limit cycles due to friction forces are investigated in [119]. In this example, the objective is to stabilize the nonlinear model at the origin.*

To build a PWP model, there are two nonlinear functions that should be approximated by PWP curves. The function $f_1(x_1) = \sin(x_1)$ *is approximated by the*

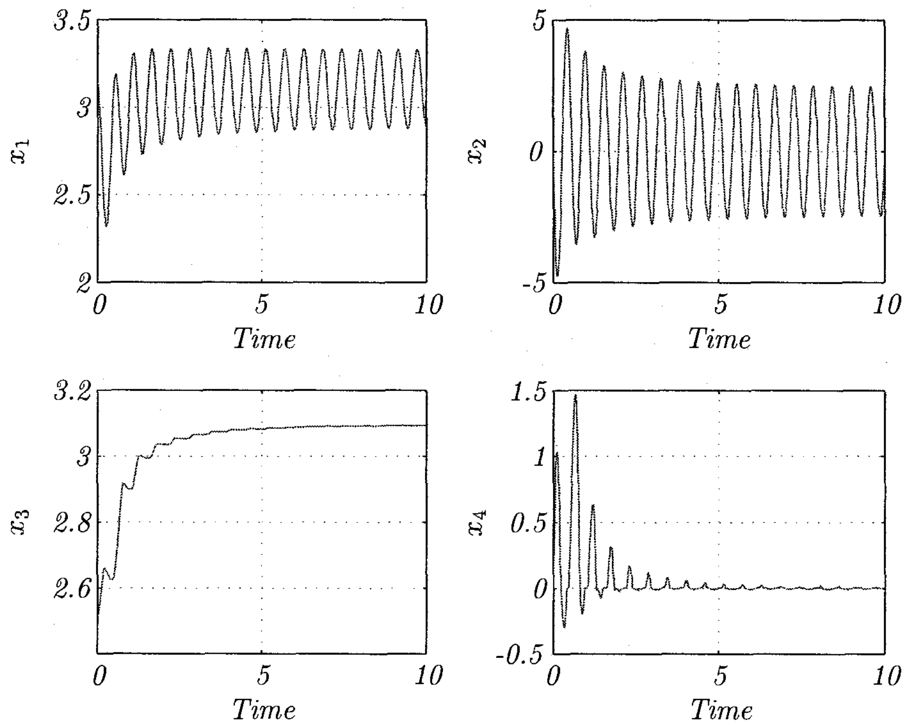

Figure 5.3: State variables of the nonlinear model - open loop

following function for $x_1 \in [-\pi, \pi]$

$$\hat{f}_1(x_1) = \begin{cases} 0.4031x_1^2 + 1.2464x_1 - 0.0211 & -\pi \leq x_1 \leq -\frac{2\pi}{7} \\ 0.908x_1 & -\frac{2\pi}{7} \leq x_1 \leq \frac{2\pi}{7} \\ -0.4031x_1^2 + 1.2464x_1 + 0.0211 & \frac{2\pi}{7} \leq x_1 \leq \pi \end{cases} \tag{5.67}$$

The nonlinear function $T_f = f_2(x_4)$ *in (5.66) is approximated by the following PWP function for* $x_4 \in [-8, 8]$

$$\hat{f}_2(x_4) = \begin{cases} -0.0057x_4^3 + 0.0873x_4^2 - 0.2472x_4 + 1.8056 & x_4 > 0 \\ -0.0057x_4^3 - 0.0873x_4^2 - 0.2472x_4 - 1.8056 & x_4 < 0 \end{cases} \tag{5.68}$$

Next, the PWP approximation of the nonlinear model (5.62)-(5.65) can be written in the strict feedback form (5.19). To start the controller synthesis procedure in subsection 5.3.2, we first consider the following system

$$\dot{x}_1 = x_2 \tag{5.69}$$

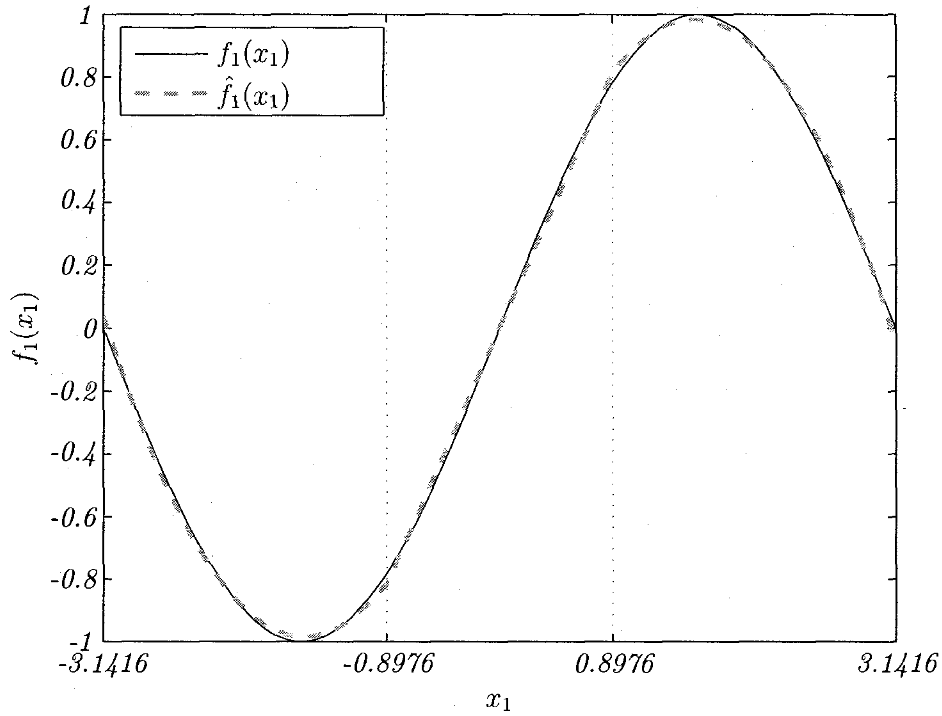

Figure 5.4: PWP approximation of $f_1(x_1)$

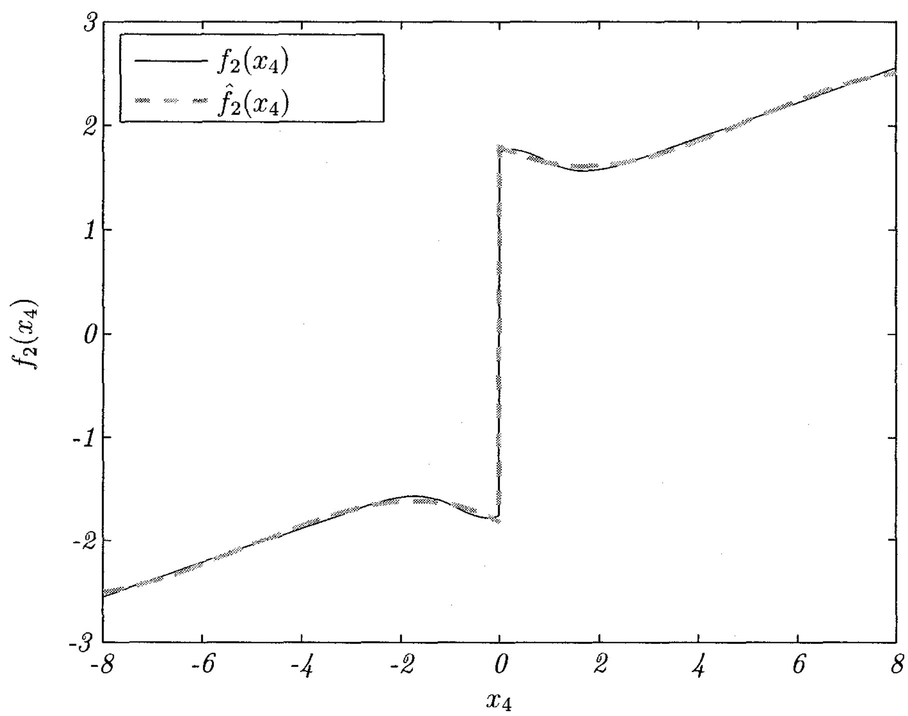

Figure 5.5: PWP approximation of $f_2(x_4)$

with

$$\mathcal{P}_{11} = \{x_1 | x_1 \in \mathbb{R}\} \tag{5.70}$$

The linear controller $x_2 = -2x_1$ *is considered in this step to make the quadratic Lyapunov function* $V_1(x_1) = \frac{1}{2}x_1^2$ *decreasing with time.*

In the second step, the following PWP system is considered

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{MgL}{I}\hat{f}_1(x_1) - \frac{K}{I}(x_1 - x_3) \end{aligned} \tag{5.71}$$

with the regions defined as

$$\mathcal{P}_{21} = \left\{ \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \middle| -\pi < x_1 < -\frac{2\pi}{7}, x_2 \in \mathbb{R} \right\} \tag{5.72}$$

$$\mathcal{P}_{22} = \left\{ \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \middle| -\frac{2\pi}{7} < x_1 < \frac{2\pi}{7}, x_2 \in \mathbb{R} \right\} \tag{5.73}$$

$$\mathcal{P}_{23} = \left\{ \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \middle| \frac{2\pi}{7} < x_1 < \pi, x_2 \in \mathbb{R} \right\} \tag{5.74}$$

Each of the following polynomials, using (5.47), describes a set that contains the common boundaries of the corresponding regions

$$E_{212}(x_1, x_2) = x_1 + \frac{2\pi}{7} \tag{5.75}$$

$$E_{223}(x_1, x_2) = x_1 - \frac{2\pi}{7} \tag{5.76}$$

Considering the Lyapunov function $V_2(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + 2x_1)^2$ *and solving the SOS feasibility problem (5.46) for the PWP system (5.71), the following controller is computed*

$$x_3 = \gamma_2(x_1, x_2) = \begin{cases} 0.26137 + 0.85161x_1 - 0.1x_2 & \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathcal{P}_{21} \\ 0.56043x1 - 0.1x2 & \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathcal{P}_{22} \\ -0.26137 + 0.85161x1 - 0.1x2 & \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathcal{P}_{23} \end{cases} \tag{5.77}$$

