On the moments of central values of modular L-functions

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ABSTRACT

On the moments of central values of modular L-functions

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The thesis studies the integer-power moments of the central values of families of modular L-functions. The two families under consideration in the thesis are those quadratic twists of a L-function associated with a cusp form and L-functions of a Hecke-basis of the space of cusp forms. Appropriate moment estimates are derived for each family. Applications of the derived estimates are given.

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NOTATIONS

We follow the standard practices in analytic number theory. When it is written $f \ll g$, it means that there exists a positive constant C such that $|f| \leq C|g|$. The notation $f \ll_a g$ means that there exists a constant C which depends only on a and $|f| \leq C|g|$.

Contents

Chapter 1

Introduction

The thesis studies the integer-power moments of the central values of families of modular L-functions. The two families under consideration in the thesis are the quadratic twists of an L-function associated with a cusp form and L-functions of a Hecke-basis of the space of cusp forms.

The plan of the thesis is the following: chapter one gives a historic survey on the subject. In chapter two, we review background material that is needed for the chapters that are to follow. The task of estimating the first moment for the quadratic family is carried out in chapter four. Here an asymptotic estimate is derived. The task of estimating the second and higher moments for the quadratic family is carried out in chapter three. Here sieve techniques are the guiding principles in deriving any reasonably good upper bound estimates. Chapter five gives applications which makes use of the moment results developed in the previous chapters. They include a non-vanishing result (section 5.1) and a zero density estimate (section 5.2). In Chapter six, we consider a different family, namely L-functions associated with a Hecke-basis of cusp forms. In such an instance, we provide asymptotic estimates on the first and second moment. For higher moments, only upper bounds are given.

1.1 Analytic Results

The moment-type results are intimately connected with the study of non-vanishing of L-functions at the critical point. Analytic results were first proved by K. Murty and R. Murty [16], and Iwaniec [6]. Earlier on, Kolyvagin [14] proved a result which states that the group of rational points on an elliptic curve E is finite if $L(1,E) \neq 0$ and the L-function $L(s,E,\chi_d)$ twisted by a suitable quadratic character has a simple zero at $s = 1$. The latter condition was subsequently proved to hold true for infinitely many discriminants d by D. Bump, S. Friedberg and J. Hoffstein [3], and K. Murty, R.Murty [16] independently. The method used in [16] is classical. They established in their paper a mean-values estimate on the derivative of L-series attached to the elliptic curve. Precisely, let $L(s,E)$ be the L-function associated with an elliptic curve E and $L(s,E,\chi_d)$ the quadratic twists of $L(s,E)$ by χ_d. Suppose $L(1,E) \neq 0$. Then over a suitable family of quadratic twists in the range of $[0,Y]$, they established

$$\sum_d L'(1,E,\chi_d) = CY\log Y + o(Y\log Y),$$

where C is non-zero and depends only on E. This theorem clearly implies the existence of infinite many discriminants such that $L(s,E,\chi_d)$ has a simple zero at $s = 1$. Iwaniec [6] subsequently gave a precise estimate on the number of non-vanishing of $L'(1,E,\chi_d)$. He showed

$$\sum_d L'(1, E, \chi_d) = \alpha Y \log Y + \beta Y + O(Y^{13/14+\epsilon}),$$

$$\sum_d |L'(1, E, \chi_d)|^4 \ll Y^{2+\epsilon}.$$

where $\alpha \neq 0$ and β only depends on E. Using these estimates and Cauchy's inequality, he then shows using the above estimates $L'(1, E, \chi_d) \neq 0$ for at least $Y^{2/3-\epsilon}$ real primitive quadratic characters. This result was later improved to $Y^{1-\epsilon}$ by Perelli and Pomykala [19].

Here the philosophy is clear: one has non-vanishing results if one is able to give good moment estimates on the specific family at hand. This idea has inspired the subsequent coming of many other papers [17, 19, 11].

In this thesis, we use this framework to study the non-vanishing of modular L-functions at the critical point (instead of the derivative of L-functions) by giving the appropriate moment estimates. The family consists of the quadratic twists of a given L-function, $L(s, f \otimes \chi_d)$. And we study not only L-functions associated with elliptic curves, but for any cusp newform of arbitary level and arbitary weight with nebentypus (i.e. $f \in S_k(\Gamma_0(N), \chi)$).

1.2 Interpolation Formula

It is a remarkable fact that the central critical values of modular L-functions are related to the Fourier Coefficients of half-integral weight cusp forms. Here one should mention the theorems of Waldspurger [22] and Kohnen [13].

To state Waldspurger's result, for every fundamental discriminant D define D_0 by

$$D_0 = \begin{cases} |D|, & \text{if } D \text{ is odd,} \\ |D|/4, & \text{if } D \text{ is even.} \end{cases}$$

Theorem 1.1. *If $f(z) = \sum_{n\geq 1} a(n)e(nz) \in S_{2k}^*(\Gamma_0(4N))$ is an even weight newform (see section 2.1 for definition) and $\delta \in \{\pm 1\}$ is the sign of the functional equation $L(s, f)$, then there exists a positive integer M with $M|N$, a Dirichlet character χ modulo $4M$, a nonzero complex number Ω_f, and a nonzero half-integral weight Hecke eigenform*

$$g_f(z) = \sum_{n=1}^{\infty} b_f(n)e(nz) \in S_{k+\frac{1}{2}}(\Gamma_0(4M), \chi)$$

with the property that there are arithmetic progressions of fundamental discriminants D coprime to $4N$ for which $\delta D > 0$ and

$$b_f(D_0)^2 = \epsilon_D \cdot \frac{L(k, f \otimes \chi_D) D_0^{k-\frac{1}{2}}}{\Omega_f}, \tag{1.1}$$

where ϵ_D is algebraic and depends only on D. For all other D with $\delta D > 0$, we have $b_f(D_0) = 0$. Moreover, there is a fixed number field K finite degree over $\mathbb{Q}$ such that the coefficients $a(n)$, $b_f(n)$ and the values of χ are in $\mathcal{O}_K$, the ring of integers of K. In addition, if $p \nmid 4N$ is prime, then

$$\lambda(p) = \chi(p)a(p),$$

where $\lambda(p)$ is the eigenvalue of $g_f(z)$ for the half-integral weight Hecke operator T_{p^2}.

In Waldspurger's theorem, the constant in (1.1) is not explicit. Kohnen's theorem [13] is more precise in the sense that the constant in his formula is explicit.

Theorem 1.2. *Let N be odd and square-free and suppose further*

$$g(z) = \sum_{n\geq 1} b(n)e(nz) \in S^*_{k+\frac{1}{2}}(\Gamma_0(4N), \chi)$$

is a Kohnen newform (see [12]). Let $f(z) = \sum_{n\geq 1} a(n)e(nz) \in S^*_{2k}(\Gamma_0(4N))$ be the unique even weight newform under Shimura's correspondence. If $l|N$ is prime, then let $\lambda_l \in \{\pm 1\}$ be the eigenvalue of the Atkin-Lehner involution

$$f|W(Q_l)(z) = \lambda_l f(z).$$

If $(-1)^k D > 0$ and D had the property $\left(\frac{D}{l}\right) = \lambda_l$ for each prime $l|N$, then

$$L(k, f\otimes \chi_d) = \frac{< f, f > \cdot \pi^k}{2^{\nu(N)}(k-1)!|D|^{k-\frac{1}{2}} < g, g >} \cdot |b(|D|)|^2.$$

For all other fundamental discriminants D with $(-1)^k D > 0$ we have $b(|D|) = 0$.

1.3 Random Matrix Theory

Modeling families of L-functions by using the choices of random matrix ensembles suggested by Katz and Sarnak [18], Keating and Snaith and other people gave very precise conjectures on the moments of central values of L-functions (see [9, 10]).

For example, the family of Dirichlet L-functions $L(s,\chi)$ as χ varies over primitive characters modulo q is a unitary family, and it is conjectured that as $q \to \infty$

$$\sum_{\chi \mod q}^{*} |L(1/2,\chi)|^{2k} \sim C_k\, q(\log q)^{k^2}$$

for positive integer k and C_k is a specified constant. Here the summation is over all primitive characters modulo q.

The family of quadratic Dirichlet L-functions $L(s,\chi_d)$, where d is a fundamental discriminant and χ_d is the associated quadratic character, is a symplectic family and it is conjectured that as $X \to \infty$

$$\sum_{|d| \leq X} L(1/2, \chi_d)^k \sim A_k\, X (\log X)^{k(k+1)/2}$$

for positive integer k and A_k is a specified constant.

The family of quadratic twists of a given newform f, $L(s, f \otimes \chi_d)$ is an orthogonal family and it is conjectured that as $X \to \infty$

$$\sum_{|d| \leq X} L(1/2, f \otimes \chi_d)^k \sim B_k\, X (\log X)^{k(k-1)/2}$$

for positive integer k and B_k is a specified constant.

The two families we study in the thesis differ from the three families presented above. There are no currently known moment-type conjectures regarding the two families in the thesis.

Chapter 2

Modular forms and L-functions

We briefly review some basic properties of classical modular forms. The reader can consult standard texts such as [7] for more details.

2.1 Classical Modular forms

In this thesis, we are dealing with modular forms of integer weights over a congruence subgroup. By saying f is a modular form of weight $k \in \mathbb{Z}$, level N and nebentypus χ, it is understood that f satisfies the following conditions:

1. f is an analytic function defined on the upper half plane $\mathbb{H}$;
2. $f(\gamma z) = (cz+d)^k \chi(d) f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$;
3. f is holomorphic at all cusps of $\Gamma_0(N)$.

Furthermore, we say f is a cusp form if it vanishes at the cusps of $\Gamma_0(N)$.

If f is a modular form, it then has a Fourier series expansion at each cusp. Especially important is the Fourier expansion of f at the cusp ∞

$$f(z) = \sum_{n\geq 0} a(n)e(nz), \quad e(z) = e^{2\pi iz}.$$

When f is a cusp form, then in the above expansion $a(0) = 0$.

Modular forms of level N, weight k and nebentypus χ form a vector space which is denoted by $M_k(\Gamma_0(N), \chi)$, and the cusp forms a subspace denoted by $S_k(\Gamma_0(N), \chi)$. The vector space $M_k(\Gamma_0(N), \chi)$ is finite dimensional. Moreover, $S_k(\Gamma_0(N), \chi)$ is a finite dimensional Hilbert space with the Petersson inner product

$$< f, g >= \int_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{g(z)}y^{k-2}dxdy.$$

Indispensable from the theory of modular forms is the concept of Hecke operators. Fix positive integers k, N. For each $n \geq 1$ the operator T_n (called nth Hecke operator) acts on the space $M_k(\Gamma_0(N), \chi)$. It is defined by the formula:

$$f|T_n(z) = \frac{1}{n}\sum_{ad=n}\chi(a)a^k\sum_{0\leq b<d} f\left(\frac{az+b}{d}\right).$$

The Hecke operators are linear, and they are multiplicative in the sense that for any $n,m \geq 1$

$$T_nT_m = \sum_{d|(n,m)} \chi(d)d^{k-1}T_{\frac{mn}{d^2}}. \tag{2.1}$$

Suppose $(n, N) = 1$. Then the operator T_n is normal on the Hilbert space $S_k(\Gamma_0(N), \chi)$ with respect to the Petersson inner product. More precisely for $f, g \in S_k(\Gamma_0(N), \chi)$ we have

$$< f|T_n, g >= \chi(n) < f, g|T_n > .$$

Thus, by standard linear algebra, one is able to find an orthonormal basis of the space $S_k(\Gamma_0(N), \chi)$ which consists of eigenfunctions for all the Hecke operators T_n with $(n, N) = 1$. If one wishes to remove the last condition $(n, N) = 1$, the theory of newforms initiated by Atkin and Lehner [1] intervenes. In this theory, one can decompose the space of cusp forms into

$$S_k(\Gamma_0(N), \chi) = S^?(N, \chi) \oplus S^*(N, \chi).$$

$S^?(N, \chi)$ is called the space of oldforms and its orthogonal complement $S^*(N, \chi)$ in $S_k(\Gamma_0(N), \chi)$ is the subspace of newforms. The space of oldforms and the space of newforms are stable under the Hecke operators T_n with $(n, N) = 1$. Therefore, each of them has an orthonormal basis consisting of eigenfunctions of the Hecke operators T_n with $(n, N) = 1$. The Hecke eigenforms of $S^*(N, \chi)$ are called newforms. One of the main result here is that: a newform is an eigenfunction for all the Hecke operators. A newform is usually normalized so that all its Fourier coefficients coincide with the eigenvalues of the Hecke operator T_n. Besides the Hecke operators T_n, a newform is also an eigenfunction for another important operator, namely the Fricke involution $\overline{W}$, which is defined by

$$\overline{W} = W \circ K$$

where

$$f|W(z) = N^{-k/2} z^{-k} f\left(\frac{-1}{Nz}\right), \tag{2.2}$$

$$f|K(z) = \overline{f(-\overline{z})}. \tag{2.3}$$

Let f be a newform and η its eigenvalue under the action of $\overline{W}$. The eigenvalue η is an important invariant in what follows in the thesis. It is complex with absolute value 1. And only in special cases, one can compute it explicitly.

Proposition 2.1. *Let χ be a primitive character of conductor N. Let f be a newform in $S_k(\Gamma_0(N), \chi)$ and η its eigenvalue by the Fricke involution. Then*

$$\eta = \tau(\overline{\chi})\lambda_N N^{-k/2}.$$

Here λ_N is the Nth Fourier coefficient of f and $\tau(\chi)$ denotes the usual Gauss sum associated with the character χ.

Proposition 2.2. *Let N be squarefree, and χ trivial. Then η is given by*

$$\eta = \mu(N)\lambda_N N^{1-k/2}.$$

Given a modular form, one may produce new modular forms by means of twisting. More precisely

Proposition 2.3. *Let $f \in M_k(\Gamma_0(N), \chi)$ be a modular form with Fourier coefficients a_n. Let N^* be the conductor of the Dirichlet character χ and let ψ be a primitive Dirichlet character modulo r. Let $f \otimes \psi$ be the function on $\mathbb{H}$ given by the Fourier expansion*

$$(f \otimes \psi)(z) = \sum_{n \geq 0} \psi(n) a_n e(nz).$$

Then $f \otimes \psi$ is also a modular form, more precisely $f \otimes \psi \in M_k(\Gamma_0(q), \chi\psi^2)$, where q is the least common multiple of N, N^*r and r^2. If f is a cusp form, then so is $f \otimes \psi$.

