# WHEN IS A PROOF A SATISFACTORY EXPLANATION FOR THE READER?

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## ABSTRACT

When is a Proof a Satisfactory Explanation for the Reader?

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Recent literature in Education and Philosophy has emphasized the importance of explanatory proofs for both mathematical practice and pedagogy. However, it remains unclear when, exactly, a proof qualifies as an explanation for a reader. This thesis investigates one potential factor in a reader's satisfaction with a proof as an explanation: the agreement between the reader's conceptual metaphors and the proof's metaphorical language. A conceptual metaphor is a cognitive mechanism, first posited in Cognitive Linguistics, used to understand abstract concepts. When reading a proof involving continuity, do a reader's conceptual metaphors for continuity influence their satisfaction with the proof as an explanation? To answer this question, we conducted a case study of four students in an undergraduate course in Analysis.

Using two semi-structured clinical interviews - conducted before and after the classroom lecture on continuity - we determined the subjects' conceptual metaphors for continuity and their satisfaction with three proofs as explanations. Before instruction, every subject appeared to understand continuity using the CONTINUITY IS GAPLESSNESS metaphor. After instruction, a new metaphor was apparent: CONTINUITY IS PRESERVATION OF CLOSENESS. The post-instruction interview presented three proofs of the same theorem, which - while mathematically equivalent - differed in their metaphorical language. The subjects were more satisfied with a proof as an explanation when it employed metaphorical language that reflected their own conceptual metaphors. Thus, the results support the conjecture that a reader's conceptual metaphors play a role in their satisfaction with a proof as an explanation. We discuss implications for teaching.

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# 1 INTRODUCTION

## 1.1 INTRODUCTION

Until recently, the Philosophy of Mathematics consisted of a quest for *foundations*. Philosophers tried to supply an infallible foundation for infallible mathematics; and when, in each case, the foundations were found to be flawed – incomplete, inconsistent, or, even worse, paradoxical! – philosophers would merely return to the drawing board and start anew. For the foundational philosopher of mathematics, mathematics was interesting because its truths were universal and necessary; the actual practice of mathematicians, on the other hand, was messy. This foundational quest painted a picture of mathematics that ignored mathematical practice.(Hersh, 1997)

That picture is changing. Recent work in the Philosophy of Mathematics has begun to attend to elements of mathematical practice, and this turn has breathed new life into a moribund field. Philosophers have finally begun to investigate varied topics, ranging from the implications of computer proofs to the nature of visualization.(Mancosu, 2008b) In the words of Mancosu (2008a), "Contemporary philosophy of mathematics offers us an embarrassment of riches."

This development in the Philosophy of Mathematics has brought it into close proximity with work in Mathematics Education. Indeed, both the philosopher of mathematics and the scholar of the teaching and learning of mathematics now have common interests. One particularly pressing topic for both domains is mathematical explanation. For the philosopher, the phenomenon of explanation is well known from the Philosophy of Science more generally. Articulating a characterization of explanation in mathematics, however, has proven difficult.(Mancosu, 2001) For the educator, explanation is central to understanding; in the classroom, the teacher wishes to explain *why*, and not just convince or demonstrate.(Hanna, 1990) Therefore, for both the philosopher and the scholar of mathematics education, the characterization of mathematical explanation is a pressing concern.

## 1.2 MOTIVATION AND OVERVIEW OF THE STUDY

The purpose of this investigation is to characterize explanation as it arises in mathematical proof.

RESEARCH QUESTION 1: When is a proof also an explanation?

The turn towards practice in the Philosophy of Mathematics has also lead philosophers to embrace a broader range of tools. Traditionally, philosophers have been wary of any appeals to psychology when reasoning about mathematics. The most famous attack against psychological explanations of aspects of mathematics was mounted by Frege against the so-called psychologism of Husserl:

> The fluctuating and indeterminate nature of these forms stands in stark contrast to the determinate and fixed nature of mathematical concepts and objects... [P]sychology should not suppose that it can contribute anything at all to the foundation of arithmetic....(Frege, 1997, p.88)

Now that the philosophy of mathematics has moved beyond foundations, the time is ripe for a return to psychological considerations. Indeed, the problem of characterizing mathematical explanation is a particularly promising domain for the deployment of tools and results from psychology. Therefore, throughout this investigation we will treat mathematical explanation as a cognitive phenomenon, and will modify our research question accordingly:

RESEARCH QUESTION 2: When is a proof a *satisfactory* explanation *for the reader*?

We will further justify this cognitive turn in Chapters 2 and 3.

Our cognitive characterization of explanatory proofs draws on two frameworks: an account of explanation developed by Ajdukiewicz (1974) and described by Sierpinska

(1994); and a characterization of mathematical cognition as fundamentally *metaphorical* by Lakoff and Núñez (2001). As a quick preview of things to come, these frameworks produce the following conjecture:

CONJECTURE A proof is a satisfactory explanation for the reader when the metaphorical content of the proof agrees with the reader's own conceptual metaphors.

Our empirical study of mathematical explanation consisted of four case studies of undergraduate students in an undergraduate Analysis class in a large, urban, North American University. In this system, students taking Analysis would have already taken several courses in Differential and Integral Calculus, including both one-variable and multi-variable Calculus courses. In this context, the research question becomes: When is a proof involving continuity a satisfactory explanation for an undergraduate student in Analysis? If our conjecture is correct, the subjects will judge a proof more explanatory if it uses metaphorical language that reflects their own conceptual metaphors for continuity. To test this conjecture, we used two semi-guided clinical interviews to determine their conceptual metaphors and their satisfaction with proofs as explanations. The results of this investigation revealed that, indeed, a similarity of the metaphorical language of a proof to the reader's own metaphors for continuity did result in their satisfaction with the proof as an explanation.

## 1.3 ORGANIZATION OF THE THESIS

This thesis consists of 7 chapters, followed by an Appendix.

In the next chapter, we will review the literature on relevant topics: mathematical explanation, as it has been treated by both educators and philosophers; the concept of continuity in mathematics, as it developed historically and as students come to understand it in the classroom; and metaphor, both in cognition generally and in mathematics especially. This Review of Literature will inform Chapter 3, the Theoretical Framework, in which we will outline the theoretical commitments that underpin this investigation.

From theory, we will move to practice: Chapter 4 will describe and justify the procedures and research instruments used to conduct interviews with four undergraduate mathematics students. Chapter 5 will present structured accounts of the four subjects' behaviour in the interviews, with a focus on their conceptual metaphors.

The next chapter, Chapter 6 will further analyse the subjects' behaviour from the point of view of the research question, applying the analytic tools presented in the Methodology chapter (4). Finally, Chapter 7 will discuss the implications of the study for our understanding of mathematical explanation, for the conceptual development of students, and for teaching.

The two questionnaires, as well as the unabridged transcripts, are included as Appendices.

# 2 REVIEW OF LITERATURE

## 2.1 INTRODUCTION

In this chapter, we will survey the literature on mathematical explanation, continuity, and metaphor. Research on explanatory proof can be found in the literature of both Philosophy and Education. This literature argues for the existence and importance of mathematical explanations, and attempts to characterize explanation. We will also survey the literature on continuity, primarily from Mathematics Education and Cognitive Science. Finally, we will review the research on Conceptual Metaphor theory, as it arose in Cognitive Linguistics and as it has been used as a framework in Mathematics Education.

## 2.2 EXPLANATION

### 2.2.1 *Mathematical Explanations Exist*

According to Mancosu (2001), we must look back to Aristotle for the first considerations of mathematical explanation. Mancosu interprets Aristotle's distinction between demonstrations "of the fact" and demonstrations "of the reasoned fact" as a distinction between explanatory and non-explanatory proofs. Moreover, Mancosu insists that this distinction informed mathematical practice over the last four centuries (Mancosu, 2008a). For instance, considerations of explanation influenced the mathematics of Guldin in the seventeenth century (Mancosu, 1996) and Bolzano and Cournot in the nineteenth (Kitcher, 1975; Mancosu, 2001). The importance of explanation for mathematical practice continues to this day. Mancosu quotes the mathematician Moddell, describing a proof that does not explain:

> Even when a proof has been mastered, there may be a feeling of dissatisfaction with it, though it may be strictly logical and convincing [...]. The argument may have been presented in such a way as to throw no light on the why and wherefore of the procedure or on the origin of the proof or why it succeeds. (quoted in Mancosu (2001))

Clearly, mathematicians recognize the existence of explanatory proofs.

Furthermore, explanation is not a peripheral goal of mathematical practice. Thurston (1994) gives a personal account of the importance of explanation in mathematical practice. He writes of the difficulties he encountered when his proofs – while non-trivial and logically valid – did not impart understanding. Due to his "idiosyncratic ... personal mental models," Thurston's proofs were not explanatory for other mathematicians. The fact that other mathematicians could not extract "personal understanding" from his proofs resulted in a "dramatic evacuation" from his field.(p.13) Explanation – or lack thereof – impacted mathematical practice. Similarly, Byers (2007) recounts an incident when mathematical practice aimed at explanation:

> These results have been proved by Richard Borcherds, using vertex algebras, but it's still not really understood what is happening. So, our problem is to understand what's going on. (Mathematician Helena Verrill quoted in Byers (2007))

Similarly, Hafner and Mancosu (2005) compile an extensive list of examples drawn from contemporary mathematics where a concern for explanation informs mathematical practice. Therefore, not only do mathematical explanations exist, but they play an important role in mathematical practice.

Within Mathematics Education, a number of authors note that proofs serve many functions, one of which is to explain (Balacheff, 2008; de Villiers, 1990; Hanna and Jahnke, 1996). Further dissecting the functions of proof, Sandborg (1997) presents a typology of mathematical explanation. After an extensive investigation of the abstracts of contemporary mathematics articles, he lists a number of ways in which the word "explanation" is used to refer to proofs, ranging from explaining why a certain method is used (Sandborg, 1998), to explaining why the result is true. This thesis is concerned with the latter sense of explanation. This bears repeating: For our purposes, an explanatory proof is one that explains *why the result is true.*

More recently, Dawson (2006), responding to the influential article of Rav (1999), argues that the practice of reproving theorems demonstrates that mathematicians use proof for many functions besides verification, one of which is explanation. Thus, it is apparent that – in attempting to characterize mathematical explanation – we are not tilting at windmills: Mathematical explanation exists and is an integral part of mathematical practice.

Explanation is especially valued in pedagogy. Generally, explanations serve a number of purposes in the classroom. Lombrozo (2006) reviews the literature on the effects of explanation on learning. Often, explanations "serve to support the broader function of guiding reasoning" (Lombrozo, 2006). In particular, Lombrozo discusses the so-called "self-explanation" effects on learning, in which self-explanations increase retention and transfer of learned knowledge; Chi (1994) demonstrates that this is also true for self-explanations in mathematics. Gopnik (2000) argues that, in general, our experience of explanation indicates that our natural drive towards theory-formation has been fulfilled – that is, we experience the "aha!" moment of explanation when we have created a coherent theory of the phenomena under consideration. In general, explanations are closely connected with understanding (Keil, 2006; Lombrozo and Carey, 2006; Sierpinska, 1994).

In the mathematics classroom, explanations are often in the form of proofs (Hersh, 1993; Davis and Hersh, 1999; Hanna, 1990, 2000). Davis and Hersh (1999) claim that the research mathematician only desires proofs that demonstrate – against the considerable evidence presented above. However, they agree with Hanna that, in the classroom, explanation reigns supreme. As Hanna (2000) puts it,

> But in the classroom, the fundamental question that proof must address is surely 'why?'. In the educational domain, then, it is only natural to view

> proof first and foremost as explanation, and in consequence to value most highly those proofs which best help to explain.

Thus, while important in mathematical practice generally, explanatory proofs are particularly important for the teaching of mathematics. The question remains, however, of when a proof is explanatory.

### 2.2.2 *Characterizing mathematical explanation*

As mentioned, the oldest attempt to characterize mathematical explanation is due to Aristotle, which inspired later attempts by Cournot and Bolzano (Mancosu, 2001). More recently, the mathematician Bouligand appealed to the notion of *causality* to characterize mathematical explanation:

> Causal Proofs: Many theorems can be given different demonstrations. The most instructive are of course those that let one understand the deep reasons of the results that one is establishing. On this matter the notion of domain of causality gives us a guide. (Bouligand, 1937, translated by Mancosu (2001))

Causal considerations are particularly common in characterizations of *scientific* explanation (Salmon, 1990). However, unlike for the natural sciences, we have no shared intuitions of causality in mathematics. Indeed, while logical inference is symmetric, causality is inherently assymetric. This is not to say, for instance, that all logical implications are necessary and sufficient; certainly, some implications are unidirectional. However, when the implication *does* go both ways, there is no unproblematic way to determine the direction of causality.

As part of a larger treatment of explanation in general, Kitcher (1981) presents a characterizing of mathematical explanation. According to Kitcher, an explanation amounts to a *theoretical unification*. While Kitcher's framework is meant to treat explanation in the sciences generally, he considers it a credit to his theory that it can also account for mathematical explanation (Kitcher, 1984, 1989). However, Sandborg (1998) objects that Kitcher's framework – while possibly descriptive of some mathematical explanations – certainly cannot account for all explanatory proofs.

The most influential philosophical account of mathematical explanation is the *characterizing property* approach of Steiner (1978). According to this view, an explanatory proof must make "reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the results depend on the property."(p.143) As Mancosu (2001) observes, this implies that some proofs will be inherently explanatory. Resnik and Kushner (1987) take issue with this implication, noting that explanation is always context-dependent – a proof is never explanatory as such, but only explanatory *relative* to the "why-questions" we wish to answer.

It is also relative to the knowledge and habits of the mind of the person asking these questions, according to the scholars of Pragmatic Logic. This relativity was present in the words of, for example, Ajdukiewizc, an early 20$^{th}$ century Polish philosopher (see Sierpinska (1994, pp. 73-74); Ajdukiewicz (1974)). Ajdukiewicz distinguished between the processes of proving and explaining in the following way. Suppose a person P is faced with a reasoning R leading from statement A to statement B. The reasoning would be an *explanation* of B for P only if P knew in advance that B was true, before it was deduced from A. If, on the other hand, P decided to accept B as true only as a result of the reasoning R and did not know before whether B is true, then this reasoning would have functioned as a *proof* of B for P (Ajdukiewicz, 1974, p. 443). This definition, however, is based on *a priori* categorization; it is not grounded in systematic observations and analyses of actual mathematical practice. Moreover, to be consistent with this theorey, we would have to give up using the word "proof" in the ordinary sense in which it is used in mathematics and speak about "reasoning functioning as proof for the person P." This would be very awkward. Sierpinska (1994) preferred to keep using the word "proof" in its ordinary mathematical meaning and defined explanation (of something accepted as a state of affairs) as the process of *founding the understanding of this something on a new basis* (p.77). She distinguished between "scientific explanation" where the new basis is more conceptual, and "didactic explanation" – used mostly in teaching – where the new basis is less conceptual (an image, a metaphor, more familiar knowledge, etc.). Our theoretical framework uses a modified version of these accounts of explanation, refashioned to predict the cognitive prerequisites for reading satisfaction with a proof as an explanation. We will describe the details in Chapter 3.

A similar point of view – relativity to the reader – is found in the more recent semiotic approaches to proof (Ernest, 2001; Rotman, 2001; Ernest, 2008; Radford, 2001; Duval, 1995). The reader of the proof cannot be separated from the proof itself, and so the reader is an essential part of the context of explanation.

Much of the research to date has considered the role of the proof's format – whether inductive, geometric, etc – at the expense of considering the reader, although recent research begun to correct that omission. Martin and Harel (1989) examine the proof-preferences of in-service mathematics teachers in undergraduate mathematics classes. They found that students valued certain proof formats over others – for instance, proofs that identify a pattern in a sequence of related examples. However, their research only considers student attitudes – not student cognition – and does not explicitly look at proofs as explanations. Hoyles (1996; 1997), following Martin and Harel, looks at the influence of proof format on student satisfaction with proofs as explanations, and finds that proof format does, indeed, play a role. Moreover, she considers the importance of student views on proof, thus involving the reader in her analysis. For instance, Healy and Hoyles (2000) identify two different notions of proof: those they would choose for themselves, and those they would expect to receive the highest marks. Reid (2002); Reid and Roberts (2004), investigating elementary- and high-school student satisfaction with proofs as explanations, again focuses primarily on proof format, but concludes that familiarity and clarity, among other factors, also played a role. Thus, recent research has begun to identify factors in the reader – familiarity, views on proof – that influence their satisfaction with proofs as explanations. Unfortunately, all the existing research on student satisfaction with proofs as explanations deals with primary- and secondary-school students; there is no research on undergraduate students.

To date, no attempt has been made to connect these empirical results to the philosophical characterizations of explanation described above. Hanna (1990), for instance, adopts Steiner (1978)'s *characterizing property* framework to generate explanatory proofs, but never verifies whether these proofs are, indeed, explanatory for the student. In this thesis, we will attempt to empirically verify whether the theoretical account of explanation used in Sierpinska (1994) and Ajdukiewicz (1974) accurately models student satisfaction with proofs as explanations – in particular, with proofs that deal with continuity.

## 2.3 CONTINUITY

While much has been written about limits (Cornu, 1991; Davis and Vinner, 1986; Robert, 3; Sierpinska, 1985; Tall and Vinner, 1981; Cottrill et al., 1996), there exists comparatively little literature on continuity. Some of the earliest theoretical reflections are in Daval and Guilbaud (1945), who write of the historical origin of the definition of continuity in methods of finding solutions to equations of the form $f(x) = c$ (e.g. $x^2 = 2$) using successive approximations. Convergence of sequences $f(x_n)$ to $f(a) = c$ for sequences $x_n$ converging to $a$ is sufficient for the technique of successive approximations to work and it is this sufficient condition that Cauchy chose as a definition condition of a function continuous at a point (Daval and Guilbaud, 1945, p. 116). The concept was thus born by putting aside the metaphor of continuous lines whose intersection must necessarily produce a point, and focusing attention on the technical details of a computational procedure used to calculate decimal expansions of radicals. The metaphor was making the existence of solutions obvious, but was useless in reasoning about objects that could not be represented graphically or numerically because of their general character, which is common in proofs.

> How was the idea of continuity born? Thanks to a reflection on an already performed mental operation, but which became clear, more precise and permanent as the mind looked at what it has been doing rather than continue doing. By becoming aware of what it has been doing, the mind could

> see it in detail; a skilled athlete can perform a perfect jump, but only an anatomist is capable of detailing the movements of the muscles necessary for the success; the mathematician is, at the same time, an athlete and an anatomist. (Daval and Guilbaud, 1945, p. 117; translated from French by Anna Sierpinska)

While the authors consider Cauchy's concept quite intuitive, they express astonishment at the word chosen to name it. "Continuity" in ordinary language, they say (p.118), evokes ideas of something that is "uninterrupted, without gaps," while the definition of continuous function – "small causes make small effects" – makes the authors think of "what is called, in mechanics, stability" (ibid.). The "continuity" metaphor fits rather well with the conclusion of the Intermediate Value Theorem[1], and this may be the reason why, for many, continuous functions are incorrectly identified with functions satisfying this conclusion. The authors quote Lebesgue as recounting that, as late as 1903, the converse of Bolzano's theorem[2] was still taught in Parisian universities. It is this inadequacy of the word "continuity" as a metaphor and the concept it denotes, that was partly at the bottom of our choice not to use this word in pre-instruction interviews with the students, where we wanted to elicit their "concept images" (see next paragraph) of the notion, rather than associations brought about by the word "continuity."

Tall and Vinner (1981) differentiate between the *concept definition* and *concept image* of continuity. The concept definition is the collection of words and symbols used by the teacher, textbook, etc to define the concept. The concept image, on the other hand, is the "total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes."(p.152) They apply this construct to continuity and find that students have a variety of preconceptions which contribute to their concept image:

1 If $f$ is continuous on $[a, b]$, and $c$ is a number between $f(a)$ and $f(b)$, then the equation $f(x) = c$ has at least one solution in $(a, b)$.

2 Bolzano's theorem is the special case of the Intermediate Value Theorem where $c = 0$, $f(a) < 0$, and $f(b) > 0$.

> The concept image derives initially from a variety of sources, for instance the colloquial usage of the term "continuous" in such phrases as "the rail is continuously welded" (meaning the track has no gaps).(p.13)

This results in a concept image that is both dynamic and reliant on a notion of gaplessness:

> The net effect on the concept image, however, is, almost certainly, a reinforcement of the intuitive idea that the graph has "no gaps" and may be drawn freely without lifting the pencil from the paper. (p. 14)

Similarly, Núñez (1998) traces the historical genesis of continuity to an intuitive, dynamic idea of gaplessness. This "natural continuity," which Núñez ascribes to Euler and other mathematicians of the eighteenth century, employs notions of motion and gaplessness. Núñez argues that developments in Analysis throughout the nineteenth century resulted in a new way of understanding continuity: as preservation of closeness. This analysis of continuity is part of a comprehensive project by Lakoff and Núñez (2001) to deploy the tools of cognitive science to understand mathematical concepts – and it is within this framework that we will conduct our investigation. In particular, they extend an approach due to Lakoff and Johnson (1980) called Conceptual Metaphor Theory.

## 2.4 CONCEPTUAL METAPHOR

A number of authors have recently explored the role of metaphor in mathematics (e.g., Pimm (1988); Lakoff and Núñez (2001); Manin (2007); Byers (2007)), and particularly in the learning of mathematics (e.g., Presmeg (1992, 1997); Sfard (1994); English (1997)). In this investigation, we employ one particular approach to metaphor: Conceptual Metaphor Theory. Lakoff and Johnson (1980) introduced conceptual metaphors to the field of Cognitive Linguistics as a tool to understand language production and understanding. A conceptual metaphor is a cognitive mechanism for understanding (abstract) conceptual domains. Technically, it is a mapping between two domains, in which the *target domain* is understood by borrowing the semantic structure of the *source* domain. We will discuss the precise details of this mechanism in Chapter 3. As mentioned above, Núñez (1998) uses the conceptual metaphor framework to analyse the concept of continuity. Empirical research has confirmed the importance of conceptual metaphors in the learning of mathematics, from arithmetic in elementary school (Chiu, 2001), to word problems (Danesi, 2007) and algebra (Ferrara, 2003) in high school. These results on conceptual metaphor, however, have yet to be replicated in university mathematics students.

Conceptual Metaphor Theory falls under the general umbrella of Embodied Cognition (Anderson, 2003; Lakoff and Núñez, 2001; Shapiro, 2007). Embodied cognition extends cognition from within the confines of the brain to include the way we use our bodies to understand and think. In many cases, conceptual metaphors "extend the structure of bodily experience [...] while preserving the inferential organization of these domains of bodily experience" (Núñez, 2008, p.162). Thus, conceptual metaphors tap into our (shared) bodily experiences to give meaning to abstract concepts – and, to mathematical concepts (Núñez et al., 1999).

One tool of embodied cognition is gesture analysis. In his seminal work, MacNeill (1992) argues that gestures are a window into thought. Spontaneously coproduced with thought, gestures reveal aspects of thought that may be hidden by deliberate speech. Recently, gesture has received considerable attention in Mathematics Education – see, for instance, the special issues of Nemirovsky et al. (2004); Edwards et al. (2009). Gesture has been found to play a role in both the teaching (Goldin-Meadow et al., 1999, 2001) and learning (Cook et al., 2008; Nemirovsky and Ferrara, 2009; Arzarello et al., 2009) of mathematics, and has been successfully deployed to identify subjects' implicit conceptualizations (Núñez, 2006; Alibali et al., 1999).

## 2.5 SUMMARY

In this chapter, we reviewed the existing literature on three topics: mathematical explanation, continuity, and metaphor. In the next chapter, we will draw on two particular frameworks from within this literature – the account of explanation in Sierpinska (1994) and Ajdukiewicz (1974), and the Conceptual Metaphor analysis of continuity in Lakoff and Núñez (2001) – to build our theoretical framework.

# 3 THEORETICAL FRAMEWORK

## 3.1 INTRODUCTION

To characterize mathematical explanation, we will combine two theoretical frameworks. First, we will adapt the account of explanation developed by Ajdukiewicz (1974) and used in the context of mathematical understanding by Sierpinska (1994). According to this account, explanation involves situating a result in a *basis of understanding* – and, to specify this basis of understanding, we will draw on the Conceptual Metaphor Theory developed by Lakoff and Johnson (1980) and deployed to study mathematics by Lakoff and Núñez (2001). Furthermore, we will introduce an institutional framework, which will prove useful in analysing our results.

## 3.2 MOTIVATION

In order to motivate a cognitive approach to mathematical explanation, let us examine a debate in the literature. In discussing explanatory proofs, both Brown (1999) and Hanna (1990) discuss proofs of the following theorem:

**Theorem 3.2.1** (Gaussian Sum). *The sum of the first* $n$ *integers is equal to half of the product of* $n$ *and* $n+1$*. That is,*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

While both authors consider the same three proofs of this theorem, they come to contradictory conclusions. Hanna claims that the proof apocryphally attributed to Gauss (which exploits symmetry) and a visual proof are both explanatory, while an inductive proof is not; Brown argues for the opposite. He writes:

> In the two number theory cases above [one of which is the theorem in question], a proof by induction is probably more insightful and explanatory than the picture-proofs. I suspect that induction – the passage from $n$ to $n + 1$ – more than any other feature, best characterizes the natural numbers. That's why a standard proof by induction is in many ways better.(Brown, 1999, p.42)

How is it that two mathematically mature authors can disagree so adamantly about the explanatory status of the proofs?

In the natural sciences, there is a general consensus about the identity of explanations; the debate revolves around the proper characterization of those explanations.(Salmon, 1990) In mathematics, however, authors cannot even decide on which proofs should qualify as explanatory. As we saw in the the Review of Literature in Chapter 2, characterizations of mathematical explanation have often been internal to mathematics – that is, "explanatory" was an inherent feature of a proof; it did not refer to a relation between the proof and the reader. The characterizing property model of Steiner (1978) and the unificatory model of Kitcher (1981) both appeal to the mathematics and ignore the mathematician. Thus, the proofs identified as explanatory by these models all fail to account for divergence in reader satisfaction with proofs as explanations. However, the example of Hanna and Brown forces us to consider the role of the mathematician – as a reader or producer of proofs – in the phenomenon of mathematical explanation.

Thus, we will assume that mathematical explanations are never explanatory as such, but only in relation to the reader. In particular, we will turn to the reader's cognition to characterize explanation. In the next two sections, we will introduce a cognitive model of mathematical explanation and an associated model of mathematical cognition

## 3.3 EXPLANATION

To characterize explanation, we will turn to the classification of types of reasonings presented by Ajdukiewicz (1974). In problem-based reasoning, the reasoner has a particular epistemic goal in relation to a statement or a state of affairs. For instance, if they are attempting decide whether or not a statement is true, then this is an example of *verifying*, a particular kind of inductive reasoning. According to Ajdukiewicz, an explanation is a problem-based reasoning that answers a 'why' question in relation to a state of affairs taken as true. In an explanation, the state of affairs to be explained – the explanandum – is already known; unlike verification, the purpose is not to increase our conviction that the state of affairs is true – which we already trust – but rather to know *why* it is true.(Sierpinska, 1994, p.74-75)

According to Sierpinska (1994, p.75), "an explanation of some state of affairs aims at founding its understanding on a different basis." Thus, in answering *why*, an explanation situates the state of affairs – say, the theorem to be explained – in a new basis of understanding. By a basis of understanding, Sierpinska and Ajdukiewicz mean a system of mental representations, although they disagree about the nature of those representations.(Sierpinska, 1994, p.49) Ajdukiewicz posits that set of representations is exhausted entirely by concepts and mental images, but Sierpinska objects that mental representations are far more varied. Nevertheless, the basic idea is the same: An explanation answers 'why' by situating the state of affairs in a system of mental representations.

For Ajdukiewicz, this account of explanation was not meant as a predictive model of explanation as it arises in practice. Rather, it is an a priori *definition*; he is classifying possible types of reasoning, not attempting to characterize a cognitive phenomenon. For our purposes, however, his account is promising as a cognitive characterization of explanation as it arises in mathematical practice.

One possible hitch: We are trying to determine precisely when a proof *is* an explanation for a reader, but Ajdukiewicz claims that proofs and explanations are mutually exclusive classes of reasoning. How then can a demonstration be both a proof *and* an explanation, simultaneously? This conflict is due to our different uses of the term "proof." Ajdukiewicz uses the term "proof" to refer to a particular type of reasoning; we are using "proof" to refer to a text produced within a mathematical practice, following certain norms. He means a proof-reasoning; we mean a proof-text. Therefore, a proof-text can serve as a proof-reasoning, or as an explanation, or as both – *depending on the purpose for which the reader is using it.* If the reader is using the proof to increase their conviction in the theorem, then the proof is also a proof in the sense of Ajdukiewicz; if the reader is using the proof to understand *why* the theorem is true, then the proof is an explanation.

Thus, we will adopt Ajdukiewicz's definition as a conjecture about mathematical explanation as it arises in the reading of proof-texts:

CONJECTURE: A proof is a satisfactory explanation for the reader when it situates the result in the reader's existing basis of understanding.

Unfortunately, this account overgenerates as a model of explanation for the reader – it predicts explanations where none exist. A reader may *want* to understand why the theorem is true, and the proof may produce a statement to the effect of, "Theorem X is true because of Proposition Y," situating the Theorem X in the basis of understanding of Proposition Y. However, this is not enough to produce an explanation in our sense. For instance, recall the experience of Thurston (1994), who writes about mathematicians who wished his proofs were explanations, but who were incapable of extracting any understanding from his proofs. Something is missing. This missing ingredient is a precise specification of the basis of understanding. For this, we turn to Conceptual Metaphor Theory.

## 3.4 CONCEPTUAL METAPHOR THEORY

Recent work by Lakoff and Núñez (2001) presents a promising and fruitful investigation of one of the cognitive prerequisites for mathematical thought: Conceptual Metaphor. First developed by Lakoff and Johnson (1980), Conceptual Metaphor Theory treats abstract concepts as fundamentally *metaphorical* and *embodied*. Not a mere literary device, a conceptual metaphor is a precise cognitive mechanism for creating meaning.

A conceptual metaphor is a relation between two cognitive domains, such that the conceptual structure of the source domain is mapped onto the target domain. Often, the source domain will be embodied – that is, it will deal with bodily experiences and sensations – in which case the mapping is called a *grounding metaphor*. Examples of embodied experiences include movement, contact, containment, etc. Figure 1 depicts the grounding metaphor between mathematical sets and the embodied experience of containment, SETS ARE CONTAINERS [1]. In the table, the source domain is listed on the left, and the arrows indicate the transfer of conceptual structure into the target domain on the right. When the source and target domain are not embodied but complex, abstract concepts, then the mapping is a called *linking metaphor*. Lakoff and Núñez (2001) argue that, through a chain of grounding and linking metaphors, the mind creates and gives meaning to advanced abstract mathematics.

Figure 1: SETS ARE CONTAINERS, adapted from Lakoff and Núñez (2001).

| CONTAINERS | $\longrightarrow$ | SETS |
|---|---|---|
| Bounded regions in space | $\longmapsto$ | Sets |
| Objects inside the bounded regions | $\longmapsto$ | Set elements |
| One bounded region inside another | $\longmapsto$ | A subset of a larger set |
| Excluded Middle: Every object $x$ is either in or out of a container $A$. | $\longmapsto$ | Excluded Middle: Every element $x$ is either in or out of a set $A$. |
| Modus Ponens: Given two containers $A$ and $B$ and an object $x$, if $A$ is in $B$ and $x$ is in $A$, then $x$ is in $B$. | $\longmapsto$ | Modus Ponens: Given two sets $A$ and $B$ and an element $x$, if $A$ is in $B$ and $x$ is in $A$, then $x$ is in $B$. |
| Modus Tollens: Given two containers $A$ and $B$ and an object $y$, if $A$ is in $B$ and $y$ is outside $B$, then $y$ is outside $A$. | $\longmapsto$ | Modus Tollens: Given two sets $A$ and $B$ and an element $y$, if $A$ is in $B$ and $y$ is outside $B$, then $y$ is outside $A$. |

Núñez (1998) maps out some of the grounding metaphors used to understand the continuity of a function. He contrasts the metaphors of Euler's *natural continuity* with those used in the Cauchy-Bolzano-Weierstrass definition of continuity. While both

1 Throughout, metaphors will be written in small caps, in the form TARGET DOMAIN IS SOURCE DOMAIN.

concepts go by the name of continuity, each presupposes a different understanding of continuity. An analysis of the conceptual metaphors reveals the structure of the ideas involved.

The concept of "natural continuity," which Núñez (1998) ascribes to Euler and other mathematicians of the eighteenth century, has two essential features (p.88):

- "It is continuity traced by motion, which takes place over time."
- "The trace of the motion is a static holistic line with no 'jumps.'"

In natural Eulerian continuity, functions are understood as dynamic curves, where the independent variable is in fictive motion. Continuity of a function is understood as gaplessness of the representative curve; this is sometimes expressed as "being able to draw the curve without lifting the pen." The corresponding metaphor, CONTINUITY IS GAPLESSNESS[2], is described in Figure 2. It is this intuition of gapless motion that is most often used to introduce continuity in first courses on Calculus.[3]

Figure 2: CONTINUITY IS GAPLESSNESS, adapted from Lakoff and Núñez (2001)

| POINTS IN SPACE | $\longrightarrow$ | REAL NUMBERS |
|---|---|---|
| Discrete point-locations | $\mapsto$ | Discrete numbers |
| Motion along a curve | $\mapsto$ | Functions |
| A curve moving over a domain of points | $\mapsto$ | Functions: mappings between sets of discrete real numbers |
| Motion without jumps | $\mapsto$ | Natural continuity |
| A curve is continuous iff you can draw it without lifting your pen | $\mapsto$ | A function is continuous iff $\lim_{x \to a} f(x)$ is equal to $f(a)$ for any $a$ in the domain. |

The arithmetization of Analysis by Bolzano, Cauchy and Weierstrass throughout the nineteenth century introduced a new metaphor for continuity (Núñez, 1998). Unlike the natural continuity of Euler, the new definition of continuity introduced by Cauchy

2 In Lakoff and Núñez (2001), "gaplessness" refers to a related, but different, metaphor. Its use here is partially inspired by the suggestion of Tall and Vinner (1981) that, for some students, the concept image of continuity is of a graph that has "no gaps."

3 Note that, while the motivating intuition is holistic, the metaphor tends towards a local conception of continuity. In natural continuity, a curve is continuous as a whole; in the CONTINUITY IS GAPLESSNESS metaphor, a function is continuous point-wise.

and Weierstrass makes no reference to fictive motion or gaplessness. Their goal was, in part, to reduce the reliance on geometric intuitions. While using the CONTINUITY IS GAPLESSNESS metaphor, a *naturally continuous* function is understood to have a graph that could be traced without jumps or gaps; on the other hand, the arithmetization of continuity allows us to consider continuity without appealing to geometric intuitions of motion or gaps. Instead, continuity is understood, statically, as preservation of closeness. A function is continuous at $a$ if we can make $f(x)$ arbitrarily close to $f(a)$ by taking $x$ sufficiently close to $a$ – or, in the words of Daval and Guilbaud (1945, p. 118), "small causes make small effects." The corresponding metaphor, CONTINUITY IS PRESERVATION OF CLOSENESS is described in Figure 3.

Figure 3: CONTINUITY IS PRESERVATION OF CLOSENESS, adapted from Lakoff and Núñez (2001)

| POINTS IN SPACE | ⟶ | REAL NUMBERS |
|---|---|---|
| Discrete point-locations | ↦ | Discrete numbers |
| Curves | ↦ | Sets of number |
| Point-locations on curves | ↦ | Numbers in sets of numbers |
| Functions as mappings from points on continuous curves to points on continuous curves | ↦ | Functions as mappings from discrete numbers in sets to discrete numbers in sets |
| A curve is continuous iff it preserves closeness between points | ↦ | A function is continuous iff it preserve closeness between numbers |

Moreover, this meaning structure is easily extended to topology. Topology is commonly described as the study of "closeness without distance."(Luk, 2005) For instance, open sets defined by distance in analysis become arbitrarily defined open sets in topology. Lakoff and Núñez (2001) even go so far as to say, "Weierstrass's preservation of closeness concept [...] is at the heart of modern topology"(p.323). The linking metaphors from the source domain of analysis into the target domain of topology are described in Figure 4.

Figure 4: OPEN SETS ARE CLOSENESS, adapted from Lakoff and Núñez (2001)

| ANALYSIS | $\longrightarrow$ | TOPOLOGY |
|---|---|---|
| Number-points | $\mapsto$ | Elements |
| $\epsilon$-neighbourhoods | $\mapsto$ | Open sets |
| Two points are close to each other | $\mapsto$ | Two elements are in an open set |
| A function is continuous $\iff$ if points are arbitrarily close in the image, then their pre-images are also close | $\mapsto$ | A mapping is continuous $\iff$ if a set is open in the image, then the pre-image of that set is also an open set |
| The function $f : \mathbb{R} \to \mathbb{R}$ preserves closeness: $\forall \epsilon, \quad \exists \delta \ni \lvert x - x_0 \rvert < \delta \implies \lvert f(x) - f(x_0) \rvert < \epsilon$ | $\mapsto$ | The function $f : X \to Y$ preserves open sets: $V$ is an open set in $Y \implies f^{-1}(V)$ is an open set in $X$ |

Therefore, the conceptual metaphor analysis of Lakoff and Núñez (2001) produces two metaphors for continuity: the dynamic CONTINUITY IS GAPLESSNESS and the static CONTINUITY IS PRESERVATION OF CLOSENESS. The latter is easily extended to a topological context by the OPEN SETS ARE CLOSENESS metaphor. By no means are these meant to exhaust the possible conceptual metaphors for continuity; according to Lakoff and Núñez, however, they were central to the historical and conceptual development of the concept. While Lakoff and Núñez derived these two metaphors for continuity from a historical and conceptual analysis of continuity, we will assume that they are also present in students' mental representations of continuity. Núñez (2006) analysed the discourse and gestures of mathematicians as they spoke about continuity, and concluded that these two metaphors were, indeed, in use. However, there is no evidence that these metaphors are used by undergraduate mathematics students. A secondary purpose of this study is to verify that these metaphors do, indeed, arise in students' understanding of continuity.

Thus, Conceptual Metaphor Theory suggests undergraduate students will understand the concept of continuity using two different conceptual metaphors. These metaphors are good candidates for the subjects' basis of understanding:

> Thus, by structuring, ordering our experience and making it 'fit in with' the existing mental structures, a metaphor is a basis of understanding. (Sierpinska, 1994, p.97)

That is, the students' basis of understanding consists, in part, of metaphorical representations of continuity.

## 3.5 SYNTHESIS

Recall our original research conjecture:

CONJECTURE: A proof is a satisfactory explanation for the reader when it situates the result in the reader's basis of understanding.

We now have sufficient theory to elaborate on our original conjecture. Since a basis of understanding is a system of mental representations, and conceptual metaphors are one particular kind of mental representation, we can rephrase our conjecture:

CONJECTURE: A proof is a satisfactory explanation for the reader when it employs conceptual metaphors that agree with those of the reader.

Of course, this is meant to characterize only one kind of explanation: explanations that explain *why* the result is true. Furthermore, it describes only one factor in readers' satisfaction with proofs as an explanation; it is quite possible that there are other mechanisms that would result in an experience of explanation upon reading a proof.

## 3.6 INSTITUTIONAL POSITIONING

In analysing the data, we encountered certain contradictions in the subjects' utterances. To explain this, we needed to adopt an institutional perspective on their behaviour. Sierpinska et al. (2008, p.292) characterize an institution as a stable "structural feature of a society," either formal or informal, that "constrains the individual behavior of its members [...] through more or less explicit and formal rules and norms." These members are referred to as 'participants,' and they "share certain values and goals and give common meaning to the basic actions of the institution."(p.292) For our purposes, since we are considering the behaviour of students registered in an analysis course in university, the institution is the *University Analysis* institution.

As members of this institution, the subjects of this study are constrained by a variety of rules and norms – norms governing proof practices, for instance, or rules about valid inferences. In their own study, Sierpinska et al. (2008, p.291) posit that the members of the institution in question positioned themselves in a variety of ways, and that this positioning determined the perspective from which they responded to the study. For our purposes, two of those institutional positions are essential:

LEARNER: when positioned as a Learner, the subjects behave as "cognitive subjects."

STUDENT: when positioned as a Student, the subjects behave as "subjects of a school institution who have to abide by its rules and norms."

By identifying the subjects institutional positioning throughout the study, we will be better able to interpret their behaviour. In Chapter 4, we will present an operationalized definition of these institutional positions.

## 3.7 SUMMARY

This chapter presented the theoretical underpinnings of this investigation: an account of explanation from Ajdukiewicz (1974) and Sierpinska (1994); and an account of (metaphorical) mental representation from Lakoff and Núñez (2001). Moreover, we presented an institutional framework that we will use to analyse the results of this investigation. To deploy these theoretical tools, we devised two semi-structured clinical interviews. In the next two chapters, we will describe and justify the methodology and present the results of the study.

# 4 METHODOLOGY

## 4.1 INTRODUCTION

This is a small-scale study, involving clinical interviews with only four subjects. The subjects were volunteers; they were not selected according to some predetermined criteria. Therefore, not only is the sample small, it is also an opportunistic sample. Hence, the study has only an exploratory value. Had this small study failed to confirm the conjecture posed in the research, the results would have been conclusive: the conjecture would be refuted. The conjecture, however, was confirmed in all four subjects. It was confirmed, but – as is often the case in exploratory studies – also refined, and therefore further research should be not only conducted on more subjects but also target a more subtle hypothesis.

At the time of the interviews, all four subjects were taking their first course in Analysis, with the same instructor and in the same class. The subjects were interviewed before and after they received instruction on the formal definition of continuity. The interviews were designed to evoke and identify the subjects' conceptual metaphors for continuity, and investigate their satisfaction with the three proofs as explanations.

In this chapter, we will first describe the two semi-structured clinical interviews, and justify their content and organization. Then we will describe the tools used to analyse and interpret the interview transcripts.

## 4.2 SEMI-STRUCTURED CLINICAL INTERVIEWS

### 4.2.1 *Subjects*

As mentioned above, the subjects were recruited, on a voluntary basis, among undergraduate students registered in the same section of an Analysis course. Students usually take this course in their second or third year of a Bachelor's program if they specialize in mathematics. There may be, however, students from other programs and other years of study taking the course in any given semester. Of the four students participating in the study, two were enrolled in the Pure and Applied Mathematics program, one was an Honours in Economics student, and one was pursuing a Bachelor's of Science degree in Computer Science (Table 1).

