Study of the coadjoint orbits of the Poincaré group in $2+1$ dimensions and their coherent states

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Abstract

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The first main objective of this thesis is to study the orbit structure of the $(2+1)$-Poincaré group $\mathbb{R}^{2,1} \rtimes SO(2,1)$ by obtaining an explicit expression for the coadjoint action. From there, we compute and classify the coadjoint orbits. We obtain a degenerate orbit, the upper and lower sheet of the two-sheet hyperboloid, the upper and lower cone and the one-sheet hyperboloid. They appear as two-dimensional coadjoint orbits and, with their cotangent planes, as four-dimensional coadjoint orbits. We also confirm a link between the four-dimensional coadjoint orbits and the orbits of the action of $SO(2,1)$ on the dual of $\mathbb{R}^{2,1}$.

The second main objective of this thesis is to use the information obtained about the structure to induce a representation and build the coherent states on two of the coadjoint orbits. We obtain coherent states on the hyperboloid for the principal section. The Galilean and the affine sections only allow us to get frames. On the cone, we obtain a family of coherent states for a generalized principal section and a frame for the basic section.

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Contents

List of symbols

- Poincaré group: $G = \mathbb{R}^{2,1} \rtimes SO(2,1)$
  - $v \in \mathbb{R}^{2,1}$
  - $\Lambda \in SO(2,1)$ (metric $\eta = \mathrm{diag}(+1,-1,-1)$)
  - Iwasawa decomposition: k a rotation, a a boost (or dilation) and n a translation
  - inner automorphism of $SO(2,1)$: $\hat{\Lambda} = m\Lambda^t m$, $m = \mathrm{diag}(1,1,-1)$
- Algebra of G: $\mathfrak{g}$, $\mathfrak{g} \ni X = (\alpha, \beta)$
  - generators of the algebra $so(2,1)$: J_0 rotation, J_1 and J_2 boosts, J_+ and J_- translations
  - α: three-vector made of coefficients of $J \in so(2,1)$
  - β: three-vector made of coefficients of $\mathbb{R}^{2,1}$
- Adjoint action matrix: $M(g)$
- Stabilizer of $X^* \in \mathfrak{g}^*$ under the coadjoint action (or little group): H_0
- Stabilizer of $k_0 \in \mathbb{R}^{(2,1)*}$ under $SO(2,1)$: S_0
- Coadjoint orbit (orbit of $X^* \in \mathfrak{g}^*$ under the coadjoint action of G): $\mathcal{O}_X$
- Orbit of $k_0 \in \mathbb{R}^{(2,1)*}$ under $SO(2,1)$: $\mathcal{O}^*$ (also denoted $\mathcal{V}_m^+$ in the hyperboloid case and $\mathcal{V}^+$ in the cone case)

- Cotangent bundle of the orbit $\mathcal{O}^*$: $T^*\mathcal{O}^*$
- Section: σ
- Unitary irreducible representation (UIR): U
- Unitary character: χ
- UIR of S_0: $L(s)$
- Action on the hyperboloid or the cone: Λ_q
- Parameter space: Γ ($\Gamma \simeq T^*\mathcal{O}^*$)
- Invariant measure on $\mathcal{O}^*$: $d\nu$
- Invariant measure on Γ: $d\mu$
- Point on the orbit: $k \in \mathcal{O}^*$
- Formal operator: A_σ
- Hilbert space: $\mathcal{H} = \mathbb{C} \otimes L^2(\mathcal{O}^*, d\nu)$

Introduction

In this thesis, we study the Poincaré group in $2 + 1$ dimensions through its coadjoint orbits structure. We first compute the coadjoint action and classify the coadjoint orbits. We then use the insight gained to build coherent states on two of those orbits, namely the upper sheet of the two-sheet hyperboloid and the upper cone.

Context

The Poincaré group is widely used in mathematical physics since it is the symmetry group of relativity. Its $3 + 1$-dimensional version is described and studied in great details by Kim and Noz [19], it is the most common in physics since our world is $3 + 1$-dimensional. The $2 + 1$-dimensional Poincaré group is studied by Gitman and Shelepin [16], they construct a UIR and obtain the orbits of the action of $SO(2,1)$ on $\mathbb{R}^{2,1}$ from work on wave equations.

Other people have studied the orbit structure of the Poincaré group. Those studies by Barut and Raczka [9] and also by Kim and Noz [19] were based on the momentum vector and performed in $3+1$ dimensions. Almorox and Prieto [6] have studied the action of $SO(2,1)$ on $\mathbb{R}^{2,1}$ by an analysis using the moment map. Here, we work with a matrix representation of the group and obtain directly those orbits by the action of the $SO(2,1)$ group matrices on some chosen vectors.

One of the main ingredients of our study is the coadjoint orbit structure of the group which is closely related to representations. This relation is outlined by Kirillov [20, 21, 22] who also stresses that the coadjoint orbits are symplectic objects. This is a useful property to define coordinates and an invariant measure on them.

In this thesis, since we base the study on a matrix representation of the group, we express the coadjoint action explicitly in this framework. This allows us to compute the coadjoint orbits along with the orbits coming from the action of $SO(2,1)$ on $\mathbb{R}^{2,1}$. We also link the two sets of orbits.

One of the tools needed to continue our research is the induced representation first introduced by Mackey [25] and developed by Kirillov [21] and others. The inducing technique allows us to obtain the representation of a group from the known representation of one of its subgroups. Thus, it is a good way to build new representations.

The representation of the Lorentz group (a subgroup of the Poincaré group) has been known for a long time from Wigner [30]. The representation of the Poincaré group is also well-known, we have already cited Kim and Noz [19] in 3+1 dimensions and Gitman and Shelepin [16] in $2+1$ dimensions.

In this thesis, the representation is obtained from the work on the coadjoint orbits. We are using the representation of the stabilizer as a starting point in the inducing technique.

Another important component of this thesis is the coherent states which were first developed in the context of quantum optics. Coherent states have been rapidly adopted and extended in other fields such as quantum mechanics and group theory. Perelomov [28] gives a mathematical development of coherent states for different groups and Klauder and Skagerstam [23] present a complete survey of the applications. In the group context, coherent states are obtained by the action of the unitary representation on an arbitrary set of vectors. This technique is used for example by Antoine and Mahara in [7], where they compute the orbits and coherent states of the affine Galilei group. The interest in coherent states lies in their applications, for example in signal and image processing [1, 8].

A lot of work has been done on the coherent states for the Poincaré group in 1+1 and 3+1 dimensions (see [1] and references therein). Here, we will focus on the $2+1$ dimensions case which is less covered in the literature. Gitman and Shelepin [16] obtain the coherent states for $SU(1,1)$ (which is isomorphic to $SO(2,1)$) in the framework of harmonic analysis. Some work is also done by de Bièvre [12] to get coherent states of semidirect product groups through orbits obtained by the moment map. Also, Bohnke [10] has constructed a tight frame on the forward cone using the $n+1$-dimensional Poincaré group with a dilation. In this work, we focus on our specific group and dimensions to perform the full computation in an explicit way and obtain the coherent states on two of the coadjoint orbits.

To summarize, the motivation of this work lies first in getting a very explicit approach to the orbits based on a matrix representation of the group, and in classifying the coadjoint orbits for the Poincaré group in 2+1 dimensions. It also consists in using the information obtained about the structure to induce a representation and build coherent states on two of those orbits.

The ultimate aim of this work would be to use the resulting coherent states in the wavelets framework. This is beyond the scope of this work, but we could apply our results to signal analysis following the principles presented in the books [11, 1, 8]. Quantization is another possible avenue. More details are given in the conclusion along with other possible openings.

Results

We first obtain a formula for the coadjoint action of the Poincaré group $G = \mathbb{R}^{2,1} \rtimes SO(2,1)$ given in a concrete matrix representation. We then compute explicitly the orbits of $SO(2,1)$ on $\mathbb{R}^{2,1}$ and the coadjoint orbits. We obtain a degenerate orbit, the upper and lower sheet of a two-sheet hyperboloid, the upper and lower cone and the one-sheet hyperboloid. The hyperboloids and the cones appear both as the first orbits and as two-dimensional coadjoint orbits. They also appear together with their cotangent space as four-dimensional coadjoint orbits. The orbits of $SO(2,1)$ on $\mathbb{R}^{2,1}$ were known in the literature, but we obtain them differently here. Moreover, we are able to link them to the four-dimensional coadjoint orbits.

The study of the coadjoint orbits allows us to get a deeper understanding of the geometry of the upper sheet of the two-sheet hyperboloid and the upper cone. We choose those two orbits because they are isomorphic to the plane. From the information obtained by computing the orbits, we are able to get a set of coordinates, an invariant measure and the induced representation of the hyperboloid and the cone. We finally compute the coherent states using this information. For the hyperboloid, we obtain nice coherent states for the principal section and frames for both the Galilean and the affine sections. For the cone, we obtain a family of coherent states for the generalized principal section and a frame for the basic section.

Organization of the thesis

Here is how the thesis is organized.

In Chapter 1, we describe the Poincaré group, its algebra and group matrix representation and we recall the Iwasawa decomposition. We also define the adjoint and coadjoint actions and then get the explicit formula to compute them in the

Poincaré case. We finally define the different orbits we are going to compute.

In Chapter 2, we use the actions defined in Chapter 1 to explicitly compute the orbit of $SO(2,1)$ on $\mathbb{R}^{2,1}$ and the coadjoint orbits arising from different initial vectors.

In Chapter 3, we give a definition of the coherent states and explain the technique that we will use to obtain them in the two next chapters.

In Chapter 4, we define coordinates and compute the invariant measure on the upper sheet of the two-sheet hyperboloid. We then get the induced representation and use it to compute the coherent states on the hyperboloid.

In Chapter 5, we go through the same process as in the previous chapter, but with the upper cone.

We finish with concluding remarks and possible openings.

Appendix A collects some basic definitions about group theory and induced representation. The other appendices contain the details of the computations needed to obtain the results.

Main contributions

The main contribution of this thesis is twofold. It first stands in the explicit computation of the coadjoint action formula for the Poincaré group. This formula is then used to compute the coadjoint orbits described in Chapter 2. Those coadjoint orbits are linked to the orbits of $SO(2,1)$ on $\mathbb{R}^{2,1}$ also computed explicitly. This explicit approach to coadjoint action based on a matrix representation of the group is new to our knowledge.

The second part of original work is the computation of the coherent states on the hyperboloid (Chapter 4) and the cone (Chapter 5) in this particular framework. The coadjoint orbits computed provide a set of coordinates and insight for the induced representation. This representation is then used to build the coherent states following a well-known technique given in [1].

Chapter 1

The $(2+1)$-Poincaré group

In this chapter, we give an extended description of the Poincaré group in $2+1$ dimensions, using its matrix representation. This will involve, among other things, studying the Lie algebra spanned by its generators, their commutation relations and the Iwasawa decomposition of the group.

We also define and compute the adjoint and coadjoint actions of the group, which will be used in Chapter 2 to compute its orbits under this action.

We finish by discussing the concept of orbits, giving the definition, the formula to compute the orbit from the group action, an isomorphism relating different orbits and the ones usually discussed in the literature (when constructing induced representations of the group).

In Appendix A.1 we collect together some basic definitions and notions from group theory.

1.1 Definition of the Poincaré group

The Poincaré group is the symmetry group of special relativity. Here, we are working in 2-space and 1-time dimensions, for which the $(2+1)$-Poincaré group is the group of all space-time symmetries.

In the present section, we describe the structure of the group, derive its Lie algebra and give the specific matrix representations of group elements and the basis of the algebra which will be used in the computation of the group actions. We also present the Iwasawa decomposition, which happens to be useful while dealing with the conical orbit (Section 2.3).

1.1.1 Description

Since we are working in a space-time of 2+1 dimensions, we write the coordinates of a point in this space-time as: (t, x, y). The Poincaré group of interest is the semidirect product, $G = \mathbb{R}^{2,1} \rtimes SO(2,1)$, an element of which can be written as:

$$G \ni g = \begin{pmatrix} \Lambda & v \\ 0 & 1 \end{pmatrix}, \tag{1.1}$$

where $v \in \mathbb{R}^{2,1}$ and $\Lambda \in SO(2,1)$, that is Λ is a 3×3 matrix such that $\Lambda \eta \Lambda^{\dagger} = \eta$, η being the metric, $\mathrm{diag}(+1, -1, -1)$. The product is given, using matrix multiplication, by:

$$g_1 g_2 = \begin{pmatrix} \Lambda_1 \Lambda_2 & \Lambda_1 v_2 + v_1 \\ 0 & 1 \end{pmatrix}. \tag{1.2}$$

The inverse of an element is thus:

$$g^{-1} = \begin{pmatrix} \Lambda^{-1} & -\Lambda^{-1} v \\ 0 & 1 \end{pmatrix}. \tag{1.3}$$

1.1.2 Matrix representation

The one-parameter subgroups of $SO(2,1)$ may be taken to be:

$$\Lambda_{J_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \Lambda_{J_1} = \begin{pmatrix} \cosh\beta & 0 & \sinh\beta \\ 0 & 1 & 0 \\ \sinh\beta & 0 & \cosh\beta \end{pmatrix}, \Lambda_{J_2} = \begin{pmatrix} \cosh\gamma & \sinh\gamma & 0 \\ \sinh\gamma & \cosh\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.4}$$

Their inverses are:

$$\Lambda_{J_0}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}, \Lambda_{J_1}^{-1} = \begin{pmatrix} \cosh\beta & 0 & -\sinh\beta \\ 0 & 1 & 0 \\ -\sinh\beta & 0 & \cosh\beta \end{pmatrix}, \Lambda_{J_2}^{-1} = \begin{pmatrix} \cosh\gamma & -\sinh\gamma & 0 \\ -\sinh\gamma & \cosh\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The six generators of the Poincaré algebra are given by:

$$J_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, J_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, J_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, P_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.5}$$

The J's are the $so(2,1)$ generators, obtained by taking the derivative of Λ_J at the identity. It is possible to write the J's and P's as 4×4 matrices following (1.1). This allows to compute the commutation relations among them:

$$\begin{aligned} &[J_0, J_1] = -J_2, \qquad &[J_0, J_2] = J_1, \qquad &[J_1, J_2] = J_0, \\ &[J_0, P_1] = P_2, \qquad &[J_0, P_2] = -P_1, \qquad &[J_1, P_0] = P_2, \\ &[J_1, P_2] = P_0, \qquad &[J_2, P_0] = P_1, \qquad &[J_2, P_1] = P_0, \end{aligned} \tag{1.6}$$

all other commutators being zero.

Remark 1.1.1 *Note that our choice of J_1 and J_2 is "unorthodox": they have been interchanged, from the way they are usually presented in the literature. This change will be helpful in the computation of the adjoint action, since it will allow us to use the standard scalar product notation.*

We also define $J_+ = J_0 + J_1$ and $J_- = J_0 - J_1$. They exponentiate to the following:

$$\Lambda_{J_+} = \begin{pmatrix} 1+\frac{u^2}{2} & \frac{u^2}{2} & u \\ -\frac{u^2}{2} & 1-\frac{u^2}{2} & -u \\ u & u & 1 \end{pmatrix}, \quad \Lambda_{J_-} = \begin{pmatrix} 1+\frac{v^2}{2} & \frac{-v^2}{2} & -v \\ \frac{v^2}{2} & 1-\frac{v^2}{2} & -v \\ -v & v & 1 \end{pmatrix}. \tag{1.7}$$

They are both translations. The commutation relations are:

$$\begin{aligned} &[J_+, J_0] = J_2, \quad [J_+, J_1] = -J_2, \quad [J_+, J_2] = J_+, \\ &[J_-, J_0] = -J_2, \quad [J_-, J_1] = -J_2, \quad [J_-, J_2] = -J_-, \\ &[J_+, J_-] = 2J_2. \end{aligned} \tag{1.8}$$

These matrices will be used for obtaining the conical orbit in Section 2.3.

1.1.3 (α, β) basis for the Lie algebra

Any element of the Lie algebra can be written as a matrix:

$$X = \begin{pmatrix} \alpha^t \cdot J & \beta \\ 0 & 0 \end{pmatrix}, \tag{1.9}$$

where α and β are three-column vectors, J is the vector $(J_0, J_1, J_2)^t$ and the product $\cdot$ is just the linear combination: $\alpha^t \cdot J = \alpha_0 J_0 + \alpha_1 J_1 + \alpha_2 J_2$. To compute the action of the group on the Lie algebra, it will be easier to work in terms of the six parameters α and β. We rewrite the element X as a column vector: $X = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

Elements in the dual of the Lie algebra will then be written as row vectors:

$$X = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow X^* = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}, \tag{1.10}$$

where α^* and β^* are themselves three-dimensional row vectors. The dual pairing in this notation is simply the scalar product, there is no need for using the trace and we choose not to use the metric in this setting.

1.1.4 Iwasawa decomposition

We present the Iwasawa decomposition of the $SO(2,1)$ group which will also be used in the case of the conical orbit (Section 2.3).

We write an element of $SO(2,1)$ as the product of three elements: $g = kan$, where k is a rotation which corresponds to Λ_{J_0}, a is a boost (or dilation) which corresponds to Λ_{J_2} and n is a translation which corresponds to $\Lambda_{J_\pm}$ depending on the case. Instead of being J_0, J_1 and J_2, the generators of the algebra are now taken to be J_0, J_2 and $J_0 \pm J_1$. This basis still generates a three-parameter group.

1.2 Group actions

We define and compute the adjoint action of the Poincaré group on its algebra as well as the coadjoint action.

1.2.1 Definition of the adjoint and coadjoint actions

The *adjoint action* is the action of a group G on its Lie algebra. For a matrix group, the adjoint action is defined by:

$$Ad(g)X = gXg^{-1}, \tag{1.11}$$

where $g \in G$ and $X \in \mathfrak{g}$. The *coadjoint action* is the action of the group on the dual $\mathfrak{g}^*$ of its Lie algebra, when the latter is considered as a vector space. Generally, the coadjoint action (denoted $Ad^\#$) is defined as in:

$$< Ad^\#(g)X_1^*, X_2 >=< X_1^*, Ad(g^{-1})X_2 >, \tag{1.12}$$

where $X_1^* \in \mathfrak{g}^*$.

1.2.2 Adjoint action

Using the definition (1.11) and the algebra element in the (α, β) basis (1.9), the adjoint action of the group G on its algebra is given by:

$$\begin{aligned} Ad(g)X &= gXg^{-1} \\ &= \begin{pmatrix} \Lambda\alpha^t \cdot J\Lambda^{-1} & -\Lambda\alpha^t \cdot J\Lambda^{-1}v + \Lambda\beta \\ 0 & 0 \end{pmatrix} \\ &\equiv \begin{pmatrix} \alpha'^t \cdot J & \beta' \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{1.13}$$

We compute the part $\Lambda\alpha^t \cdot J\Lambda^{-1}$ for a generic element of the group $\Lambda = \Lambda_{J_0}\Lambda_{J_1}\Lambda_{J_2}$ in order to *extract* the action as a linear combination of the J's. After a few manipulations, we get:

$$\Lambda\alpha^t \cdot J\Lambda^{-1} = (m(\Lambda^{-1})^t m\alpha)^t \cdot J, \tag{1.14}$$

where the matrix m is:

$$m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{1.15}$$

The details of this manipulation are given in Appendix B.1. Note it is because of this computation that the unorthodox definition of J_1 and J_2 is needed (see Remark 1.1.1). For convenience, we set the following notation: $m(\Lambda^{-1})^t m \equiv \hat{\Lambda}^{-1}$ and, then, $\hat{\Lambda} = m\Lambda^t m$. This is an inner automorphism of the group.