2.2 Hecke L-functions

Consider $f \in M_k(\Gamma_0(N), \chi)$, so f has the Fourier expansion at ∞

$$f(z) = \sum_{n \geq 0} a_n e(nz).$$

One defines its Hecke L-function to be the Dirichlet series

$$L(s, f) = \sum_{n \geq 1} \frac{a_n}{n^s}$$

and the completed L-function

$$\Lambda(s, f) = \left(\frac{\sqrt{N}}{2\pi} \right)^s \Gamma(s) L(s, f).$$

Bounding a_n trivially (see section 2.4), $L(s, f)$ is absolutely convergent for $\mathrm{Re}(s) > \frac{k}{2}$. Regarding the analyticity of $L(s, f)$, Hecke proved

Theorem 2.1 (Hecke). *With notations above, $L(s, f)$ has a meromorphic continuation to the whole complex plane and the completed L- function satisfies the functional equation*

$$\Lambda(s, f) = i^k \Lambda(k - s, f|W).$$

Moreover, $L(s, f)$ is entire if f is a cusp form, and otherwise it has only a simple pole at $s = k$.

Recall W is the operator defined earlier (2.2). The proof is very much like Riemann's proof of the functional equation of $\zeta(s)$.

Proof. By the definition of the gamma function we have for $n \geq 1$,

$$\left(\frac{\sqrt{N}}{2\pi} \right)^s \Gamma(s) n^{-s} = \int_0^\infty e^{-2\pi n y / \sqrt{N}} y^s \frac{dy}{y},$$

so in the region of absolute convergence we have the representation

$$\Lambda(s,f)=\int_0^\infty \left(f\left(\frac{iy}{\sqrt{N}}\right)-a_0\right)y^s\frac{dy}{y}.$$

Here we split the integral into the part from 1 to ∞ and the part from 0 to 1 and we transform the latter as follows

$$\begin{aligned}\int_0^1 \left(f\left(\frac{iy}{\sqrt{N}}\right)-a_0\right)y^s\frac{dy}{y} &= \int_1^\infty \left(f\left(\frac{i}{y\sqrt{N}}\right)-a_0\right)y^{-s}\frac{dy}{y}\\ &= i^k\int_1^\infty \left(f|W\left(\frac{iy}{\sqrt{N}}\right)-a_0\right)y^{k-s}\frac{dy}{y}.\end{aligned}$$

Adding both parts we obtain the integral representation

$$\Lambda(s,f)=\int_1^\infty \left(f\left(\frac{iy}{\sqrt{N}}\right)-a_0\right)y^s\frac{dy}{y}+i^k\int_1^\infty \left(f|W\left(\frac{iy}{\sqrt{N}}\right)-a_0\right)y^{k-s}\frac{dy}{y}.$$

Since $f-a_0$ decays exponentially at infinity, the meromorphic continuation follows, and the functional equation is then clear. □

Corollary 2.1. *Let f be a normalized newform in $S_k(\Gamma_0(N),\chi)$ with Fourier expansion*

$$f(z)=\sum_{n\geq 1}\lambda_n e(nz).$$

Then the Hecke L-function of f has an Euler product expansion

$$L(s,f)=\prod_p\left(1-\lambda_p+\chi(p)p^{k-1-2s}\right)^{-1}$$

and has analytic continuation to an entire function. The completed L-function

$$\Lambda(s,f)=\left(\frac{\sqrt{N}}{2\pi}\right)^s\Gamma(s)L(s,f)$$

satisfies the functional equation

$$\Lambda(s, f) = i^k \overline{\eta} \Lambda(k - s, \overline{f})$$

Proof. The functional equation is a direct consequence of Theorem 2.1 and the fact f is an eigenfunction of $\overline{W}$ with eigenvalue η. Using (2.1), we may write

$$\sum_{n\geq 1} \frac{T_n}{n^s} = \prod_p \left(1 - T_p p^{-s} + \chi(p) p^{k-1-2s}\right)^{-1}.$$

The Euler product of $L(s, f)$ now follows by the above expansion and together with the fact $f|T_n = \lambda_n f$, $n \geq 1$. $\square$

Proposition 2.3 tells us how to get new modular forms by twisting the old one with characters. The following proposition tells us when one may get newforms by twisting.

Proposition 2.4. *Let f be newform in $S_k(\Gamma_0(N), \chi)$ and ψ a primitive Dirichlet character modulo r with $(r, N) = 1$ Then $f \otimes \psi$ is a newform of level Nr^2 and the Hecke L-function of $f \otimes \psi$ is entire and polynomially bounded in vertical strips. Moreover, the completed L-function satisfies the functional equation*

$$\Lambda(s, f \otimes \psi) = i^k w \overline{\eta_f} \Lambda(k - s, \overline{f} \otimes \overline{\psi})$$

where η_f is the eigenvalue of f for the operator $\overline{W}$, and the root number w depends only on χ and ψ, namely

$$w = \chi(r)\psi(N)\frac{\tau(\psi)^2}{r}.$$

2.3 Approximate Functional Equation

The approximate functional equation gives an analytic expression for $L(s, f)$ inside the critical strip $0 \leq \mathrm{Re}(s) \leq k$.

Theorem 2.2. *Let $f = \sum a_n e(nz)$ be a newform in $S_k(\Gamma_0(N), \chi)$. Let $G(u)$ be any function which is holomorphic and bounded in the strip $-2k < \mathrm{Re}\, u < 2k$, even, and normalized by $G(0) = 1$. Let $X > 0$. Then for any $s = \sigma + it$ in the strip $0 \leq \sigma \leq k$ we have*

$$L(s, f) = \sum_{n \geq 1} \frac{a_n}{n^s} V_s\left(\frac{2\pi n}{X\sqrt{N}}\right) + i^k \overline{\eta} \left(\frac{\sqrt{N}}{2\pi}\right)^{k-2s} \sum_{n \geq 1} \frac{\overline{a_n}}{n^{k-s}} V_{k-s}\left(\frac{2\pi n X}{\sqrt{N}}\right) \quad (2.4)$$

where $V_s(y)$ is a smooth function defined by

$$V_s(y) = \frac{1}{2\pi i} \int_{(\alpha)} y^{-u} \frac{\Gamma(s+u)}{\Gamma(s)} G(u) \frac{du}{u} \quad (2.5)$$

with a fixed α satisfying $1 + k/2 < \alpha < 2k$.

Proof. Consider the integral

$$I(X, s, f) = \frac{1}{2\pi i} \int_{(\alpha)} X^u \Lambda(s+u, f) G(u) \frac{du}{u}$$

The integral exists because $\Lambda(\sigma + it, f)$ decays exponentially at infinity for fixed σ. By the same reason, one can move the line of integration to $\mathrm{Re}(u) = -\alpha$. Thus

$$\Lambda(s, f) = I(X, s, f) - \frac{1}{2\pi i} \int_{(-\alpha)} X^u \Lambda(s+u, f) G(u) \frac{du}{u}$$

where $\Lambda(s, f)$ comes from the residue of the simple pole of $G(u)/u$ at $u = 0$. We now perform a change of variable in the above integral and invoke the functional equation for f, this yields

$$\Lambda(s,f) = I(X,s,f) + i^k \overline{\eta} I(X^{-1}, k-s, \overline{f}). \tag{2.6}$$

Since α is chosen so that the Dirichlet series defining $L(s,f)$ is absolutely convergent in such a region, we have

$$I(X,s,f) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \sum_{n\geq 1} \frac{a_n}{n^s} \frac{1}{2\pi i} \int_{(\alpha)} \left(\frac{2\pi n}{\sqrt{N}X}\right)^{-u} \Gamma(s+u) G(u) \frac{du}{u}$$

$$I(X^{-1}, k-s, \overline{f}) = \left(\frac{\sqrt{N}}{2\pi}\right)^{k-s} \sum_{n\geq 1} \frac{\overline{a_n}}{n^{k-s}} \frac{1}{2\pi i} \int_{(\alpha)} \left(\frac{2\pi n X}{\sqrt{N}}\right)^{-u} \Gamma(k-s+u) G(u) \frac{du}{u}.$$

Put them back in (2.6) and divide both sides by $(\sqrt{N}/2\pi)^s \Gamma(s)$, the theorem is proved. $\square$

For a suitable test function $G(u)$, both sums in (2.4) are effectively limited to the terms with $n \ll |s|$. We shall see this with a particular choice of $G(u)$

$$G(u) = \cos\left(\frac{\pi u}{4kA}\right)^{-4kA} \quad \text{with} \quad A > 2k.$$

Proposition 2.5. *With notations above. The derivatives of $V_s(y)$ satisfies*

$$y^a V_s^a(y) = \delta_a + O\left(\frac{y}{|s|+1}\right)^A$$

$$y^a V_s^a(y) \ll \left(1 + \frac{y}{|s|+1}\right)^{-A}$$

where $\delta_0 = 1$, $\delta_a = 0$, if $a > 0$. The implied constant depends only on a and A.

Proof. We have the formula

$$y^a V_s^a(y) = \frac{1}{2\pi i} \int_{(\alpha)} y^{-u} (-u)^a \frac{\Gamma(s+u)}{\Gamma(s)} G(u) \frac{du}{u}.$$

If $a = 0$ this becomes

$$V_s(y) = \frac{1}{2\pi i}\int_{(a)} y^{-u}\frac{\Gamma(s+u)}{\Gamma(s)}G(u)\frac{du}{u}.$$

Since A is chosen so that $G(u)$ is holomorphic with $-A \leq \mathrm{Re}\, u \leq A$, we may move the line of integration to $(-A)$, thus

$$V_s(y) = 1 + \frac{1}{2\pi i}\int_{(-A)} y^{-u}\frac{\Gamma(s+u)}{\Gamma(s)}G(u)\frac{du}{u}.$$

where the main term 1 comes from the residue of the simple pole of $G(u)/u$ at $u = 0$. Using the bounds

$$G(u) \ll e^{-\pi|u|}$$

$$\frac{\Gamma(s+u)}{\Gamma(s)} \ll (|s|+1)^{\mathrm{Re}\, u}\exp(\frac{\pi}{2}|u|)$$

we conclude

$$V_s(y) = 1 + \mathrm{O}\left(\frac{y}{|s|+1}\right)^A.$$

If $a > 0$, one sees this by using the same contour,

$$y^a V_s^a(y) = \int_{(-A)} = \mathrm{O}\left(\frac{y}{|s|+1}\right)^A.$$

The proof for the second assertion is similar, one shifts the contour to the line (A) in this case. □

Using the same ideas as used in the proof of Theorem 2.2, one may derive an approximate functional equation for the mth power of $L(s, f)$ where m is any positive integer. Such a result is needed later.

Theorem 2.3. *With the notations above, for any given positive integer m,*

$$L^m(s,f) = \sum_{n\geq 1} \frac{b_n}{n^s} V_s\left(\frac{(2\pi)^m n}{XN^{m/2}}\right) + (i^k\overline{\eta})^m \left(\frac{\sqrt{N}}{2\pi}\right)^{m(k-2s)} \sum_{n\geq 1} \frac{\overline{b_n}}{n^{k-s}} V_{k-s}\left(\frac{(2\pi)^m nX}{N^{m/2}}\right) \tag{2.7}$$

where

$$b_n = \sum_{k_1 k_2 \cdots k_m = n} a_{k_1} a_{k_2} \cdots a_{k_m} \tag{2.8}$$

$$V_s(y) = \frac{1}{2\pi i} \int_{(\alpha)} y^{-u} \left(\frac{\Gamma(s+u)}{\Gamma(s)}\right)^m G(u) \frac{du}{u} \tag{2.9}$$

with a fixed α satisfying $1 + k/2 < \alpha < 2k$.

2.4 Bounds for Fourier Coefficients

Let $f = \sum a_n e(nz)$ be a cusp form in $S_k(\Gamma_0(N), \chi)$. We are interested in estimating the size of a_n. We begin by stating a criterion for the cusp form. See chapter five of [7].

Lemma 2.1. *Suppose f is a modular form for the group Γ. Then f is a cusp form if and only if $(\operatorname{Im} z)^{k/2}|f(z)|$ is bounded in the upper-half plane.*

By the Parseval identity and the lemma above

$$\sum_n |a_n|^2 e^{-4\pi ny} = \int_0^1 |f(z)|^2 dx \ll y^{-k},$$

whence

$$\sum_{n\leq N} |a_n|^2 \ll y^{-k} e^{4\pi Ny}$$

for any $y > 0$. Choosing $y = N^{-1}$, we obtain

Theorem 2.4.

$$\sum_{n\leq N}|a_n|^2 \ll N^k.$$

Remark 2.1. The upper bound above is the best possible, for one can prove using the Rankin-Selberg method that

$$\sum_{n\leq N}|a_n|^2 \sim cN^k,$$

where c is a positive constant depending on f. The theorem also shows that for any individual coefficient it yields

$$a_n \ll n^{k/2}.$$

In fact, we have the bounds(originally known as the Ramanujan-Petersson conjecture)

$$a_n \ll \tau(n)n^{\frac{k-1}{2}} \tag{2.10}$$

by the work of Deligne [4]. Here $\tau(n)$ is the number of divisors of n.

Using Cauchy's inequality, we deduce the following estimate from Theorem 2.4

Corollary 2.2. *For any $N \geq 1$ we have*

$$\sum_{n\leq N}|a_n| \ll N^{\frac{k+1}{2}}.$$

The bound can be improved if we drop the absolute value to allow cancellation between the terms.

Theorem 2.5. *For any real α and $N \geq 1$ we have*

$$\sum_{n \leq N} a_n e(\alpha n) \ll N^{k/2} \log 2N$$

where the implied constant depends only on f (not on α).

We also need the following result in chapter three, the proof of which can be found in [6] (Lemma 1).