The Analysis course is not the first time students hear about continuity of functions; they would have already encountered the concept in the prerequisite Calculus courses. Calculus courses normally focus on techniques of solving a limited number of typical problems, based on intuitive or procedural understanding of concepts. It is only in Analysis courses that students are expected to understand a formal definition of continuity of a function at a point and its use in proofs. Students must understand proofs of theorems about continuity presented in lectures and textbooks well enough to be able to produce proofs of certain simpler statements themselves in assignments and tests. It is at this point of learning about continuity of functions that issues regarding proofs – such as how well a proof explains why the conclusion is necessarily true – start to matter for students. This is the reason why we chose to interview students in an Analysis course rather than students in a Calculus course.

| NAME | YEAR | ACADEMIC PROGRAM |
|---|---|---|
| Y | 2nd | B.Sc. Pure and Applied Mathematics |
| P | 3rd | B.Sc. Pure and Applied Mathematics |
| K | 3rd | B.A Honours Economics |
| J | 1st | B.Sc. Computer Science |

Table 1: Interview Subjects

Each pre- and post-instruction interview lasted approximately 30 minutes.

### 4.2.2 *Pre-Instruction Interview*

The first interview was conducted before the subjects were introduced to a formal definition of continuity in the course. Its purpose was to determine their conceptual metaphors for continuity before their instruction in the formal definition. It consisted of four questions – presented on paper – each of which introduced some information about a function and asked a series of sub-questions about that function. Continuity was never explicitly mentioned in the questions or by the interviewer, unless first mentioned by the subject; the interview questions were designed to spontaneously evoke a discussion of continuity.

We will examine each question in turn, justifying its design and explaining its purpose. The entire questionnaire, as presented to the subjects, is included as Appendix A. For simplicity, we will refer to Sub-question n of Question m as "Question m.n," and the function in Question m as "$f_m$" or "Function m."

*Question 1*

Question 1 only consisted of one sub-question:

1. The table below presents a fragment of the table of approximate values of some function $f : \mathbb{R} \mapsto \mathbb{R}$. Do you think the equation $f(x) = 0$ has a solution? Why or why not?

| $x$ | $f(x)$ |
|---|---|
| 1.3 | −0.4215907 |
| 1.4 | −0.0567776 |
| 1.5 | 0.3709375 |
| 1.6 | 0.8696576 |
| 1.7 | 1.4481457 |
| 1.8 | 2.1158368 |

This question was motivated by the following corollary of the Intermediate Value Theorem, sometimes called Bolzano's theorem or the Darboux property theorem.

**Theorem 4.2.1** (Bolzano). *If a real function* $f$ *is continuous on a closed interval* $[a, b]$, *and* $f(a) < 0, f(b) > 0$, *then there exists* $c \in (a, b)$ *such that* $f(c) = 0$.

From the point of view of the CONTINUITY IS GAPLESSNESS metaphor, this theorem says that, in going from negative to positive values or vice versa, a continuous function necessarily passes through zero. In the mindset of the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor, the theorem says that a continuous function that has values in the neighbourhood of zero on both sides of it, necessarily admits also the value zero itself. Since the table of values in Question 1 lists negative values of the function for $x < 1.4$, and positive values for $x > 1.5$, if the function is continuous, then the equation $f(x) = 0$ has a solution. The information in the question, however, does not guarantee the continuity of the function. We expected students who have developed at least an intuitive notion of continuity – and were therefore aware that not every function is continuous everywhere in its domain – to offer this kind of conditional response to the question. We expected the question to provoke them to speak about continuity in a language consistent with their conceptual metaphors associated with this notion. It could be the gaplessness metaphor or the closeness metaphor, or something yet different. By avoiding using, in this interview, the word "continuity," we expected to obtain the subjects' more "natural" or "genuine" metaphorical language about continuity. The word "continuity" is already a metaphor and our particular use of it in an informal phrase might have influenced the subjects to speak in a similar language. We would have thus obtained a mirror image of our own language, rather than students' spontaneous language, reflecting their own thinking about continuity.

To further engage the subjects in discussing continuity – or in case they had not thought about continuity in responding to Question 1 – a second question was proposed.

*Question 2*

This sub-question introduced a discontinuous function that is defined by a complicated polynomial function for $x \in \mathbb{Q}$, and defined as 1 for $x \notin \mathbb{Q}$.

$$f(x) = \begin{cases} (0.01 \cdot x^3 + 0.2 \cdot x^2 + 1)(x^2 - 2) & \text{for } x \in \mathbb{Q} \\ 1 & \text{for } x \notin \mathbb{Q} \end{cases}$$

The values of the polynomial at x's listed in the table of values in Question 1 (which were all rational) were approximately equal to the y values in the table; in fact, the values in the table were obtained using the polynomial in Question 2. Therefore, the function in Question 1 might be just this polynomial, and therefore continuous. The continuity, however, of this complicated polynomial would have to be deduced from the theorem that all polynomials are continuous, not guessed by visualizing the graph or recognizing the polynomial as one of the canonical examples of continuous functions (such as, e.g. the quadratic polynomial). This methodological strategy was intended to avoid habitual reactions on the part of the subjects and their production of lexicalized metaphors. Taken out of their comfort zone – of course $f(x) = x^2$ is continuous! – the subjects would be more likely to produce novel metaphors of their own invention than recite familiar mathematical phrases.

Another feature of the function in Question 2 is its resemblance with the Dirichlet function (0 on rational numbers and 1 on irrational numbers), which is a canonical example of an everywhere discontinuous function. Therefore, assuming that students were familiar with this example, it was expected that, if subjects had not made an association with continuity in Question 1, they would make it at this point. They might revise their affirmative answer to this question to produce a conditional one, with a discussion of the possible assumptions under which the answer would be affirmative.

To guide the subjects towards thinking in the above directions, several more specific sub-questions were asked in Question 2.

1. Could this function be the function that Question 1 talks about? (You can use a calculator.)

   This question set up a possible comparison between $f_1$ and $f_2$. It was hoped that, for subjects who took it for granted that $f_1$ was continuous, the comparison between $f_1$ and $f_2$ would evoke a reflection on continuity.

2. Is $\sqrt{2}$ a root of the equation $f(x) = 0$? Why or why not?

This question was meant to draw out a possible difference between $f_1$ and $f_2$. In conjunction with Question 2.3, furthermore, it was meant to elicit a discussion of continuity, since $f_2$ fails to be continuous at $\sqrt{2}$.

3. Is 1.414213562 a good approximation of a root of $f(x) = 0$? Why or why not?

   This question attempted to evoke a discussion of continuity by asking about approximation, since approximation was central to the development of the concept of continuity. (See Chapter 2) When a function has a discontinuity, we cannot use nearby values to approximate the value of the function at the point of discontinuity. In this case, although 1.414213562 is very close to $\sqrt{2}$, the function differs considerably at those values: $f_2(1.414213562)$ is nearly zero while $f(\sqrt{2}) = 1$. This question is conducive to thinking (and talking) about continuity in terms of closeness, since a number is a good approximation for a nearby value when the values of the function are also close.

*Question 3*

The function in Question 3 is defined similarly to the function in Question 2 – with the sole difference that, for $x \notin \mathbb{Q}$, the function is equal to 0. In particular, $f_3(\sqrt{2}) = 0$. Thus, $f_3$ is continuous at $x = \sqrt{2}$ – unlike $f_2$, which is discontinuous. If Question 2 failed to provoke a discussion of continuity, it was hoped that Question 3 would.

1. Could this function be the function that Question 1 talks about? (You can use a calculator.)

2. Is $\sqrt{2}$ a root of the equation $f(x) = 0$? Why or why not?

3. Is 1.414213562 a good approximation of a root of $f(x) = 0$? Why or why not?

   The purpose of Questions 3.1, 3.2 and 3.3 was similar to Questions 2.1, 2.2 and 2.3: To draw attention to a value for which continuity is an issue, and elicit a discussion of continuity by asking about approximations to that value.

4. What is the main difference between the function in Question 2 and the function in Question 3?

At first glance, $f_2$ and $f_3$ are nearly identical. However, the difference between $f_2$ and $f_3$ for $x \notin \mathbb{Q}$ results in a difference in the continuity of the functions at $x = \sqrt{2}$. This question was designed to draw attention to this difference, and – once again – inspire a discussion of continuity.

*Question 4*

This question introduced a piecewise linear function, linear on $(-\infty, 7.8)$ and on $[7.8, \infty)$, and gave some values of the function.

| $x$ | $f(x)$ |
|---|---|
| 7.5 | −0.25 |
| 7.6 | −0.20 |
| 7.7 | −0.15 |
| 7.8 | 0.10 |
| 7.9 | 0.15 |
| 8.0 | 0.20 |
| 8.1 | 0.25 |
| 8.2 | 0.30 |

The function is discontinuous at $x = 7.8$; if the linear function on $(-\infty, 7.8)$ were extended to include 7.8, $f_4(7.8)$ would equal $-0.1$, while $f_4(7.8)$ is given as 0.1 in the table of values. This is a much simpler function than the previous ones and the discontinuity is also simpler than in Question 2. The subjects must have seen this type of local discontinuity in their Calculus courses. If the subjects failed to think about continuity in the previous questions, this one was almost explicitly pointing in this direction, and asking them to rethink their answers. One may ask why we did not directly ask Question 4, without going through Questions 1, 2 and 3. We wanted to provoke the subjects to think about the *concept* of continuity rather than about the word "continuity." This word is a metaphor in itself carrying visual images and intuitions that support the feeling of "understanding" but are not technically useful in proofs. This research was about the role of conceptual metaphors in understanding *proofs* and not just in understanding the 'idea" of continuity. This is why Question 4 was again requiring the subjects to look back at the functions in the previous questions, and compare and discuss their properties.

1. Can you sketch a graph of this function?

2. Does this function have a limit at $x = 7.8$?

   Questions 4.1 and 4.2 were meant to draw visual attention to the discontinuity at $x = 7.8$. By drawing the two linear parts of $f_4$, it was hoped that the discontinuity would be visually obvious to the subjects.

3. What, if anything, does this function have in common with the function in Question 2?

   At first glance, $f_2$ and $f_4$ are quite different. The latter is easily graphed; the former, not. The latter is given by a table of values; the former, by an analytic expression. In both cases, however, they are defined differently on two sets: the latter on $(-\infty, 7.8)$ and on $[7.8, \infty)$, and the former on $\mathbb{Q}$ and $\neg\mathbb{Q}$. Furthermore, this results in a discontinuity: at 7.8 for the latter, and at $\sqrt{2}$ for the former. Thus, this question was meant to draw attention to the functions' discontinuity.

4. Compare this function with the function in Question 3.

   Unlike $f_2$, $f_3$ is continuous at $\sqrt{2}$. However, $f_3$ is discontinuous nearly everywhere else. On the other hand, $f_4$ is discontinuous at a point, but continuous everywhere else. This function was meant to draw attention to the difference between continuity at a point (local continuity) and continuity on the function's domain (global continuity).

### 4.2.3 *Post-Instruction Interview*

The second interview was conducted after the subjects had been introduced to formal definitions of continuity in the course: first to the neighbourhoods definition, and then to the $\epsilon - \delta$ definition. Its purpose was to determine their satisfaction with three proofs as explanations, and to further explore their conceptual metaphors for continuity. It consisted of:

- three proofs of the same theorem,

- a sequence of questions on the proofs,
- a sequence of questions on the subject's understanding of continuity, and
- three functions – given both by their analytic definitions and by their graphs – for which the subjects had to discuss continuity.

In the next sections, we will describe and justify the design and content of each of these elements. See Appendix B for the questionnaire as it was presented to the subjects.

*Three Proofs*

The three proofs all demonstrated the following theorem:

**Theorem 4.2.2** (Composition Preserves Continuity). *If* $g$ *is continuous, and* $f$ *is continuous on the range of* $g$*, then* $f(g(x))$ *is continuous.*

In order to test the conjecture that the explanatory power of a proof is related to its metaphorical content, we varied the proofs' metaphorical content while preserving their mathematical content. Each proof followed the following structure:

1. Definition of Continuity
2. Restatement of the result to be proved
3. Reasoning from the continuity of f
4. Reasoning from the continuity of g
5. Deduction of the result to be proved
6. Conclusion

Thus, the proofs were formally identical. By maintaining the same underlying logical structure for each proof, we controlled for two of the factors identified by Reid and Roberts (2004): proof format and type of reasoning. Furthermore, the identical logical structure helped maintain a consistent length for all three proofs: Proof 1 and 3 had 14 lines, and Proof 1 had 15 lines.

The three proofs differed in the metaphorical language they used to discuss continuity. Proof 1 used the static language of the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor; Proof 2 used the dynamic language of the CONTINUITY IS GAPLESSNESS metaphor; and Proof 3 was the proof presented during the lectures of MATH 364, using neighbourhoods. We will refer to these as the Static Proof, Dynamic Proof, and Neighbourhoods Proof, respectively. Note that, since neighbourhoods are a formalization of the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor, both the Static Proof and the Neighbourhoods Proof have similar metaphorical content, but differ in the formality of their mathematical terminology. See Table 2 for a comparison of the proofs' language.

We purposefully left the first two proofs at a slightly informal level for two reasons. First, the proofs were left at a conceptual level to draw attention to the semantic aspects of the proof over the syntactic (algebraic) aspects. Second, the $\epsilon - \delta$ formalism was avoided in order to maintain a high degree of clarity and simplicity, which are known factors in reader satisfaction with a proof as an explanation (Reid and Roberts, 2004). We feared that excessive formalism would unnecessarily reduce the clarity of the proofs, focusing the students' attention on the local details of the algebraic manipulations and preventing the students from thinking globally about the structure of the proof.

Similar concerns for simplicity motivated our choice of theorem. We chose a theorem that is often cited as an example of a result that is easier to prove using neighbourhoods. Thus, any subjects who chose the Dynamic Proof did not do it for reasons of simplicity.

A preliminary analysis of the transcripts from the pre-instruction interview, and the existing literature on students' conceptions of continuity (e.g., Tall and Vinner (1981)), both suggested that the subjects were using the CONTINUITY IS GAPLESSNESS metaphor. Since this metaphor is deployed in the Dynamic Proof, it was predicted that students would have a preference for this proof. Furthermore, it was expected that the subjects would have a slight bias towards the first proof they read; the first proof would prime a particular conceptual metaphor, which might then interfere with the activation of

| PROOF | DEFINITION | CONCEPTUAL METAPHOR |
|---|---|---|
| 1 | Recall that "$f$ is continuous at $a$" means that we can make $f(x)$ as **close** as we want to $f(a)$ by taking $x$ **close** enough to $a$. | CONTINUITY IS PRESERVATION OF CLOSENESS |
| 2 | Recall that "$f$ is continuous at $a$" means that we can **move** $f(x)$ as near to $f(a)$ as we want, by **moving** $x$ sufficiently near to $a$. | CONTINUITY IS GAPLESSNESS |
| 3 | Recall that "$f$ is continuous at $a$" means that whenever $W$ is a neighbourhood of $f(a)$, there exists a neighbourhood $U$ of $a$ such that $f(U) \subset W$. | CONTINUITY IS PRESERVATION OF CLOSENESS |

Table 2: Comparison of the language used in the three proofs. Metaphorical language is in bold.

the requisite conceptual metaphors for the subsequent proofs. Thus, the subjects were presented with the Static Proof first, to avoid amplifying any pre-existing bias toward the Dynamic Proof.

*Response to the Proofs*

This section of the interview attempted to determine the subjects' satisfaction with the three proofs as explanations.

1. Which proof does the best job of explaining why $f(g(x))$ is continuous? Why?

   This question determined the most explanatory proof for the subject. The question was carefully worded to restrict consideration to how well the proofs explained the *theorem* under consideration. That is, the question was not asking about how well the proofs explained – for instance – the definition of continuity, or the structure of the proof.

   If a subject chose the Neighbourhoods Proof, they were asked their preference between the Static and Dynamic Proofs. This was done to rule out the possibility that they were choosing the Neighbourhoods Proof based on familiarity (from class), one of the factors identified by Reid and Roberts (2004).

2. What does this proof explain to you?

   This question was meant to evoke a discussion, in the subject's own words, of the *"Why"* of the explanatory proof. This would allow insight into both the subject's conceptual metaphors for continuity, and their notion of explanation.

3. Which proof is worst at explaining why $f(g(x))$ is continuous? Why?

   This question, like the first, was meant to determine the subject's satisfaction with the proofs as explanations.

4. What do you mean by "explanatory"?

   Recall that Sandborg (1997) identified seven different usages of explanation in mathematics. This raises the issue: Was the subject identifying proofs that explained the result? Or were they, for example, identifying proofs that explained the definition of continuity? The purpose of this question was to establish how the subject had interpreted the questions about explanation.

5. Imagine that a student who hasn't taken a class in Analysis asks you why $f(g(x))$ is continuous. How would you explain it to them?

   Robert (3) suggests that asking a subject how they would explain a notion to a 14 or 15 year old would better evoke a discussion of their concept image versus the concept's formal definition.[1] Inspired by her suggestion, this question was meant to inspire the subject to reflect on their conceptual metaphors for continuity, and not just the formal definition.

*Questions about Continuity*

Since continuity was explicitly mentioned in the three proofs, the continuous cat was out of the bag. Thus, unlike in the pre-instruction interview, the post-instruction interview asked directly about the subjects' understanding of continuity. The purpose

1 Roberts was investigating the notion of a convergent sequence. Her research is discussed in Tall (1992)

of these questions was to determine the subject's conceptual metaphors for continuity after they had received instruction in the concept's formal definition.

1. How do you understand continuity of a function?

2. Can you give an example of a function that is continuous? Why is it continuous?

3. Can you give an example of a function that is not continuous? Why is it not continuous?

4. Complete these sentences:

    a) Continuity is like...

    b) Continuity is *not* like...

Each of these questions attempts to illicit a metaphorical response from the subject. By asking about "understanding," the subject was encouraged to speak in their own words, which is conducive to metaphorical discourse; likewise for completing the two sentences in Question 4. By asking the subjects to justify their choice of examples, Questions 2 and 3 were meant to encourage a discussion of what the subject attended to when determining continuity.

*Three Functions*

In this section, the subjects were asked to determine if each of a list of three functions was or was not continuous. They were supplied with the analytic definition and the graph of each function.

1.

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

This function is continuous everywhere except at $x = 0$. The function's oscillations lend themselves to a Dynamic interpretation; the very word "oscillation," from the Latin word for *swing*, evokes thoughts of real physical motion.

However, understanding the discontinuity at 0 is probably easier with the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor than with the CONTINUITY IS

Gaplessness metaphor. For instance, one could imagine an ideal mathematician with an infinite amount of time, drawing the infinite oscillations; thus, although discontinuous, the function could still be "drawn." Also, there is no gap at $x = 0$; the function is defined at 0, and the $f(x)$ often equals zero as $x \to 0$. On the other hand, the oscillations prevent the function from remaining "arbitrarily close" to 0; Continuity is Preservation of Closeness attends to the proper aspect of this particular function to understand its discontinuity at 0. Thus, the question's intention was to determine the ease with which the subject could deploy this metaphor – or if they had it at all. The question might fail to achieve this methodological goal with subjects dismissing continuity of the function at 0, by saying that the function has no limit at 0 and therefore cannot be continuous at this point. In this case, the subject would be asked to elaborate, in the hope of eliciting their conceptual metaphors related to continuity.

2.

$$f(x) = e^x$$

This function is everywhere continuous. The simplicity and monotonicity of the graph were meant to suggest a Dynamic discussion of the function's behaviour. Indeed, this is the type of function that instructors use as an intuition pump for the "you can draw it without lifting your hand" conception of continuity. The question's purpose was to determine the stability of the subject's Continuity is Preservation of Closeness metaphor, if it has appeared in their response to the previous question.

3.

$$f(x) = \frac{1}{x} \qquad x \neq 0$$

This function is continuous everywhere on its domain. When Tall and Vinner (1981) asked undergraduate math students if this function was continuous on its domain, the large majority (35 out of 41) said it was not, and from this the authors concluded "that most students have a concept image which does not allow 'gaps' in the picture"(p.166). Indeed, reasoning with a dynamic CONTINUITY IS GAPLESSNESS metaphor might lead the subject to conclude that the function is discontinuous on its domain, since it makes a "jump" at 0. This question was designed to create an opportunity for bringing the two conceptual metaphors – GAPLESSNESS and PRESERVATION OF CLOSENESS – into direct conflict. Again, this was only an opportunity and not a necessity, because the subject might just state that continuity at 0 of the function cannot be discussed since the function is not defined at this point, without reference to any metaphor on which his or her understanding of continuity is based.

## 4.3 ANALYTICAL TOOLS

To interpret the results of the interviews described above, we must analyse the transcripts from both a conceptual and an institutional perspective, since the subjects responded as both cognitive subjects (Learners) and subjects of an institution (in this case – Calculus and Analysis students). To begin, we must extract the subjects' *conceptual metaphors* for continuity from their responses to the interviews. This requires tools borrowed from Conceptual Metaphor Theory. Furthermore, we must determine the subject's institutional *positioning*, using tools from the theoretical perspective developed by Sierpinska et al. (2008).

### 4.3.1 *Conceptual Metaphor Analysis*

As detailed in Chapter 3, Conceptual Metaphor Theory describes a mechanism – conceptual metaphor – for transferring the structure of one domain into another. The concept of continuity, as used in mathematics, is typically understood using two different metaphors: CONTINUITY IS GAPLESSNESS and CONTINUITY IS PRESERVATION OF CLOSENESS. We will use two tools for identifying the subjects' conceptual metaphors for continuity: analysing their utterances about continuity for metaphorical content;

and studying the gestures they coproduce while talking about continuity, to determine their underlying conceptualization of continuity . In this section, for each metaphor, we will present an operational definition of how a subject would behave if they were deploying the metaphor.

CONTINUITY IS GAPLESSNESS: This metaphor construes continuity in terms of fictive motion and gaplessness. When thinking with this metaphor, a subject's language will use dynamic vocabulary. For instance, functions and variables will be said to move: "as x moves towards a. . . ." A continuous function moves smoothly, without jumps, taking on every point in its path. Typical language will talk of functions that "jump" and of variables that "move towards" a value. Subjects using this metaphor will use dynamic gestures: sweeping motions of the hand that *iconically* represent the function's fictive motion.

CONTINUITY IS PRESERVATION OF CLOSENESS: This metaphor construes continuity in terms of closeness. A subject with this metaphor will use static vocabulary. For instance, functions will be described as if their values are motionless: "if you *take* x close enough to a. . . ." With a function that is continuous at a, elements in the image will be close to f(a) if the elements in the domain are sufficiently close to a. Thus, subjects will talk about values that are "close to" or "around" each other. Gestures using this metaphor will be primarily *indexical*, indicating the (static) location of imagined values. When the gestures are iconic, they will be static; the gesture will represent the graph as a completed object, and not as a dynamic process.

A metaphor may lose its conceptual content, Sierpinska (1994, p. 97) notes, and become a "lexicalized" or "conventional" metaphor.

> Our language is full of such "lexicalized" or "conventional" metaphors. This is also true for the mathematical language. For example, expressions such as "convergent sequence," "limit of a function f(x) when x tends to a" are remindful of the metaphors used by the creators of Calculus and Analysis to describe the newly identified notions. (Sierpinska, 1994, p.97)

These metaphors, however, are dead – a product of the speaker's habits and not of their thought. We will take care to distinguish conventional metaphors from genuine conceptual metaphors.

Using these descriptions, we will determine a subject's conceptual metaphor for continuity.

### 4.3.2 *Institutional Positioning*

Recall from Chapter 3 that, according to Sierpinska et al. (2008), the participants in an institution take on particular *positions*. From this perspective, the subjects of this study are also participants in an institution – the institution of "University Analysis" – and so also take on various positions throughout the study. By identifying their positioning at a given time, we can determine how they are relating to the tasks and goals of the interview. Two positions are especially important for the analysis of the interviews: *Student* and *Learner*.

Based on Sierpinska et al. (2008), Hardy (2009) developed an operational definition of these two positions relative to the College Calculus institution. We will adapt this operationalization to identify the subjects' positioning in the Undergraduate Analysis institution.

STUDENT: When responding to a question in the interview, a subject takes the position of Student relative to the Undergraduate Analysis institution if they aim at producing a discourse about continuity perceived as promoted in the course and likely to be rewarded by good grades. For instance, the subject would refer to a textbook or classroom lecture when explaining the definition of continuity, and would appeal to authority in writing and reading proofs.

LEARNER: A subject is positioned as a *Learner* when they "behave as a cognitive subject interested in knowing [Analysis]; his or her goal in the institution is to learn" (Hardy, 2009). Unlike the Student, who is concerned with evaluations, the Learner is concerned with understanding. "Learners would try to justify everything they do, because for them, theoretical justification is an essential component of doing mathematics; furthermore, they would question the institution's explanations and their own, because critical thinking is for them a learning strategy." (ibid.) For instance, if asked to discuss the continuity of a particular function, a Learner would not be content quoting a result from a textbook or lecture. Rather, they would actively deploy their understanding to explain the function's continuity, and use the question as an opportunity to deepen that understanding. Furthermore, a Learner would approach the reading and writing of proofs as an opportunity to deepen their understanding of the concepts involved and the result to be proved.

These operational definitions allow us to determine the subjects' positioning, and thus to interpret the goals and motivations of their responses during the clinical interviews.

## 4.4 TRANSCRIPT CODING

To apply these operational definitions for conceptual metaphors and institutional positioning, we applied a coding scheme; see Table 3. This scheme was used to identify salient passages in the transcripts. The results of this coding are presented in the next chapter.

## 4.5 SUMMARY

In this chapter we described and justified the elements of the two semi-structured clinical interviews, one conducted before instruction, and the other after instruction on continuity of functions in an Analysis course. Furthermore, we described the tools used to analyse the data collected from these interviews. These tools will allow us to identify the subjects' shifting conceptual metaphors and institutional positioning throughout the interviews. In the next two chapters, we will describe and analyse the results of the interviews.

| CODE | CATEGORY | EXAMPLES |
|---|---|---|
| DL | Dynamic Language: | "...increasing..." "...goes from...all the way to..." |
| SL | Static Language: | "...take a value to be..."<br>"...the function has the value..." |
| CL | Closeness Language: | "...x is sufficiently close to a..."<br>"...all the values around a..." |
| GL | Gaplessness Language: | "...all the values in between..."<br>"...there won't be any gaps..." |
| DG | Dynamic Gesture: | Iconic gestures that represent a function or variable's fictive motion. |
| SG | Static Gesture: | Iconic gestures that represent the complete graph of a function; indexical gestures. |
| SP | Student Positioning: | "...just like in class..."<br>"...this would get the best mark..." |
| LP | Learner Positioning: | "...the way I understand it ..."<br>"...in my way of thinking ..." |

Table 3: Transcript Coding Scheme

# 5 RESULTS

## 5.1 INTRODUCTION

In this chapter we will reconstruct the eight student interviews – two interviews with each of the four subjects – into a cohesive and coherent narrative. By presenting a narrative reconstruction of the interviews, we allow for a quick overview of each subject's behaviour. Unabridged transcripts are included in Appendix C.

For the sake of clarity, throughout the reconstructions of the pre-instruction interviews, we will use "Function 1" or $f_1$ to refer to the function described in Question 1. Also, we will refer to the sub-questions posed in Question 1 as Question 1.1, etc. (Mutatis mutandis for Questions 2, 3 and 4.)

## 5.2 SUBJECT Y'S PRE-INSTRUCTION BEHAVIOUR

Subject Y is in her second year of a B.Sc. degree in Pure and Applied Mathematics.

### 5.2.1 *Question 1*

*Question 1.1: Do you think the equation* $f(x) = 0$ *has a solution? Why or why not?*

The questioner begins by asking Subject Y whether the function described in Question 1, $f_1$, will equal zero for any value of $x$. Her immediate reaction is to say no.(6) Next, she notes the behaviour of the function around $x = 1.4$ and $x = 1.5$:

> Because it's increasing up until one point eight but then it's decreasing [when $x$ goes from 1.4 to 1.3]. But there could be a number between 1.4 and 1.5 that would equal zero. (10)

However, while the behaviour of the table of values leads her to believe that $f_1(x) = 0$ for some value of $x$ between 1.4 and 1.5, she is concerned that "it could also not" (14) since, "maybe it's an irrational, or maybe it [...] just doesn't exist kind-of-thing." In the end, she says she "couldn't tell because [she doesn't] know the function." (18)

### 5.2.2 *Question 2*

*Question 2.1: Could this function be the function that Question 1 talks about? (You can use a calculator.)*

When asked to determine if Function 2 ($f_2$) could be identical to $f_1$, Y choose to check if the values of both functions agree. She first chooses to calculate the value of $f_2$ for $x = 1.5$, since – although the question no longer concerns finding roots of the function – she believes that the function may have a root near that value: "if it were to be zero, it would probably be around [$x = 1.5$]."(27) After a bit of difficulty with the calculator, she finally verifies that $f_1$ and $f_2$ agree for $x = 1.5$.

However, the tone of her voice indicates that she remains unsure that the functions are identical (48), and she decides to try another value. This time she wants to check "one [value] that's not close to" (50) the first value, and chooses 1.3 and 1.8 since "there's a significant difference in their answer" (56) from the original choice of 1.5. She verifies that $f_1$ and $f_2$ agree for $x = 1.8$. Upon being told that the two functions agree for every value in the table of values, she remains unconvinced that they are identical since she "doesn't really know about this part [i.e. the irrational numbers]." (86)

*Question 2.2: Is $\sqrt{2}$ a root of the equation $f(x) = 0$? Why or why not?*

In trying to determine if $x = \sqrt{2}$ is a root of $f_2(x) = 0$, Y first attends to the upper part of the function's definition, which would indeed equal zero for $x = \sqrt{2}$. However, when prompted by the interviewer – "So what kind of number is root two?" (105) – she quickly corrects herself, recognizing the restriction of $x \in \mathbb{Q}$:

> Oh, no, it's not [equal to zero]! Because if it was root two, then that's an irrational, so that would have to equal one. (106)

*Question 2.3: Is* $1.414213562$ *a good approximation of a root of* $f(x) = 0$*? Why or why not?*

The notion of an approximation to a root poses difficulties for Y. Again, she begins by attending to $(x^2 - 2)$:

> I would say yes [$x = 1.414213562$ is a good approximation to a root], but only because this [$(x^2 - 2)$] would make it zero. (132)

She quickly states that, while the value 1.414213562 is "an approximation of the irrational" (138) number $\sqrt{2}$, it "still wouldn't work" (142) as an approximation of a root since:

> it's only an approximation [of $\sqrt{2}$]" (144) [and] if it's exactly root two, it would be zero. But that doesn't count because we have a restriction [for $x \in \mathbb{Q}$]. (146)

Thus, while "[the value of the function] might get close [to zero]" for $x = 1.414213562$, it still would not be an approximation of a root since "this is an approximation of root two, so therefore it still wouldn't be zero" because "of that bottom part [i.e. the restriction]." (150)

### 5.2.3 *Question 3*

*Question 3.1: Could this function be the function that Question 1 talks about? (You can use a calculator.)*

Y immediately states that $f_3$ could very well be $f_1$, recalling the calculations she carried out for $f_2$ and noting that $f_2$ and $f_3$ are defined identically for $x \in \mathbb{Q}$.

*Question 3.2: Is* $\sqrt{2}$ *a root of the equation* $f(x) = 0$*? Why or why not?*

Y states that, since $f_3$ is equal to zero for $x \notin \mathbb{Q}$, $x = \sqrt{2}$ is a root of $f_3(x) = 0$.

*Question 3.3: Is* $1.414213562$ *a good approximation of a root of* $f(x) = 0$*? Why or why not?*

After a very short hesitation, Y states that $x = 1.414213562$ is a good approximation of a root of $f_3(x) = 0$. In this case, the value is "close enough to root two" so it is "kinda like a representation of our irrational, which would produce zero."(170)

*Question 3.4: What is the main difference between the function in Question 2 and the function in Question 3?*

The main difference between $f_2$ and $f_3$, according to Y, is "the irrationals" (172) and how "we can produce an f of x equals zero" for $f_3$ because of how it is defined for $x \notin \mathbb{Q}$. After a pause, the interviewer asks if there were differences in terms of how well 1.414213562 approximated a root. In response, Y states that "[$f_3$] allowed us to find an approximation of a root, while [$f_2$] didn't." (180) She did not draw the connection between the existence of approximations at a point, and continuity at that point.

### 5.2.4 *Question 4*

*Question 4.1 Can you sketch a graph of this function?*

Y has considerable difficulty drawing the piecewise-defined Function 4 [$f_4$]. Her first sketch connects the two (disjoint) linear sections into a single straight line. Once her attention is drawn to the regular change in $f_4$ for each incremental change in x, she notices an abnormality in the behaviour of $f_4$ between $x = 7.7$ and 7.8:

> The increment between [each value for x in the table of values] is point five [0.05] from seven point five to seven point seven, and then from eight point two to seven point eight it's the same. But then there's a difference between seven point seven and seven point eight. (210)

She did not know how to incorporate this information into her graph, but recognized that something was "a little awkward" (212) with it.

*Question 4.2: Does this function have a limit at $x = 7.8$?*

When trying to determine if $f_4$ has a limit at $x = 7.8$, Y first wonders about the significance of the intervals:

> [The interval $(-\infty, 7.8)$] is open, and [the interval $[7.8, \infty)$] is closed, and I don't know if that means anything.

According to Y, a function has a limit if "it's approaching something." (220) She notes that $f_4(7.8) = 0.1$, and concludes by stating, "I'm going to say no,"(226) it doesn't have a limit at $x = 7.8$, "because there's a definite value to it."

However, she returns to her concerns over the form of the intervals and comments on the "awkward gap"(236) in the value of $f_4$ between $x = 7.7$ and $x = 7.8$, both of which leave her unsure about the (non-) existence of a limit.

When she is reminded that a function must have a "limit from the left" and a "limit from the right" which should be equal in order to have a limit at a point (237), she recognizes the two pieces of the function do not coincide at $x = 7.8$:

> So then [the answer] is no [the function does not have a limit at $x = 7.8$]. If [the values of $f_4$] were positive values from seven point nine [to] eight point two, and from seven point seven to seven point five [the values of $f_4$] were also positive values, and they were equal, and they were both [...] approaching [the value of the function at] seven point eight, and they were the same, then I would say yes [it has a limit]. But since [the values of the function on $[7.5, 7.7]$] are negative, then no [there is no limit]. (242)

*Question 4.3: What, if anything, does this function have in common with the function in Question 2?*

At first, Y sees no grounds on which to compare $f_2$ and $f_4$. When, after a period of reflection, she is prompted by, "How about... roots?", she responds with:

> Yeah, I was thinking that... That's what I was trying to think of. I'm looking at [$f_4$] like how we looked at [$f_2$] before. And if there was an f x equals zero, it would be in there [, between seven point seven and seven point eight]. (250-254)

Further utterances suggest that Y thinks that $f_4$ might, in fact, equal zero for some value between $x = 7.7$ and $x = 7.8$. While she recognizes that it might not equal zero "because of the way that [the value of the function] is reacting"(261) between $x = 7.7$ and $x = 7.8$, she nevertheless defends the probable existence of a root of

$f_4(x) = 0$. When encouraged to consider carefully the implications of the predictable, linear behaviour of the function, she responds by observing:

> if this [segment of the function] is linear, then there are infinitely many values on this line. [...] So therefore there could be the value of the root on that line – that would produce something, like, we don't know from where. So [...] there could be roots in this function. (284-290)

Here, the appeal seems to be to the mysteries of the infinite: There are an infinite number of points, so we can't know what they're all doing! Next, she discusses the possible irrationality of the root:

> [I]f we were to look at a root in the form of a decimal, it wouldn't produce something nice like [the values for x in the table of values]. It would have a bunch of awkward numbers, and it would be an approximation. [...I]f we were just looking at a general root, in decimal form, it would have a bunch of numbers, and if we looked at it, it would have an approximation because we couldn't really write it down." (294-298)

Now Y is conflating the inability to express a (possibly irrational) number as a finite decimal expansion, with the inability to predict the behaviour of a function at that number. She recalls the difficulties with using 1.414213562 as an approximation for $\sqrt{2}$, and seems to fear that $f_4$ will be similarly baffling in terms of the (non-)existence of roots.

But she remains concerned about importance of the linearity of $f_4$. Since $f_4$ is linear, "it has the same slope"(302) except for at $x = 7.8$ when it no longer "follow[s ...] the pattern"(304) and there's a "gap."(308). Now, Y observes that the interval $(-\infty, 7.8)$ goes "up until seven point eight, so there's no space where it wouldn't be included."(318) She seems to be expressing the fact that there is "no space" on $(-\infty, 7.8)$ that isn't "included" in the linear pattern. This observation allows her to draw a new, improved graph of $f_4$ that incorporates the gap between $x = 7.7$ and $x = 7.8$. Still Y laments, "It just seems that the gap is just so awkward."(326)

Upon receiving encouragement to redraw the graph of $f_4$ with careful attention to the placement of the points around $x = 7.8$, she draws a (properly disjoint) piecewise linear function. Finally, "because it's so much easier to see when you break [the graph] up into pieces"(340), Y realizes that "there would be no point where [$f_4$] is zero"(338) since the value of $f_4$ "would be pretty close to negative point one"(346) as x approaches 7.8 from the left. Indeed, as x moves from $(-\infty, 7.8)$ to $[7.8, \infty)$, the function "goes up, to point one, so there's a gap in our graph between almost negative point one and point one."(362)

*Question 4.4: Compare this function with the function in Question 3.*

Asked to compare $f_4$ to $f_3$, Y notes spontaneously that $f_4$ and $f_2$ share a lack of roots: "in question two we couldn't have a zero,"(374) and "[the function in] Question Four is more related to [the function in] Question Two, in that [the function in] Question Two didn't have a root for f of x equals zero."(388)

On the other hand, "[the function in] question three does"(388) have a root ($x = \sqrt{2}$), so "question three allows us to have an approximation, but question four doesn't."(394) In $f_4$, "[the values of the function] can't go there" – they can't go to zero – "while [$f_3$] is allowing us to" have $f_3(x) = 0$. Thus, "[$f_4$] is closer related to question two than it is to question three."

## 5.3 SUBJECT Y'S POST-INSTRUCTION BEHAVIOUR

### 5.3.1 *Response to the Proofs*

After Y reads the proofs, the interviewer asks for her initial impression. She spontaneously raises the issue of explanation:

> You can tell how [Proof 1] was more explaining it to people that would never have taken Analysis. (24) [...] This is how you would see it if you didn't really get things. It's really straightforward. (30)"

On the other hand, she states that Proof 3 is both "the kind of proof that our teacher gave [her] in class"(26) and "closer to [...] how [she] would look at continuous functions."(28)

*Question 1: Which proof does the best job of explaining why* $f(g(x))$ *is continuous?*

In response to the first question, she dismisses the first proof: "I didn't really like the first one"(36) since it "didn't really make a good enough argument for [her]."(38) Proof 1 reminds her of her first experiences with continuity: "When you learned of continuous functions in high school, it's really just like [Proof 1]."(48)

Furthermore, she comments on the lack of proofs in her pre-Analysis experiences with continuity:

> There's really never a proof in high school. They're always just like, 'It's continuous just because it is.'(50) And you're almost just told it's continuous.(52)

At first she says Proof 2 is the most explanatory.(40) She changes her mind, though, and chooses Proof 3:

> I liked the third one.(42) [...] Because that one was a lot more concrete.(45) [...] The third one is more like 'This is exactly why it is the way it is.'(54) [...] For some reason, when they talk about neighbourhoods for me, I can visualize it more so.(56)

However, she adds a caveat: "But that just might be because that's how I had it explained to me [in class]."(46)

She adds that Proof 3 is preferable to Proof 2 because of its power to evoke mental imagery:

> [When Proof 2] is like, 'We want $f(g(x))$ to move towards $f(g(a))$,' it doesn't really... I don't really see that.(58) [...] I see it better in this kinda sense [used in Proof 3].(60)

*Question 2: What does this proof explain to you?*

For Y, Proof 3 "gives you a recall [of what it means] when f is continuous at a."(64) She states that the proof "goes in nice steps"(66) starting with the fact that "you know when f is continuous, so then it tells you, 'You know this, and then this, and this.' So it's like a series of steps."(68)

When asked about her second-favourite proof, Y chooses Proof 2. Although it "just doesn't make as much sense to [her]"(72) when Proof 2 talks about "how we can show to move one [variable] towards the other,"(72) nevertheless it "is convincing," She remarks:

> It's just that I don't visualize it, and I don't understand it as well as I would understand something really concrete, as in the third one.(76) [...] Because [neighbourhoods], that was clearly defined to me in class. But this "moving something"... Whenever my teachers get very visual and try to explain it like, "This is gonna go towards this," well, I need to see it, and I need to know why it does that.(78)

She gives the example of limits:

> So when we got really visual, like as far as limits too, it was like, "This is going towards..." I was like, 'No. Let me do it my own way.'(84) [...] Teachers try to talk about it, and I'm like, 'No, draw the graph out and show me.'(86)

Instead, she prefers explicit theorems:

> I'm the kind of person that needs to know the theorems of why it does this.(82) [...] When we do limits in Analysis, there was a theorem and a way that you can *show* it.(88)

*Question 4: What do you mean by "explanatory"?*

In response to the question, Y declares that a proof is explanatory when "it's giving [her] as much detail as possible."(92) When prompted by the interviewer to complete the sentence, "So explanation is like ...," she continues by saying:

> The amount of detail, or the justification. Bigger, fluffier proofs that just give you every little thing.(96) [...] A lot of proofs, I've noticed, will just show part of things, and it won't really give you a reference to, like, 'Where did *this* come from? Where did *this* come from?' I guess a really good explanatory proof would be like, 'Because we know this, we know this.' Giving every little detail so you completely get the picture.(98)

She remarks, however, that explanation is very personal:

> But then again, it's dependent on which way people like to see things. Because a lot of people would probably prefer a proof like the first one, actually. Because [Proof 1] is a little bit easier to understand [for other people]. But in my sense, it's not [easier to understand]. (92)

However, she contradicts her own characterization of explanation by noting that, "So even though the third one gives a little bit less information, or less clear detail, for me it's easier to see."(94)

*Question 5: Imagine that a student who hasn't taken a class in Analysis asks you why* $f(g(x))$ *is continuous. How would you explain it to them?*

At first, Y suggests using Proof 1 to explain the result.(100) She changes her mind and suggests Proof 3, however, and its use of neighbourhoods. While Proof 3 is intimidating on paper for Y, it becomes clear when she can visualize the neighbourhoods.(122)

> The idea of a neighbourhood is talked about in other classes, too. And it's very visual, and that's why I think I understood it. Because it's very like, 'Here's your thing [said while placing her hands close together to indicate a point] and *here's* your neighbourhood [said moving hands in a circular motion around initial point].'(100) [...] The neighbourhood is the little parentheses around it.(102) [...] It's very visual and it's very easy to see. [...] And the word neighbourhood helps people to get it. Like, 'Oh, ok, well it's in this little area!'(104)

She hesitates, however, noting that she "wouldn't know how to explain that."(104)

When asked by the interviewer about the language of Proof 1 – "By taking x close enough to $a$" – she remarks that "that kinda sounds like a neighbourhood to me."(106) On the other hand, the language of Proof 2 – "moving x sufficiently near to a" – is more difficult to visualize, even though all three proofs "are all saying the same thing." She explains that "when [she] reads examples, or the teachers talk, there are just certain things that [she] do[esn't] get." She explains that in those cases, she has to "find a way to figure it out on my own, a different way to see it."(114)

### 5.3.2 *Continuity*

*Question 1: How do you understand continuity of a function?*

The subject identifies two notions of continuity:

> When we were younger, [continuity] was that there's no holes, skips, or jumps. We were told that, and it was very, 'Here's a graph. Is it continuous or not?'(128)

She states that, in Analysis, the notion is more complicated:

> And then when we got a little bit older [...], like right now in the class, our teacher is comparing a lot of functions to the way that, the way that... I don't know. It's getting not so basic.(130)

She suggests that these two notions of continuity arise in different contexts:

> [Now, in Analysis,] there's a lot of proofs. [...] It used to be so simple. Look at a graph. Is it continuous?(132)

Furthermore, the new notion is more difficult. While "it used to be so simple," the new notion of continuity "is getting a little bit shaky."(132) While reading proofs, she explains that she has to constantly refer back to the initial definition of continuity; in fact, she reveals that she uses a sticky note to mark the definition in her textbook.(134) When proving results for the assignments in her Analysis course, she has to refer back to the initial definition of continuity.(136)

Her remarks suggest that she is unsure of her ability to characterize continuity without referring to the textbook definition:

> I could give an example, but I don't think it would be [...] sufficient enough. I guess as far as proving something on an assignment or a test, I don't think I could do that well enough unless I memorized the proofs.(138) [...] I just review [the definition], I look back. I use it a lot, and then it will stick. (140)

*Question 2: Can you give an example of a function that is continuous? Why is it continuous?*

The subject offers $y = x$ as an example of a continuous function. When asked to explain why, she says, "Going back to the way we learned in high school, there's no holes, skips or jumps."(148)

*Question 3: Can you give an example of a function that is not continuous? Why is it not continuous?*

The subject explains that step functions are not continuous "because there are points where it may not necessarily be included."(152) She gives the example of a function that may be defined for $x$ from 1 to 2, and again for $x$ greater than 3, but undefined between 2 and 3.(154)

*Question 4: Complete these sentences:*

1. "Continuity is like..."
2. "Continuity is *not* like..."