Remark 1.2.1 *By computing $\hat{\Lambda} = m\Lambda^t m$ for the three one-parameter subgroups, we get the following:*

$$\hat{\Lambda}_{J_0} = m\Lambda^t_{J_0}m = \Lambda_{J_0}, \; \hat{\Lambda}_{J_1} = m\Lambda^t_{J_1}m = \Lambda_{-J_1}, \; \hat{\Lambda}_{J_2} = m\Lambda^t_{J_2}m = \Lambda_{J_2}. \tag{1.16}$$

We can also work out the following by direct computation:

$$-\Lambda\alpha^t \cdot J\Lambda^{-1}v = -(J\cdot v)\hat{\Lambda}^{-1}\alpha, \tag{1.17}$$

where $J\cdot v$ is the matrix (J_0v, J_1v, J_2v), recalling that v is three-column vector. We write it down explicitly here for later use:

$$J\cdot v = \begin{pmatrix} 0 & v_2 & v_1 \\ -v_2 & 0 & v_0 \\ v_1 & v_0 & 0 \end{pmatrix}. \tag{1.18}$$

The details of the computation are presented in Appendix B.2.

We have thus obtained the transformation of the parameters α and β:

$$\begin{aligned} \alpha'\cdot J &= \Lambda\alpha\cdot J\Lambda^{-1} \\ &= \hat{\Lambda}^{-1}\alpha\cdot J, \\ \beta' &= -\Lambda\alpha\cdot J\Lambda^{-1}v + \Lambda\beta \\ &= \Lambda\beta - (J\cdot v)\hat{\Lambda}^{-1}\alpha. \end{aligned}$$

We can rewrite the transformation as a 6×6 matrix $M(g)$:

$$\begin{pmatrix}\alpha' \\ \beta'\end{pmatrix} = M(g)\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \equiv \begin{pmatrix}\hat{\Lambda}^{-1} & 0 \\ -(J\cdot v)\hat{\Lambda}^{-1} & \Lambda\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}. \tag{1.19}$$

The adjoint action of $g=(\Lambda, v)$ on $X=(\alpha,\beta)^t$ is then written in a matrix form:

$$Ad(g)X = M(g)X.$$

1.2.3 Coadjoint action

We now define and compute the coadjoint action in the six-parameter space of α^* and β^*. In this notation, equation (1.12) reads:

$$< X_1^*, Ad(g^{-1})X_2 > = X_1^* M(g^{-1})X_2 = < Ad^{\#}(g)X_1^*, X_2 >, \tag{1.20}$$

where $M(g)$ is defined in (1.19). The coadjoint action is then:

$$Ad^{\#}(g)X^* = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} M(g^{-1}). \tag{1.21}$$

The matrix $M(g^{-1})$ is easily obtained from $M(g)$ using $g^{-1} = (\Lambda^{-1}, -\Lambda^{-1}v)$ as in (1.3):

$$M(g^{-1}) = \begin{pmatrix} \hat{\Lambda} & 0 \\ (J \cdot (\Lambda^{-1}v))\hat{\Lambda} & \Lambda^{-1} \end{pmatrix}. \tag{1.22}$$

We also compute the inverse of the matrix $M(g)$:

$$M(g)^{-1} = \begin{pmatrix} \hat{\Lambda} & 0 \\ \Lambda^{-1}(J \cdot v) & \Lambda^{-1} \end{pmatrix}. \tag{1.23}$$

By direct computation for the different one-parameter subgroups Λ_{J_0}, Λ_{J_1} and Λ_{J_2} (which is sufficient since $SO(2,1)$ is a three-parameter group), we get that:

$$(J \cdot (\Lambda^{-1}v))\hat{\Lambda} = \Lambda^{-1}(J \cdot v). \tag{1.24}$$

Then, $M(g^{-1})$ and $M(g)^{-1}$ are the same as expected. We will use $M(g)^{-1}$ to define the coadjoint action. This gives:

$$\boxed{Ad^{\#}(g)X^* = \begin{pmatrix}\alpha^*\hat{\Lambda} + \beta^*\Lambda^{-1}(J \cdot v) & \beta^*\Lambda^{-1}\end{pmatrix},} \tag{1.25}$$

where Λ and v are fixed by the choice of g, an element of the group. The choice of X^* varies for the different cases.

1.3 Orbits

We define here the two types of orbits studied in the following. We give the explicit formula to obtain one of them for the Poincaré group. We also give an isomorphism relating these two types of orbits. We finish by presenting the orbits already known in the literature.

1.3.1 Definitions of the orbits

The theoretical basis and fundamental definitions regarding orbits are given in Appendix A.1.2. We define and describe here the two particular types of orbits studied in this thesis.

The first type is the orbit obtained from the action of $SO(2,1)$ on $\mathbb{R}^{(2,1)*}$. Generally, in the semidirect product group setting where $G = V \rtimes S$, it would be the orbit coming from the action of S on the dual of V. Those orbits are used to obtain representations in the induced representation method. We thus call them *representation generating orbits* in the following.

The second kind of orbit is obtained from the action of G on the dual of its algebra ($\mathfrak{g}^*$), that is under the coadjoint action. It is then called the *coadjoint orbit.* Coadjoint orbits are always even dimensional and have the structure of symplectic manifolds. In this sense, they model classical phase spaces, a characteristic which is useful also for defining coordinates.

1.3.2 Formula for the representation generating orbit

We have just defined the representation generating orbit. Here is how we will compute it for the Poincaré group.

To get the action of the $SO(2,1)$ part of the group on the dual of $\mathbb{R}^{2,1}$, we just multiply the row-vector $X^* = \begin{pmatrix}\gamma_0 & \gamma_1 & \gamma_2\end{pmatrix}$ by the subgroup matrices. Here is the result:

$$\begin{aligned}
X^*\Lambda_{J_0} &= \begin{pmatrix}\gamma_0 & \gamma_1\cos\alpha + \gamma_2\sin\alpha & -\gamma_1\sin\alpha + \gamma_2\cos\alpha\end{pmatrix}, \\
X^*\Lambda_{J_1} &= \begin{pmatrix}\gamma_0\cosh\beta + \gamma_2\sinh\beta & \gamma_1 & \gamma_0\sinh\beta + \gamma_2\cosh\beta\end{pmatrix}, \\
X^*\Lambda_{J_2} &= \begin{pmatrix}\gamma_0\cosh\gamma + \gamma_1\sinh\gamma & \gamma_0\sinh\gamma + \gamma_1\cosh\gamma & \gamma_2\end{pmatrix}.
\end{aligned} \tag{1.26}$$

The vector X^* will be specified for different cases in the next chapter.

1.3.3 Isomorphism

In the case of a semidirect product group, there is an isomorphism relating some coadjoint orbits and the orbits of the action of S on V^* which is given in [1]. It is based on the following definitions:

- the group $G = V \rtimes S$, V a vector space, $S \subset GL(V)$;
- H_0 stabilizer of $(k_0, 0) \in \mathfrak{g}^*$ under coadjoint action, $k_0 \in V^*$;
- $\mathcal{O}_{(k_0,0)}$ orbit of $(k_0, 0) \in \mathfrak{g}^*$ under the coadjoint action of G;
- $T^*\mathcal{O}^*$ cotangent bundle of the orbit of k_0 in V^* under S.

The equation 10.49 in [1] gives the following sequence of isomorphisms:

$$\Gamma = G/H_0 \simeq \mathcal{O}_{(0,k_0)} \simeq T^*\mathcal{O}^*. \tag{1.27}$$

In Chapter 2, we will explicitly check this isomorphism for some particular vectors k_0. We will see that $\mathcal{O}_{(0,k_0)}$ are the four-dimensional orbits.

1.3.4 Orbits studied in the literature

One last thing we would like to mention about the Poincaré group is that the representation generating orbits are well-known in $3+1$ dimensions. In Barut and Raczka [9] and also in Kim and Noz [19], they obtain the orbits for the action of $SO(3,1)$ on $\mathbb{R}^{3,1}$ from the momentum vector $p^2 = -m^2 = -p_0^2 + p_1^2 + p_2^2 + p_3^2$. This gives:

- a one-sheet hyperboloid for $m^2 < 0$;
- a cone for $m^2 = 0$, upper and lower for $p_0 > 0$, $p_0 < 0$;
- the origin (degenerate orbit) for $m^2 = 0$, $p_0 = 0$;

- a two-sheet hyperboloid for $m^2 > 0$, upper and lower for $p_0 > 0$, $p_0 < 0$.

Note that they are three-dimensional objects in this case.

Almorox and Prieto [6] have obtained similar two-dimensional orbits for the action of $SO(2,1)$ on $\mathbb{R}^{2,1}$ from an analysis using the moment map.

In the next chapter, we will perform those computations using a different, very explicit, approach. Moreover, we will compute the coadjoint orbits and link them to those representation generating orbits.

We have now every tool needed to work out the orbits. That is a precise definition of the group and a useful way of writing the adjoint and coadjoint actions for the computations.

Chapter 2

Computation of the different orbits

In this chapter, we compute explicitly the representation generating orbit and coadjoint orbits. We use different three-vectors k_0 as initial vectors of $\mathbb{R}^{2,1}$ for the orbits under the action of $SO(2,1)$ and both $\begin{pmatrix} 0 & k_0 \end{pmatrix}$ and $\begin{pmatrix} k_0 & 0 \end{pmatrix}$ six-vectors for the coadjoint action. We will see that those orbits are linked to each other. We also provide a graphical representation of the orbits.

We use the formula obtained in (1.25) for the coadjoint action and the bijection between the orbit of a point and the quotient of the group G by the stabilizer of this point described in Appendix A.1.2 to obtain the coadjoint orbits. We also check that the isomorphism (1.27) is actually verified in each of our cases.

Note that the details of the computations are presented in Appendix C. We present here the results and the discussion.

2.1 Degenerate orbit

The first case is simple. We start with the initial vector $k_0 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$. Since the vectors $\begin{pmatrix} 0 & k_0 \end{pmatrix}$ and $\begin{pmatrix} k_0 & 0 \end{pmatrix}$ are the same, we only have one coadjoint orbit.

2.1.1 Representation generating orbit

Using the set of equations (1.26) with $X^* = k_0 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$, we simply obtain the origin, that is a degenerate orbit.

2.1.2 Coadjoint orbit

As mentioned above, the fixed vector is the same for the cases $\begin{pmatrix} 0 & k_0 \end{pmatrix}$ and $\begin{pmatrix} k_0 & 0 \end{pmatrix}$, that is: $\begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. The orbit we obtain is again degenerate; it is only the point at the origin. In this case, the stabilizer is the whole group G.

2.1.3 Isomorphism

We study here the isomorphism (1.27) for the case $k_0 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$. We have the following objects:

- the stabilizer H_0 is G;
- the quotient G/H_0 is the origin;
- the representation generating orbit $\mathcal{O}^*$ is the origin;
- its cotangent bundle $T^*\mathcal{O}^*$ is trivial.

We can see that the coadjoint orbit $\Gamma = G/H_0$ is isomorphic to the cotangent bundle of the representation generating orbit, they are both the origin.

This case is not very interesting, but it is still part of the complete classification. We now move to non-trivial situations.

2.2 Two-sheet hyperboloid orbit

We study now the orbits emerging from the initial vector $k_0 = \begin{pmatrix} \pm m & 0 & 0 \end{pmatrix}$, where $m > 0$ is the mass.

2.2.1 Representation generating orbit

We plug in the vector $X^* = k_0 = \begin{pmatrix} \pm m & 0 & 0 \end{pmatrix}$ in the equation (1.26) for the representation generating orbit. The details of the computation are shown in Appendix C.1.

For $k_0 = \begin{pmatrix} m & 0 & 0 \end{pmatrix}$, we get the upper sheet of the two-sheet hyperboloid with its vertex at $q_0 = m$. For $k_0 = \begin{pmatrix} -m & 0 & 0 \end{pmatrix}$, we get the lower sheet of the two-sheet hyperboloid with its vertex at $q_0 = -m$. They are presented in Figure 2.1.

2.2.2 Four-dimensional coadjoint orbit

We now compute the coadjoint orbit from the coadjoint action given by (1.25). We fix the vector $\begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} = \begin{pmatrix} 0 & k_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \pm m & 0 & 0 \end{pmatrix}$. The explicit computation appears in Appendix C.1.

The stabilizer H_0 is the rotation and the time translation. The quotient of the group by the stabilizer leaves the two boosts and the two space translations to generate the coadjoint orbit. We thus have the upper or lower sheet of the two-sheet hyperboloid (depending on the sign in $\pm m$) with the space plane, that is a four-dimensional coadjoint orbit.

The equation of the hyperboloid is $q_0^2 - q_1^2 - q_2^2 = m^2$. It is the same hyperboloid as the orbit shown in Figure 2.1. The hyperboloids have their vertex at $\pm m$ and for any mass they have the same cone as an asymptote.

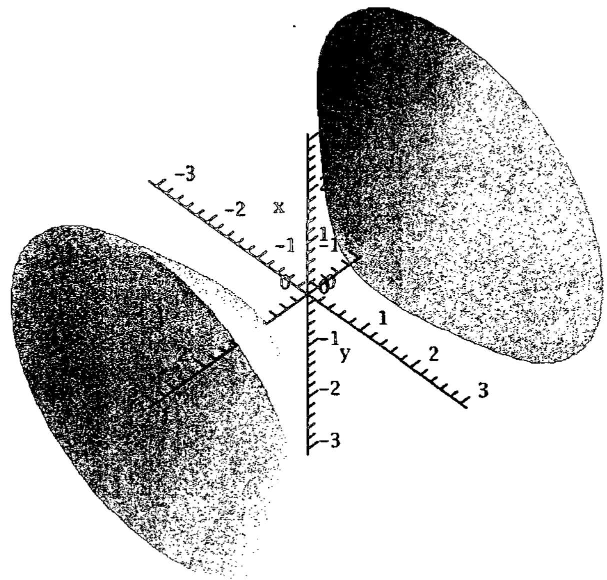

Figure 2.1: Two-sheet hyperboloid with $m = 1$

2.2.3 Two-dimensional coadjoint orbit

We compute again the coadjoint orbit from the coadjoint action given by (1.25), but we use the fixed vector: $\begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} = \begin{pmatrix}k_0 & 0\end{pmatrix} = \begin{pmatrix}\pm m & 0 & 0 & 0 & 0 & 0\end{pmatrix}$ instead. The details are in Appendix C.1.

The stabilizer is made of the rotation and the three translations. We are left only with the two boosts to generate the coadjoint orbit. We thus get the upper and lower sheet of a two-sheet hyperboloid with m and $-m$ respectively. This is different than the previous case since we are left with a two-dimensional structure instead of a four-dimensional one.

2.2.4 Isomorphism

We try to link the representation generating orbit and the four-dimensional coadjoint orbit through the isomorphim (1.27).

- The stabilizer H_0 is Λ_{J_0}, v_0;

- the quotient G/H_0 gives one sheet of the two-sheet hyperboloid with the xy-plane;

- the representation generating orbit $\mathcal{O}^*$ is one sheet of the two-sheet hyperboloid;

- its cotangent bundle $T^*\mathcal{O}^*$ is then a sheet of hyperboloid with a cotangent plane.

Since we can easily map a plane to another plane, we can see that the four-dimensional coadjoint orbit is isomorphic to the cotangent bundle of the representation generating orbit.

2.3 Cone orbit

In this section, we study the orbits arising from the initial vector $k_0 = \begin{pmatrix} \pm 1 & 1 & 0 \end{pmatrix}$.

2.3.1 Representation generating orbit

Once again, we use equation (1.26) with the vector $X^* = k_0 = \begin{pmatrix} \pm 1 & 1 & 0 \end{pmatrix}$ this time. The computation is given explicitly in Appendix C.2.

We obtain the upper cone for the vector $k_0 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$ and the lower cone for $k_0 = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}$. They appear in Figure 2.2.

2.3.2 Four-dimensional coadjoint orbit

We now compute the coadjoint orbit for the vector $\begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} = \begin{pmatrix} 0 & k_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \pm 1 & 1 & 0 \end{pmatrix}$. The computations are in Appendix C.2 and make use of the Iwasawa decomposition of $SO(2,1)$ presented in Section 1.1.4. This decomposition is a more natural basis for this particular vector.

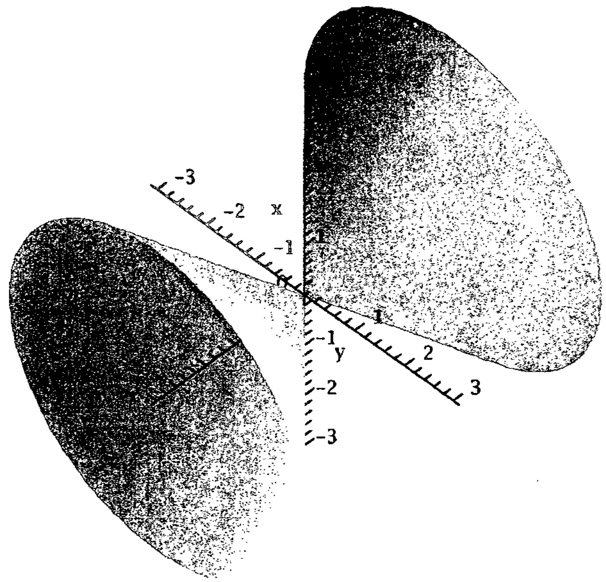

Figure 2.2: Cone

The stabilizer H_0 is made of a translation $\Lambda_{J_\pm}$ (or n in the Iwasawa decomposition) and the vector $x = \mp t$. The orbit is then generated by a rotation and a boost which gives the cone with the plane generated by the y-axis and the axis $x = \pm t$. This is a four-dimensional orbit.

The cone equation is $q_0^2 - q_1^2 - q_2^2 = 0$. It is the upper cone for $q_0 > 0$ and the lower one for $q_0 < 0$. It is the same as the cone shown in Figure 2.2. This cone is actually the limiting cone of the two-sheet hyperboloid in the massless limit.

2.3.3 Two-dimensional coadjoint orbit

If we start with the vector $\begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} = \begin{pmatrix}k_0 & 0\end{pmatrix} = \begin{pmatrix}\pm 1 & 1 & 0 & 0 & 0 & 0\end{pmatrix}$ instead, we get a two-dimensional orbit. The details are in Appendix C.2.

The initial vector in this case is stabilized by the n translation of $SO(2,1)$ and also by all of the $\mathbb{R}^{2,1}$ translations. The quotient thus leaves a rotation and a boost to generate the orbit, a two-dimensional cone.

2.3.4 Isomorphism

We again have a look at the isomorphism (1.27).

- The stabilizer H_0 is the translation n and the vector $x = \mp t$;
- the quotient G/H_0 is the cone and the plane generated by the y-axis and the axis $x = \pm t$;
- the representation generating orbit $\mathcal{O}^*$ is the cone;
- its cotangent bundle $T^*\mathcal{O}^*$ is the cone with a cotangent plane.

Here also, we can map planes to each other to show that our four-dimensional coadjoint orbit is isomorphic to the cotangent bundle of the representation generating orbit.

2.4 One-sheet hyperboloid orbit

We now present the study of the orbits originating from the initial vector $k_0 = \begin{pmatrix} 0 & m & 0 \end{pmatrix}$.

2.4.1 Representation generating orbit

We use the equation (1.26) with the vector $X^* = k_0 = \begin{pmatrix} 0 & m & 0 \end{pmatrix}$ to obtain the representation generating orbit. The detailed computation is in Appendix C.3.

We obtain the one-sheet hyperboloid cutting the space plane at the circle of radius m. It is shown in Figure 2.3.

2.4.2 Four-dimensional coadjoint orbit

Using the fixed vector $\begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} = \begin{pmatrix} 0 & k_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & m & 0 \end{pmatrix}$, we compute the coadjoint orbit in Appendix C.3

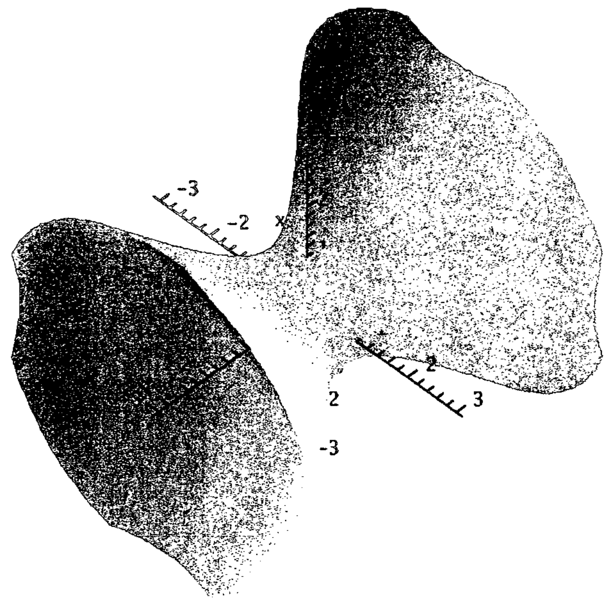

Figure 2.3: One-sheet hyperboloid with $m = 1$

The stabilizer is the boost in the x-direction (Λ_{J_1}) and the x-translation (v_1). The orbit is then generated by the rotation and the y-boost to which we add the t- and y-translations. Geometrically, this is the one-sheet hyperboloid with the ty-plane.