For the next step, the following PWP system is considered

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{MgL}{I}\hat{f}_1(x_1) - \frac{K}{I}(x_1 - x_3)\\ \dot{x}_3 &= x_4\end{aligned} \tag{5.78}$$

with the following regions

$$\mathcal{P}_{31} = \left\{ \left[\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right] \middle| -\pi < x_1 < -\frac{2\pi}{7}, x_2 \in \mathbb{R}, x_3 \in \mathbb{R} \right\} \tag{5.79}$$

$$\mathcal{P}_{32} = \left\{ \left[\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right] \middle| -\frac{2\pi}{7} < x_1 < \frac{2\pi}{7}, x_2 \in \mathbb{R}, x_3 \in \mathbb{R} \right\} \tag{5.80}$$

$$\mathcal{P}_{33} = \left\{ \left[\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right] \middle| \frac{2\pi}{7} < x_1 < \pi, x_2 \in \mathbb{R}, x_3 \in \mathbb{R} \right\} \tag{5.81}$$

and the following polynomials for the common boundaries

$$E_{312}(x_1, x_2, x_3) = x_1 + \frac{2\pi}{7} \tag{5.82}$$

$$E_{323}(x_1, x_2, x_3) = x_1 - \frac{2\pi}{7} \tag{5.83}$$

Considering the Lyapunov function $V_3(x_1, x_2, x_3) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + 2x_1)^2 + \frac{1}{2}(x_3 - \gamma_2(x_1, x_2))^2$ *and solving the corresponding SOS feasibility problem for the PWP system (5.78), the following controller is computed*

$$x_4 = \gamma_3(x_1, x_2, x_3) = \begin{cases} 2.1731 - 77.5789x_1 - 75.8241x_2 - 80x_3 & \left[\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right] \in \mathcal{P}_{31} \\ -80x_1 - 75.8241x_2 - 80x_3 & \left[\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right] \in \mathcal{P}_{32} \\ -2.1731 - 77.5789x_1 - 75.8241x_2 - 80x_3 & \left[\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right] \in \mathcal{P}_{33} \end{cases} \tag{5.84}$$

For the next step, the following PWP system is considered

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{MgL}{I}\hat{f}_1(x_1) - \frac{K}{I}(x_1 - x_3) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{\hat{f}_2(x_4)}{J} + \frac{K}{J}(x_1 - x_3) + \frac{1}{J}u
\end{aligned} \tag{5.85}$$

with the following regions

$$\mathcal{P}_{41} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| -\pi < x_1 < -\frac{2\pi}{7}, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}, x_4 > 0 \right\} \tag{5.86}$$

$$\mathcal{P}_{42} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| -\frac{2\pi}{7} < x_1 < \frac{2\pi}{7}, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}, x_4 > 0 \right\} \tag{5.87}$$

$$\mathcal{P}_{43} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| \frac{2\pi}{7} < x_1 < \pi, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}, x_4 > 0 \right\} \tag{5.88}$$

$$\mathcal{P}_{44} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| -\pi < x_1 < -\frac{2\pi}{7}, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}, x_4 < 0 \right\} \tag{5.89}$$

$$\mathcal{P}_{45} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| -\frac{2\pi}{7} < x_1 < \frac{2\pi}{7}, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}, x_4 < 0 \right\} \tag{5.90}$$

$$\mathcal{P}_{46} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| \frac{2\pi}{7} < x_1 < \pi, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}, x_4 < 0 \right\} \tag{5.91}$$

and the following polynomials for the common boundaries

$$E_{412}(x_1, x_2, x_3, x_4) = x_1 + \frac{2\pi}{7} \tag{5.92}$$

$$E_{414}(x_1, x_2, x_3, x_4) = x_4 \tag{5.93}$$

$$E_{423}(x_1, x_2, x_3, x_4) = x_1 - \frac{2\pi}{7} \tag{5.94}$$

$$E_{425}(x_1, x_2, x_3, x_4) = x_4 \tag{5.95}$$

$$E_{436}(x_1, x_2, x_3, x_4) = x_4 \tag{5.96}$$

$$E_{445}(x_1, x_2, x_3, x_4) = x_1 + \frac{2\pi}{7} \tag{5.97}$$

$$E_{456}(x_1, x_2, x_3, x_4) = x_1 - \frac{2\pi}{7} \tag{5.98}$$

The structure of the regions of the PWP system (5.85) is shown in Figure 5.2. This

$$
\begin{tikzcd}
x_1 & & & & \mathcal{P}_{11} \arrow[dll] \arrow[d] \arrow[drr] & & & \\
(x_1, x_2) & & \mathcal{P}_{21} \arrow[d] & & \mathcal{P}_{22} \arrow[d] & & \mathcal{P}_{23} \arrow[d] & \\
(x_1, x_2, x_3) & & \mathcal{P}_{31} \arrow[dl] \arrow[d] & & \mathcal{P}_{32} \arrow[dl] \arrow[dr] & & \mathcal{P}_{33} \arrow[d] \arrow[dr] & \\
(x_1, x_2, x_3, x_4) & \mathcal{P}_{41} & \mathcal{P}_{44} & \mathcal{P}_{42} & & \mathcal{P}_{45} & \mathcal{P}_{43} & \mathcal{P}_{46}
\end{tikzcd}
$$

Figure 5.6: The structure of the regions of the PWP system (5.85)

figure shows that, for example, the image of the region $\mathcal{P}_{42}$ on the (x_1, x_2) space is a subset of $\mathcal{P}_{22}$.

Considering the Lyapunov function $V_4(x_1, x_2, x_3, x_4) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + 2x_1)^2 + \frac{1}{2}(x_3 - \gamma_2(x_1, x_2))^2 + \frac{1}{2}(x_4 - \gamma_3(x_1, x_2, x_3))^2$ and solving the SOS feasibility problem (5.49) for the PWP system (5.85), the following PWA controller is computed

$$
u = \begin{cases}
737.0 - 9786x_1 - 11040x_2 - 1311x_3 - 162.7x_4 & \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \in \mathcal{P}_{41} \\
-10610x_1 - 11040x_2 - 13110x_3 - 162.7x_4 & \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \in \mathcal{P}_{42} \\
-736.2 - 9786x_1 - 11040x_2 - 13110x_3 - 162.7x_4 & \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \in \mathcal{P}_{43} \\
736.2 - 9786x_1 - 11040x_2 - 13110x_3 - 162.5x_4 & \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \in \mathcal{P}_{44} \\
-10610x_1 - 11040x_2 - 13110x_3 - 162.5x_4 & \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \in \mathcal{P}_{45} \\
-737.0 - 9786x_1 - 11040x_2 - 13110x_3 - 162.5x_4 & \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \in \mathcal{P}_{46}
\end{cases} \tag{5.99}
$$

Fig. 5.7 shows the states of the nonlinear system in feedback connection with the

PWA controller (5.99) with the initial condition $x_0 = [\pi\ 0\ 0.8\pi\ 0]^T$*. The system converges to the origin in 4 seconds. However, by examining the transient changes of the control input in Fig. 5.8, we realize that the input is very large. Another possible controller is a PWP controller. Next, a PWP controller of third order in* x_1*, first order in* x_2*, first order in* x_3 *and third order in* x_4 *is designed. Considering the Lyapunov function* $V_4(x_1,x_2,x_3,x_4) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2+2x_1)^2 + \frac{1}{2}(x_3-\gamma_2(x_1,x_2))^2 + \frac{1}{2}(x_4-\gamma_3(x_1,x_2,x_3))^2$ *and solving the SOS feasibility problem (5.49) for the PWP system (5.85), the following PWP controller is computed*

$$\begin{aligned} u =& 11.46 + 832.0x_1 - 44.25x_2 - 169.9x_1^2 + 23.74x_1x_2 - 10.66x_1^3 - 10.42x_1^2x_2 \\ & - 1490x_3 + 24.93x_1x_3 - 10.94x_1^2x_3 - 17.53x_4 - 0.08227x_1x_4 - 0.3943x_2x_4 \\ & - 0.4231x_3x_4 + 0.01364x_4^2 + 0.1917x_1^2x_4 + 0.3465x_1x_2x_4 + 0.3670x_1x_3x_4 \\ & - 0.008023x_1x_4^2 - 0.01344x_2x_4^2 - 0.01427x_3x_4^2 - 0.001407x_4^3 \\ & \mathit{for} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \in \mathcal{P}_{41} \end{aligned} \tag{5.100}$$

$$\begin{aligned} u =& 966.8x_1 - 70.87x_2 - 0.0071x_1^2 - 0.009167x_1x_2 - 4.058x_1^3 - 3.835x_1^2x_2 - 1518x_3 \\ & - 0.01287x_1x_3 - 4.11x_1^2x_3 - 17.92x_4 - 0.6414x_1x_4 - 0.6152x_2x_4 - 0.6545x_3x_4 \\ & + 0.009x_4^2 + 0.06297x_1^2x_4 + 0.1x_1x_2x_4 + 0.11x_1x_3x_4 - 0.01316x_1x_4^2 \\ & - 0.01344x_2x_4^2 - 0.01427x_3x_4^2 - 0.001407x_4^3 \\ & \mathit{for} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \in \mathcal{P}_{42} \end{aligned} \tag{5.101}$$