The subject struggles to find the right word. She suggests that continuity is like "non-stopping" and "non-interfered"(160), all while sliding her hand forward in a smooth motion.(163) She states that continuity is "very flowing, and one-streamed."(170)

On the other hand, she continues, continuity is not like "segments,"(166) "broken pieces, and fragments, because when [she] look[s] at discontinuity, there's a skip or a jump or something, so that's really choppy."(170)

### 5.3.3 *Functions: Explain why each of these functions is or is not continuous on its domain.*

Next, the interviewer asks Y – for each of the following three functions – whether or not it is continuous on its domain, and why.

1.

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

While Y notes that she recognizes this function from class (174), and is "pretty sure that it's continuous until it gets to zero,"(176) she can't initially remember why.

> I remember that we were talking about that [function], and how it just comes and it gets really small, and all the sine curves get closer and closer. But then it gets to zero.(176) [. . . ] Because at [x =] zero, [the function's value] is zero.(178)

While saying this, Y traces the path of the function in the air with her finger, wiggling her finger up and down to indicate the oscillations near zero.(177) She concludes that the function "is [continuous], but not at zero."(180)

2.

$$f(x) = e^x$$

The subject immediately states that the function is continuous.(184) When asked for a justification, she describes the graph:

> Because when you look at the right side, it just keeps going until infinity. And then this part [the left side] kind of goes to zero but it never really gets there. So I would say that it would be continuous.(188)

When the interviewer presses her further, she explains:

> Yeah. I don't know if that [reasoning] is good enough. But like I said, when we were told to look at graphs, or continuous functions, it was like, 'Ok, does this graph look continuous?' And I'm like, 'Ok!'(190)

3.

$$f(x) = \frac{1}{x} \qquad x \neq 0$$

Y first observes that "[i]t has two continuous parts,"(196) on either side of the y-axis:

> When you look at the right side, this will always go up, and it's never going to reach the y-axis.(198) [...] And then [...] on the left side it goes down.(200)

While saying this, Y mimics the shape of the graph with a sweeping motion of her hand.

She next attends to the center of the graph, "this part in the middle where it's never going to meet."(202) This part leads her to say, "it's not continuous,"(202) although she admits that she "do[es]n't know how to explain it" and she "do[es]n't know what the correct definition is."(202) Nevertheless, she concludes that "as a whole, it's not [continuous]."(202)

## 5.4 SUBJECT P'S PRE-INSTRUCTION BEHAVIOUR

Subject P is in his third year of a B.Sc. in Pure and Applied Mathematics.

### 5.4.1 *Question 1*

*Question 1.1: Do you think the equation* $f(x) = 0$ *has a solution? Why or why not?*

After looking at the table of values, P concludes that $f(x) = 0$ will have a solution, citing the Intermediate Value Theorem:

> I think that it has [a solution], by the Intermediate Value Theorem. If f of a is negative and f of b is positive, then there is a zero in between. (6)

He states that the solution will be when x is "between one point four and one point five," (10) presumably because $f_1(1.4) < 0$ and $f_2(1.5) > 0$.

### 5.4.2 *Question 2*

*Question 2.1: Could this function be the function that Question 1 talks about? (You can use a calculator.)*

At first P has difficulty understanding the question. When prompted by the interviewer to check if the two functions agree for $x = 1.3$, P attempts to verify that, for $x = 1.3$ and $x = 1.4$, the value of the function in Question 2 will have the same sign as the function in Question 1.(20) He first observes that the first factor in the analytic expression in Question 2 – $(0.01 \cdot x^3 + 0.2 \cdot x^2 + 1)$ – will be positive since "there are [only] pluses so it cannot be minus."(20) Next he examines the second term of $f_2$ – $(x^2 - 2)$ – and wonders "if x squared, out of these two values, is less than two."(20) After using the calculator to square 1.3 and 1.4, he concludes that "[t]hose look like they're satisfying one point three and one point four," since $f_2(1.3) < 0$ and $f_2(1.4) > 0$, as required by $f_1$. He then observes that $f_1(1.5)$ and $f_2(1.5)$ will also agree in sign.

Thus, he concludes that "I guess, yeah [$f_2$ could be $f_1$]."(20) However, he does not want to calculate the exact value of $f_2$: "Do I have to calculate everything?"(20) The interviewer tells him that it is his prerogative, but reassures him that the values of $f_2$ do, indeed, match those of $f_1$.(21, 23)

However, he remains unsure. He is concerned by the restriction of $f_2$ to the rationals: "this thing here [the definition of $f_2$ for $x \notin \mathbb{Q}$]. It's confusing me."(30, 32) He wonders aloud if "the Intermediate Value Theorem" is "only for R,"(24) remarking that "if it's only for R, then it might be"(24) the same function – emphasizing the word "might" to indicate his doubt. He concludes that, since "we can always find rationals"(44) between any two numbers, the two functions "can be"(44) identical.

*Question 2.2: Is $\sqrt{2}$ a root of the equation $f(x) = 0$? Why or why not?*

At first, P states that $x = \sqrt{2}$ is "definitely"(48) a root, looking at the $(x^2 - 2)$ in the definition of $f_2$. When prompted to consider the rationality of $\sqrt{2}$, he changes his mind and states that, when $x = \sqrt{2}$, "f of x cannot be zero, because it's one by definition."(58)

*Question 2.3: Is* $1.414213562$ *a good approximation of a root of* $f(x) = 0$*? Why or why not?*

P thinks that it is a good approximation, explaining:

> If it's close enough to the square root of two, and it's a rational, then we have this square is close to two.(60) [...] That number minus two is going to be very small, so it's going to be close to zero.(64)

When asked if the function will be identical to zero for any value near $x = 1.414213562$, he responds, "No, I don't think so."(74)

### 5.4.3 *Question 3*

*Question 3.1: Could this function be the function that Question 1 talks about? (You can use a calculator.)*

P replies that, since $f_2$ and $f_3$ are identical for $x \in \mathbb{Q}$ and differ only when "it's irrational"(82, 88), the function in Question 3 "can be the function"(85) in Question 1.

*Question 3.2: Is* $\sqrt{2}$ *a root of the equation* $f(x) = 0$*? Why or why not?*

In response, P immediately exclaims, "Yes! [...] Because the root of two is irrational."(90)

*Question 3.3: Is* $1.414213562$ *a good approximation of a root of* $f(x) = 0$*? Why or why not?*

P asks if the number in this question is the same as the number in Question 2.3.(92) When the interviewer reassures him that it is, P explains:

> I guess the first answer would be, it's going to be close to zero, because it's rational. But it's never actually going to *be* zero, because this is not an irrational number. [...] Square root of two is irrational, and this guy is rational, so it wouldn't be [a good approximation.](96)

When the interviewer remarks that the number is very close to the square root of two (97), P states that "the answer is going to be close to zero, but it's not actually going to be zero"(98), while $f_3(\sqrt{2})$ equals zero.(106)

*Question 3.4: What is the main difference between the function in Question 2 and the function in Question 3?*

P's first response is that "the main difference" is that "when x is irrational, we have different values" for $f_2$ and $f_3$.(108) When asked how this would change the function,(109) P is unsure; he's "not so good at seeing that"(110) so he says, "I don't know how it changes."(112)

He then spontaneously wonders about the functions' zeros:

> We have five zeros in the first one?(114) [...] How many solutions do we have for this equation [for $f_2$ when $x \in \mathbb{Q}$]?(116) [...] I think it's five (118) [...] because the highest [power] is x to the five.(120) [...] But I don't know if they're complex, or...(122)

He concludes by returning to the question, stating, "I don't know what's different."(124)

### 5.4.4 *Question 4*

*Question 4.1 Can you sketch a graph of this function? Question 4.2: Does this function have a limit at $x = 7.8$?*

In his first sketch, P connects the two linear segments as a single linear function. When asked if the function has a limit at $x = 7.8$, he states that "it has a limit, because [it has a limit] from both sides."(150)

The interviewer prompts P to examine the table of values to determine the behaviour of the function as x approaches 7.8. After determining, with the interviewer's help, that the function has the same slope on both intervals (153-161), P asks, "Okay, so we have a gap here, or no?"(162) He then answers himself: "It's not continuous"(172) at $x = 7.8$, "so it doesn't have a limit."(180)

*Question 4.3: What, if anything, does this function have in common with the function in Question 2?*

P answers that neither $f_4$ nor $f_2$ "have a limit"(196) – "at seven point eight" for $f_4$, and at "the square root of two" for $f_2$.(198) He explains further that at $f_2(\sqrt{2})$, "it jumps to one."(200) Similarly, at $f_4(7.8)$, there is a "a gap"(202) or – he corrects himself – a "discontinuity."(204)

*Question 4.4: Compare this function with the function in Question 3.*

P's first response attends to the functions' zeros:

> I don't know here [$f_3$] if the zero is going to be rational or irrational. It might be that both of them are rational, so.... Between them, they'll be rational or irrational.... It's not a very promising answer.(212)

Then the interviewer reminds P of his answer to Question 4.3, which appealed to $\lim_{x\to\sqrt{2}} f_2(x)$ and $\lim_{x\to 7.8} f_4(x)$. This prompts him to consider $\lim_{x\to\sqrt{2}}$ for $f_3(x)$, and he spontaneously mentions continuity:

> Well, I guess here it's going to be continuous, because it's defined everywhere.(216) [...] At the square root of two. It's going to be zero, so at that point it's going to be [continuous]. But here [at $x = 7.8$ for $f_4$], it's going to be discontinuous.(218)

## 5.5 SUBJECT P'S POST-INSTRUCTION BEHAVIOUR

### 5.5.1 *Response to the Proofs*

After reading the first two proofs, P remarks that he had a tough time differentiating them.(14) After reading all three proofs, he asks to re-read Proof 2, since he "didn't get this one."(18) He re-reads the proof, expresses his understanding – "Okay!"(20) – and the guided interview begins.

*Question 1: Which proof does the best job of explaining why* $f(g(x))$ *is continuous?*

P responds that Proof 3 is the most explanatory proof.(28) He cites the use of neighbourhoods, which allow him to "visualize how that is happening."(30)

Asked to compare the first two proofs, P says that Proof 1 is more explanatory. He singles out the phrase "far towards" in Proof 2, stating that he has not been exposed to such a phrase:

> It's just in terms of 'far towards,' which I haven't met right now."(36) [...] It's not a problem. [...] It's the first time seeing a proof like that. So that's why I think I had to read it again.(38)

*Question 2: What does this proof explain to you?*

When asked *what* Proof 3 explains, P states that "it explains where the functions are, and I can picture it in my head. I can picture the coordinate system."(54) He then proceeds to recreate in his own words the proof's argument for why the theorem is true:

> We are trying to prove that the neighbourhood... We are trying to prove that the neighbourhood... Oh, I lost it... G of a, that the neighbourhood of e is actually in the neighbourhood of f.(58) [...] Which is in the neighbourhood of U, which is in the neighbourhood of... which is a subset of W.(59)

He concludes by pointing to the statement of the theorem at the end of the proof. When asked if this is what the proof explains, he says, "Yes, I would say yes. Yes."(66)

*Question 3: Which proof is worst at explaining why* $f(g(x)$ *is continuous? Why?*

The worst, for P, was Proof 2.(68) According to P, this proof lacks mathematical formalism: "Because there are [...] not a lot of mathematical entities here."(70) Furthermore, he "is used to seeing everything in terms of definitions in proofs" since he is preparing for his final exam.(72)

When the interviewer observes that both Proof 1 and Proof 2 are equally informal (73), P replies that he "perceiv[es] more familiar things in Proof One than Proof Two."(74)

*Question 4: What do you mean by "explanatory"?*

P's response is immediate and precise: an explanatory proof "shows me why a certain theorem works."

*Question 5: Imagine that a student who hasn't taken a class in Analysis asks you why* $f(g(x))$ *is continuous. How would you explain it to them?*

Although P prefers Proofs 1 and 3 for himself, he would use Proof 2 to explain the result to someone else.(84)

He recalls his own first exposure to Analysis, when he had to translate the formal proofs into more informal arguments:

> Because even I had to translate the proofs in the book, in terms of this, to explain to myself.(86) [...] At the beginning, I had to translate what those neighbourhoods mean. I had to explain to myself with a picture in front of me, in terms... in a way like proof number two [the Dynamic Proof].(88)

For instance, he recalls how he first translated the neighbourhood definition of continuity:

> So, for example, the first sentence of the proof [Proof 3]: 'Recall that f is continuous which means that whenever W is a neighbourhood of f of a, there exists a neighbourhood of U, of a, such that whenever f of U is a subset of W." So now I can perfectly understand what that means.(92) [...] But at the beginning I had to translate what W neighbourhood of f of a is.(94) [...] Which means you can move, in terms of numbers, around f of a, as close to f of a, by moving sufficiently near to a. So approaching a, to a from both sides.(96)

Thus, when he was first exposed to the definition of continuity in Analysis "one month ago or two weeks ago"(98), he had to translate the language of Proof 3 into the language of Proof 2 in order to gain understanding.

However, after sufficient experience, the new definition became more natural:

> And after that, after a while, you get used to specific notations, a specific way how to prove something or how to understand something. So the more words you have, then it becomes more confusing.(86)

### 5.5.2 *Continuity*

*Question 1: How do you understand continuity of a function?*

P's understanding involves the "epsilon-delta definition."(110)

> I understand it in terms of the [epsilon-delta] definition. (106) Which helps me translate it to the definition with the neighbourhoods.

His understanding, however, quickly becomes visual: "And after that I can just picture it out."(112) He continues by describing the definition in his own words:

> Which means that whenever, however close you go to... For any neighbourhood of f of... of... some value, a. You can always find x around a so that it's going to be in the middle. [...] The one neighbourhood has to influence the other neighbourhood. (112)

*Question 2: Can you give an example of a function that is continuous? Why is it continuous?*

P suggests $f(x) = x^2$ as an example of a continuous function. When asked why it is continuous, he again appeals to visual considerations:

> Well, that particular one because I could first picture it in my head.(118)
> [...] There are no discontinuities.(120)

He recognizes that this is not a mathematically sufficient argument, laughing at his visual rationale and reassuring the interviewer, "But I can prove it, I guess."(122)

*Question 3: Can you give an example of a function that is not continuous? Why is it not continuous?*

P proposes $f(x) = 1/x$ as a discontinuous function, and offers two different justifications. First, he observes that "it's not continuous because it's not defined at zero."(128) Furthermore, he notes:

If you approach zero from left and you approach zero from right, you have two different values, which means that it's discontinuous because it doesn't approach a specific number.(130)

*Question 4: Complete these sentences:*

1. "Continuity is like..."

   P says, "It's like... a line"(134) – particularly a line without "any holes in it" or any "gaps."(136)

2. "Continuity is *not* like..."

   P cites another example of a discontinuous function: Continuity "is not like tangent of x, the graph of tangent of x"(140) because "it's discontinuous at [...] pi over two and three pi over two."(144) When asked why he's looking upwards as he describes the graph of tan x, he again recounts his visualization: "I'm looking up because I'm perceiving... I have the graph of tangent in my head."(146)

### 5.5.3 *Functions: Explain why each of these functions is or is not continuous on its domain.*

1.

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

First, P recognizes this function as a possible test question, exclaiming, "Man, I have to know this for the final!"(152)

He begins to appraise the function's continuity by determining "what's happening as x approaches zero."(154) He struggles, however, and becomes frustrated – "Ah, I don't know!"(156) – so the interviewer prompts him to consider the function for values other than zero.(161) He responds immediately in the affirmative:

> I can say it's continuous [when $x \neq 0$]. First, seeing the graph. And second of all, this function, one over x, is not continuous... like, the critical point should be only at zero.(164)

Prompted by the interviewer to return to his original consideration of the function as x approaches zero, P is unsure:

> Well, by looking at the graph I'd say it's continuous.(170) [...] Although I don't know what's happening around there [$x = 0$].(172) [...] But I don't know if the picture can tell us about every graph [...] I think it's going to be continuous.(176)

Now the interviewer directs P to consider the rapid oscillations of the function around $x = 0$. The interviewer says, "As x goes towards zero [...] sine is going to keep going up to one, and then down to negative one, and then up to one... over and over again." When P does not respond, the interviewer continues, "And as you see from the graph, it oscillates more and more often as you get close to zero.(187) [...] It's going from one to negative one.(189)"

At this point, P asks, "But it's going to take all the values in between, right?"(192) The interviewer responds in the affirmative, and P states:

> So when it's going to be very close to zero, then it's going to be one on this side – or minus one, I don't know – and on that side. So it's going to be different than zero. So limit as this goes to zero is going to be one or minus one, and from the right as well. But f at zero is zero. So I guess it's going to be discontinuous at zero.(194)

P concludes by remarking, "I'm going to remember this example. You'll have to tell me after if this is... [correct]."(196)

2.

$$f(x) = e^x$$

P immediately states that the function is continuous, "because of the graph."(202) He apologizes, however, for using a visual justification, saying, "Sorry."(202)

3.

$$f(x) = \frac{1}{x} \qquad x \neq 0$$

After clarifying that the domain does not include zero,(206-209) P concludes that the function is continuous "because of the graph."(212) When questioned about what information he extracts from the graph, he amends his statement, saying, "Well, I don't have to look at the graph."(216) Instead, he looks at the equation, and in particular at the denominator, which he notes is "going to get bigger and bigger [...] but with a minus sign" as x approaches zero from the left.(216, 218) Similarly as x approaches from the right: "And with minus infinity and plus infinity, it's the same."(222) From this he concludes, "So that means that you can take any value as close to zero as you want, and it's going to be defined there, except at zero."(220)

## 5.6 SUBJECT K'S PRE-INSTRUCTION BEHAVIOUR

Subject K is in his third year of a B.A. Honours degree in Economics.

### 5.6.1 *Question 1*

*Question 1.1: Do you think the equation* $f(x) = 0$ *has a solution? Why or why not?*

Subject K responds to the question by appealing implicitly to Bolzano's theorem: "Well it does [equal zero], because it goes from negative to positive, so it crosses zero at some point."(2) When asked to explain this inference, he specifies that continuity is a necessary assumption, so that "if it's a continuous function, it has to be [zero]."(4)

He does not know for sure, however, that the function in question is continuous:

> Well, I don't know [if the function is continuous], actually, from looking at this [table of values]. But I assume that, if it's continuous, then there will be a solution. (6)

### 5.6.2 *Question 2*

*Question 2.1: Could this function be the function that Question 1 talks about? (You can use a calculator.)*

After some initial difficulty with the operation of the calculator – his initial calculations produce values that do not correspond to the table of values (14-18) – he is convinced by the interviewer that, if calculated properly, the function in Question 2 will produce the values from Question 1.

However, he remains unconvinced that Function 2 is identical to Function 1:

> Well, there's a one there [in the definition of $f_2$], for non-rationals. (26) So if [$f_1$] is continuous (28) [...] then you'll have a value of one between every one of [the rational values of $f_2$].(32)

The implication seems to be that the discontinuous behaviour of $f_2(x)$ for irrational values of x – "then you'll have a value of one between every one of them"(32) – does not accord with his assumption that $f_1$ is continuous.(4, 28)

Although he remains unsure – "I guess. I don't know."(34) – he concludes, nevertheless, that $f_2$ "could be"(34) the function $f_1$ from Question 1.

*Question 2.2: Is $\sqrt{2}$ a root of the equation $f(x) = 0$? Why or why not?*

At first K states that $\sqrt{2}$ is a root of $f(x) = 0$, noticing the $(x^2 - 2)$ factor in the upper half of the definition of $f_2$ and explaining, "Yeah, square root of two, squared, minus two, is zero."(52) However, this only applies for $x \in \mathbb{Q}$. When the interviewer draws his attention to this fact, he changes his response, exclaiming, "Ooh! Then it doesn't work (56) [...] because then [$f_2$] is going to be equal to one"(58) since the square root of two "is irrational."(60)

*Question 2.3: Is 1.414213562 a good approximation of a root of $f(x) = 0$? Why or why not?*

In response to the question, K states that "yes [the number is a good approximation of a root], because then it's a rational number."(64) And since the number – unlike $\sqrt{2}$ proper – is rational, "then it's not one. [The function] is going to be zero [at the number] – or close to zero."(66) While he recognizes that 1.414213562 is not close to an actual root – since "when it's actually the square root of two,"(72) the function "will be one"(70) – he maintains that it is a good approximation because "you can get close, but not zero."(74)

### 5.6.3 *Question 3*

*Question 3.1: Could this function be the function that Question 1 talks about? (You can use a calculator.)*

Recalling that the values of the function in Question 1 matched the values from Question 2, K reasons that they will match again for Question 3: "It was the same function [in Question 2]. I mean, you said [$f_2$] does match [the values of $f_1$]. So I'm going to assume that it does match [for $f_3$]."(82) Thus, he concludes, "It could be that function, yeah."(84)

*Question 3.2: Is $\sqrt{2}$ a root of the equation $f(x) = 0$? Why or why not?*

When asked if $\sqrt{2}$ is a root, the subject responds by saying that "it's one of them.(90) [...] Because any irrational number will give you zero.(92)"

*Question 3.3: Is 1.414213562 a good approximation of a root of $f(x) = 0$? Why or why not?*

K does not think that 1.414213562 is a good approximation, because "any irrational number will actually give you zero, will be the root for your equation.(98) [...] [The number 1.414213562] is a close approximation, but you can get an exact answer if you plug in any irrational.(102)"

*Question 3.4: What is the main difference between the function in Question 2 and the function in Question 3?*

The first difference that K notices is in the definition: "Well, it's defined differently. For any irrational number, you can get one for the first [function], and you can get zero for the second [function]."(106)

When asked how that changes the function, he states, "This function [$f_2$] is actually discontinuous (108) [...] at any irrational number.(112)" When asked about $f_3$, he first responds, "This one is continuous."(116) He immediately qualifies that claim:

> Actually, no. It will be discontinuous for anything but root two (118) [...] because [...] it's going to be either positive or negative, and it's going to jump back to zero for any [irrational number].(120)

The only value for which $f_3$ is continuous, K states, is "for x is equal to root two."(126)

### 5.6.4 *Question 4*

*Question 4.1 Can you sketch a graph of this function?*

After some initial confusion about what he was supposed to sketch – he started by sketching the intervals of the domain before realizing, "Oh! I have to sketch this function!"(140) – K produces a sketch of $f_4$. The graph does not reflect the behaviour of $f_4$ on $(7.7, 7.8)$, so the interviewer prompts further reflection by asking, "If that pattern [of linear increase from 7.5 to 7.7] kept going, would would it be at seven point eight?(153) [...] Is that reflected in your graph?(157)"

K responds, remarking that if the linear function defined on $(-\infty, 7.8)$ extended to include 7.8, then "It's [value would be] negative zero point one.(156)" He corrects his graph to take this into account.

*Question 4.2: Does this function have a limit at $x = 7.8$?*

K states that the function does not have a limit at $x = 7.8$:

> Because you're going to have a limit going(164) [...] from plus infinity to seven point eight (166) [...] which is going to be zero point ten.(168) [...] And you have a limit from negative infinity, which is going to have a limit of negative zero point ten.(170)

Thus, K concludes that the limit "does not exist."(172)

*Question 4.3: What, if anything, does this function have in common with the function in Question 2?*

After some reflection on $f_2$ and $f_4$, K notes that "they're both discontinuous." Clarifying, he states, "Well, this function $[f_2]$ is discontinuous at that point $[x = \sqrt{2}]$, and that function [$f_4$] is discontinuous at seven point eight." When prompted further, he also notes that $f_2$ is nonlinear, while $f_4$ is linear.

*Question 4.4: Compare this function with the function in Question 3.*

Comparing $f_4$ to $f_3$, K remarks again on the issue of continuity:

> This one [$f_3$] is continuous at root two, but it's discontinuous everywhere else.(196) [. . . ] This is the other way around [for $f_4$]. It's continuous everywhere, but discontinuous at seven point eight.(200)

Furthermore, he states that, while $f_4$ is linear, "this [$f_3$] looks like a nonlinear function."

The interviewer then asks him if he could draw $f_3$. The subject attempts a sketch, asking for the table of values from Question 1.(204) He quickly notes, however, that he "can't draw a line for this (212) [. . . ] because there will be jumps here to zero all the time. There will be holes.(216)" He decides to mark the irrational "jumps" with a series of dots along $y = 1$. (At this point, he has begun to draw $f_2$ instead of $f_3$.)

When asked to compare this sketch and the sketch of $f_4$, he states that one difference is that $f_4$ "is continuous so [he] can assume there's a continuous line between every one of these points." This allows him to "draw [$f_4$] in two parts" while he "can't draw the other one."(230)

When asked where $f_4$ is continuous, he states, "This side, and that side,"(226) pointing to the two linear parts of $f_4$. The interviewer responds by asking, "What about the function as whole? Is the whole function continuous?" (227, 231) To this, K replies, "Well, continuity is a local property."(232) He mentions that in optimization, a subject he has studied before, he had to "show at least local continuity, or you can show that it's continuous at every point."(242) Even when it's continuous at every point, however, "it's a local property still."(244)

### 5.7.1 *Response to the Proofs*

*Question 1: Which proof does the best job of explaining why* $f(g(x))$ *is continuous?*

After reading the proofs, Subject K spontaneously remarks that each proof is "more or less the same."(10) However, he adds, "[Proof 3] is better, though.(16) I liked this one [Proof 3] better.(18) When asked why, he explains that it's "shorter" (22), "is less wordy,"(24) and "there's more math symbols."(24) Thus, K states that "[Proof 3] is easier to understand."(29) He contrasts the wording of Proof 2 and Proof 3:

> It says, "move x sufficiently far towards a." I'm not sure what this means.(30) [...] But if you say if it's in the neighbourhood of a, then that sounds better.(32) [...] It sounds more precise.(34)

In response to Question 1, K states that the three proofs differ very little, since "they all seem to explain exactly the same thing."(40) In particular, he states that Proof 1 and Proof 2 "are almost the same,"(42) while Proof 3 "is exactly the same thing, but using math symbols."(44) When the interviewer reminds K that he had contrasted the use of language by each proof, K says this make no difference in his satisfaction with the proofs as explanations.

*Question 2: What does this proof explain to you?*

K draws on the language of both Proof 1 and Proof 2 in articulating what the proofs explained to him:

> Well, you can get as close as you want to f of x by moving x close a. Close to f of a. And you repeat the same thing for the domain of g. You can move as close as you want to g of a, which is f of a let's say, so you can get as close as you want to g of x.

*Question 3: Which proof is worst at explaining why* $f(g(x)$ *is continuous? Why?*

K states that Proof 2 is the least explanatory, drawing attention to the wording of the proof: "I don't understand what exactly 'sufficiently far towards a' means."(58)

Contrasting the use of "sufficiently close" in Proof 1 and "sufficiently far towards" in Proof 2,(62) he notes that the word "far" has connotations of moving away, since "far is going away from something."(64)

*Question 4: What do you mean by "explanatory"?*

At first, K is frustrated with this question, exclaiming, "Well... explain!(72) [...] Well, it's just explain. That's what it means exactly.(74)" He then offers an expanded characterization: "Explaining what happened, and why it is so."(76) An explanatory proof "explains the statement of the theorem"(86) or "a particular result."(80) In the case of the three proofs he has just read, they explain "that f of g of x is continuous."

*Question 5: Imagine that a student who hasn't taken a class in Analysis asks you why* $f(g(x))$ *is continuous. How would you explain it to them?*

K responds that he would use a proof similar to Proof 1. He finds Proof 1 preferable to Proof 3 because the latter is too technical:

> He might not be familiar with some constructs, or some terminology or symbols that you would use in Analysis.(92) [...] Like set theory, for example, or subsets, or neighbourhoods.(94)

On the other hand, he prefers Proof 1 to Proof 2 because Proof 1 "is easier to read"(102) due to the choice of language.

### 5.7.2 *Continuity*

*Question 1: How do you understand continuity of a function?*

In response to the question, K recites a definition of continuity:

> Continuity is, at any point of a function, if you move a little bit away, or, yeah, a little bit away from that point, there's going to be a value of that function in the range, close enough.

*Question 2: Can you give an example of a function that is continuous? Why is it continuous?*

K states that constant functions are continuous because they are defined on their domain and they preserve closeness:

> A constant function is continuous.(108) [...] Because for every value of x, there's a value of y, and for every value of x close to that x, there's a value of y in the neighbourhood of f of x.(110)

*Question 3: Can you give an example of a function that is not continuous? Why is it not continuous?*

K first notes that "not continuous, it means there's a break somewhere."(112) As an example, he offers "a function, let's say, where y is equal to one when it's rational, or zero when it's irrational." When asked why this would be discontinuous, he explains, "Because every rational point contains an irrational really close to it, so the value of the function will jump by one all the time." Moving his hand up and down to represent the function, he notes, "It will be up and down all the time, at every point."

*Question 4: Complete these sentences:*

1. "Continuity is like..."

   At first K cannot complete the sentence. Eventually he offers, "Continuity, it makes me think you can draw a function with a pen, without lifting."(124) Thus, continuity is "like a line, a curve."(126)

2. "Continuity is *not* like..."

   K states that discontinuity makes him "think of a line with a break in it."(136) No words come to mind, however.

   He continues to explain that he usually "think[s] graphically,"(146) even when reading proofs: "For continuity, I would picture a curve, a neighbourhood for the domain and a neighbourhood for the range."(150) While saying this, he makes little circles in the air with his index finger, meant to represent neighbourhoods.

5.7.3 *Functions: Explain why each of these functions is or is not continuous on its domain.*

1.

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

K states that the first function is discontinuous, "because around zero [...] it's going to be very often here or there,"(160) said while pointing to the top and bottom of the graph. "So at zero it would be discontinuous,"(168) while "everywhere else it should be continuous"(172) since "the line looks continuous,"(178) "it looks like a curve."(182) Asked what it means for a graph to "look like a curve," he says, "Ummm, it's continuous. You can draw it continuously."(184) While saying this, he moves his hand in a wave-like motion. When asked to show with his hand what a *dis*continuous function would look like, he makes a horizontal motion, then a vertical motion, followed by a continuation of the original horizontal motion (189) – something very similar to a step function.

2.

$$f(x) = e^x$$

K immediately states that the function is "continuous everywhere." When asked why, he explains, "It's continuous everywhere because, for any neighbourhood of your range, your function is going to be there for any neighbourhood of x."(200) He states that he thinks of this when he sees the function, but he also remembers having seen the function before: "Plus, I *know* that e to the x is continuous."(202) The interviewer asks if this was the case for the previous function, and K responds that – while he'd seen the function before – he couldn't "remember exactly"(206) where the previous function was continuous.

3.

$$f(x) = \frac{1}{x} \qquad x \neq 0$$

After a bit of reflection, K states that "it's discontinuous at zero."(210) When the interviewer reminds him that 0 is not part of the domain, he corrects himself, saying, "Then it is continuous (216) [...] because it's continuous everywhere except zero, so if that's not part of the domain, then it's continuous.(218)"

When asked how he knew in the first place that the function was "continuous everywhere except zero," he responds, "It's the same [as the last function]. For any neighbourhood of x, you have a function that exists in a neighbourhood of y."(220) The interviewer prompts further, asking how he gets that information about the function, K explains that he looks at "usually the picture"(222) and also "if there's any restrictions on the domain." He adds, "Or if I don't know, you plug it in the definition of continuity and check." For the functions in question, however, "these are known functions, so you don't have to do that."(228)

The interviewer asks when he would otherwise turn to the definition of continuity, and K states, "Well, if you have to write a proof, you have to use the definition."(234) Also, "if you want to use continuity somewhere else, then you need to use the definition."(240) When looking at "a simple function like this,"(242) applying the definition would not be his first approach. "But if it's something complex that you can't tell, then you would put in the definition to check."(242)

## 5.8 SUBJECT J'S PRE-INSTRUCTION BEHAVIOUR

Subject J is in his first year of a B.Sc. degree in Computer Science.

### 5.8.1 *Question 1*

*Question 1.1: Do you think the equation* $f(x) = 0$ *has a solution? Why or why not?*

In response to the question, J begins by remarking on the function's apparent rate of change, saying, "If this goes down, I might think that the rate at which it's decreasing doesn't decrease."(84) He then shifts his attention to the function's domain:

It would seem that f of x equals zero does have a solution, because it's from R and any point from R must have an image.(8) [...] Otherwise it wouldn't be a function.(10)

The interviewer intervenes and asks, "So, x could be zero – but is f of x going to be zero?" In response, J notes that "it depends on if the function is surjective or not"(12) and, when pushed to make a guess, he spontaneously raises the issue of continuity: "Well, if it is continuous, it would have [a solution]. Between one point four and one point five, it would be zero."(14) When asked to explain further, he elaborates with, "Because f of x equals zero is between f of one point four and f of one point five."(16)

### 5.8.2 *Question 2*

*Question 2.1: Could this function be the function that Question 1 talks about? (You can use a calculator.)*

After struggling to calculate particular values of $f_2$ – "I forgot to put the brackets"(28) – the interviewer reveals that the values of $f_2$ do, indeed, match those from the table of values for $f_1$.(33) With this information, the subject states that $f_2$ could be $f_1$ since they have the same domain: "because if it's in Q or not in Q, it's in R."(36)

*Question 2.2: Is $\sqrt{2}$ a root of the equation $f(x) = 0$? Why or why not?*

Subject J immediately recognizes the irrationality of $\sqrt{2}$, saying, "It is not [a root], because the square root of two is irrational. So it [the value of the function] would be one."(40)

*Question 2.3: Is 1.414213562 a good approximation of a root of $f(x) = 0$? Why or why not?*

Although J recognizes the value of the function is never actually zero (47), he states that 1.414213562 "would be a good approximation, depending on what kind of precision you want."(44) He explains that "[$(x^2 - 2)$] would tend to zero [...], so that would be close to f of x equals zero."(44)

### 5.8.3 *Question 3*

*Question 3.1: Could this function be the function that Question 1 talks about? (You can use a calculator.)*

J's answer is short: "Yeah. Well, that's the same [as in Question 2]."(52)

*Question 3.2: Is* $\sqrt{2}$ *a root of the equation* $f(x) = 0$*? Why or why not?*

Again, J's answer is quick and direct: "Yes [it's a root], because the square root of two is irrational, so it would be zero."(54)

*Question 3.3: Is* 1.414213562 *a good approximation of a root of* $f(x) = 0$*? Why or why not?*

At first J sees no difference between this question and the analogous one for $f_2$: "Yeah, well, the same as the question before. [...] This term will tend to zero, so it will be close to zero." Then the interviewer prompts him with, "And is it close to an actual root?"(57) In response, he remarks that "square root of two is an actual root, and this is close to the square root of two."(60)

*Question 3.4: What is the main difference between the function in Question 2 and the function in Question 3?*

The main difference that J notes is that "in the function in Question Two, the root of two wasn't actually a root of zero, whereas here [in Question Three] it is a root of zero."(62) The interviewer asks how this changes the way he thinks about the functions,(63) and J remarks that it changes the functions' "use."(64)

> You can use this function different, because you can use this function to approximate the square root of two when it tends to zero, whereas you couldn't use the other function.(64)

### 5.8.4 *Question 4*

*Question 4.1 Can you sketch a graph of this function?*

After quickly sketching the function, he remarks, "it should be about like that, because that half [on $[7.8, \infty)$] is continuous and the bottom half [on $(-\infty, 7.8)$] is continuous."(70)

He expresses concern about the behaviour of the function between 7.7 and 7.8, saying, "Assuming it continues between those two values at the same rate. [...] Between seven point seven and seven point eight. Because it could do something else."(72,76)

*Question 4.2: Does this function have a limit at $x = 7.8$?*

The subject states that he does not think $f_4$ has a limit at $x = 7.8$, "because if you go from the left and from the right, there's a discontinuity. So it doesn't have a limit at that point."(82)

*Question 4.3: What, if anything, does this function have in common with the function in Question 2?*

Both $f_2$ and $f_4$ are discontinuous, according to J: "There is a discontinuity point at... Around square root of two it is discontinuous. [...] And here [for $f_4$], well, it is discontinuous at seven point eight."(84, 86)

*Question 4.4: Compare this function with the function in Question 3.*

J notes that, while $f_3$ is continuous at $\sqrt{2}$ – unlike $f_2$ – it is still discontinuous elsewhere:

> In this case, the function would be continuous at the square root of two. Although, when it is other irrationals – like the square root of three – I think it wouldn't be continuous [...], because this term [$(x^2 - 2)$] might not tend to zero when it goes to square root of three, for example.

Thus, J remarks that $f_3$ is discontinuous at $\sqrt{3}$ "or any other irrational."(94)

### 5.9.1 *Response to the Proofs*

The interviewer begins by introducing the result to be proved. When asked if he recalls it from class, J says that he does.(16)

After reading the proofs, his first impression is that Proof 3 is "more going towards neighbourhoods and more advanced concepts that you'd see in Analysis, whereas the other two proofs could be understood by people who maybe haven't taken Analysis and are in Cegep."(24)

*Question 1: Which proof does the best job of explaining why* $f(g(x))$ *is continuous?*

In response to the question, J selects Proof 3, but adds a caveat: "Assuming you know the definition of a neighbourhood."(28) The interviewer asks which proof would be more explanatory if the reader did not know the definition of a neighbourhood.(29) J chooses Proof 2, and justifies his choice:

> It's referring to the distance between the two, and how you can get them close.(34)

*Question 2: What does this proof explain to you?*

J attempts to reconstruct the reasoning of Proof 3:

> It explains that, whatever interval you take in the domain of f of g of x, you can find an interval...there is an interval...Actually, it's the other way around.(38)

He stops himself and restarts by considering an interval in the range:

> If you take an interval of the range of [...] f of g of x, [...] you can always find an interval in the domain of f of g of x such that the image of the points in the domain will be contained in the other interval.(38)

While saying this, he is gesturing in the air, using his fingers to represent the intervals in the domain and range.(39-48)

When asked what Proof 1 explains to him, he states:

> You would have a function, and whenever you would take a point that would be close to some point a, you can always take it so close such that the image of that point would also be very close to f of a.(52)

*Question 3: Which proof is worst at explaining why* $f(g(x)$ *is continuous? Why?*

J states that Proof 2 is the worst at explaining why $f(g(x))$ is continuous. He explains that he takes issue with the expression "move far towards," finding it "a bit strange."(58)

The interviewer asks if it is the word "move" that he finds problematic. J responds affirmatively, stating that "moving is strange because that's not really a defined concept."(60)

*Question 4: What do you mean by "explanatory"?*

J states that an explanation "would be something to make sure that [...] what you're trying to demonstrate is clear."(64) He adds that there should be "no doubt about whether or not it is correct."(66) When the interviewer asks about the difference between these two characterizations, J states that there are different kinds of explanations:

> That would explain the concept of, say, continuous in this case. But that wouldn't explain how to do the proof. [...] That's two different things.(68, 70)

*Question 5: Imagine that a student who hasn't taken a class in Analysis asks you why* $f(g(x))$ *is continuous. How would you explain it to them?*

J says that he would model his explanation on Proof 1:

> It would look like proof number one a lot. It would be, like, in any interval, when you do a square...it would contain U and W, let's say. You would find that the function is in that square.(76)

When asked to explain what he means by a "square," he reformulates his explanation:

> You would have a continuous function. [...] And when you take some point a... and some point, let's call it b... If you take the interval that is here, you would have a plus something that is very small, and a minus something, the same thing. And you would find that, if you would take that interval to be very small, you would have that the function is always in both intervals at the same time.

He simultaneously draws a graph of a function. When the interviewer asks if the "squares" are the intervals around a and b, J responds, "Well, yeah, that's it. When the e is very small."