The equation of the one-sheet hyperboloid is $q_0^2 - q_1^2 - q_2^2 = -m^2$. It is the same as the one in Figure 2.3.

We remark that the orbit generators are the same as for the cone, but they are not acting on the same vector, thus giving a different orbit.

2.4.3 Two-dimensional coadjoint orbit

We now use the fixed vector $X_0^* = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} = \begin{pmatrix}k_0 & 0\end{pmatrix} = \begin{pmatrix}0 & m & 0 & 0 & 0 & 0\end{pmatrix}$ to get a coadjoint orbit. The details are in Appendix C.3.

The stabilizer for this vector is the boost in the x-direction and the three $\mathbb{R}^{2,1}$ translations. We thus obtain the rotation and the y-boost as the orbit generators. Geometrically, we can see it as the one-sheet hyperboloid. This is a two-dimensional structure.

2.4.4 Isomorphism

We check the isomorphism (1.27) for the vector $k_0 = \begin{pmatrix} 0 & m & 0 \end{pmatrix}$.

- The stabilizer H_0 is the boost Λ_{J_1} and the translation v_1;
- the quotient G/H_0 is the rotation Λ_{J_0} and the boost Λ_{J_2} with the translations v_0 and v_2;
- the representation generating orbit $\mathcal{O}^*$ is the one-sheet hyperboloid;
- its cotangent bundle $T^*\mathcal{O}^*$ is the one-sheet hyperboloid with its cotangent plane.

The isomorphism between our four-dimensional coadjoint orbit and the cotangent bundle of the representation generating orbit is again verified.

2.5 Summary

We present a summary of the coadjoint orbits obtained in Table 2.1. It is interesting to remark that both the one-sheet and two-sheet hyperboloid have the cone orbit as an asymptote.

The initial vectors k_0 that we have used cover all the cases; that is a purely time, a purely space and a mixed time-space initial vector. We retrieve the same orbits presented in the literature and given in Section 1.3.4.

Now let us examine the isomorphism (1.27) studied in details in Sections 2.1.3, 2.2.4, 2.3.4 and 2.4.4:

$$\Gamma = G/H_0 \underbrace{\simeq}_{1} \mathcal{O}_{(0,k_0)} \underbrace{\simeq}_{2} T^*\mathcal{O}^*.$$

Fixed vector	Stabilizer	Orbit generators	Geometry
$\begin{pmatrix}0, & 0\end{pmatrix}$	whole G	nothing	origin (degenerate)
$\begin{pmatrix}k_0, & 0\end{pmatrix}$ $k_0 = \begin{pmatrix}\pm m & 0 & 0\end{pmatrix}$	Λ_{J_0} v_0, v_1, v_2	$\Lambda_{J_1}, \Lambda_{J_2}$	2-sheet hyp. [upper (+) and lower (-)]
$\begin{pmatrix}0, & k_0\end{pmatrix}$ $k_0 = \begin{pmatrix}\pm m & 0 & 0\end{pmatrix}$	Λ_{J_0} v_0	$\Lambda_{J_1}, \Lambda_{J_2}$ v_1, v_2	2-sheet hyp. + xy-plane [upper (+) and lower (-)]
$\begin{pmatrix}k_0, & 0\end{pmatrix}$ $k_0 = \begin{pmatrix}\pm 1 & 1 & 0\end{pmatrix}$	$\Lambda_{J_\pm}$ v_0, v_1, v_2	$\Lambda_{J_0}, \Lambda_{J_2}$	cone [upper (+) and lower (-)]
$\begin{pmatrix}0, & k_0\end{pmatrix}$ $k_0 = \begin{pmatrix}\pm 1 & 1 & 0\end{pmatrix}$	$\Lambda_{J_\pm}$ $x = \mp t$	$\Lambda_{J_0}, \Lambda_{J_2}$ $x = \pm t, v_2$	cone + plane [upper (+) and lower (-)]
$\begin{pmatrix}k_0, & 0\end{pmatrix}$ $k_0 = \begin{pmatrix}0 & m & 0\end{pmatrix}$	Λ_{J_1} v_0, v_1, v_2	$\Lambda_{J_0}, \Lambda_{J_2}$	1-sheet hyp.
$\begin{pmatrix}0, & k_0\end{pmatrix}$ $k_0 = \begin{pmatrix}0 & m & 0\end{pmatrix}$	Λ_{J_1} v_1	$\Lambda_{J_0}, \Lambda_{J_2}$ v_0, v_2	1-sheet hyp. + ty-plane

Table 2.1: Coadjoint orbits of the group $G = \mathbb{R}^{2,1} \rtimes SO(2,1)$

The first isomorphism (1) has been used to compute the coadjoint orbit. The other isomorphism (2) links the four-dimensional coadjoint orbit and the cotangent bundle of the representation generating orbit. We have seen that we can connect planes together to confirm that this isomorphism is verified.

This ends the first big segment and main contribution of this thesis. We have obtained different coadjoint orbits for the Poincaré group and linked them with the representation generating orbit.

Chapter 3

Coherent states: definition and method

Besides the orbits, the coherent states are the important objects in this thesis. We give here a short introduction before explaining the technique that will be used to obtain them in Sections 4.4 and 5.4.

3.1 Introduction and background definitions

A good introductory book about coherent states and their applications is by Klauder and Skagerstam [23], Perelomov [28] also gives a good mathematical introduction to the topic.

Coherent states are basically continuously labeled quantum states. They form an overcomplete family of vectors in the Hilbert space, such that an arbitrary vector can be written as a (possibly infinite) sum of coherent states.

The notion of coherent states was introduced to study the link between classical and quantum mechanics. They have been named in the context of quantum optics, but quickly spread to other fields such as condensed matter physics, atomic physics, nuclear physics and mathematical physics (for example in harmonic analysis and quantization theory).

The canonical coherent states have some specific properties. For example, they obey the minimal uncertainty property, that is they saturate Heisenberg's inequality: $<\Delta Q><\Delta P>=\frac{\hbar}{2}$ instead of $<\Delta Q><\Delta P>\geq\frac{\hbar}{2}$.

Another fundamental property of canonical coherent states η is that they satisfy the resolution of the identity:

$$\int|\eta><\eta|=I. \tag{3.1}$$

This is the property that we will check for the states created in Chapters 4 and 5.

It is possible to obtain generalizations of the canonical coherent states.

Definition 3.1.1 *We start from a separable Hilbert space $\mathfrak{h}$, a locally compact space X and a measure on X, $d\nu$. We define the vectors $|x>\in\mathfrak{h}$. We assume the following properties:*

- *The mapping $x\mapsto|x>$ is weakly continuous (for each $|\phi>\in\mathfrak{h}$, $<x|\phi>$ is continuous in the topology of X).*
- *The resolution of the identity: $\int_X|x><x|d\nu(x)=I_{\mathfrak{h}}$ holds in the weak sense. That is, for any $|\phi>,|\psi>\in\mathfrak{h}$, we have:*

$$\int_X<\phi|x><x|\psi>d\nu(x)=<\phi|\psi>.$$

The vectors $|x>$ satisfying these properties form a family of generalized coherent states. *If $I_{\mathfrak{h}}$ is replaced by a bounded operator with a bounded inverse, we obtain a* frame.

One way of building canonical coherent states is by acting on a fixed vector with the unitary square-integrable representation of a locally compact group $|g>=U(g)|\psi>$. A generalization of this technique involve working on a homogeneous space of the group. This is how we are going to define our coherent states in the next chapters. This technique is widely used in the literature, see [1, 3, 7] for example. It is explained in details in Section 3.2.

The interest of coherent states for mathematical physicists lies in their numerous applications. For example, they are used in signal and image processing through the wavelet framework, see [1, 8]. They are also used in quantization, see [2, 4, 15].

3.2 Getting coherent states for a semidirect product group

We present here the method that will be used in Chapters 4 and 5 to obtain the coherent states. We actually follow the technique described in [1], §10.3. This technique is used to build coherent states on homogeneous spaces $Y = G/H$. This is our case, since coadjoint orbits have been defined as a quotient $(\mathcal{O}_X = G/H_0)$. This method of building coherent states actually generalizes the one described by Perelomov [28].

We need a Hilbert space and suitable coordinates on our structure, as well as an invariant measure; it is $\mathcal{H} = \mathbb{C} \otimes L^2(\mathcal{O}^*, d\nu)$. We first get a UIR from the little group representation using the induced representation method, this is described in Appendix A.2. We check if this representation is square-integrable. If it is not (which will be the case), we write the coset decomposition of the group and take a quotient. Here, the coadjoint orbit structure will be helpful to write the decomposition. In order to undo this quotient, we define a suitable section: $\sigma : \mathcal{O}_X \to G$. This is a key step and this choice can lead to coherent states, frames or no coherent states at all.

Once we have all these ingredients, we take a set of vectors η in the Hilbert space and act with the UIR on it. From this new set of vectors η_σ, we define a formal operator $A_\sigma = \int_\Gamma |\eta_\sigma >< \eta_\sigma| d\mu$. We then integrate the formal operator in order to check the resolution of the identity. The integral is written $I_{\phi,\psi} =< \phi|A_\sigma \psi >$, where $\phi, \psi \in \mathcal{H}$. If we have the resolution of the identity, that is $I_{\phi,\psi} =< \phi|\psi >$, then we define the coherent states as this set of vectors transformed by the UIR η_σ and, possibly, normalized.

If the section chosen does not allow us to achieve the resolution of the identity, we can still find some bounds on the operator A_σ and get frames. Instead of:

$$\int < \phi|\eta >< \eta|\psi >=< \phi|\psi >,$$

we would have:

$$0 < A < \phi|\psi > \leq \int < \phi|\eta >< \eta|\psi > \leq B < \phi|\psi > < \infty.$$

That means that the operator and its inverse are bounded. If $A = B$, that is the resolution of the identity is satisfied, then we have a tight frame.

We now have all the necessary definitions and an efficient method to compute the coherent states in the next two chapters.

Chapter 4

Coherent states on the upper sheet of the two-sheet hyperboloid

In this chapter, we first define a set of coordinates and an invariant measure on the upper sheet of the two-sheet hyperboloid. We then compute the induced representation and describe a set of sections that are used to finally compute the coherent states as described in Section 3.2.

We obtain the coherent states for the principal section in 4.4. For the Galilean section in 4.5, we can get bounds using a special form of η, we thus have a frame. The affine section in 4.6 also leads to a frame if we have a special θ in the section and some restrictions on η.

From now on, we call hyperboloid the upper sheet of the two-sheet hyperboloid as obtained in Section 2.2.

4.1 Coordinates and measure

We first define a set of coordinates using the symplectic structure of the four-dimensional orbit. The hyperboloid is seen as the coordinate space and its cotangent plane is the momentum space. We then compute the invariant measure under the coadjoint action.

4.1.1 Coordinates

We set the following space coordinates on the hyperboloid: $q = k_0\Lambda^{-1}$, where $k_0 = (m, 0, 0)$ and Λ is the $SO(2,1)$ part of the group element g used in the coadjoint action which generates the orbit. We can check that $q_0^2 - q_1^2 - q_2^2 = m^2$ is verified. We also define the momentum coordinates on the cotangent plane: $p = q(J \cdot v) = (-q_1v_2 + q_2v_1, q_0v_2 + q_2v_0, q_0v_1 + q_1v_0)$. The normal to the plane is $n_p = (q_0, q_1, -q_2)$. This gives the following constraint equation for the plane: $q_0p_0 + q_1p_1 - q_2p_2 = 0$.

Using the coadjoint action equation (1.25), we compute the prime coordinates:

$$(q', p') = (q, p)M(g)^{-1} = (q\hat{\Lambda} + p\Lambda^{-1}(J \cdot v), p\Lambda^{-1}). \tag{4.1}$$

We remark from the definition that p depends on the point q to which it is attached. We thus need to transform the p coordinate in order to compute the invariants. We postulate the following:

$$p = \tilde{p}\Lambda_{q\Lambda}, \tag{4.2}$$

where Λ_q is a pure boost. The general form for those boosts is:

$$\Lambda_q = \frac{1}{m}\begin{pmatrix} q_0 & q_1 & q_2 \\ q_1 & m + \frac{q_1^2}{m+q_0} & \frac{q_1q_2}{m+q_0} \\ q_2 & \frac{q_1q_2}{m+q_0} & m + \frac{q_2^2}{m+q_0} \end{pmatrix}; \tag{4.3}$$

obtained from [5].

We then rewrite the following:

$$\begin{aligned}
p' &= p\Lambda^{-1} \\
&= \tilde{p}\Lambda_{q\Lambda}\Lambda^{-1} \\
&= \tilde{p}\Lambda_{q\Lambda}\Lambda^{-1}\Lambda_q^{-1}\Lambda_q \\
&= \tilde{p}R\Lambda_q \qquad (4.4) \\
&= \tilde{p}'\Lambda_q,
\end{aligned}$$

where we have defined $R = \Lambda_{q\Lambda}\Lambda^{-1}\Lambda_q^{-1}$. We can check that R is actually a rotation by applying it to the vertex of the hyperboloid:

$$\begin{aligned}
(m,0,0)R &= (m,0,0)\Lambda_{q\Lambda}\Lambda^{-1}\Lambda_q^{-1} \\
&= q\Lambda\Lambda^{-1}\Lambda_q^{-1} \\
&= q\Lambda_q^{-1} \\
&= (m,0,0). \qquad (4.5)
\end{aligned}$$

We see that the vertex of the hyperboloid is stabilized by the matrix R which is thus a rotation. It is possible to characterize the angle of this rotation. It is called the Wigner angle and is worked out in [26] using the isomorphim of $SO(2,1)$ with $SL(2,\mathbb{R})$.

4.1.2 Invariant measure

We want to compute the invariant measure on the hyperboloid in our set of coordinates. From (4.1), we write dq_1' and dq_2' replacing q_0' using the constraint $q_0^2 - q_1^2 - q_2^2 = m^2$. We get that:

$$\frac{dq_1' \wedge dq_2'}{q_0'} = \frac{dq_1 \wedge dq_2}{q_0} \qquad (4.6)$$

is invariant. We denote this measure $d\nu$. We remark that the measure does not depend on the mass. Since all the hyperboloids have the same shape (except for the position of the vertex on the time axis), it is normal that the measure be the same on each of them.

In (4.4), we have defined $\tilde{p}' = \tilde{p}R$, R being a rotation, then:

$$d\tilde{p}'_1 \wedge d\tilde{p}'_2 = d\tilde{p}_1 \wedge d\tilde{p}_2 \tag{4.7}$$

is easily seen to be invariant.

Finally, the invariant measure on the whole orbit is:

$$d\mu = \frac{d\tilde{p}_1 \wedge d\tilde{p}_2 \wedge dq_1 \wedge dq_2}{q_0}. \tag{4.8}$$

There is another way to obtain coordinates and an invariant measure, the coset decomposition. It is presented in Appendix D.

The measure obtained from the right coset decomposition is the same measure as the one obtained from the coadjoint orbit structure.

4.2 Induced representation

We describe here how the induced representation is obtained for the hyperboloid. This follows the method explained in Appendix A.2.

4.2.1 Tools

We recall here the different objects we will use in the following:

- $S_0 = \Lambda_{J_0}$ (the rotation) is the stabilizer of $V^* \ni k_0 = (m, 0, 0)$;
- $\mathcal{O}^*$ the orbit of k_0 in $\mathbb{R}^{(2,1)*}$ under $SO(2,1)$ is the hyperboloid ($q_0^2 - q_1^2 - q_2^2 = m^2$);
- $d\nu(q)$ is the invariant measure on $\mathcal{O}^*$, the hyperboloid, $d\nu = \frac{dq_1 \wedge dq_2}{q_0}$ as presented in Section 4.1.2;

- H_0 is the stabilizer of $(0, k_0) \in \mathfrak{g}^*$ under the coadjoint action, it is the rotation and time translation;

- $\mathcal{O}_{(0,k_0)}$ is the orbit of $(0, k_0) \in \mathfrak{g}^*$ under the coadjoint action of G, it is the hyperboloid and the space plane here;

- $T^*\mathcal{O}^*$ is the cotangent bundle of the orbit $\mathcal{O}^*$, here it is the hyperboloid with its cotangent plane;

- $\Gamma = G/H_0$ is the hyperboloid and the space plane;

- $d\mu(q,p)$ is the invariant measure on Γ, that is $d\mu = \frac{d\tilde{p}_1 \wedge d\tilde{p}_2 \wedge dq_1 \wedge dq_2}{q_0}$, where q_i, p_i are the natural coordinates and $p = \tilde{p}\Lambda_{q\Lambda}$.

We now have everything we need to obtain the induced representation.

4.2.2 Representation

We follow the procedure described in Appendix A.2 in order to obtain the induced representation.

The unitary irreducible representation (UIR) χL of $V \rtimes S_0$ carried by an Hilbert space $\mathfrak{k}$ is:

$$(\chi L)(v, s) = \exp[-i < k_0; v >]L(s).$$

In our case, S_0 being only the rotation, we need a one-dimensional representation. It is written as $e^{in\theta}$, where $n \in \mathbb{Z}$. The Hilbert space is thus $\mathfrak{k} = \mathbb{C}$, because we get a complex phase.

Now, we want to induce a representation of the Poincaré group from χL. We start from the coset decomposition: $(v, s) = (0, \Lambda_k)(\Lambda_k^{-1}v, s_0)$, where Λ_k is a pure boost and $s_0 = \Lambda_{J_0}$ is a rotation. We then act on the left part (which represents $\mathcal{O}^*$):

$$(v, s)(0, \Lambda_p) = (0, \Lambda_{sp})(\Lambda_{sp}^{-1}v, \Lambda_{sp}^{-1}s\Lambda_p). \tag{4.9}$$

We obtain the following cocycles:

$$h : G \times \mathcal{O}^* \to V \rtimes S_0, \quad h((x,s),p) = (\Lambda_{sp}^{-1}x, h_0(s,p));$$
$$h_0 : S \times \mathcal{O}^* \to S_0, \quad h_0(s,p) = \Lambda_{sp}^{-1} s \Lambda_p. \tag{4.10}$$

We need to compute the cocycles for the inverse group element, we get:

$$h((v,s)^{-1},p) = (-\Lambda_{s^{-1}p}^{-1} s^{-1} v, \Lambda_{s^{-1}p}^{-1} s^{-1} \Lambda_p), \tag{4.11}$$

where $h_0(s^{-1},p) = \Lambda_{s^{-1}p}^{-1} s^{-1} \Lambda_p$ is a rotation (see Appendix E.1). This gives the UIR:

$$(\chi L)(h((v,s)^{-1},p)) = \exp[-i < k_0; -\Lambda_{s^{-1}p}^{-1} s^{-1} v >] L(h_0(s^{-1},p)). \tag{4.12}$$

We now have to rewrite the argument $< k_0; -\Lambda_{s^{-1}k}^{-1} s^{-1} v >$. First, we recall the action of Λ_k in both $\mathbb{R}^{2,1}$ and its dual.

Definition 4.2.1 *If v, $v_0 \in \mathbb{R}^{2,1}$ are 3-column vectors, k, $k_0 \in \mathbb{R}^{(2,1)*}$, 3-row vectors, then the 3×3 boost matrix Λ acts in the following way:*

- $k_0 \Lambda_k = k, \qquad k_0 = k \Lambda_k^{-1}, \qquad k_0 \Lambda_k^{-1} = \bar{k}, \qquad k_0 = \bar{k} \Lambda_k;$
- $\Lambda_v v_0 = v, \qquad v_0 = \Lambda_v^{-1} v, \qquad \Lambda_v^{-1} v_0 = \bar{v}, \qquad v_0 = \Lambda_v \bar{v};$

where $\bar{k} = (k_0, -\mathbf{k})$.