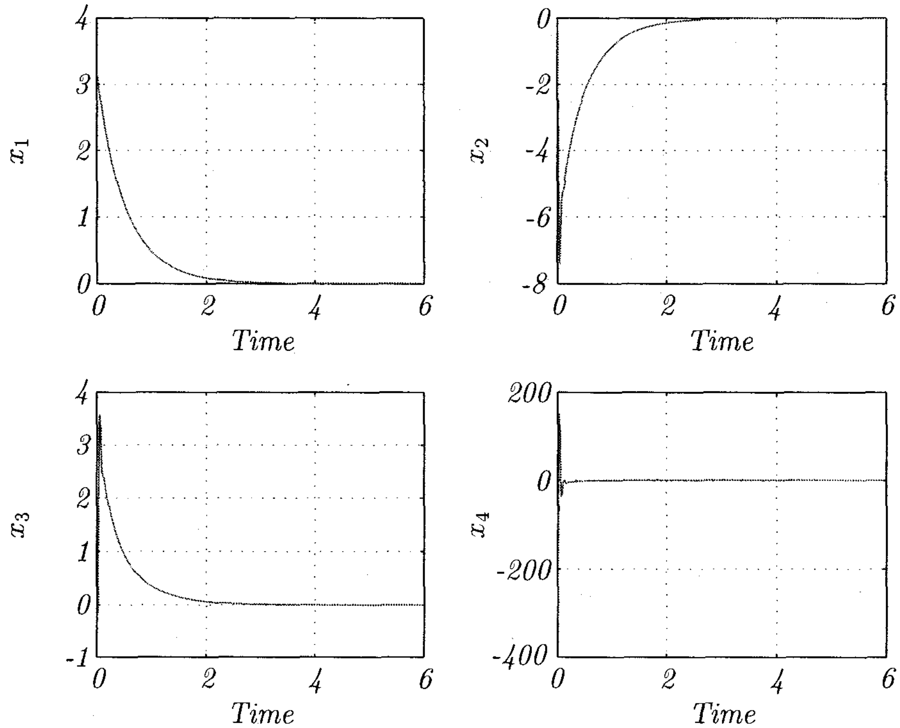

Figure 5.7: State variables of the nonlinear model - PWA controller

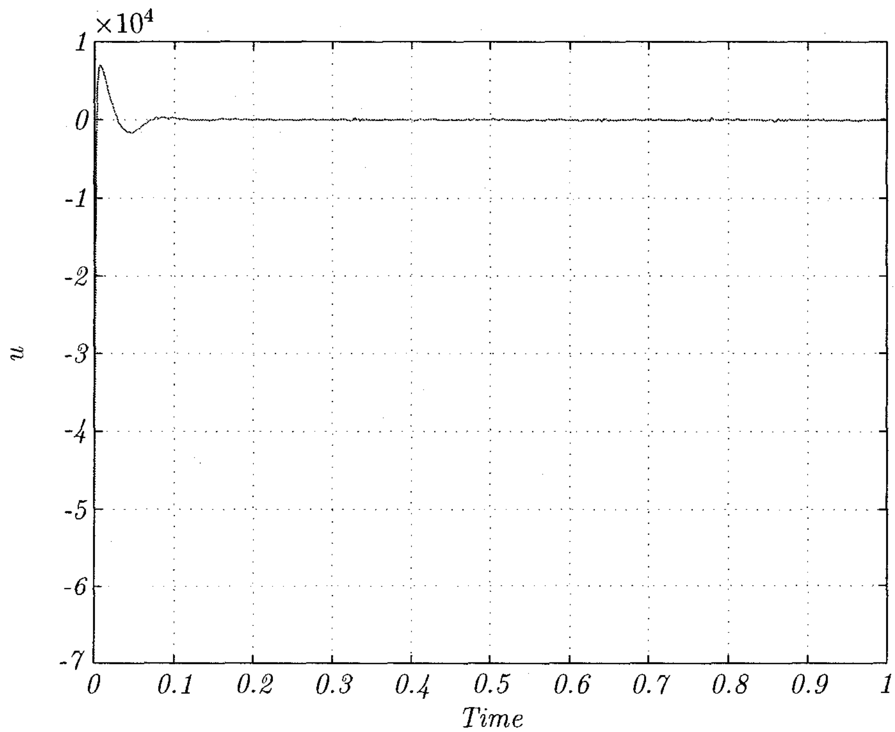

Figure 5.8: Control input - PWA controller

$$
\begin{aligned}
u =& -10.69 + 832.0x_1 - 44.25x_2 + 169.9x_1^2 - 23.75x_1x_2 - 10.66x_1^3 - 10.42x_1^2x_2 \\
& - 1490x_3 - 24.95x_1x_3 - 10.95x_1^2x_3 - 17.54x_4 - 0.63x_1x_4 - 0.2946x_2x_4 \\
& - 0.3062x3 * x4 + 0.01371x_4^2 - 0.4293x_1^2x_4 - 0.2568x_1x_2x_4 - 0.279x_1x_3x_4 \\
& - 0.0184x_1x_4^2 - 0.01344x_2x_4^2 - 0.01427x_3x_4^2 - 0.001407x_4^3 \\
& \mathit{for} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \in \mathcal{P}_{43}
\end{aligned}
\tag{5.102}
$$

$$
\begin{aligned}
u =& 10.69 + 832x_1 - 44.25x_2 - 169.9x_1^2 + 23.74x_1x_2 - 10.66x_1^3 - 10.42x_1^2x_2 \\
& - 1490x_3 + 24.93x_1x_3 - 10.94x_1^2x_3 - 17.36x_4 + 0.634x_1x_4 + 0.2984x_2x_4 \\
& + 0.31x_3x_4 - 0.01362x_4^2 - 0.4328x_1^2x_4 - 0.26x_1x_2x_4 - 0.2825x_1x_3x_4 \\
& - 0.01852x_1x_4^2 - 0.01349x_2x_4^2 - 0.01431x_3x_4^2 + 0.001045x_4^3 \\
& \mathit{for} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \in \mathcal{P}_{44}
\end{aligned}
\tag{5.103}
$$

$$
\begin{aligned}
u =& 966.8x_1 - 70.87x_2 - 0.0071x_1^2 - 0.009167x_1x_2 - 4.058x_1^3 - 3.8347x_1^2x_2 \\
& - 1518x_3 - 0.01287x_1x_3 - 4.11x_1^2x_3 - 17.74x_4 + 0.6477x_1x_4 + 0.6215x_2x_4 \\
& + 0.6612x_3x_4 - 0.008877x_4^2 + 0.0624x_1^2x_4 + 0.09985x_1x_2x_4 + 0.1086x_1x_3x_4 \\
& - 0.01324x_1x_4^2 - 0.01349x_2x_4^2 - 0.01431x_3x_4^2 + 0.001045x_4^3 \\
& \mathit{for} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \in \mathcal{P}_{45}
\end{aligned}
\tag{5.104}
$$

$$
\begin{aligned}
u =& -11.47 + 832x_1 - 44.25x_2 + 169.9x_1^2 - 23.75x_1x_2 - 10.66x_1^3 - 10.42x_1^2x_2 \\
& - 1490x_3 - 24.95x_1x_3 - 10.95x_1^2x_3 - 17.35x_4 + 0.08544x_1x_4 + 0.3976x_2x_4 \\
& + 0.4266x_3x_4 - 0.01358x_4^2 + 0.1946x_1^2x_4 + 0.3493x_1x_2x_4 + 0.37x_1x_3x_4 \\
& - 0.008x_1x_4^2 - 0.01349x_2x_4^2 - 0.01431x_3x_4^2 + 0.001x_4^3 \\
& \mathit{for} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \in \mathcal{P}_{46}
\end{aligned}
\tag{5.105}
$$

Fig. 5.9 shows the states of the nonlinear system in feedback connection with the PWP controller with the initial condition $x_0 = [\pi\ 0\ 0.8\pi\ 0]^T$. *The system converges to the origin a bit slower in comparison to the case of the PWA controller. However, Fig. 5.10 shows that in this case the control input is much smaller.*

5.5 Conclusions

In this chapter, the strict feedback form for PWP systems was introduced. Using backstepping, controller synthesis for this large class of PWP systems was formulated as an SOS program, which is a convex optimization problem. The synthesis problem was addressed in two cases: SOS Lyapunov functions for PWP systems with discontinuous vector fields and PWP Lyapunov functions for PWP systems with continuous vector fields. One of the main advantages of the proposed method is that it addresses PWP systems with discontinuous vector fields regardless of possible attractive sliding modes.