The interviewer remarks that J's explanation only clarifies the continuity of one function, and asks, "So how would you explain why f of g of x is continuous? Composition like that?" In response, J states:

> Well, by knowing that, you could take – say this [the function in his original explanation] is f of x – you could do that for f of x and g of x. So, by take g of x sufficiently close to some point, you could get f of that point to get sufficiently close to the image of that point. [...] So both of them would be in the same square. (92, 94)

### 5.9.2 *Continuity*

*Question 1: How do you understand continuity of a function?*

J first gives the "intuitive definition we've all seen," that "the function doesn't have jumps."(96) He continues with a definition involving neighbourhoods:

> And now it's more like always in a neighbourhood, such that you can always... that the image is... that you can find a neighbourhood such that all the images of the other neighbourhood are in it.(98)

*Question 2: Can you give an example of a function that is continuous? Why is it continuous?*

J suggests $f(x) = x^2$, and explains why it is continuous:

Because x squared is always in the neighbourhood. Like, if you take x minus c, say, you see that if this is smaller than some number, you can always find a number such that x squared minus c squared will also be smaller than another number, that would be depending on c.(102)

*Question 3: Can you give an example of a function that is not continuous? Why is it not continuous?*

J suggests "some function that would be defined piecewise, like f equals one if x is negative and f equals two if x is positive."(104) He states that such a function would be discontinuous at $x = 0$. When asked why, he states, "Because the function would jump."(106) He continues by saying:

> When you try to make some interval that is centered at zero, and then the image would be, say, one. And you take some interval of one that wouldn't include two. Every time you go to the right of zero, you couldn't find any number that would be... that wouldn't have it's image to be two.(106)

*Question 4: Complete these sentences:*

1. "Continuity is like..."

   J responds, "It would be a line without jumps."

2. "Continuity is *not* like..."

   For J, continuity is not like "anything that has holes, or is defined piecewise such that the images don't touch together."(112)

5.9.3 *Functions: Explain why each of these functions is or is not continuous on its domain.*

1.

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

J responds, "That is definitely not continuous."(116) He justifies his answer by constructing two sequences. First, he considers the sequence $1/2\pi n$, and notes that "the sequence tends to zero."(116) The value of the function at each number in this sequence is always zero: "Sine of one over one over two pi n [that is, $\sin(1/(1/2\pi n))$], that will tend... that will always be zero."(116)

Next, he considers the sequence $1/(2\pi n + \pi)$, and states that "when you put it in the function, that would always give one."(116) He concludes by observing that "both [sequences] tend to zero," but "both those functions would be far."(116)

> So if you take n sufficiently large, you will always find that no matter how close you try to get to zero, you always have two different values that are always bigger than [...] a positive number that you can fix.(116)

Thus, J explains why the function is discontinuous at $x = 0$.

2.

$$f(x) = e^x$$

J states that this function is continuous:

> That one is continuous, because whenever you have, like, x minus c, if you take that to be smaller than some number such that you would be in some interval, you can always find another interval in your y-axis such that e to the x minus e to the c would be in that interval.(118)

3.

$$f(x) = \frac{1}{x} \qquad x \neq 0$$

J states that the function is continuous; his reasoning is similar to the one he used for $f(x) = e^x$:

> You can always find some neighbourhood of x such that it is in a neighbourhood of y. And since x cannot be zero, you cannot include that as your point for the interval. Therefore you can always find a neighbourhood such that zero is not included. So it will never actually go across the zero to be... to have such a large interval on y.

He observes that the function would not be continuous if 0 were in the domain:

> Because on[. . . ] any interval centred at x equals zero, the function goes to minus infinite and plus infinite on [. . . ] each side.(122) [. . . ] So it wouldn't be in a finite interval.(124)

## 5.10 SUMMARY

In this chapter, we reported the behaviour of each subject in the two interviews. In the next chapter, we will analyse their behaviour using the tools developed in Chapter 4.

# 6 ANALYSIS OF THE RESULTS

## 6.1 INTRODUCTION

In this chapter, we will analyse the subjects' behaviour during the interviews. First, we will comment generally on their responses, especially as they relate to their understanding of continuity. Next, we will analyse the subjects' transcripts by applying the operationalized definitions of conceptual metaphors and institutional positioning that we developed in Chapter 4. Finally, we will analyse the subjects' notions of explanation.

## 6.2 GENERAL REFLECTIONS

The four subjects all share an unexpected interpretation of what constituted a "good approximation of a root of $f(x) = 0$." For the subjects, a good approximation produces a value that is close to zero; it does not, however, have to be close to an actual root. Thus, they understand the questions about approximations as asking about values that behave similarly to roots of $f(x) = 0$, but are not necessarily close to actual roots. This might have obscured the relationship between a continuous function and its usefulness in approximating roots. Any function – continuous or otherwise – that has values near zero can have "good approximations to a root" in their sense; but a continuous function is particularly well suited to approximating an actual root.

Three of the four subjects, however, spontaneously mention continuity during the pre-instruction interview. Only Subject Y never mentions continuity. Subjects J and K raise concerns about continuity almost immediately, while answering Question 1. The remaining subject, P, first brings up the concept during Question 4, when evaluating the limit of $f_4$ as x approaches 7.8.

Both Subjects Y and J – who differ completely in their conceptual metaphors, as we shall see – respond to Question 3.3 by stating that $f_2$ and $f_3$ differ in their "use" in finding an approximation – but this only after stating that 1.414213562 is a good approximation of a root in both Question 2 and Question 3. It seems that only when presented with both ways in which a number can approximate a root – by being close to an actual root, and by mapping to a value close to zero – did they realize the importance of the latter interpretation. By then, perhaps, it was too late for them to recognize the role that continuity plays in approximation, and this may explain why they did not mention the concept.

Two subjects have difficulty with the direction of the flow of information in the definition of continuity. When describing their understanding of continuity, they start with an interval in the domain and find an interval in the range; the definition of continuity, on the other hand, starts in the range and moves to the domain. Subject J has a false start when explaining what the Neighbourhoods Proof explains to him:

> It explains that, whatever interval you take in the domain of f of g of x, you can find an interval... there is an interval... Actually, it's the other way around. If you take an interval in the range [...](38)

At first, J reasons from the domain to the range, before correcting himself and starting over. Later, when discussing the continuity of $f(x) = e^x$, he makes the same mistake and does not correct himself:

> Ok, that one is continuous, because whenever you have, like, x minus c, if you take that to be smaller than some number such that you would be in some interval, you can always find another interval in your y-axis such that e to the x minus e to the c would be in the interval.(118)

Here, J starts with an interval in the domain ("like, x minus c") and then tries to find an interval that contains the image ("you can always find another interval in your y-axis"). He reasons from the domain to the range. Similarly, when K thinks of continuity, he starts with an interval in the domain and finds an interval in the range. Asked why a constant function is continuous, he responds:

> Because for every value of x, there's a value of y, and for every value of x close to that x, there's a value of y in the neighbourhood of f of x.(110)

However, earlier he expressed an understanding of the proper order of the definition:

> [A function is continuous when] you can get as close as you want to f of x by moving x close to a.(54)

Both Subject J and K are aware of the proper definition, but sometimes express it in the opposite order.

## 6.3 INSTITUTIONAL POSITIONING

Throughout the interviews, the subjects' utterances suggest shifting institutional positioning. For instance, when Subject Y describes a preference for proofs that "give you a reference [to a textbook theorem],"(98) or worries that her own proof would not be "a very reasonable proof, or sufficient enough,"(138) she is positioning herself as the Student who is expected to follow perceived standards of mathematical discourse. At other times, however, she positions herself as a Learner:

> So when [teachers] got really visual [...] I was like, "No, let me do it my own way."(84)

> When I read proofs, or when I read examples and stuff, or the teachers talk, there's just certain things that I don't get. So I just skip it and find a way to figure it out on my own, like a different way to see it. (114)

As we will see, this fluid transition between institutional positions is evident in every subject.

Subjects' responses while positioned as Learners are of particular interest. There is a link between a subject's institutional positioning and their rapport with explanation.

As Students, the subjects are required to demonstrate and convince – in a classroom exchange with an instructor, on an assignment, or on an exam – using the norms and rules of the institution. As Learners, on the other hand, the subjects are driven to understand, that is, to explain to themselves why the result is true. Thus, only as Learners will they authentically engage with proofs as opportunities for explanation. Furthermore, when positioned as Students they are likely to reproduce the discourse of the classroom or the textbook, rather than their own internal metaphorical discourse; the Students' metaphorical language mirrors institutional norms rather than their conceptual understanding. When the subjects are positioned as Learners, we can determine their personal conceptual metaphors and evaluations of explanation. Therefore, in the analysis of subjects' conceptual metaphors that follows, we will attempt to distinguish their responses as Students from their responses as Learners, and pay particular attention to their behaviour as Learners when evaluating our conjecture.

## 6.4 CONCEPTUAL METAPHORS FOR CONTINUITY

In this section, we will identify each subject's dominant conceptual metaphor for continuity.

### 6.4.1 *Subject Y*

Throughout both the pre- and post-instruction interviews, Y shows a preference for dynamic language and considerations of gaplessness. At times, she speaks as though using the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor; these times, however, are all primarily when she is positioned as a Student. During the pre-instruction interview, furthermore, she never explicitly discusses continuity.

*As Student*

As a Student, Subject Y sometimes discusses continuity in terms of closeness. When first discussing the three proofs in the post-instruction interview, Y says she prefers the Neighbourhoods Proof because "this is the kind of proof that our teacher gave us in class."(26) This appeal to classroom norms indicates that she is positioned as a Student. She continues by stating that this proof expresses how she understands continuity:

> It's a little bit closer [. . . ] to how I would look at continuous functions.(28)

Furthermore, she claims that, for her, a proof in terms of neighbourhoods is more "concrete" and that she can "visualize it more so." However, she then clarifies these statements:

> I see it better in this kinda sense. There's this neighbourhood, and then you know that this is existing – because that was a theorem – so then like this is also true, this neighbourhood of this. So then you know f of g of x is continuous.(60)

This quote suggests that Y connects neighbourhoods and continuity insofar as they are related by a *theorem* – that is, by the authority of the institution. Similarly, discussing her ability to construct proofs involving continuity, she delegates the authority to the textbook:

> I have to have a sticky note where [the textbook defines continuity].(134) We have a lot of questions in the assignments that are like, "Prove that something something, given f of x is continuous." So I obviously have to remember what that means, that f of x is continuous. So I go back to that sticky note, and then I'm like, "Ok, this is what that theorem told me."(136)

Subject Y is describing her ability to apply the formal definition of continuity when she has positioning herself as a Student. Thus, she does speak of continuity in terms of static neighbourhoods and closeness – but only as a Student, in the context of the University-Analysis institution.

*As Learner*

When positioned as a Learner, on the other hand, Subject Y consistently employs the dynamic language of the CONTINUITY IS GAPLESSNESS metaphor. From the beginning of the pre-instruction interview, she speaks of functions in terms of gaplessness: she states that $f_1$ probably has a root "between 1.4 and 1.5." She seems to be assuming that $f_1$ does not have any "gaps." Later, she complains that $f_4$ has "that awkward gap" that leads her to "have no idea" if the function has a limit as x approaches 7.8.(236) She remains fixated on the notion of a "gap" when trying to determine if $f_4$ is ever equal to zero:

> If [the function] is linear, then there are infinitely many values on this line.(284) [...] [The linear function defined on $(-\infty, 7.8)$ continues] up until seven point eight. So there's no space where it wouldn't be included.(318) [...] It just seems that that gap is so awkward.(316) What if we were looking at seven point seven nine nine nine nine nine... You know what I mean?(342)

She remains perturbed by the fact that there is a gap in the graph of the function, and even goes so far as to suggest we look at 7.799999... for a possible zero – hidden right at the edge of the interval, perhaps. Furthermore, her language throughout the pre-instruction interview is dynamic: "it's increasing [...] but then it's decreasing"(10), "it's approaching something"(220), "it would go..."(348), "it wouldn't go past"(356), etc.

In the post-instruction interview, Subject Y employs similar metaphorical language. She explains how she understands continuity:

> When we learned it, when we were younger, it was that there's no holes, skips or jumps.(128) [...] And then when we got a little bit older, as we learned it, it starts to get, like right now in the class, our teacher is comparing a lot of continuous functions to the way that, the way that... I don't know. It's getting not so basic.(130)

When the institutional discourse surrounding continuity transitioned from an informal discussion of jumps to one expressed in terms of closeness, she was only able to follow as a Student; as a Learner, she still understands continuity in terms of gaplessness. This is further supported by her examples of continuous and discontinuous functions:

> [A line is continuous because] there's no holes, skips, or jumps.(148) [...] Step functions (150) [are not continuous] because there are points where it may not necessarily be included. [...] There are certain parts that are not necessarily included, and it skips, and it goes in jumps.(154)

Thus, for Y, the essential quality of a continuous function is its lack of gaps – points that are not included. Furthermore, continuity is intimately tied to fictive motion:

> [Continuity is like...] non-stopping, or non-interfered.(160) [...] Because, when we said "Continuity is like...," I think, very flowing and one-streamed.(170)

While saying this, she slides her hand in a smooth motion, representing the dynamic "non-interfered" motion of a continuous function. Later, when discussing the continuity of the function $f(x) = 1/x, x \neq 0$, she represents the function with a dynamic gesture, saying, "This will always go up, and it's never going to reach the y-axis."(198) She concludes that the function is discontinuous because of the gap: "So then there's this part in the middle, where it's never going to meet."(202)

Y's discourse, therefore, is consistently dynamic and based on considerations of gaplessness. As described in the Theoretical Framework, the CONTINUITY IS GAPLESSNESS metaphor often conflates the function with its representation as a graph. It was only the arithmetization of Analysis, according to Núñez (1998), that divorced the concept of continuity of a function from the concept of a *naturally continuous* curve. If Y understands continuity with the CONTINUITY IS GAPLESSNESS metaphor, then this may explain why she failed to mention continuity during the pre-instruction interview. When a function is difficult to draw, then it is equally difficult to understand continuity as "drawn without jumps." Since the functions in the pre-instruction interview were selected to prevent easy graphical representation, Y could not even begin to understand the continuity of the functions in relation to their representation as 'curves.'

Therefore, as a Learner, Subject Y demonstrates a consistent use of the dynamic CONTINUITY IS GAPLESSNESS metaphor.

### 6.4.2 *Subject P*

Subject P employs a variety of metaphorical phrases, mixing the static and the dynamic, gaplessness and closeness. During the pre-instruction interview, he did not mention continuity until the last question.

*As Student*

When discussing continuity, Subject P only indicates that he is positioned as a Student when faced with the first function in the post-instruction interview, exclaiming, "Man, I have to know this for the final!"(152) Elsewhere, though, P describes how, when faced when an unfamiliar definition, he had to "explain to [him]self" so that he could "understand something."(86) Thus, it seems reasonable to conclude that for most of the interview, P has positioned himself as a Learner, engaging with the tasks as opportunities for understanding.

*As Learner*

In the pre-instruction interview, Subject P discusses continuity primarily using the CONTINUITY IS GAPLESSNESS metaphor. For instance, when asked in Question 1.1 if the function has a zero, he appeals to the Intermediate Value Theorem and decides that "there is a zero *in between*"(6) $f(1.4) < 0$ and $f(1.5) > 0$; that is, there are no gaps between $f(1.4)$ and $f(1.5)$. He does not (explicitly) consider the function's continuity, however; it seems that he is committing the same fallacious reasoning discussed by Daval and Guilbault, where Bolzano's gaplessness is thought to presuppose continuity. Elsewhere, he talks of functions' fictive motion: "This is increasing here(156)[...] At the square root of two, it jumps to one."(200) When discussing continuity, he considers gaps and gaplessness: "Ok, so we have a gap here, or no?(162) [...] We have [...] a gap?(202) [...] It's going to be continuous, because it's defined everywhere."(216)

By the time of his post-instruction interview, Subject P moves fluidly between the two metaphors. He states that, until very recently, he had to translate between metaphors to understand textbook proofs:

> Because even I had to translate the proofs in the book, in terms of [neighbourhoods], to explain to myself. [...] At the beginning, I had to translate what those neighbourhoods mean [...] in a way like proof number two [the dynamic proof].

This ability to translate between metaphors is evident throughout his post-instruction interview. He employs, at times, the dynamic language of gaplessness; at others, the static language of closeness; and, often, a mixture of the two. Consider his discussion of $f(x) = 1/x$. (205-223) He combines, in one utterance, dynamic language such as, "when it's going to approach to zero," with static, closeness language such as, "take any value as close to zero as you want."(220) He applies both metaphors; he is able to translate between the CONTINUITY IS GAPLESSNESS and the CONTINUITY IS PRESERVATION OF CLOSENESS metaphors.

Indeed, we witness this shift between metaphors when he attempts to determine the continuity of $f(x) = \sin(1/x)$ at $x = 0$. He appears to begin by using the dynamic CONTINUITY IS GAPLESSNESS metaphor:

> I'm thinking about what's happening as x approaches zero.(154) [...] As it's coming from the left, it's going up and down.(175) [...] But it's going to take all the values in between, right?(192)

He thus struggles, using this metaphor, to interpret the meaning of continuity in this context:

> I don't know what's happening around [$x = 0$]. (172)

Indeed, there are no "gaps" in this function, since it is defined (in the questionnaire) as zero for $x = 0$; gaplessness is uninformative. Furthermore, an idealized hand could draw a graph of the function – the infinite oscillations present no problem to a hand with infinite time – so the function does not exhibit fractured (fictive) motion. However, after prompting from the interviewer, P transitions to a CONTINUITY IS PRESERVATION OF CLOSENESS metaphor, and nearly instantly resolves the question of continuity:

> So when it's going to be very close to zero, then it's going to be one on this side – or minus one, I don't know – and on that side. So it's going to be different than zero. [...] But f at zero is zero. So I guess it's going to be discontinuous at zero.(194)

Notice the use of the static language of closeness: "very *close* to zero," it's going to *be* one on this side," etc.

Therefore, Subject P's pre-instruction reveals a reliance on the CONTINUITY IS GAPLESSNESS metaphor. His post-instruction interview, however, indicates an ability to apply both the CONTINUITY IS PRESERVATION OF CLOSENESS and CONTINUITY IS GAPLESSNESS metaphors.

### 6.4.3 *Subject K*

In the pre-instruction interview, Subject K immediately raises the issue of continuity. However, he almost always expresses himself using the dynamic language of CONTINUITY IS GAPLESSNESS. By the time of the post-instruction interview, he has undergone a marked transition, and primarily understands continuity in terms of closeness.

*As Student*

Subject K only overtly positions himself as a Student toward the end of the post-instruction interview. In discussing the continuity of $f(x) = e^x$, he begins by speaking of neighbourhoods: "It's continuous everywhere because, for any neighbourhood of your range, your function is going to be there for any neighbourhood of x."(200) He then clarifies this by saying, "Yeah, plus I know that e to the x is continuous."(202) When asked if this means he has been told previously by someone that the function is continuous, he responds, "Yeah."(204) In this case, K is relying on the authority of whoever taught him about the continuity of $f(x) = e^x$. However, by explicitly contrasting the authority of his instruction and his own explanation of the function's continuity, K demonstrates his awareness of the different positions he can take.

*As Learner*

Subject K begins, in the pre-instruction interview, with a dynamic conception of continuity in terms of gaplessness. Asked if $f_1(x) = 0$ has a root, he concludes that it does, "because it goes from negative to positive, so it crosses zero at some point."(3) This utterance combines dynamic language – "goes from," – with an implicit deployment of gaplessness – "so it crosses zero at some point." When prompted, he recognizes that he has assumed continuity: "Well, if it's a continuous function, it has to be [zero at some point]."(5) He continues with this dynamic language of gaplessness later, when he explains that is impossible to graph $f_3$ because "there will be jumps here to zero all the time. There will be holes."(217) The graph of $f_4$, on the other hand, is easier to draw, since it "is continuous so I can assume there's a continuous line between every one of these points."(225) For K, a "continuous line" is one that can be drawn freely between points – that is, a dynamic gapless line.

By the time of the post-instruction interview, however, K shows a strong preference for the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor. In describing continuity, he speaks of closeness rather than gaps: "Well, you can get as close as you want to f of x by moving x close to a."(54) Note that he still speaks of "moving" the variable; his language is still dynamic. This juxtaposition of dynamic language and closeness is repeated later:

> Continuity is, at any point of a function, if you move a little bit away, or, yeah, a little bit away from that point, there's going to be a value of that function in the range, close enough.(106)

He repeats this discourse when explaining why $f(x) = \sin(1/x)$ is not continuous at $x = 0$:

> Because around zero, you're going to have the function, it's going to be very often either here or there. I don't know what the value is, but...(160)

[...] And as it approaches zero, it's going to be very often away from zero.(162)

Here, K uses considerations of closeness to identify a discontinuity: "around zero," "away from zero." This statement was paired with a series of static, indexical gestures, with which he pointed out the highest and lowest points of the function's oscillations. This static gesturing suggests that – while he is still using dynamic language – his thought is actually static; if he were thinking dynamically, the co-produced gesture would have mirrored that dynamism. This suggests that his dynamic language is only a lexicalization of the dynamic metaphor. He has adopted the dynamic discourse of the CONTINUITY IS GAPLESSNESS metaphor, but abandoned its conceptual content.

The language of closeness is repeated when he describes why a constant function is continuous (110); when he describes how he pictures continuity using neighbourhoods (150); and when explaining the continuity of $f(x) = e^x$, and again for $f(x) = 1/x, x \neq 0$.(193-221)

On only two occasions in the post-instruction interview does K's discourse suggest a consideration of gaplessness. When asked what continuity makes him think of, he replies, "It makes me think you can draw a function with a pen, without lifting."(124) Of course, this is a classic articulation of the CONTINUITY IS GAPLESSNESS metaphor. Later, he explains that, to determine if a function is continuous from its graph, he checks if "you can draw it continuously."(184) This suggests a line drawn without gaps.

However, although he uses dynamic language and sometimes appeals to gaplessness, he shows a strong preference throughout the post-instruction interview for the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor. As mentioned, his use of static, indexical gestures instead of dynamic gestures suggests that his dynamic language is a lexicalization of the CONTINUITY IS GAPLESSNESS metaphor. Therefore, his dominant metaphor during the post-instruction interview is CONTINUITY IS PRESERVATION OF CLOSENESS.

### 6.4.4 *Subject J*

By the time of the post-instruction interview, Subject J shows – of all the subjects — the strongest preference for the Continuity is Preservation of Closeness metaphor. He confidently deploys it to understand and explain the continuity of a variety of functions.

*As Student*

Subject J does not overtly indicate his institutional positioning. However, we can speculate that he is primarily positioned as a Learner. For instance, when discussing his first introduction to continuity as the property that "the function doesn't have jumps," he describes the definition as "the intuitive definition that we've all seen."(96) On the other hand, when he explains the continuity of a function, he consistently applies the formal definition of continuity, speaking either of neighbourhoods or intervals.(38, 52, 86, 102, 106, 116, 120) This suggests that he has internalized the distinction between the formal and intuitive definitions; for J, the definitions differ in terms of the intuitions and understanding they generate, and not in their institutional normativity. Thus, throughout the interviews he uses the discourse of understanding – and, as such, has positioned himself as a Learner.

*As Learner*

By the time of the second interview, Subject J demonstrates a marked preference for the Continuity is preservation of closeness metaphor. However, this preference was not as pronounced in the first interview. In fact, during the pre-instruction interview, J only mentions continuity twice – at the beginning and the end – using a mixture of metaphors. First, when asked to determine if $f_1(x) = 0$ has a root, he remarks, "If this goes down, I might think that the rate at which it's decreasing doesn't decrease." He then concludes:

> Well, if it is continuous it would have one. Between one point four and one point five it would be zero. [...] Because f of x equals zero is between f of one point four and f of one point five.(14,16)

Here, he appeals first to the fictive motion ("goes down," "decreasing") of the function. Then he exploits the gaplessness of the graph, since $f_1(x) = 0$ would lie between $f_1(1.4)$ and $f_1(1.5)$ if there were no gaps in the graph of $f_1$. Later J discusses the continuity of $f_2$. He begins with dynamic language – "Because if you go from the left..."(82) – but then makes a comment which reveals considerations of closeness: "There is a discontinuity point at... around square root of two, it is discontinuous."(84) He speaks of a discontinuity *around* $\sqrt{2}$, an odd usage which may indicate that he is considering the interval close to the discontinuity. While certainly not conclusive, this may indicate a nascent metaphorical understanding of continuity in terms of closeness.

In the post-instruction interview, however, J speaks and gestures almost exclusively using the static language of closeness. First, J expresses a preference for Proof 1 because "it's referring to the distance between the two, and how you can get them close."(34) Then, when answering Question 2 and expressing what Proof 3 explains to him, he speaks statically of closeness:

> If you take an interval [...] you can always find an interval [...] all the points in that domain will be contained in the other interval [...](38)

The terms "take" and "find" are static terms, expressing being and not becoming. Also, as discussed in Chapter 3, "contained" is an embodied notion used to understand sets and closeness. Furthermore, while saying this, J makes iconic gestures in the air to indicate intervals along imaginary x- and y-axes.(39-48) These gestures are static and communicate closeness. Later, when explaining continuity, he says, "if you would take that interval to be very small."(86) Again, this utterance is both static – "take," not "make" – and related to closeness, since the interval is necessarily "small."

On a few occasions, he appeals to "jumps"(106,108) or "holes"(112) to explain discontinuities; this is the language of gaplessness. Even here, though, his language is static – "you take some interval..."(106) – and he appeals to notions of closeness – "you couldn't find any number that [...] wouldn't have its image to be two,"(106) and thus not far.

Throughout the post-instruction interview, J's reasoning about continuity applies variations of the formal definition, and speaks of closeness.(38, 52, 86, 102, 106, 116, 120) Indeed, the majority of J's language is static, and his reasoning primarily involves considerations of closeness. Therefore, between the pre- and post-instruction interview, the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor becomes dominant for J.

### 6.4.5 *Summary of Subjects' Conceptual Metaphors*

This experiment captured the evolution of the subjects' conceptual metaphors. Between the pre- and the post-instruction interviews, some acquired a new metaphor (e.g. Subject Y) and others adjusted the importance of the metaphors they already had (e.g. Subject J). Thus, the meaning and importance of a subject's conceptual metaphors are not fixed; they are constantly changing as the subject receives new instruction and is exposed to new mathematical contexts.

During the second interview – at that slice of time – each subject, positioned as a Learner, had a particular meaning structure for continuity that deployed both metaphors to varying degrees and in various relations to each other. It is this snapshot from the post-instruction interview that we will use to interpret their satisfaction with the proofs as explanations. During the post-instruction interview, Subject Y primarily uses the CONTINUITY IS GAPLESSNESS metaphor, and this discourse while positioned as a Learner is disconnected from her discourse while positioned as a Student. Subject P translates freely between the two metaphors; he maintains both metaphors, but has drawn connections between the two, and is able to draw on either. Subjects K and J predominantly use the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor. Subject K retains certain lexicalized dynamic phrases that no longer seem to carry metaphorical content – dead metaphors. Both K and J exhibit a unified discourse, unchanged by their institutional positioning. The subjects' preferences for one metaphor over the other are summarized in Table 4.

| SUBJECT | DOMINANT METAPHOR | REPRESENTATIVE QUOTE |
|---|---|---|
| Y | GAPLESSNESS | "When you look at the right side, this will always go up, and it's never going to reach the y-axis. [...] "It's not continuous" because of "this part in the middle where it's never going to meet." |
| P | Translates between metaphors | "Because even I had to translate the proofs in the book, in terms of [neighbourhoods], to explain to myself. [...] At the beginning, I had to translate what those neighbourhoods mean [...] in a way like proof number two [the dynamic proof]." |
| K | PRESERVATION OF CLOSENESS (with lexicalized GAPLESSNESS) | $f(x) = \sin(1/x)$ isn't continuous at $x = 0$ "because around zero [...] it's going to be very often here or there" [pointing to the top and bottom of the graph] |
| J | PRESERVATION OF CLOSENESS | "So, by taking g of x sufficiently close to some point, you could get f of that point to get sufficiently close to the image of that point." |

Table 4: Summary of the subjects' conceptual metaphors for continuity.

## 6.5 NOTIONS OF EXPLANATION

In this section we will analyse the subjects' notions of explanation. We will begin by examining their explicitly stated characterizations. Then we will summarize their evaluations of the three proofs as explanations.

### 6.5.1 *Subjects' Discussions of Explanation*

All four subjects interpret "explaining" as "explaining why the result is true" – the type of explanation that the study hoped to characterize. For instance, Subject K states that, in this case, an explanatory proof is "explaining a particular result (79) [...] that f of g of x is continuous."(81) When asked what their preferred proof explained, every subject reconstructed the reasoning of the proof in their own words; they recounted the *why* of the proof's result. (Y 64-69; P 45-66K 54; J 35-52) Only Subject J commented on the possibility of other kinds of explanation:

> [Another proof] would explain the concept of, say continuous in this case. But that wouldn't explain how to do the proof.(68) [...] That's two different things.(70)

The subjects' explicit characterizations of explanatory proofs are largely contradictory and incoherent – as was to be expected, considering this study was motivated by the difficulty of characterizing explanation. Subject Y advocates "as much detail as possible."(92) An explanatory proof, for her, "makes sure that you get every detail."(98) Both Subject K and Subject P reiterated the function of an explanatory proof. For K, an explanatory proof is one that is good at "explaining what happened, and why it is so"(76); for P, an explanatory proof "shows me why a certain theorem works." Subject J, on the other hand, lists clarity and conviction as defining aspects of an explanatory proof.(64,66)

However, none of these characterizations are sufficient to explain their own choice of proof. It seems that Y's explicit criterion does not unambiguously distinguish between the three proofs – since they are structurally equivalent, they present nearly the same amount of detail. Indeed, she comments, "So even though the third one gives a little bit less information, or less clear detail, for me it's easier to see."(94) Subjects K and P only present tautologies; in short, they state that an explanation explains. Finally, J's criteria are mixed. Reid and Roberts (2004) identify clarity as a factor in reader acceptance of a proof as an explanation. Conviction, however, is a completely different phenomenon, and cannot account for P's own satisfaction with the three proofs as explanations.

### 6.5.2 *Subjects' Choice of Explanatory Proofs*

See Table 5 for a summary of the subjects' choice of the most explanatory proof for themselves and for others.

| SUBJECT | Proofs as Explanations | |
|---|---|---|
| | MOST TO LEAST SATISFACTORY | FOR OTHERS |
| Y | Neighbourhoods, then Dynamic, then Static | Neighbourhoods |
| P | Neighbourhoods, then Static, then Dynamic | Dynamic |
| K | Neighbourhoods, then Static, then Dynamic | Static |
| J | Neighbourhoods, then Static, then Dynamic | Static |

Table 5: Subjects' satisfaction with the proofs as explanations

When asked which proof does a better job of explaining why the result is true, every subject chooses the Neighbourhoods Proof. Subject Y, however, indicates that she was positioned as a Student. She remarks that she thinks the Neighbourhoods Proof is more explanatory, "but that might be because that's how I had it explained to me.(46) [...] Furthermore, she prefers that proof because it allows her to refer back to previous results: "...and then you know that this is existing, because that was a theorem...."(60) In choosing the Neighbourhoods Proof, she is relying on the authority of the teacher – "that's how I had it explained to me" – and the textbook – "because that was a theorem." Thus, she is positioned as a Student, not a Learner. In fact, her immediate response is to select the Dynamic Proof as the most explanatory; it is only upon reflection that she modifies her choice, choosing the Neighbourhoods Proof over the Dynamic Proof. She was unique in identifying the Dynamic Proof as the second most explanatory.

For Subjects P, K and J, the Static Proof is the second most explanatory proof, while the Dynamic Proof was the least explanatory. When asked why, both P and J mention aspects of the language of the proof. J remarks that the Static Proof is "referring to the distance between the two and how you can get them close," while the Dynamic Proof is "not really referring to something that is close."(34) P has a more negative appraisal:

> Number Two [Dynamic Proof] I had a little bit of a problem, only because it's very...ah, like...It's just in terms of "far towards," which I haven't met right now.(36) [...] It's not a problem. I just need...It's, like, the first time confronting...It's the first time seeing a proof like that. So that's why I think I had to read it again.(38)

Subject K similarly comments on the use of "far towards:"

> Because this one says sufficiently close, and this one says sufficiently far towards a.(60)

And Subject J also criticizes the dynamic language:

> Well, moving is strange because that's not really a defined concept.(60)

Even Subject Y, for whom the Dynamic Proof was the second most explanatory, expresses concern, objecting, "[Neighbourhoods was clearly defined to me in my class, but this moving something...."(78) Indeed, it may be that familiarity – one of the factors identified by Reid and Roberts (2004) – was particularly influential in this case, prejudicing the subjects against the Dynamic Proof.

When asked how they would explain the result to a student who has not taken Analysis, three of the four subjects choose proofs that corresponded to the proofs they identified as most explanatory for themselves.

Subjects K and J, who preferred the Neighbourhoods and Static Proofs, both say they would use the Static Proof, since the student might not have experience with neighbourhoods.

Subject Y says she would use the Neighbourhoods Proof. She justifies her choice by claiming that "the idea of a neighbourhood is talked about in other classes too, and it's very visual."(100) However, neighbourhoods are generally not discussed in mathematics classes before Analysis, and their use in other classes is largely unrelated to their technical use in mathematics. This inconsistency suggests that perhaps she is positioned as a Student. While positioned as a Student, she may feel pressure to choose a proof with a more "mathematical" appearance. Indeed, Y behaves often throughout the interviews as a Student, and this positioning probably explains her choice of the Neighbourhoods Proof as both the most explanatory for her and for others.

Unlike Subjects K, J and Y, Subject P chooses the Dynamic Proof to explain to others. He is the only subject who chooses a proof for others that did not correspond to their own preferred explanatory proof. When asked why he would use this proof, he says:

> Because even I had to translate the proofs in the book, in terms of [neighbourhoods], to explain to myself. And after that, after a while, you get used to specific notations, a specific way how to prove something or how to understand something.(86) [...] But again, I'm going to say that at the beginning I had to translate what those neighbourhoods mean. I had to explain to myself with a picture in front of me, in terms... in a way like proof number two [i.e. the Dynamic Proof].(88)

He describes how, when first learning the formal definition of continuity, he translated from static to dynamic language:

> I had to translate what 'W neighbourhood of f of a' is [...], which means you can move, in terms of numbers, around f of a, as close to f of a, by moving x sufficiently near to a. So approaching a [...] from both sides – which, in the Proof Three, [is] said 'x is in the neighbourhood of a.'(96)

Thus, P recognizes the translatability of the two metaphors; with sufficient mental gymnastics, he can translate his reasoning from CONTINUITY IS PRESERVATION OF CLOSENESS to CONTINUITY IS GAPLESSNESS, and vice-versa. In selecting a proof to explain to another, he chooses a proof that would have explained to himself as he understood continuity a few weeks ago, before he began to acquire the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor.

Therefore, while all four subjects identify the Neighbourhoods Proof as the most explanatory for themselves, only three make that choice while positioned as Learners. Those three – Subjects P, K and J – also all identify the Static Proof as the second most explanatory for themselves. Subject Y, on the other hand, chooses the Neighbourhoods Proof while positioned as a Student; as a Learner, she chooses the Dynamic Proof. Thus, accounting for the influence of familiarity on the subjects' choice of the Neighbourhoods Proof, we can summarize subjects' satisfaction with the proofs as explanations like this: Subjects P, K and J preferred proofs that deployed the static CONTINUITY IS PRESERVATION OF CLOSENESS metaphor; Subject Y preferred the proof that deployed the CONTINUITY IS GAPLESSNESS metaphor.

## 6.6 SUMMARY

In this chapter, we applied the analytic tools developed in Chapter 4 to the subjects' interview transcripts. We traced the evolution of their conceptual metaphors for continuity and documented their implicit and explicit notions of explanation – all while remaining attentive to their positioning within the University-Analysis Institution.

We are finally ready to discuss the connections between explanation and conceptual metaphor. In the next chapter, we will use these results to answer our motivating research question: Do the reader's conceptual metaphors affect their acceptance of a proof as an explanation?

# 7 DISCUSSION AND CONCLUSIONS

## 7.1 INTRODUCTION

In this chapter we present the conclusions and implications of the study. First, using the data collected in the clinical interviews, we answer our original research question and draw general conclusions about conceptual metaphor and mathematical explanation. Next, we discuss the implications of these results – for mathematical explanation, for mathematical practice more generally, and for mathematical pedagogy. We conclude with a brief discussion of directions for future research.

## 7.2 CONCLUSIONS

Now that we have identified the subjects' conceptual metaphors and analysed their notion of explanation, we can finally answer our initial research question. Table 6 synthesizes the results of the study. It shows each subject's dominant metaphor for continuity and their choice of most explanatory proof for him- or herself and – if different – for others. Are a reader's conceptual metaphors a factor in their satisfaction with proofs as explanations? More precisely, is a reader more likely to accept a proof as an explanation if the proof's metaphorical language for continuity mirrors their own conceptual metaphors for continuity?

On a first analysis, the results do, indeed, show a correlation between each subject's dominant conceptual metaphor and their choice of explanatory proof. Subject Y uses the dynamic CONTINUITY IS GAPLESSNESS metaphor; she identifies the Dynamic Proof as the most explanatory. Subject P translates between both metaphors; he identifies the Static Proof as most explanatory for himself, and the Dynamic Proof as most explana-

| SUBJECT | DOMINANT METAPHOR | EXPLANATORY PROOF |
|---|---|---|
| Y | Gaplessness | Dynamic |
| P | Translates metaphors | Dynamic for others/ Static for self |
| K | Preservation of Closeness (lexicalized Gaplessness) | Static |
| J | Preservation of Closeness | Static |

Table 6: Relations between subjects' conceptual metaphors and their choice of explanatory proof.

tory for others. Subjects K and J both primarily use the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor; they identify the Static Proof as most explanatory for themselves. These results are positive evidence that a reader's conceptual metaphors are a factor in their satisfaction with a proof as an explanation. Thus, on a crude level, we can answer our research question in the affirmative: A proof involving continuity is more explanatory for the reader when it uses metaphorical language that reflects the reader's own conceptual metaphors for continuity. It appears that a reader is satisfied with a proof as an explanation when it situates the result in their metaphorical basis of understanding for the pertinent concepts.

## 7.3 DISCUSSION

### 7.3.1 *Conceptual Metaphors*

The behaviour of the subjects supports our assumption in the theoretical framework that proofs are never explanatory in and of themselves; they are explanatory for a particular reader, at a particular moment. Consider, for instance, Subject Y's reflection on when a proof is explanatory:

> But then again, it's kinda dependent on which way people like to see things. Because a lot of people would probably prefer a proof like the first one, actually. Because this one is a little bit more, like, it's easier to understand, I guess. But in my sense, it's not.(92)

Indeed, as a reader's conceptual metaphors develop over time, their satisfaction with a proof as an explanation may also change. Subject P, who translates back and forth between metaphors, chooses the Static Proof as the most explanatory for himself; for others, however, he chooses the Dynamic Proof. Recall his justification:

> Because even I had to translate the proofs in the book, in terms of [neighbourhoods], to explain to myself. [...] At the beginning, I had to translate what those neighbourhoods mean [...] in a way like proof number two [the dynamic proof].

P recognizes that in the past – before his acquisition of the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor at some point between the pre- and post-instruction interviews – a dynamic proof would have been explanatory for him.

Indeed, only a dynamic proof would have sufficed; he "had to" translate from the static language of neighbourhoods to the dynamic language of fictive motion. Although P claims that he no longer requires the dynamic metaphor, his behaviour indicates that he still relies on both, and struggles to switch between them. For instance, when he is unable to understand the discontinuity of $f(x) = \sin(1/x)$ in terms of gaplessness, it takes considerable prompting on the part of the interviewer (151-196) before he considers the function in terms of preservation of closeness. For P, the two metaphors – while connected – are still in conflict; they both vie for his attention and he only switches between metaphors with difficulty. This coexistence of the two metaphors explains P's identification of both the Static and Dynamic proofs as explanatory. Therefore, conceptual metaphors, and hence readers' assessments of proofs as an explanations, vary both at one time as a reader accommodates multiple metaphors, and over time as these metaphors evolve for the reader.

The results of this research allow us to draw a number of more general conclusions about students' metaphorical thought. While there is literature on these particular conceptual metaphors for continuity that documents their existence in both the history of mathematics (Lakoff and Núñez, 2001) and the thought of contemporary mathematicians (Núñez, 2006), this is the first study, as far as we know, that has explored their use by students. Moreover, beyond merely documenting their existence, the present case studies reveal the richness of the evolution of these metaphors as the subjects struggle to reach a conceptual equilibrium. As described in Chapter 5, the subjects' dominant conceptual metaphors evolve – sometimes drastically – between the pre- and post-instruction interviews. A student may acquire a new metaphor or gradually abandon an old one, and throughout the process will have to accommodate two inconsistent metaphors.

This accommodation results in diverse meaning structures among the Subjects. Subject Y maintains two disconnected and parallel metaphorical discourses for continuity, each deployed according to her institutional positioning. She has accommodated the two metaphors only in that they coexist; they remain independent and dissociated. Comparably, Subject P also uses two distinct discourses, although his discourses – while parallel much like Y's – are connected by his ability to translate between metaphors. While he has not unified his understanding of continuity, he nevertheless is trying actively to reach an equilibrium between the divergent metaphors. Subjects K and J, on the other hand, have attained a degree of equilibrium, and demonstrate a unified discourse – the discourse of CONTINUITY IS PRESERVATION OF CLOSENESS [1]. The subjects demonstrate that the accommodation of multiple metaphors can be accomplished in a number of ways.

The subjects' dominant metaphors affect their success in the interview tasks. When asked if $f(x) = 1/x, x \neq 0$, is continuous on its domain, Subjects P, K and J – reasoning with the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor – determine correctly that it is, indeed, continuous. On the other hand, Subject Y – reasoning with the CONTINUITY IS GAPLESSNESS metaphor – states incorrectly that the function is discontinuous at $x = 0$, "this part in the middle where it's never going to meet"(202). This confirms Tall and Vinner (1981)'s suggestion that student failure to recognize the continuity of this function on its domain is related to having a concept image "which does not allow 'gaps.'"(p.166)

The nature of the subjects' (metaphorical) conceptual evolution indicates a certain incompatibility between the two metaphors: P translates with difficulty between

1 Recall that, while Subject K retains certain conventional mathematical phrases from the CONTINUITY IS GAPLESSNESS metaphor, these are a lexicalization of the metaphor – the metaphor is dead.

metaphors; Y cannot translate at all, and her behaviour remains splintered along institutional lines; and both K and J abandon the conceptual content of the CONTINUITY IS GAPLESSNESS metaphor when they acquire the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor. This points to a difficulty with the introduction of continuity as it is currently done. It may be that students are poorly served by the "intuitive" meaning of continuity that, while present in the history of the concept, is notably absent from the meaning of continuity as it is used in mathematics today. As Daval and Guilbaud (1945, p.119) point out, the intuitive notion of gaplessness is formalized as *completeness*, not continuity, and the conflation of these concepts both historically (by mathematicians) and today (by students) is the source of much confusion. Freudenthal (1963) warns against the pedagogical practice of antididactical inversion: the introduction of a concept as it is used today, without reference to, or grounding in, its historical or conceptual origins. Sierpinska (1988) gives the example of the teaching of the function concept:

> The most fundamental conception of a function is that of a relationship between variable magnitudes. If this is not developed, representations such as equations and graphs lose their meaning and become isolated from one another. (Sierpinska, 1988, p.572)

Thus, she concludes, "[i]ntroducing functions to young students by their elaborate modern definition is a didactical error - an antididactical inversion."