Proposition 2.6. *Let α be real and ψ be a periodic function of period r. We then have*

$$\sum_{n \leq N} a_n \psi(n) e(\alpha n) \ll \Psi N \log N,$$

where

$$\Psi = \frac{1}{r} \sum_{a \pmod r} \left| \sum_{b \pmod r} \psi(b) e\left(\frac{ab}{r}\right) \right|.$$

Moreover, if $|\psi| \leq 1$ and s is a positive integer then we have

$$\sum_{\substack{n \leq N \\ (n,s)=1}} a_n \psi(n) e(\alpha n) \ll \tau(s) r^{1/2} N \log N$$

and

$$\sideset{}{^\flat}\sum_{\substack{n \leq N \\ (n,s)=1}} a_n \psi(n) e(\alpha n) \ll \tau(s) r^{1/2} N (\log N)^7$$

where the last summation is over squarefree positive integers.

Chapter 3

Bounding the Second and Higher moments

We prove upper bounds for the power moments of the family $L(s, f \otimes \chi_d)$. In Section 3.1, the concept of the large sieve is introduced. Using such techniques, we then give upper bounds for all the even moments of the family $L(s, f \otimes \chi_d)$. This is achieved in Section 3.2. In Section 3.3, a refined estimate for the second moment is derived.

3.1 Large Sieve inequalities

Large sieve inequalities are inequalities of the type

$$\sum_{x \in \mathfrak{X}} \left| \sum_{n \leq N} a_n x(n) \right|^2 \leq C(\mathfrak{X}, N) \|a\|^2$$

for any complex numbers a_n, where $\|a\|^2 = \sum |a_n|^2$.

Let $\mathfrak{X}$ consist of all primitive Dirichlet characters of modulus $q \leq Q$, our first example of large sieve type of inequalities is the following theorem of Bombieri and Davenport [2]

Theorem 3.1. *For any complex numbers a_n with $M < n \leq M+N$, we have*

$$\sum_{q\leq Q} \frac{q}{\phi(q)} \sum_{\chi \pmod q}^{*} \left| \sum_{M<n\leq M+N} a_n \chi(n) \right|^2 \leq (Q^2 + N - 1)\, \|a\|^2$$

where inner sum is over all primitive characters (mod q).

There are remarkable consequences of this theorem.

If we restrict $\mathfrak{X}$ to primitive quadratic characters with conductors at most Q, here is a powerful result of Heath-Brown [5].

Theorem 3.2. *For any complex numbers a_n,*

$$\sum_{|d|\leq Q} \left| \sum_{n\leq N} a_n \chi_d(n) \right|^2 \ll Q^{\epsilon} N^{1+\epsilon} (Q+N) \max_{n\leq N} |a_n|^2.$$

The implied constant depends only on ϵ.

As an application, we shall in Section 3.3 use Heath-Brown's inequality to prove an upper bound of correct order of magnitude for the second moment.

The Fourier coefficients of cusp forms, or better the eigenvalues of Hecke operators in the space of cusp forms, are analogues of Dirichlet characters. Let $\mathcal{F}$ be an orthonormal basis of $S_k(\Gamma_0(q), \chi)$. Let $f(z) = \sum a_f(n) e(nz)$ be the Fourier expansion of f at ∞. The following large sieve inequality [21] is used in Chapter Five.

Theorem 3.3. *Let $k \geq 2$. Then for any complex numbers a_n we have*

$$\frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f\in\mathcal{F}} \left| \sum_{n\leq N} a_n a_f(n) n^{\frac{1-k}{2}} \right|^2 \ll \left(1 + \mathrm{O}\left(\frac{N\log N}{qk}\right)\right) \|a\|^2.$$

The implied constant is absolute.

3.2 A crude estimate

Let f be a newform in $S_k(N,\chi)$. As mentioned in the introduction, the family of L-functions of interest consists of quadratic twists of f. More precisely, let d be a fundamental discriminant such that $(d,N) = 1$. We denote by χ_d the quadratic character of conductor $|d|$. In view of Proposition 2.4, the twisted $f \otimes \chi_d$ is a newform in $S_k(\Gamma_0(Nd^2), \chi)$ and $L(s, f \otimes \chi_d)$ is entire. The critical strip of $L(s, f \otimes \chi_d)$ is centered at $\mathrm{Re}\, s = k/2$. Summing over all such d, we are interested in estimating

$$\sum_{|d|\le Q}^{*} |L(k/2, f \otimes \chi_d)|^{2m}, \tag{3.1}$$

where m is any positive integer.

Before stating our result, a few remarks on the expected magnitude of (3.1) are appropriate. If one assumes the Lindelöf hypothesis, which says

$$L(k/2, f \otimes \chi_d) \ll_\epsilon d^\epsilon, \tag{3.2}$$

one then gets

$$\sum_{|d|\le Q} |L(k/2, f \otimes \chi_d)|^{2m} \ll_\epsilon Q^{1+\epsilon}. \tag{3.3}$$

Of course, the Lindelöf hypothesis for modular L-functions is currently not known to be true. Instead if we use the Ramanujan-Petersson bound (2.10) for the Fourier coefficients of f and the approximate functional equation, a simple computation shows

$$L(k/2, f \otimes \chi_d) \ll_{f,\epsilon} d^{1/2+\epsilon}.$$

Summing over d,

$$\sum_{|d| \leq Q} |L(k/2, f \otimes \chi_d)|^{2m} \ll_{f,\epsilon} Q^{m+1+\epsilon}.$$

We prove in this section the following

Theorem 3.4. *For any $t \in \mathbb{R}$,*

$$\sum_{q \leq Q} \sum_{\psi(q)}^{*} |L(k/2 + it, f \otimes \psi)|^{2m} \ll (Q^2 + Q^m(|t| + k + 1)^m)(Q(|t| + k + 1))^{\epsilon}.$$

The implied constant depends on ϵ and f. The inner sum is over all primitive characters ψ with conductor $q \leq Q$ and $(q, N) = 1$.

An immediate corollary of the theorem above is

Corollary 3.1.

$$\sum_{|d| \leq Q}^{*} |L(k/2, f \otimes \chi_d)|^{2m} \ll \begin{cases} Q^{2+\epsilon}, & m = 1; \\ Q^{m+\epsilon}, & m \geq 2. \end{cases}$$

The strategy for the proof of the Theorem is to write $L^m(k/2+it, f \otimes \chi)$ in two finite sums using an approximate functional equation. The large sieve inequality is then used to bound the square of such sums.

Proof. Let

$$f(z) = \sum_{n \geq 1} a_n e(nz)$$

be the Fourier expansion of f at ∞. By Theorem 2.3, we may write

$$L^m(k/2+it, f\otimes\psi) \tag{3.4}$$

$$= \sum_{n\geq 1} \frac{b_n\psi(n)}{n^{k/2+it}} V_{k/2+it}\left(\frac{(2\pi)^m n}{N^{m/2}q^m}\right) + \omega_q \left(\frac{\sqrt{N}q}{2\pi}\right)^{-2imt} \sum_{n\geq 1} \frac{\overline{b_n\psi(n)}}{n^{k/2-it}} V_{k/2-it}\left(\frac{(2\pi)^m n}{N^{m/2}q^m}\right) ; \tag{3.5}$$

where

$$\omega_q = \left(i^k \overline{\eta_f}\chi(q)\psi(N)\frac{\tau^2(\psi)}{q}\right)^m$$

$$b_n = \sum_{k_1\cdots k_m = n} a_{k_1}\cdots a_{k_n}$$

$$V_s(y) = \frac{1}{2\pi i}\int_{(\alpha)} y^{-u}\left(\frac{\Gamma(s+u)}{\Gamma(s)}\right)^m G(u)\frac{du}{u}.$$

We now make the choice for $G(u)$

$$G(u) = \left[\cos\left(\frac{\pi u}{4kA}\right)\right]^{-4kmA}, \quad \text{with} \quad A > 2k$$

and deduce as in Proposition 2.5

$$V_s(y) \ll \left(1 + \frac{y}{(|s|+1)^m}\right)^{-A}. \tag{3.6}$$

Now let

$$B = Q^D(|t|+k+1)^D \quad \text{with} \quad D = \frac{2+2mA}{2A-\epsilon}.$$

In view of the approximate functional equation (3.4) and Cauchy's inequality

$$
\begin{aligned}
&\sum_{q\le Q}\ \sideset{}{^*}\sum_{\psi(q)} |L(k/2+it, f\otimes\psi)|^{2m} \\
&\ll \sum_{q\le Q}\ \sideset{}{^*}\sum_{\psi(q)} \left|\sum_{n\ge 1} \frac{b_n\psi(n)}{n^{k/2+it}} V_{k/2+it}\left(\frac{(2\pi)^m n}{N^{m/2}q^m}\right)\right|^2 && \text{(first term)} \\
&+ \sum_{q\le Q}\ \sideset{}{^*}\sum_{\psi(q)} \left|\sum_{n\ge 1} \frac{\overline{b_n\psi(n)}}{n^{k/2-it}} V_{k/2-it}\left(\frac{(2\pi)^m n}{N^{m/2}q^m}\right)\right|^2. && \text{(second term)}
\end{aligned}
$$

For the first term, truncate the sum up to B and use the large sieve inequality (Theorem 3.1)

$$
\begin{aligned}
&\sum_{q\le Q}\ \sideset{}{^*}\sum_{\psi(q)} \left|\sum_{n\le B} \frac{b_n\psi(n)}{n^{k/2+it}} V_{k/2+it}\left(\frac{(2\pi)^m n}{N^{m/2}q^m}\right)\right|^2 \\
&\ll (Q^2+B)\sum_{n\le B}\frac{|b_n|^2}{n^k} \\
&\ll (Q^2+B)B^{\epsilon} \\
&= (Q^2+Q^D(|t|+k+1)^D)(Q(|t|+k+1))^{D\epsilon}.
\end{aligned}
$$

In bounding the above coefficients b_n, we resorted to the Ramanujan-Petersson bound (2.10) and the elementary fact $\tau_k(n)\ll n^{\epsilon}$. In the remaining range $n>B$, we rely on (3.6) to get

$$\begin{aligned}
&\sum_{q\le Q}\sideset{}{^*}\sum_{\psi(q)}\left|\sum_{n>B}\frac{b_n\psi(n)}{n^{k/2+it}}V_{k/2+it}\left(\frac{(2\pi)^m n}{N^{m/2}q^m}\right)\right|^2\\
&\ll\sum_{q\le Q}\sideset{}{^*}\sum_{\psi(q)}\left|\sum_{n>B}\frac{|b_n|}{n^{k/2}}\left(1+\frac{(2\pi)^m n}{N^{m/2}q^m(|k/2+it|+1)^m}\right)^{-A}\right|^2\\
&\ll\sum_{q\le Q}\sideset{}{^*}\sum_{\psi(q)}\left|\sum_{n>B}\frac{|b_n|}{n^{k/2}}\left(\frac{n}{q^m(|t|+k+1)^m}\right)^{-A}\right|^2\\
&\ll\sum_{q\le Q}\sideset{}{^*}\sum_{\psi(q)}\left|q^{mA}(|t|+k+1)^{mA}\sum_{n>B}n^{-1/2-A+\epsilon}\right|^2\\
&\ll Q^{2+2mA+D-2DA+2D\epsilon}(|t|+k+1)^{2mA+D-2DA+2D\epsilon}\\
&\ll Q^{D+D\epsilon}(|t|+k+1)^{D+D\epsilon}.
\end{aligned}$$

Choose $A=2/\epsilon$, so $D=m+\mathrm{O}(\epsilon)$. This finishes the proof. □

3.3 The Second Moment

From Corollary 3.1, we know

$$\sideset{}{^*}\sum_{|d|\le Q}|L(k/2,f\otimes\chi_d)|^2\ll Q^{2+\epsilon}.$$

The expected value for the second moment(in fact for any moment) is $Q^{1+\epsilon}$ if one is willing to assume the Lindelöf hypothesis. That being said, we shall prove in this section the following

Theorem 3.5. *Let $k\ge 2$ and σ be fixed in $[k/2,k-1/2]$, the estimate*

$$\sideset{}{^*}\sum_{|d|\le Q}|L(\sigma+it,f\otimes\chi_d)|^2\ll\left(Q+(Q(|t|+1))^{k+1-2\sigma}\right)\left(Q(|t|+1)\right)^{\epsilon}$$

holds. The implied constant depends only on ϵ.

The crux of the proof is Heath-Brown's large sieve inequality(Theorem 3.2). We mention also that such an inequality is applied in other contexts [19, 5].

We introduce notations before the proof. Let

$$S(Q, f, s) = \sum_{Q<|d|\leq 2Q}^{*} |L(s, f\otimes\chi_d)|^2.$$

In what follows we study the sum $S(Q, f, s)$. It is straightforward to check that the theorem follows if we can prove the same bound for $S(Q, f, s)$. Let $\nu(\sigma, f)$ be the inf of the $\nu\in\mathbb{R}$ for which

$$S(Q, f, s) \ll (Q + (Q(|t|+1))^{k+1-2\sigma})(Q(|t|+1))^{\nu}. \tag{3.7}$$

We begin with a lemma.