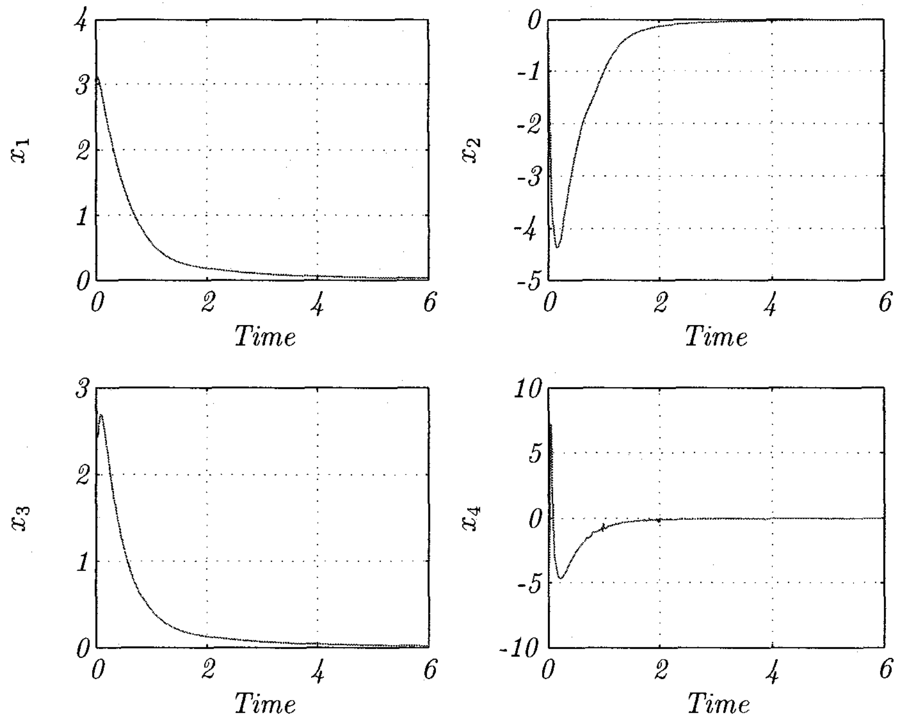

Figure 5.9: State variables of the nonlinear model - PWP controller

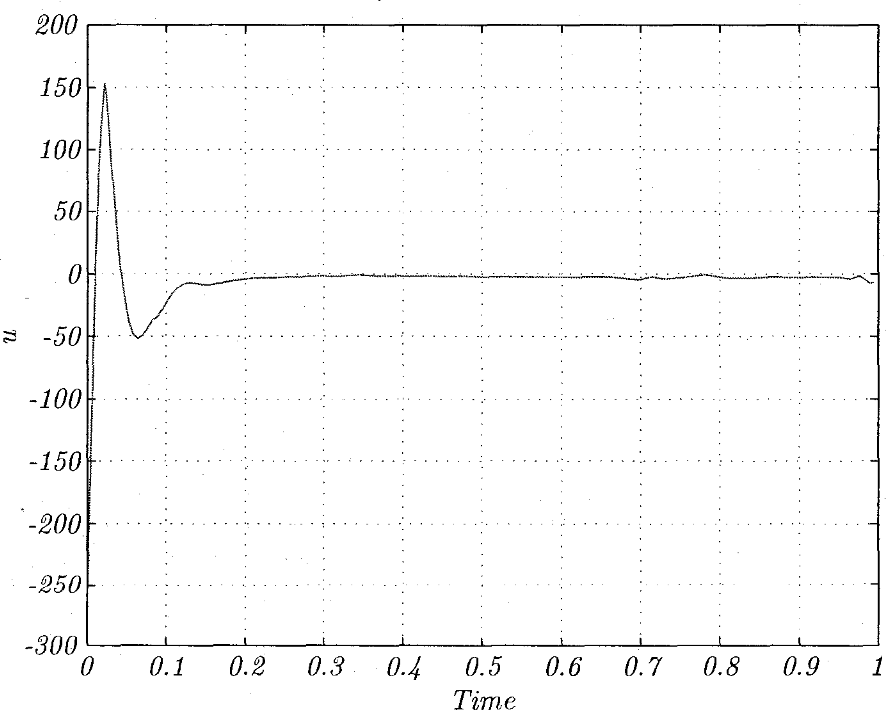

Figure 5.10: Control input - PWA controller

Chapter 6

Sampled-Data Piecewise Affine Systems: A Time-Delay Approach

This chapter addresses stability analysis of sampled-data piecewise-affine (PWA) systems consisting of a continuous-time plant in feedback connection with a discrete-time emulation of a continuous-time state feedback controller. The sampled-data system is considered as a continuous-time system with a variable delay. Conditions under which the trajectories of the sampled-data closed-loop system will converge to an attracting invariant set are then presented. It is also shown that when the sampling period converges to zero, these conditions coincide with sufficient conditions for non-fragility of the stabilizing continuous-time PWA state feedback controller.

6.1 Introduction

The research work on continuous-time PWA systems has concentrated on Lyapunov-based controller synthesis methods [56, 66, 96, 101, 103]. However, none of these approaches would be applicable directly to controller synthesis for computer-controlled or sampled-data PWA systems. This is the scenario mostly encountered in applications given the flexibility of control implementation in a microprocessor.

Although linear sampled-data systems are a well-studied matter [27], controller emulation for systems with possible discontinuities at the switching, such as sampled-data PWA systems, has not had many research contributions. In fact, only recently these systems have started to be addressed in the literature in references such as [6, 62, 63, 99, 122, 123, 129]. The approach by [123] established that, under certain conditions, the controllable subspaces of a continuous-time switched linear system and its discrete-time counterpart are the same. Canonical forms of switched linear systems based on controllability are presented in [122]. The reference [129] considers stability analysis of switched systems that can switch between a set of continuous-time plants and a set of discrete-time plants but does not handle sampled-data systems involving a cascade of a discrete-time system between a sample-and-hold and a continuous-time system. Furthermore, it does not address controller design. The approach by [6, 62, 63] was probably the first where the term "sampled-data PWA systems" is used, although the systems described in this work do not posses the typical structure of a continuous-time plant being controlled by a discrete-time controller. The problem addressed in [6, 62, 63] is one where the controller is continuous-time and the switching events are the ones controlled by the system logic inside a computer. In other words, in these systems it is assumed that the designer has command over the switching times of the system, which does not occur often in practice. For this class of systems, reference [6] presents a probabilistic analysis of controllability. The preliminary study of [62, 63] is interesting as it highlights important limitations of current discrete-time PWA control methodologies when applied to the control of a physical continuous-time system. As mentioned in [62] unexpected phenomena such as chattering can occur, depending on the switching times. This increases the interest in studying computer implementations of controllers designed in continuous-time. Reference [99] addresses the classical structure of a sampled-data system whereby the system is continuous-time and the controller is being implemented (emulated) in discrete-time inside a computer. However, the sampling time must be constant.

This chapter departs considerably from previous research by addressing stability analysis of sampled-data PWA systems using a time delay approach. In fact, the discrete-time PWA controller is seen as a continuous-time PWA controller with a delay that varies with time. Using a Lyapunov-Krasovskii functional, LMI conditions are derived as sufficient conditions for convergence of the sampled-data PWA system trajectories to an attracting invariant set. One of the advantages of the proposed method is that it can be applied to sampled-data PWA systems with variable sampling time as opposed to [99] that deals with a constant sample time. A very important and interesting property of the LMI conditions proposed in this chapter is that when the sampling time converges to zero, these conditions coincide with LMI conditions for non-fragility of the continuous-time PWA controller. Therefore, to implement a continuous-time PWA controller in discrete-time, it is required that the controller be robust to variations in the controller parameters. This in itself is a very interesting result.

The chapter starts by the stability analysis of the sampled-data system when a continuous-time controller is emulated in discrete-time. A numerical example is included to show the performance of the proposed method. Finally, the chapter closes by stating the conclusions.

6.2 Stability of Sampled-Data PWA Systems

Consider a PWA controller of the following form

$$u(t) = K_i x(t) + k_i, \;\; x(t) \in \mathcal{R}_i \tag{6.1}$$

for the PWA system

$$\dot{x} = A_i x + a_i + Bu, \quad \text{for } x \in \mathcal{R}_i \tag{6.2}$$

with the region $\mathcal{R}_i$ defined as

$$\mathcal{R}_i = \{x | E_i x + e_i \succ 0\}, \tag{6.3}$$

The closed-loop system is assumed to be asymptotically stable. It is also assumed that the vector field of the open loop PWA system (3.10) with $u(t) = 0$ is continuous across the boundaries of two or more regions and $a_i = 0$ for $i \in \mathcal{I}(0)$.

If the PWA controller (6.1) is implemented as a digital controller and is connected to the PWA system (6.2) through a sample-and-hold, the closed-loop system can be described by

$$\dot{x}(t) = A_i x(t) + a_i + B(K_j x(t_k) + k_j), \tag{6.4}$$

for $x(t) \in \mathcal{R}_i$ and $x(t_k) \in \mathcal{R}_j$ where t_k for $k \in \mathbb{N}$ is the sampling time and $t_k \leq t < t_{k+1}$. The closed-loop system (6.4) can be rewritten as

$$\dot{x}(t) = A_i x(t) + a_i + B(K_i x(t_k) + k_i) + Bw, \tag{6.5}$$

for $x(t) \in \mathcal{R}_i$ and $x(t_k) \in \mathcal{R}_j$ where

$$w(t) = (K_j - K_i)x(t_k) + (k_j - k_i), \;\; x(t) \in \mathcal{R}_i, \;\; x(t_k) \in \mathcal{R}_j \tag{6.6}$$

The input $w(t)$ is a result of the fact that $x(t)$ and $x(t_k)$ are not necessarily in the same region.

Following [82], the time elapsed since the last sampling time will be denoted by

$$\rho(t) := t - t_k, \;\; t_k \leq t < t_{k+1} \tag{6.7}$$

and τ_M (τ_D) is defined as the maximum (minimum) interval between sampling times.

$$\tau_D \leq t_{k+1} - t_k \leq \tau_M, \forall k \in \mathbb{N} \tag{6.8}$$

Consider a Lyapunov-Krasovskii functional of the form

$$V(x_s, \rho) := V_1(x) + V_2(x_s) + V_3(x_s, \rho) \tag{6.9}$$

where

$$x_s(t) \quad := \quad \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}, \; t_k \le t < t_{k+1}$$

$$\begin{aligned} V_1(x) &:= x^{\mathrm{T}} P x \\ V_2(x_s) &:= \int_{-\tau_M}^{0} \int_{t+r}^{t} \dot{x}^{\mathrm{T}}(s) R \dot{x}(s) ds dr \\ V_3(x_s, \rho) &:= (\tau_M - \rho)(x(t) - x(t_k))^{\mathrm{T}} X (x(t) - x(t_k)) \end{aligned}$$

and P, R and X are positive definite matrices. Therefore, the Lyapunov-Krasovskii functional $V(x_s, \rho)$ is positive definite. At the sampling times, $V(x_s, \rho)$ does not increase because $V_2(x_s)$ and $V_3(x_s, \rho)$ are non-negative right before each sampling time and they become zero right after the sampling time [82]. It can be shown that $V(x_s, \rho)$ satisfies the following inequality

$$\lambda_{\min}(P)\|x\|^2 \le V(x_s, \rho) \le \sigma_a \|x_s\|^2 + \sigma_b \tag{6.10}$$

where

$$\begin{aligned} \sigma_a &= \lambda_{\max}(P) + 2(\tau_M - \rho)\lambda_{\max}(X) + \frac{\tau_M^2}{2}\lambda_{\max}(\bar{R}), \\ \sigma_b &= \frac{\tau_M^2}{2}\lambda_{\max}(\bar{R}), \end{aligned}$$

where $\lambda_{\min}(.)$ and $\lambda_{\max}(.)$ mean the minimum and maximum eigenvalues of a matrix, respectively, and

$$\bar{R} = \arg\max_{i,j} \lambda_{\max}(\bar{R}_{ij}) \tag{6.11}$$

$$\bar{R}_{ij} = \begin{bmatrix} A_i^{\mathrm{T}} \\ K_j^{\mathrm{T}} B^{\mathrm{T}} \\ a_i^{\mathrm{T}} + k_j^{\mathrm{T}} B^{\mathrm{T}} \end{bmatrix} R \begin{bmatrix} A_i & BK_j & a_i + Bk_j \end{bmatrix} \tag{6.12}$$

The main result of this chapter is now presented.