We also need to recall the dual action in the pairing writing, it was originally: $< Ad^{\#}(g) X_1^*, X_2 > = < X_1^*, Ad(g^{-1}) X_2 >$. In the case of interest here, we rewrite this as: $< k_1 \Lambda_{ks}; v_2 > = < k_1; \Lambda_{s^{-1}k}^{-1} v_2 >$. On the LHS, k is in the dual, while on the RHS, k in the original vector space. We take the transpose of the argument of Λ and the inverse of both group elements Λ and s.

We can then rewrite the argument as follows:

$$\begin{aligned} < k_0; -\Lambda_{s^{-1}k}^{-1} s^{-1} v > &= - < k_0 \Lambda_{ks}; s^{-1} v > \\ &= - < ks; s^{-1} v > \\ &= -kss^{-1} v = -kv = - < k; v > . \end{aligned} \tag{4.13}$$

The UIR is finally written this way:

$$\left({}^{\chi^L}U(v,s)\phi\right)(k) = \exp[i < k; v >]L(h_0(s^{-1},k))^{-1}\phi(s^{-1}k). \tag{4.14}$$

The UIR we will be using in the following is:

$$\boxed{\left({}^{\chi^L}U(v,s)\phi\right)(k) = \exp[i < k; v >]\exp[-in\theta(k,s)]\phi(s^{-1}k),} \tag{4.15}$$

where k is a point on the hyperboloid, $n \in \mathbb{Z}$ and θ is the rotation parameter.

4.2.3 Square-integrability

We now check if this representation is square-integrable over the full group G. First of all, here is the definition of square-integrability:

Definition 4.2.2 *A representation is said to be square-integrable if* $\exists\, \eta \in \mathcal{H}, \eta \neq 0$ *such that:*

$$\int_G | < U(g)\eta|\phi > |^2 dm(g) < \infty, \quad \forall \phi \in \mathcal{H}, \tag{4.16}$$

where $dm(g)$ is the measure on G.

The integration on the full group is the integration on the parameters of the three translations, the rotation and the two boosts.

We can write the following to start:

$$\begin{aligned} < U(g)\eta|\phi > &= < e^{ix\cdot v}e^{-in\theta}\eta|\phi > \\ &= \int_{\mathcal{V}_m^+} \eta^*(s^{-1}x)e^{in\theta}e^{-ix\cdot v}\phi(x)\frac{d\mathbf{x}}{x_0}, \end{aligned} \tag{4.17}$$

$$\begin{aligned} < U(g)\eta|\phi >^* &= < \phi|U(g)\eta > \\ &= \int_{\mathcal{V}_m^+} \phi^*(y)e^{iy\cdot v}e^{-in\theta'}\eta(s^{-1}y)\frac{d\mathbf{y}}{y_0}, \end{aligned} \tag{4.18}$$

where $\mathcal{V}_m^+$ is the hyperboloid. We also need the integral definition of the δ function in two dimensions (from the Fourier transform):

$$\delta(\mathbf{x}-\mathbf{y}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} e^{i\mathbf{v}\cdot(x-\mathbf{y})}d\mathbf{v}. \tag{4.19}$$

The detailed computation of the integral is given in Appendix E.2.

We obtain that the UIR is not square-integrable on the whole group. We will then need to work on the quotient.

4.3 Quotient to phase-space and choice of sections

We describe here the quotient that we take in order to have a square-integrable representation as well as a few sections we will use to undo this quotient.

4.3.1 Quotient

In order to have a square-integrable representation, we take the quotient to the phase-space. We follow the left quotient decomposition presented in Appendix D.2:

$$(\Lambda, v) = \left(\Lambda_{Rq}, (0, p_1, p_2)^t\right) \left(R, (a, 0, 0)^t\right). \tag{4.20}$$

We redefine $\tilde{q} = Rq = (q_0, q_1 \cos t - q_2 \sin t, q_1 \sin t + q_2 \cos t)$. We work with $(\Lambda_{\tilde{q}}, (0, p_1, p_2)^t)$ which represents the hyperboloid and the space plane orbit.

Even if we take the quotient, we will use the natural coordinates and the invariant measure associated to it (given in Section 4.2.1) instead of the measure obtained from the left quotient decomposition.

4.3.2 Sections

There are several possible sections we can choose to undo the quotient, that is to go from the orbit to the group. Here are the ones we will use.

The most simple is the Galilean section:

$$\sigma_0 : \Gamma \to G\,, \quad \sigma_0(\mathbf{q}, \mathbf{p}) = ((0, \mathbf{p}), \Lambda_q). \tag{4.21}$$

It can be used to obtain a generic section:

$$\sigma : \Gamma \to G\,, \quad \sigma(\mathbf{q}, \mathbf{p}) = \sigma_0(\mathbf{q}, \mathbf{p})((f(\mathbf{q}, \mathbf{p}), \mathbf{0}), R(\mathbf{q}, \mathbf{p})). \tag{4.22}$$

where f is a scalar function and R is a rotation. From there, we can choose f to be able to solve the integral.

A particular class of such sections are the affine sections for which $f(\mathbf{q},\mathbf{p}) = \phi(\mathbf{q})+\mathbf{p}\cdot\theta(\mathbf{q})$ and $R(\mathbf{q},\mathbf{p}) = R(\mathbf{q})$. Moreover, here we can choose $\phi = 0$. Actually, this affine section appears as a constraint on f in the computation for the generic section.

We also define the principal section:

$$\sigma_{\mathcal{P}} : \Gamma \to G\ , \quad \sigma_{\mathcal{P}}(q,p) = (\Lambda_q p, \Lambda_q). \tag{4.23}$$

Here is the list of the sections we will be using in the following:

- the principal section: $\sigma_{\mathcal{P}}(q,p) = (\Lambda_q p, \Lambda_q)$ in 4.4;
- the Galilean section: $\sigma_0(\mathbf{q},\mathbf{p}) = ((0,\mathbf{p})^t, \Lambda_q)$ in 4.5;
- the generic section: $\sigma(\mathbf{q},\mathbf{p}) = \sigma_0(\mathbf{q},\mathbf{p})((f(\mathbf{q},\mathbf{p}),\mathbf{0}), R(\mathbf{q},\mathbf{p}))$, which will become the affine section: $\sigma_{aff} = ((\mathbf{p}\cdot\theta(\mathbf{q}),\mathbf{0})\Lambda_q + (0,\mathbf{p}), \Lambda_q R(\mathbf{q}))$ in 4.6.

The principal section leads to a nice result. We obtain coherent states for a general set of vectors. For the Galilean and affine sections, we obtain frames, that is we are able to get bounds on the integral, provided a special form of η.

4.4 Coherent states for the principal section

We perform the computations to obtain coherent states on the hyperboloid. We follow the method outlined in Section 3.2. Here, we are using the principal section.

4.4.1 Definition of the set of vectors

We recall the definition of the principal section (4.23): $\sigma_{\mathcal{P}}(q,p) = (\Lambda_q p, \Lambda_q)$. We choose a set of vectors (vector-valued functions) η in the Hilbert space $\mathcal{H} = \mathbb{C} \otimes$

$L^2(\mathcal{V}_m^+, \frac{dq_1 \wedge dq_2}{q_0})$. Those vectors are transformed by the UIR (4.15) in the following way:

$$\begin{aligned}
(\eta_{\sigma_{\mathcal{P}}(q,p)})(k) &= (U(\sigma_{\mathcal{P}}(q,p))\eta)(k) \\
&= (U(\Lambda_q p, \Lambda_q)\eta)(k) \\
&= e^{ik\cdot\hat{p}}e^{-in\theta}\eta(\Lambda_q^{-1}k),
\end{aligned} \tag{4.24}$$

where $\hat{p} = \Lambda_q p$ and k is an arbitrary point on the hyperboloid.

The formal operator is as follows:

$$A_{\sigma_{\mathcal{P}}} = \int_\Gamma |\eta_{\sigma_{\mathcal{P}}(q,p)} >< \eta_{\sigma_{\mathcal{P}}(q,p)}| \frac{1}{q_0} d\mathbf{q}d\mathbf{p}, \tag{4.25}$$

where Γ is the four-dimensional orbit, that is the hyperboloid and the space plane.

The change of coordinate $p \to \hat{p} = \Lambda_q p$ gives a complicated Jacobian which is hard to work with. Instead, we try the change of coordinate $k \to X(k)$ by rewriting the dot product in the exponential:

$$\begin{aligned}
k \cdot \hat{p} &= k \cdot (\Lambda_q p) \\
&= k\eta\Lambda_q p \\
&= k\Lambda_q^{-1}\eta p \\
&= (k\Lambda_q^{-1}) \cdot p \\
&= X(k) \cdot p,
\end{aligned} \tag{4.26}$$

where $\eta = \mathrm{diag}(1, -1, -1)$ is the metric governing the dot product here. We also use the fact that $\eta\Lambda_q = \Lambda_q^{-1}\eta$ and define $X(k) = k\Lambda_q^{-1}$.

We compute the Jacobian for the change of coordinate $k \to X(k)$:

$$|J| = \frac{1}{mk_0}(q_0k_0 - q_1k_1 - q_2k_2) = \frac{1}{mk_0} q \cdot k. \tag{4.27}$$

This is the change of coordinate we will use. We then write $e^{ik\cdot\hat{p}} = e^{iX(k)\cdot p}$ in the integral.

4.4.2 Integration of the formal operator

We want to see under what conditions the formal operator $A_{\sigma_\mathcal{P}}$ satisfies the resolution of the identity. We thus compute the integral:

$$I_{\phi,\psi} \;=\; <\phi|A_{\sigma_\mathcal{P}}\psi>. \tag{4.28}$$

The details are in Appendix E.3.

We obtain that:

$$I_{\phi,\psi} \;=\; \int_{\mathcal{V}_m^+} \phi^*(k)\mathcal{A}_{\sigma_\mathcal{P}}(k)\psi(k)\frac{d\mathbf{k}}{k_0}, \tag{4.29}$$

where

$$\mathcal{A}_{\sigma_\mathcal{P}}(k) \;=\; (2\pi)^2\int_{\mathcal{V}_m^+} |\eta(\Lambda_q^{-1}k)|^2\frac{m}{q\cdot k}\frac{d\mathbf{q}}{q_0}. \tag{4.30}$$

4.4.3 Rewriting of the vector argument

We need to rewrite $|\eta(\Lambda_q^{-1}k)|^2$ as a function of q in order to perform the integral and evaluate $\mathcal{A}_{\sigma_\mathcal{P}}(k)$.

We have the following by definition or simple computation:

- $\Lambda_k k_0 = k, \quad k_0 = \Lambda_k^{-1}k, \quad \Lambda_k^{-1}k_0 = \bar{k}, \quad k_0 = \Lambda_k\bar{k};$
- $\Lambda_k\Lambda_q k_0 = \tilde{R}\Lambda_q\Lambda_k k_0$, where $\tilde{R}$ is a rotation;
- $\Lambda_q^{-1} = \Lambda_{\bar{q}};$
- $\overline{\Lambda_k^{-1}q} = \Lambda_k\bar{q}.$

Remark 4.4.1 *The second item expresses the fact that $\Lambda_k\Lambda_q$ applied to k_0 and $\Lambda_q\Lambda_k$ applied to k_0 differ only by a rotation. Note that this is not true if applied to some other vector.*

We can thus rewrite:

$$\begin{aligned}
|\eta(\Lambda_q^{-1}k)|^2 &= |\eta(\Lambda_q^{-1}\Lambda_k(m,0,0)^t)|^2 \\
&= |\eta(\tilde{R}\Lambda_k\Lambda_{\bar{q}}(m,0,0)^t))|^2 \\
&= |\eta(\Lambda_k\bar{q})|^2 \\
&= |\eta(q')|^2, \qquad (4.31)
\end{aligned}$$

where we define $q' = \Lambda_k\bar{q}$. We have also set that $|\eta|^2$ is invariant under rotation, that is $|\eta(Rq)|^2 = |\eta(q)|^2$. This means that it is a function of the 0^{th} (time) component only.

We thus compute the 0^{th} component of the argument $q' = \Lambda_k\bar{q}$:

$$q'_0 = (\Lambda_k\bar{q})_0 = \frac{1}{m}k \cdot q, \qquad (4.32)$$

where $k \cdot q = k_0q_0 - k_1q_1 - k_2q_2$.

4.4.4 Evaluation of the integral

We return to the evaluation of the integral (4.30) for $\mathcal{A}_{\sigma_\mathcal{P}}(k)$. We use the fact that $q'_0 = \frac{1}{m}q \cdot k$ and that $\frac{d\mathbf{q}}{q_0} = \frac{d\mathbf{q}'}{q'_0}$ (since this is an invariant measure) to write:

$$\mathcal{A}_{\sigma_\mathcal{P}}(k) = (2\pi)^2 \int_{\mathcal{V}_m^+} |\eta(q'_0)|^2 \frac{1}{q'_0} \frac{d\mathbf{q}'}{q'_0}. \qquad (4.33)$$

We recall that η is square-integrable and that $q'_0 \geq m > 0$. We can then see that $\mathcal{A}_{\sigma_\mathcal{P}}(k)$ is actually a constant with respect to k (it only depends on q'). Then,

$$I_{\phi,\psi} = \mathcal{A}_{\sigma_\mathcal{P}} < \phi|\psi > . \qquad (4.34)$$

4.4.5 Resulting coherent states

The resulting coherent states are the vectors:

$$(\eta_{\sigma_\mathcal{P}(q,p)})(k) = e^{ik\cdot\Lambda_q p}e^{-in\theta}\eta(\Lambda_q^{-1}k). \qquad (4.35)$$

They have to be normalized by $\sqrt{\mathcal{A}_{\sigma_\mathcal{P}}}$ in (4.33) in order to have the resolution of the identity.

4.5 Coherent states for the Galilean section

We perform once again the computations to obtain coherent states on the hyperboloid following the method outlined in Section 3.2. Here, we are using the Galilean section.

We will not obtain coherent states, but we will get a frame for a restricted type of η.

4.5.1 Definition of the set of vectors

The Galilean section (4.21) is written $\sigma_0(\mathbf{q},\mathbf{p}) = ((0,\mathbf{p})^t, \Lambda_q)$. We take a set of vectors (vector-valued functions) η in the Hilbert space $\mathcal{H} = \mathbb{C} \otimes L^2(\mathcal{V}_m^+, \frac{dq_1 \wedge dq_2}{q_0})$. We transform it by the UIR (4.15):

$$\begin{aligned}
(\eta_{\sigma_0(\mathbf{q},\mathbf{p})})(k) &= (U(\sigma_0(\mathbf{q},\mathbf{p}))\eta)(k) \\
&= (U((0,\mathbf{p}),\Lambda_q)\eta)(k) \\
&= e^{ik\cdot(0,\mathbf{p})^t} e^{-in\theta}\eta(\Lambda_q^{-1}k),
\end{aligned} \tag{4.36}$$

where k is an arbitrary point on the hyperboloid.

The formal operator is written:

$$A_{\sigma_0} = \int_\Gamma |\eta_{\sigma_0(\mathbf{q},\mathbf{p})} >< \eta_{\sigma_0(\mathbf{q},\mathbf{p})}| \frac{1}{q_0} d\mathbf{q} d\mathbf{p}, \tag{4.37}$$

where Γ is the four-dimensional orbit.

4.5.2 Integration of the formal operator

We want to see if, and under what conditions, the formal operator A_{σ_P} satisfies the resolution of the identity. We thus compute the following integral:

$$I_{\phi,\psi} \quad = \quad < \phi | A_{\sigma_0} \psi > . \tag{4.38}$$

All the details of the computation are given in Appendix E.4.

Assuming again that $|\eta|^2$ is rotation-invariant, we get the following:

$$I_{\phi,\psi} = \int_{\mathcal{V}_m^+} \phi^*(k) \mathcal{A}_{\sigma_0}(k) \psi(k) \frac{d\mathbf{k}}{k_0}, \tag{4.39}$$

where

$$\mathcal{A}_{\sigma_0}(k) \quad = \quad (2\pi)^2 \int_{\mathcal{V}_m^+} |\eta(\Lambda_k \bar{q})|^2 \frac{1}{k_0} \frac{d\mathbf{q}}{q_0}. \tag{4.40}$$

4.5.3 Estimation of the integral

We would like to obtain that $\mathcal{A}_{\sigma_0}$ is independant of k. Actually, there is no solution. A simple guess $\eta = \sqrt{k_0}$ would give $\mathcal{A}_{\sigma_0}$ independant of k, but η would then not be square-integrable.

We can try to find some function $f(k, q)$ such that $\eta = f(k, q)\sqrt{k_0}$ would be square-integrable and $\mathcal{A}_{\sigma_0}$ would be independant of k. Unfortunately, it is not possible because then we would need $f = f(q)$ and η would not be square-integrable anymore.

We can also try to bound $\mathcal{A}_{\sigma_0}(k)$ to get a frame. As a bound for any η, we only have the condition $0 < \frac{1}{k_0} \leq \frac{1}{m}$. The inverse of the operator is not bounded.

We thus need to find a suitable η to obtain a bound on the operator. We can write $\eta = f(k, q)\sqrt{k_0}$ where f is such that both η and f are square-integrable. Moreover, suppose $f(k, q)$ is such that we have bounds on the operator: $0 < a \leq \mathcal{A}_{\sigma_0}(k) \leq b < \infty$. We would thus have a frame for the Galilean section under these conditions.

4.5.4 Resulting frames

As described in Definition 3.1.1, the set of vectors:

$$\sigma_{\sigma_0} = \{\eta_{\sigma_0(\mathbf{q},\mathbf{p})} | (\mathbf{q},\mathbf{p}) \in \mathbb{R}^4\} \subset \mathcal{H} \tag{4.41}$$

constitutes a frame for a suitable function f as described above.

4.6 Coherent states for the affine section

We now start from the generic section to obtain coherent states on the hyperboloid following the method outlined in Section 3.2. We will be lead to the affine section during the integration process.

We will again get a frame for a particular affine section and a restricted type of η.

4.6.1 Definition of the set of vectors

We start from a generic section:

$$\sigma(\mathbf{q},\mathbf{p}) = (\Lambda_q(f(\mathbf{q},\mathbf{p}),\mathbf{0})^t + (0,\mathbf{p})^t, \Lambda_q R(\mathbf{q},\mathbf{p})), \tag{4.42}$$

where f is a scalar function and R is a rotation.

We take a set of vectors (vector-valued functions) η in the Hilbert space $\mathcal{H} = \mathbb{C} \otimes L^2(\mathcal{V}_m^+, \frac{dq_1 \wedge dq_2}{q_0})$. We transform it by the UIR (4.15):

$$\begin{aligned}
(\eta_{\sigma(\mathbf{q},\mathbf{p})})(k) &= (U(\sigma(\mathbf{q},\mathbf{p}))\eta)(k) \\
&= (U(\hat{p}, \Lambda_q R(\mathbf{q},\mathbf{p}))\eta)(k) \\
&= e^{ik\cdot\hat{p}} e^{-i\theta} \eta(R^{-1}(\mathbf{q})\Lambda_q^{-1} k),
\end{aligned} \tag{4.43}$$

where $\hat{p} = \Lambda_q(f(\mathbf{q},\mathbf{p}),\mathbf{0})^t + (0,\mathbf{p})^t$. Since we know the matrix form of Λ_q, we can rewrite $\hat{p} = \frac{1}{m} f(\mathbf{q},\mathbf{p}) q + (0,\mathbf{p})^t$.