Lemma 3.1. *The formula*

$$L(s, f\otimes\chi_d) = \sum_{n\geq 1} a_n\chi_d(n)n^{-s}e^{-n/X} - \frac{1}{2\pi i}\int_{(\alpha)} L(w, f\otimes\chi_d)\Gamma(w-s)X^{w-s}dw$$

is valid for $1/2\leq\alpha<\sigma=\operatorname{Re}s$

Proof. Consider the integral

$$\frac{1}{2\pi i}\int_{(\alpha)} L(w, f\otimes\chi_d)\Gamma(w-s)X^{w-s}dw, \quad 1/2\leq\alpha<\sigma$$

which is well-defined in view of the bound

$$\Gamma(x+iy) \ll_x e^{-|y|}$$

Moving the line of integration to $(k/2)$. We have a simple pole at $w=s$ with the residue $L(s, f\otimes\chi_d)$ and

$$\frac{1}{2\pi i}\int_{(k/2)} L(w, f\otimes\chi_d)\Gamma(w-s)X^{w-s}dw = L(s, f\otimes\chi_d)\exp(-n/X)$$

by the Mellin inversion formula. □

Proof of the Main Theorem. In view of the lemma above, we have by Cauchy's inequality

$$\begin{aligned}|L(s, f\otimes\chi_d)|^2 \ll_\sigma &\left|\sum_{n\geq 1} a_n\chi_d(n)n^{-s}e^{-n/X}\right|^2\\ &+X^{2(\alpha-\sigma)}\int_{-\infty}^{\infty}|L(\alpha+iu, f\otimes\chi_d)|^2 e^{-|u-t|}du.\end{aligned}$$

Summing over d,

$$\begin{aligned}S(Q, f, s) \ll_\sigma &\sum_{Q<|d|\leq 2Q}\left|\sum_{n\geq 1} a_n\chi_d(n)n^{-s}e^{-n/X}\right|^2\\ &+X^{2(\alpha-\sigma)}\int_{-\infty}^{\infty}S(Q, f, \alpha+iu)e^{-|u-t|}du. \qquad (3.8)\end{aligned}$$

From the functional equation(Proposition 2.4) for $L(w, f\otimes\chi_d)$, we have the bound

$$|L(\alpha+iu, f\otimes\chi_d)| \ll (Q(|u|+1))^{k-2\alpha}|L(k-\alpha-iu, \overline{f}\otimes\chi_d)|.$$

Thus running over d and using (3.7)

$$\begin{aligned}&S(Q, f, \alpha+iu)\\ \ll&(Q(|u|+1))^{2(k-2\alpha)}S(Q, \overline{f}, k-\alpha-iu)\\ \ll_{\epsilon,\alpha}&(Q(|u|+1))^{2(k-2\alpha)}(Q+(Q(|u|+1))^{-k+1+2\alpha})(Q(|u|+1))^{\nu(k-\alpha,\overline{f})+\epsilon}.\end{aligned}$$

This leads to a bound for the second term of (3.8)

$$X^{2(\alpha-\sigma)}\int_{-\infty}^{\infty} S(Q,f,\alpha+iu)e^{-|u-t|}du \tag{3.9}$$
$$\ll(Q(|t|+1))^{2(k-2\alpha)}(Q+(Q(|t|+1))^{-k+1+2\alpha})(Q(|t|+1))^{\nu(k-\alpha,\bar{f})+\epsilon}.$$

To bound the first term of (3.8), one resorts to Theorem 3.2. In view of Heath-Brown's theorem and bounds for the Fourier coefficients (2.10), one has

$$\sum_{Q<|d|\le 2Q}\left|\sum_{N<n\le 2N} a_n\chi_d(n)n^{-s}e^{-n/X}\right|^2 \ll_\epsilon Q^\epsilon(Q+N)N^{k-2\sigma+\epsilon}. \tag{3.10}$$

Now for large $N_0(\gg X\log Q$ for instance)

$$\sum_{Q<|d|\le 2Q}\left|\sum_{n\ge 1} a_n\chi_d(n)n^{-s}e^{-n/X}\right|^2$$
$$=\sum_{Q<|d|\le 2Q}\left|\sum_{n\le N_0} a_n\chi_d(n)n^{-s}e^{-n/X}\right|^2+\sum_{Q<|d|\le 2Q}\left|\sum_{n> N_0} a_n\chi_d(n)n^{-s}e^{-n/X}\right|^2$$
$$=\sum_{Q<|d|\le 2Q}\left|\sum_{n\le N_0} a_n\chi_d(n)n^{-s}e^{-n/X}\right|^2+\mathrm{O}(Q).$$

We break the interval$[1,N_0]$ into $\mathrm{O}(\log N_0)$ subintervals of the type $N<n\le 2N$. Set $N_0=X\log^2 Q(|t|+1)$. Apply (3.10) to each $N<n\le 2N$

$$\sum_{Q<|d|\le 2Q}\left|\sum_{n\le N_0} a_n\chi_d(n)n^{-s}e^{-n/X}\right|^2$$
$$\ll_\epsilon Q^\epsilon N_0^{k-2\sigma+\epsilon}(Q+N_0)\log N_0$$
$$\ll_\epsilon (Q+X^{k+1-2\sigma})(Q(|t|+1)X)^\epsilon.$$

Put the above estimate and (3.9) in (3.8) to get

$$S(Q,f,\sigma+it)\ll_{\epsilon,\alpha}(Q+X^{k+1-2\sigma})(Q(|t|+1)X)^\epsilon \tag{3.11}$$
$$+(Q(|t|+1))^{2(k-2\alpha)}(Q+(Q(|t|+1))^{-k+1+2\alpha})(Q(|t|+1))^{\nu(k-\alpha,\bar{f})+\epsilon}.$$

If $k/2+\delta \le \sigma \le k-1/2$. Let $\alpha = k-\sigma$, so $1/2 \le \alpha < \sigma$. Set $X = (Q(|t|+1))^{1+\delta}$.

$$\begin{aligned} S(Q,f,s) \ll_{\epsilon,\sigma} & (Q+(Q(|t|+1))^{k+1-2\sigma})(Q(|t|+1))^{(2+\delta)\epsilon+\delta} \\ & + (Q(|t|+1))^{\nu(\sigma,\overline{f})-2\delta(2\sigma-k)+\epsilon}(Q+(Q(|t|+1))^{k+1-2\sigma}) \\ \ll_{\epsilon,\sigma} & (Q+(Q(|t|+1))^{k+1-2\sigma})\big((Q(|t|+1))^{(2+\delta)\epsilon+\delta} \\ & + (Q(|t|+1))^{\nu(\sigma,\overline{f})-2\delta(2\sigma-k)+\epsilon}\big), \end{aligned}$$

whence

$$\nu(\sigma,f) \le \max(A, \nu(\sigma,\overline{f})+B), \quad A=(2+\delta)\epsilon+\delta, \quad B=\epsilon-2\delta(2\sigma-k). \tag{3.12}$$

One also has by running through the same argument

$$\nu(\sigma,\overline{f}) \le \max(A, \nu(\sigma,f)+B), \quad A=(2+\delta)\epsilon+\delta, \quad B=\epsilon-2\delta(2\sigma-k). \tag{3.13}$$

If $\nu(\sigma,\overline{f}) > \nu(\sigma,f)$; choose $\delta=\sqrt{\epsilon}$, (3.13) gives

$$\nu(\sigma,f) < \nu(\sigma,\overline{f}) \le A \to 0.$$

If $\nu(\sigma,\overline{f}) \le \nu(\sigma,f)$, with $\delta=\sqrt{\epsilon}$, (3.12) gives

$$\nu(\sigma,f) \le A \to 0.$$

This finishes the case $k/2+\delta \le \sigma \le k-1/2$.

The second case being $k/2 \le \sigma \le k/2+\delta$. In this case, let $\alpha = k/2-\delta$, $X=(QT_t)^{1+\delta}$. So $1/2 \le \alpha < \sigma$,

$$
\begin{aligned}
S(Q,f,s) \ll_{\epsilon,\sigma} & \, (Q + (Q(|t|+1))^{k+1-2\sigma})(Q(|t|+1))^{(2+\delta)\epsilon+\delta} \\
& + (Q(|t|+1))^{2\delta+\nu(k/2+\delta,\overline{f})+\epsilon-(2\sigma-k)-2\delta(\sigma-\alpha)}(Q + (Q(|t|+1))^{1-2\delta}) \\
& \leq (Q + (Q(|t|+1))^{k+1-2\sigma}) \\
& \cdot \left((Q(|t|+1))^{(2+\delta)\epsilon+\delta} + (Q(|t|+1))^{2\delta+\nu(k/2+\delta,\overline{f})+\epsilon}\right).
\end{aligned}
$$

Whence

$$
\nu(\sigma, f) \leq \max((2+\delta)\epsilon + \delta, 2\delta + \nu(k/2+\delta, \overline{f}) + \epsilon) \to 0.
$$

The theorem follows. □

Chapter 4

The Asymptotic of the First Moment

We prove an asymptotic formula for the first moment of the family $L(s, f \otimes \chi_d)$. The techniques used in the proof are those of Iwaniec [6]. In his paper, Iwaniec did calculation for the derivative of L-series attached to an elliptic curve. In our case, we do calculation on L-series associated with newforms in $S_k(\Gamma_0(N), \chi)$. Our argument more or less follows Iwaniec's original argument except our family of quadratic twists differs from his.

4.1 Background

Let f be a newform in $S_k(\Gamma_0(N), \chi)$, where χ is a primitive Dirichlet character mod N. Let $r = \mathbf{ord}(\chi)$ if the order of χ is even and $r = 2\mathbf{ord}(\chi)$ if the order of χ is odd. Denote

$D: \quad \{0 < d \equiv m^r \pmod{4N} : \text{for some } m \text{ prime to } 4N$

and $\nu_p(d) = 1 \text{ or } r+1, \ \forall p | d\}$;

$D^* : \quad \{d \in D : d \ r\text{th powerfree}\}$.

It is clear $d \in D^* \Rightarrow d$ is a fundamental discriminant. Let χ_d be the quadratic with conductor d.

Let

$$f(z) = \sum_{n \geq 1} a_n n^{(k-1)/2} e(nz)$$

be the Fourier expansion of f at the cusp ∞ and define the corresponding L-function as

$$L(s, f) = \sum_{n \geq 1} \frac{a_n}{n^s}.$$

Remark 4.1. We have normalized the Fourier coefficients of f in such a way that it has the effect of putting the critical line of the L-function at $\operatorname{Re} s = 1/2$.

The twisted L series

$$L(s, f \otimes \chi_d) = \sum_{n \geq 1} \frac{a_n \chi_d(n)}{n^s}$$

admits an analytic continuation to the entire complex plane with the functional equation given by

$$\begin{aligned} \Lambda(s, f \otimes \chi_d) &= \left(\frac{\sqrt{N} d}{\pi}\right)^s \Gamma\left(\frac{s + (k-1)/2}{2}\right) \Gamma\left(\frac{s + (k+1)/2}{2}\right) L(s, f \otimes \chi_d) \\ &= \omega \Lambda(1 - s, \overline{f} \otimes \chi_d). \end{aligned}$$

A simple computation shows that

$$\omega = i^k \overline{\eta_f},$$

where η_f is the eigenvalue of Atkin-Lehner involution(see section 2.2). $L(s, f\otimes\chi_d)$ has an Euler product expansion

$$L(s, f\otimes\chi_d) = \prod_p \left(1 - \frac{\alpha_p\chi_d(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta_p\chi_d(p)}{p^s}\right)^{-1},$$

and it satisfies the Ramanujan-Petersson conjecture (i.e. $|\alpha_p| = |\beta_p| = 1$ for all $p \nmid N$ and $|\alpha_p|, |\beta_p| \leq 1$ otherwise).

The symmetric square of f is defined as

$$L(s, sym^2 f) = \frac{L(2s, \chi^2)}{L(s, \chi)} \sum_{n\geq 1} \frac{a_n^2}{n^s}.$$

It is known that $L(s, sym^2 f)$ is entire and does not vanish on the line $\mathrm{Re}\, s = 1$.

The main theorem of this chapter is

Theorem 4.1. *Suppose*

$$\left(\frac{L(1, sym^2 f)}{|L(1, sym^2 f)|}\right)^2 \neq -i^k\overline{\eta_f}\left(\frac{L(2, \chi^2)}{|L(2, \chi^2)|}\right)^2.$$

Then it is true

$$\sum_{\substack{d\in D^* \\ d\leq Q}} L(1/2, f\otimes\chi_d) = CQ + \mathrm{o}(Q),$$

where $C = C_f$ is a constant depending only on f.

The strategy of the proof of the theorem is first to use the approximate functional equation (Theorem 2.2) to show

$$L(1/2, f\otimes\chi_d) = \mathcal{A}(d\sqrt{N}, \chi_d) + \omega\,\overline{\mathcal{A}(d\sqrt{N}, \chi_d)}, \tag{4.1}$$

where

$$\mathcal{A}(X,\chi_d) = \sum_{n\geq 1} \frac{a_n \chi_d(n)}{n^{1/2}} V\left(\frac{2\pi n}{X}\right).$$

Here V is as defined in (Theorem 2.2). The plan is then to bound $\mathcal{A}(d\sqrt{N},\chi_d)$. Due to exponential decay of the smooth function V, the estimate $\mathcal{A}(X,\chi_d) \ll X^{1/2}$ is true. Whence

$$L(1/2, f\otimes\chi_d) = \mathcal{A}(X,\chi_d) + \mathrm{O}(dX^{-1/2}), \tag{4.2}$$

for any $X > 0$.