Theorem 6.1. *For the sampled-data PWA system (6.5), assume there exist symmetric positive matrices P, R, X and matrices N_i for $i = 1, \dots, M$ such that*

- *for all $i \in \mathcal{I}(0)$,*

$$\Omega_i + \tau_M M_{1i} + \tau_M M_{2i} < 0 \tag{6.13}$$

$$\begin{bmatrix} \Omega_i + \tau_M M_{1i} & \tau_M \begin{bmatrix} N_i \\ 0 \end{bmatrix} \\ \tau_M \begin{bmatrix} N_i^T & 0 \end{bmatrix} & -\tau_M R \end{bmatrix} < 0 \tag{6.14}$$

- *for all $i \notin \mathcal{I}(0)$, $\bar{\Lambda}_i \succ 0$,*

$$\overline{\Omega}_i + \tau_M \overline{M}_{1i} + \tau_M \overline{M}_{2i} < 0 \tag{6.15}$$

$$\begin{bmatrix} \overline{\Omega}_i + \tau_M \overline{M}_{1i} & \tau_M \begin{bmatrix} N_i \\ 0 \\ 0 \end{bmatrix} \\ \tau_M \begin{bmatrix} N_i^T & 0 & 0 \end{bmatrix} & -\tau_M R \end{bmatrix} < 0 \tag{6.16}$$

where

$$\Omega_i = \begin{bmatrix} \Psi_i & \begin{bmatrix} P \\ 0 \end{bmatrix} B \\ B^T \begin{bmatrix} P & 0 \end{bmatrix} & -\gamma I \end{bmatrix},$$

$$\begin{aligned} \Psi_i = & \begin{bmatrix} P \\ 0 \end{bmatrix} \begin{bmatrix} A_i & BK_i \end{bmatrix} + \begin{bmatrix} A_i^T \\ K_i^T B^T \end{bmatrix} \begin{bmatrix} P & 0 \end{bmatrix} \\ & - \begin{bmatrix} I \\ -I \end{bmatrix} X \begin{bmatrix} I & -I \end{bmatrix} \\ & - N_i \begin{bmatrix} I - I \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} N_i^T + \eta I_{2n \times 2n}, \end{aligned}$$

$$M_{1i} = \begin{bmatrix} A_i^T \\ K_i^T B^T \\ B^T \end{bmatrix} R \begin{bmatrix} A_i & BK_i & B \end{bmatrix},$$

$$M_{2i} = \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} X \begin{bmatrix} A_i & BK_i & B \end{bmatrix} + \begin{bmatrix} A_i^T \\ K_i^T B^T \\ B^T \end{bmatrix} X \begin{bmatrix} I & -I & 0 \end{bmatrix},$$

$$\overline{\Omega}_i = \begin{bmatrix} \overline{\Psi}_i & \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} B \\ B^T \begin{bmatrix} P & 0 & 0 \end{bmatrix} & -\gamma I \end{bmatrix},$$

$$\begin{aligned} \overline{\Psi}_i = & \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} A_i & BK_i & Bk_i + a_i \end{bmatrix} \\ & + \begin{bmatrix} A_i^T \\ K_i^T B^T \\ k_i^T B^T + a_i^T \end{bmatrix} \begin{bmatrix} P & 0 & 0 \end{bmatrix} - \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} X \begin{bmatrix} I & -I & 0 \end{bmatrix} \\ & - \begin{bmatrix} N_i \\ 0 \end{bmatrix} \begin{bmatrix} I - I & 0 \end{bmatrix} - \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} \begin{bmatrix} N_i^T & 0 \end{bmatrix} \\ & + \begin{bmatrix} \eta I_{2n \times 2n} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} E_i^T & 0 \\ 0 & 0 \\ e_i^T & 1 \end{bmatrix} \bar{\Lambda}_i \begin{bmatrix} E_i & 0 & e_i \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

$$\overline{M_{1i}} = \begin{bmatrix} A_i^T \\ K_i^T B^T \\ k_i^T B^T + a_i^T \\ B^T \end{bmatrix} R \begin{bmatrix} A_i & BK_i & Bk_i + a_i & B \end{bmatrix},$$

$$\overline{M_{2i}} = \begin{bmatrix} I \\ -I \\ 0 \\ 0 \end{bmatrix} X \begin{bmatrix} A_i & BK_i & Bk_i + a_i & B \end{bmatrix} + \begin{bmatrix} A_i^T \\ K_i^T B^T \\ k_i^T B^T + a_i^T \\ B^T \end{bmatrix} X \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix}$$

Let there be constants Δ_K and Δ_k such that

$$\|w\| \leq \Delta_K \|x(t_k)\| + \Delta_k \tag{6.17}$$

Define

$$\mu_\theta = \frac{\sqrt{\gamma}\Delta_k}{\sqrt{\theta\eta} - \sqrt{\gamma}\Delta_K} \tag{6.18}$$

and the region

$$\Phi_\theta = \{x_s | \ \|x_s\| \leq \mu_\theta\} \tag{6.19}$$

for some positive constant $\theta < 1$ that verifies

$$\Delta_K < \sqrt{\frac{\theta\eta}{\gamma}} \tag{6.20}$$

Then, all the trajectories of the system (6.5) in $\mathcal{X}$ converge to the following invariant set

$$\Omega = \{x_s | \ V(x_s, \rho) \leq \sigma_a \mu_\theta^2 + \sigma_b\} \tag{6.21}$$

□

Proof. The proof is divided into two parts.

1. First, it is shown that the inequalities (6.13), (6.14), (6.15) and (6.16) are sufficient conditions for the following inequality to hold

$$\dot{V}(x_s, \rho) \leq -\eta x_s^{\mathrm{T}} x_s + \gamma w^{\mathrm{T}} w \tag{6.22}$$

for $t_k < t < t_{k+1}$. Since $V_1(x) = x^{\mathrm{T}}Px$, one has

$$\dot{V}_1(x) = \dot{x}^{\mathrm{T}}Px + x^{\mathrm{T}}P\dot{x} \tag{6.23}$$

$V_2(x_s)$ can be written in the following form

$$V_2(x_s) = \int_{-\tau_M}^{0} g(t,r)dr \tag{6.24}$$

where

$$g(t,r) = \int_{t+r}^{t} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)ds \tag{6.25}$$

Thus, since $\dot{\rho} = 1$ for $t_k < t < t_{k+1}$,

$$\dot{V}_2(x_s) = \int_{-\tau_M}^{0} \frac{\partial}{\partial t}g(t,r)dr \tag{6.26}$$

The expression

$$\frac{\partial}{\partial t}g(t,r) = \dot{x}^{\mathrm{T}}(t)R\dot{x}(t) - \dot{x}^{\mathrm{T}}(t+r)R\dot{x}(t+r) \tag{6.27}$$

then yields

$$\dot{V}_2(x_s) = \tau_M\dot{x}^{\mathrm{T}}(t)R\dot{x}(t) - \int_{t-\tau_M}^{t} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)ds \tag{6.28}$$

From (6.8) one has $\rho \leq \tau_M$ and considering the fact that R is positive definite, this leads to

$$\dot{V}_2(x_s) \leq \tau_M\dot{x}^{\mathrm{T}}(t)R\dot{x}(t) - \int_{t-\rho}^{t} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)ds \tag{6.29}$$

Since R is positive definite, for any matrix $N_i \in \mathbb{R}^{n\times 2n}$ one has

$$\begin{bmatrix} \dot{x}^{\mathrm{T}}(s) & x_s^{\mathrm{T}}(t)N_i \end{bmatrix} \begin{bmatrix} R & -I \\ -I & R^{-1} \end{bmatrix} \begin{bmatrix} \dot{x}(s) \\ N_i^{\mathrm{T}}x_s(t) \end{bmatrix} \geq 0 \tag{6.30}$$

and therefore

$$-\dot{x}(s)^{\mathrm{T}}R\dot{x}(s) \leq x_s^{\mathrm{T}}(t)N_iR^{-1}N_i^{\mathrm{T}}x_s(t) - 2x_s^{\mathrm{T}}(t)N_i\dot{x}(s) \tag{6.31}$$

Integrating both sides from $t-\rho$ to t and using (6.7) yields,

$$\begin{aligned} -\int_{t-\rho}^{t} \dot{x}(s)^{\mathrm{T}} R \dot{x}(s) ds \ \leq \ & \rho x_s^{\mathrm{T}}(t) N_i R^{-1} N_i^{\mathrm{T}} x_s(t) \\ & -2x_s^{\mathrm{T}}(t) N_i \begin{bmatrix} I & -I \end{bmatrix} x_s(t) \end{aligned} \tag{6.32}$$

It follows from (6.29) and (6.32) that

$$\begin{aligned} \dot{V}_2(x_s) \ \leq \ & \tau_M \dot{x}^{\mathrm{T}} R \dot{x} + \rho x_s^{\mathrm{T}} N_i R^{-1} N_i^{\mathrm{T}} x_s \\ & -2x_s^{\mathrm{T}} N_i \begin{bmatrix} I & -I \end{bmatrix} x_s \end{aligned} \tag{6.33}$$

For $V_3(x_s, \rho)$, since $\dot{\rho} = 1$ for $t_k < t < t_{k+1}$, one can write

$$\dot{V}_3(x_s, \rho) = -(x(t) - x(t_k))^{\mathrm{T}} X (x(t) - x(t_k)) + 2(\tau_M - \rho)(x(t) - x(t_k))^{\mathrm{T}} X \dot{x}(t) \tag{6.34}$$