We will use this as a change of coordinate $p \longrightarrow \hat{p}$. We compute the Jacobian:

$$|J| = 1 + \frac{1}{m} q_1 \frac{\partial f}{\partial p_1} + \frac{1}{m} q_2 \frac{\partial f}{\partial p_2} \tag{4.44}$$

We want to choose f such that the Jacobian is independant of p to be able to perform directly the integral giving a δ function. This leads to the affine section:

$$f(\mathbf{q}, \mathbf{p}) = \mathbf{p} \cdot \theta(\mathbf{q}) + \phi(\mathbf{q}), \tag{4.45}$$

where θ is a two-vector function and ϕ is a scalar function. The resulting Jacobian is:

$$|J| = 1 + \frac{1}{m} \mathbf{q} \cdot \theta. \tag{4.46}$$

We define the formal operator:

$$A_\sigma = \int_\Gamma |\eta_{\sigma(\mathbf{q},\mathbf{p})} >< \eta_{\sigma(\mathbf{q},\mathbf{p})}| \frac{d\mathbf{q}d\mathbf{p}}{q_0}. \tag{4.47}$$

4.6.2 Integration of the formal operator

We want to integrate the formal operator in order to see if it satisfies the resolution of the identity. The details of the integration are given in Appendix E.5. Note that we restrain $R(q, p) = R(q)$.

We obtain:

$$I_{\phi,\psi} = \int_{\mathcal{V}_m^+} \phi^*(k) \mathcal{A}_\sigma(k, q) \psi(k) \frac{d\mathbf{k}}{k_0}: \tag{4.48}$$

where

$$\mathcal{A}_\sigma(k, q) = (2\pi)^2 \int_{\mathcal{V}_m^+} |\eta(\hat{\Lambda}_q^{-1} k)|^2 \frac{1}{k_0} \frac{m}{m + \mathbf{q} \cdot \theta} \frac{d\mathbf{q}}{q_0} \tag{4.49}$$

and $\hat{\Lambda}_q = \Lambda_q R(\mathbf{q})$.

4.6.3 Rewriting of the vector argument

We need to rewrite the argument of η as in 4.4.3. The process is exactly the same except that we have a supplementary rotation $R(\mathbf{q})$ which disappears under the assumption that $|\eta|^2$ is rotation-invariant.

The integral (4.49) is then rewritten:

$$\mathcal{A}_\sigma(k,q) = (2\pi)^2 \int_{\mathcal{V}_m^+} |\eta(q_0')|^2 \frac{1}{k_0} \frac{m}{m + \mathbf{q}\cdot\theta} \frac{d\mathbf{q}}{q_0}, \tag{4.50}$$

where $q_0' = \frac{1}{m} k \cdot q$.

4.6.4 Estimation of the integral and resulting frames

In order to obtain a result, we need to choose a particular value for the θ function. If we set $\theta = \left(\frac{1}{q_1}, \frac{1}{q_2}\right)$, the integral (4.50) reads:

$$\mathcal{A}_\sigma(k,q) = (2\pi)^2 \int_{\mathcal{V}_m^+} |\eta(q_0')|^2 \frac{1}{k_0} \frac{m}{m+2} \frac{d\mathbf{q}}{q_0}. \tag{4.51}$$

Recalling that $\frac{d\mathbf{q}}{q_0}$ is the invariant measure, we are in the same situation as for the Galilean section. Then, we can write $\eta = f(k,q)\sqrt{k_0}$ with f such that both η and f are square-integrable. Moreover, we suppose that $f(k,q)$ is such that we have bounds on the operator: $0 < a \leq \mathcal{A}_\sigma(k) \leq b < \infty$. We would thus have a frame for the affine section with a particular θ under these conditions.

We have built a set of coherent states on the hyperboloid using the principal section. We have obtained a frame for the very simple Galilean section under certain conditions on the initial vector set. We have also obtained a frame for a particular form of the affine section under restrictions on η.

Chapter 5

Coherent states on the upper cone

In this chapter, we compute the coherent states on the upper cone (denoted by cone only in the following). We use the same method as in Chapter 4 for the hyperboloid.

We define a set of natural coordinates and an invariant measure on the cone. We also present another set of coordinates which happens to be useful for the computations. We compute the induced representation and describe a set of sections that is used to finally compute the coherent states as described in Section 3.2.

In 5.4, we work with a generalized principal section, we obtain coherent states. In 5.5, we use the basic section to obtain a frame under some conditions on the initial vector.

5.1 Coordinates and measure

We start by defining a set of coordinates on the cone and its cotangent plane. We compute the invariant measure on this structure. We also present another set of coordinates related to the projection of the cone which will be useful in the computations for the coherent states.

5.1.1 Natural coordinates

We use the natural coordinate $q = (q_0, q_1, q_2)$ on the cone embedded in a three-dimensional space. It satisfies $q_0^2 - q_1^2 - q_2^2 = 0$.

The p coordinate is on a cotangent plane to the cone. Here, we will use $p = (p_0, p_0, p_2)$ as the coordinate on the plane obtained in Section 2.3.

Once again, the cotangent plane, hence the p coordinate, is attached to the cone at a point q. In the hyperboloid case, p was changed to $\tilde{p}$ by a pure boost. We need an equivalent transformation here. Therefore, we define $p = \tilde{p}\Lambda_{q\Lambda}$, where Λ_q is such that $(1,1,0)\Lambda_q = q$. We then have $q\Lambda_q^{-1} = (1,1,0)$. It is possible to obtain a matrix representation of Λ_q. The computation is presented in Appendix F.1. The result is:

$$\Lambda_q = \frac{1}{q_0 + q_1}\begin{pmatrix} 1 + q_0^2 + q_0 - q_1 & q_0q_1 - 1 - q_0 + q_1 & q_2(1 + q_0) \\ q_0q_1 - 1 - q_0 + q_1 & 1 + q_1^2 + q_0 - q_1 & -q_2(1 - q_1) \\ q_2(1 + q_0) & -q_2(1 - q_1) & q_0 + q_1 + q_2^2 \end{pmatrix}. \tag{5.1}$$

5.1.2 Invariant measure

From the natural coordinates, we compute $q' = q\hat{\Lambda}$ (note that $q' = q\hat{\Lambda} + p\Lambda^{-1}(J \cdot v)$, but we study the measure only on the cone, that is without translations v). The invariant measure is $\frac{dq_1 \wedge dq_2}{q_0}$ when there are no translations.

We also have:

$$\begin{aligned} p' &= p\Lambda^{-1} \\ &= \tilde{p}\Lambda_{q\Lambda}\Lambda^{-1} \\ &= \tilde{p}\Lambda_{q\Lambda}\Lambda^{-1}\Lambda_q^{-1}\Lambda_q \\ &= \tilde{p}R\Lambda_q \\ &= \tilde{p'}\Lambda_q, \end{aligned}$$

where we have set $R = \Lambda_{q\Lambda}\Lambda^{-1}\Lambda_q^{-1}$ and $\tilde{p'} = \tilde{p}R$.

We check that R is a rotation on the cone. It should keep the value of q_0 constant. We cannot use the condition $(1,0,0)R = (1,0,0)$ because $(1,0,0)$ is not a point on the cone. We thus check the condition $(1,1,0)R = (1,a,b)$, where $a^2 + b^2 = 1$.

$$\begin{aligned}
(1,1,0)R &= (1,1,0)\Lambda_{q\Lambda}\Lambda^{-1}\Lambda_q^{-1} \\
&= q\Lambda\Lambda^{-1}\Lambda_q^{-1} \\
&= q\Lambda_q^{-1} \\
&= (1,1,0).
\end{aligned}$$

This means that R is a rotation. We have $\tilde{p'} = \tilde{p}R$, then $d\tilde{p_1} \wedge d\tilde{p_2}$ is invariant.

Finally, the invariant measure on the cone is:

$$d\mu = \frac{dq_1 \wedge dq_2 \wedge d\tilde{p_1} \wedge d\tilde{p_2}}{q_0}. \tag{5.2}$$

5.1.3 Projection coordinates

We present here another representation of the cone coadjoint orbit. This is based on the projection of the cone on a plane.

We represent an element of the cone orbit with its cotangent plane by $g = (R_\theta, \lambda; \mathbf{b})$, where R_θ is from (1.4) $\Lambda_{J_0} = \begin{pmatrix} 1 & 0 \\ 0 & R_\theta \end{pmatrix}$, λ is the dilation factor (actually $\lambda = \cosh\gamma + \sinh\gamma$ from Λ_{J_2} in (1.4)) and $\mathbf{b}$ is a two-vector characterizing the cotangent plane. The product of two elements is:

$$gg' = (R_\theta, \lambda; \mathbf{b})(R_{\theta'}, \lambda'; \mathbf{b}') = (R_{\theta+\theta'}, \lambda\lambda'; \mathbf{b} + \lambda R_\theta \mathbf{b}'). \tag{5.3}$$

It is obtained from the matrix product of two elements of a semidirect product group and a direct computation for the rotation and the boost.

We will use the angle θ and the dilation parameter $\lambda \equiv \cosh\gamma + \sinh\gamma$ as the coordinates on the cone. In this setting, the coordinates on the plane are b_1 and b_2 from $\mathbf{b}$.

We then have another representation of Λ_q, the action on the cone. Besides the matrix given in (5.1), we can use $\Lambda_q \equiv \lambda R_\theta$. It will act on the two-vector $(1,0)$ to take it to the projection of q on a plane, that is (q_1, q_2).

5.2 Induced representation

We now describe how the induced representation of the Poincaré group is obtained. This follows the method given in Appendix A.2.

5.2.1 Tools

We list here the different objects we will use in the following:

- $S_0 = n$ (a translation) is the stabilizer of $V^* \ni k_0 = (1,1,0)$;
- $\mathcal{O}^*$ the orbit of k_0 in $\mathbb{R}^{(2,1)*}$ under $SO(2,1)$ is the cone $(q_0^2 - q_1^2 - q_2^2 = 0)$;
- $d\nu(q)$ is the invariant measure on $\mathcal{O}^*$, the cone, hence $d\nu = \frac{dq_1 \wedge dq_2}{q_0}$ as presented in Section 5.1.2;
- H_0 is the stabilizer of $(0, k_0) \in \mathfrak{g}^*$ under the coadjoint action, where $k_0 = (1,1,0)$, hence it is the translation n and the translation $v = (t, -t, 0)$;
- $\mathcal{O}_{(0,k_0)}$ is the orbit of $(0, k_0) \in \mathfrak{g}^*$ under the coadjoint action of G, here it is the cone and the plane (p_0, p_0, p_2);
- $T^*\mathcal{O}^*$ is the cotangent bundle of the orbit $\mathcal{O}^*$, here the cone and its cotangent plane;
- $\Gamma = G/H_0$ from (1.27), it is then the cone and the plane (p_0, p_0, p_2);
- $d\mu(q,p)$ is the invariant measure on Γ, $\frac{d\tilde{p}_1 \wedge d\tilde{p}_2 \wedge dq_1 \wedge dq_2}{q_0}$, where q_i, p_i are the natural coordinates and $p = \tilde{p}\Lambda_{q\Lambda}$.

We now have everything we need to obtain the induced representation.

5.2.2 Representation

We follow the procedure described in Appendix A.2 in order to obtain the induced representation.

We associate a unitary character χ to $V = \mathbb{R}^{2,1}$ in the same way as before:

$$\chi(x) = \exp(-i < k_0; x >), \tag{5.4}$$

where $k_0 = (1, 1, 0)$.

Let $s \mapsto L(s)$ be a UIR of S_0 carried by a Hilbert space $\mathfrak{k}$. Here, S_0 is the translation (n in the Iwasawa decomposition). Then, $L(s)$ is a one-dimensional unitary representation: $e^{it\rho}$, where $t \in \mathbb{R}$ and ρ is the translation parameter. The UIR χL of $V \rtimes S_0$ carried by $\mathfrak{k}$ is:

$$(\chi L)(x, s) = \exp[-i < k_0; x >]e^{it\rho}. \tag{5.5}$$

The Hilbert space is $\mathfrak{k} = \mathbb{C}$.

Now, we want to induce a representation of the Poincaré group $G = \mathbb{R}^{2,1} \rtimes SO(2, 1)$ from χL. From the coset decomposition, $(x, s) = (0, \Lambda_k)(\Lambda_k^{-1}x, s_0)$ (where Λ_k is the transformation on the cone and $s_0 = n$) we act on $(0, \Lambda_k)$ which represents the cone $\mathcal{O}^*$:

$$(x, s)(0, \Lambda_p) = (0, \Lambda_{sp})(\Lambda_{sp}^{-1}x, \Lambda_{sp}^{-1}s\Lambda_p). \tag{5.6}$$

We obtain the following cocycles:

$$h : G \times \mathcal{O}^* \to V \rtimes S_0, \quad h((x, s), p) = (\Lambda_{sp}^{-1}x, h_0(s, p));$$
$$h_0 : S \times \mathcal{O}^* \to S_0, \quad h_0(s, p) = \Lambda_{sp}^{-1}s\Lambda_p. \tag{5.7}$$

They look the same as for the hyperboloid due to the notation, actually Λ and S_0 are different matrices.

The UIR is again written this way:

$$\left({}^{\chi^L}U(x,s)\phi\right)(k) = \exp[i < k; x >]L(h_0(s^{-1},k))^{-1}\phi(s^{-1}k). \tag{5.8}$$

The writing is the same as in the hyperboloid case except that the objects are geometrically different.

Finally, the UIR we will be using is written:

$$\boxed{({}^{\chi^L}U(v,s)\phi)(k) = \exp[i < k; v >]e^{-it\rho(k,s)}\phi(s^{-1}k).} \tag{5.9}$$

It is similar to the UIR obtained for the hyperboloid in (4.15), but here k is a point on the cone, $t \in \mathbb{R}$ and ρ is a translation parameter.

5.2.3 Square-integrability

We now check if this representation is square-integrable over the full group G. We have defined square-integrability in Definition 4.2.2.

We write the following:

$$\begin{aligned} < U(g)\eta|\phi > &= < e^{ix\cdot v}e^{-in\theta_x}\eta|\phi > \\ &= \int_{\mathcal{V}^+} \eta^*(s^{-1}x)e^{in\theta_x}e^{-ix\cdot v}\phi(x)\frac{d\mathbf{x}}{x_0}, \end{aligned} \tag{5.10}$$

$$\begin{aligned} < U(g)\eta|\phi >^* &= < \phi|U(g)\eta > \\ &= \int_{\mathcal{V}^+} \phi^*(y)e^{iy\cdot v}e^{-in\theta_y}\eta(s^{-1}y)\frac{d\mathbf{y}}{y_0}. \end{aligned} \tag{5.11}$$

We will need the integral definition of the delta function (4.19).

The detailed computation is given in Appendix F.2. We obtain that the UIR is not square-integrable on the whole group. We therefore need to work on the quotient.

5.3 Quotient to phase-space and choice of sections

Since the UIR is not square-integrable, we need to take a quotient and work only on the four-dimensional orbit. We also need to define some suitable sections in order to go back to the full group.

5.3.1 Quotient

We take the quotient to the phase-space to have a square-integrable representation. We follow the left quotient decomposition:

$$(\Lambda, v) = \left(\Lambda_k, \left(\frac{p_0+p_1}{2}, \frac{p_0+p_1}{2}, p_2\right)^t\right)(n, (t, -t, 0)^t). \tag{5.12}$$

Note that in the Iwasawa decomposition Λ_k is ka a product of a rotation and a boost and n is the translation.

Since getting the left quotient decomposition measure is pretty hard in this case (because the cone coordinates are angles), we will use the natural coordinates and the invariant measure associated to it listed in Section 5.2.1.

5.3.2 Sections

Later, we will need to fix a section in order to be able to undo the quotient taken above. We describe here the different possible choices.

By analogy with the hyperboloid case, we define a basic section:

$$\sigma_0 : \Gamma \to G\ , \quad \sigma_0(q, p) = \left(\left(\frac{p_0+p_1}{2}, \frac{p_0+p_1}{2}; p_2\right)^t, \Lambda_q^{-1}\right), \tag{5.13}$$

where Λ_q is ka product of a rotation and a boost, its matrix form is given in (5.1). Then, we can get a generic section $\sigma : \Gamma \to G$ from there:

$$\begin{aligned} \sigma(\mathbf{q}, \mathbf{p}) &= \sigma_0(q, p)((f, -f, 0)^t, N) \\ &= \left(\Lambda_q^{-1}(f, -f, 0)^t + \left(\frac{p_0+p_1}{2}, \frac{p_0+p_1}{2}, p_2\right)^t, \Lambda_q^{-1}N\right), \end{aligned} \tag{5.14}$$

where $f = f(q, p)$ is a scalar function and $N = N(q, p)$ is a translation.

There is also the generalized principal section which is defined paralleling the principal section for the hyperboloid:

$$\sigma_{\mathcal{P}} : \Gamma \to G, \quad \sigma_{\mathcal{P}}(q, p) = (\Lambda_q^\alpha, \Lambda_q^\beta p). \tag{5.15}$$

We have added some freedom with the α and β exponents. We will get some constraint on those exponents when computing the coherent states.

In 5.4, we will work with the generalized principal section which leads to a nice result. We obtain a family of coherent states for a general set of vectors. With the basic section, presented in 5.5, we are only able to obtain a frame under some restrictions.

5.4 Coherent states for the generalized principal section

We now move to the computation of the coherent states. We start with the generalized principal section. We follow the method described in Section 3.2.

5.4.1 Definition of the set of vectors

We recall the definition of the generalized principal section:

$$\sigma_{\mathcal{P}}(q, p) = (\Lambda_q^{\alpha}, \Lambda_q^{\beta} p). \tag{5.16}$$

We start with a set of square-integrable vectors η in the Hilbert space $\mathcal{H} = \mathbb{C} \otimes L^2(\mathcal{V}^+, \frac{dq_1 \wedge dq_2}{q_0})$. We transform them using the UIR given in (5.9):

$$(\eta_{\sigma_{\mathcal{P}}})(k) = e^{i<k;\hat{p}>} e^{-it\rho} \eta(\Lambda_q^{-\alpha} k), \tag{5.17}$$

where $\hat{p} = \Lambda_q^{\beta} p$ and k is an arbitrary point on the cone.

The formal operator is defined as follows:

$$A_{\sigma_{\mathcal{P}}} = \int_{\Gamma} |\eta_{\sigma_{\mathcal{P}}} >< \eta_{\sigma_{\mathcal{P}}}| \frac{d\mathbf{q} d\mathbf{p}}{q_0}. \tag{5.18}$$

5.4.2 Integration of the formal operator

The integral of the formal operator $I_{\Phi,\Psi} =< \Phi|A_{\sigma_\mathcal{P}}\Psi >$ is given in Appendix F.3.

We work out the Jacobian of the change of coordinate $p \rightarrow \hat{p} = \Lambda_q^\beta p$. To do this, we need to use the projection coordinates defined in Section 5.1.3. We thus rewrite: $\hat{p} = \Lambda_q^\beta p \equiv \lambda^\beta R_{\beta\theta}\mathbf{p}$, where $\mathbf{p}$ is the projection of p on the punctured plane. Then, we have that: $\hat{p}_1 = \lambda^\beta \cos\beta\theta p_1 - \lambda^\beta \sin\beta\theta p_2$ and $\hat{p}_2 = \lambda^\beta \sin\beta\theta p_1 + \lambda^\beta \cos\beta\theta p_2$ which gives the Jacobian $|J| = \lambda^{2\beta}$.

When rewriting the integral with this Jacobian (see the details in Appendix F.3), we obtain:

$$I_{\Phi,\Psi} = \int_{\mathcal{V}^+} \Phi^*(k)\mathcal{A}_{\sigma_\mathcal{P}}(k)\Psi(k)\frac{dk}{k_0}, \tag{5.19}$$

where

$$\mathcal{A}_{\sigma_\mathcal{P}}(k) = (2\pi)^2 \int_{\mathcal{V}^+} |\eta(\Lambda_q^{-\alpha}k)|^2 \frac{1}{\lambda^{2\beta}}\frac{1}{k_0}\frac{d\mathbf{q}}{q_0}. \tag{5.20}$$

5.4.3 Rewriting of the vector argument

In order to rewrite the argument of η, we use the projection setting and the isomorphism of the cone with the punctured plane.

We redefine the point k to be the initial point $(1,0)^t$ on which the rotation R_ϕ and the dilation τ act, that is $k \rightarrow \mathbf{k} = \tau R_\phi(1,0)^t$. We also have that $k_0 = \tau$, the time component only depends on the dilation. We can also obtain this from $k_0^2 = k_1^2 + k_2^2 = \tau^2\cos^2\phi + \tau^2\sin^2\phi$. Similarly, we write Λ_q as λR_θ, then $\Lambda_q^{-\alpha} = \lambda^{-\alpha}R_{-\alpha\theta}$.