However, our results suggest that – in the case of continuity – such an inversion may be didactically appropriate. Unlike in the case of functions, the historical origins of continuity are so disconnected from its modern usage that a historically exact introduction may be counterproductive. Y's halted conceptual development and P's struggles to translate between coexisting metaphors are evidence that understanding continuity in terms of gaplessness can be an obstacle[2] to the acquisition of the modern understanding. Note that we are not suggesting a purely formal, axiomatic introduction – only that the informal discussion should refer to closeness, and save the discus-

2 Here we're using the term "obstacle" in the colloquial sense; however, this may also be an instance of an *epistemological obstacle* in the sense of Brousseau (1997). See also Bachelard (1938).

sion of gaplessness for the introduction of completeness. Recall that the questions on approximation in the pre-instruction interview failed, for most students, to evoke a discussion of approximation. However, as Daval and Guilbaud (1945) note, the need to approximate functions was central to the development of the modern definition of continuity – many methods of approximation only work when the function is continuous. And so, perhaps, students should be introduced to continuity as it arose in the historical context of approximation, and not, as is commonly done, in the context of what Núñez (1998) calls *natural continuity*: the understanding of continuity in terms of gaplessness that was common in the time of Euler.

An unexpected phenomenon was the wide range of the influence of institutional positioning. For some subjects, their positioning made little observable difference in their behaviour. Subjects K and J have appropriated the institutional discourse; their discourse while positioned as Learners mirrors their discourse as Students, and reproduces the normative discourse of the University Analysis institution. They do not noticeably change the nature of their discourse at any point of their post-instruction interview; they consistently apply the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor, whether they have positioned themselves as Students or as Learners. Similarly, Subject P is able to translate between metaphors, and so his metaphorical discourse while positioned as a Learner is not considerably different from his discourse as a Student. On the other hand, Subject Y has not internalized the institutional discourse; while she can reproduce the static institutional discourse of neighbourhoods and closeness, her own authentic thinking as a Learner is dynamic. Thus, her behaviour changes radically depending on how she positions herself, and her applications of the two conceptual metaphors remain largely disconnected.

It may be that such a fractured discourse has implications for a subject's conceptual evolution. As discussed above, both Subject P and Subject Y exhibit two distinct metaphorical discourses. However, Subject P's translation between metaphors indicates his struggle to integrate both metaphors – his attempt to move beyond the disequilibrium of working with multiple metaphors. On the other hand, Subject Y's behaviour remains utterly disconnected, only deploying each metaphor while occupying the associated institutional position, and this prevents her from confronting the conflict between her metaphors and evolving beyond this disconnect. Meanwhile, Subjects K and J – whose discourses are unified across institutional positions – possess a unified understanding of continuity. Indeed, the harmonization of a subject's conceptions of continuity while positioned as a Learner and as a Student may be predictive of their consolidation of the concept of continuity.[3] This relation between harmonization of discourse across institutional positions and concept consolidation may explain the different degrees to which the subjects have acquired the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor. These results underscore the importance of an institutional framework, even when studying putatively cognitive phenomena like mathematical explanation and metaphor.

### 7.3.2 *Research Instruments*

This study attempted a cognitive characterization of a phenomenon that has, until now, been approached otherwise – behaviourally (Reid, 2002; Reid and Roberts, 2004; Healy and Hoyles, 2000; Hoyles, 1997) or mathematically (Steiner, 1978; Kitcher, 1981; Hanna, 1990). There is no literature on the cognition of mathematical explanation; indeed, historically, the reader of the proof has been ignored or downplayed. As such, we needed to create new research instruments. In particular, we had to develop a sequence of proofs that, while mathematically identical, differed metaphorically, and mathematical tasks that evoke metaphorical discourse about continuity.

In general, the research instruments were successful. Although the proofs were logically equivalent, they elicited diverse responses from the subjects; indeed, every subject was able to identify a most and a least explanatory proof. Thus, altering the metaphorical content while controlling the mathematical content was a successful methodology. Generally, though, this is difficult to accomplish. One possibility – especially for more mathematically sophisticated subjects – might be to exploit categorical equivalences. For instance, the category of (sober) Topological Spaces is categorically

3 See, for instance, (Herschkowitz et al., 2001) for an extended discussion of concept consolidation.

equivalent to the category of (spatial) Frames. This equivalence allows for the "translation" of a proof in one category into an equivalent proof in the other. (Johnstone, 1983) Were the current research extended to graduate students or research mathematicians, this could provide a good corpus of mathematically equivalent proofs that differ metaphorically.

Moreover, the mathematical tasks were quite effective at evoking a variety of metaphorical responses. Consider one small example: The gestural responses of Y and K to the first function on the post-instruction interview, $f(x) = \sin(1/x)$. Subject Y responded dynamically: She wiggled her finger up and down, representing the fictive motion of the function's oscillations, while saying, "It just comes and it gets really small, and all the sine curves get closer and closer."(176) Subject K, on the other hand, responded statically: He made a series of deictic gestures, pointing to the top and bottom of the function's graph, while saying, "Because around zero, you're going to have the function, it's going to be very often either here or there."(160) The task managed to evoke two very different gestural responses – responses that typified the two conceptual metaphors under consideration. These evoked gestures are reminiscent of gestures documented by Núñez (2006) in professors of mathematics. Moreover, the close connection between the subjects' gestures and their metaphorical speech corroborates the extensive literature on language-gesture coproduction (MacNeill, 1992; Alibali et al., 1999).

However, two aspects of the research instruments proved problematic. First, the use of the phrase "move sufficiently far towards" in the dynamic proof presented difficulties. Two of the subjects – Subject K (60-64) and Subject J (36) – expressed discomfort with the phrase, particularly with the word "far." This unusual phrasing was alien to the subjects. If we were to use this proof again, we would shorten "move sufficiently far towards" to "move sufficiently towards." Second, and more problematic, was the informality of the proofs. While the Static and Dynamic proofs were modelled on the formal Neighbourhoods proof and preserved all essential inferences at a conceptual level, they may not have been sufficiently formal to qualify as proofs within the University Analysis institution. The course on Analysis is designed to introduce students to formal proof; the institutional norms require the use of formal algebraic methods when proving. Indeed, at least two of the subjects are explicitly aware that the first two proofs violate institutional norms: J complains about the Dynamic Proof, saying, "Moving is strange because that's not really a defined concept,"(60) and K prefers the Neighbourhoods Proof because "there's more math symbols."(24) As discussed in Chapter 4, we purposefully reduced the level of formality of the first two proofs to increase simplicity and clarity – but this reduction in formality may have been excessive. While the literature shows that mathematical proofs are seldom fully formalized (Azzouni, 2004; Rav, 2007; Aberdein, 2008) and many of the formal deductive steps of a proof are often implicit (Duval, 2000), there remains a baseline of required proof formality in mathematical practice – and that baseline should be maintained in Mathematics Education research (Balacheff, 2008). It may be that these first two proofs did not meet that baseline of formality.

### 7.3.3 *Explanation*

The subjects' choices of proofs confirmed two findings in the literature. First, institutional factors influence student proof preference. Healy and Hoyles (2000) investigated the relation between student views on proof and their preference for various proofs. She identified two widespread notions of proof: "those about arguments they considered would receive the best mark and those about arguments they would adopt for themselves" (p.46). Healy and Hoyles, however, do not offer a theoretical explanation of these results. Viewed from an institutional perspective, these two categories suggest the institutional positions of Learner and Student: as Students, the subjects identified those proofs that would receive the best mark; and as Learners, they identified those proofs they would adopt for themselves. When thus analysed, the observation of Healy and Hoyles (2000) anticipates one of our own results: Subject Y selects proofs that conform to the norms of the University Analysis institution. In any investigation of proof, it is necessary to take into account the institutional factors. Second, previous research identified familiarity as a factor in students' judgements of proof explanatoriness (Healy and Hoyles, 2000; Reid and Roberts, 2004) This study confirmed the influence of familiarity. All subjects chose the most familiar proof – the Neighbourhoods Proof they had seen in class – and one even commented on the effect of her familiarity with the proof.(Subject Y, 46) Thus, for the purposes of investigating explanation, it is important to have a set of proofs that are equally familiar to the reader.

The results of this thesis conjure up another characterization of mathematical explanation in the literature: the "characterizing property" treatment of Steiner (1978). Recall that Steiner argues that an explanatory proof must make use of a characterizing property of the concepts involved. For Steiner, this characterizing property is entirely mathematical, unrelated to the reader of the proof. A corollary of this characterization is that certain proofs are unconditionally explanatory – a consequence that has been widely criticized (Resnik and Kushner, 1987). We may be able to salvage Steiner's approach, however, if we define the characterizing property in relation to the reader of the proof. Indeed, our account of explanation presented in this thesis also depends on a characterizing property, albeit a cognitive one: the reader's conceptual metaphors. If the reader characterizes a continuous function using the property of, say, gaplessness, then – to use Steiner's jargon – a proof will be explanatory when it "makes reference to [that] characterizing property" of continuity; mutatis mutandis for preservation of closeness. It may be, then, that a cognitivized version of Steiner's account can avoid the absolutism that doomed it in its original form.

The compatibility between Steiner's account and the conceptual metaphor account of explanation suggests that conceptual metaphors may play a role in mathematical explanation, not just for students, but generally. Indeed, the same mechanisms may play a role in judgements of explanatoriness by mathematicians, teachers and students. Recall Thurston (1994), who recounts the difficulties that other mathematicians had when trying to understand his proofs. He writes:

> My personal mental models and structures are similar in character to the kinds of models groups of mathematicians share – but they are often different models.

As a result, his "idiosyncratic...personal mental models" made it difficult to for other mathematicians to understand his proofs. Note that if we replace "personal mental models" with "conceptual metaphors," we have a restatement of our characterization of explanation. It may be that linking metaphors, which connect two abstract domains, are a part of Thurston's so-called mental models – in which case, the characterization defended in this thesis would extend to research mathematicians, too. In any case, the effect of conceptual metaphors on explanation among research mathematicians is a promising line of inquiry for future research.

Moreover, while conceptual metaphors are a factor in a reader's satisfaction with a proof as an explanation, a proof may also be a factor in the creation of new conceptual metaphors for a reader. Thurston (1994) and Dawson (2006) discuss how mathematicians read proofs to develop a deeper understanding of the concepts involved – that is, the reading of the proofs engenders a new conceptualization. This may also be the case with conceptual metaphor. A proof may reveal logical connections between two distinct mathematical domains; a reader of that proof could then exploit those connections to create a new conceptual metaphor linking those domains. The proof reveals logical connections; the reader creates conceptual connections. Although this study did not directly investigate the causes of the subjects' conceptual evolution, a quote from Subject P is suggestive:

> Because even I had to translate the proofs in the book, in terms of this, to explain to myself. *And after that, after a while, you get used to specific notations, a specific way how to prove something or how to understand something.*(p.86, emphasis added)

By reading proofs, P acquired a new way "how to understand something": in this case, he began to understand continuity as PRESERVATION OF CLOSENESS. Not only did P's conceptual metaphors affect his satisfaction with proofs as explanations, but also his reading of proofs affected his conceptual metaphors.

Thus, there may be a two-way interaction between conceptual metaphor and the reading of proof. Let us consider a consequence of this suggested interaction between proof and conceptual metaphor. A new proof that connected two previously remote domains would be non-explanatory when it first appears; nobody would possess a conceptual metaphorical linking the domains. In time, however, the proof would generate the requisite linking metaphor in a reader – and the proof, in turn, would become explanatory for that reader. While suggested by the data, however, this coupling between conceptual metaphors and the reading and writing of proofs remains a conjecture.

### 7.3.4 *Educational Implications*

The implications of this study for the teaching of mathematics are twofold. First, teachers should attend to metaphorical aspects of both their instruction and the discourse of their students. Second, when selecting a proof for use in the classroom, they should carefully consider the implicit conceptual metaphors deployed by the proof. We will consider each of these in turn.

A student's metaphorical discourse reveals their underlying metaphorical thought. Thus, when a teacher pays attention to their students' metaphorical discourse – manifested explicitly in gesture and speech, or implicitly in reasoning – they are then better prepared to diagnose misconceptions. For instance, even if the student can successfully use neighbourhoods to prove results involving continuity, they have not fully internalized the concept if their speech and gesture still express the dynamic CONTINUITY IS GAPLESSNESS metaphor. Consider Subject Y, who is able to replicate the institutional discourse of neighbourhoods, but who still thinks dynamically: Her misconceptions are concealed by her successful proving practices, and are only revealed by a subtle analysis of her metaphorical discourse. By attending to this discourse, a teacher can gain insight into a student's thought.

Furthermore, a teacher should carefully modulate their own metaphorical language. For instance, a teacher's use the lexicalized dynamic expression "the limit of the function as x approaches a" – a conventional mathematical phrase – may, for the student, connote actual motion. This is not to say that the teacher should avoid using such phrases. Certainly, these phrases are part of the mathematical lexicon and must be acquired by the student; in fact, the connotation of motion may be quite helpful in some contexts. However, when the notion of fictive motion is no longer appropriate, the teacher must be aware of possible conflicts between their intended meaning and the student's metaphorical interpretation. Moreoever, as discussed above, the intuitive introduction of continuity in terms of gaplessness may be more hindrance than help; the discourse of gaplessness belongs to completeness, not continuity.

Finally, explanatory proofs are used in the classroom to impart understanding – and the relation between conceptual metaphor and explanation provides one tool for identifying appropriate classroom proofs. By attending to their students' conceptual metaphors, a teacher can select proofs that mirror these metaphors – or, if a students' metaphors are inappropriate, at the very least recognize the source of their disagreement over the explanatoriness of the proofs.

## 7.4 FUTURE WORK

The success of the present study suggests a number of possible avenues for future research. The most obvious suggestion is to expand the present case study into a large-scale quantitative experiment that tests the relation between students' conceptual metaphors and their satisfaction with proofs as explanations. The behaviour of the four students studied presents strong support for a correlation between metaphor and mathematical explanation, but that support will remain anecdotal until these results are reproduced with a more significant sample size.

Furthermore, future studies should test subject satisfaction with proofs as explanations at different stages of their conceptual development. It would have been interesting to see how the subjects of this case study would have judged the Static and Dynamic proofs if they had been asked *before* they had been exposed to the formal definition of continuity. Exposing a student to proofs on multiple occasions, however, raises concerns of familiarity, and we would have to devise a proper methodology for the diachronic study of a subject's satisfaction with proofs as explanations. Finally, future studies should test the relation between conceptual metaphor and explanation for concepts other than continuity – the real numbers, for instance. By studying a larger sample size, tracking the evolution of subjects' satisfaction with proofs, and testing a variety of concepts, we could lend further credence to the correlation between a reader's conceptual metaphors and their satisfaction with proofs as explanations.

In this study, three of the four students acquired the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor – the normative metaphor for University-Analysis. However, the remaining subject, Subject Y, remained bound to the CONTINUITY IS GAPLESSNESS metaphor. It is unclear why she did not acquire the new static metaphor, although we suggest above that this was related to a discord between Y's discourse while positioned as a Learner and as a Student. Is there a didactic intervention that would encourage Subject Y to internalize the CONTINUITY IS PRESERVATION OF CLOSENESS metaphor? What is the relation between the evolution of a student's conceptual metaphors and classroom instruction, homework tasks, etc.? These remain open questions, ripe for investigation.

The results of the current study suggest possible investigations that extend beyond the student. As discussed above, the same mechanism discovered here may be partially responsible for mathematical explanation among mathematicians, both as teachers and researchers. As teachers, mathematicians are encouraged to select explanatory proofs for the classroom, but it remains unknown how they actually select these proofs. Can and should they account for possible differences between the proofs that explain for them and the proofs that explain for their students? As researchers, on the other hand, mathematicians seek out explanatory proofs to gain insight into a result. Further investigation is required to determine if a research mathematician's conceptual metaphors are a factor in their satisfaction with proofs as explanations.

# APPENDIX

# A PRE-INSTRUCTION QUESTIONNAIRE

## A.1 QUESTION 1

The table below presents a fragment of the table of approximate values of some function $f : \mathbb{R} \mapsto \mathbb{R}$.

| $x$ | $f(x)$ |
|---|---|
| 1.3 | −0.4215907 |
| 1.4 | −0.0567776 |
| 1.5 | 0.3709375 |
| 1.6 | 0.8696576 |
| 1.7 | 1.4481457 |
| 1.8 | 2.1158368 |

1. Do you think the equation $f(x) = 0$ has a solution? Why or why not?

Consider the function $f : \mathbb{R} \mapsto \mathbb{R}$ defined by the following rule:

$$f(x) = \begin{cases} (0.01 \cdot x^3 + 0.2 \cdot x^2 + 1)(x^2 - 2) & \text{for } x \in \mathbb{Q} \\ 1 & \text{for } x \notin \mathbb{Q} \end{cases}$$

1. Could this function be the function that Question 1 talks about? (You can use a calculator.)
2. Is $\sqrt{2}$ a root of the equation $f(x) = 0$? Why or why not?
3. Is 1.414213562 a good approximation of a root of $f(x) = 0$? Why or why not?

Consider the function $f : \mathbb{R} \mapsto \mathbb{R}$ defined by the following rule:

$$f(x) = \begin{cases} (0.01 \cdot x^3 + 0.2 \cdot x^2 + 1)(x^2 - 2) & \text{for } x \in \mathbb{Q} \\ 0 & \text{for } x \notin \mathbb{Q} \end{cases}$$

1. Could this function be the function that Question 1 talks about? (You can use a calculator.)
2. Is $\sqrt{2}$ a root of the equation $f(x) = 0$? Why or why not?
3. Is 1.414213562 a good approximation of a root of $f(x) = 0$? Why or why not?
4. What is the main difference between the function in Question 2 and the function in Question 3?

## A.4 QUESTION 4

The table below contains some values of a piecewise linear function $f : \mathbb{R} \mapsto \mathbb{R}$. One piece of the function is linear on $(-\infty, 7.8)$, and the other is linear on $[7.8, \infty)$.

| x | f(x) |
|---|---|
| 7.5 | −0.25 |
| 7.6 | −0.20 |
| 7.7 | −0.15 |
| 7.8 | 0.10 |
| 7.9 | 0.15 |
| 8.0 | 0.20 |
| 8.1 | 0.25 |
| 8.2 | 0.30 |

1. Can you sketch a graph of this function?

2. Does this function have a limit at $x = 7.8$?

3. What, if anything, does this function have in common with the function in Question 2?

4. Compare this function with the function in Question 3.

# B

# POST-INSTRUCTION QUESTIONNAIRE

## B.1 EXPLANATORY PROOF

Consider three proofs of the following theorem:

> *If* $g$ *is continuous, and* $f$ *is continuous on the range of* $g$, *then* $f(g(x))$ *is continuous.*

**Theorem B.1.1.** *Take any two functions* $g$ *and* $f$ *defined on the real numbers. If* $g$ *is continuous, and* $f$ *is continuous on the range of* $g$*, then* $f(g(x))$ *is continuous.*

*Proof 1.* Recall that "$f$ is continuous at $a$" means that we can make $f(x)$ as close as we want to $f(a)$ by taking $x$ close enough to $a$.

Take any $a$ in the domain of g. We need to show that we can make $f(g(x))$ as close as we want to $f(g(a))$, by taking $x$ close enough to $a$.

Say we want $f(g(x))$ to be so close to $f(g(a))$ that they only differ by some arbitrary number. Since $f$ is continuous, we can do this by making $g(x)$ sufficiently close to $g(a)$.

Now, we need to make sure that we can make $g(x)$ sufficiently close to $g(a)$. However, $g$ is also continuous, so, by taking $x$ sufficiently close to $a$, we can make $g(x)$ as close as we want to $g(a)$.

Thus, if $x$ is sufficiently close to $a$, then $f(g(x))$ will be arbitrarily close to $f(g(a))$.

Therefore, $f(g(x))$ is continuous for all $a$ in the domain of g. In other words, $f(g(x))$ is continuous.

□

**Theorem B.1.2.** *Take any two functions* $g$ *and* $f$ *defined on the real numbers. If* $g$ *is continuous, and* $f$ *is continuous on the range of* $g$*, then* $f(g(x))$ *is continuous.*

*Proof 2.* Recall that "$f$ is continuous at $a$" means that we can move $f(x)$ as near to $f(a)$ as we want, by moving $x$ sufficiently near to $a$.

Take any $a$ in the domain of $g$. We need to show that we can move $f(g(x))$ towards $f(g(a))$ until they differ by any arbitrary distance by moving $x$ far enough towards $a$.

Say we want $f(g(x))$ to be moved towards $f(g(a))$ until they differ by less than a certain distance. Since $f$ is continuous, we can do this by moving $g(x)$ sufficiently far towards $g(a)$.

Now, we need to make sure that we can move $g(x)$ sufficiently far towards $g(a)$. However, $g$ is also continuous, so, by moving $x$ sufficiently far towards $a$, we can move $g(x)$ towards $g(a)$ until they differ by any distance we want.

Thus, if we move $x$ sufficiently far towards $a$, then we will have moved $f(g(x))$ arbitrarily near to $f(g(a))$.

Therefore, $f(g(x)$ is continuous for all $a$ in the domain of $g$. In other words, $f(g(x))$ is continuous. □

**Theorem B.1.3.** *Take any two functions* $g$ *and* $f$ *defined on the real numbers. If* $g$ *is continuous, and* $f$ *is continuous on the range of* $g$*, then* $f(g(x))$ *is continuous.*

*Proof 3.* Recall that "$f$ is continuous at $a$" means that whenever $W$ is a neighbourhood of $f(a)$, there exists a neighbourhood $U$ of $a$ such that $f(U) \subset W$.

Take any $a$ in the domain of $g$. We need to show that for any neighbourhood $W$ of $f(g(a))$, there is a neighbourhood $U$ of $a$ such that $f(g(U)) \subset W$.

First, since $f$ is continuous, there always exists a neighbourhood $V$ of $g(a)$ such that $f(V) \subset W$.

Next, since $g$ is continuous, there also always exists a neighbourhood $U$ of $a$ such that $g(U) \subset V$.

Thus, for any neighbourhood $W$ of $f(g(a))$, there exists a neighbourhood $U$ of $a$ such that

$$f(g(U)) \subset f(V) \subset W$$

Therefore, $f(g(x))$ is continuous for all $a$ in the domain of $g$. In other words, $f(g(x))$ is continuous.

□

### B.1.1 *Response to the Proofs*

1. Which proof does the best job of explaining why $f(g(x))$ is continuous? Why?
2. What does this proof explain to you?
3. Which proof is worst at explaining why $f(g(x))$ is continuous? Why?
4. What do you mean by "explanatory"?
5. Imagine that a student who hasn't taken a class in Analysis asks you why $f(g(x))$ is continuous. How would you explain it to them?

1. How do you understand continuity of a function?
2. Can you give an example of a function that is continuous? Why is it continuous?
3. Can you give an example of a function that is not continuous? Why is it not continuous?
4. Complete these sentences:
   a) Continuity is like...
   b) Continuity is *not* like...

Explain why each of these functions is or is not continuous on its domain.

1.

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

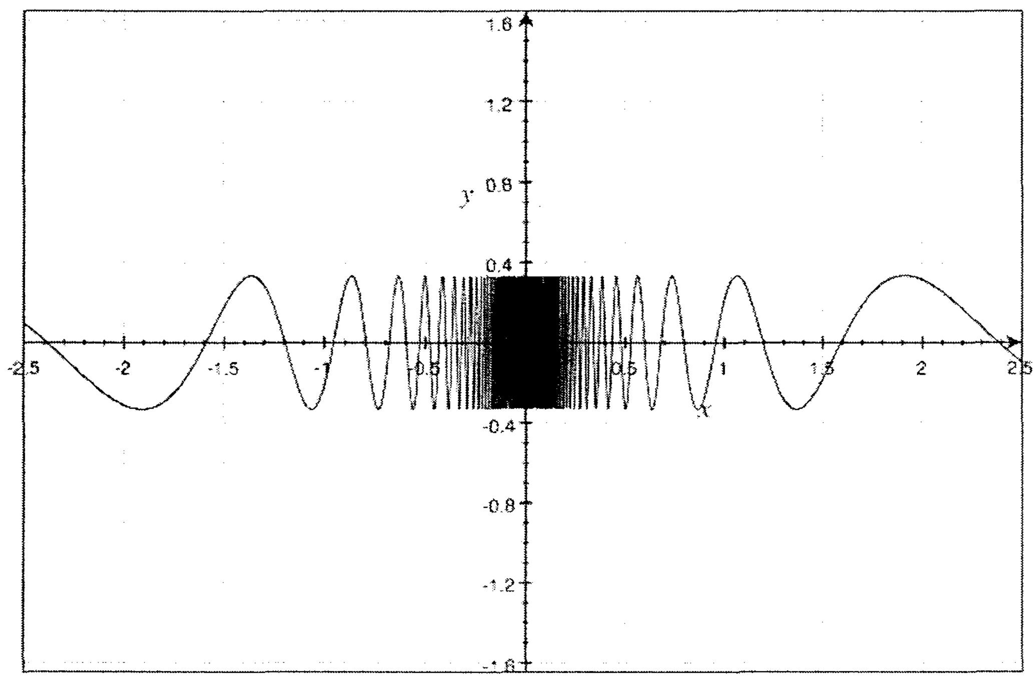


Figure 5: Graph of $f(x) = \sin(1/x), f(0) = 0$.

2.

$$f(x) = e^x$$

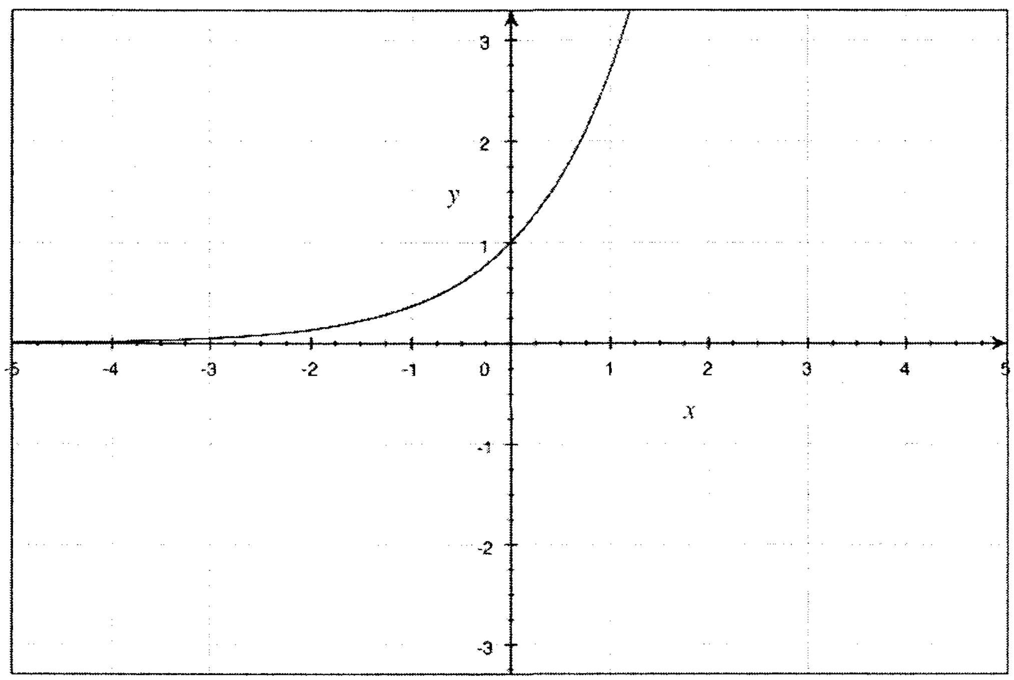


Figure 6: Graph of $f(x) = e^x$.

3.

$$f(x) = \frac{1}{x} \qquad x \neq 0$$

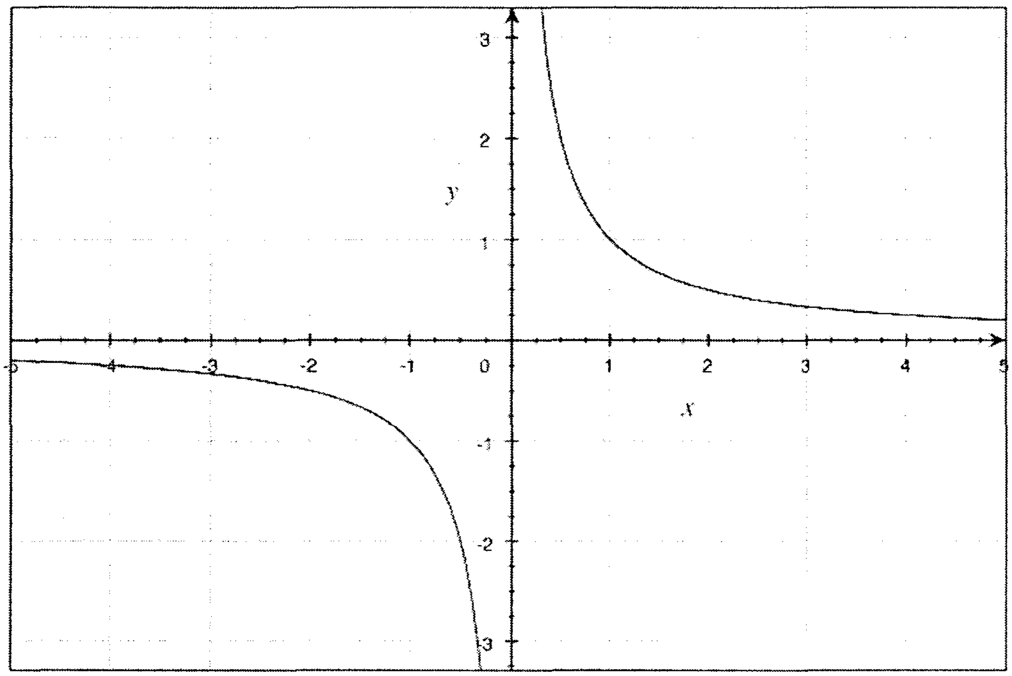


Figure 7: Graph of $f(x) = 1/x, x \neq 0$.

# C

# INTERVIEW TRANSCRIPTS

## C.1 SUBJECT Y

### C.1.1 *Pre-Instruction Interview*

1. T: Can you say your name?
2. Y: [omitted]
3. T: Ok, thank you Y[omited]. So, we'll start with question one. The table below presents a fragment of the table of approximate values of some function f from the real numbers to the real numbers. Ok, so when x is at 1.3, that's what f of x is... that's 1.4... So the question is, do you think the equation f(x)=0 has a solution? Why or why not?
4. Y: Um....
5. T: So, is there a spot where if x is a certain, whatever number, f of x is going to be zero?
6. Y: Ok, I would say... no?
7. T: Ok. And why do you say that?
8. Y: Because... ok wait, I'm looking at the digits.. 1, 2, 3... actually, well, it doesn't look like there's any sort of... oh, maybe...
9. T: And why do you say maybe?
10. Y: Oh, no, I lied... Because... well, kind of... Because it's increasing up until like 1.8 but then it's decreasing this way. But, there could be a number between 1.4 and 1.5 that would equal.
11. T: Ok. It would equal... f of x would be zero there?
12. Y: Yeah.
13. T: Ok, and you think that because at 1.8 it's fairly large, and then it decreases...
14. Y: Yeah, and then it goes back. But it could also not.
15. T: Ok. And why do you say it could not?
16. Y: Because there couldn't... maybe it's like an irrational, or maybe it just... it just doesn't exist kind of thing?
17. T: And you can't...?
18. Y: Well, I couldn't tell because I don't know the function.
19. T: Ok. Question 2. Here we have another function. So consider the function f from the real numbers to the real numbers. And this isn't the same f we had before necessarily. Defined by the following rule - so here we do we have a rule. So if x is rational, then it's this.
20. Y: Ok, and then if x is irrational...
21. T: And if x is not rational, irrational, then it's defined as 1.
22. Y: Ok
23. T: So, the first question is: Could this function be the function that Question 1 talks about? You can use a calculator - to check values, for instance. So if you wanna...

24. (Moving papers.)
25. Y: Ok.
26. T: Ok, so can you just tell me what you're doing here?
27. Y: Oh, well ok, I picked 1.5 to choose, because if it were to be zero, it would probably be around there, so I'm just trying to, like, see kind-of-thing. And that one's easier to do, I think.
28. T: So you're going to plug it into this formula that we have...
29. Y: Into this function for whatever x is. But I generally tend to screw up when I do calculations like this, so we'll see. (Writing.)
30. Y: OK, oh.
31. T: X to the three... Push shift... (Giving advice on using the calculator. Calculator noises.)
32. Y: ... Ooops.
33. T: You might want to check this value here. It's 2.25 minus 2, so it's probably going to be positive.
34. Y: Oh, right.
35. T: Ok. So what is 1.5 supposed to be?
36. Y: Oh, I might have done that wrong.
37. T: Oh, you're supposed to multiply it by point two-five.
38. Y: Oh, perfect. See, I make silly mistakes.... Still wrong. (Laughter.)
39. T: Well, it's supposed to work out.
40. Y: I'm sure it is! But...
41. T: Well, that looks about right... two-point-two-five times zero-point-two is... this is probably maybe zero point... zero point four five.
42. Y: Yeah. Perfect!
43. T: So let's try that one more time.
44. Y: Ok.... Yup!
45. T: Perfect! So what did you find?
46. Y: I found that this... well, it follows for one point five.
47. T: Ok, one point five it works. So after doing one point five, what do you think about the question? Do you think that this function could be the function that question one talks about.
48. Y: (Unsure.) It could be.
49. T: Ok. Do you want to try one more value, maybe?
50. Y: Yeah, probably one that's not close to it though.
51. T: So if you think if we try another value it won't match up? It just matches up for this one...
52. Y: Well, it could. But, it could also... you know what I mean? So I would pick personally, if I had to check, I would pick these three.
53. T: So you're pointing out one point three, one point five...
54. Y: ... one point five and one point eight.
55. T: Ok, and why would you pick those three?
56. Y: Because I, well, I'd just pick them because there's a significant difference in their answer?

57. T: Oh, yeah. So that's a place where there's a big change in the value of the function?
58. Y: Yeah, yeah.
59. T: Ok, so let's try one more.
60. Y: Ok.
61. T: Ok, so the first value we tried was one point five, and now we're going to try...
62. Y: One point eight.
63. T: Ok, great.
64. Y: (incomprehensible mumbling. Writing.) Ok, let's see. (Calculator noises.) Ok.
65. T: Ok, I think there might be an extra zero there... let's try this one once more... (Calculator noises.)
66. Y: Why do I keep doing that? (Laughter.)
67. T: I don't know! (Laughter.)
68. Y: Ok.
69. T: Ok, so what did we find?
70. Y: That it also follows for one point eight.
71. T: Ok, so when you plugged in one point eight for the function in Question Two, you got the expected value from Question One.
72. Y: Ok so this could be, yeah, this could be a function of this, of these values.
73. T: Ok, what if I told you that all these values match up.
74. Y: Ok.
75. T: Which is true. If you plug in all these values...
76. Y: .. that's what I'm thinking...
77. T: ... that's what would you come up if you were using the function in question 2.
78. Y: Ok
79. T: So given that information, how do you feel about that function...
80. Y: ... following this?
81. T: ... being the same as that function there, yeah.
82. Y: Um.... Ok wait. I'm looking at it and this part matches...
83. T: Which part?
84. Y: Well, this, like, this would make sense if it was like this.
85. T: For the rationals?
86. Y: Yeah, but I don't really know about this part.
87. T: So you're worried about...
88. Y: About the irrational.
89. T: So can we tell how it acts for the irrationals from Question One?
90. Y: Um, I guess, I don't really know how to use this, like, the information that was given in Question One to see if that would...
91. T: Yeah.
92. Y: But I'm assuming it would.
93. T: Ok.

94. Y: But that's just an assumption.

95. T: Yeah. I see, you're saying you can't tell. That's totally an ok answer.

96. Y: Ok. (Relieved.)

97. T: Ok, Question Two. Is... the square root of two a root of the equation f x equals zero? Why or why not?

98. Y: Um....

99. T: Do you know what "a root of the equation" means?

100. Y: Eh-eh. (Meaning no.)

101. T: It means a value for x where it's going to be f of x equals zero.

102. Y: Ok. Well, I wouldn't.... I don't know.... Ok. So, it would be like if this were x, right?

103. T: Yeah.

104. Y: Then it would be here. (Pointing to $x^2 - 2$)

105. T: Uh-huh. (Yes.) So what kind of number is root two?

106. Y: Oh, no, it's not! Because if it was root two, then that's an irrational, so that would have to equal one.

107. T: Oh, ok.

108. Y: Is that good enough for a "Why not" answer?

109. T: Yeah, that's a good answer. So, would it make f x equal to zero?

110. Y: No.

111. T: No. F of x ends up being....?

112. Y: One.

113. T: Ok. So let's go look back over here. Remember when I asked you if you thought this one had a solution, f of x equals zero.

114. Y: Ok.

115. T: Right. So do you remember about where it's going to be?

116. Y: Yeah. Between one point four and... one point five.

117. T: Ok. And do you know approximately what root two is? You can use your calculator.

118. Y: Ok. Um.... Oh, yeah, 1.4.

119. T: Ok, so it's like right in between the two.

120. Y: Yup.

121. T: Ok.... So, let's say it wasn't.... say you had gone to put it in the top part. What do you think it would have ended up being. Like, let's say you didn't have this restriction on the rationals.

122. Y: yeah, then we'd only have the top part. Well, then this would be zero.

123. T: Which part?

124. Y: X squared minus two would be zero. Wouldn't it? Because root two turns into two, minus two, which is zero. So then it would be zero. But we have this bottom restriction. Therefore it's not.

125. T: And so it's the bottom restriction that keeps it from being...

126. Y: Exactly.

127. T: Ok, cool. Is one point four one blah... umm, which is, you know, an approximation to root two, so that's close to root two. Is that a good approximation of a root of f of x equals zero? Why or why not?

128. Y: Umm...
129. T: Ok, so what they're asking is...
130. Y: Yeah.
131. T: For here, let's say this does have a root, somewhere, where it equals zero. Is one point one four two blah blah going to be close to it?
132. Y: I would say yes, but only because this would make it zero.
133. T: Ok.
134. Y: So if we could find, like... yeah.
135. T: The part where it says x squared minus two?
136. Y: Because that's what produced our zero. Which would make f of x equal zero.
137. T: So if you put this in would it be exactly zero?
138. Y: No, because it's an irra... well, that's an approximation of the irrational.
139. T: It's an approximation, so it won't be exactly...
140. Y: Yeah, but it's close, it's close enough. But then that still wouldn't... really be...
141. T: It still wouldn't...?
142. Y: It still wouldn't work.
143. T: Because?
144. Y: Because it still wouldn't be zero, because it's only an approximation.
145. T: Right. And what happens when it's exactly root two?
146. Y: Well, if it's exactly root two, it would be zero. But that doesn't count because we have a restriction.
147. T: Oh, ok. So what would the function... like, put out. If you put in, root two, what would the function...
148. Y: One.
149. T: And so... you're saying this might get it close to zero.
150. Y: It might get close, but... only, yeah. See, if we didn't have that bottom part, and we plugged in root, then it would be zero. But this is an approximation of root two, so therefore it still wouldn't be zero.
151. T: Ok, so it's an approximation of root two, but not an approximation of a root where it equals zero. Is that what you're saying?
152. Y: Yeah.
153. T: Ok. Question 3. Almost done. Consider the function f from R to R. So this is another function. Defined by the following rule. F of x equals... and the top part is the same as before.
154. Y: Ok.
155. T: And now the bottom, when x is irrational it equals to zero. Before it was one.
156. Y: Exactly.
157. T: Could this function be the function that question one talks about?
158. Y: Yes. Because we already proved the top part.
159. T: Yeah. How did we prove it?
160. Y: By plugging in f of one point five and f of one point eight, finding out the values are equal.
161. T: And plus I told you all the other ones.

162. Y: Yeah! (Laughter.)

163. T: We pretended we did them all.

164. Y: Yeah, we pretended I knew them all! (Laughter)

165. T: In theory! (Laughter) Um... ok, Question Two. Is square root of two a root of the equation f x equals zero?

166. Y: Yes, because for the irrational numbers it produces zero, which is what we have as root two.

167. T: Ok, cool. Three. Is one point four one four two... which is close to root two. Is that a good approximation of a root of f x equals zero? Why or why not?

168. Y: Ah... I would still say... well, NO I would say yes.

169. T: Ok.

170. Y: Because now that this is, like, close enough to root two, which is kinda like a representation of our irrational, which would produce zero.

171. T: Ok, good answer. Four. What is the main difference between the function in question two and the function in question three.

172. Y: The irrationals.

173. T: Ok, and what about them?

174. Y: In that we can produce an f of x equals zero because this allows us to.

175. T: Ok.

176. Y: And... in the other one, it didn't let us do that.

177. T: Ok.

178. Y: Well, we didn't know that. Because, uh, the root didn't let us.

179. T: Ok, and how about in terms of how you answered question three. There was a difference there also.

180. Y: Oh yeah, because this was a better approximation of our root, because we were allowed to use, like, it was like an approximation of root two, which can produce zero. So it kind of allowed us to find an approximation of a root, while the other one didn't.

181. T: Ok, great. Last question. The table below contains some values of a piecewise linear function, f from the reals to the reals. One piece of the function is linear from minus infinity to seven point eight, and the other point is linear from seven point eight to infinity. So linear means, like, a straight line function, right?

182. Y: Um-hum (Yes)

183. T: And piece-wise means it's defined in two different parts. Kinda like before... where we had two cases. So, that's piecewise defined.

184. Y: Ok.

185. T: So when it's on minus infinity, seven point eight, there's one definition. And when it's on seven point eight, infinity, there's another definition.