4.2 Proof of the Asymptotic Formula

In view of 4.1,

$$\sum_{\substack{d\in D^* \\ d\leq Q}} L(1/2, f\otimes\chi_d) = \sum_{\substack{d\in D^* \\ d\leq Q}} \mathcal{A}(d\sqrt{N},\chi_d) + \omega \sum_{\substack{d\in D^* \\ d\leq Q}} \overline{\mathcal{A}(d\sqrt{N},\chi_d)}. \tag{4.3}$$

We first analyze the sum $\sum_d \mathcal{A}(d\sqrt{N},\chi_d)$. The treatment of the second sum is identical. As was done in Iwaniec's paper [6], we relax the condition that d is rth powerfree by introducing the factor $\sum_{a^r|d}\mu(a)$, then split the sum according to whether $a\leq A$ or $a > A$ and in the latter case we return to rth powerfree numbers by extracting rth power divisors of $a^r d$. Thus

$$\sum_{\substack{d\in D^* \\ d\leq Q}} \mathcal{A}(d\sqrt{N},\chi_d) = S + R$$

where

$$S = \sum_{\substack{a \leq A \\ (a,4N)=1}} \mu(a) \sum_{\substack{d \in D \\ a^r d \leq Q}} \mathcal{A}(a^r d\sqrt{N}, \chi_{a^r d}),$$

$$R = \sum_{(b,4N)=1} \sum_{\substack{a>A \\ a|b}} \mu(a) \sum_{\substack{d \in D^*, \\ b^r d \leq Q}} \mathcal{A}(b^r d\sqrt{N}, \chi_{b^r d}).$$

4.2.1 Treating R term

We treat R first. Using Iwaniec's identity (Page 370, [6])

$$\mathcal{A}(X, \chi_{b^r d}) = F(b)\mathcal{A}\left(\frac{X}{kl}, \chi_d\right),$$

with

$$F(b) = \sum_{\substack{k|b \\ l|b}} \alpha_k \beta_l \chi_d(kl) \frac{\mu(k)\mu(l)}{(kl)^{1/2}}.$$

We deduce using (4.2) and the fact $F(b) \ll \tau(b)^2$

$$\begin{aligned}
&\sum_{\substack{d \in D^*, \\ b^r d \leq Q}} \mathcal{A}(b^r d\sqrt{N}, \chi_{b^r d}) \\
&= \sum_{\substack{d \in D^*, \\ b^r d \leq Q}} F(b)\mathcal{A}\left(\frac{b^r d\sqrt{N}}{kl}, \chi_d\right) \\
&= \sum_{\substack{d \in D^*, \\ b^r d \leq Q}} F(b)\left(L(1/2, f \otimes \chi_d) + \mathrm{O}(d^{1/2} b^{-r/2} k^{1/2} l^{1/2})\right) \\
&= \sum_{\substack{d \in D^*, \\ b^r d \leq Q}} F(b) L(1/2, f \otimes \chi_d) + \mathrm{O}(b^{-2r+\epsilon} Q^{3/2}) \\
&\ll \left(\sum_{\substack{d \in D^*, \\ b^r d \leq Q}} |L(1/2, f \otimes \chi_d)|^4\right)^{1/4} \left(\sum_{\substack{d \in D^*, \\ b^r d \leq Q}} |F(b)|^{4/3}\right)^{3/4} + \mathrm{O}(b^{-2r+\epsilon} Q^{3/2}).
\end{aligned}$$

We have applied Hölder's inequality the last line. Resorting to Corollary 3.1 gives the bound

$$\ll (Q^{5/4} b^{-5r/4} + Q^{3/2} b^{-2r}) Q^{\epsilon}.$$

Thus

$$R \ll \sum_{(b,4N)=1} \left(\sum_{\substack{a>A \\ a|b}} 1 \right) (Q^{5/4} b^{-5r/4} + Q^{3/2} b^{-2r}) Q^{\epsilon} \ll (Q^{5/4} A^{-5r/4+1} + Q^{3/2} A^{-2r+1}) Q^{\epsilon}.$$

4.2.2 Treating S term

Recall

$$\begin{aligned} S &= \sum_{\substack{a\le A \\ (a,4N)=1}} \mu(a) \sum_{\substack{d\in D \\ a^r d\le Q}} \mathcal{A}(a^r d\sqrt{N}, \chi_{a^r d}) \\ &= \sum_{\substack{a\le A \\ (a,4N)=1}} \mu(a) \sum_{\substack{d\in D \\ a^r d\le Q}} \sum_{\substack{n\ge 1 \\ (n,a)=1}} \frac{a_n \chi_d(n)}{n^{1/2}} V\left(\frac{2\pi n}{a^r d\sqrt{N}}\right). \end{aligned}$$

Now write $n = hl^2 m$ such that $h|4N^{\infty}$, $(4N, lm) = 1$ and m is squarefree. For n written this way and $d \in D$ we have $\chi_d(n) = \chi_d(m)$ subject to $(d, l) = 1$. The last condition is detected by the usual Möbius inversion giving

$$S = \sum_{\substack{a\le A \\ (a,4N)=1}} \mu(a) \sum_{\substack{n=hl^2 m \\ (n,a)=1}} \frac{a_n}{n^{1/2}} \sum_{q|l} \mu(q) \sum_{\substack{qd\in D \\ a^r qd\le Q}} \chi_{qd}(m) V\left(\frac{2\pi n}{a^r qd\sqrt{N}}\right).$$

Next by means of Gauss sums we write for squarefree m,

$$\chi_d(m) = \overline{\epsilon_m} m^{-1/2} \sum_{2|r|<m} \chi_{Nr}(m) e\left(\frac{\overline{4N} rd}{m}\right),$$

where $\epsilon_m = 1$ if $m \equiv 1 \mod 4$, $\epsilon_m = i$ if $m \equiv -1 \mod 4$ and $\overline{4N}4N \equiv 1 \mod m$. This gives

$$S = \sum_{\substack{a\le A \\ (a,4N)=1}} \mu(a) \sum_{\substack{n=hl^2 m \\ (n,a)=1}} \frac{a_n}{n^{1/2}} \sum_{q|l} \mu(q) \sum_{2|r|<m} \chi_{Nrq}(m) \sum_{d},$$

where

$$\sum_d = \sum_{\substack{qd\in D\\ a^r qd\le Q}} V\left(\frac{2\pi n}{a^r qd\sqrt{N}}\right) e\left(\frac{\overline{4N}rd}{m}\right).$$

Next put $\Delta = \min(1/2, a^r q Q^{-1+\epsilon})$ and split the sum $S = S_0 + S_1 + S_2$, according to the conditions $r = 0, 0 < |r| < \Delta m, \Delta m \le r < m/2$ respectively. For S_2, we change variable so

$$\sum_d = \sum_{\substack{y\equiv 0 \pmod{a^2 q}\\ y/a^r\in D, y\le Q}} V\left(\frac{2\pi n}{y\sqrt{N}}\right) e\left(\frac{\overline{4N}ry}{a^r qm}\right)$$

We now break the sum up over congruences classes modulo $4Na^r q$. The number of such classes we denote by $\gamma(4N)$ which is bounded by $4N$. Thus

$$\sum_d = \sum_{\nu \pmod{4Na^r q}} \sum_{\substack{y\equiv\nu \pmod{4Na^r q}\\ y/a^r\in D, y\le Q}} V\left(\frac{2\pi n}{y\sqrt{N}}\right) e\left(\frac{\overline{4N}ry}{a^r qm}\right).$$

To estimate the inner sum, we invoke Lemma two in Iwaniec's paper [6] which states that if $g(x)$ satisfies $g^{(j)}(t) \ll (|t| + X)^{-j}$ for all $j \ge 1$ and α real, then

$$\sum_{y\equiv\nu \pmod{Y}} g(y)e(\alpha y) \ll \frac{X}{Y}\left(\frac{Y}{X\|\alpha Y\|}\right)^j$$

for all $j \ge 2$ provided αY is not an integer. Here $\|x\|$ denotes the distance between x and its nearest integer. To prove the lemma, we use Poisson's summation to get

$$\sum_{y\equiv\nu \pmod{Y}} g(y)e(\alpha y) = \frac{1}{Y}\sum_{u=-\infty}^{\infty} e(\frac{u\nu}{Y})\hat{g}(\alpha - \frac{u}{Y})$$

where $\hat{g}(y)$ denotes the Fourier transform of $g(x)$. We have $\hat{g}(y) \ll X(Xy)^{-j}$ by the partial integration j times, whence the lemma follows by trivial summation over u. In our case, $g(y) = V\left(\frac{2\pi n}{y\sqrt{N}}\right)$ and $\alpha = \frac{\overline{4N}r}{a^r qm}$; also we may take $X = Q+n$. Hence

$$\sum_d \ll \gamma(4N)\frac{Q+n}{4Na^r q}\left(\frac{4Na^r q}{(Q+n)\Delta}\right)^j .$$

By choosing large j, it is clear now $S_2 \ll 1$.

To estimate S_1, we sum first over m to get

$$S_1 = \sum_{\substack{a\leq A \\ (a,4N)=1}} \mu(a) \sum_{\substack{h,l \\ (hl^2,a)=1}} a_h h^{-1/2} l^{-1} \sum_{q|l} \mu(q) \sum_{\substack{dq\in D \\ a^r dq\leq Q}} \sum_{r\geq 1} \sum_m , \tag{4.4}$$

where

$$\sum_m = \sum_{\substack{m>\frac{r}{\Delta},(m,4Na)=1 \\ m \text{ squarefee}}} a_{l^2 m} m^{-1} \chi_{Nqr}(m)\overline{\epsilon_m} V\left(\frac{2\pi n}{a^r dq\sqrt{N}}\right) e\left(\frac{\overline{4N}rd}{m}\right).$$

Now write $e(\frac{\overline{4N}rd}{m}) = e(\frac{rd}{4Nm})e(\frac{-\overline{m}rd}{4N})$ and let

$$\psi(m) = \overline{\epsilon_m}\chi_{Nrq}(m)e(\frac{-\overline{m}rd}{4N})$$
$$g(t) = t^{-1}V\left(\frac{2\pi hl^2 t}{a^r dq\sqrt{N}}\right)e(\frac{rd}{4Nt}).$$

Then

$$\sum_m = \sum_{\substack{m>\frac{r}{\Delta},(m,4Na)=1 \\ m \text{ squarefee}}} a_{l^2 m}\psi(m)g(m).$$

Now $\psi(m)$ has absolute value 1 and period $4Nrq$, so using Proposition 2.6, one gets the estimate

$$\sum_{\substack{m\leq x,(m,Ma)=1 \\ m \text{ squarefee}}} a_{l^2 m}\psi(m) \ll \tau(l)\tau(4Na)(qr)^{1/2}x^{1/2+\epsilon}.$$

Partial summation now gives

$$\sum_m \ll \tau(l)\tau(4Na)q^{1/2}\left(r^{1/2}|g(r/\Delta)|(r/\Delta)^{1/2+\epsilon} + r^{1/2}\int_{r/\Delta}^{\infty} |g'(t)|t^{1/2+\epsilon}dt\right).$$

It is easy to see now inside (4.4)

$$\sum_{d,r,m} \ll h^{-1} l^{-2} a^{r/2} q Q^{1/2+\epsilon},$$

and so

$$S_1 \ll A^{r/2+1} Q^{1/2+\epsilon}.$$

Now we compute the main term S_0. When $r = 0$, only the term with $m = 1$ contributes to the sum, and

$$S_0 = \sum_{\substack{a \le A \\ (a,4N)=1}} \mu(a) \sum_{\substack{n=hl^2 \\ (n,a)=1}} \frac{a_n}{n^{1/2}} \sum_{q|l} \mu(q) \sum_d,$$

where

$$\sum_d = \sum_{\substack{dq \in D \\ a^r dq \le Q}} V\left(\frac{2\pi n}{a^r q d \sqrt{N}}\right).$$

The sum over d, as with S_2, breaks into classes $\mod 4Na^r q$. Each class contributes

$$\frac{Q}{4Na^r q} \int_0^1 V\left(\frac{2\pi n}{tQ\sqrt{N}}\right) dt + \mathrm{O}\left((1 + \frac{n}{Q})^{-2}\right).$$

There are total $\gamma(4N)$ such classes. Thus

$$S_0 = Q\frac{\gamma(4N)}{4N} \sum_{n=hl^2} \frac{a_n}{n^{1/2}} \frac{\phi(l)}{l} \sum_{\substack{a \le A \\ (a,4Nl)=1}} \frac{\mu(a)}{a^r} \int_0^1 V\left(\frac{2\pi n}{tQ\sqrt{N}}\right) dt + \mathrm{O}(AQ^{1/2+\epsilon}).$$

Now writing

$$\sum_{\substack{a \le A \\ (a,4Nl)=1}} \frac{\mu(a)}{a^r} = \frac{1}{\zeta(r)} \prod_{p|4Nl} (1 - \frac{1}{p^r})^{-1} + \mathrm{O}(A^{1-r})$$

in S_0 gives

$$S_0 = CQ \int_0^1 B(tQ\sqrt{N}) dt + \mathrm{O}\left((AQ^{1/2} + A^{1-r}Q)Q^\epsilon\right) \tag{4.5}$$

with

$$C = \frac{\gamma(4N)}{4N\zeta(r)} \prod_{p|4N} (1 - \frac{1}{p^r})^{-1}$$

$$B(x) = \sum_{n=hl^2} \frac{b_n}{n^{1/2}} V\left(\frac{n}{x}\right)$$

$$b_n = a_n \prod_{\substack{p|n \\ p\nmid 4N}} (1 - \frac{1}{p})(1 - \frac{1}{p^r})^{-1}.$$

Returning to the definition of $V(x)$ as an inverse Mellin transform, we get

$$B(x) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma(s+k/2)}{\Gamma(k/2)} L(s+1/2) (\frac{x}{2\pi})^s \frac{ds}{s}$$

where

$$L(s) = \sum_{n=hl^2} \frac{b_n}{n^s} = \sum_{h|4N^\infty} \frac{a_h}{h^s} \sum_{(l,4N)=1} \frac{b_{l^2}}{l^{2s}}.$$

Now $L(s)$ differs from $L(2s, sym^2 f)/L(4s, \chi^2)$ by a product of Euler factors which converges absolutely for $\mathrm{Re}\, s \geq 1/4$. It follows that $L(s)$ is analytic in this range and has polynomial growth in $\mathbf{Im}(s)$. Moving the line of integration to $\mathbf{Re}(s) = -1/4$, we get

$$B(x) = L(1/2) + \mathrm{O}(x^{-1/4}).$$

Substituting this in (4.5)

$$S_0 = CL(1/2)Q + \mathrm{O}\left((AQ^{1/2} + A^{1-r}Q + Q^{3/4})Q^\epsilon\right).$$

Putting all the estimates together gives

$$\sum_{\substack{d\in D^* \\ d\leq Q}} \mathcal{A}(d\sqrt{N}, \chi_d) = S_0 + S_1 + S_2 + R$$

$$= CL(1/2)Q + \mathrm{O}\left((A^{r/2+1}Q^{1/2} + Q^{3/4} + A^{1-r}Q + A^{1-5r/4}Q^{5/4} + A^{1-2r}Q^{3/2})Q^\epsilon\right).$$

If we let $A = Q^{1/3r}$, a simple computation shows

$$\sum_{\substack{d \in D^* \\ d \leq Q}} \mathcal{A}(d\sqrt{N}, \chi_d) = CL(1/2)Q + \mathrm{O}\left(Q^{\frac{2+5r}{6r}+\epsilon}\right).$$

Thus the theorem is proved in view of equation (4.3) and we have

$$\sum_{\substack{d \in D^* \\ d \leq Q}} L(1/2, f \otimes \chi_d) = CQ\left(L(1/2) + \omega\overline{L(1/2)}\right) + \mathrm{O}\left(Q^{\frac{2+5r}{6r}+\epsilon}\right).$$

Chapter 5

Applications

5.1 Non-vanishing Results

One may the apply the moment theorems proved heretofore to get non-vanishing results. Recall that f is a newform in $S_k(\Gamma_0(N), \chi)$, where χ is a primitive Dirichlet character modulo N. Let $r = \mathbf{ord}(\chi)$ if the order of χ is even and $r = 2\mathbf{ord}(\chi)$ if the order of χ is odd. Define

$$\begin{aligned} D &: \{ 0 < d \equiv \nu^r \pmod{4N} : \text{for some } \nu \text{ prime to } 4N \\ &\quad \text{and } \nu_p(d) = 1, r+1, \forall p | d \} \\ D^* &: \{ d \in D : d \; r\text{th powerfree} \} \\ N(Q) &: \sharp \{ d \leq Q : d \in D^*, L(k/2, f \otimes \chi_d) \neq 0 \}. \end{aligned}$$

So $N(Q)$ counts the number of quadratic twists $L(s, f \otimes \chi_d)$ which does not vanish at $k/2$. By the Cauchy-Schwarz inequality

$$N(Q) \geq \frac{\left| \sum_{\substack{d \leq Q \\ d \in D^*}} L(k/2, f \otimes \chi_d) \right|^2}{\sum_{\substack{d \leq Q \\ d \in D^*}} |L(k/2, f \otimes \chi_d)|^2}.$$

Thus in view of Theorem 3.5 and Theorem 4.1, we have

$$N(Q) \gg_{\epsilon,f} Q^{1-\epsilon}.$$

Remark 5.1. It is believed (unproved) that there is positive proportion of non-vanishing (i.e. $N(Q) \gg Q$) in such a family.