From (6.23), (6.33) and (6.34), it follows that a sufficient condition for (6.22) is the following inequality

$$\begin{aligned} & \dot{x}^{\mathrm{T}} P x + x^{\mathrm{T}} P \dot{x} + \tau_M \dot{x}^{\mathrm{T}} R \dot{x} + \rho x_s^{\mathrm{T}} N_i R^{-1} N_i^{\mathrm{T}} x_s \\ & -2x_s^{\mathrm{T}} N_i \begin{bmatrix} I & -I \end{bmatrix} x_s \\ & -x_s^{\mathrm{T}} \begin{bmatrix} I \\ -I \end{bmatrix} X \begin{bmatrix} I & -I \end{bmatrix} x_s \\ & +2(\tau_M - \rho) x_s^{\mathrm{T}} \begin{bmatrix} I \\ -I \end{bmatrix} X \dot{x} \\ & +\eta x_s^{\mathrm{T}} x_s - \gamma w^{\mathrm{T}} w \leq 0 \end{aligned} \tag{6.35}$$

For $i \in \mathcal{I}(0)$, one has

$$\dot{x} = \begin{bmatrix} A_i & BK_i \end{bmatrix} x_s + Bw, \tag{6.36}$$

for $x(t) \in \mathcal{R}_i$ and $x(t_k) \in \mathcal{R}_j$. Replacing $\dot{x}$ from (6.36) into (6.35) yields

$$
\begin{aligned}
& x_s^{\mathrm{T}} \left(\begin{bmatrix} P \\ 0 \end{bmatrix} \begin{bmatrix} A_i & BK_i \end{bmatrix} + \begin{bmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}} B^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} P & 0 \end{bmatrix} \right. \\
& + \tau_M \begin{bmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}} B^{\mathrm{T}} \end{bmatrix} R \begin{bmatrix} A_i & BK_i \end{bmatrix} + \rho N_i R^{-1} N_i^{\mathrm{T}} \\
& - N_i \begin{bmatrix} I & -I \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} N_i^{\mathrm{T}} \\
& - \begin{bmatrix} I \\ -I \end{bmatrix} X \begin{bmatrix} I & -I \end{bmatrix} \\
& + (\tau_M - \rho) \begin{bmatrix} I \\ -I \end{bmatrix} X \begin{bmatrix} A_i & BK_i \end{bmatrix} \\
& \left. + (\tau_M - \rho) \begin{bmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}} B^{\mathrm{T}} \end{bmatrix} X \begin{bmatrix} I & -I \end{bmatrix} + \eta I \right) x_s \\
& + x_s^{\mathrm{T}} \begin{bmatrix} P \\ 0 \end{bmatrix} B w + w^{\mathrm{T}} B^{\mathrm{T}} \begin{bmatrix} P & 0 \end{bmatrix} x_s \\
& + \tau_M x_s^{\mathrm{T}} \begin{bmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}} B^{\mathrm{T}} \end{bmatrix} R B w + \tau_M w^{\mathrm{T}} B^{\mathrm{T}} R \begin{bmatrix} A_i & BK_i \end{bmatrix} x_s \\
& + (\tau_M - \rho) x_s^{\mathrm{T}} \begin{bmatrix} I \\ -I \end{bmatrix} X B w + (\tau_M - \rho) w^{\mathrm{T}} B^{\mathrm{T}} X \begin{bmatrix} I & -I \end{bmatrix} x_s \\
& + \tau_M w^{\mathrm{T}} B^{\mathrm{T}} R B w - \gamma w^{\mathrm{T}} w < 0
\end{aligned}
\tag{6.37}
$$

Since (6.37) is affine in ρ, if it holds for $\rho = 0$ and $\rho = \tau_M$ then it is satisfied for any $\rho \in [0, \tau_M]$. For $\rho = 0$, the inequality (6.37) can be written as (6.13). Using Schur complement for $\rho = \tau_M$, the inequality (6.37) can be converted to (6.14).

For $i \notin \mathcal{I}(0)$, one has

$$\dot{x} = \begin{bmatrix} A_i & BK_i & a_i + Bk_i \end{bmatrix} \bar{x}_s + Bw, \;\; x \in \mathcal{R}_i \tag{6.38}$$

where

$$\bar{x}_s = \begin{bmatrix} x_s \\ 1 \end{bmatrix} \tag{6.39}$$

It follows from (2.62) that

$$\begin{bmatrix} x^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} E_i^{\mathrm{T}} & 0 \\ e_i^{\mathrm{T}} & 1 \end{bmatrix} \bar{\Lambda}_i \begin{bmatrix} E_i & e_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} > 0, \;\; x \in \mathcal{R}_i \tag{6.40}$$

where $\bar{\Lambda}_i \succ 0$. Using (6.38) and (6.40), a sufficient condition for (6.37) when $x \in \mathcal{R}_i$ with $i \notin \mathcal{I}(0)$ can be written as

$$\begin{aligned}
\bar{x}_s^{\mathrm{T}} \Bigg(& \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} A_i & BK_i & a_i + Bk_i \end{bmatrix} + \begin{bmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}} B^{\mathrm{T}} \\ a_i^{\mathrm{T}} + k_i^{\mathrm{T}} B^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} P & 0 & 0 \end{bmatrix} \\
& + \tau_M \begin{bmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}} B^{\mathrm{T}} \\ a_i^{\mathrm{T}} + k_i^{\mathrm{T}} B^{\mathrm{T}} \end{bmatrix} R \begin{bmatrix} A_i & BK_i & a_i + Bk_i \end{bmatrix} + \rho \begin{bmatrix} N_i \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} N_i^{\mathrm{T}} & 0 \end{bmatrix} \\
& - \begin{bmatrix} N_i \\ 0 \end{bmatrix} \begin{bmatrix} I & -I & 0 \end{bmatrix} - \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} \begin{bmatrix} N_i^{\mathrm{T}} & 0 \end{bmatrix} - \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} X \begin{bmatrix} I & -I & 0 \end{bmatrix} \\
& + (\tau_M - \rho) \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} X \begin{bmatrix} A_i & BK_i & a_i + Bk_i \end{bmatrix} \\
& + (\tau_M - \rho) \begin{bmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}} B^{\mathrm{T}} \\ a_i^{\mathrm{T}} + k_i^{\mathrm{T}} B^{\mathrm{T}} \end{bmatrix} X \begin{bmatrix} I & -I & 0 \end{bmatrix} + \eta \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \Bigg) \bar{x}_s
\end{aligned}$$

$$+\bar{x}_s^{\mathrm{T}}\begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} Bw + w^{\mathrm{T}}B^{\mathrm{T}}\begin{bmatrix} P & 0 & 0 \end{bmatrix}\bar{x}_s$$

$$+\tau_M\bar{x}_s^{\mathrm{T}}\begin{bmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}}B^{\mathrm{T}} \\ k_i^{\mathrm{T}}B^{\mathrm{T}} + a_i^{\mathrm{T}} \end{bmatrix} RBw + \tau_M w^{\mathrm{T}}B^{\mathrm{T}}R\begin{bmatrix} A_i & BK_i & Bk_i + a_i \end{bmatrix}\bar{x}_s$$

$$+(\tau_M-\rho)\bar{x}_s^{\mathrm{T}}\begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} XBw + (\tau_M-\rho)w^{\mathrm{T}}B^{\mathrm{T}}X\begin{bmatrix} I & -I & 0 \end{bmatrix}\bar{x}_s$$

$$+\tau_M w^{\mathrm{T}}B^{\mathrm{T}}RBw - \gamma w^{\mathrm{T}}w + \bar{x}_s^{\mathrm{T}}\begin{bmatrix} E_i^{\mathrm{T}} & 0 \\ 0 & 0 \\ e_i^{\mathrm{T}} & 1 \end{bmatrix}\bar{\Lambda}_i\begin{bmatrix} E_i & 0 & e_i \\ 0 & 0 & 1 \end{bmatrix}\bar{x}_s < 0 \qquad (6.41)$$

Inequality (6.15) is equivalent to (6.41) for $\rho = 0$ and using Schur complement, inequality (6.16) is equivalent to (6.41) for $\rho = \tau_M$. Since (6.41) is affine in ρ, inequalities (6.15) and (6.16) imply that (6.41) is satisfied for any $\rho \in [0, \tau_M]$.

In conclusion, (6.22) is satisfied for $t_k < t < t_{k+1}$, $k = 0, 1, 2, \ldots$ and any $x \in \mathcal{R}_i$, $i = 1, 2, \ldots, M$.

2. In the second part of the proof, it will be shown that for a given for $0 < \theta < 1$, Ω is an attracting invariant set. For any $x_s \notin \Omega$, one has

$$V(x_s, \rho) > \sigma_a\mu_\theta^2 + \sigma_b \qquad (6.42)$$

It follows from (6.10) that $\|x_s\| > \mu_\theta$ and therefore (6.18) leads to

$$\sqrt{\theta\eta}\|x_s\| > \sqrt{\gamma}(\Delta_K\|x_s\| + \Delta_k) \qquad (6.43)$$

It follows from (6.17) and (6.43) that

$$\theta\eta x_s^{\mathrm{T}}x_s > \gamma w^{\mathrm{T}}w \qquad (6.44)$$

The inequality (6.22) can be written as

$$\dot{V}(x_s,\rho) \leq -(1-\theta)\eta x_s^{\mathrm{T}} x_s - \theta\eta x_s^{\mathrm{T}} x_s + \gamma w^{\mathrm{T}} w \tag{6.45}$$

for $0 < \theta < 1$. Therefore it follows from (6.44) that

$$\dot{V}(x_s,\rho) \leq -(1-\theta)\eta x_s^{\mathrm{T}} x_s \tag{6.46}$$

and from $\|x_s\| > \mu_\theta$, one has

$$\dot{V}(x_s,\rho) \leq -(1-\theta)\eta\mu_\theta^2, \text{ for } t_k < t < t_{k+1} \tag{6.47}$$

Therefore $V(x_s,\rho)$ decreases between the sampling times for $\|x_s\| > \mu_\theta$. As it was mentioned earlier, $V(x_s,\rho)$ also decreases at each sampling time. Therefore there is a finite time t^θ such that $x_s(t^\theta) \in \Phi_\theta$ and therefore from (6.18), one has $V(x_s(t^\theta),\rho) \leq \sigma_a\mu_\theta^2 + \sigma_b$, which means $x_s(t^\theta) \in \Omega$. Therefore, Ω is an attracting invariant set.