186. Y: Ok.

187. T: Ok. Here are the values. Take a look at those.

188. Y: (Pause) Ok.

189. T: Ok. Can you sketch the graph of this function.

190. Y: Yeah...

191. T: Ok, can you do that right here?

192. Y: Uh, does it have to be nice?

193. T: No no no no, just to give an idea.

194. Y: (Pause. Writing.) Ok, that's supposed to go like that... It's kinda a sketch..

195. T: Ok, and so one piece of the function... Ok, let's see if this matches up with the values. So, at seven point five...

196. Y: ... it was...

197. T: ... it was supposed to be negative. Ok, that's good. At seven point six it's negative. At seven point seven it's negative. At seven point eight it's supposed to be...

198. Y: ...it was kinda, it was right... yeah. It kinda, kinda, I kinda messed up right around here.

199. T: Ok, so you think that it matches up, the two parts match up? Look at the values...

200. Y: What do you mean, the... Oh. Oh, ok.

201. T: This is supposed to be, sorry, this is supposed to be zero point one. Which, I think, is what you did.

202. Y: Yeah.

203. T: So here, when x is increasing by point one in each step, it's going up by minus zero point zero five. And then between seven point seven and seven point eight, it goes up by... it goes from negative all the way to positive.

204. Y: Uh-huh. (Yes.)

205. T: Is it the same step between...

206. Y: No. Not between... well, because, yeah. This would go, to make this to go to zero is point one five, and to go up again is point one oh, so that's like a two point five difference between these two.

207. T: Between...

208. Y: ... seven point seven and seven point eight.

209. T: Ok, so... the step between, umm...

210. Y: The increments between them is like, point fives from seven point five to seven point seven, and then from eight point two to seven point eight it's the same. But then there's like a difference in between seven point seven and seven point eight.

211. T: Ok. Is that reflected in your graph?

212. Y: Um... kind of, but that's why, maybe that's why it got a little awkward.

213. T: Ok, ok cool. Ok. Two. Does this function have a limit at x equals seven point eight?

214. Y: Um... ok, let's see.

215. T: Ok, so what are you looking for when you're looking at the values.

216. Y: Ok, well, right now I'm looking at the information right there, and this.

217. T: So you're looking at the information in the intervals.

218. Y: Yeah, because that one's open, and that one's closed, and I don't know if that means anything. And then I'm looking at the value at seven point eight.

219. T: Ok. And so, to know if the function has a limit, what do we have to look at?

220. Y: That it's it, it's like approaching something.

221. T: Ok. And in this case what should it approach?

222. Y: In this case it should approach... Well, x equals seven point eight would be point one oh.

223. T: Uh-hum. (Yes)

224. Y: So it would have to be... approaching... I don't know how to look at that. Ok... (pause)... Well, it can't... Hum... It's kinda awkward. Especially along... (Inaudible mumbling.) Ok, wait. X equals seven point eight would be point one oh. There's a value for it.

225. T: Ok

226. Y: So, I'm going to say... Noooo?

227. T: Ok. So we have a value for the function, but it doesn't have a limit.

228. Y: Yeah.

229. T: And so why do you say it doesn't have a limit?

230. Y: Ok, well, part of me is saying because there's like there's a definite value to it, and part of me is saying because... like... that if it's a piece, right, so like this part could probably show us, like it's not including seven point eight.

231. T: From... point... from the left?

232. Y: From negative infinity, yeah. But then this part... like, I don't know if that means anything.

233. T: Well, this means that...

234. Y: ... it's included.

235. T: Yeah, like in the, like, the definition for the, you know, like the second case like we had before.

236. Y: Oh, yeah, ok. I don't know. I would say... But then there's that awkward gap, too. I'm... I really have no idea. I'm gonna say no.

237. T: Ok. One thing people look at when they look at a limit is the limit from the left and the limit from the right...

238. Y: ... to see if they add up.

239. T: Uh-huh. Could you use that in this case to... examine...

240. Y: Well, yeah, because the limit from the left would be looking at seven point seven, and the right would be.... Oh, then no, it doesn't.

241. T: Ok, when you look at seven point seven...

242. Y: well, seven point seven, the value would be negative point one five, and at seven point nine it's positive point one five... and negative... they don't equal each other... Oh! Oh yeah, so then it's no. Because then it would be, if it was like... If THESE were positive values, from seven point nine from eight point two were positive values, and from seven point seven to seven point five were also positive values, and they were equal, and they were both increasing... approaching seven point eight, and they were the same, then I would say yes. But since these are negative, then no.

243. T: Ok. Cool.

244. Y: But that's just my guess.

245. T: Ok. Three. What, if anything, does this function have in common with the the function in question two. Here's the function from question two.

246. Y: Ok.

247. T: So. Compare this function to that function here, yeah.

248. Y: Hummm... I have no idea.

249. T: Ok. Here's one thing you could compare it about. How about... roots? A place where f x equals zero.

250. Y: Yeah, I was thinking that...

251. T: So what did we find for question 2?

252. Y: Well... That's what I was trying to think of... Because if I... I'm looking at it, like, how we looked at it before. And like, if there was an f x equals zero, it would be like, in there.

253. T: Ok, and... so between seven point seven and seven point eight in question four?

254. Y: Yeah.

255. T: Ok, so, do you think there's going to be a place where f x equals zero in there?

256. Y: I think that there could be, because... See, in a way it could be because the way these increments are going...

257. T: Between eight point two and seven point nine?

258. Y: Yeah, and then between seven point five and seven point seven. How they're both five, point oh five. But then again, it's such an, it's like... The increments between x itself are point one, and since... Like, it might exist, but you know... because... It COULD be between seven point seven and seven point eight where there would be an f of x equals zero.

259. T: Uh-huh.

260. Y: But it's not necessarily true because of the way that this is reacting.

261. T: Because between seven point seven and seven point eight...

262. Y: Yeah.

263. T: Ok, and it tells you that the function is linear, so it's like a straight line.

264. Y: Ok, so then...

265. T: Between seven point seven...

266. Y: Oh! Then it probably will be...

267. T: Probably...?

268. Y: ... will be.

269. T: Probably will be, yeah? And where do you think it's gonna be? Let's see, going from seven point five to seven point six, it's going up by zero point five, seven point six to seven point seven it's going up by zero point five. So where will it be zero, do you think? If we continue with that pattern.

270. Y: I would say... about... Ok, well seven point seven and something. So, like, seven...

271. T: Let's say it kept on going in the same pattern. This went from seven point seven to seven point eight and the same pattern kept going. What would the value be at seven point eight?

272. Y: Well, if the same pattern kept going, then it would be at negative point ten.

273. T: Uh-huh. Negative point ten?

274. Y: Yeah. Because if this was still...

275. T: Right.

276. Y: Like, if this wasn't here. Like all the values that are I've already written. Because, it's just like adding a point five.

277. T: And it's, they tell you the function is linear.

278. Y: It's linear, exactly.

279. T: So you know it's probably going to keep the same pattern...

280. Y: Yeah, because that's what it means to be linear.

281. T: Ok. And so.. and so, to go back to the question. In terms of roots, in Question 2 we found that root two wasn't a root because of this definition.

282. Y: Exactly.

283. T: And here... how do you think it compares?

284. Y: (Pause.) Well, I would say since it's linear, that there could be... like, we don't know that, you know, that's not linear, right. So then... if this one IS linear, then there are infinitely many values on this line.

285. T: Yeah.

286. Y: So therefore there could be the value of the root on that line.

287. T: Ok.

288. Y: That would produce something, like, we don't know from where. Because it goes from negative infinity to positive infinity.

289. T: Right.

290. Y: So it could be, that there could be roots in this function.

291. T: What about the fact that linear means, like, it's a straight line?

292. Y: Uh-huh.

293. T: So, do you think you could use the fact that it's a straight line, that it's going to keep going with the same slope. Maybe we could use that to predict if there's going to be any more roots? Because you have, like, a drawing here...

294. Y: Yeah... well... I'm trying to think about it. Because, if we were to look at a root in like in the form of a decimal it wouldn't produce something nice like this.

295. T: Right.

296. Y: it would have, like, a bunch of awkward numbers, and it would be an approximation.

297. T: To get... a root? In question four?

298. Y: Yeah, if we were just looking at a general root, in decimal form, it would have a bunch of numbers, and if we looked at it, it would have an approximation because we couldn't really write it down.

299. T: You wouldn't be able to write down the root for question four?

300. Y: Like, a general root. Like, how we... like, how THAT was just an approximation, but...

301. T: Like 1.4 and root two, it's just an approximation.

302. Y: Exactly. Um... I don't really know. Because.... Well, if it was linear and it used, it produced, it has like the same slope...

303. T: Uh-huh.

304. Y: Well, ok, it would make so much more sense to look at this thing if this followed, you know, the pattern.

305. T: From seven point eight to eight point two.

306. Y: Yeah.

307. T: But it's a different pattern. Is that what you're saying?

308. Y: Well, like, the gap at least, from seven point seven to seven point eight.

309. T: Ok...

310. Y: Because that's what's kinda throwing me off as far as its being linear.

311. T: Right. And so... they tell you that it's piecewise...

312. Y: ...exactly...

313. T: ... so there's two different parts to it. And so ALMOST all the way up to seven point, like all the way up until you reach it, it's going to keep doing this pattern. That we have between seven point seven...

314. Y: And then from, starting from seven point eight on, it's going to have this certain pattern.

315. T: Um-hum.

316. Y: Yeah, and since this one's open and that one's including, it means it's following...

317. T: It means that seven point eight, the new pattern starts.

318. Y: Exactly. Or it's like, up until seven point eight. So there's no space where it wouldn't be included. Where it's, like, not... (Pause.) I don't know. I'm trying to think of, like...

319. T: What I'm going to do is, I'm going to put a new graph.

320. Y: Ok

321. T: And we're going to try to do a nice graph. And it's not going to be to scale....

322. Y: ... but at least we'll be able to see it.

323. T: Right, because it's gonna be, it was a little bit squished together, you know. This is going to be seven point nine... (Drawing.) and then we have to go from... (Someone walks in. "Hi!") Ok, so let's put up all the values on here. And... Ok, so let's try to connect the dots. About there, yeah...

324. Y: Yeah...

325. T: And then these here... around here...

326. Y: See, just about... (Inaudible.) It just seems that that gap is just so awkward.

327. T: Yeah.

328. Y: Ok.

329. T: Ok. Four. Compare this function, this one here in question four, to the function in question three.

330. Y: Ok. Ok, like before?

331. T: Yeah, like before. Remember in this one, what did we find out about the root.

332. Y: Eh... it could produce x equals zero.

333. T: Ok.

334. Y: Wait, now I'm starting to think more about the other one.

335. T: Yeah? What about it?

336. Y: Could I see that graph again?

337. T: Yeah, sure.

338. Y: Ok, because... there would be no point where it's zero.

339. T: Ok. Going by the rule?

340. Y: Yeah. Because it's so much easier to see when you break it up into pieces.

341. T: Right.

342. Y: So there would be no point where this would be zero, because... Humm.... Well, actually, because if we're only looking at these kinds of things... What if we were looking at seven point seven nine nine nine nine... you know what I mean? There could be increments between that, that would like approach from like there, but that's not necessarily true.

343. T: Ok.

344. Y: I personally don't think that there would be a point where it could be zero.

345. T: Ok, how about if we looked at seven point seven, and said... ok, if we went as close to seven point eight as possible, how high could the value get? Because we know the rule, right? So we can know about as far as it's going to go. From seven point six, seven point seven...

346. Y: Oh, yeah, it would be... Oh.... That's true. Because if this was like seven point seven nine nine whatever, it would be pretty close to negative point one.

347. T: Ok.

348. Y: Oh! Well, yeah. If it's saying it is, then it would go to like... it would go to like... there. Wait, my bad.... I can't....

349. T: So, it's like seven point seven... at like, right at seven point eight... So there's seven point eight.

350. Y: Ok, wait. We're saying that seven point... right here... it couldn't go...

351. T: Could it be around...?

352. Y: Well, because this was... wait, now my graph is messed up. Oh, that's because this should have been here.

353. T: Oh, ok.

354. Y: That makes sense now. Yeah, because then it would only go to, like, there.

355. T: Ok, and so you're drawing, you're saying that when it's almost at seven point eight, it's going to be around...

356. Y: ... it wouldn't go past... point one oh. Negative point one oh.

357. T: Ok. Oh, ok, so this is a bit different from what we said before.

358. Y: Yeah, because then it would be like, it's not including seven point eight, which means it's not going to approach... Like, if we were to continue the first part of the function, then seven point eight WOULD be negative point one.

359. T: Right.

360. Y: But since we can't, it's gonna be, like, ALMOST negative point one.

361. T: Ok. So what happens at seven point eight?

362. Y: Well, then, it goes up, to point one. So there's a gap in our graph between almost negative point one and point one.

363. T: Ok, and so... yeah?

364. Y: And so there's just like this gap right here. So we could have... well, we wouldn't have an f of x equals.

365. T: It would not be f of x equals zero, is that what you said?

366. Y: Yeah, because there's no point that could get us there.

367. T: Ok, so let's compare this to Question Three again.

368. Y: Yeah. I'm looking at...

369. T: Oh yeah, Question Three.

370. Y: Ok. But then in three we said... that there was... hum.

371. T: So in three we said root two IS a root. Because...

372. Y: Exactly. Because our function, our parameters allow it.

373. T: Ok.

374. Y: But... I'm trying to see how that would relate to THIS function. Because here we know, well I'm assuming, that there would be no point where it would be zero.

375. T: Ok.

376. Y: But in this, in Question Three we could have a zero, and in Question Two we couldn't have a zero.

377. T: Right.

378. Y: So I don't really, I'm not really seeing... how this one kind of goes into this one.

379. T: Ok, how about if we look in terms of approximating. So here, this is close to root two, so as we get close to root two, you said...?

380. Y: It would be a good approximation.

381. T: Ok. And how about in terms of question four. Because these are kind of like approximations.

382. Y: Ok. Well, then...

383. T: So, in terms of a root. Question Three was, is one point four one a good approximation of a root of f of x equals zero? So if that's the same question here... Is, say, seven point seven or seven point eight, are those a good approximations of a root of this function?

384. Y: I'm going to say... No.

385. T: Uh-huh. Is there a root for this function?

386. Y: No.

387. T: Is there a place where it equals zero?

388. Y: No. Because I'm thinking that this one, Question Tour, is more related to Question Two, in that Question Two didn't have a root for f of x equals zero, and Question Three does.

389. T: Ok.

390. Y: Because this one kind of allowed us to have this approximation.

391. T: Ok.

392. Y: But then this one...

393. T: Question allowed you to have an approximation.

394. Y: Yeah, exactly, Question Three allows us to have an approximation, but Question Four doesn't exactly. Well, because I'm thinking that it couldn't, it couldn't go... past...

395. T: Past...?

396. Y: Negative point one. And then seven point eight starts there. It's not like it would...

397. T: Starts where?

398. Y: Starts at point one oh. So there isn't, they can't go there. While this one, Question Three is allowing us to. If that makes any sense...

399. T: Yeah.

400. Y: Yeah. I'm just trying to think of how it would be... related, in the sense of... (Pause.) Yeah, I think it's closer related to Question Two than it is to Question Three.

401. T: Ok.

402. Y: Ok..

403. T: Ok.

404. Y: Yeah.

405. T: Ok. Thanks so much!

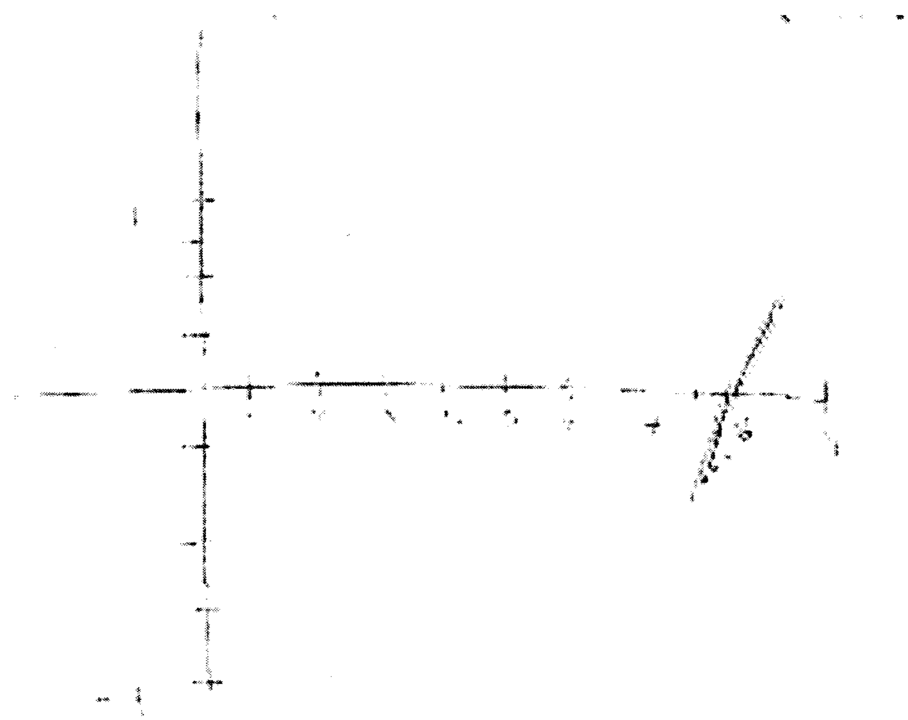

Figure 8: Y's first sketch of $f_4$

### C.1.2 *Post-Instruction Interview*

1. T: Ok, can you say your name please?
2. Y: [omitted]
3. T: [omitted]. Like last time, whatever you think, just say it out loud. Can you write your name just there?
4. Y: Yeah.
5. T: Great. Thanks. Ok, to start off, we're going to look at three proofs of the same theorem.
6. Y: Ok
7. T: Of the same result. And the result is, if g is continuous, and f is continuous on the range of g, then f of g of x is also continuous.
8. Y: Ok.
9. T: Ok? So there's going to be three proofs. You just read each proof through, and then I'm going to ask you some questions.
10. Y: Ok
11. T: And as you're reading them, if you have any thoughts about the proof, you can say them out loud, or ask me any questions.
12. Y: Ok. Cool.
13. T: So here's the first proof.
14. Y: Ok. (Pause while reading.) Ok.

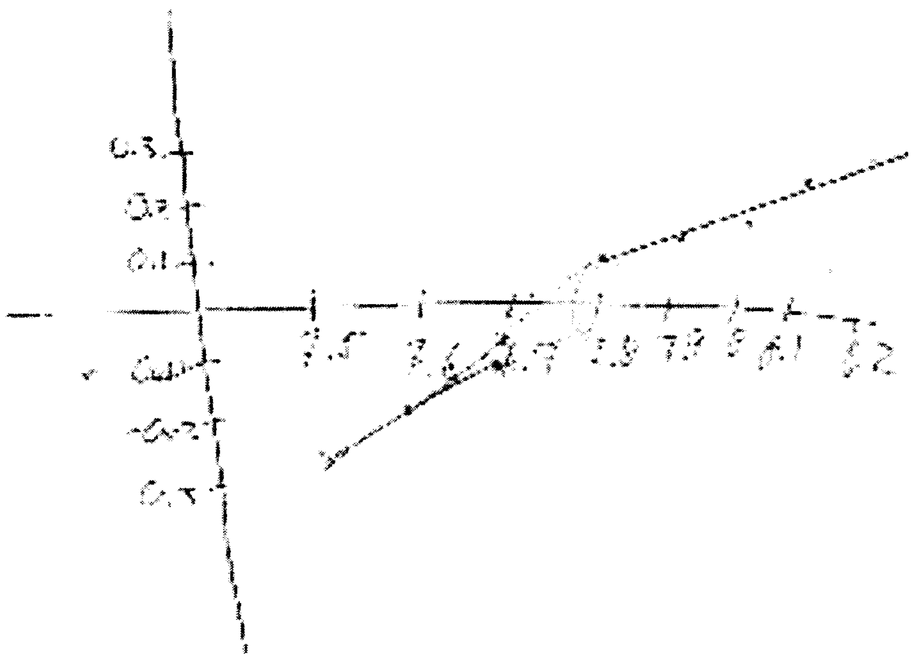

Figure 9: Y's second sketch of $f_4$

15. T: Ok. Do you have any questions about it?
16. Y: Not really.
17. T: Ok, here's the second proof.
18. Y: (Reading.) Ok.
19. T: Ok. Questions?
20. Y: No.
21. T: And here's the last one.
22. Y: (Reading.) OK.
23. T: Ok. So we have those three proofs. What did you think?
24. Y: Um... you can tell how that one, the first one, was a lot more, like, explaining it to people that would never have taken Analysis.
25. T: Ok.
26. Y: And then it kinda just leads it... Because this is the kind of proof that our teacher gave us in class.
27. T: Which one? Proof number three?
28. Y: The one here. It's a little bit closer to what, to how I would look at continuous functions.
29. T: Ok
30. Y: But the first one is more like, this is kinda how you would see it if you didn't really get things. It's really straight-forward.
31. T: Ok.

32. Y: And then it just starts getting into, like, going into neighbourhoods.

33. T: To neighbourhoods?

34. Y:Yeah.

35. T: Ok. So, which proof does the best job of explaining why f of g of x is continuous?

36. Y: Ummm.... I would say... I didn't really like the first one, for example.

37. T: Ok. You didn't really like Proof One?

38. Y: Yeah, not really. It kinda just, it didn't really make a good enough argument for me.

39. T: Ok, so when you say it wasn't a good... you didn't find it convincing?

40. Y: No, not really. So I would have to say, probably the second one.

41. T: The second one?

42. Y: Well, let me see it again. (Reading.) Actually, no, I liked the third one.

43. T: Ok

44. Y: Because that one was a lot more concrete.

45. T: Which one is? The third one is more concrete?

46. Y: yeah, but that just might be because that's how I had it explained to me.

47. T: Ok.

48. Y: Like when you learned of continuous functions, in like, like, high school, it's really just like this.

49. T: Proof one?

50. Y: Or more so, there's really never a proof in high school. They're always just like, "It's continuous because it is."

51. T: Ok.

52. Y: And you're almost just kinda told it's continuous.

53. T: And you think that proof one was like that?

54. Y: Kinda. And then the third one is more like, 'This is exactly why it is the way it is.'

55. T: And why do you think it's more exact... what about proof three is more exact?

56. Y: Well because it... For some reason when they talk about neighbourhoods for me, it's a lot, I can like visualize it more so.

57. T: Ok.

58. Y: Versus this one where it's like, "We want f of g of x to move towards f of g of a." Like, it doesn't really... I don't really see that.

59. T: You don't really see that.

60. Y: I see it better like in this kinda sense. There's this neighbourhood, and then you know that this is existing - because that was a theorem - so then like this is also true, this neighbourhood of this. So then you know f of g of x is continuous.

61. T: Ok. So looking at proof three, which you said was best at explaining why. What does it explain to you? Like, in your own words.

62. Y: What do you mean?

63. T: Well, you looked at the proof, and you said it explains...

64. Y: Well, well yeah. It starts to tell you, like well, explaining what just - cause it gives you a recall, when f is continuous at a. So you know this part, you know what f is a continuous function as a whole is.

65. T: Ok.

66. Y: And then they try to take you, well ok you're going to look at g. And you know that, see, ok... (reading.) A neighbourhood W of f of g of a, yeah... (reading.) Yeah, see, it's very like, it just goes in nice steps.

67. T: Ok.

68. Y: Like that's kinda how I was seeing it. Like, ok you know when f is continuous, so then it tells you, "You know this, and then this, and this." So it's like a series of steps.

69. T: Ok. And then you said your second favourite was...

70. Y: The second one. Because that one was not as, kinda, unconvincing as the first one. This one at least kinda goes into... Ok, the first one was generally...

71. T: ... (Reading.)

72. Y: yeah, like it was ok, but I don't really like how they just show, like, how we can show to move one towards the other. Ok, maybe it is convincing, but it just doesn't make as much sense to me.

73. T: It doesn't make as much sense to you.

74. Y: No.

75. T: Ok. So then which proof is the worst, you said?

76. Y: Yeah, I don't like the first one. Because I can't really... Like I said, it might not be that it's a bad proof, it's just that I don't visualize it, and I don't understand it as well as much as I would understand something really concrete, as in the third one.

77. T: With neighbourhoods.

78. Y: Cause that's like, that was clearly defined to me in my class. But this "moving something"... Whenever my teachers get very visual, or don't really, they try to explain it all like, "This is gonna go towards this, and whatever." Well, I need to see it, and I need to know, like, why it does that.

79. T: Ok, so when they start talking about something moving towards something else... What happens?

80. Y: Well, I think about it, but part of my brain just shuts off. I'm like, Ok! I don't really...

81. T: You're not able to picture it?

82. Y: yeah, exactly. I'm not really... I'm the kind of person that needs to know the theorems of why it does this. And then I'm like, "Well, ok, from this, then this..."

83. T: Ok

84. Y: So when we got really visual, like as far as limits too, it was like "this is going towards..." I was like "NO. Like, let me do it my own way."

85. T: And how would you do it your own way?

86. Y: Like, I would... Like, teachers try to talk about it. And I'm like, "No, draw the graph out, and show me."

87. T: Right

88. Y: That's how I would see. Or when we do limits in Analysis, there was a theorem and a way that you can show it.

89. T: Ok.

90. Y: It wasn't just like, "Well, let's just kinda think about it." No, I need to see it, and I need to do it. 9:08

91. T: Right. Ok, cool. Umm, so, what do you mean by, what do you understand by "explanatory." When I say, 'which proof does the best job of explaining?' how did you understand that word?

92. Y: I would say it's more like... it's giving me as much detail as possible. But then again, it's kinda dependent on which way people like to see things. Because a lot of people would probably prefer a proof like the first one, actually. Because this one is a little bit more, like, it's easier to understand, I guess. But in my sense, it's not.

93. T: Ok

94. Y: So even though the third one gives a little bit less information, or less clear detail, for me it's easier to see.

95. T: Ok, so explanation is like...

96. Y: The amount of detail, or the justification. Like bigger, fluffier proofs that just give you every little thing. Like, "this is because of this. Ok, well let's go back and look at this, and see why this happens.

97. T: Ok, so it goes back and explains...

98. Y: Yeah, makes sure that you get every detail. Because a lot of proofs, I've noticed, it'll just show part of the things, and it won't really give you a reference, to like "where did this come from, where did this come from." I guess a really good explanatory proof would be like, "because we know this, we know this. Because we know this, we know this." Giving every little detail so you completely get the picture.

99. T: Ok. Imagine that a student who hasn't taken a class in analysis asks you why f of g of x is continuous. How do you explain it to them?

100. Y: Well, I would try to use the first proof, but I personally don't... wouldn't... I don't know. Because if they've never taken Analysis before... But, even something like the third one, like the idea of a neighbourhood is talked about in other classes too. And it's very visual and that's why I think I understood it. Because it's very like "Here's your thing, and here's your neighbourhood.

101. T: Ok, so you're putting your hand close together, like that's the point.

102. Y: Yeah, and the neighbourhood is the little parentheses around it where they...

103. T: And you're putting your hands like little parantheses.

104. Y: Yeah, exactly. So that's kinda, it's very visual and it's very easy to see. I think, because you're like, you know where your point is, and then, ok, well you know there has to be this neighbourhood. So it's kinda like people can see it. And the word neighbourhood kinda just helps people to, like, get it. Like, "oh, ok, well it's in this little area! And then you go into this..." But I wouldn't know how to explain that.

105. T: How about this expression here: "By taking x close enough to a..." What does that make you think of? That's in proof one.

106. Y: Well I would say... That kinda sounds like a neighbourhood to me.

107. T: Yeah?

108. Y: Because... ok, wait. [Reading the proof to herself.] Yeah, well I would see, like, I don't know if I'm right but I would see f of a being that point in the middle, and like f of x is like in that neighbourhood. I don't know if that's right but that's how I would look at it.

109. T: How about in Proof Two where it says, it talks about "Moving f of x as near to f of a as we want." And "moving x sufficiently near to a." How do you think about that?

110. Y: See, that one's kind of the same. They all kinda go in the say way. They're all saying the same thing... I didn't understand, like I couldn't visualize that one. I don't know why.

111. T: You couldn't visualize x...

112. Y: moving x...

113. T: ... moving x, visually, towards a.

114. Y: Yeah, there's certain things that I don't... Like, when I read proofs, or when I read examples and stuff, or the teachers talk, there's just certain things that I don't get. So I just skip it and find a way to figure it out on my own, like a different way to see it.

115. T: Right.

116. Y: Because each teacher has their own way of seeing it. If I ask another student or if I ask another teacher who I know knows that subject, and I'm like, "Can you explain this to me again?" And then I'm like, OK. So if I saw this, if I saw the second proof in a textbook, I would look somewhere else online to explain it to me.

117. T: Oh yeah, for a different proof?

118. Y: Yeah, because I wouldn't, I wouldn't be able to... like, I don't see it like that.

119. T: Ok, and so, to get back to the question, a student hasn't taken Analysis but wants to know why it's true. So, maybe they have some math, maybe they've taken calculus or something. So you said at first you might use proof three?

120. Y: Yeah, well, when you start getting into, like, "There exists a neighbourhood U of A... " [Reading.] Well, see, when you look at it on paper, it's kind of intimidating for me. Because I'm like, ok I have to remember what's included in what.

121. T: When you look at what on paper? The whole proof?

122. Y: Yeah, well like, because I have to, I kinda have to, like I said, I have to visualize it. I have to see what I'm talking about. Because, f of U is in W, ok and then I'm like, f of g o U is in W. And then I have to remember what's in what, and what they're talking... which neighbourhood is of which. But then it's a little easier, because you can see that this part belongs to this part, and then why this... Yeah, I'd probably explain it this way.

123. T: OK. With Proof Three, with neighbourhoods.

124. Y: Yeah.

125. T: OK. The next question is, "How do you understand continuity of a function?" 15:00

126. Y: What does that mean? What do you...

127. T: What does that mean to you? When a function is continuous? What does that mean?

128. Y: Well, when we learned it, when we were younger, it was that there's no holes, skips, or jumps. We were told that, and it was very, "Here's a graph. Is it continuous or not?" And you're like, "Well, no! Because there's skips, there's holes... there's points of discontinuity.

129. T: Ok.

130. Y: And then when we got a little bit older, as we learned it, it starts to get, like right now in the class, our teacher is comparing a lot of continuous functions to the way that, the way that... I don't know. It's getting not so basic.

131. T: Ok.

132. Y: There's a lot of proofs. And when I do the homework I'm constantly finding myself reviewing the proofs over and over, because it's getting a little bit shaky. It used to be so simple. Look at a graph. Is it continuous? And now it's just like, I have no idea! I've got to read it again, because I'm like, "Hold on!"

133. T: Ok. So you go back and read the proof again?

134. Y: yeah, or like the theorems, or I have to sticky note where... I'm always referring to where they give you the first theorem, "f of x is continuous when... blah blah blah."

135. T: And blah blah blah, that's the definition?

136. Y: Yeah. And then I look at the proof, and then I'm like, "ok, because..." We have a lot of questions in the assignments that are like, "Prove that something something, given f of x is continuous." So I obviously have to remember what that means, that f of x is continuous. So I go back to that sticky note, and then I'm like, "OK, this is what that theorem told me." And then I go back and do the problem.

137. T: OK. So do you think you could try explaining in your own words what "f of x is continuous" means?

138. Y: Probably not. Like, I could give an example, but I don't think it would be a very... reasonable proof, or sufficient enough. You know what I mean? I guess as far as proving something on an assignment or a test, I don't think I could do that well enough unless I memorized the proofs.

139. T: So you'd have to memorize the definition of continuity?

140. Y: Yeah. I just review, I look back. I use it a lot, and then it will stick. I can't just look at it once and then remember.

141. T: Ok, so you said that you could give me an example. Can you give me an example of a function that is continuous?

142. Y: That's continuous? Just any function?

143. T: Any function.

144. Y: Um... y equals x?

145. T: ok. So that's like...

146. Y: a line.

147. T: Ok, and why is that continuous?

148. Y: Well, going back to the way we learned in high school, there's no holes, skips, or jumps.

149. T: Can you give an example of a function that's not continuous?

150. Y: Step functions.

151. T: Ok. And why are those not continuous?

152. Y: Well, when we were doing those in statistics, we showed that it was discontinuous because there are points where it may not necessarily be included.

153. T: What might not be included?

154. Y: Well, if you're saying, "This function equals one from zero to two." Or, "This function equals x from three to something." There are certain parts that are not necessarily included, and it skips, and it goes in jumps.

155. T: Ok. And why isn't that continuous?

156. Y: Because there are skips and jumps.

157. T: Ok, complete these sentences. "Continuity is like..."

158. Y: I don't know... [pause]

159. T: Anything. What words come to mind? It doesn't necessarily have to be, "Continuity is like..." Just, what does that make you think of?

160. Y: I don't know what the word is. There's a word that I'm looking for... it's like "non-stopping" or "non-interferred." It just goes.

161. T: And as you're doing that, you're moving your hand forward.

162. Y: yeah.

163. T: Sliding your hand forward.

164. Y: yeah. Un-interfered, I guess. Continuity is like an un-interfered line, or an un-interfered... I don't know what I'm trying to say. I don't know what that word is that I'm thinking of. Continuity is like... [pause] I don't know! I can't think of it.

165. T: Ok. Well, keep trying to think of it, and we'll see later. Now, "Continuity is not like..."

166. Y: [pause] It's really hard! [pause] Segments?

167. T: Ok.

168. Y: I would say... Because... that's tricky!

169. T: So what are you thinking about?

170. Y: Well, I'm thinking ... Because, when we said, "Continuity is like...", I think, very flowing, and like "one-streamed," I guess. And then, "Continuity is not like..." segments, and broken pieces, and framents - like sentence fragments. Because when I look at discontinuity, there's a skip or a jump or something. So that's really choppy.

171. T: Ok

172. Y: So, Continuity is not like choppy lines, I guess.

173. T: Ok, cool. Let's look at three functions, and you can tell me if it is or is not continuous on its domain. This is the first function. It's sine one over x when x isn't zero. And then when x is zero, the function is equal to zero.

174. Y: Well, this one we had in class.

175. T: That's ok. So, is it or is it not continuous on its domain?

176. Y: Ummm.... I'm pretty sure that it's continuous until it gets to zero. I don't remember why, but I remember that we were talking about that, and how it just comes and it gets really small, and all the sine curves get closer and closer. But then when it gets to zero... Because...

177. T: Ok, you're moving, you're wiggling your finger up and down...

178. Y: Yeah, as it gets closer in the middle. Because at zero, it's zero.

179. T: Yeah.

180. Y: Yeah, I don't think it's continuous. I mean, it is, but not at zero.

181. T: Ok.

182. Y: I think.

183. T: Ok. Number two. F of x is equal to e to the x. Exponential function. Is this continuous or not continuous?

184. Y: Yes.

185. T: It is continuous?

186. Y: I think so.

187. T: Why or why not?

188. Y: I would say it is, because when you look at the right side, it just keeps going until infinity. And then this part kind of goes to zero but it never really gets there. So I would say that it would be continuous.

189. T: Why? Just because that's the way the function... goes?

190. Y: Yeah. I don't know if that's good enough. But like I said, when we were told to look at graphs, or continuous functions, it was like, "Ok, does this graph look continuous?" And I'm like, "Ok!"

191. T: Right, back in Calculus, when you first learnt it.

192. Y: Yeah, it was just like, "is this continuous?" And they'd give you a graph, and....

193. T: And, looking at this... [pointing to function two]

194. Y: I would say, yeah!

195. T: Last one. F of x equals one over x. X doesn't equal zero. Is this continuous or not continuous on its domain?

196. Y: It has two continuous parts.

197. T: Which ones?

198. Y: When you look at the right side, this will always go up, and it's never going to reach the y axis. And then this one is also going to do the same.

199. T: The left side is going to go down...

200. Y: Yeah, on the left side it goes down.

201. T: And as you're saying this, you're sliding your hand in from the right, and sliding in from the left...

202. Y: yeah, so then there's this part in the middle, where it's never going to meet. So I would say it's not continuous. Because, well, I don't know how to explain it. I don't know what the correct definition is, but it has two continuous parts. If you were to look for x is greater than zero, this part would be continuous. And if you would like when x is less than zero, then this part would be continuous. But as a whole, it's not.

## C.2 SUBJECT P

### C.2.1 *Pre-Instruction Interview*

1. T: Can you say your name?

2. P: [omitted]

3. T: Perfect. So here's Question One. Here, I'll read it to you. The table below presents a fragment of the table of approximate values of some function f, from the reals to the reals.

4. P: Ok.

5. T: So, when x is one point three, that's what f of x is. When x is one point eight, that's what it is. [Pointing to the table.] So the question is, "Do you think the equation f of x equals zero has a solution? Why or why not?"

6. P: I think that it has, by the Intermediate Value Theorem. If f of a is negative and f of b is positive, then there is a zero in between.

7. T: Ok.

8. P: So, it has. Do I have to write it down?

9. T: No, just talking is good. Where do you think it's going to be...? At what value of x?

10. P: Well, it's going to be in between one point four and one point five.

11. T: Ok. You think that because of the intermediate value theorem.

12. P: Yes, that's what I said.

13. T: Ok. Question two. Consider the function f, defined from the reals to the reals - this might be a different function, maybe - defined by the following rule. So when x is rational, it's defined by this. [Pointing at the definition.] And when it's irrational, it's one. Could this function be the function that question one talks about? You can use a calculator.

14. P: Wait, I didn't get it. I was thinking about something else. Defined by the following... when x is... [Reading.] Ok.

15. T: Ok?

16. P: Could this function... [Reading.]

17. T: For instance, when x is one point three, do you think this [the equation in Question Two] would give you the right value?

18. P: One point three... One point three minus...[Using calculator.] Well, it can, I guess.

19. T: So, what are you checking there, when you're looking at it?

20. P: I'm looking at if we can have here... Definitely we cannot have a minus, because there are pluses so it cannot be minus. And here I'm looking if x squared, out of these two values, is less than two. And if it's less than two, then this is going to be minus, the second part. So minus multiplied by a plus is going to be minus. But I don't know... One point four... [Using calculator.] Those look like they're satisfying one point three and one point four. And one point five it will be, because one point five squared is more than two. So I guess, yeah. Do I have to calculate everything?

21. T: Well, if you want. I'll tell you, if you put the values in, it matches up exactly.

22. P: Ok.

23. T: So, if you put one point three, you get this value. So do you think this function could be the same function from Question One?

24. P: Well, I don't remember, though, that theorem, that if these are in the rationals... No, those are only for R, right? The Intermediate Value Theorem? If it's only for R, then it might be.

25. T: Uh-huh.

26. P: But...

27. T: R... the reals, you mean.

28. P: The reals, yeah yeah yeah. It might be, I guess

29. T: It might be.

30. P: Like this, this thing here... It's...

31. T: Where it says, when x is irrational?

32. P: Yeah, it's confusing me. I have to check, so... [...]

33. T: So, does this tell you anything about the irrationals, in Question One?

34. P: I don't get it. No. About the irrationals?

35. T: Do they talk about the irrationals in Question One?

36. P: No, I can't figure this out. Maybe you can help me... You could ask me something else.

37. T: No, I'm just wondering.

38. P: Ok.

39. T: Because you're saying, you're confused by the fact that it says that f of x is one when x is irrational.

40. P: Uh-huh.

41. T: That you're not sure if it's the same. So I was just wondering, because it doesn't really talk about the irrationals here [in Question One]. So it's tough to know, because these are all rational numbers here.

42. P: When it's one, then it will be here...

43. T: When x is one?

44. P: No, the function will be one when the function is irrational. So it's going to be one here. So, it's irrational between one point six and one point seven, but we can always find rationals, so it can be.

45. T: It can be. Ok, that's a good answer. Question Two. Is the square root of two a root of the equation f of x equals zero? Why or why not?

46. P: [...]

47. T: So that means, if x is the square root of two...

48. P: yeah yeah yeah. So definitely it is.

49. T: Why?

50. P: Because square root of two squared is two. Two minus two is zero. So multiplied by anything else is going to be zero.

51. T: Ok. So one thing you might want to look at is, is the square root of two rational?

52. P: Not, it's... Square root of two? Irrational.

53. T: And so, does that change your answer? Remember, it's defined differently.

54. P: Oh! Ok ok ok... Stupid! [Laughing.] So of what, the equation of what, we don't know the formula...

55. T: No no, of this equation here. It tells you the equation of the function. It's for this function here.

56. P: Oh yeah. So it cannot be, because it's one.

57. T: Oh ok. So when x is the square root of two...

58. P: ...then f of x cannot be zero, because it's one by definition.

59. T: Ok, by definition. Question Three. Is one point four one four two one three five six two... So, that's very close to the square root of two. Is that a good approximation of a root of f of x equals zero? Why or why not?

60. P: If it's close enough to the square root of two, and it's a rational, then we have this square is close to two... So it's a good approximation, yes

61. T: And why will it be a good approximation? What happens to the function?

62. P: It's going to be rational, right? This number?

63. T: Yeah.

64. P: This number, squared, is going to be close to two. So two minus two is going to be a very small number... That number minus two is going to be very small, so it's going to be close to zero.

65. T: It's going to be close to zero. Ok. But will it be close to a place where f of x actually is zero?

66. P: [...]

67. T: For instance, you're saying it's going to be really close to zero. Will the function ever be zero near there?

68. P: Sorry? I lost it again.

69. T: That's ok. So you're saying this is very close to the square root of two. And then when you square it and subtract it, it's going to be close to zero...

70. P: Yeah.

71. T: It's going to be close to zero. Do you think it'll ever, the function will ever actually be zero in that area, around this number? Will it ever actually be equal exactly to zero.

72. P: No. Actually zero?

73. T: Yeah, actually zero.

74. P: No, I don't think so.

75. T: Ok. This is a new function now. Consider the function f, from the reals to the reals, defined by the following rule. So here the top part is the same, when x is rational. But now, when x is irrational, the function is equal to zero.

76. P: Ok.

77. T: Before it was equal to one, now it's zero. Could this function be the function that Question One talks about?

78. P: I don't know. I'm confused. Tired and sick and confused. [Laughing.] And more confused by the fact that I'm so stupid that I don't understand anything. [Laughing.]