With all this information, we are able to write:

$$\begin{aligned} |\eta(\Lambda_q^{-\alpha}k)|^2 &= |\eta(\lambda^{-\alpha}R_{-\alpha\theta}\tau R_\phi(1,0)^t)|^2 \\ &= |\eta(\lambda^{-\alpha}\tau R_{\phi-\alpha\theta}(1,0)^t)|^2 \\ &= |\eta(\lambda^{-\alpha}\tau(1,0)^t)|^2. \end{aligned} \tag{5.21}$$

where we have considered that $|\eta|^2$ is rotation-invariant.

We rewrite the integral (5.20):

$$\mathcal{A}_{\sigma_{\mathcal{P}}}(k) = (2\pi)^2 \int_{\mathcal{V}^+} |\eta(\lambda^{-\alpha}\tau(1,0)^t)|^2 \frac{1}{\lambda^{2\beta}} \frac{1}{\tau} \frac{d\mathbf{q}}{q_0}. \tag{5.22}$$

We can see that $|\eta|^2$ depends only on the length of the vector $\lambda^{-\alpha}\tau$, we thus change the variable to $q_0' = q_1' = \lambda^{-\alpha}\tau$ (and $q_2' = 0$).

5.4.4 Evaluation of the integral

We recall that the measure $\frac{d\mathbf{q}}{q_0}$ is invariant. Also, provided $2\beta = -\alpha$, we have $\tau\lambda^{2\beta} = q_0'$ too, then we get the resolution of the identity because the operator is now written:

$$\mathcal{A}_{\sigma_{\mathcal{P}}} = (2\pi)^2 \int_{\mathcal{V}^+} |\eta(q_0')|^2 \frac{1}{q_0'} \frac{d\mathbf{q}'}{q_0'} \tag{5.23}$$

and does not depend on k. Also, recalling that η is square-integrable and $q_0' > 0$, for an initial vector η in the domain of the unbounded operator, multiplication by $\frac{1}{q_0'}$, we have:

$$\begin{aligned} I_{\Phi,\Psi} &= \mathcal{A}_{\sigma_{\mathcal{P}}} \int_{\mathcal{V}^+} \Phi^*(k)\Psi(k)\frac{dk}{k_0} \\ &= \mathcal{A}_{\sigma_{\mathcal{P}}} < \Phi|\Psi > . \end{aligned} \tag{5.24}$$

5.4.5 Resulting coherent states

The vectors $\eta_{\sigma_{\mathcal{P}}}$ are coherent states for the section:

$$\sigma_{\mathcal{P}}(q,p) = (\Lambda_q^{-2\mu}, \Lambda_q^{\mu} p), \tag{5.25}$$

that is:

$$(\eta_{\sigma_{\mathcal{P}}})(k) = e^{i<k;\Lambda_q^{\mu}p>} e^{-it\rho} \eta(\Lambda_q^{2\mu} k) \tag{5.26}$$

are a family of coherent states for $\mu \in \mathbb{Z}$ and a suitable η with the normalization by $\sqrt{\mathcal{A}_{\sigma_{\mathcal{P}}}}$ given in (5.23). This is a promising new result.

5.5 Coherent states for the basic section

We now try to work with the basic section (5.13). We again follow the method described in Section 3.2. We will obtain a frame under some conditions on η.

5.5.1 Definition of the set of vectors

We recall the definition of the basic section:

$$\sigma_0 : \Gamma \to G \, , \quad \sigma_0(q,p) = \left(\left(\frac{p_0+p_1}{2}, \frac{p_0+p_1}{2}, p_2 \right)^t, \Lambda_q^{-1} \right), \tag{5.27}$$

where Λ_q is the matrix acting on the cone.

We start with a set of square-integrable vectors η in the Hilbert space $\mathcal{H} = \mathbb{C} \otimes L^2(\mathcal{V}^+, \frac{dq_1 \wedge dq_2}{q_0})$. We transform them using the UIR given in (5.9):

$$(\eta_{\sigma_0})(k) = e^{i<k;(\frac{p_0+p_1}{2}, \frac{p_0+p_1}{2}, p_2)>} e^{-it\rho} \eta(\Lambda_q k), \tag{5.28}$$

where k is an arbitrary point on the cone.

The formal operator is defined as follows:

$$A_{\sigma_0} = \int_\Gamma |\eta_{\sigma_0} >< \eta_{\sigma_0}| \frac{d\mathbf{q}d\mathbf{p}}{q_0}. \tag{5.29}$$

5.5.2 Integration of the formal operator and rewriting of the vector argument

The integral of the formal operator $I_{\Phi,\Psi} =< \Phi | A_{\sigma_0} \Psi >$ is given in Appendix F.4. We obtain:

$$I_{\phi,\psi} \;\; = \;\; \int_{\mathcal{V}^+} \phi^*(k) A_{\sigma_0}(k) \psi(k) \frac{dk}{k_0}, \tag{5.30}$$

where

$$A_{\sigma_0}(k) = (2\pi)^2 \int_{\mathcal{V}^+} |\eta(\Lambda_q k)|^2 \frac{1}{k_0} \frac{d\mathbf{q}}{q_0}. \tag{5.31}$$

We again need to rewrite the argument of η using the projection coordinates as in Section 5.4.3. This gives:

$$\begin{aligned}
|\eta(\Lambda_q k)|^2 &= |\eta(\lambda R_\theta \tau R_\phi (1,0)^t)|^2 \\
&= |\eta(\lambda \tau R_{\phi+\theta}(1,0)^t)|^2 \\
&= |\eta(\lambda \tau (1,0)^t)|^2,
\end{aligned} \tag{5.32}$$

where we have considered that $|\eta|^2$ is rotation-invariant.

Hence, the integral (5.31) can be written:

$$A_{\sigma_0}(k) = (2\pi)^2 \int_{\mathcal{V}^+} |\eta(\lambda\tau(1,0)^t)|^2 \frac{1}{\tau} \frac{d\mathbf{q}}{q_0}. \tag{5.33}$$

5.5.3 Estimation of the integral and resulting frame

We can set that $\eta(\lambda, \tau) = f(\lambda, \tau)\sqrt{\tau}$ where f is such that both η and f are square-integrable. Moreover, if we suppose that $f(\lambda, \tau)$ is such that we have bounds on the operator: $0 < a \leq A_{\sigma_0}(k) \leq b < \infty$, we have a frame for the basic section.

Under these particular conditions, the integral (5.31) is bounded and the vectors (5.28) form a frame.

We have built a family of coherent states on the cone using a generalized principal section. We have also obtained a frame under certain conditions for the basic section.

Conclusion

Summary of the results

The main results of this thesis are divided in two parts.

We have first studied the orbit structure of the Poincaré group in $2+1$ dimensions. From a matrix representation of the group, we have computed a formula for the adjoint and coadjoint action. This has allowed us to obtain and classify the coadjoint orbits of the group. We have also directly calculated the representation generating orbits. We have obtained a degenerate orbit, the two-sheet hyperboloid, the upper and lower cones and the one-sheet hyperboloid; they all also appear as two-dimensional coadjoint orbits. Moreover, the hyperboloids and the cones appear with their cotangent plane as four-dimensional coadjoint orbits. Finally, the representation generating orbits and the four-dimensional coadjoint orbits were linked.

Despite the fact that those representation generating orbits were known, they have been obtained here in a different way. The explicit computation of the coadjoint orbits also represents new work.

The other part of the thesis concerns coherent states. Using the information obtained from the coadjoint orbit structure of the upper sheet of the two-sheet hyperboloid and of the upper cone, we have defined coordinates and an invariant measure on them. We have also induced a representation and computed the coherent states on each of them. For the hyperboloid, we have obtained coherent states for the principal section and a frame for the Galilean and the affine sections. For the cone, we have obtained a family of coherent states for a generalized principal section and a frame for the basic section.

The way the coherent states are obtained also represent an original contribution for our particular group especially in the matrix representation. Indeed, even if the $1+1$ and $3+1$-dimensional cases have been studied in details, the $2+1$-dimensional case was neglected.

Further work

Here are some possible continuation and new directions of the work done in this thesis.

It would be interesting to study quantization from those coherent states following the ideas in [27, 14, 24] and using [2].

We could also work out the induced representation and the coherent states for the one-sheet hyperboloid which is isomorphic to the cylinder.

The complexification of the hyperboloid using the method developed by Hall and Mitchell [17] for the complexified sphere would also be something to explore. We can define coherent states from there and compare them to the ones obtained in Chapter 4.

One of the most promising applications is in the wavelets scheme. We can apply our results to signal and image analysis using the books [11, 1, 8].

It is possible to apply the process to other semidirect product groups like the Jacobi group ($SL(2,\mathbb{R}) \ltimes H(\mathbb{R})$, H being the Heisenberg group). We could again compute the coadjoint action and classify the coadjoint orbits from a matrix representation of this group.

Further research could also be about obtaining a general framework for coadjoint orbits of semidirect product groups.

There are two more possible generalizations:

- Extend the procedure to a superstructure (G becoming a supergroup), that is apply the orbit method and get the symplectic structure for a supergroup. El Gradechi [13] has extended the geometric quantization to a supersymplectic supermanifold.
- Generalize the orbit method for infinite-dimensional structure using diffeomorphism group, Iglesias (in a joint work with Donato) has explanations on his website [18].

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Appendix A

Theoretical background

In this appendix, we present some background knowledge regarding group theory and the induced representation method.

A.1 Group theory

We introduce the generic definitions for Lie groups and Lie algebras as well as the dual algebra and the semidirect product group structure. We also present concepts related to orbits and some basic notions about differential and symplectic geometry.

A.1.1 A few definitions

A Lie group G has a structure of differentiable manifold. The multiplication and the inverse are smooth maps. Here, we use a matrix representation.

The differentiation at identity of each one-parameter subgroup gives the Lie algebra generators. We can see the algebra $\mathfrak{g}$ as the tangent space at the identity. The multiplication law of the algebra is called Lie bracket (or commutator) and satisfies:

- bilinearity: $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$ and $[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z]$, $\forall\ X,\ Y,\ Z \in \mathfrak{g}$ and $\forall\ \alpha,\ \beta \in \mathbb{K}$, here $\mathbb{K} = \mathbb{R}$;

- antisymmetry: $[X, Y] = -[Y, X]$, $\forall\ X,\ Y \in \mathfrak{g}$;

- Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, $\forall\ X,\ Y,\ Z \in \mathfrak{g}$.

The dual algebra is denoted $\mathfrak{g}^*$. The dual pairing is a bilinear form such that: $<;>$: $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$. In a particular basis, we have $< X_i^*; X_j >= \delta_{ij}$. For matrix groups, we have that $< A; B >= tr(AB)$. When we are working with vectors, this is simply the scalar product (with or without the metric of the group).

We are working with a semidirect product group, the Poincaré group, which is described in details in Chapter 1. The general structure of such a group is: $G = V \rtimes S$, where V is a real vector space and S is a subgroup of $GL(V)$.

One last useful definition is the definition of a Hilbert space. A Hilbert space is a generalization of a Euclidean space, it is a complex linear vector space with a scalar product $< \cdot, \cdot >$. In this thesis, we mainly use $L^2(X, d\mu)$, that is the space of square-integrable functions on X with respect to an invariant measure $d\mu$.

A.1.2 Orbits

We have a group space X. The group G acts on the space by the bijective map $x \mapsto gx$. The *orbit* of a fixed point x of the set X under the action of the group G is the set $Gx = \{gx \in X | g \in G\}$. The *stabilizer* (also named *little group*) of a point x is the set $H_x = \{h \in G | hx = x\}$.

The group G acts *faithfully* on X if $\bigcap_{x \in X} H_x = e$.

The group G acts *freely* on X if $H_x = e, \forall x \in H$.

If $\forall x, y \in X$ there $\exists g \in G$ such that $y = gx$, G is said to act *transitively* on X.

There is a bijection between the orbit of a point x and the quotient of the group G by the stabilizer of this point: $Gx \simeq G/H_x$. We are using this bijection to obtain our coadjoint orbits in Chapter 2.

A.1.3 Differential and symplectic geometry

We describe the coadjoint orbits in Section 1.3.1. Since those orbits actually have a natural symplectic structure, we introduce some notions of symplectic geometry here. We also present some notions of differential geometry.

A *symplectic vector space* is a real vector space V with a bilinear, antisymmetric and nondegenerate form Ω.

A *symplectic manifold* is an even dimensional manifold with a closed antisymmetric two-form. The tangent space at any point of the manifold is a symplectic vector space. The coadjoint orbits are even dimensional and some of them have the structure of a symplectic manifold.

The *cotangent bundle*, denoted $T^*\mathcal{O}^*$, of a smooth manifold $\mathcal{O}^*$ is the vector bundle of all the cotangent spaces at every point on the manifold. It may be described also as the dual bundle to the tangent bundle ($T\mathcal{O}^*$).

We can put coordinates on a cotangent bundle structure. Usually, coordinates on the manifold are q_i and coordinates on the cotangent space are p_j. The invariant measure can be computed using these coordinates.

The *projection map* (π), as the name tells, is a map from a manifold to a submanifold. The quotient of a group by one of its subgroups (for example the little group) can be seen as a kind of projection. That is to say that we restrict ourselves to some subgroup.

The *section* (σ) allows us to go back to the original manifold (or group). It is a way to undo the projection or the quotient. Precisely, we have that $\pi(\sigma(U)) = U$, where U is an open set of the manifold. There is a choice involved there which may change the issue of the computation. We need to choose sections for the computation of the coherent states in 4.3.2 and 5.3.2.

A.2 Induced representation method

The induced representation method is a canonical method to construct a representation of a group. It has been introduced by Mackey [25]. The idea is to obtain the representation of a group from the already known representation of one of its subgroups. We briefly discuss general facts about representation theory and then present the method as it is used in Sections 4.2 and 5.2.

A *linear representation* is basically a map T from a locally compact group G to the set of bounded linear functions on a separable Hilbert space $\mathfrak{h}$. It satisfies:

$$T(g_1 g_2) = T(g_1)T(g_2)\ \forall g_1, g_2 \in G, \quad T(e) = \mathbb{I},$$

where e is the identity element of G and $\mathbb{I}$ is the identity in the Hilbert space.

A *unitary representation* is such that T is a unitary operator. That is T obeys $T^* = T^{-1}$. An *irreducible representation* has no nontrivial invariant subspaces. A unitary irreducible representation has both of those properties and is denoted UIR.

As stated earlier, the principle underlying the theory of induced representations is that you can induce the representation of a group from an already known representation of one of its subgroups. Namely, this subgroup is the little group or stabilizer.

The method that is used to get representations for the computation of the coherent states in Chapters 4 and 5 follows [1], §10.2.4.

We have that $d\nu$ is the invariant measure on the orbit Γ and the Hilbert space is $\mathcal{K} = L^2(\Gamma, d\nu)$.

We first associate a unitary character χ to $V = \mathbb{R}^{2,1}$ in this way:

$$\chi(v) = \exp(-i < k_0; v >), \tag{A.1}$$

where $v \in \mathbb{R}^{2,1}$ and k_0 is the initial vector which determines the case.

Let $s \mapsto L(s)$ be a UIR of S_0, the little group, carried by a Hilbert space $\mathcal{K}$.

The UIR χL of $V \rtimes S_0$ carried by $\mathcal{K}$ is then:

$$(\chi L)(v, s) = \exp[-i < k_0; v >]L(s). \tag{A.2}$$

Now, we want to induce a representation of $G = V \rtimes S$ from χL. From the coset decomposition: $(v, s) = (0, \Lambda_k)(\Lambda_k^{-1}v, s_0)$ (Λ_k is the action on the hyperboloid or the cone), we act on the left part (which represents $\mathcal{O}^*$, the hyperboloid or the cone):

$$(v, s)(0, \Lambda_p) = (0, \Lambda_{sp})(\Lambda_{sp}^{-1}v, \Lambda_{sp}^{-1}s\Lambda_p), \tag{A.3}$$

where $p \in \mathcal{O}^*$. We obtain the following cocycles:

$$\begin{aligned} h : G \times \mathcal{O}^* \to V \rtimes S_0, \qquad & h((v, s), p) = (\Lambda_{sp}^{-1}v, h_0(s, p)); \\ h_0 : S \times \mathcal{O}^* \to S_0, \qquad & h_0(s, p) = \Lambda_{sp}^{-1}s\Lambda_p. \end{aligned} \tag{A.4}$$

The UIR is then written this way:

$$\left({}^{\chi L}U(v, s)\phi\right)(k) = \exp[i < k; v >]L(h_0(s^{-1}, k))^{-1}\phi(s^{-1}k). \tag{A.5}$$

The orbit method is a natural continuation of the induced representation method, it has been built by Kirillov [20]. We do not treat it here.

Appendix B

Detailed computations for the definition of the adjoint and coadjoint actions

In this appendix, we give the details of the manipulations done in order to obtain the actions in Section 1.2.

B.1 Rewriting of equation (1.14)

We show here how the equality (1.14) ($\Lambda\alpha^t \cdot J\Lambda^{-1} = (m(\Lambda^{-1})^t m\alpha)^t \cdot J$) is obtained in the computation of the adjoint action.

We want to write $\Lambda\alpha^t \cdot J\Lambda^{-1} = \alpha'^t \cdot J$, where $\alpha^t \cdot J = \alpha_0 J_0 + \alpha_1 J_1 + \alpha_2 J_2$. We perform the computation for each element of the linear combination J_i. We first assume $\Lambda J_i \Lambda^{-1} = \sum a_{ij} J_j$, then we compute the part $\Lambda J_i \Lambda^{-1}$ for a generic element of the group $\Lambda = \Lambda_{J_0}\Lambda_{J_1}\Lambda_{J_2}$ in order to extract this linear combination. We have used *Maple* to perform the matrix multiplications.

We obtain the following (using the definition of J_i given in (1.5)):

$$\Lambda J_0 \Lambda^{-1} = \Lambda^{-1}_{00} J_0 + \Lambda^{-1}_{01} J_1 - \Lambda^{-1}_{02} J_2,$$
$$\Lambda J_1 \Lambda^{-1} = \Lambda^{-1}_{10} J_0 + \Lambda^{-1}_{11} J_1 - \Lambda^{-1}_{12} J_2,$$
$$\Lambda J_2 \Lambda^{-1} = -\Lambda^{-1}_{20} J_0 - \Lambda^{-1}_{21} J_1 + \Lambda^{-1}_{22} J_2.$$

If we collect those results in a matrix, we can see that:

$$\Lambda J \Lambda^{-1} = \begin{pmatrix} \Lambda^{-1}_{00} & \Lambda^{-1}_{01} & -\Lambda^{-1}_{02} \\ \Lambda^{-1}_{10} & \Lambda^{-1}_{11} & -\Lambda^{-1}_{12} \\ -\Lambda^{-1}_{20} & -\Lambda^{-1}_{21} & \Lambda^{-1}_{22} \end{pmatrix} \begin{pmatrix} J_0 \\ J_1 \\ J_2 \end{pmatrix}. \tag{B.1}$$

This is where we need to introduce the matrix m in order to fix the signs:

$$m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{B.2}$$

We reintroduce α and write this result as:

$$\Lambda \alpha^t \cdot J \Lambda^{-1} = (\alpha^t m \Lambda^{-1} m) \cdot J. \tag{B.3}$$

The last step is to rewrite $\alpha^t m \Lambda^{-1} m$ as something times α in order to be able to write the coadjoint action in a matrix form. We explicitly compute the expression and get that $\alpha^t m \Lambda^{-1} m = m(\Lambda^{-1})^t m \alpha$.

We thus have rewritten:

$$\Lambda \alpha^t \cdot J \Lambda^{-1} = (m(\Lambda^{-1})^t m \alpha)^t \cdot J. \tag{B.4}$$

B.2 Rewriting of equation (1.17)

We now show how the equality (1.17) ($-\Lambda \alpha^t \cdot J \Lambda^{-1} v = -(J \cdot v)\hat{\Lambda}^{-1} \alpha$) is obtained in the computation of the adjoint action.