5.2 Zero Density Estimate

Let f be a newform in $S_k(\Gamma_0(N), \chi)$, $k \geq 2$. Let χ_d be the quadratic character of conductor $|d|$, $(d, N) = 1$. Writing

$$N(\sigma, T, d) = \#\{\rho = \beta + i\gamma : L(\rho, f \otimes \chi_d) = 0, \beta \geq \sigma, |\gamma| \leq T\}$$

It is well known in the literature that ([8], section 5.3)

$$N(k/2, T, d) = \frac{T}{\pi} \log \frac{Nd^2T^2}{(2\pi e)^2} + O\left(\log Nd^2(|T| + k)\right).$$

Extending this result, we prove in this section

Theorem 5.1. *Let $\epsilon > 0$. Then*

$$\sum_{|d| \leq Q}^{*} N(\sigma, d, T) \ll Q^{\frac{2k+2-4\sigma}{k+2-2\sigma}} T^{\frac{2k+3-4\sigma}{k+2-2\sigma}} (QT)^{\epsilon}.$$

uniformly for $\frac{k}{2} \leq \sigma \leq \frac{k+1}{2}$. And the constant only depends on ϵ

The proof is to follow the outline given in Montgomery's book [15] in which he derived various zero density estimates for the Dirichlet L-functions.

For $\sigma > \frac{k+1}{2}$, $L(s, f \otimes \chi_d)$ has an Euler product of degree two satisfying the Ramanujan conjecture. Let

$$L(s, f\otimes\chi_d)=\sum_{n\geq 1} a_n\chi_d(n)n^{-s}, \qquad \frac{1}{L(s, f\otimes\chi_d)}=\sum_{n\geq 1} b_n\chi_d(n)n^{-s}$$

$$M_x(s,\chi_d)=\sum_{n\leq x} b_n\chi_d(n)n^{-s}, \qquad L(s, f\otimes\chi_d)M_x(s,\chi_d)=\sum_{n\geq 1} c_n\chi_d(n)n^{-s}.$$

Lemma 5.1. *Let $s=\sigma+it$, $y>0$. For $\frac{k}{2}<\sigma<\frac{k+1}{2}$,*

$$\begin{aligned}
&\exp(-1/y)+\sum_{n>x} c_n\chi_d(n)n^{-s}\exp(-n/y)\\
=\quad & L(s, f\otimes\chi_d)M_x(s,\chi_d)\\
+&\frac{1}{2\pi}\int_{-\infty}^{\infty} L(k/2+it+iu, f\otimes\chi_d)M_x(k/2+it+iu,\chi_d)y^{k/2-\sigma+iu}\Gamma(k/2-\sigma+iu)du.
\end{aligned}$$

Proof. It is not hard to see

$$c_1=1, \quad c_n=0, \quad 2\leq n\leq x.$$

By the Mellin inversion formula, one has

$$\begin{aligned}
&\exp(-1/y)+\sum_{n>x} c_n\chi_d(n)n^{-s}\exp(-n/y)\\
=&\sum_{n\geq 1} c_n\chi_d(n)n^{-s}\exp(-n/y)\\
=&\frac{1}{2\pi i}\int_{(2k)} L(s+w, f\otimes\chi_d)M_x(s+w,\chi_d)y^w\Gamma(w)dw.
\end{aligned}$$

If $\frac{k}{2}<\sigma<\frac{k+1}{2}$, shifting the contour to the line $(k/2-\sigma)$, we pick the residue $L(s, f\otimes\chi_d)M_x(s,\chi_d)$ at $w=0$. $\square$

Let $R=R(Q,T,x,y)$ be the number of discriminants $d\leq Q$, $(d,N)=1$, for which $L(s, f\otimes\chi_d)$ has a zero in the square

$$\sigma\leq \mathrm{Re}(s)<\sigma+1/\log QT, \quad t\leq \mathrm{Im}(s)<t+1/\log QT, \tag{5.1}$$

with $\frac{k}{2} + \frac{1}{\log QT} \le \sigma \le \frac{k+1}{2}$, $|t| \le T$.

Remark 5.2. The plan for the proof of Theorem 5.1 is to bound each $R(Q, T, x, y)$ and sum over all such squares. Also, notice the assumption $\sigma \ge \frac{k}{2} + \frac{1}{\log QT}$ is valid, since otherwise the theorem is trivial.

There are essentially two things to consider in the analysis for $R(Q, T, x, y)$.

Lemma 5.2. *With notations above,*

1. *There are $R_1 \gg R/\log y$ values of d as above, with corresponding zeros $\rho = \beta + i\gamma$ in the square (5.1) for which*

$$\left| \sum_{U<n\le 2U} c_n \chi_d(n) n^{-\rho} \exp(-n/y) \right| \gg 1/\log y$$

2. *There are $R_2 \gg R$ values of d as above, with corresponding zeros $\rho = \beta + i\gamma$ in the square such that*

$$\left| \int_{-A\log(QT)}^{A\log(QT)} L(k/2 + i\gamma + iu, f \otimes \chi_d) M_x(k/2 + i\gamma + iu, \chi_d) y^{k/2-\beta+iu} \right.$$
$$\left. \cdot \Gamma(k/2 - \beta + iu) du \right| \gg 1.$$

Proof. Let $\rho = \beta + i\gamma$ be a zero of $L(s, f \otimes \chi_d)$ in the square (5.1). In view of Lemma 5.1, by choosing large A and y, two things can happen

1.
$$\left| \sum_{x<n\le y^2} c_n \chi_d(n) n^{-\rho} \exp(-n/y) \right| \gg 1,$$

2.
$$\left| \int_{-A\log(QT)}^{A\log(QT)} L(k/2 + i\gamma + iu, f \otimes \chi_d) M_x(k/2 + i\gamma + iu, \chi_d) y^{k/2-\beta+iu} \right.$$
$$\left. \cdot \Gamma(k/2 - \beta + iu) du \right| \gg 1.$$

Now if we divide the interval $[x, y^2]$ into dyadic subintervals, the first condition is equivalent to **1'**: There exists a $U \in [x, y^2]$ such that

$$\left| \sum_{U < n \leq 2U} c_n \chi_d(n) n^{-\rho} \exp(-n/y) \right| \gg 1/\log y.$$

Recall $R = R(Q, T, x, y)$ is the number of discriminants $d \leq Q$, $(d, N) = 1$, for which $L(s, f \otimes \chi_d)$ has a zero in the square

$$\sigma \leq \mathrm{Re}(s) < \sigma + 1/\log QT, \quad t \leq \mathrm{Im}(s) < t + 1/\log QT.$$

Let R_1,R_2 represent the number of discriminants for which the conditions **1'** and 2 are true. Clearly $R \leq R_1 + R_2$. From this inequality, Lemma 5.2 is clear. ◻

We are now ready to prove the main theorem

Proof. First consider the second case in Lemma 5.2. We have

$$\begin{aligned} y^{\sigma - k/2}/\log QT &\ll \int_{-A\log(QT)}^{A\log(QT)} |L(k/2 + i\gamma + iu, f \otimes \chi_d) M_x(k/2 + i\gamma + iu, \chi_d)| \, du \\ &\ll \int_{t - A\log(QT)}^{t+1+A\log(QT)} |L(k/2 + iu, f \otimes \chi_d) M_x(k/2 + iu, \chi_d)| \, du. \end{aligned}$$

Summing the inequality over R_2 discriminants and applying Cauchy-Schwarz inequality we get

$$
\begin{aligned}
&Ry^{\sigma-k/2}/\log QT \\
&\ll \int_{t-A\log(QT)}^{t+1+A\log(QT)} \sideset{}{^*}\sum_{|d|\le Q} |L(k/2+iu, f\otimes\chi_d)|\,|M_x(k/2+iu,\chi_d)|\,du \\
&\le \int_{t-A\log(QT)}^{t+1+A\log(QT)} \left(\sideset{}{^*}\sum_{|d|\le Q} |L(k/2+iu, f\otimes\chi_d)|^2\right)^{1/2} \\
&\cdot \left(\sideset{}{^*}\sum_{|d|\le Q} |M_x(k/2+iu,\chi_d)|^2\right)^{1/2} du \\
&\le \left(\int_{t-A\log(QT)}^{t+1+A\log(QT)} \sideset{}{^*}\sum_{|d|\le Q} |L(k/2+iu, f\otimes\chi_d)|^2\,du\right)^{1/2} \\
&\cdot \left(\int_{t-A\log(QT)}^{t+1+A\log(QT)} \sideset{}{^*}\sum_{|d|\le Q} |M_x(k/2+iu,\chi_d)|^2\,du\right)^{1/2}. \qquad (5.2)
\end{aligned}
$$

Recall in Section 3.3, it is proved that

$$
\sideset{}{^*}\sum_{|d|\le Q} |L(k/2+iu, f\otimes\chi_d)|^2 \ll_\epsilon (Q(|u|+1))^{1+\epsilon}.
$$

Thus

$$
\int_{t-A\log(QT)}^{t+1+A\log(QT)} \sideset{}{^*}\sum_{|d|\le Q} |L(k/2+iu, f\otimes\chi_d)|^2\,du \ll (QT)^{1+\epsilon}. \qquad (5.3)
$$

The second term in (5.2) can be dealt with by means of Heath-Brown's large sieve inequality, as presented in Theorem 3.2. We split the interval $[1,x]$ into ranges of the form $V < n \le 2V$. Thus

$$\int_{t-A\log(QT)}^{t+1+A\log(QT)} \sum_{|d|\le Q}^{*} |M_x(k/2+iu,\chi_d)|^2\,du$$

$$\ll \sum_V \int_{t-A\log(QT)}^{t+1+A\log(QT)} \sum_{|d|\le Q}^{*} \left| \sum_{V<n\le 2V} b_n\chi_d(n)n^{-k/2-iu}\right|^2 du$$

$$\ll (\log x)\log QT \max_{t-A\log QT\le u\le t+1+A\log QT} \sum_{|d|\le Q}^{*} \left| \sum_{V<n\le 2V} b_n\chi_d(n)n^{-k/2-iu}\right|^2$$

$$\ll (QTx)^{\epsilon}(Q+x). \tag{5.4}$$

Putting (5.3) and (5.4) in (5.2) gives

$$R \ll (QTx)^{\epsilon}(QT)^{1/2}(Q+x)^{1/2}y^{k/2-\sigma}. \tag{5.5}$$

Consider now the first case in Lemma 5.2. Assume $y \le (QT)^c$ for some constant c. By partial summation and Cauchy-Schwarz

$$(\log QT)^{-2} \ll \left| \sum_{U<n\le 2U} c_n\chi_d(n)n^{-\rho}\exp(-n/y)\right|^2$$

$$= \left| \sum_{U<n\le 2U} c_n\chi_d(n)n^{-s}\exp(-n/y)n^{s-\rho}\right|^2$$

$$= \left| (2U)^{s-\rho} \sum_{U<n\le 2U} c_n\chi_d(n)n^{-s}\exp(-n/y) \right.$$

$$\left. - \int_U^{2U} \sum_{U<n\le V} c_n\chi_d(n)n^{-s}\exp(-n/y)dV^{s-\rho}\right|^2$$

$$\ll \left| \sum_{U<n\le 2U} c_n\chi_d(n)n^{-s}\exp(-n/y)\right|^2 + \int_U^{2U} \sum_{U<n\le V} \left|c_n\chi_d(n)n^{-s}\exp(-n/y)\right|^2 dV/V,$$

for R_1 discriminants. Summing over d, one gets by applying Heath-Brown's large sieve inequality

$$R/(\log QT)^3 \ll \sum_{|d|\le Q}^{*} \left| \sum_{U<n\le 2U} c_n \chi_d(n) n^{-s} \exp(-n/y) \right|^2$$
$$+ \int_U^{2U} \sum_{|d|\le Q}^{*} \sum_{U<n\le V} \left| c_n \chi_d(n) n^{-s} \exp(-n/y) \right|^2 dV/V,$$
$$\ll (QTU)^{\epsilon}(Q+U)U^{2(k/2-\sigma)} \exp(-U/y).$$