79. T: No, you're doing well! You're giving good answers!

80. P: Umm, wait a second now. That needs to be zero. [Quietly.] This can be zero... [...]

81. T: For instance, this top part is the same as... The top part in Question Three is the same as it was...

82. P: Ok, it's the same!

83. T: Yeah, it's the same. The only thing that's changed now is when x is irrational. Now it's zero. Before it was one.

84. P: Irrational... When x is irrational... So, yes, it can.

85. T: It can be the function.

86. P: Yes.

87. T: Ok.

88. P: Because it's irrational.

89. T: Two. Is the square root of two a root of the equation f of x equals zero? Why or why not?

90. P: Yes, now I guess! [Laughing.] Because the root of two is irrational.

91. T: Ok, three.

92. P: Is this the same number as before?

93. T: Same number as before, yeah.

94. P: Ok.

95. T: Close to the square root of two. Is it a good approximation of a root of f of x equals zero? Why or why not?

96. P: I guess the first answer would be, it's going to be close to zero, because it's rational. But it's never actually going to be zero, because this is not an irrational number. And only square root of two - I mean, not only square root of two - but square root of two is a... Is it only that one? Square root of two is irrational, and this guy is rational, so it wouldn't be... [...]

97. T: This is very close to the square root of two, though.

98. P: Well, the answer is going to be close to zero, but it's not actually going to be zero.

99. T: Well it be zero close by, for the function?

100. P: [...]

101. T: I mean, is there an x near to this number where f of x equals zero?

102. P: In rationals, or in...?

103. T: In rationals, or irrationals. Any x.

104. P: Yes, there is [one] close.

105. T: Where?

106. P: Square root of two!

107. T: Ah. Ok. Square root of two gives zero. Ok, Question Four. What is the main difference between the function in Question Two and the function Question Three?

108. P: Well I guess when x is irrational, we have different values. That's the main difference.

109. T: And how does that change the nature of the function?

110. P: I'm not so good at seeing that...

111. T: Oh yeah?

112. P: I don't know how it changes... But actually... This is for all x irrational, right?

113. T: Yeah.

114. P: So, I don't know here... We have five zeros in the first one?

115. T: Five zeros?

116. P: Five roots? How many solutions do we have for this equation?

117. T: I don't know. How many do you think?

118. P: Well, I think it's five...

119. T: Why do you think it's five?

120. P: Because the highest is x to the five.

121. T: Ah, ok, yeah.

122. P: The highest power. So I guess... But I don't know if they're complex, or...

123. T: So they might be complex, or... And what about in Question Three? You think there might be at most five roots in Question Two. What about in Question Three?

124. P: No no no no... here? Well, I think the same. Or it doesn't make sense. [...] The only thing that's different? I don't know what's different.

125. T: Well you mentioned that, when x is irrational in Question Two, f is equal to one. So that's a difference.

126. P: Oh yeah. Ok. [...]

127. T: Last one. The table below contains some values of a piecewise linear function f. One piece of the function is linear on negative infinity to seven point eight, and the other is linear from seven point eight to infinity.

128. P: [...]

129. T: Do you know what piecewise linear means?

130. P: Yeah.

131. T: So here are the values. At seven point eight, it's zero point one. At eight point two, it's zero point three.

132. P: Ok.

133. T: So these are the values...

134. P: These are the values given for some function.

135. T: For some function. And it's piecewise linear.

136. P: Ok.

137. T: Can you sketch a graph of this function?

138. P: Can I sketch a graph?

139. T: Yeah, try to sketch a graph.

140. P: On this paper?

141. T: Yeah, sure.

142. P: Five... [Drawing.]

143. T: It doesn't have to be to scale.

144. P: Yeah yeah... [...] Seven... [Drawing.] Seven point seven... Ok, I'm going to take this seven point seven. Seven point five to seven point seven... Eight here? What's happening here.... I don't know.

145. T: What's happening where? Seven point eight?

146. P: Yeah. So I don't know if this function... Does the function have a limit... [Reading.]

147. T: That's the next question. Does this function have a limit at x equals seven point eight.

148. P: Ah, seven point eight is here! Oh, sorry sorry sorry sorry sorry... Then it's completely different. I didn't see the seven point eight here. So seven point eight here, it's zero point ten here. Wait wait...

149. T: Positive zero point ten.

150. P: So it's going to be something like... Sorry about that... [Drawing.] Is that working? I guess it's working. So this function, I would say... They're linear functions? Well, it has a limit, because from both sides... [...]

151. T: Does it match up from both sides?

152. P: [...] Yes.

153. T: Let's see. Going from eight point two to seven point eight. Starts off at zero point three.

154. P: Ok.

155. T: How about from the other direction? A linear function. So at seven point five, it's negative zero point two five.

156. P: This is increasing here. And this one here, it's..?

157. T: It's... every point one in x, it increases by zero point zero five.

158. P: And here it's the same. [Looking at the interval from 7.8 to 8.2]

159. T: So the question is: Does it have a limit at seven point eight?

160. P: [...]

161. T: You can try counting, maybe, between seven point five and seven point six.... Negative zero point two five... Negative zero point two... negative zero point one five...

162. P: Ok, so we have a gap here, or no?

163. T: Well, if you keep the pattern going, what's going to happen after seven point seven? Seven point five, seven point six, seven point...

164. P: What do you mean, what's going to happen?

165. T: Like here. Seven point five, it's negative zero point two five. At seven point six, it's negative two...

166. P: Ok, so there we have a difference of... what's that?

167. T: Zero point zero five.

168. P: And then... zero point twenty... [...]

169. T: Does it seem to continue, the pattern?

170. P: No!

171. T: Ah. So what happens here?

172. P: I don't know what happens. It's not continuous!

173. T: It's not continuous?

174. P: Yeah.

175. T: At what point.

176. P: At zero point eight.

177. T: At... seven point eight?

178. P: Seven point eight! Sorry.

179. T: So, what does that mean about...

180. P: ...so it doesn't have a limit! Finally, yes.

181. T: Ok.

182. P: Oy vey.

183. T: It's tricky! Don't worry. Ok, Question Three. What, if anything, does this function have in common with the function in Question Two?

184. P: Is your professor going to hear these answers? [Laughing.]

185. T: [Laughing.] It's ok!

186. P: She's going to find me and kill me!

187. T: No, that's why you signed the paper - it's anonymous. You're just Student Number One.

188. P: Oh my god!

189. T: No, you're doing well. You have good answers.

190. P: What does this function... [Reading.]

191. T: What, if anything, does this function here have in common with the function in Question Two.

192. P: [...]

193. T: Remember, this is the one where the irrationals go to one. This number is close to zero. And root two goes to one.

194. P: [...]

195. T: What do they have in common?

196. P: They don't have a limit?

197. T: Where?

198. P: Well, this one at seven point eight, and this one at the square root of two.

199. T: Ok.

200. P: Because at the square root of two, it jumps to one.

201. T: Ok. And here, we have...?

202. P: We have... I don't know how to say it. A gap? No...

203. T: Sure.

204. P: What do you say, discontinuity?

205. T: Ok. Question Four. Compare this function to the function in Question Three.

206. P: [...]

207. T: So, this one here, this piece-wise linear one, with this one. Remember, for this one, when x was irrational, the function was... So how do these compare?

208. P: What do you mean, how do these compare?

209. T: For instance, before, when you were looking at the function in Question Two, you said, the function in Question Four and the function in Question Two were similar because...

210. P: ...there's no limit.

211. T: ... at seven point eight and root two, there's no limit. It's a discontinuity. So how about for Question Three and Question Four?

212. P: Well, they... I don't know here [Question Three] if the zero is going to be rational or irrational. It might be that both of them are rational, so... Between them, they'll be rational or irrational. It's not a very promising answer.

213. T: Uh-huh.

214. P: [...]

215. T: How about, before you looked - for Question Two and Question Four - at the limit as you got close to seven point eight, as you got close to the square root of two. What happens...

216. P: ... well, I guess here it's going to be continuous, because it's defined everywhere.

217. T: At what point?

218. P: At the square root of two. It's going to be zero, so at that point it's going to be. But here, it's going to be discontinuous. [In Question Four, for x=7.8]

219. T: Ok. So that's a difference between the two?

220. P: Yes, I would say that.

221. T: Ok.

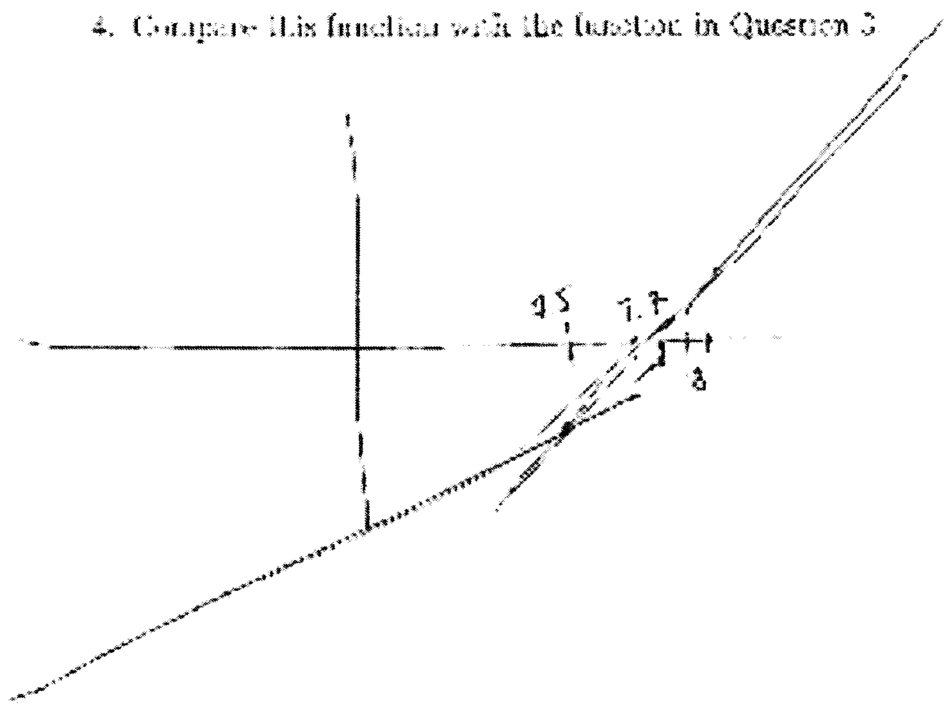

Figure 10: P's sketch of $f_4$

### C.2.2 *Post-Instruction Interview*

1. T: So, just like last time, whatever you think, just say it out loud. No wrong answers - I just want to know what you're thinking. Okay. So, consider three proofs of the following theorem, of this theorem. If g is continuous, and f is continuous on the range of g, then f of g of x is also continuous. Okay?
2. P: Okay.
3. T: So you know this theorem?
4. P: Yeah. So I have to say three proofs?
5. T: No, I'm going to give you three proofs...
6. P: ...okay okay okay okay...
7. T: Read through them, and then I'll have a few questions after.
8. P: Okay.
9. T: Here's proof number one.
10. P: [Reading.] Okay.
11. T: Okay? Any questions about that one?
12. P: No.
13. T: Here's proof number two.
14. P: [Reading.] I have to read them pretty carefully. I don't really see any difference between them. Should I read them again?
15. T: You can look at them all again. But I'll have you look at the third one. Why don't you take a look at that.
16. P: [Reading.]
17. T: Ok. So here are the three proofs. [Setting out the three proofs.] Do you want to look at them a bit more?
18. P: Yeah. [Reading.] I didn't get this one.
19. T: Which one? Number two?
20. P: Yeah. [Re-reading Proof 2.] Okay!
21. T: Ok. First question. Which proof does the best job of explaining why f of g of x is continuous?
22. P: [...]
23. T: And why does it do the best job.
24. P: I'm blank. You're asking me, or for...?
25. T: Yeah, for you.
26. P: Number three.
27. T: Number three.
28. P: Yes. Because I can visualize it the best. The proof. Because we have the neighbourhoods.
29. T: Ok. And so what do you visualize when you read number three?
30. P: I don't know if they're correct or not - I guess they are - but when you have... When the proof, or the definition of continuity, is given in terms of neighbourhoods, I can actually, in a way, visualize how that is happening. So I guess that's why I would see this is the easiest for me.
31. T: Number three.

32. P: Yeah.

33. T: How about for the other two? Which one does a...

34. P: This is the better one, because it follows the definition as well.

35. T: Number one?

36. P: Number one. Yeah, sorry. And number two I had a little bit of a problem, only because it's very... ah, like... It's just in terms of "far towards," which I haven't met right now. I guess.

37. T: And so, when you hear those words...

38. P: It's not a problem. I just needed... It's, like, the first time confronting... It's the first time seeing a proof like that. So that's why I think I had to read it again.

39. T: This one? Number two.

40. P: Yeah, number two. I think I got stuck with number two.

41. T: Ok.

42. P: Even the first time.

43. T: So, looking back at number three. So that's the one that explains the most for you...

44. P: Uh-huh.

45. T: What does it explain to you?

46. P: [...]

47. T: So when you read, it explains something to you. What does it explain to you?

48. P: You want me to refer to the proof, or...?

49. T: No, not necessarily. Just in your own words. So I said, "Which one explains why, the best." What does that mean to you, when I said that? What were you thinking in your head when you heard that question?

50. P: [...] [Laughing.] I don't... I don't know.

51. T: Do you understand the question?

52. P: No. Sorry. [Laughing.]

53. T: Oh, don't be sorry! So this proof, you said it explains something....

54. P: Ok! It explains how the funct... It just explains where the functions are, and I can picture it in my head. I can picture the coordinate system. And I guess that's why.

55. T: Ok.

56. P: And so I can see that... We have this definition where f is continuous.

57. T: Uh-huh.

58. P: And after that, the both definitions for f and for g. And... We are trying to prove that the neighbourhood... We are trying to prove that the neighbourhood... Oh, I lost it... G of a, that the neighbourhood of e is actually in the neighbourhood of f, which is going to be in the neighbourhood of W.

59. T: Ok.

60. P: Yeah, U... Which is in the neighbourhood of U, which is in the neighbourhood of... which is a subset of W.

61. T: Ok, so you're pointing at this last line of math, here. [In Proof 3]

62. P: Yes.

63. T: "f of g of U is in the subset of..."

64. P: Yes.

65. T: "... subset of W." Ok. And so, when you read the proof, that's what it explains to you? Is that what you're saying? It explains why this is true, this line?

66. P: Yes, I would say yes. Yes. At this point, I just went quickly, but yeah.

67. T: Ok. Which proof is worst at explaining why f of g of x is continuous?

68. P: Worst? Number two, I would say.

69. T: Number two?

70. P: Well... Worst. Only because I haven't... Because there are too many... not a lot of mathematical entities here.

71. T: Ok.

72. P: So I... Right now, I'm used to - because I'm preparing for my final exam - I'm used to seeing everything in terms of definitions in proofs. So maybe that's why.

73. T: And compared to... Number Two compared to Number One. It's the same amount of language. Both of them don't have very much... mathematics... like, very much mathematical...

74. P: Yeah, I guess. [...] I guess I'm perceiving more familiar things in Proof One than Proof Two.

75. T: Ok. So when I said, "Which proof does the best job of explaining why?" What do you understand by "explanatory." That word: "explain." What do you understand when you hear that word?

76. P: Well, I guess it shows me why a certain theorem works.

77. T: Ok.

78. P: I guess.

79. T: And so if a proof is an explanatory proof, it's one that shows why the theorem works?

80. P: Yup. I guess.

81. T: Now, imagine that a student - who hasn't taken a class in Analysis - asks you why f of g of x is continuous.

82. P: Ok.

83. T: How would you explain it to them.

84. P: I would go with Number Two, I guess.

85. T: Proof number two?

86. P: Proof number two, sorry. Because even I had to translate the proofs in the book, in terms of this, to explain to myself. And after that, after a while, you get used to specific notations, a specific way how to prove something or how to understand something. So the more words you have, then it becomes more confusing.

87. T: Ok.

88. P: But again, I'm going to say that, at the beginning, I had to translate what those neighbourhoods mean. I had to explain to myself with a picture in front of me, in terms... in a way like proof number two.

89. T: Ok, then... so first, you were understanding in terms of, like, number two. Like what parts of number two? Can you give me a sentence? Something like... [...]

90. P: Something like what?

91. T: Because you said you had to translate things like proof number three...

92. P: Ok, for example, here it says, ugh... g... So, for example, the first sentence of the proof: "Recall that f is continuous which means that whenever W is a neighbourhood of f of a, there exists a neighbourhood of U, of a, such that whenever f of U is a subset of W." So now I can perfectly understand what that means.

93. T: Ok.

94. P: But at the beginning I had to translate what W neighbourhood of f of a is.

95. T: Right.

96. P: Which means that you can move, in terms of numbers, around f of a, as close to f of a, by moving x sufficiently near to a. So approaching a, to a from both sides. Which, in the Proof Three, said x is in the neighbourhood of a.

97. T: Ok. And so when you first read proofs like proof three...

98. P: Yeah, well at the beginning, when I was first studying... I mean, I guess that would one month ago, or two weeks ago. [Laughter.] So I had to try to understand it this way. Explain it to myself. But after that I guess you get used to it, to following the rule, kind of. It becomes pretty easy.

99. T: Ok, great. So, a couple more questions. We're done with these.

100. P: Nice! I thought you were going to tell me what's wrong with everything I said.

101. T: [Laughter.] No no no! "You didn't see this, and this, and this..."

102. P: [Laughter.]

103. T: No, you had good answers. Now a couple questions about continuity now.

104. P: Ok.

105. T: How do you understand continuity of a function?

106. P: I understand continuity of a function that... well, I understand it in terms of the definition.

107. T: Ok.

108. P: So, should I tell it, or...?

109. T: Well, which definition?

110. P: Ah, epsilon-delta definition. Delta or sigma? Delta.

111. T: Epsilon-delta?

112. P: Epsilon-delta definition. Which helps me translate it to the definition with the neighbourhoods. And after that I can just picture it out. I mean... for... Which means that whenever, however close you go to... For any neighbourhood of f of... of.. some value, a. You can always find x around a so that it's going to be in the middle. Which means, in a way, that... Yeah, that's it. They have to... The one neighbourhood has to influence the other neighbourhood.

113. T: Ok.

114. P: For our function to be continuous.

115. T: Right. Can you give me an example of a function that's continuous?

116. P: Ah, yes. F of x equals x squared.

117. T: Ok. And why is that continuous?

118. P: [...] Ho! Well, that particular one because I could first picture it in my head.

119. T: Right.

120. P: There are no discontinuities.

121. T: Right.

122. P: So... I would say that's why! [Laughing.] But I can prove it, I guess...

123. T: If you had to.

124. P: I hope! Yeah.

125. T: Ok. Can you give me an example of a function that's not continuous?

126. P: Uh... One over x.

127. T: Ok. And why is that not continuous?

128. P: Well, it's not continuous because it's not defined at zero.

129. T: Ok.

130. P: Right? And so... if you approach zero from left, and you approach zero from right, you have two different values. Which means that it's discontinuous because it doesn't approach a specific number.

131. T: Ok. Complete these sentences. "Continuity is like..."

132. P: [...] Uh. What's the question? Whatever?

133. T: Anything. Anything you want to put in there. It doesn't have to be exactly right. Just whatever you think of.

134. P: Ok, it's like... ugh... a line.

135. T: Ok.

136. P: Not with any... how do you call it? With any... holes in it. It's not holes, but... [...] gaps! I guess. I would say.

137. T: Anything else?

138. P: No! [Laughing.]

139. T: Ok. "Continuity is not like..."

140. P: [...] ...is not like tangent of x. The graph of tangent of x. Yes.

141. T: Ok.

142. P: Yes. That's my example of what a discontinuous function is.

143. T: Why? Why do you say that?

144. P: Well, because it's discontinuous at... at pi over two? Is it? I think so. [Laughing.] Maybe it's not. At pi over two, and three pi over two.

145. T: And you're looking up...

146. P: Well, I'm looking up because I'm perceiving.... I have the graph of tangent in my head. So I'm always... you know...

147. T: Trying to picture it. Ok. I have three functions now.

148. P: Ok.

149. T: So explain why each of these functions is, or is not continuous on its domain.

150. P: Ok.

151. T: So here's the first one. F of x equals sine of one over x, if x isn't equal to zero. And the function is equal to zero, if x is equal zero. So if x is zero, the function's zero, and everywhere else it's sine one over x.

152. P: [...] Man, I have to know this for the final! Can you give me a second? [...]

153. T: What are you thinking about now?

154. P: I'm thinking about what's happening as x approaches zero.

155. T: Ok.

156. P: So, when x approaches zero, it's going to be sine of... [...] Ah, I don't know.

157. T: You don't know?

158. P: No.

159. T: Ok.

160. P: I'd need time to think about this. And I think it's pretty straightforward, but I just cannot see it right now.

161. T: Ok. Well, how about other than at x equals zero? At other places?

162. P: Ok, it's continuous. Yeah.

163. T: Ok.

164. P: I can say it's continuous. First, seeing the graph. And second of all, this function, one over x, is not continuous... like, the critical point should be only at zero.

165. T: Ok. So, you were talking about "as x approaches zero."

166. P: Yes.

167. T: That's what you were saying. And so, what happens to the sine function as one over x goes to...

168. P: Well, that's what I'm trying to figure out. But I think that I'll need more time to figure that out. Which I don't know why it's like that, but...

169. T: Yeah. Well, how about if we just look at the graph. Because that can tell us...

170. P: Well, by looking at the graph I'd say it's continuous.

171. T: Yeah?

172. P: Yes. Although I don't know what's happening around there.

173. T: Ok.

174. P: Although it's... Yeah.

175. T: See, as it's coming from the left, it's going up and down. First up to one, and then negative one. And from the right, it's the same thing.

176. P: No no no, I know what's happening around there. But I don't know if the picture can tell us about every graph. So... yeah, when it's going to zero, I don't know what... This is going to zero. This is going to be sine of a much bigger and bigger number. Which is... I think it's going to be continuous.

177. T: Yeah.

178. P: I would say yes.

179. T: How about if I were to tell you this... So, as x goes towards zero, one over x is going to get bigger and bigger. And then sine is going to keep going up to one, and then down to negative one, and then up to one... over and over again.

180. P: Sorry?

181. T: So, x is going to zero....

182. P: Yes.

183. T: So this one over x...

184. P: ...is going to be bigger and bigger, ok.

185. T: Is going to be bigger and bigger. But as one over x goes towards infinity, sine of one over x will keep on taking the values one, down to negative one, one, negative one. It will do that an infinite number of times. It will oscillate.

186. P: Uh-huh.

187. T: And as you see from the graph, it oscillates more and more often as you get close to zero. So at zero, when x is zero, f of x is zero.

188. P: Uh-huh.

189. T: But everywhere around it, you have sine of one over x, and it's going from one to negative one.

190. P: Uh-huh.

191. T: So does that tell you anything about the continuity?

192. P: [...] But it's going to take all the values in between, right?

193. T: Uh-huh, yeah.

194. P: So... So when it's going to be very close to zero, then it's going to be one on this side - or minus one, I don't know - and on that side. So it's going to be different than zero. So limit as this goes to zero is going to be one or minus one, and from the right as well. But f at zero is zero. So I guess it's going to be discontinuous at zero.

195. T: Ok.

196. P: I'm going to remember this example. You'll have to tell me after if this is... [...]

197. T: Ok. Function number two. F of x is equal to e to the x.

198. P: Ok.

199. T: Continuous or discontinuous?

200. P: Continuous.

201. T: Why?

202. P: Because of the graph. Sorry.

203. T: Yeah, that's a good answer! I mean, however you think.

204. P: Yeah.

205. T: Ok. Number three. F of x equals one over x. X does not equal zero. Is this continuous or discontinuous on it's domain?

206. P: [...] On it's domains?

207. T: Yeah.

208. P: And the domain is all reals?

209. T: All reals except for zero.

210. P: Ah! Continuous.

211. T: Continuous? Why?

212. P: Because of the graph.

213. T: Because of the graph? Yeah?

214. P: Yeah.

215. T: And what do you see when you look at the graph? What tells you that it's continuous.

216. P: Well, I don't have to look at the graph. I can see that as x approaches from the left, then it's going to be...

217. T: So you're pointing at the...

218. P: Denominator. Which is going to get bigger and bigger.

219. T: You're pointing at the function.

220. P: Yeah. So when x is going to get bigger and bigger... Um... Yeah, when it's going to approach to zero, the denominator is going to get smaller and smaller - oh, sorry — so the whole function is going to get bigger but with a minus sign. So that means that you can take any value as close to zero as you want, and it's going to be defined there. Except at zero.

221. T: Ok.

222. P: And with minus infinity and plus infinity, it's the same. So...

223. T: Ok. Ok. So that's all the questions.

## C.3 SUBJECT K

### C.3.1 *Pre-Instruction Interview*

1. T: We're studying the learning of Analysis. There are four sections. First, the table below presents a fragment of the table of approximate values of some function f from the reals to the reals. So when x is one point three, the function is negative zero point four. One point four, negative zero point zero five. Zero point three... all the way up to one point eight. So the question is, do you think the equation f of x equals zero has a solution? Why or why not?
2. K: Well it does, because it goes from negative to positive, so it crosses zero at some point.
3. T: Ok. And so why do you think that, when it goes from negative to positive, it has to cross zero?
4. K: Well, if it's a continuous function, it has to be [zero].
5. T: Ok, and you think that f of x is a continuous function? Why do you think it's a continuous function?
6. K: Well, I don't know, actually, from looking at this. But I assume that, if it's continuous, then there will be a solution.
7. T: Ok, cool. Question 2. Consider the function f, from the reals to the reals, defined by the following rule. When x is rational, it's this thing at the top. And when x is irrational, it's one.
8. K: Well... [laughing] I have to calculate this?
9. T: Well, give it at a try, and then I'll... Could this function be the function that question one talks about? You can use a calculator.
10. K: I don't have a calculator.
11. T: I have one. Don't worry - I'm prepared. [...] So, you can try a value and then we'll talk about it.
12. K: [Using calculator.]
13. T: You're just calculating some values?
14. K: [...] Umm... no.
15. T: It should work out.
16. K: Really?
17. T: Yeah.
18. K: I get six!
19. T: Ok...
20. K: Well, I don't know how to use your calculator. [joking]
21. T: Yeah, I feel this is kind of a messy calculator. Ok, I promise you that if you all these values in here, you would get all of those values.
22. K: Ok.
23. T: Ok, and so we'll go from there. So considering that these values match up, could this function be this function?
24. K: [...]
25. T: So what are you thinking now?
26. K: Well, there's a one there, for non-rationals.
27. T: Ok.
28. K: So if it's continuous... [...]

29. T: So if what's continuous?
30. K: The function is continuous.
31. T: Which one?
32. K: This one. [pointing at Function 1] Then you'll have a value of one between every one of them.
33. T: Ok.
34. K: I guess. I don't know. It could be.
35. T: I could be, yeah. Does this tell you anything about the behaviour when x is irrational? [pointing at the table of values in Question 1]
36. K: No.
37. T: No, ok. And what about in Question 2? Does it tell you anything about when x is irrational?
38. K: What's Question 2?
39. T: The function in Question 2.
40. K: Yeah....
41. T: So, you know the behaviour of this function when x is irrational.
42. K: Yeah, right.
43. T: Ok, so Question 2. Is the square root of two a root of the equation f of x equals zero? Why or why not?
44. K: [...]
45. T: So, when x is the square root of two - does that make the function zero? Why or why not?
46. K: [...]
47. T: What are you looking [at] here?
48. K: Yeah.
49. T: Yeah, it will?
50. K: It's because of here.
51. T: Where? X squared minus two?
52. K: Yeah. If x is positive, if x is... Yeah, it's square root of two squared, minus two is zero.
53. T: And so this will make the whole thing... What about the fact that the square root of two is irrational?
54. K: [...] Ok, yeah.
55. T: Does that change your answer?
56. K: [...] Ooh! Then it doesn't work.
57. T: Why not?
58. K: Because then it's going to be equal to one.
59. T: Ok. Because... square root of two is...
60. K: ...is irrational, yeah.
61. T: And so it's going to be one, which is not zero.
62. K: Yeah.
63. T: Ok, so what about one point four one two one three five six two... which is pretty close to the square root of two. Is that a good approximation of a root of f of x equals zero? Why or why not?

64. K: Then yes, because then it's a rational number.

65. T: Ok, and so, it's...

66. K: And then it's not one. It's going to be zero - or close to zero.

67. T: Ok, it's going to be close to zero. Is this number here - one point four one etc etc - is it close to a number that, when it goes into the function, actually is zero?

68. K: I'm sorry?

69. T: So like, you said this number - one point four one four... – is close to zero, when it goes in the function. Is there a number nearby that actually goes to zero, exactly. Like, equals, not just close?

70. K: Um... I don't think so. Because it'll be one.

71. T: When will it be one?

72. K: When it's actually the square root of two.

73. T: Ok, so when it's actually...

74. K: So, you can get close, but not zero.

75. T: Ok, cool. Question three. Consider the function f, from the reals to the reals - not necessarily the same f - defined by the following rule. So the top part's the same.

76. K: Yeah.

77. T: But now the bottom, it says - when x is irrational - it's zero now, the function. Before it was one.

78. K: Ok.

79. T: So the question is, Could this function be the function that Question One talks about? You can use a calculator.

80. K: [using calculator]

81. T: And here are the values from Question One, just to remind you.

82. K: Yeah, but it was the same function. I mean, you said it does match. So I'm going to assume that it does match.

83. T: Yeah, so the values match, yeah. And so, considering that the values match, the values from the table...

84. K: It could be that function, yeah.

85. T: Ok. And do you think it might be?

86. K: [...]

87. T: There's no right answer. I'm just curious.

88. K: Yeah, it could be.

89. T: Ok, question two. Is the square root of two a root of the equation f of x equals zero? Why or why not?

90. K: [...] Umm, it's one of them.

91. T: And why do you say that?

92. K: Because any irrational number will give you zero.

93. T: Ok.

94. K: Yeah.

95. T: And so there's nothing special about root two, it's just because it's irrational...

96. K: Yeah.

97. T: Ok, question three. Is one point four one etc etc, which is close to the square root of two, is that a good approximation of a root of f of x equals zero? Why or why not?
98. K: Well, no. Because then any irrational number will be actually, will give you zero, will be the root for your equation.
99. T: Any irrational?
100. K: Yeah.
101. T: Ok, so, how does that affect your answer?
102. K: Well, this answer is a close approximation, but you can get an exact answer if you plug in any irrational.
103. T: Ok, so it's a good approximation, but it's not exact. And you can do better, is what you're saying.
104. K: Yeah, with an irrational.
105. T: Ok, four. What is the main difference between the function in Question Two and the function in Question Three?
106. K: Well, it's defined differently. For any irrational number, you get one for the first one, and you get zero for the second one.
107. T: Ok. And how does that change the function?
108. K: You're going to have... This function is actually discontinuous...
109. T: Which one? Question Two?
110. K: Yeah.
111. T: Where?
112. K: At any irrational number.
113. T: Ok... any rational or irrational?
114. K: Irrational.
115. T: And what about Question Three?
116. K: This one is continuous.
117. T: Where?
118. K: Ahhh, actually... no, it will be discontinuous for anything but root two.
119. T: Ok, and why is that?
120. K: Well, because you're going to have, it's going to be either positive or negative, and it's going to jump back to zero for any... [...]
121. T: Ok, and so you're moving your hand up and down. So you're saying, it's going to be at a certain value, and then for an irrational it's going to jump to zero?
122. K: It's going to jump to zero.
123. T: Ok. Is it continuous anywhere?
124. K: It's continuous at zero.
125. T: For what? For x is equal to zero?
126. K: Yeah. Oh no, sorry, for x is equal to root two.
127. T: Oh, ok. But otherwise it's not going to be... continuous...
128. K: Yeah.
129. T: Ok, Question Four. The table below contains some values of a piecewise linear function f from the reals to the reals. One piece of the function is linear on the interval from negative infinity to seven point eight. The other is linear from seven point eight to infinity, and the second interval contains seven point eight.

130. K: Uh-huh. [Yes.]

131. T: So here they are. Seven point five, seven point six... it's negative here, zero point two five. And then all the way at eight point two it's zero point three. Ok?

132. K: Yeah.

133. T: Can you sketch a graph of this function?

134. K: [...] And so from minus infinity to seven point eight... [...] It's going to be something like that... [...] And it's going to be like this. [...] Ok, yeah.

135. T: And so... where's seven point five on here?

136. K: Seven point five?

137. T: Yeah.

138. K: Right here.

139. T: And what's the value of the function there?

140. K: [...] Oh! I have to sketch this function.

141. T: Yeah.

142. K: Ohhh, ok.

143. T: These values are the values of this function.

144. K: Ok [...] seven point five is gonna be negative a quarter... [..] It's not going to be to scale.

145. T: That's ok.

146. K: Seven point six is going to be... [...] Ok, it's a linear function so it's going to be something like... this.

147. T: Ok.

148. K: [...]

149. T: And so my question about this graph is... Let's look at the pattern in the values of this function.

150. K: Ok.

151. T: So at seven point five, it's negative zero point two...

152. K: ...five.

153. T: At seven point six, it's negative zero point two zero. At seven point seven, it's negative zero point one five. If that pattern kept going, what would it be at seven point eight?

154. K: Ummm..

155. T: Because it's linear, so it's going to have the same slope.

156. K: It's negative zero point one.

157. T: Is that reflected in your graph?

158. K: No. So it's going to be something like... this. And... at seven point eight, it's going to be... here.

159. T: Ok, so now you have a function that has the two parts.

160. K: Yeah.

161. T: Ok, cool. So the question is, does this function have a limit at x equals seven point eight?

162. K: Umm, no.

163. T: Why not?

164. K: Because you're going to have a limit going this way that's going to be...
165. T: Going which way?
166. K: From plus infinity, to seven point eight.
167. T: Ok.
168. K: Which is going to be zero point ten.
169. T: Ok.
170. K: And you have a limit from negative infinity, which is going to have a limit of negative zero point ten.
171. T: Ok, and so what does that mean about the limit at seven point eight?
172. K: It does not exist.
173. T: Ok. Because it has two different limits, from both directions...
174. K: ... yeah.
175. T: What, if anything, does this function have in common with the function in Question Two? So this is Question Two. [Moving paper.] The one where, if x is irrational, the function is equal to one.
176. K: [...]
177. T: Umm, and we said that root two is not a root of the equation, f of x equals zero. So what do these two have in common?
178. K: Well, they're discontinuous!
179. T: Ok.
180. K: At some point. Well, this function is discontinuous at that point, and that function is discontinuous at seven point eight?
181. T: The one in Question Four is discontinuous at which point?
182. K: At seven point eight.
183. T: And the one in Question Two is discontinuous at... you said many points?
184. K: At, uh, yeah many points.
185. T: And so they have that in common. And what about differences?
186. K: [...]
187. T: You said, first of all, that this one has one discontinuity, and this one has lots.
188. K: Yeah.
189. T: Are there any other differences?
190. K: Well, this one is nonlinear.
191. T: Ok. So the one in Question Two is non-linear. And what about Question Four?
192. K: That one's linear.
193. T: Ok, cool. Compare this function in Question Four to the function in Question Three.
194. K: [...]
195. T: So, this is the one where f is zero for irrational x, and root two was a root of the equation f of x equals zero. So how does the one in Question Three compare to the function in Question Four?
196. K: Well this one is continuous at root two, but it's discontinuous everywhere else.
197. T: Ok. The one in Question Three?
198. K: Yeah.

199. T: What about Question Four?

200. K: This is the other way around. It's continuous everywhere, and discontinuous at seven point eight.

201. T: Ok, cool. Any other differences between the two?

202. K: And... this looks like a nonlinear function.

203. T: And this is linear. How about drawing it? Would you be able to draw this, do you think?

204. K: This? [The function from Question Three] Can I have the table from the other function?

205. T: Yeah, sure.

206. K: Let's see here. [...]

207. T: What I'm curious about is how you're going to draw the ones where x is irrational.

208. K: Well, we'll see...

209. T: We'll see.

210. K: We'll try. [laughing]

211. T: Ok. [laughing]

212. K: [...] Ok, so that's negative zero point four... [...] So here... [...] So here... [...] I can't draw a line for this.

213. T: Why not?

214. K: Because there will be jumps to, uh... this is question number one, right?

215. T: Yeah.

216. K: Or two, or three. Because there will be jumps here to zero all the time. There will be holes.

217. T: There will be holes in the...

218. K: I can put a dot there for a value of a rational number.

219. T: For the rational. And what about the irrational?

220. K: Uh, it will be... the first one will be one.

221. T: Will be one. For Question Two, it will be one.

222. K: And so it will be dots here. Let's say this is... So here is one, so there will be dots here. [Sound of marking dots.]

223. T: Ok. So how does drawing this compare to drawing... Drawing the functions in Question Two and Question Three, how does that compare to drawing the function in Question Four?

224. K: Well, this is continuous so I can assume there's a continuous line between every one of these points.

225. T: What's continuous?

226. K: This side, and that side.

227. T: So, the two segments of Question Four are continuous. OK. What about the function as a whole?

228. K: The function as a whole?

229. T: Yeah.

230. K: Well, I can draw this in two parts. I can't draw the other one.

231. T: But what about the continuity of the function as a whole, I meant. So you said it's continuous here and there. Is the whole function continuous?

232. K: Well, continuity is a local property, you know?
233. T: I don't know, you tell me!
234. K: Well, as far as I know, it's a local property.
235. T: Ok, so you can only have continuity at a single point.
236. K: Yeah.
237. T: Ok, great.
238. K: Ok.
239. T: Anything else you want to say?
240. K: No, that's it. You made me think. [End of recording.]
241. T: [Recording restarts, mid-conversation] Do you mind if I record this, actually? So you said you deal with local continuity when you need optimization...
242. K: Yeah, we do a lot of optimization. So we have to show at least local continuity, or... You can show that it's continuous at every point.
243. T: Ok. So you can show that a function is continuous at every point, but that's still just local.
244. K: Yeah, it's a local property still.

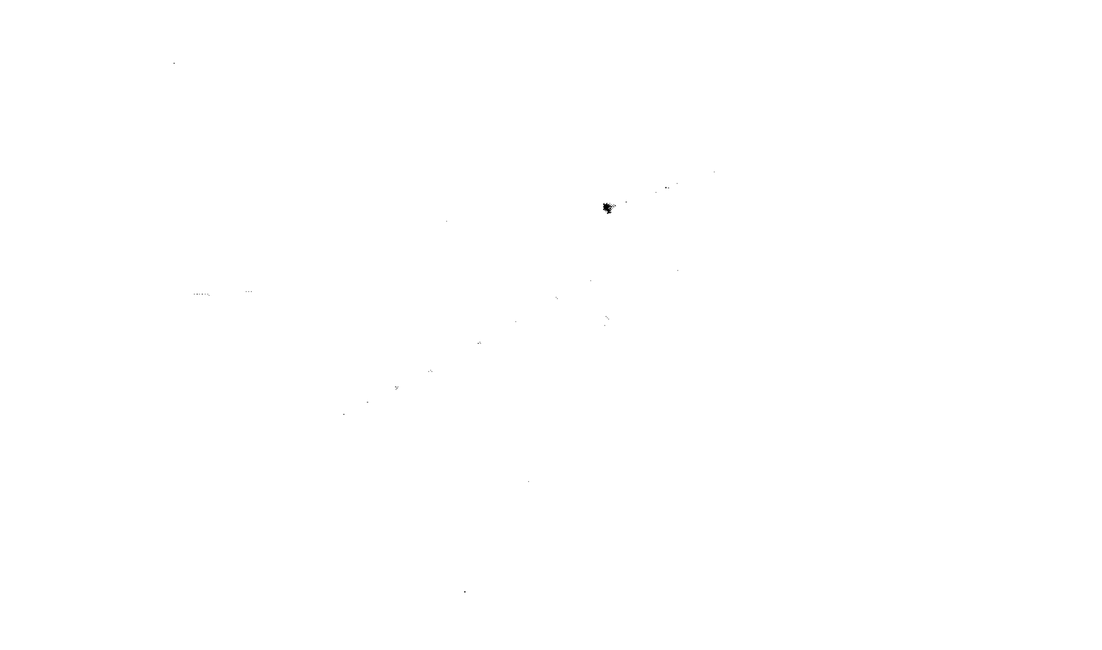

Figure 11: K's sketch of $f_4$

### c.3.2 *Post-Instruction Interview*

1. T: You remember the drill: Whatever you're thinking, just say it out loud. No right answers. We're just interested in what you're thinking. And if you have any questions, feel free to ask.
2. K: Yeah.
3. T: Ok, cool. Can you say your name?

4. K: [omitted]
5. T: So, we're going to start off with three proofs of the same theorem. And the theorem is this one: If g is continuous, and f is continuous on the range of g, then f of g of x is continuous. Ok? So we're going to have three different proofs of these. I'm going to show them to you one at a time. And I'm going to ask a couple of questions about them afterwards. Ok?
6. K: Yeah.
7. T: So here's the first.
8. K: [Reading.] Ok.
9. T: Ok. Any questions about that one?
10. K: No, this seems to be all right. It makes sense.
11. T: Ok. Here's proof number two.
12. K: [Reading.] Ok. It's more or less the same.
13. T: Ok. Here's proof number three.
14. K: [Reading.] Yeah.
15. T: Ok.
16. K: It's more or less the same thing. This one's better, though.
17. T: Yeah? Which one?
18. K: This one. I liked this one better.
19. T: Number three? You liked number three best.
20. K: Yeah.
21. T: Ok. Why is that?
22. K: Well, this one is shorter.
23. T: Number three is shorter, yeah.
24. K: And it's less wordy, and it's more symbols.
25. T: It's more...?
26. K: There's more math symbols.
27. T: Oh, more symbols.
28. K: So it's easier to understand.
29. T: It's easier to understand.
30. K: Like here. It says, "move x sufficiently far towards a." I'm not sure what this means.
31. T: Ok.
32. K: But if you say if it's in the neighbourhood of a, then that sounds better.
33. T: It sounds better.
34. K: It sounds more precise.
35. T: Ok. And so, in theorem two, this section.
36. K: Which part.
37. T: "Move x sufficiently far towards a." You say that's imprecise
38. K: Uh-huh.
39. T: Ok, cool. Which proof out of the three does the best job of explaining why f of g of x is continuous?
40. K: [...] They all seem to explain exactly the same thing.