□

Remark 6.1. *The upper bound for $\|w\|$ defined in (6.17) can be obtained as*

$$\begin{aligned}\Delta_K &= \max_{i,j=1,\dots,M} \|K_i - K_j\| \\ \Delta_k &= \max_{i,j=1,\dots,M} \|k_i - k_j\|\end{aligned} \tag{6.48}$$

Note that for the case where $K_i = K_j$ and $k_i = k_j$, $\Delta_K = \Delta_k = 0$ and (6.20) is automatically satisfied. In this case $w = 0$ and $\mu_\theta = 0$.

Remark 6.2. *For $\tau_M \to 0$ and*

$$\bar{P} = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}, \; N_i = \begin{bmatrix} -PBK_i + I \\ -I \end{bmatrix}, \; X = (\beta - 2)I \tag{6.49}$$

where $\beta > \max(\eta, 2)$ and

$$\eta_c = \eta + \frac{\eta\beta}{\beta - \eta}, \tag{6.50}$$

the inequalities (6.13), (6.14), (6.15) and (6.16) are reduced to the following inequalities for all $i \in \mathcal{I}(0)$

$$\begin{bmatrix} (A_i + BK_i)^T P + P(A_i + BK_i) + \eta_c I & PB \\ B^T P & -\gamma I \end{bmatrix} < 0 \tag{6.51}$$

and for $i \notin \mathcal{I}(0)$

$$\begin{bmatrix} (\bar{A}_i + \bar{B}\bar{K}_i)^T \bar{P} + \bar{P}(\bar{A}_i + \bar{B}\bar{K}_i) + \bar{E}_i^T \bar{\Lambda}_i \bar{E}_i + \eta_c \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} & \bar{P}\bar{B} \\ \bar{B}^T \bar{P} & -\gamma I \end{bmatrix} < 0 \tag{6.52}$$

Conditions (6.51) and (6.52) are sufficient conditions for input to state stability of the continuous-time PWA system (6.2) with the following condition satisfied for $V(x) = x^T P x$

$$\dot{V}(x) < -\eta_c x^T x + \gamma w^T w \tag{6.53}$$

This result establishes that the continuous-time PWA controller should satisfy a very important property: non-fragility. In other words, if there exists an error w in the implementation of the continuous-time PWA controller (6.1) as shown in the following

$$u(t) = K_i x(t) + k_i + w(t) \tag{6.54}$$

and the norm of w is bounded, the norm of the state vector $x(t)$ remains bounded.

6.3 Numerical Example

Example 6.1. *A state space model was built for an experimental two degrees of freedom helicopter in [38]. In this example, a simplified version of the pitch model of the experimental helicopter (Fig. 6.1) is considered. This model is described by*

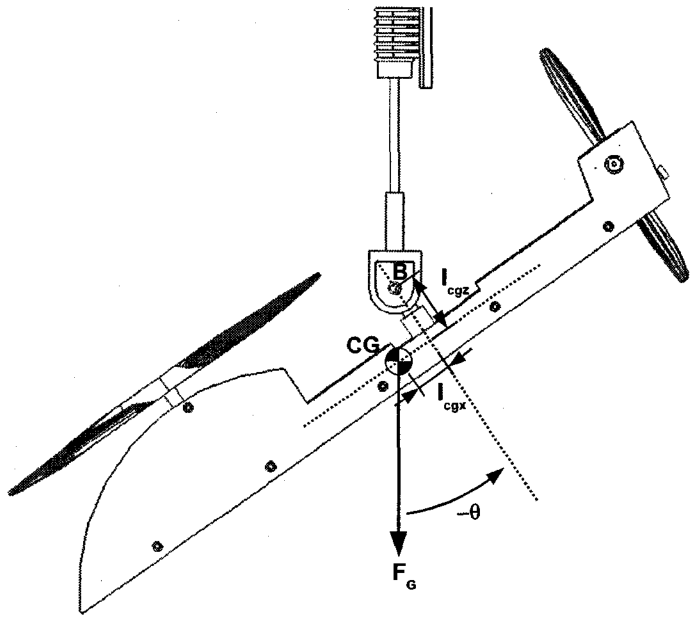

Figure 6.1: Pitch model for the experimental helicopter [38]

Table 6.1: Parameters of the helicopter model

Parameter	*Value*	*Unit*
I_{yy}	*0.0283*	kgm^2
m_{heli}	*0.9941*	*kg*
l_{cgx}	*0.0134*	*m*
l_{cgz}	*0.0289*	*m*
F_{kM}	*0.0003*	*Nm*
F_{vM}	*0.0041*	*Nm/rad/s*
g	*9.81*	m/s^2

the following equations

$$\begin{aligned}\dot{x}_1 =& x_2 \\ \dot{x}_2 =& \frac{1}{I_{yy}}(-m_{heli}l_{cgx}g\cos(x_1) - m_{heli}l_{cgz}g\sin(x_1) - F_{kM}\,\mathrm{sgn}(x_2) - F_{vM}x_2 + u)\end{aligned} \tag{6.55}$$

where x_1 and x_2 represent pitch angle and pitch rate, respectively. The values of the parameters are shown in Table 6.1.

The PWA approximation of the following nonlinear function in (6.55)

$$f(x_1) = -m_{heli}l_{cgx}g\cos(x_1) - m_{heli}l_{cgz}g\sin(x_1) \tag{6.56}$$

is then computed based on a uniform grid in x_1. *The resulting approximation is shown in Figure 6.1. A PWA model is obtained by replacing* $f(x_1)$ *by* $\hat{f}(x_1)$ *in (6.55). The PWA model is described by the following equations*

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 5.3058 & -0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ 22.2968 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \qquad \textit{for } x \in \mathcal{R}_1$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -8.1786 & -0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -3.1208 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \qquad \textit{for } x \in \mathcal{R}_2$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -10.5751 & -0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -4.6265 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \qquad \textit{for } x \in \mathcal{R}_3$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1.9210 & -0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -12.4780 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \qquad \textit{for } x \in \mathcal{R}_4$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 10.7980 & -0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -29.2108 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \qquad \textit{for } x \in \mathcal{R}_5$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 5.3058 & +0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ 22.2968 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \qquad \textit{for } x \in \mathcal{R}_6$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -8.1786 & +0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -3.1208 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \qquad \textit{for } x \in \mathcal{R}_7$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -10.5751 & +0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -4.6265 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \qquad \textit{for } x \in \mathcal{R}_8$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1.9210 & +0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -12.4780 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \qquad \textit{for } x \in \mathcal{R}_9$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 10.7980 & +0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -29.2108 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \qquad \textit{for } x \in \mathcal{R}_{10}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *and*

$$\begin{aligned}
\mathcal{R}_1 =&\{x| -\pi < x_1 < -\frac{3\pi}{5}, x_2 > 0\} & \mathcal{R}_2 =&\{x| -\frac{3\pi}{5} < x_1 < -\frac{\pi}{5}, x_2 > 0\} \\
\mathcal{R}_3 =&\{x| -\frac{\pi}{5} < x_1 < \frac{\pi}{5}, x_2 > 0\} & \mathcal{R}_4 =&\{x|\frac{\pi}{5} < x_1 < \frac{3\pi}{5}, x_2 > 0\} \\
\mathcal{R}_5 =&\{x|\frac{3\pi}{5} < x_1 < \pi, x_2 > 0\} & \mathcal{R}_6 =&\{x| -\pi < x_1 < -\frac{3\pi}{5}, x_2 < 0\} \\
\mathcal{R}_7 =&\{x| -\frac{3\pi}{5} < x_1 < -\frac{\pi}{5}, x_2 < 0\} & \mathcal{R}_8 =&\{x| -\frac{\pi}{5} < x_1 < \frac{\pi}{5}, x_2 < 0\} \\
\mathcal{R}_9 =&\{x|\frac{\pi}{5} < x_1 < \frac{3\pi}{5}, x_2 < 0\} & \mathcal{R}_{10} =&\{x|\frac{3\pi}{5} < x_1 < \pi, x_2 < 0\}
\end{aligned} \tag{6.57}$$

The following PWA controller is then designed to stabilize the origin ($x_1 = x_2 = 0$) *for the PWA system (6.57) using the backstepping method in subsection 5.3.1.*

$$\begin{aligned}
u =& -0.2919x_1 - 0.1092x_2 - 0.6313, \quad \textit{for } x \in \mathcal{R}_1 \\
u =& 0.0900x_1 - 0.1092x_2 + 0.0887, \quad \textit{for } x \in \mathcal{R}_2 \\
u =& 0.1579x_1 - 0.1092x_2 + 0.1314, \quad \textit{for } x \in \mathcal{R}_3 \\
u =& -0.1961x_1 - 0.1092x_2 + 0.3538, \quad \textit{for } x \in \mathcal{R}_4 \\
u =& -0.4475x_1 - 0.1092x_2 + 0.8278, \quad \textit{for } x \in \mathcal{R}_5 \\
u =& -0.2919x_1 - 0.1092x_2 - 0.6319, \quad \textit{for } x \in \mathcal{R}_6 \\
u =& 0.0900x_1 - 0.1092x_2 + 0.0881, \quad \textit{for } x \in \mathcal{R}_7 \\
u =& 0.1579x_1 - 0.1092x_2 + 0.1308, \quad \textit{for } x \in \mathcal{R}_8 \\
u =& -0.1961x_1 - 0.1092x_2 + 0.3532, \quad \textit{for } x \in \mathcal{R}_9 \\
u =& -0.4475x_1 - 0.1092x_2 + 0.8272, \quad \textit{for } x \in \mathcal{R}_{10}
\end{aligned}$$

Using Theorem 6.1, a sampling time for discrete time implementation of the proposed PWA controller can be computed so that the closed loop sampled data system converges to a bounded invariant set. In this example, we consider η *and* γ *as optimization parameters. However, to provide a larger upper bound on* Δ_K*, we require that* $\eta > \gamma$ *and* $\gamma > 1$*. Now, solving an optimization problem to maximize* τ_M *subject*