We already have obtained that $\Lambda\alpha^t \cdot J\Lambda^{-1} = (m(\Lambda^{-1})^t m\alpha)^t \cdot J = (\hat{\Lambda}^{-1}\alpha)^t \cdot J$, then:

$$-\Lambda\alpha^t \cdot J\Lambda^{-1}v = -((\hat{\Lambda}^{-1}\alpha)^t \cdot J)v. \tag{B.5}$$

We first rename the vector $-\hat{\Lambda}^{-1}\alpha = W = (W_0, W_1, W_2)$. We then expand the dot product with J. We recall that the vector $J = (J_0, J_1, J_2)$ is made of matrices. This gives $(W_0 J_0 + W_1 J_1 + W_2 J_2)$. We apply this linear combination to the vector v using the explicit expression of J's given in (1.5). This gives:

$$\begin{pmatrix} W_1 v_2 + W_2 v_1 \\ -W_0 v_2 + W_2 v_0 \\ W_0 v_1 + W_1 v_0 \end{pmatrix}. \tag{B.6}$$

We compare this result to $-(J \cdot v)\hat{\Lambda}^{-1}\alpha$, that is the matrix $J \cdot v$ given in (1.18) applied to the vector W. We obtain:

$$\begin{pmatrix} v_2 W_1 + v_1 W_2 \\ -v_2 W_0 + v_0 W_2 \\ v_1 W_0 + v_0 W_1 \end{pmatrix}. \tag{B.7}$$

We see that the two ways of writing this give the same vector. We have thus:

$$-\Lambda\alpha^t \cdot J\Lambda^{-1}v = -(J \cdot v)\hat{\Lambda}^{-1}\alpha. \tag{B.8}$$

Appendix C

Computations for the orbits

We give here the detailed computations regarding the different orbits presented in Chapter 2.

C.1 Computations for the two-sheet hyperboloid orbit

In Section 2.2, we have set $k_0 = \begin{pmatrix} \pm m & 0 & 0 \end{pmatrix}$. We compute the representation generating orbit and the two coadjoint orbits for this case.

Representation generating orbit

To get the action of $SO(2,1)$ on the dual of $\mathbb{R}^{2,1}$, we just multiply the row-vector $X^* = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \end{pmatrix}$ by the subgroup matrices. Here is the result for $X^* = k_0 = \begin{pmatrix} \pm m & 0 & 0 \end{pmatrix}$:

$$X^* \Lambda_{J_0} = \begin{pmatrix} \pm m & 0 & 0 \end{pmatrix}; \tag{C.1a}$$

$$X^* \Lambda_{J_1} = \begin{pmatrix} \pm m \cosh\beta & 0 & \pm m \sinh\beta \end{pmatrix}; \tag{C.1b}$$

$$X^* \Lambda_{J_2} = \begin{pmatrix} \pm m \cosh\gamma & \pm m \sinh\gamma & 0 \end{pmatrix}. \tag{C.1c}$$

We have that the two boosts are acting. The vectors (C.1b) and (C.1c) are two hyperbolas. This gives the upper sheet of the two-sheet hyperboloid for $+m$ and the lower sheet of the two-sheet hyperboloid for $-m$.

Coadjoint orbits

From the coadjoint action definition:

$$Ad^{\#}(g)X^* = \begin{pmatrix}\alpha^*\hat{\Lambda} + \beta^*\Lambda^{-1}(J \cdot v) & \beta^*\Lambda^{-1}\end{pmatrix}$$

with the vector $\begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} = \begin{pmatrix}0 & k_0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & \pm m & 0 & 0\end{pmatrix}$, we get that:

$$Ad^{\#}(g)X_0^* = \begin{pmatrix}(\pm m, 0, 0)\Lambda^{-1}(J \cdot v) & (\pm m, 0, 0)\Lambda^{-1}\end{pmatrix}. \tag{C.2}$$

The second part $(\pm m, 0, 0)\Lambda^{-1}$ is very similar to the representation generating orbit computation. It is again generated by the two boosts, this represents the two-sheet hyperboloid.

For the first part, the simplest way to see the geometry is to use the bijection with the quotient by the stabilizer. The matrix $J \cdot v$ is defined in (1.18). We want to solve:

$$(\pm m, 0, 0)\Lambda^{-1}(J \cdot v) = 0, \quad (\pm m, 0, 0)\Lambda^{-1} = (\pm m, 0, 0). \tag{C.3}$$

Using the second one, the first equation becomes: $(\pm m, 0, 0)(J \cdot v) = 0$, which is rewritten, using (1.18):

$$\pm m(0, v_2, v_1) = 0. \tag{C.4}$$

So, the time translation v_0 is the stabilizer. The quotient leaves the two space translations to generate the orbit, that is the space plane. We remark that solving $(\pm m, 0, 0)\Lambda^{-1} = (\pm m, 0, 0)$ gives the rotation Λ_{J_0} as the stabilizer and, thus, the two boosts as the orbit generators.

We thus have a four-dimensional orbit composed of the two-sheet hyperboloid and the space plane.

Now, if we use $\begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} = \begin{pmatrix}k_0 & 0\end{pmatrix} = \begin{pmatrix}\pm m & 0 & 0 & 0 & 0 & 0\end{pmatrix}$ instead, we get that the coadjoint action is:

$$Ad^{\#}(g)X_0^* = \begin{pmatrix}(\pm m, 0, 0)\hat{\Lambda} & 0\end{pmatrix}. \tag{C.5}$$

From the expression of $\hat{\Lambda}$ for the one-parameter subgroups given in (1.16), we can see that we have again something similar to the representation generating orbit, that is an orbit generated by the two boosts which is the two-sheet hyperboloid, a two-dimensional orbit.

C.2 Computations for the cone orbit

For Section 2.3, we have $k_0 = \begin{pmatrix}\pm 1 & 1 & 0\end{pmatrix}$. We compute the representation generating orbit and the two coadjoint orbits for this case.

Representation generating orbit

To get the action of $SO(2,1)$ on the dual of $\mathbb{R}^{2,1}$, we just multiply the row-vector $X^* = \begin{pmatrix}\gamma_0 & \gamma_1 & \gamma_2\end{pmatrix}$ by the subgroup matrices. Here is the result for $X^* = k_0 = \begin{pmatrix}\pm 1 & 1 & 0\end{pmatrix}$:

$$X^*\Lambda_{J_0} = \begin{pmatrix}\pm 1 & \cos\alpha & -\sin\alpha\end{pmatrix}, \tag{C.6a}$$

$$X^*\Lambda_{J_1} = \begin{pmatrix}\pm\cosh\beta & 1 & \pm\sinh\beta\end{pmatrix}, \tag{C.6b}$$

$$X^*\Lambda_{J_2} = \begin{pmatrix}\pm\cosh\gamma + \sinh\gamma & \pm\sinh\gamma + \cosh\gamma & 0\end{pmatrix}. \tag{C.6c}$$

From (C.6a), we see that if we cut the orbit at $t = \pm 1$, we have a circle. From (C.6b), we see that if we cut the orbit at $x = 1$, we have an hyperbola. From (C.6c), we see that if we cut the orbit at $y = 0$, we have a straight line. Moreover, in all cases, we have that $t^2 - x^2 - y^2 = 0$. This thus gives the upper and lower cone depending on the sign for the time component.

Coadjoint orbits

From the definition of the coadjoint action:

$$Ad^{\#}(g)X^* = \begin{pmatrix}\alpha^*\hat{\Lambda} + \beta^*\Lambda^{-1}(J \cdot v) & \beta^*\Lambda^{-1}\end{pmatrix}$$

with the vector $\begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} = \begin{pmatrix}0 & k_0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & \pm 1 & 1 & 0\end{pmatrix}$, we get:

$$Ad^{\#}(g)X^* = \begin{pmatrix}(\pm 1, 1, 0)\Lambda^{-1}(J \cdot v) & (\pm 1, 1, 0)\Lambda^{-1}\end{pmatrix}. \tag{C.7}$$

We want to compute the stabilizer of this action in order to compute the quotient G/H_0 which is isomorphic to the coadjoint orbit. We must then solve the following:

$$(\pm 1, 1, 0)\Lambda^{-1} = (\pm 1, 1, 0), \quad (\pm 1, 1, 0)(J \cdot v) = (0, 0, 0). \tag{C.8}$$

The second equation gives, using (1.18), $(-v_2, \pm v_2, \pm v_1 + v_0) = 0$. It is solved by the vector $v = (t, \mp t, 0)$.

For the first equation, we get that $(1, 1, 0)$ is stabilized by $\Lambda_{J_+}^{-1}$ and $(-1, 1, 0)$ is stabilized by $\Lambda_{J_-}^{-1}$, the Λ's are defined in (1.7). They are both translations (n in the Iwasawa decomposition as described in 1.1.4).

The coadjoint orbit is given by the quotient of the group by the stabilizer just obtained. That is:

- $SO(2,1)/n$ is ka in the Iwasawa decomposition, that is a rotation (Λ_{J_0}) and a boost (Λ_{J_2}) acting on the vector $(\pm 1, 1, 0)$;
- $\mathbb{R}^{2,1}/(t, \mp t, 0) = (0, 0, v_2) \times (t, \pm t, 0)$ which is the plane generated by the y-axis and the axis $x = \pm t$.

The first part is simply given by:

$$(\pm 1, 1, 0)\Lambda_{J_0} = m(\pm 1, \cos\alpha, -\sin\alpha),$$
$$(\pm 1, 1, 0)\Lambda_{J_2} = m(\pm\cosh\gamma + \sinh\gamma, \cosh\gamma \pm \sinh\gamma, 0). \tag{C.9}$$

This is a circle above (below) the xy-plane and a half-line, they generate the cone.

We thus have a four-dimensional orbit made of the cone and the plane generated by the y-axis and the axis $x = \pm t$.

We now compute the stabilizer of the vector $\begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} = \begin{pmatrix}k_0 & 0\end{pmatrix} = \begin{pmatrix}\pm 1 & 1 & 0 & 0 & 0 & 0\end{pmatrix}$ in order to compute the quotient G/H_0 which is isomorphic to another coadjoint orbit:

$$Ad^{\#}(g)X^* = \begin{pmatrix}(\pm 1, 1, 0)\hat{\Lambda} & 0\end{pmatrix}. \tag{C.10}$$

We need to solve $(\pm 1, 1, 0)\hat{\Lambda} = (\pm 1, 1, 0)$. It is again stabilized by the n translation ($\hat{\Lambda}_{J_-}$ for $(1, 1, 0)$ and $\hat{\Lambda}_{J_+}$ for $(-1, 1, 0)$). All the $\mathbb{R}^{2,1}$ translations also stabilize the vector. The quotient leaves the rotation and the boost to generate the cone, a two-dimensional orbit.

C.3 Computations for the one-sheet hyperboloid orbit

For Section 2.4, the initial vector is $k_0 = \begin{pmatrix}0 & m & 0\end{pmatrix}$. We compute the representation generating orbit and the two coadjoint orbits for this case.

Representation generating orbit

To get the action of $SO(2,1)$ on the dual of $\mathbb{R}^{2,1}$, we just multiply the row-vector $X^* = \begin{pmatrix}\gamma_0 & \gamma_1 & \gamma_2\end{pmatrix}$ by the subgroup matrices. Here is the result for $X^* = k_0 =$

$\begin{pmatrix}0 & m & 0\end{pmatrix}$:

$$
\begin{aligned}
X^*\Lambda_{J_0} &= \begin{pmatrix}0 & m\cos\alpha & -m\sin\alpha\end{pmatrix}, && \text{(C.11a)}\\
X^*\Lambda_{J_1} &= \begin{pmatrix}0 & m & 0\end{pmatrix}, && \text{(C.11b)}\\
X^*\Lambda_{J_2} &= \begin{pmatrix}m\sinh\gamma & m\cosh\gamma & 0\end{pmatrix}. && \text{(C.11c)}
\end{aligned}
$$

We have the rotation and one boost acting. The vector (C.11a) is a circle and the vector (C.11c) is a hyperbola. This gives the one-sheet hyperboloid.

Coadjoint orbits

From the definition:

$$
Ad^{\#}(g)X^* = \begin{pmatrix}\alpha^*\hat{\Lambda} + \beta^*\Lambda^{-1}(J\cdot v) & \beta^*\Lambda^{-1}\end{pmatrix},
$$

using the vector $\begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} = \begin{pmatrix}0 & k_0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0 & m & 0\end{pmatrix}$, we get the coadjoint action:

$$
Ad^{\#}(g)X^* = \begin{pmatrix}(0,m,0)\Lambda^{-1}(J\cdot v) & (0,m,0)\Lambda^{-1}\end{pmatrix}. \tag{C.12}
$$

We want to get the stabilizer by solving:

$$
(0,m,0)\Lambda^{-1} = (0,m,0), \qquad (0,m,0)(J\cdot v) = 0. \tag{C.13}
$$

The boost in the x-direction (Λ_{J_1}) solves the first part. The second part is $(-v_2, 0, v_0) = 0$, which means that the x-translation (v_1) is in the stabilizer.

The quotient tells us that the coadjoint orbit is generated by the rotation and the y boost to which we add the time and y translations. This is a four-dimensional orbit composed of the one-sheet hyperboloid and the ty-plane.

Now, if we use $X_0^* = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} = \begin{pmatrix}k_0 & 0\end{pmatrix} = \begin{pmatrix}0 & m & 0 & 0 & 0 & 0\end{pmatrix}$, we get:

$$
Ad^{\#}(g)X^* = \begin{pmatrix}(0,m,0)\hat{\Lambda} & 0\end{pmatrix}. \tag{C.14}
$$

The stabilizer here is simply Λ_{J_1} with all the $\mathbb{R}^{2,1}$ translations. The orbit is then generated by the rotation and the boost acting on $(0, m, 0)$, this is the two-dimensional one-sheet hyperboloid.

Appendix D

Coset decomposition measures for the hyperboloid

We use the coset decomposition to get another set of coordinates and invariant measure on the hyperboloid. We first present the decomposition and then the coordinates and measure for both the right and left decomposition.

We know that the stabilizer leading to the hyperboloid is the rotation and the time-translation (see Section 2.2.2). We wish to decompose the group in its time and spatial parts to take the quotient. The spatial part will give the coordinates of the coadjoint orbit.

First, we recall the group product:

$$g_1 g_2 = \begin{pmatrix} \Lambda^{(1)}\Lambda^{(2)} & \Lambda^{(1)}v^{(2)} + v^{(1)} \\ 0 & 1 \end{pmatrix}. \tag{D.1}$$

We also need the decomposition of a Lorentz transformation in a rotation and a pure boost. This is given in [29]:

$$\Lambda = R\Lambda_q \quad \text{or} \quad \Lambda = \Lambda_{Rq} R, \tag{D.2}$$

where R is a rotation:

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \tag{D.3}$$

and Λ_q is a pure boost, see (4.3):

$$\Lambda_q = \begin{pmatrix} q_0 & q_1 & q_2 \\ q_1 & 1 + \frac{q_1^2}{1+q_0} & \frac{q_1 q_2}{1+q_0} \\ q_2 & \frac{q_1 q_2}{1+q_0} & 1 + \frac{q_2^2}{1+q_0} \end{pmatrix}. \tag{D.4}$$

Here, we consider the case $m = 1$ only. We now work out both the left and right invariant measure.

D.1 Right coset decomposition

We have the following equality:

$$(R, (a, 0, 0)^t)(\Lambda_q, (0, p_1, p_2)^t) = (\Lambda, v) = (R\Lambda_q, (v_0, v_1, v_2)^t). \tag{D.5}$$

We compute the product on the left-hand side using (D.1) and we solve to get the value of a, p_1 and p_2. We obtain the following: $a = v_0$, $p_1 = v_1 \cos t + v_2 \sin t$, $p_2 = v_2 \cos t - v_1 \sin t$.

The coordinates are simply q_0, q_1 and q_2 appearing in Λ_q for the hyperboloid and p_1 and p_2 given above for the cotangent plane.

To get the right invariant measure, we act by a generic element $g^{(0)}$ on the quotient $G/H_R = (\Lambda_q, (0, p_1, p_2)^t)$ to get the double prime coordinates p'' and q''. We then compute dp'', dq'' and the right invariant measure.

We need to rewrite $\Lambda^{(0)}$ in the polar decomposition. We use (D.3) and (D.4) to obtain $\Lambda^{(0)} = R^{(0)}\Lambda_q^{(0)}$. The double prime coordinates are extracted from:

$$(\Lambda'', v'') = (\Lambda_q, (0, p_1, p_2)^t)(R^{(0)}\Lambda_q^{(0)}, v^{(0)}). \tag{D.6}$$

From there, we compute the invariant measure:

$$
\begin{aligned}
dp''_1 \wedge dp''_2 &= dp_1 \wedge dp_2 + \text{some } dq \text{ terms}, \\
\frac{dq''_1 \wedge dq''_2}{q''_0} &= \frac{dq_1 \wedge dq_2}{q_0}, \\
\frac{dq''_1 \wedge dq''_2 \wedge dp''_1 \wedge dp''_2}{q''_0} &= \frac{dq_1 \wedge dq_2 \wedge dp_1 \wedge dp_2}{q_0}.
\end{aligned} \tag{D.7}
$$

D.2 Left coset decomposition

We now use the other decomposition $\Lambda = \Lambda_{Rq}R$. We write:

$$
(\Lambda_{Rq}, (0, p_1, p_2)^t)(R, (a, 0, 0)^t) = (\Lambda, v) = (\Lambda_{Rq}R, (v_0, v_1, v_2)^t). \tag{D.8}
$$

We redefine $\tilde{q} = Rq = (q_0, q_1 \cos t - q_2 \sin t, q_1 \sin t + q_2 \cos t)$. From (D.8), we obtain that $a = \frac{v_0}{q_0}$, $p_1 = v_1 - \frac{v_0}{q_0}\tilde{q}_1$, $p_2 = v_2 - \frac{v_0}{q_0}\tilde{q}_2$.

We again rewrite $\Lambda^{(0)}$ in the polar decomposition $\Lambda^{(0)} = \Lambda_{\tilde{q}^{(0)}} R^{(0)}$, where $\tilde{q}^{(0)} = R^{(0)} q^{(0)}$.

We act on the left quotient by the left to get the prime coordinates:

$$
(\Lambda', v') = (\Lambda^{(0)}, v^{(0)})(\Lambda_{\tilde{q}}, (0, p_1, p_2)^t). \tag{D.9}
$$

The decomposition of Λ' is $\Lambda_{\tilde{q'}} R'$, where $\tilde{q'} = R'q'$. We extract q' and p' from (Λ', v') and compute the dq' and dp' to obtain the left invariant measure:

$$
\begin{aligned}
q'_0 dp'_1 \wedge dp'_2 &= q_0 dp_1 \wedge dp_2 + \text{some } dq \text{ terms}, \\
\frac{dq'_1 \wedge dq'_2}{q'_0} = \frac{d\tilde{q'}_1 \wedge d\tilde{q'}_2}{\tilde{q'}_0} &= \frac{d\tilde{q}_1 \wedge d\tilde{q}_2}{\tilde{q}_0} = \frac{dq_1 \wedge dq_2}{q_0}, \\
dq'_1 \wedge dq'_2 \wedge dp'_1 \wedge dp'_2 &= dq_1 \wedge dq_2 \wedge dp_1 \wedge dp_2.
\end{aligned} \tag{D.10}
$$

This decomposition also leads to an invariant two-form. Using $\tilde{q}_1$ and $\tilde{q}_2$ as coordinates, we have computed:

$$
dp'_1 \wedge d\tilde{q'}_1 + dp'_2 \wedge d\tilde{q'}_2 = dp_1 \wedge d\tilde{q}_1 + dp_2 \wedge d\tilde{q}_2. \tag{D.11}
$$

Appendix E

Computations for the coherent states on the hyperboloid

We give here the detailed computations regarding the induced representation and the coherent states on the hyperboloid given in Chapter 4.