41. T: Yeah?
42. K: Pretty much. These two are almost the same.
43. T: One and two.
44. K: Yeah. And this one is exactly the same thing, but using math symbols.
45. T: Oh, ok. And so you think they explain the same thing about why f of g of x is continuous.
46. K: Yeah.
47. T: And so you mentioned that the language is different. And so, even though the language is different, they're still explaining the same thing?
48. K: Yup.
49. T: Ok. And what does the proof explain to you? You said they all explain why f of g of x is continuous.
50. K: What do you mean?
51. T: Well, the question was, which of the three is the best at explain why f of g of x is continuous.
52. K: Yeah.
53. T: So, what do they explain about f of g of x being continuous? What do they tell you about it?
54. K: Well, you can get as close as you want to f of x by moving x close a. Close to f of a. And you repeat the same thing for the domain of g. You can move as close as you want to g of a, which is f of a let's say, so you can get as close as you want to g of x.
55. T: Ok cool. So you said they're all pretty good at explaining why - they all say the same thing. Is there one that's worse at explaining why f of g of x is continuous?
56. K: Um, I think the second one.
57. T: Yeah? Why is that?
58. K: Um, like this part. I don't understand what exactly "sufficiently far towards a" means.
59. T: [short interruption when someone enters the room.]
60. K: Yeah, because this one says sufficiently close, and this one says sufficiently far towards a.
61. T: Right. And what's the difference, for you?
62. K: Well sufficiently close, is close to a. Sufficiently far, it means it's going somewhere... far towards something.
63. T: And you're moving your hand from the left to the right through the air as you're saying this...
64. K: Yeah, like far is going away from something.
65. T: Ah! Oh, ok ok ok... And so, here, you see...
66. K: ...you're moving towards a, and here you're moving from somewhere.
67. T: Ok, ok. And so, does that make one of these worse at explaining why f of g of x is continuous?
68. K: Yeah.
69. T: Which one?
70. K: This one.

71. T: Oh ok, Number Two. Ok. Because that "sufficiently far" is unclear to you. Ok, so the question was, Which proof does the best, or the worst, at explaining why f of g of x is continuous. What do you understand by the word explanatory, or explain? What do you understand by that?
72. K: Well... explain!
73. T: Explain.
74. K: Well, it's just explain. That's what it means exactly.
75. T: Yeah, ok.
76. K: Explaining what happened, and why it is so.
77. T: Why it is so... And so what part are they... So I said, so you said, This proof does a really good job of explaining. Ok, well, what is it explaining?
78. K: [...]
79. T: Is it explaining a particular word, or a particular result?
80. K: It's explaining a particular result.
81. T: Which result?
82. K: That f of g of x is continuous.
83. T: Ok. So when you hear, "Which proof does the best job of explaining?" you mean it explains the theorem.
84. K: Yes.
85. T: Ok. And so, by reading the proof, if it's a proof that explains to you...
86. K: Yeah, if it's a proof, that explains the statement of the theorem.
87. T: Ok, and for the proof to explain, that makes you understand why it's true?
88. K: Yup.
89. T: Ok, so now imagine that a student who hasn't taken a class in Analysis - but maybe has taken other math classes - asks you why f of g of x is continuous. How would you explain it to them?
90. K: I would probably do something like this. Like the first one.
91. T: Like the first one. Why? Why would you pick that one?
92. K: Because he might not be familiar with some constructs, or some terminology or symbols you would use in Analysis.
93. T: Ok. Like?
94. K: Like set theory, for example. Or subsets. Or neighbourhoods.
95. T: Right. And you point to number three...
96. K: Yeah.
97. T: Because those are used in number three?
98. K: Yeah.
99. T: Ok. And so you'd pick number one over number two?
100. K: Yeah.
101. T: Yeah. And why is that?
102. K: Well, number two is a bit easier to read... Oh, number one, sorry!
103. T: Number one is easier to read. Because you find the words confusing there?
104. K: Yeah.

105. T: Ok, great. So those are the questions on that... Now I have a couple of questions on continuity. How do you understand continuity of a function?

106. K: Continuity is, at any point of a function, if you move a little bit away, or, yeah, a little bit away from that point, there's going to be a value of that function in the range, close enough.

107. T: Can you give an example of a function that's continuous?

108. K: A constant function is continuous.

109. T: Ok. And why is it continuous?

110. K: Because for every value of x, there's a value of y, and for every value of x close to that x, there's a value of y in the neighbourhood of f of x.

111. T: What about a function that's not continuous?

112. K: Not continuous, it means there's a break somewhere.

113. T: Can you give me an example?

114. K: A function, let's say, where y is equal to one when it's rational, or zero when it's irrational.

115. T: And why is that one not continuous?

116. K: Because every rational point contains an irrational really close to it, so the value of the function will jump by one all the time.

117. T: And you're moving your hand up and down, because that's what the function's doing?

118. K: Yeah, it will be up and down all the time, at every point.

119. T: Ok. Ok, complete these sentences: "Continuity is like..."

120. K: [...]

121. T: Dot dot dot... [laughter]

122. K: I dunno.

123. T: Ok, well, what does continuity make you think of?

124. K: Continuity, it makes me think you can draw a function with a pen, without lifting.

125. T: Ok. And so, it's like...

126. K: It's like a line. A curve.

127. T: Anything else?

128. K: [...]

129. T: Ok. "Continuity is not like..."

130. K: [...] That is a hard question!

131. T: Yeah. It makes you think.

132. K: Lemme think...

133. T: But whatever comes into your mind. Like, what things pop into your mind? "Continuity is not like..."

134. K: [...]

135. T: So when you think of continuity... Someone says, "Oh, this is a continuous function." What's the first thing... You don't even see it yet. You just think... You just hear "continuity." What pops into your mind?

136. K: I usually think of a line with a break in it.

137. T: For what?

138. K: For, uh, a function, let's say.

139. T: Yeah, but, when they say which word? What makes you think of a break in it... Discontinuity?

140. K: Yeah, discontinuity.

141. T: Oh, discontinuity. So that's what you'd see in your head.

142. K: Yeah.

143. T: You'd expect to see something like that. So, like, a break. And how about if they said, "continuity" - if something was continuous. What would you picture in your head?

144. K: Something like a line or a curve.

145. T: And would any words come to your mind?

146. K: No, it's usually just... I think graphically.

147. T: So when you're thinking about... like, proofs for instance. You say you think graphically. Do you think graphically when you're reading through the proofs?

148. K: Sort of, yeah.

149. T: So, what kind of things would you picture?

150. K: Well, for continuity I would picture a curve, a neighbourhood for the domain and a neighbourhood for the range.

151. T: And you're doing a little circle with your finger around the different points...

152. K: Yeah.

153. T: That's how you see a neighbourhood?

154. K: Yeah.

155. T: Ok. I have three functions now. Can you explain why each of the functions is or is not continuous? So tell me if it's continuous or not, and why. On it's domain.

156. K: Ok.

157. T: Here's the first one. So, when x is not zero - so everywhere but zero - it's the function sine one over x. And at the point x equals zero, it's equal to zero exactly. So first, is it or is it not continuous?

158. K: [...] This one is non-continuous.

159. T: Non-continuous. And why is that?

160. K: Because around zero, you're going to have the function, it's going to be very often either here or there. I don't know what the value is, but...

161. T: Ok, and you're pointing to the top and the bottom [of the graph].

162. K: Yeah. And as it approaches zero, it's going to be very often away from zero.

163. T: Very often away from zero. And so at zero, it's not continuous - is that what you're saying? Or just the function as a whole? Or what?

164. K: At zero. [...]

165. T: Because you said, when x is equal to zero, it's going to jump from the... And you were pointing to the top and the bottom.

166. K: Yeah. When you get closer to zero.

167. T: Ok. What happens when you get closer to zero.

168. K: So at zero it would be discontinuous.

169. T: And is there anywhere else where it's discontinuous?

170. K: Umm... [...]
171. T: Because just before now, you said "close to zero."
172. K: Yeah, yeah, yeah... That's what I'm looking at now. [...] Everywhere else it should be continuous.
173. T: Ok. And why is it continuous?
174. K: Everywhere else it's continuous because... [...] I dunno. Because it's continuous! [laughter]
175. T: [laughter] Because! Because it's continuous!
176. K: [laughter] I don't remember, actually.
177. T: Ok, but how are you judging that? Are you looking at the...
178. K: I'm looking at the line. And the line looks continuous.
179. T: At the graph?
180. K: Yeah.
181. T: And so... Because you can see the way it looks? And what kinds of things about it tell you that it's continuous?
182. K: I dunno. It looks like a curve.
183. T: It looks like a curve. And what do you mean by curve?
184. K: Um... it's continuous. You can draw it continuously.
185. T: And you're moving your hand back and forth...
186. K: Yeah.
187. T: Ok. And so, show me with your hand what something would look like if it was discontinuous.
188. K: It would probably look something like this.
189. T: Ok. So you move your hand, first in a straight line, and then you move straight up, and then you go in a straight line again.
190. K: Yup.
191. T: Ok, like a jump or something. And you don't see that here?
192. K: No.
193. T: Ok. Here's number two: f of x is equal to e to the x. So, the exponential function.
194. K: Uh-huh [yes].
195. T: Here's the picture. So is that continuous or discontinuous?
196. K: That's continuous everywhere.
197. T: Continuous everywhere.
198. K: Yeah.
199. T: Ok. Why?
200. K: It's continuous everywhere because, for any neighbourhood of your range, your function is going to be there for any neighbourhood of x.
201. T: Ok, ok. And that's how you tell, when you look at the function? You think of that?
202. K: Yeah. Plus, I know that e to the x is continuous.
203. T: Ah! You know that, because someone told you that already.
204. K: Yeah.
205. T: What about the one before? Have you seen that one before?

206. K: Yeah, I have, but I don't remember exactly where it's... I know it's discontinuous at zero, but I don't remember if it's continuous everywhere else.

207. T: Ok. What about this one. F of x equals one over x, and x doesn't equal zero. Is that continuous or discontinuous?

208. K: What's the domain of this function?

209. T: X is the reals, except for zero.

210. K: [...] Well, it's discontinuous at zero.

211. T: Ok. When x equals zero?

212. K: Yeah.

213. T: Ok. But the function says, x doesn't equal zero. That's not part of the domain.

214. K: Oh! That's not part of the domain?

215. T: Yeah.

216. K: [...] Then it is continuous.

217. T: Ok. And why?

218. K: Because it's continuous everywhere except zero, so if that's not part of the domain, then it's continuous.

219. T: And why is it continuous everywhere but zero? How do you know that?

220. K: It's the same. For any neighbourhood of x, you have a function that exists in a neighbourhood of y.

221. T: Ok. You just know that about the function, the definition here? Or you can tell from looking at the picture? What do you look to to get that information?

222. K: Well, usually the picture.

223. T: You look to the picture.

224. K: And usually I see if there's any restrictions on the domain.

225. T: So you look to the picture. You look for... You see if it's like a curve. And then you see if there's any restr...

226. K: Or if I don't know, you plug it in the definition of continuity and check.

227. T: Ah, ok.

228. K: But these ones are known functions, so you don't have to do that.

229. T: Right. But if you didn't know it, then you would have to rely on...

230. K: Yup.

231. T: ...rely on the...

232. K: ... on the definition.

233. T: Ok. And so, when do you go to the definition? Is that the only time you use the definition?

234. K: Well, if you have to write a proof, you have to use the definition.

235. T: Ah! For a proof, you need the definition...

236. K: If you need to write a formal proof, yeah...

237. T: ...you need the definition. And if you need to check it, you need the definition.

238. K: Yeah.

239. T: And is there any other time when you'd use the definition?

240. K: Well, if you want to use continuity somewhere else, then you need to use the definition.
241. T: But for answering questions like this... That wouldn't be one of the first tools you'd use.
242. K: Well, if it's a simple function like this, no. But if it's something complex that you can't tell, then you would put in the definition to check.
243. T: Ok, cool.

## C.4 SUBJECT J

### C.4.1 *Pre-Instruction Interview*

1. T: Ok, can you say your name?
2. J: [omitted]
3. T: Ok, here's Question One. All the read the question out loud.
4. J: Ok.
5. T: The table below presents a fragment of the table of approximate values of some function f, from the real numbers to the real numbers. Ok? So when x is one point three, the function is negative zero point four blah blah blah. When x is one point eight, the function is two point one one etcetera etcetera. Ok?
6. J: Ok.
7. T: So the question is: Do you think the equation f of x equals zero has a solution? Why or why not?
8. J: [...] Umm... If this goes down, I might think that the rate at which it's decreasing doesn't decrease. And, uh... Until... It would seem that f of x equals zero does have a solution, because it's from R, and any point from R must have an image. So... [...]
9. T: Ok.
10. J: Otherwise it wouldn't be a function.
11. T: So, x could be zero. But is f of x going to be zero? Is there a number where...
12. J: Well, it depends if the function is surjective or not.
13. T: Ok. And by looking at these values here, can you make a guess? Because you have a couple examples of what the values are.
14. J: Well, if it is continuous, it would have one. Between one point four and one point five it would be zero.
15. T: Ok. And why do you think it is? You're saying that if it is continuous, it will be [zero]. And how do you know that from looking at the values?
16. J: Because f of x equals zero is between f of one point four and f of one point five.
17. T: Ok, great. Question Two. Consider the function f from the reals to the reals - not necessarily the same function - defined by the following rule.
18. J: Yeah, ok.
19. T: So, when x is rational, it's this times that. [Pointing to the analytic expression.] And when x is irrational, it's one.
20. J: Ok.
21. T: Could this function be the function that Question One talks about? You can use a calculator to check.
22. J: Ugh...
23. T: I have a calculator right here.

24. J: Oh ok. [...] [Using calculator.]
25. T: Here, use x to the y. [Explaining the use of the calculator.]
26. J: Ugh... it didn't work.
27. T: I think we forgot to put the brackets.
28. J: Yeah, I forgot to put the brackets, yeah.
29. T: It does work.
30. J: Yeah, ok.
31. T: It does match up. You can trust me on that.
32. J: Ok yeah, ok.
33. T: So, all these values here, do work here.
34. J: They do work in there. Ok.
35. T: So the question: Could this function here be the function that Question One talks about? Is it possible?
36. J: Uh, yeah. Because if it's in Q or not in Q, it's in R.
37. T: Yeah, ok. Um...
38. J: Yeah.
39. T: Ok, Question Two. Is the square root of two a root of the equation f of x equals zero? Why or why not?
40. J: It is not, because the square root of two is irrational. So it would be one.
41. T: Ok, it would be one. Is one point four one four two one three five six two a good approximation of a root of f of x equals zero? Why or why not?
42. J: [...]
43. T: So this number is close to root two, ok?
44. J: Yeah. [...] I would assume that, if this would tend to zero - if that were square root of two - so that would be close to f of x equals zero. So it would be a good approximation, depending on what kind of precision you want.
45. T: Ok. And so you would get really close to f of x equals zero. Would it be close to a number which is, which does map to zero?
46. J: Ugh... Yeah, because this would be finite, and this would start to get really small. So it would be close to zero.
47. T: Ok. Will it ever actually be zero?
48. J: No.
49. T: No. Ok. Consider this function, f from the reals to the reals, defined by the following rule. So the first part is the same, but when x is irrational, it's now zero.
50. J: Ok.
51. T: Ok? Could this function be the function that Question One talks about? You can use a calculator.
52. J: Yeah. Well, that's the same.
53. T: Ok. Is the square root of two a root of the equation f of x equals two? Why or why not?
54. J: Yes, because the square root of two is irrational, so it would be zero,
55. T: Ok. Is one point four one four two one three five six two a good approximation of a root of f of x equals zero? Why or why not?

56. J: Yeah, well, the same as the question before. It will... This term will tend to zero, so it will be close to zero.
57. T: Ok. And is it close to an actual root?
58. J: [...]
59. T: So...
60. J: Well, yeah, square root of two is an actual root, and this is close to the square root of two. So...
61. T: Ok. What is the main difference between the function in Question Two and the function in Question Three?
62. J: In the function in Question Two, the root of two wasn't actually a root of zero, whereas here it is a root of zero.
63. T: Ok. And does that change the way you think of the function? Are the functions different in some way?
64. J: Uh... [...] Well it makes the... the use. You can use this function different, because you can use this function to approximate the square root of two when it tends to zero, whereas you couldn't use the other function.
65. T: Ok. Question Four. The table contains some values of a piecewise linear function from the reals to the reals. One piece of the function is linear from negative infinity to seven point eight, not including seven point eight. And the other is linear on seven point eight to infinity, but including seven point eight. Here is the table of values.
66. J: [...] Ok.
67. T: Can you sketch a graph of this function?
68. J: Uh... [Sketching.]
69. T: You can take up as much space as you'd like.
70. J: Ok. [...] It should be about like that. Because that half is continuous, and the bottom half is continuous.
71. T: Ok.
72. J: Assuming it continues between those two values at the same rate.
73. T: Ok, so you're saying... So it's going to look like this. So you're going to have two linear functions, here.
74. J: Yup.
75. T: And you're saying if it continues at the same rate...
76. J: Yeah, between seven point seven and seven point eight. Because it could do something else.
77. T: Ok. And, the circle here means it doesn't include that point?
78. J: Yeah it doesn't include, actually, seven point eight. That line is seven point eight.
79. T: Ok, I got it. Does this function have a limit at x equals seven point eight?
80. J: No, it doesn't.
81. T: And why not?
82. J: Because if you go from the left and from the right, there's a discontinuity. So it doesn't have a limit at that point.
83. T: Ok. What, if anything, does this function have in common with the function in Question Two?
84. J: There is a discontinuity point at... around square root of two, it is discontinuous.
85. T: And here?

86. J: And here, well, it is discontinuous at seven point eight.

87. T: Ok. Compare this function with the function in Question Three.

88. J: Um... In this case, the function would be continuous at the square root of two. Although, when it is other irrationals - like the square root of three - I think it wouldn't be continuous.

89. T: And so, but at root two, Question Three is continuous, but elsewhere it might not be continuous?

90. J: Well, yeah, at other points that are not the square root of two.

91. T: And why is that? Why do you say that?

92. J: Because, uh, this term might not tend to zero when it goes to square root of three, for example.

93. T: Ok.

94. J: Or any other irrational.

95. T: Ok. Great!

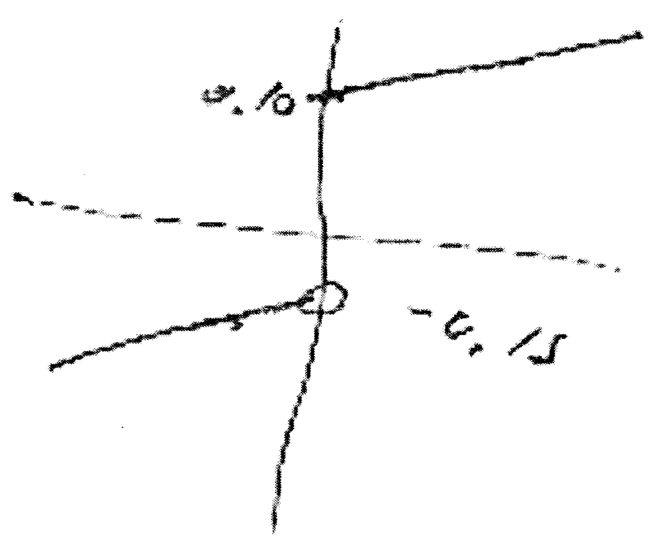

Figure 12: J's sketch of $f_4$

### C.4.2 *Post-Instruction Interview*

1. T: Can you say your name?
2. J: [omitted]
3. T: Ok. So we did the first interview a couple of weeks ago.
4. J: Yup.
5. T: And now this is the follow-up interview. To start, I'm going to give you three proofs of the same theorem.
6. J: Ok,
7. T: So they're all proving the same thing. And I'm just going to ask you to read them one-by-one.
8. J: Ok.
9. T: And if you have any questions, just let me know. If you have any comments, feel free to say them. And then I'll have some questions afterwards.
10. J: Ok.
11. T: Ok?
12. J: Ok.
13. T: So, this is the theorem, the result. If g is continuous, and f is continuous on the range of g, then f of g of x is continuous.
14. J: Yes, ok.
15. T: So this is something you've seen before in class, right?
16. J: Yeah.
17. T: So here's the first proof.
18. J: [Reading.] Oh, ok.
19. T: Ok? Here's proof number two.
20. J: [Reading.] Ok.
21. T: Here's proof number three.
22. J: [Reading.] Ok.
23. T: Ok. So, what did you think?
24. J: Umm, well, that one is more going towards neighbourhoods and more advanced concepts that you'd see in Analysis. Whereas the other two proofs could be understood by people who maybe haven't taken Analysis and are in Cegep, or something like that.
25. T: You said the third one has some concepts that you're learning in Analysis. Which concepts?
26. J: The concept of neighbourhood. And a little bit of set theory, although that is seen in Cegep also.
27. T: Ok. Which of the proofs does the best job of explaining why the result is true? Explaining why f of g of x is continuous?
28. J: I would say it would be this one, assuming you know the definition of a neighbourhood. [Pointing at Proof 3]
29. T: Number three? Ok. And what if somebody didn't know the definition of a neighbourhood?
30. J: Then it would be number one or two... I'll just check real quick... [...] [Reading.] I'd say proof number one explains better, without the definition of a neighbourhood.

31. T: This one here?
32. J: No, this one. That's number one.
33. T: Oh yeah, right. Ok. And why do you think it explains it better?
34. J: It's referring to the distance between the two, and how you can get them close. Whereas the other is talking about a distance, and not really referring to something that is close.
35. T: Ok, cool. So, looking at this proof here...
36. J: ...yup...
37. T: ...number three. What does it explain to you?
38. J: It explains that, whatever interval you take in the domain of f of g of x, you can find an interval... there is an interval... Actually, it's the other way around. If you take an interval of the range of x... f of g of x, excuse me, you can always find an interval in the domain of f of g of x such that the image of all the points in that domain will be contained in the other interval.
39. T: Ok. And as you're doing that, you're holding your fingers up in the air, kind of like a vertical line, sort of like the range, to show.... So what are you picturing as you're doing this?
40. J: That's actually... they would be like the function.
41. T: Ok.
42. J: And you would put... They called it W in the proof. You would have, like, W on the y-axis, and you would have, then... they called it U, on the x-axis.
43. T: So you're picturing the graph?
44. J: Yeah.
45. T: And you're picturing the domain...
46. J: ...and the range, yeah.
47. T: ... and the range, on the x- and the y-axis
48. J: Yeah.
49. T: Ok, cool. And how about for this one here?
50. J: Yeah?
51. T: In proof number one, when you look at that, what does that explain to you?
52. J: That is, you would have a function, and whenever you would take a point that would be close to some point a, you could always take it so close such that the image of that point would also be very close to f of a.
53. T: Ok. So which proof is the worst at explaining why f of g of x is continuous?
54. J: I would think that it would be proof number two.
55. T: Proof number two.
56. J: Yeah.
57. T: And so why is that one worse than the others?
58. J: Because they talk about how the two differ, but they don't really explain... Like, the wording is strange. Like, they move something far towards. And the wording is a bit strange.
59. T: And so, the word... is it moving? Is that the word that you find strange?
60. J: Well, moving is strange because that's not really a defined concept. I don't know, I think it's not exactly precise in wording.

61. T: Ok. So, when I said, "Which proof explains best." Explains. What do you understand by the word "explanatory."

62. J: [...]

63. T: Like, when I asked you that question, what did you understand by explain?

64. J: That would be something to make sure that you know that, ummm, what you're trying to demonstrate is clear.

65. T: Yeah.

66. J: And there is no doubt about whether or not it is correct.

67. T: Ok. So if you had a proof that was very logical, it was step-by-step, you followed every step and got to the answer and you could tell that it was definitely true... But you didn't really know why you were doing it like that. Would that explain to you?

68. J: Well, that would explain the concept of, say, continuous in this case. But that wouldn't explain how to do the proof.

69. T: Oh, ok.

70. J: That's two different things.

71. T: Ok. So understanding why it's true, and understanding how to do the proof - those are different things. Is that what you're saying?

72. J: Yeah!

73. T: So, imagine that a student, who hasn't taken a class in Analysis, is wondering why f of g of x is continuous. So they know what continuous means, kind of - maybe they've taken calculus or something like that.

74. J: Ok, yeah.

75. T: How would you explain it to them, in your own words.

76. J: It would look like proof number one a lot. It would be, like, in any interval, when you do a square... like... it would contain U and W, let's say. You would find that the function is in that square.

77. T: What do you mean, a square?

78. J: Actually, maybe that's, umm...

79. T: Can you draw it for me?

80. J: Ok, yeah.

81. T: I like how, when you talk about it, you hold your hands in the air, like you see the function...

82. J: Yeah!

83. T: And you put your right hand where the domain is on the x-axis, and your left hand where the range is...

84. J: ...yeah.

85. T: I like that.

86. J: Yes, basically. Ok, I will explain this. You would have a continuous function. It would be defined like that. [Indicating his drawing.] And when you take some point a... and some point, let's call it b... If you take the interval that is here, you would have a plus something that is very small, and a minus something, the same thing. And you would find that, if you would take that interval to be very small, you would have that the function is always in both intervals at the same time.

87. T: Ok. And so, by the square, you mean... This is one side of the square, and this is... [Gesturing to the intervals around a and b.]

88. J: Well, yeah, that's it. When the e is very small.

89. T: Ok, and so, this explains the continuity of one function.

90. J: Yeah.

91. T: So how would you explain why f of g of x is continuous? Composition like that?

92. J: Well, by knowing that, you could take - say this is f of x - you could do that for f of x and g of x. So, by taking, g of x sufficiently close to some point, you could get f of that point to get sufficiently close to the image of that point.

93. T: Ok.

94. J: So, both of them would be in the same square.

95. T: Ok. How do you understand continuity of a function? We've talked about this, but maybe you could just say it again. So when you think of continuity of a function - how do you understand that?

96. J: That is the parameter that the function doesn't have jumps. Then it's continuous. That's the intuitive definition we've all seen.

97. T: Ok.

98. J: And now it's more like being always in a neighbourhood, such that you can always... that the image is... that you can find a neighbourhood such that all the images of the other neighbourhood are in it.

99. T: Ok. Can you give an example of a function that's continuous?

100. J: Oh, yeah. F of x equals x squared is continuous.

101. T: OK. Why is it continuous?

102. J: Because, uh, x squared is always in the neighbourhood. Like, if you take x minus c, say, you see that if this is smaller than some number, you can always find a number such that x squared minus c squared will also be smaller than another number, that would be depending on c.

103. T: Can you give an example of a function that's not continuous?

104. J: Yup. It would be some function that would be defined piecewise, like f equals one if x is negative and f equals two if x is positive. Then the function wouldn't be continuous at zero.

105. T: And why not?

106. J: Because the function would jump. So that, when you try to make some interval that is centered at zero, and then the image would be, say, one. And you take some interval of one that wouldn't include two. Every time you go to the right of zero, you couldn't find any number that would be... that wouldn't have its image to be two.

107. T: Ok. Complete these sentences. "Continuity is like...?" So what does that make you think of. What is continuity like?

108. J: Uh... it would be a line without jumps.

109. T: Ok. Anything else?

110. J: Not really.

111. T: Ok. "Continuity is not like...?"

112. J: Um... Anything that has holes, or is defined piecewise such that the pieces don't touch together.

113. T: Ok. I have three functions for you to look at. So I want you to explain why each function is, or is not, continuous on its domain. Ok?

114. J: Ok.

115. T: The first is this one. So this one is defined piecewise. So when x isn't zero, it's sine one over x. And then when x is zero, the function is equal to zero.

116. J: Ok. [...] Um... that is definitely not continuous, because... Well, at zero it's not continuous. Because if you take sine of x, if you take something... x to be some sequence that would tend to... that would be, like, two pi n. If it's one over two pi n, then the sequence tends to zero. But sine of two pi n tends to... Well, sine of one over one over two pi n, that will tend... that will always be zero. But if you take another sequence that would be, say, one over two pi n plus pi. That would give something that would always give... When you put it in the function, that would always give one. But both tend to zero. So when you... Both those functions would be far. So if you take n sufficiently large, you will always find that no matter how close you try to get to zero, you always have two different values that are always bigger than any... than a number... than a positive number that you can fix.

117. T: Ok. Next function. This is f of x is equal to e to the x.

118. J: Ok, that one is continuous, because whenever you have, like, x minus c, if you take that to be smaller than some number such that you would be in some interval, you can always find another interval in your y-axis such that e to the x minus e to the c would be in that interval.

119. T: Ok. And, last one. F of x is one over x, and x is never equal to zero.

120. J: Umm... [...] That would be continuous. Because, when you... The same way, when you take... You can always find some neighbourhood of x such that it is in a neighbourhood of y. And since x cannot be zero, you cannot include that as your point for the interval. Therefore you can always find a neighbourhood such that zero is not included. So it will never actually go across the zero to be... to have such a large interval on y.

121. T: And when you're doing that, you're moving your hand from the top and the bottom.

122. J: Yeah. That's... If, actually, x equals zero were somewhere in the domain, it wouldn't be continuous at x equals zero, because on the interval... any interval centred at x equals zero, the function goes to minus infinity and plus infinity on both, on each side.

123. T: Ok.

124. J: So it wouldn't be in a finite interval.

125. T: Ok. Ok, great. So, those are the questions for today!

# BIBLIOGRAPHY

Aberdein, A. (2008). The informal logic of mathematical proof. In Hersh, R., editor, *18 Unconventional Essays on the Nature of Mathematics*, chapter 4. Springer. (Cited on page 116.)

Ajdukiewicz, K. (1974). *Pragmatic Logic*. D. Reidel Publishing Company. (Cited on pages 2, 9, 11, 15, 16, 17, and 25.)

Alibali, M., Bassok, M., Solomon, K., Suc, S. E., and Goldin-Meadow, S. (1999). Illuminating mental representations through speech and gesture. *Psychological Science*, 10:327–333. (Cited on pages 14 and 115.)

Anderson, M. L. (2003). Embodied cognition: A field guide. *Artificial Intelligence*, 149. (Cited on page 14.)

Arzarello, F., Paola, D., Robutti, O., and Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, 70:97–110. (Cited on page 14.)

Azzouni, J. (2004). The derivator-indicator view of mathematical practice. *Philosophia Mathematica*, 12. (Cited on page 116.)

Bachelard, G. (1938). *La formation de l'esprit scientifique*. Vrin. (Cited on page 112.)

Balacheff, N. (2008). The role of the researcher's epistemology in mathematics education: an essay on the case of proof. *ZDM*, 40. (Cited on pages 6 and 116.)

Bouligand, G. (1937). *Struture des Theories*. Hermann. (Cited on page 8.)

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Kluwer. (Cited on page 112.)

Brown, J. (1999). *Philosophy of Mathematics: An Introduction to the World of Proofs and Pictures*. Routledge. (Cited on pages 16 and 17.)

Byers, W. (2007). *How Mathematicians Think: Using Ambiguity, Contradiction, and Paradox to Create Mathematics*. Princeton University Press. (Cited on pages 6 and 13.)

Chi, M. e. a. (1994). Eliciting self-explanations improves understanding. *Cognitive Science*, 18:439–477. (Cited on page 7.)

Chiu, M. M. (2001). Using metaphors to understand and solve arithmetic problems: Novices and experts working with negative numbers. *Mathematical Thinking and Learning*, 3(2 and 3):93–124. (Cited on page 14.)

Cook, S. W., Mitchell, Z., and Goldin-Meadow, S. (2008). Gesturing makes learning last. *Cognition*, 106. (Cited on page 14.)

Cornu, B. (1991). Limits. In Tall, D., editor, *Advanced Mathematical Thinking*. Kluwer. (Cited on page 11.)

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., and Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process schema. *Journal of Mathematical Behavior*, 5:167–192. (Cited on page 11.)

Danesi, M. (2007). A conceptual metaphor framework for the teaching of mathematics. *Studies in Philosophy of Education*, 26. (Cited on page 14.)

Daval, R. and Guilbaud, G. T. (1945). *Le Raisonnement Mathematique*. Presses Universitaires de France. (Cited on pages 11, 12, 22, 112, and 113.)

Davis, P. and Hersh, R. (1999). *The Mathematical Experience*. Mariner Books. (Cited on page 7.)

Davis, P. and Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior*, 5:281–303. (Cited on page 11.)

Dawson, J. W. (2006). Why do mathematicians re-prove theorems? *Philosophia Mathematica*, 14. (Cited on pages 7 and 118.)

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, 24(17-24). (Cited on page 6.)

Duval, R. (1995). *Semiosis et pensee humaine*. Peter Lang. (Cited on page 10.)

Duval, R. (2000). Ecriture, raisonnment et devouverte de la demonstration en mathematiques. *Recherches en Didactique des Mathematiques*, 20:135–170. (Cited on page 116.)

Edwards, L., Radford, L., and Arzarello, F. (2009). Special Issue: Gestures and multimodality in the construction of mathematical meaning. *Educational Studies in Mathematics*, 70(2). (Cited on page 14.)

English, L. D., editor (1997). *Mathematical reasoning: Analogies, metaphors, and images*. Lawrence Erlbaum Associates. (Cited on page 13.)

Ernest, P. (2001). A semiotic perspective of mathematical activity. In *Proceedings of PME 26*. (Cited on page 10.)

Ernest, P. (2008). Towards a semiotics of mathematical text: Part 1, 2, and 3. *For the Learning of Mathematics*, 28(1, 2, 3). (Cited on page 10.)

Ferrara, F. (2003). Metaphors as vehicles of knowledge: An exploratory analysis. In Pateman, N., Dougherty, B., and Zilliox, J., editors, *Proceedings of PME 27*. (Cited on page 14.)

Frege, G. (1997). *The Frege Reader*. Blackwell. (Cited on page 2.)

Freudenthal, H. (1963). Enseignement des mathematiques modernes ou enseignement moderne des mathematiques? *L'Enseignement Mathematique*, 9:28–44. (Cited on page 112.)

Goldin-Meadow, S., Kim, S., and Singer, M. (1999). What the teachers' hands tell the students' minds about math. *Journal of Educational Psychology*, 91:720–730. (Cited on page 14.)

Goldin-Meadow, S., Nusbaum, H., Kelly, S. D., and Wagner, S. M. (2001). Explaining math: Gesturing lightens the load. *Psychological Science*, 12:516–522. (Cited on page 14.)

Gopnik, A. (2000). Explanation as orgasm and the drive for causal knowledge: the function, evolution, and phenomenology of the theory-formation system. In Keil, F. and Wilson, R., editors, *Explanation and Cognition*. MIT Press. (Cited on page 7.)

Hafner, J. and Mancosu, P. (2005). The varieties of mathematical explanation. In Mancosu, P., Jorgensen, K. F., and Pedersen, S. A., editors, *Visualization, Explanation and Reasoning Styles in Mathematics*, volume 327. Kluwer. (Cited on page 6.)

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange*, 21. (Cited on pages 2, 7, 11, 16, and 114.)

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, 44. (Cited on page 7.)

Hanna, G. and Jahnke, H. (1996). Proof and proving. In Bishop, A., editor, *International Handbook of Mathematics Education*, pages 877–908. Kluwer. (Cited on page 6.)

Hardy, N. (2009). *Students' models of the knowledge to be learned about limits in college level Calculus courses. The influence of routine tasks and the role played by institutional norms.* PhD thesis, Concordia University. (Cited on page 42.)

Healy, L. and Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, 31. (Cited on pages 10, 114, 116, and 117.)

Herschkowitz, R., Schwartz, B., and Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education*, 32:195–222. (Cited on page 114.)

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, 24:389–399. (Cited on page 7.)

Hersh, R. (1997). *What Is Mathematics, Really?* Oxford University Press. (Cited on page 1.)

Hoyles, C. (1996). Beyond the classroom: The curriculum as a key factor in students' approaches to proof. In Pothier, Y., editor, *Proceedings of the Annual Meeting of the Canadian Mathematics Education Study Group*, pages 7–25. (Cited on page 10.)

Hoyles, C. (1997). The curricular shaping of students' approach to proof. *For the Learning of Mathematics*, 17:7–16. (Cited on pages 10 and 114.)

Johnstone, P. (1983). The point of pointless topology. *Bulletin of the American matheamtical Society*, 8. (Cited on page 115.)

Keil, F. (2006). Explanation and understanding. *Annual Review of Psychology*, (57):227–254. (Cited on page 7.)

Kitcher, P. (1975). Bolzano's ideal of algebraic analysis. *Studies in the History and Philosophy of Science*, 6(229-269). (Cited on page 5.)

Kitcher, P. (1981). Explanatory unification. *Philosophy of Science*, 48:507–531. (Cited on pages 8, 17, and 114.)

Kitcher, P. (1984). *The Nature of Mathematical Knowledge*. Oxford University Press. (Cited on page 8.)

Kitcher, P. (1989). Explanatory unification and the causal structure of the world. In Kitcher, P. and Salmon, W., editors, *Scientific Explanation*, Minnesota Studies in the Philosophy of Science. University of Minnesota Press. (Cited on page 8.)

Lakoff, G. and Johnson, M. (1980). *Metaphors We Live By*. University of Chicago Press. (Cited on pages 13, 16, and 19.)

Lakoff, G. and Núñez, R. (2001). *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*. Basic Books. (Cited on pages 3, 13, 14, 15, 16, 19, 20, 21, 22, 23, 25, and 110.)

Lombrozo, T. (2006). The structure and function of explanations. *TRENDS in Cognitive Science*, 10(10). (Cited on page 7.)

Lombrozo, T. and Carey, S. (2006). Functional explanation and the function of explanation. *Cognition*, 99:167–204. (Cited on page 7.)

Luk, H. S. (2005). The gap between secondary school and university mathematics. *International Journal of Mathematical Education in Science and Technology*, 36(2):161–174. (Cited on page 22.)

MacNeill, D. (1992). *Hand and Mind: What Gestures Reveal About Thought*. Chicago University Press. (Cited on pages 14 and 115.)

Mancosu, P. (1996). *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*. Oxford University Press. (Cited on page 5.)

Mancosu, P. (2001). Mathematical explanation: Problems and prospects. *Topoi*, 20:97–117. (Cited on pages 1, 5, 6, 8, and 9.)

Mancosu, P., editor (2008b). *The Philosophy of Mathematical Practice*. Oxford University Press. (Cited on page 1.)

Mancosu, P. (Summer 2008a). Explanation in mathematics. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. (Cited on pages 1 and 5.)

Manin, Y. (2007). *Mathematics as Metaphor*. American Mathematical Society. (Cited on page 13.)

Martin, G. and Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education*, 20(1):41–51. (Cited on page 10.)

Nemirovsky, R., Borba, M., and DiMattia, C. (2004). PME Special Issue: Bodily activity and imagination in mathematics learning. *Educational Studies in Mathematics*, 57(3). (Cited on page 14.)

Nemirovsky, R. and Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics*, 70:159–174. (Cited on page 14.)

Núñez, R. (1998). What did Weierstrass really define? *Mathematical Cognition*, 4(2):85–101. (Cited on pages 13, 14, 20, 21, 93, and 113.)

Núñez, R. (2006). Do real numbers really move? language, thought, and gesture: The embodied cognitive foundations of mathematics. In Hersh, R., editor, *18 Unconventional Essays on the Nature of Mathematics*. Springer. (Cited on pages 14, 23, 110, and 115.)

Núñez, R. (2008). Conceptual metaphor and cognitive foundations of mathematics: Actual infinity and human imagination. Draft for future publication in the Journal of Pragmatics. (Cited on page 14.)

Núñez, R., Edwards, L., and Matos, J. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics*, 39:45–65. (Cited on page 14.)

Pimm, D. (1988). Mathematical metaphor. *For the Learning of Mathematics*, 8(1). (Cited on page 13.)

Presmeg, N. C. (1992). Prototypes, metaphors, metonymies, and imaginative rationality in high school mathematics. *Educational Studies in Mathematics*, 23:595–610. (Cited on page 13.)

Presmeg, N. C. (1997). Reasoning with metaphors and metonymies in mathematics learning. In English, L. D., editor, *Mathematical reasoning: Analogies, metaphors, and images*. Lawrence Erlbaum Associates. (Cited on page 13.)

Radford, L. (2001). On the relevance of semiotics in mathematics education. In *Discussion Group on Semiotics and Mathematics Education at the 25th PME International Conference*. (Cited on page 10.)

Rav, Y. (1999). Why do we prove theorems. *Philosophia Mathematica*, 7:5–41. (Cited on page 7.)

Rav, Y. (2007). A critique of a formalist-mechanist version of the justification of arguments in mathematician's proof practices. *Philosophia Mathematica*, 15:291–320. (Cited on page 116.)

Reid, D. (2002). Elements in accepting an explanation. *Journal of Mathematical Behavior*, 20. (Cited on pages 10 and 114.)

Reid, D. and Roberts, R. (2004). Adult learners' criteria for explanation. *ZDM*, 36. (Cited on pages 10, 34, 35, 37, 103, 105, 114, and 117.)

Resnik, M. and Kushner, D. (1987). Explanation, independence and realism in mathematics. *British Journal for the Philosophy of Science*, 38. (Cited on pages 9 and 117.)

Robert, A. (3). L'acquisition de la notion de convergence des suites numeriques dans l'enseignement superieur. *Recherches en Didactique des Mathematiques*, (307-341). (Cited on pages 11 and 37.)

Rotman, B. (2001). *Mathematics as Sign: Writing, Imagining, Counting*. Stanford University Press. (Cited on page 10.)

Salmon, w. (1990). *Four Decades of Scientific Explanation*. University of Minnesota Press. (Cited on pages 8 and 17.)

Sandborg, D. (1997). *Explanation in Mathematical Practice*. PhD thesis, University of Pittsburgh. (Cited on pages 6 and 37.)

Sandborg, D. (1998). Mathematical explanation and the theory of why-questions. *British Journal for the Philosophy of Science*, 49. (Cited on pages 7 and 8.)

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics*, 14:44–54. (Cited on page 13.)

Shapiro, L. (2007). The embodied cognition research programme. *Philosophy Compass*, 2/2. (Cited on page 14.)

Sierpinska, A. (1985). Obstacles epistemologiques relatifs a la notion de limite. *Recherches en Didactique des Mathematiques*, 6:5–67. (Cited on page 11.)

Sierpinska, A. (1988). Epistemological remarks on functions. In *Proceedings of the Twelth International Conference for the Psychology of Mathematics Education*, pages 568–575. Psychology of Mathematics Education. (Cited on page 112.)

Sierpinska, A. (1994). *Understanding in Mathematics*. Number 2 in Studies in Mathematics Education. Falmer. (Cited on pages 2, 7, 9, 11, 15, 16, 18, 24, 25, and 41.)

Sierpinska, A., Bobos, G., and Knipping, C. (2008). Sources of students' frustration in pre-university level, prerequisite mathematics courses. *Instructional Science*, 36:289–320. (Cited on pages 24, 25, 40, and 42.)

Steiner, M. (1978). Mathematical explanation. *Philosophical Studies*, 34. (Cited on pages 9, 11, 17, 114, and 117.)

Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In Grouws, D. A., editor, *Handbook for Research on Mathematics Teaching and Learning*, chapter 20, pages 495–511. MacMillan. (Cited on page 37.)

Tall, D. and Vinner, S. (1981). Concept image and concept definition in mathematics with particulare reference to limits and continuity. *Educational Studies in Mathematics*, pages 151–169. (Cited on pages 11, 12, 21, 35, 39, and 111.)

Thurston, W. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society*, 30:161–177. (Cited on pages 6, 19, and 118.)