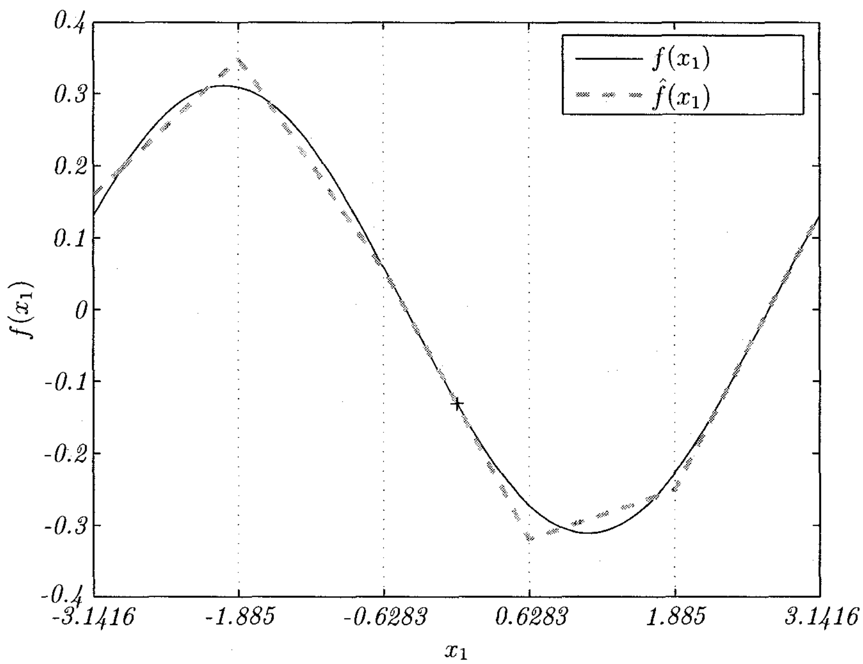

Figure 6.2: PWA function - Helicopter model

to the constraints of Theorem 6.1 and $\eta > \gamma > 1$*, one has*

$$\tau_M^\star = 0.1465, \ \eta = 4.2403, \ \gamma = 4.2403 \tag{6.58}$$

$$P = \begin{bmatrix} 30.4829 & 2.4706 \\ 2.4706 & 4.4771 \end{bmatrix}, \ R = \begin{bmatrix} 44.9622 & 9.0745 \\ 9.0745 & 3.1994 \end{bmatrix}, \tag{6.59}$$

$$X = \begin{bmatrix} 499.9799 & 11.6429 \\ 11.6429 & 24.1825 \end{bmatrix} \tag{6.60}$$

Figure 6.1 shows the trajectories of the nonlinear model (6.55) in feedback connection with the continuous time PWA controller. The trajectories of a sampled data PWA controller with a sampling time of 0.1465 *second is shown in Figure 6.1.*

6.4 Conclusions

This chapter has presented stability results for closed-loop sampled-data PWA systems under state feedback. PWA sampled-data systems were considered as delay

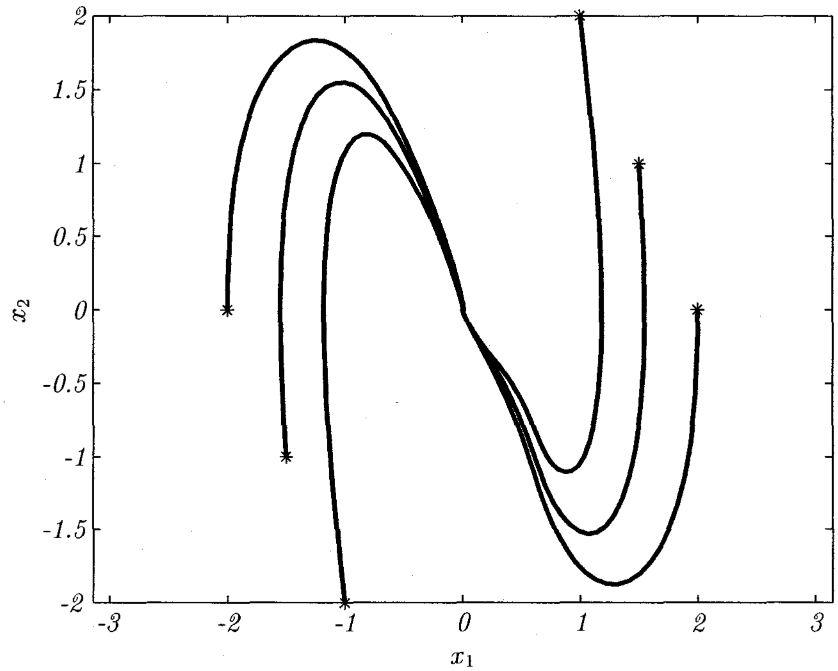

Figure 6.3: Trajectories of the nonlinear Helicopter model - continuous time PWA controller

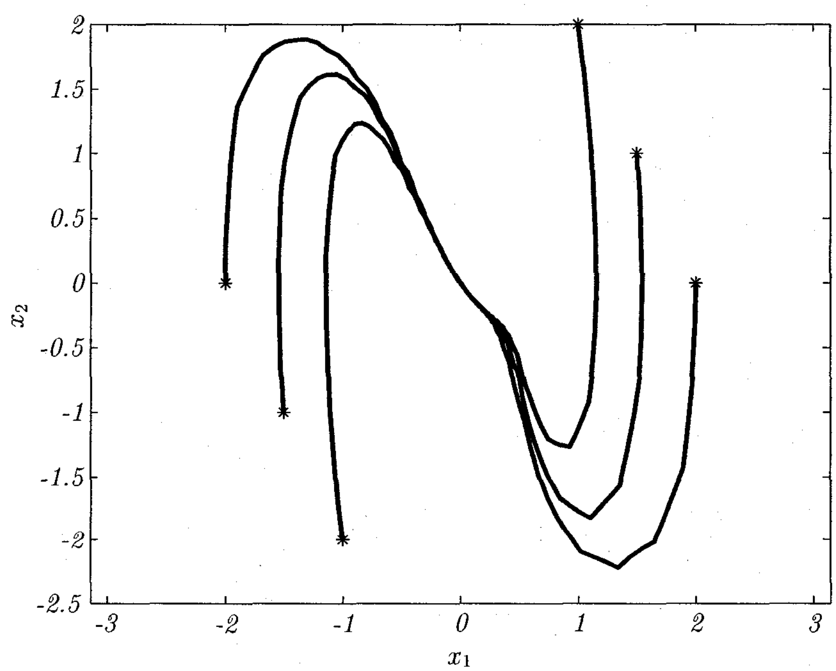

Figure 6.4: Trajectories of the nonlinear Helicopter model - sampled data PWA controller

systems with variable delay. The result for PWA systems is equivalent to the non-fragility of the continuous-time PWA controller when the sampling time converges to zero.

Chapter 7

Conclusions

The contributions of this thesis are summarized in this chapter. Potential extensions of the proposed methods are then discussed. Finally, open problems for future research are presented. The focus of this thesis has been to develop efficient *computational* controller synthesis methods for PWP/PWA systems. In the following, the fundamental questions raised in Chapter 1 are revisited considering the contributions of this work:

- How can PWA controllers be designed to keep the performance of linear controllers in a neighborhood of the equilibrium point, while guaranteeing a larger region of attraction?

  Chapter 3 proposed a two-step synthesis method to achieve both local performance and global stability for nonlinear systems that can be *bounded* by PWA differential inclusions. In this method, a local robust linear controller is first designed for a neighborhood of the desired equilibrium point to satisfy a local performance requirement. The local linear controller is then extended to a PWA controller to globally stabilize the nonlinear system. The proposed method iscast as a set of BMIs and is not a convex problem. An open problem for future research is: Can the PWA extension of linear controllers be formulated as a convex problem?

- Is it possible to formulate the PWA/PWP controller synthesis as a convex optimization problem?

  There are two contributions in this work toward answering this question: PWA controller synthesis for PWA slab system in chapter 4 and PWP controller synthesis for PWP systems in strict feedback form in chapter 5.

  - In chapter 4, an interesting duality relation was revealed in the LMIs describing sufficient conditions for the stability of PWA slab differential inclusions. This concept was then employed to find a duality relation for the L_2-gain design. As a result, the definition of the regions of a PWA slab system was extended, the L_2-gain controller design was formulated as a set of LMIs and this design method was extended to PWA slab systems with an output that is also a PWA function of the state. The new method presented in chapter 4 enables stability and performance analysis, as well as controller synthesis, for PWA slab systems as a solution of convex optimization problems. The new concept of *dual parameter set* was the basis of the development of convex controller synthesis for PWA slab systems. However, the dual parameter set of PWA slab system does not necessarily define a PWA system. The open problem is: What is the dual of a PWA system?
  - In chapter 5, the strict feedback form for PWP systems was introduced. Backstepping controller synthesis for this large class of PWP systems was formulated as an SOS program, which is a convex optimization problem. The synthesis problem was addressed in two cases: SOS Lyapunov functions for PWP systems with discontinuous vector fields and PWP Lyapunov functions for PWP systems with continuous vector fields. One of

the main advantages of the proposed method is that it addresses PWP systems with discontinuous vector fields regardless of possible attractive sliding modes. A very useful extension of the proposed method is backstepping controller synthesis for PWP differential inclusions.

- For a sampled-data implementation of a continuous-time PWA controller, how large can the sampling time be?

  Chapter 6 presented stability analysis results for closed-loop sampled-data PWA systems under state feedback. These results were obtained by considering PWA sampled-data systems as delay systems with a variable delay. In chapter 6, it is assumed that a PWA controller is given and then the maximum sampling time is computed as the solution of a convex problem. An interesting extension would be a convex PWA controller synthesis method to guarantee stability of the closed-loop sampled-data system for a given sampling time.

Based on the previous observation, the proposed extensions of the current research are as follows:

1. To develop a backstepping PWP controller synthesis method for PWP differential inclusions in strict feedback form

2. To develop a convex PWA controller synthesis method to guarantee stability of the closed-loop sampled-data system for a given sampling time

Finally, two fundamental open problems to be solved are

1. Can general PWP/PWA controller synthesis be converted to a convex problem?

2. What is the dual of a PWA system?

Analysis and synthesis of PWS systems thus seems to be a very rich field of study.

$\sim$

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