Since $x \le U \le y^2$, one gets the estimate

$$R \ll (QTy)^{\epsilon}(Qx^{2(k/2-\sigma)} + y^{k+1-2\sigma}). \tag{5.6}$$

In view of (5.5) and (5.6)

$$R \ll (QTy)^{\epsilon}\left((QT)^{1/2}(Q+x)^{1/2}y^{k/2-\sigma} + Qx^{2(k/2-\sigma)} + y^{k+1-2\sigma}\right).$$

Setting

$$x = Q, \quad y = Q^{\frac{2}{k+2-2\sigma}} T^{\frac{1}{k+2-2\sigma}}$$

gives

$$R \ll Q^{\frac{2k+2-4\sigma}{k+2-2\sigma}} T^{\frac{k+1-2\sigma}{k+2-2\sigma}} (QT)^{\epsilon}.$$

Summing over squares as defined by (5.1), the theorem is proved. $\square$

Chapter 6

Other Moment Results

Instead of averaging over a family of quadratic characters, we average in this chapter over a different family, namely a Hecke basis of the space of cusp forms. More precisely, let $\mathcal{H}_k(p)$ be the set of an orthonormal basis of cusp forms of weight k for the congruence subgroup $\Gamma_0(p)$ with p prime. Let ψ be a primitive character with conductor r prime to p. While dealing with forms in $\mathcal{H}_k(p)$, it is often convenient to introduce harmonic weights that arise from the Petersson norm of f. So for what follows, we adopt the notation

$$\sum_{f\in\mathcal{H}_k}^{h} = \sum_{f\in\mathcal{H}_k} \frac{1}{\omega_f}$$

where

$$\omega_f := \frac{(4\pi)^{k-1}}{\Gamma(k-1)} < f, f >_p .$$

The main results of the chapters are

Theorem 6.1. *Let $\mathcal{H}_k(p)$ be the set of an orthonormal basis of cusp forms of* *weight k with $k \equiv 0 \pmod 4$. Set*

$$D(m) = \sum_{f \in \mathcal{H}_k(p)}^{h} L(k/2, f \otimes \psi) a_f(m).$$

If $p \nmid m$, we have the asymptotic formula as $k \to \infty$

$$D(m) = m^{k/2-1} \left(\psi(m) - \omega \frac{p-1}{p+1} \overline{\psi(pm)} + \omega \frac{\overline{\psi(m)}}{p+1} \right) + \mathrm{O}(m^{\frac{k-1}{2}} k^2 e^{-k})$$

and if $p \mid m$, then

$$D(m) = m^{k/2-1} \left(\psi(m) - \omega \frac{p(p-1)}{p+1} \overline{\psi(m/p)} + \omega \frac{\overline{\psi(m)}}{p+1} \right.$$
$$\left. + \omega \frac{\overline{\psi(m)}}{(p+1)p^{k-1}} \right) + \mathrm{O}(m^{\frac{k-1}{2}} k^2 e^{-k}).$$

Here $\omega = \psi(p) \frac{\tau(\psi)^2}{r}$.

Theorem 6.2. *With the same assumptions above and in addition if the character ψ be quadratic, then*

$$\sum_{f \in \mathcal{H}_k(p)}^{h} L(k/2, f \otimes \psi)^2 = (\log k) \frac{\phi(r)}{2r} \left(\frac{p^2 + 2 + p^{-k}}{p+1} \right) + \mathrm{O}(1).$$

Theorem 6.3. *With the same assumptions as in Theorem 6.1, we have for any positive integer m,*

$$\sum_{f \in \mathcal{H}_k(p)}^{h} L(k/2, f \otimes \psi)^{2m} \ll (\log k)^{2m},$$

where the implied constant is absolute.

We begin by stating necessary lemmas. The proof of the above theorems follow in the ensuing sections.

Proposition 6.1 (Petersson's Formula). *Let $N \geq 1$, $k \geq 2$. Let $\mathcal{H}_k(N)$ be any Hecke basis of $S_k(\Gamma_0(N))$*

$$(nm)^{\frac{1-k}{2}} \sum_{f \in \mathcal{H}_k(N)}^{h} a_f(n)a_f(m) = \delta(m,n) + 2\pi i^{-k} \sum_{\substack{c>0 \\ c \equiv 0 \pmod N}} \frac{S(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Here $S(m,n;c)$ is the Kloosterman sum,

$$S(m,n;c) = \sum_{d \pmod c}^{*} e\left(\frac{md + n\bar{d}}{c}\right)$$

and J_{k-1} is the Bessel function of order $k-1$.

The proof is standard (see [8] Chapter 14 for instance).

Bounding the Bessel function and Kloosterman sum trivially, one obtains

Lemma 6.1. *If k is large, and $mn \ll k^2$ then*

$$(nm)^{\frac{1-k}{2}} \sum_{f \in \mathcal{H}_k(N)}^{h} a_f(n)a_f(m) = \delta(m,n) + \mathrm{O}(e^{-k}).$$

Furthermore, if p is prime and if we restrict to newforms of level p, then we have

$$(nm)^{\frac{1-k}{2}} \sum_{f \in S^*(p)}^{h} a_f(n)a_f(m) = \begin{cases} \frac{p}{p+1}\delta(m,n) + \mathrm{O}(e^{-k}) & p \nmid n \text{ or } p \nmid m \\ \frac{p-1}{p+1}\delta(m,n) + \mathrm{O}(e^{-k}) & p | n \text{ and } p | m \end{cases}$$

Proof. The first assertion follows by bounding the Bessel functions and the Kloosterman sum trivially. One may see [20] for a proof. For the second assertion, one has the Hecke-algebra decomposition

$$S_k(\Gamma_0(p)) = S_k^*(\Gamma_0(p)) \oplus 2S_k(\Gamma_0(1)).$$

More precisely, one can find a basis of $S_k(\Gamma_0(p))$ consisting of newforms of level p and forms $f(pz)$ and $f(z)$ where $f(z)$ is a newform of level 1. Thus

$$\begin{aligned} &\sum_{f \in S_k(p)}^{h} a_f(n)a_f(m) \\ &= \sum_{f \in S_k^*(p)}^{h} a_f(n)a_f(m) + \frac{1}{p+1}\sum_{f \in S_k(1)}^{h} a_f(n)a_f(m) + \frac{1}{p+1}\sum_{f \in S_k(1)}^{h} a_f(n/p)a_f(m/p). \end{aligned}$$

where the coefficients $\frac{1}{p+1}$ come from the Petersson norm change from the group $\Gamma_0(p)$ to $\Gamma_0(1)$ (i.e. $< f, f >_p = (p+1) < f, f >_1$). Applying Petersson's formula and the first part of the lemma, the second assertion follows. $\square$

Lemma 6.2. *With $f \in \mathcal{H}_k(p)$, $k \equiv 0 \pmod 4$ and m a positive integer*

$$\begin{aligned} L(k/2, f \otimes \psi)^m = &\sum_{n \geq 1} a_f(n)\psi(n)n^{-k/2}V_{k,m}\left(\left(\frac{2\pi}{\sqrt{pr}}\right)^m n\right) \\ &+\omega_m \eta_f^m \sum_{n\geq 1} a_f(n)\overline{\psi(n)}n^{-k/2}V_{k,m}\left(\left(\frac{2\pi}{\sqrt{pr}}\right)^m n\right) \end{aligned} \tag{6.1}$$

where $\omega_m = (\psi(p)\frac{\tau(\psi)^2}{r})^m$ and the weights $V_{k,m}$ satisfy

$$V_{k,m}(y) = \begin{cases} 1 + O(e^{-k}) & y \leq \frac{k^m}{2^m e^{m+4}}; \\ O(1) & \frac{k^m}{2^m e^{m+4}} < y \leq 2^{m+2}k^m; \\ O(\frac{k^m}{ye^k}) & n > 2^{m+2}k^m. \end{cases}$$

Proof. The approximate functional equation (6.1) is a direct consequence of Theorem 2.2 where the weight is defined by

$$V_{k,m}(y) = \frac{1}{2\pi i}\int_{(\alpha)} y^{-u}\left(\frac{\Gamma(k/2+u)}{\Gamma(k/2)}\right)^m \frac{du}{u}.$$

We have

$$\begin{aligned} |V_{k,m}(y)| &\leq \frac{1}{2\pi}\int_{(\alpha)} y^{-u}\left(\frac{|\Gamma(k/2+u+1)|}{\Gamma(k/2)}\right)^m \frac{|du|}{|(u+k/2)^m u|} \\ &\leq y^{-\alpha}\left(\frac{\Gamma(k/2+\alpha+1)}{\Gamma(k/2)}\right)^m. \end{aligned}$$

In the case $n > 2^{m+2}k^m$, choose $\alpha = k$. So

$$|V_{k,m}(y)| \leq \left(\frac{(2k)^m}{y}\right)^k \ll \frac{k^m}{y}\left(\frac{(2k)^m}{y}\right)^{k-1} \ll \frac{k^m}{ye^k}.$$

In the case $y < \frac{k^m}{2^m e^{m+4}}$, one moves the line of integration to $-k/2+1$. Picking the residue 1 at $u=0$ and the remaining term is bounded by $\frac{y^{k/2-1}}{\Gamma(k/2)^m} \ll e^{-k}$. In the last case $\frac{k^m}{2^m e^{m+4}} < y \leq 2^{m+2}k^m$, one simply set $\alpha = 1$. $\square$

6.1 The First Moment

We first consider the case $p \nmid m$. Recall

$$D(m) = \sum_{f \in \mathcal{H}_k(p)}^h L(k/2, f \otimes \psi) a_f(m).$$

In view of (6.1)

$$\begin{aligned} D(m) &= \sum_{f \in \mathcal{H}_k(p)}^h \sum_{n \geq 1} a_f(m) a_f(n) \psi(n) n^{-k/2} V_{k,1}\left(\frac{2\pi n}{\sqrt{p}r}\right) \\ &+ \omega \sum_{f \in \mathcal{H}_k(p)}^h \sum_{n \geq 1} \eta_f a_f(m) a_f(n) \overline{\psi(n)} n^{-k/2} V_{k,1}\left(\frac{2\pi n}{\sqrt{p}r}\right) \end{aligned} \tag{6.2}$$

For the first sum in (6.2), we first truncate the sum to the range $\ll k$.

$$\begin{aligned} &\sum_{f \in \mathcal{H}_k(p)}^h \sum_{n \geq 1} a_f(m) a_f(n) \psi(n) n^{-k/2} V_{k,1}\left(\frac{2\pi n}{\sqrt{p}r}\right) \\ &= \sum_{n \leq \frac{4\sqrt{p}rk}{\pi}} n^{-k/2} \psi(n) V_{k,1}\left(\frac{2\pi n}{r}\right) \sum_{f \in \mathcal{H}_k(p)}^h a_f(m) a_f(n) + O(m^{\frac{k-1}{2}} k^2 e^{-k}) \end{aligned}$$

Now applying Petersson's formula to get

$$\begin{aligned} &\sum_{n \leq \frac{4\sqrt{p}rk}{\pi}} n^{-1/2} \psi(n) V_{k,1}\left(\frac{2\pi n}{\sqrt{p}r}\right) m^{\frac{k-1}{2}} \left(\delta(m,n) + O(e^{-k})\right) + O(m^{\frac{k-1}{2}} k^2 e^{-k}) \\ &= m^{k/2-1} \psi(m) V_{k,1}\left(\frac{2\pi m}{\sqrt{p}r}\right) + O(m^{\frac{k-1}{2}} k^2 e^{-k}) \\ &= m^{k/2-1} \psi(m) + O(m^{\frac{k-1}{2}} k^2 e^{-k}). \end{aligned}$$

For the second sum in (6.2), we first truncate the sum over n to $\ll k$ then separate the forms according to f is a newform of level p or of the form $f(dz)$, $d = 1, p$ where $f(z)$ comes from level 1. In each case, by Proposition 2.2, η_f is given by

$$\eta_f = \begin{cases} -a_f(p)p^{1-k/2} & f \in S_k^*(p) \\ 1 & f \in S_k(1). \end{cases}$$

Thus

$$\begin{aligned}
&\omega \sum_{f\in\mathcal{H}_k(p)}^h \sum_{n\geq 1} \eta_f a_f(m)a_f(n)\overline{\psi(n)}n^{-k/2}V_{k,1}\left(\frac{2\pi n}{\sqrt{p}r}\right) \\
=&\omega \sum_{f\in\mathcal{H}_k(p)}^h \sum_{n\leq \frac{4\sqrt{p}rk}{\pi}} \eta_f a_f(m)a_f(n)\overline{\psi(n)}n^{-k/2}V_{k,1}\left(\frac{2\pi n}{\sqrt{p}r}\right) + \mathrm{O}(m^{\frac{k-1}{2}}k^2e^{-k}) \\
=&\omega(A+B+C) + \mathrm{O}(m^{\frac{k-1}{2}}k^2e^{-k}).
\end{aligned}$$

Where A, B, C denotes respectively the sum over forms of level p, 1 and shifts on forms of level 1.

$$\begin{aligned}
A =& \sum_{f\in S_k^*(p)}^h \sum_{n\leq \frac{4\sqrt{p}rk}{\pi}} \eta_f a_f(m)a_f(n)\overline{\psi(n)}n^{-k/2}V_{k,1}\left(\frac{2\pi n}{\sqrt{p}r}\right) \\
=& -p^{1-k/2} \sum_{n\leq \frac{4\sqrt{p}rk}{\pi}} n^{-k/2}\overline{\psi(n)}V_{k,1}\left(\frac{2\pi n}{\sqrt{p}r}\right) \sum_{f\in S_k^*(p)}^h a_f(p)a_f(m)a_f(n)
\end{aligned}$$