E.1 Rotation in the cocycle

In the computation of the cocycle for the inverse element (4.11), we need to study the following combination $\Lambda^{-1}_{s^{-1}q} s^{-1} \Lambda_q$. We check here that it is a rotation by applying it to the vertex of the hyperboloid. We recall that $\Lambda_q (m, 0, 0)^t = q$.

$$\begin{aligned} \Lambda^{-1}_{s^{-1}q} s^{-1} \Lambda_q (m, 0, 0)^t &= \Lambda^{-1}_{s^{-1}q} s^{-1} q \\ &= (m, 0, 0)^t. \end{aligned} \tag{E.1}$$

So, $\Lambda^{-1}_{s^{-1}q} s^{-1} \Lambda_q$ is actually a rotation. The angle θ of this rotation is the Wigner angle. It is possible to characterize it using the group isomorphism between $SO(2,1)$ and $SL(2, \mathbb{R})$. The angle depends on a point on the hyperboloid (q) and the generator of the transformation (s^{-1}, an element of $SO(2,1)$).

E.2 Checking square-integrability of the UIR

Here is the detailed computation of the integral outlined in Section 4.2.3.

We write $\int_G$ as $\int_V \int_S$. Since, x and y are on the hyperboloid, the integration of the exponential is only in two dimensions.

$$
\begin{aligned}
&\int_G | < U(g)\eta|\phi > |^2 dm(g) \\
&= \int_V \int_S \int_{\mathcal{V}_m^+} \int_{\mathcal{V}_m^+} \eta^*(s^{-1}x)e^{in\theta}e^{-ix\cdot v}\phi(x)\phi^*(y)e^{iy\cdot v}e^{-in\theta'}\eta(s^{-1}y)\frac{\mathbf{dx}}{x_0}\frac{\mathbf{dy}}{y_0}dvds \\
&= (2\pi)^2 \int_{\mathbb{R}} \int_S \int_{\mathcal{V}_m^+} \int_{\mathcal{V}_m^+} \eta^*(s^{-1}x)e^{in(\theta-\theta')}\phi(x)\phi^*(y)\eta(s^{-1}y)\delta(\mathbf{y}-\mathbf{x})\frac{\mathbf{dx}}{x_0}\frac{\mathbf{dy}}{y_0}dsdv_0 \\
&= (2\pi)^2 \int_{\mathbb{R}} \int_S \int_{\mathcal{V}_m^+} (\eta^*(s^{-1}x)\phi(x))(\phi^*(x)\eta(s^{-1}x))\frac{1}{x_0}\frac{\mathbf{dx}}{x_0}dsdv_0 \\
&= (2\pi)^2 \int_{\mathbb{R}} dv_0 \int_S \int_{\mathcal{V}^+} |\eta^*(s^{-1}x)\phi(x)|^2 \frac{1}{x_0}\frac{\mathbf{dx}}{x_0}ds
\end{aligned}
$$

We then have the integral of a positive quantity on the translation v_0 which is infinite. Then, the UIR is not square-integrable.

E.3 Computation of the integral for the principal section

In Section 4.4, we need to compute the integral (4.28). We give the details of the integration.

$$
\begin{aligned}
I_{\phi,\psi} &= <\phi|A_{\sigma_{\mathcal{P}}}\psi> \\
&= \int_{\Gamma}\int_{\mathcal{V}_m^+}\int_{\mathcal{V}_m^+} \phi^*(k)e^{iX(k)\cdot p}e^{-in\theta}\eta(\Lambda_q^{-1}k)\eta^*(\Lambda_q^{-1}k')e^{in\theta'}e^{-iX(k')\cdot p}\times \\
&\qquad\qquad \times\psi(k')\frac{d\mathbf{k}}{k_0}\frac{d\mathbf{k}'}{k_0'}\frac{d\mathbf{q}d\mathbf{p}}{q_0} \\
&= (2\pi)^2\int_{\mathcal{V}_m^+}\int_{\mathcal{V}_m^+}\int_{\mathcal{V}_m^+} \phi^*(k)e^{-in\theta}\eta(\Lambda_q^{-1}k)\eta^*(\Lambda_q^{-1}k')e^{in\theta'}\psi(k')\times \\
&\qquad\qquad \times\delta(X(k)-X(k'))\frac{d\mathbf{k}}{k_0}\frac{d\mathbf{k}'}{k_0'}\frac{d\mathbf{q}}{q_0} \\
&= (2\pi)^2\int_{\mathcal{V}_m^+}\int_{\mathcal{V}_m^+}\int_{\mathcal{V}_m^+} \phi^*(k)e^{-in\theta}\eta(\Lambda_q^{-1}k)\eta^*(\Lambda_q^{-1}k')e^{in\theta'}\psi(k')\times \\
&\qquad\qquad \times\delta(X(k)-X(k'))\frac{d\mathbf{k}}{k_0}\frac{m\,dX(k')}{q\cdot k'}\frac{d\mathbf{q}}{q_0} \\
&= (2\pi)^2\int_{\mathcal{V}_m^+}\int_{\mathcal{V}_m^+} \phi^*(k)|\eta(\Lambda_q^{-1}k)|^2\psi(k)\frac{m}{q\cdot k}\frac{d\mathbf{k}}{k_0}\frac{d\mathbf{q}}{q_0}
\end{aligned}
$$

We have used the fact that if $X(k)=X(k')$, then $k=k'$ and also $\theta=\theta'$.

E.4 Computation of the integral for the Galilean section

In Section 4.5, we need to compute the integral (4.38). We give the details of the integration.

$$
\begin{aligned}
I_{\phi,\psi} &= <\phi|A_{\sigma_0}\psi> \\
&= \int_\Gamma \int_{\mathcal{V}_m^+} \int_{\mathcal{V}_m^+} \phi^*(k) e^{i(k_1p_1+k_2p_2)} e^{-in\theta} \eta(\Lambda_q^{-1}k)\eta^*(\Lambda_q^{-1}k') e^{in\theta'} \times \\
&\qquad\qquad \times e^{-i(k_1'p_1+k_2'p_2)}\psi(k') \frac{d\mathbf{k}}{k_0}\frac{d\mathbf{k}'}{k_0'}\frac{dqdp}{q_0} \\
&= (2\pi)^2 \int_{\mathcal{V}_m^+}\int_{\mathcal{V}_m^+}\int_{\mathcal{V}_m^+} \phi^*(k) e^{-in\theta}\eta(\Lambda_q^{-1}k)\eta^*(\Lambda_q^{-1}k')e^{in\theta'}\psi(k') \times \\
&\qquad\qquad \times \delta(k-k')\frac{d\mathbf{k}}{k_0}\frac{d\mathbf{k}'}{k_0'}\frac{d\mathbf{q}}{q_0} \\
&= (2\pi)^2 \int_{\mathcal{V}_m^+}\int_{\mathcal{V}_m^+} \phi^*(k)\eta(\Lambda_q^{-1}k)\eta^*(\Lambda_q^{-1}k)\psi(k)\frac{1}{k_0}\frac{d\mathbf{k}}{k_0}\frac{d\mathbf{q}}{q_0} \\
&= \int_{\mathcal{V}_m^+} \phi^*(k)\mathcal{A}_{\sigma_0}(k)\psi(k)\frac{d\mathbf{k}}{k_0},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A}_{\sigma_0}(k) &= (2\pi)^2 \int_{\mathcal{V}_m^+} |\eta(\Lambda_q^{-1}k)|^2 \frac{1}{k_0}\frac{d\mathbf{q}}{q_0} \\
&= (2\pi)^2 \int_{\mathcal{V}_m^+} |\eta(\Lambda_k\bar{q})|^2 \frac{1}{k_0}\frac{d\mathbf{q}}{q_0}.
\end{aligned} \tag{E.2}
$$

In the second line of (E.2), we have used the work done in Section 4.4.3 (for the principal section) to rewrite the argument of η.

We have also again used the fact that if $X(k) = X(k')$, then $k = k'$ and also $\theta = \theta'$.

E.5 Computation of the integral for the affine section

In Section 4.6, we need to compute the integral of the formal operator. We give the details of the integration here.

$$
\begin{aligned}
I_{\phi,\psi} &= <\phi|A_\sigma\psi> \\
&= \int_\Gamma\int_{\mathcal{V}_m^+}\int_{\mathcal{V}_m^+}\phi^*(k)e^{i\mathbf{k}\cdot\hat{p}}e^{-in\theta}\eta(\hat{\Lambda}^{-1}(q,p)k)\eta^*(\hat{\Lambda}^{-1}(q,p)k')e^{in\theta'}\times \\
&\qquad\qquad \times e^{-i\mathbf{k}'\cdot\hat{p}}\psi(k')\frac{d\mathbf{k}}{k_0}\frac{d\mathbf{k}'}{k_0'}\frac{1}{q_0}d\mathbf{q}d\mathbf{p} \\
&= \int_\Gamma\int_{\mathcal{V}_m^+}\int_{\mathcal{V}_m^+}\phi^*(k)e^{i\mathbf{k}\cdot\hat{p}}e^{-in\theta}\eta(\hat{\Lambda}^{-1}(q,p)k)\eta^*(\hat{\Lambda}^{-1}(q,p)k')e^{in\theta'}\times \\
&\qquad\qquad \times e^{-i\mathbf{k}'\cdot\hat{p}}\psi(k')\frac{m}{m+\mathbf{q}\cdot\theta}\frac{d\mathbf{k}}{k_0}\frac{d\mathbf{k}'}{k_0'}\frac{1}{q_0}d\mathbf{q}d\hat{\mathbf{p}} \\
&= (2\pi)^2\int_{\mathcal{V}_m^+}\int_{\mathcal{V}_m^+}\int_{\mathcal{V}_m^+}\phi^*(k)e^{-in\theta}\eta(\hat{\Lambda}^{-1}(q,p)k)\eta^*(\hat{\Lambda}^{-1}(q,p)k')e^{in\theta'}\times \\
&\qquad\qquad \times\psi(k')\frac{m}{m+\mathbf{q}\cdot\theta}\delta(\mathbf{k}-\mathbf{k}')\frac{d\mathbf{k}}{k_0}\frac{d\mathbf{k}'}{k_0'}\frac{1}{q_0}d\mathbf{q} \\
&= (2\pi)^2\int_{\mathcal{V}_m^+}\int_{\mathcal{V}_m^+}\phi^*(k)|\eta(\hat{\Lambda}^{-1}(q,p)k)|^2\psi(k)\frac{m}{m+\mathbf{q}\cdot\theta}\frac{1}{k_0}\frac{d\mathbf{k}}{k_0}\frac{1}{q_0}d\mathbf{q}
\end{aligned}
$$

We have that $\hat{\Lambda}(q,p) = \Lambda_q R(\mathbf{q})$ and, for $p \longrightarrow \hat{p}$ in the affine section, $|J| = 1 + \frac{1}{m}\mathbf{q}\cdot\theta$.

Appendix F

Computations for the coherent states on the cone

We give here the detailed computations regarding the representation of the cone and its coherent states as presented in Chapter 5.

F.1 Matrix representation of Λ_q for the cone

We would like to get an explicit matrix representation of Λ_q in terms of q.

Since Λ_q is the matrix which brings the initial vector $(1, 1, 0)$ to the point q on the cone (that is $q_0^2 - q_1^2 - q_2^2 = 0$), it has the following properties:

1. $\Lambda_q(1, 1, 0)^t = q$ **and** $(1, 1.0)\Lambda_q = q$;

2. $\det \Lambda_q = 1$;

3. $\Lambda_q \eta \Lambda_q^t = \eta$, where η is the metric of signature $(1, -1, -1)$.

The last two properties reflect the fact that $\Lambda_q \in SO(2, 1)$.

Here is how we proceed. Using the first property, we write a matrix with some unknowns. We then compute the third property and get a system of equations to solve. Once they are solved, we plug in the values and check that the determinant is 1.

The starting point is:

$$\begin{pmatrix} \frac{1}{2}(q_0 - q_1) + B & \frac{1}{2}(q_0 + q_1) - B & (1-c)q_2 + C \\ \frac{1}{2}(q_0 + q_1) - B & \frac{1}{2}(-q_0 + q_1) + B & cq_2 - C \\ (1-d)q_2 + D & dq_2 - D & A \end{pmatrix}. \tag{F.1}$$

While solving, we can rewrite $C' = C - cq_2$ and $D' = D - dq_2$. This gives:

$$\Lambda_q = \begin{pmatrix} \frac{1}{2}(q_0 - q_1) + B & \frac{1}{2}(q_0 + q_1) - B & q_2 + C' \\ \frac{1}{2}(q_0 + q_1) - B & \frac{1}{2}(-q_0 + q_1) + B & -C' \\ q_2 + D' & -D' & A \end{pmatrix}. \tag{F.2}$$

After solving for the third property, we get that:

- $A = q_0 - q_1 + 1$;
- $B = \frac{1}{q_0+q_1}\left(1 + \frac{q_0^2}{2} + \frac{q_1^2}{2} + q_0 - q_1\right)$;
- $C' = D' = \frac{q_2}{q_0+q_1}(1 - q_1)$.

This way the determinant is 1 and the matrix satisfies $\Lambda_q \eta \Lambda_q^t = \eta$. Moreover, we remark that it is symmetric.

We can rearrange things to write:

$$\Lambda_q = \frac{1}{q_0 + q_1}\begin{pmatrix} 1 + q_0^2 + q_0 - q_1 & q_0q_1 - 1 - q_0 + q_1 & q_2(1+q_0) \\ q_0q_1 - 1 - q_0 + q_1 & 1 + q_1^2 + q_0 - q_1 & -q_2(1-q_1) \\ q_2(1+q_0) & -q_2(1-q_1) & q_0 + q_1 + q_2^2 \end{pmatrix}. \tag{F.3}$$

The inverse of this matrix is:

$$\Lambda_q^{-1} = \frac{1}{q_0 + q_1}\begin{pmatrix} 1 + q_0^2 + q_0 - q_1 & -q_0q_1 + 1 + q_0 - q_1 & -q_2(1+q_0) \\ -q_0q_1 + 1 + q_0 - q_1 & 1 + q_1^2 + q_0 - q_1 & -q_2(1-q_1) \\ -q_2(1+q_0) & -q_2(1-q_1) & q_0 + q_1 + q_2^2 \end{pmatrix}.$$

It acts on $k_0 = (1, 1, 0)$ in a strange way, but it acts on $\bar{k}_0 = (1, -1, 0)$ as follows:

$$\Lambda_q^{-1}(1, -1, 0)^t = \bar{q}, \qquad (1, -1, 0)\Lambda_q^{-1} = \bar{q}^t, \tag{F.4}$$

which means that the inverse matrix acts on the image of k_0 under parity to give the image of q under parity.

F.2 Checking square-integrability of the UIR

We check that the UIR defined in (5.9) is not square-integrable.

The integration on the full group is the integration on the parameters of the three translations, the rotation and the two boosts. We write $\int_G$ as $\int_V \int_S$ and since x and y are on the cone, the integration of the exponential is only in two dimensions.

$$\begin{aligned}
&\int_G | < U(g)\eta|\phi > |^2 dm(g) \\
&= \int_V \int_S \int_{\mathcal{V}^+} \int_{\mathcal{V}^+} \eta^*(s^{-1}x) e^{in\theta_x} e^{-ix\cdot v} \phi(x) \phi^*(y) e^{iy\cdot v} e^{-in\theta_y} \eta(s^{-1}y) \frac{\mathbf{dx}}{x_0} \frac{\mathbf{dy}}{y_0} dv ds \\
&= (2\pi)^2 \int_{\mathbb{R}} \int_S \int_{\mathcal{V}^+} (\eta^*(s^{-1}x)\phi(x))(\phi^*(x)\eta(s^{-1}x)) \frac{1}{x_0} \frac{\mathbf{dx}}{x_0} ds dv_0 \\
&= (2\pi)^2 \int_{\mathbb{R}} dv_0 \int_S \int_{\mathcal{V}^+} |\eta^*(s^{-1}x)\phi(x)|^2 \frac{1}{x_0} \frac{\mathbf{dx}}{x_0} ds
\end{aligned}$$

We have the integral of a positive quantity on the translation v_0 which is infinite. Then, the UIR is not square-integrable.

F.3 Computation of the integral for the principal section

We compute in details the integral for the coherent states on the cone with the principal section as in 5.4.

$$\begin{aligned}
I_{\Phi,\Psi} &= \int_\Gamma \int_{\mathcal{V}^+} \int_{\mathcal{V}^+} \Phi^*(k) e^{ik\cdot\hat{p}} e^{-it\rho} \eta(\Lambda_q^{-\alpha} k) \eta^*(\Lambda_q^{-\alpha} k') e^{it\rho'} e^{-ik'\cdot\hat{p}} \Psi(k') \times \\
&\qquad \times \frac{dk}{k_0} \frac{dk'}{k'_0} \frac{d\mathbf{q}}{q_0} d\mathbf{p} \\
&= \int_\Gamma \int_{\mathcal{V}^+} \int_{\mathcal{V}^+} \Phi^*(k) e^{ik\cdot\hat{p}} e^{-it\rho} \eta(\Lambda_q^{-\alpha} k) \eta^*(\Lambda_q^{-\alpha} k') e^{it\rho'} e^{-ik'\cdot\hat{p}} \Psi(k') \times \\
&\qquad \times \frac{dk}{k_0} \frac{dk'}{k'_0} \frac{d\mathbf{q}}{q_0} \frac{d\hat{p}}{\lambda^{2\beta}} \\
&= (2\pi)^2 \int_{\mathcal{V}^+} \int_{\mathcal{V}^+} \int_{\mathcal{V}^+} \Phi^*(k) e^{-it\rho} \eta(\Lambda_q^{-\alpha} k) \eta^*(\Lambda_q^{-\alpha} k') e^{it\rho'} \Psi(k') \times \\
&\qquad \times \delta(k-k') \frac{dk}{k_0} \frac{dk'}{k'_0} \frac{d\mathbf{q}}{q_0} \frac{1}{\lambda^{2\beta}} \\
&= (2\pi)^2 \int_{\mathcal{V}^+} \int_{\mathcal{V}^+} \Phi^*(k) \eta(\Lambda_q^{-\alpha} k) \eta^*(\Lambda_q^{-\alpha} k) \Psi(k) \frac{dk}{k_0} \frac{1}{k_0} \frac{d\mathbf{q}}{q_0} \frac{1}{\lambda^{2\beta}}.
\end{aligned}$$

We recall that ρ depends on the group element (q,p) and on the point where we are acting k (or k').

F.4 Computation of the integral for the basic section

We compute in details the integral for the coherent states on the cone with the basic section as in Section 5.5.

$$\begin{aligned}
I_{\phi,\psi} &= <\phi|A_{\sigma_0}\psi> \\
&= \int_{\Gamma} <\phi|\eta_{\sigma_0}><\eta_{\sigma_0}|\psi> \frac{d\mathbf{q}d\mathbf{p}}{q_0} \\
&= \int_{\Gamma}\int_{\mathcal{V}+}\int_{\mathcal{V}+} \phi^*(k)e^{i[p_0(k_0-k_1)/2+p_1(k_0-k_1)/2+p_2k_2]}e^{-it\rho}\eta(\Lambda_q k)\eta^*(\Lambda_q k') \times \\
&\qquad \times e^{-it\rho'}e^{-i[p_0(k_0'-k_1')/2+p_1(k_0'-k_1')/2+p_2k_2']}\psi(k')\frac{dk}{k_0}\frac{dk'}{k_0'}\frac{d\mathbf{q}d\mathbf{p}}{q_0} \\
&= (2\pi)^2\int_{\mathcal{V}+}\int_{\mathcal{V}+}\int_{\mathcal{V}+} \phi^*(k)e^{-it\rho}\eta(\Lambda_q k)\eta^*(\Lambda_q k')e^{-it\rho'}\delta(k-k')\frac{dk}{k_0}\frac{dk'}{k_0'}\frac{d\mathbf{q}}{q_0} \\
&= (2\pi)^2\int_{\mathcal{V}+}\int_{\mathcal{V}+} \phi^*(k)|\eta(\Lambda_q k)|^2\psi(k)\frac{1}{k_0}\frac{dk}{k_0}\frac{d\mathbf{q}}{q_0}
\end{aligned}$$