Since p is the level, we have $a_f(m)a_f(p) = a_f(pm)$. And applying Lemma 6.1

$$\begin{aligned}
&= -p^{1-k/2}\frac{p-1}{p+1} \sum_{n\leq \frac{4\sqrt{p}rk}{\pi}} n^{-k/2}\overline{\psi(n)}V_{k,1}\left(\frac{2\pi n}{\sqrt{p}r}\right)(pmn)^{\frac{k-1}{2}}\delta(n,pm) + \mathrm{O}(m^{\frac{k-1}{2}}k^{1/2}e^{-k}) \\
&= -m^{k/2-1}\frac{p-1}{p+1}\overline{\psi(pm)}V_{k,1}\left(\frac{2\pi\sqrt{pm}}{r}\right) + \mathrm{O}(m^{\frac{k-1}{2}}k^{1/2}e^{-k}) \\
&= -m^{k/2-1}\frac{p-1}{p+1}\overline{\psi(pm)} + \mathrm{O}(m^{\frac{k-1}{2}}k^{1/2}e^{-k}).
\end{aligned}$$

$$\begin{aligned} B &= \frac{1}{p+1} \sum_{f \in S_k(1)}^h \sum_{n \leq \frac{4\sqrt{p}rk}{\pi}} \eta_f a_f(m) a_f(n) \overline{\psi(n)} n^{-k/2} V_{k,1}\left(\frac{2\pi n}{\sqrt{p}r}\right) \\ &= \frac{1}{p+1} \sum_{n \leq \frac{4\sqrt{p}rk}{\pi}} n^{-k/2} \overline{\psi(n)} V_{k,1}\left(\frac{2\pi n}{\sqrt{p}r}\right) \sum_{f \in S_k(1)}^h a_f(m) a_f(n) \\ &= \frac{1}{p+1} \sum_{n \leq \frac{4\sqrt{p}rk}{\pi}} n^{-1/2} m^{\frac{k-1}{2}} \overline{\psi(n)} V_{k,1}\left(\frac{2\pi n}{\sqrt{p}r}\right) \delta(n,m) + O(m^{\frac{k-1}{2}} k^{1/2} e^{-k}) \\ &= \frac{1}{p+1} m^{k/2-1} \overline{\psi(m)} V_{k,1}\left(\frac{2\pi m}{\sqrt{p}r}\right) + O(m^{\frac{k-1}{2}} k^{1/2} e^{-k}) \\ &= \frac{1}{p+1} m^{k/2-1} \overline{\psi(m)} + O(m^{\frac{k-1}{2}} k^{1/2} e^{-k}). \end{aligned}$$

The contribution from C is zero because $p \nmid m$. Putting things together, we have

$$D_m = m^{k/2-1}\left(\psi(m) - \omega\frac{p-1}{p+1}\overline{\psi(pm)} + \omega\frac{1}{p+1}\overline{\psi(m)}\right) + O(m^{\frac{k-1}{2}} k^2 e^{-k}),$$

with $\omega = \psi(p)\frac{\tau(\psi)^2}{r}$.

For the case $p|m$, the calculation is similar. Using the approximate functional equation and truncating the sum to $\ll k$, one is left with

$$D(m) = m^{k/2-1}\psi(m) + \omega(A+B+C) + O(m^{\frac{k-1}{2}} k^2 e^{-k})$$

Where A, B, C denote respectively contributions from sums of the forms of level p, 1 and shifts on forms of level 1. They can be computed using Lemma 6.1. We have

$$A = -m^{k/2-1}\frac{p(p-1)}{p+1}\overline{\psi(m/p)}$$
$$B = m^{k/2-1}\frac{1}{p+1}\overline{\psi(m)}$$
$$C = m^{k/2-1}\frac{1}{(p+1)p^{k-1}}\overline{\psi(m)}.$$

Remark 6.1. One may carry out the same analysis for the case $\mathcal{H}_k(1)$(Hecke basis of level one). And because every $f \in \mathcal{H}_k(1)$ is also a newform, we have $a_f(1) = 1, \forall f$. So computing $D(1)$ we get a true asymptotic formula for the first moment. One gets

$$D(1) = \sum_{f\in\mathcal{H}_k}^{h} L(k/2, f\otimes\psi) = 1 + \frac{\tau(\psi)^2}{r} + O(k^2e^{-k}).$$

6.2 The Second Moment

The approximate function equation for $L(k/2, f\otimes\psi)^2$ with ψ quadratic and is

$$L(k/2, f\otimes\psi)^2 = \sum_{n\geq 1} b_n n^{-k/2}\psi(n)V_{k,2}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^2 n\right)$$
$$+\eta_f^2\sum_{n\geq 1} b_n n^{-k/2}\overline{\psi(n)}V_{k,2}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^2 n\right)$$

where

$$b_n = \sum_{lm=n} a_f(l)a_f(m).$$

Thus

$$
\begin{aligned}
&\sum_{f\in\mathcal{H}_k(p)}^h L(k/2, f\otimes\psi)^2\\
=&\sum_{f\in\mathcal{H}_k(p)}^h \sum_{l,m\geq 1} a_f(l)a_f(m)(lm)^{-k/2}\psi(lm)V_{k,2}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^2 lm\right)\\
+\eta_f^2&\sum_{f\in\mathcal{H}_k(p)}^h \sum_{l,m\geq 1} a_f(l)a_f(m)(lm)^{-k/2}\overline{\psi(lm)}V_{k,2}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^2 lm\right)\\
=&\sum_{lm\leq\frac{2\sqrt{p}rk}{\pi}} (lm)^{-k/2}\psi(lm)V_{k,2}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^2 lm\right)\sum_{f\in\mathcal{H}_k(p)}^h a_f(l)a_f(m)\\
+&\sum_{lm\leq\frac{2\sqrt{p}rk}{\pi}} (lm)^{-k/2}\overline{\psi(lm)}V_{k,2}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^2 lm\right)\sum_{f\in\mathcal{H}_k(p)}^h \eta_f^2 a_f(l)a_f(m) + \mathrm{O}(k^3e^{-k})\\
=&I + II + \mathrm{O}(k^3e^{-k}).
\end{aligned}
$$

For the first sum we apply Lemma 6.1 to get

$$
\begin{aligned}
I =& \sum_{lm\leq\frac{2\sqrt{p}rk}{\pi}} (lm)^{-1/2}\psi(lm)V_{k,2}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^2 lm\right)(\delta(m,n)+\mathrm{O}(e^{-k}))\\
=& \sum_{l^2\leq\frac{2\sqrt{p}rk}{\pi}} l^{-1}\psi(l^2)V_{k,2}\left(\left(\frac{2\pi l}{\sqrt{p}r}\right)^2\right) + \mathrm{O}(ke^{-k})\\
=& \sum_{\substack{l^2\leq\frac{2\sqrt{p}rk}{\pi}\\ (l,r)=1}} l^{-1}V_{k,2}\left(\left(\frac{2\pi l}{\sqrt{p}r}\right)^2\right) + \mathrm{O}(ke^{-k})\\
=&\frac{\phi(r)}{2r}\log k + \mathrm{O}(1).
\end{aligned}
$$

For the second sum, as before we separate the sum according to whether f is a newform of level p, a form of level 1 or a shift of a form of level one. Thus

$$II = A + B + C.$$

$$
\begin{aligned}
A &= \sum_{lm \leq \frac{2\sqrt{p}rk}{\pi}} (lm)^{-k/2}\psi(lm)V_{k,2}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^2 lm\right) \sum_{f\in S_k^*(p)}^h \eta_f^2 a_f(l)a_f(m) \\
&= p^{2-k} \sum_{lm \leq \frac{2\sqrt{p}rk}{\pi}} (lm)^{-k/2}\psi(lm)V_{k,2}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^2 lm\right) \sum_{f\in S_k^*(p)}^h a_f(pl)a_f(pm) \\
&= \frac{p(p-1)}{p+1} \sum_{lm \leq \frac{2\sqrt{p}rk}{\pi}} (lm)^{-1/2}\psi(lm)V_{k,2}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^2 lm\right) \delta(l,m) + \mathrm{O}(ke^{-k}) \\
&= \frac{p(p-1)}{p+1} \sum_{\substack{l^2 \leq \frac{2\sqrt{p}rk}{\pi} \\ (l,r)=1}} l^{-1}V_{k,2}\left(\left(\frac{2\pi l}{\sqrt{p}r}\right)^2\right) + \mathrm{O}(ke^{-k}) \\
&= \frac{p(p-1)}{p+1}\frac{\phi(r)}{2r}\log k + \mathrm{O}(1).
\end{aligned}
$$

$$
\begin{aligned}
B &= \frac{1}{p+1} \sum_{lm \leq \frac{2\sqrt{p}rk}{\pi}} (lm)^{-k/2}\psi(lm)V_{k,2}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^2 lm\right) \sum_{f\in S_k(1)}^h a_f(l)a_f(m) \\
&= \frac{1}{p+1} \sum_{lm \leq \frac{2\sqrt{p}rk}{\pi}} (lm)^{-1/2}\psi(lm)V_{k,2}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^2 lm\right) \delta(l,m) + \mathrm{O}(ke^{-k}) \\
&= \frac{1}{p+1} \sum_{\substack{l^2 \leq \frac{2\sqrt{p}rk}{\pi} \\ (l,r)=1}} l^{-1}V_{k,2}\left(\left(\frac{2\pi l}{\sqrt{p}r}\right)^2\right) + \mathrm{O}(ke^{-k}) \\
&= \frac{1}{p+1}\frac{\phi(r)}{2r}\log k + \mathrm{O}(1).
\end{aligned}
$$

$$
\begin{aligned}
C &= \frac{1}{p+1} \sum_{lm \leq \frac{2\sqrt{p}rk}{\pi}} (lm)^{-k/2} \psi(lm) V_{k,2}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^2 lm\right) \sum_{f \in S_k(1)}^h a_f(l/p) a_f(m/p) \\
&= \frac{1}{p+1} p^{1-k} \sum_{\substack{lm \leq \frac{2\sqrt{p}rk}{\pi} \\ p|l, p|m}} (lm)^{-1/2} \psi(lm) V_{k,2}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^2 lm\right) \delta(l,m) + \mathrm{O}(ke^{-k}) \\
&= \frac{1}{p+1} p^{1-k} \sum_{\substack{l^2 \leq \frac{2\sqrt{p}rk}{\pi} \\ p|l, (l,r)=1}} l^{-1} V_{k,2}\left(\left(\frac{2\pi l}{\sqrt{p}r}\right)^2\right) + \mathrm{O}(ke^{-k}) \\
&= \frac{1}{(p+1)p^k} \frac{\phi(r)}{2r} \log k + \mathrm{O}(1).
\end{aligned}
$$

Collecting results, we have

$$
\sum_{f \in \mathcal{H}_k(p)}^h L(k/2, f \otimes \psi)^2 = (\log k) \frac{\phi(r)}{2r} \left(\frac{p^2 + 2 + p^{-k}}{p+1}\right) + \mathrm{O}(1).
$$

6.3 The Higher Moments

The large sieve inequality (Theorem 3.3) for the cusp forms comes in handy when dealing with the upper bound for the higher even moments. In our setting, the inequality becomes

$$
\sum_{f \in \mathcal{H}_k(p)}^h \left| \sum_{n \leq N} a_n a_f(n) n^{\frac{1-k}{2}} \right|^2 \ll \left(1 + \mathrm{O}\left(\frac{N \log N}{k}\right)\right) \|a\|^2 \qquad (*)
$$

where a_n is any sequence of complex numbers.

The mth power of the L-function can be written using the approximate functional equation

$$L(k/2, f\otimes\psi)^m = \sum_{n\geq 1} b_n\psi(n)n^{-k/2}V_{k,m}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^m n\right)$$
$$+\omega_f \sum_{n\geq 1} b_n\overline{\psi(n)}n^{-k/2}V_{k,m}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^m n\right)$$

where

$$b_n = \sum_{l_1\cdot l_2\cdots l_m=n} a_f(l_1)a_f(l_2)\cdots a_f(l_m).$$

Thus by Cauchy-Schwarz, we have

$$\sum_{f\in\mathcal{H}_k}^h L(k/2, f\otimes\psi)^{2m}$$
$$\ll \sum_{f\in\mathcal{H}_k}^h \left|b_n\psi(n)n^{-k/2}V_{k,m}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^m n\right)\right|^2 + \sum_{f\in\mathcal{H}_k}^h \left|b_n\overline{\psi(n)}n^{-k/2}V_{k,m}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^m n\right)\right|^2$$

We first treat the first sum. The analysis for the second sum is identical.

$$\sum_{f\in\mathcal{H}_k}^h \left|b_n\psi(n)n^{-k/2}V_{k,m}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^m n\right)\right|^2$$
$$=\sum_{f\in\mathcal{H}_k}^h \left|\sum_{\substack{l_i\geq 1\\ 1\leq i\leq m}} a_f(l_1)\cdots a_f(l_m)(l_1\cdots l_m)^{k/2}\psi(l_1\cdots l_m)V_{k,m}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^m l_1\cdots l_m\right)\right|^2$$
$$=\sum_{f\in\mathcal{H}_k}^h \left|\sum_{l_1 l_2\cdots l_m\leq \frac{2^{m/2}rk}{\pi}} a_f(l_1)\cdots a_f(l_m)(l_1\cdots l_m)^{k/2}\psi(l_1\cdots l_m)\right.$$
$$\left.\cdot V_{k,m}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^m l_1\cdots l_m\right)\right|^2 + \mathrm{O}(k^{2m+1}e^{-2k})$$
$$\leq \sum_{f\in\mathcal{H}_k}^h \left|\sum_{l_1\leq \frac{2^{m/2}rk}{\pi}} a_f(l_1)(l_1)^{-k/2}\psi(l_1)\right|^2 \sum_{f\in\mathcal{H}_k}^h \left|\sum_{l_2\leq \frac{2^{m/2}rk}{\pi l_1}} a_f(l_2)(l_2)^{-k/2}\psi(l_2)\right|^2$$
$$\cdots\sum_{f\in\mathcal{H}_k}^h \left|\sum_{l_m\leq \frac{2^{m/2}rk}{\pi l_1\cdots l_{m-1}}} a_f(l_m)(l_m)^{-k/2}\psi(l_m)V_{k,m}\left(\left(\frac{2\pi}{\sqrt{p}r}\right)^m l_1\cdots l_m\right)\right|^2 + \mathrm{O}(k^{2m+1}e^{-2k}).$$

Using (*) to bound each term in the product gives

$$\sum_{f\in\mathcal{H}_k}^{h} L(k/2, f\otimes\psi)^{2m} \ll (\log k)^{2m}.$$

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